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by

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MIXED HP-FINITE ELEMENT METHOD FOR LINEAR ELASTICITY WITH WEAKLY IMPOSED SYMMETRY III: STABILITY ANALYSIS IN 3D

WEIFENG QIU* AND LESZEK DEMKOWICZ†

Abstract. The paper presents a generalization of Arnold-Falk-Winther elements for three dimensional linear elasticity, to meshes with elements of variable order. The generalization is straightforward but the stability analysis involves a non-trivial modification of involved interpolation operators. The analysis addresses only the h-convergence.

Key words. elasticity, mixed formulation, hp elements

AMS subject classifications. 65N30, 65L12

1. Introduction. Linear elasticity is a classical subject, and it has been studied for a long time. The paper focuses on the so-called dual–mixed formulation with weakly imposed symmetry that may be derived by considering stationary points of the generalized Hellinger-Reissner functional [16]. We restrict ourselves to the static case only and, for the sake of simplicity, we assume that the body is fixed on the whole boundary. We look for stress tensor $\sigma \in H(\text{div}, \Omega; \mathbb{M})$, displacement vector $u \in L^2(\Omega; \mathbb{V})$, and infinitesimal rotation $p \in L^2(\Omega; \mathbb{K})$ satisfying

$$\begin{aligned} \int_\Omega (A\sigma : \tau + \text{div}\tau \cdot u + \tau : p) d\boldsymbol{x} = 0, \quad \tau \in H(\text{div}, \Omega; \mathbb{M}), \\ \int_\Omega \text{div}\sigma \cdot v d\boldsymbol{x} = \int_\Omega f \cdot v d\boldsymbol{x}, \quad v \in L^2(\Omega; \mathbb{V}), \\ \int_\Omega \sigma : q d\boldsymbol{x} = 0, \quad q \in L^2(\Omega; \mathbb{K}). \end{aligned} \tag{1.1}$$

The first equation represents a relaxed form of the Hooke's law combined with Cauchy geometrical relations, the second one represents the equilibrium equations (in a strong form), and the third one enforces the symmetry of the stress tensor. We refer to the next section for a detailed description of energy spaces: $H(\text{div}, \Omega; \mathbb{M})$, $L^2(\Omega; \mathbb{V})$ and $L^2(\Omega; \mathbb{K})$. The operator A denotes the compliance tensor (operator) mapping stress tensor into strain tensor. The operator is bounded, symmetric, uniformly positive definite, and it preserves the symmetry of the tensor.

The traditional motivation for studying the formulation (1.1) comes from handling nearly incompressible materials. Our interest in the subject stems from a study of a class of (visco)elastic vibration problems for structures with large material contrast, see [17] for a motivating example.

A number of authors have developed approximation schemes based on formulation (1.1), among others see [4, 5, 12, 1, 2, 3, 13, 9, 15, 18, 19, 20, 21]. For a brief description of these methods, we refer to the introduction in [5]. We also refer to the recent work of Cockburn, Gopalakrishnan and Guzman [8] who have developed a new mixed method for linear elasticity using a hybridized version of (1.1).

The work presented in this paper is based on the mixed finite element methods developed by Arnold, Falk and Winther in [4, 12, 5].

The ultimate goal of this work is to lay down theoretical foundations for, and implement a fully automatic hp-adaptive Finite Element (FE) method based on a generalization of the AFW element to meshes with variable order. The generalization builds on the exact grad-curl-div sequence that holds for hp meshes, see [10, 11] and it is rather straightforward. The formulation is easily accommodated in a general hp code supporting the exact sequence.

The h convergence analysis presented in [5] for meshes with arbitrary but uniform polynomial order, does not however generalize immediately to elements with variable order.

With the proof of p and hp convergence as an ultimate goal, our initial efforts start with a less ambitious goal of proving first stability and convergence for uniform h-refinements of meshes of variable order.

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At the first glance, a generalization of the techniques from [5] seems to be easy. But, as we have shown in [17], the (natural generalization of) canonical projection operators defined in [4] do not commute with divergence operator on meshes with variable order, a property essential in the proof of discrete stability. We have resolved this problem by invoking the Projection Based (PB) interpolation operators in [17]. Unfortunately, the PB operators do not commute with an algebraic operator S_{n-2}, introduced in [4], another essential construction in the AFW proof of discrete stability. We have resolved the problem by designing a new, special operator $\tilde{W}_h$ in [17] that satisfies the commutativity property, as needed. Unfortunately, we managed to prove well-definedness of operator $\tilde{W}_h$ only for polynomial orders $0 \leq p \leq 3$, and only for two space dimensions, see [17]. In this contribution, we resolve the problem by designing new PB operators and a a variant of $\tilde{W}_h$, a new operator $\overline{\Pi}^{1,-}_{\tilde{r},h}$, discussed in the text.

An outline of the paper is the follows. Section 2 introduces notations. In Section 3, we define the involved finite element spaces. Section 4 recalls the mixed formulation of linear elasticity with weakly imposed symmetry and states the Brezzi conditions for the stability. In Section 5, we establish all technical results needed for proving the stability. We construct the new PB operators and the operator $\overline{\Pi}^{1,-}_{\tilde{r},h}$. Finally, in Section 6, we prove the Brezzi conditions.

2. Notations. In this section, we introduce some basic notations. We define $\mathbb{M}$ to be the space of 3×3 real matrices, and $\mathbb{V}$ to be $\mathbb{R}^3$. For any 3×3 real matrices A, B, we define

$$A : B = \mathrm{tr}(AB^{\top}).$$

We denote by $\mathbb{S}$ and $\mathbb{K}$ the subspaces of symmetric and anti-symmetric matrices in $\mathbb{R}^{3\times 3}$. Each anti-symmetric matrix can be identified with a vector in $\mathbb{V}$ given by the mapping $\mathrm{vec} : \mathbb{K} \to \mathbb{V}$:

$$\mathrm{vec}\begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Ω is a domain in $\mathbb{R}^3$. For any vector space $\mathbf{X}$ with inner product, we denote by $L^2(\Omega; \mathbf{X})$ the space of square-integrable vector fields on Ω with values in $\mathbf{X}$. In the paper, $\mathbf{X}$ will be $\mathbb{R}$, $\mathbb{V}$, $\mathbb{M}$, or $\mathbb{K}$. When $\mathbf{X} = \mathbb{R}$, we will write $L^2(\Omega)$. The norm associated with $L^2(\Omega; \mathbf{X})$, denoted by $\|\cdot\|_{L^2(\Omega;\mathbf{X})}$, is obtained by taking the square root of the sum of (squared) L^2 norms of individual components of the vector fields on Ω.

Notice that, for scalar-valued functions, norm $\|\cdot\|_{L^2(\Omega)}$ coincides with the standard L^2-norm. The corresponding Sobolev space of order m, which is the subspace of $L^2(\Omega; \mathbf{X})$ consisting of functions with all partial derivatives of order less than or equal to m in $L^2(\Omega; \mathbf{X})$, is denoted by $H^m(\Omega; \mathbf{X})$. The norm associated with $H^m(\Omega; \mathbf{X})$, denoted by $\|\cdot\|_{H^m(\Omega;\mathbf{X})}$, equals to the square root of the sum of (squared) L^2-norms of all partial derivatives with order less than or equal to m, for all components of vector fields on Ω. When $\mathbf{X} = \mathbb{R}$, $\|\cdot\|_{H^m(\Omega)}$ coincides with the standard H^m-norm for scalar-valued functions.

The spaces $H(\mathrm{curl}, \Omega), H(\mathrm{div}, \Omega)$ are defined by

$$\begin{aligned} H(\mathrm{curl}, \Omega) &= \{\boldsymbol{u} \in L^2(\Omega; \mathbb{V}) : \mathrm{curl}\boldsymbol{u} \in L^2(\Omega; \mathbb{V})\} \\ H(\mathrm{div}, \Omega) &= \{\boldsymbol{v} \in L^2(\Omega; \mathbb{V}) : \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\} \end{aligned}$$

with the norms,

$$\begin{aligned} \|\boldsymbol{u}\|_{H(\mathrm{curl},\Omega)} &= (\|\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{V})} + \|\mathrm{curl}\boldsymbol{u}\|^2_{L^2(\Omega;\mathbb{V})})^{1/2}, \quad \boldsymbol{u} \in H(\mathrm{curl}, \Omega) \\ \|\boldsymbol{v}\|_{H(\mathrm{div},\Omega)} &= (\|\boldsymbol{v}\|^2_{L^2(\Omega;\mathbb{V})} + \|\mathrm{div}\boldsymbol{v}\|^2_{L^2(\Omega)})^{1/2}, \quad \boldsymbol{v} \in H(\mathrm{div}, \Omega). \end{aligned}$$

We extend the definitions of ∇ to $\mathbb{V}$-valued functions, curl and div to $\mathbb{M}$-valued functions by applying these operators row-wise. The space $H(\mathrm{curl}, \Omega; \mathbb{M}), H(\mathrm{div}, \Omega; \mathbb{M})$ are defined by

$$\begin{aligned} H(\mathrm{curl}, \Omega; \mathbb{M}) &= \{\sigma \in L^2(\Omega; \mathbb{M}) : \mathrm{curl}\sigma \in L^2(\Omega; \mathbb{M})\} \\ H(\mathrm{div}, \Omega; \mathbb{M}) &= \{\sigma \in L^2(\Omega; \mathbb{M}) : \mathrm{div}\sigma \in L^2(\Omega; \mathbb{V})\} \end{aligned}$$

with the norms,

$$\|\sigma\|_{H(\mathrm{curl},\Omega;\mathbb{M})} = (\|\sigma\|^2_{L^2(\Omega;\mathbb{M})} + \|\mathrm{curl}\sigma\|^2_{L^2(\Omega;\mathbb{M})})^{1/2}$$
$$\|\sigma\|_{H(\mathrm{div},\Omega;\mathbb{M})} = (\|\sigma\|^2_{L^2(\Omega;\mathbb{M})} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega;\mathbb{V})})^{1/2}.$$

$\mathcal{P}_r(\Omega)$ denotes the space of polynomials on Ω with degree less than or equal to r. When r is a negative integer, $\mathcal{P}_r(\Omega) = \{0\}$. $\mathcal{P}_r(\Omega;\mathbb{V}) = [\mathcal{P}_r(\Omega)]^3$. Throughout this paper, we assume that r is a nonnegative integer.

3. Finite element spaces.

3.1. Finite element spaces on a single tetrahedron. Let T be an arbitrary tetrahedron in $\mathbb{R}^3$. We denote by $\triangle_k(T)$, the union of k-dimensional subsimplexes of T. We denote by $\triangle(T)$, the union of all subsimplexes of T.

For any $r \in \mathbb{Z}_+ := \{n \in \mathbb{Z} : n \geq 0\}$, we introduce

$$\mathcal{P}_r\Lambda^3(T) := \mathcal{P}_r(T), \mathcal{P}_r\Lambda^2(T) := \mathcal{P}_r(T;\mathbb{V}),$$

$$\mathring{\mathcal{P}}_r\Lambda^2(T) := \{\omega \in \mathcal{P}_r\Lambda^2(T) : \forall F \in \triangle_2(T), \omega\cdot\boldsymbol{n}|_F = 0\},$$

$$\mathcal{P}_r^-\Lambda^2(T) := \mathcal{P}_{r-1}(T;\mathbb{V}) + \boldsymbol{x}\mathcal{P}_{r-1}(T),$$

$$\mathring{\mathcal{P}}_r^-\Lambda^2(T) := \{\omega \in \mathcal{P}_r^-\Lambda^2(T) : \forall F \in \triangle_2(T), \omega\cdot\boldsymbol{n}|_F = 0\},$$

$$\mathcal{P}_r^-\Lambda^1(T) := \mathcal{P}_{r-1}(T;\mathbb{V}) + \boldsymbol{x}\times\mathcal{P}_{r-1}(T;\mathbb{V}),$$

$$\mathring{\mathcal{P}}_r^-\Lambda^1(T) := \{\omega \in \mathcal{P}_r^-\Lambda^1(T) : \forall F \in \triangle_2(T), \omega - (\omega\cdot\boldsymbol{n})\boldsymbol{n}|_F = 0\}.$$

$$\mathring{\mathcal{P}}_r\Lambda^0(T) := \{u \in \mathcal{P}_r(T) : u|_{\partial T} = 0\}. \tag{3.1}$$

Here $\boldsymbol{n}$ is a normal unit vector on F. For F, an arbitrary face of T, we introduce

$$\mathcal{P}_r^-\Lambda^1(F) := \mathcal{P}_{r-1}(F;\mathbb{R}^2) + \boldsymbol{y}\mathcal{P}_{r-1}(F). \tag{3.2}$$

Here $\boldsymbol{y}$ denote any orthogonal coordinates on F. In [4, 12], spaces in (3.1,3.2) are defined in the language of exterior calculus. Here we rewrite them in the standard language of calculus. Please refer to [4] and [12] for a detailed correspondence.

We denote by $\tilde{r}$ a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \Delta(T)$ and $e \subset f$ then $\tilde{r}(e) \leq \tilde{r}(f)$. We introduce now formally the FE spaces of variable order.

DEFINITION 3.1.

$$\mathcal{P}_{\tilde{r}}\Lambda^3(T) := \mathcal{P}_{\tilde{r}(T)}\Lambda^3(T) = \mathcal{P}_{\tilde{r}(T)}(T),$$

$$\mathcal{P}_{\tilde{r}}\Lambda^2(T) := \{\omega \in \mathcal{P}_{\tilde{r}(T)}\Lambda^2(T) : \forall F \in \triangle_2(T),\, \omega\cdot\boldsymbol{n}|_F \in \mathcal{P}_{\tilde{r}(F)}(F)\},$$

$$\mathcal{P}_{\tilde{r}}^-\Lambda^2(T) := \{\omega \in \mathcal{P}_{\tilde{r}(T)}^-\Lambda^2(T) : \forall F \in \triangle_2(T),\, \omega\cdot\boldsymbol{n}|_F \in \mathcal{P}_{\tilde{r}(F)-1}(F)\},$$

$$\mathcal{P}_{\tilde{r}}^-\Lambda^1(T) := \{\omega \in \mathcal{P}_{\tilde{r}(T)}^-\Lambda^1(T) : \forall F \in \triangle_2(T),\, \omega - (\omega\cdot\boldsymbol{n})\boldsymbol{n}|_F \in \mathcal{P}_{\tilde{r}(F)}^-\Lambda^1(F);$$

$\forall \boldsymbol{t} \in \triangle_1(T), \omega\cdot\boldsymbol{t} \in \mathcal{P}_{\tilde{r}(e)-1}(e)\}$ *where* $\boldsymbol{t}$ *is a tangential vector on* e.

REMARK 3.2. *In the definition of* $\mathcal{P}_{\tilde{r}}^-\Lambda^1(T)$, $\omega - (\omega\cdot\boldsymbol{n})\boldsymbol{n}|_F$ *is a tangential vector field on* F. *So* $\omega - (\omega\cdot\boldsymbol{n})\boldsymbol{n}|_F$ *can be considered as a two-component vector field.*

DEFINITION 3.3. *We define* $\mathcal{P}_{\tilde{r}}\Lambda^3(T;\mathbb{V}) := \mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})$. *We also define* $\mathcal{P}_{\tilde{r}}\Lambda^2(T;\mathbb{V})$, $\mathcal{P}_{\tilde{r}}^-\Lambda^2(T;\mathbb{V})$, *and* $\mathcal{P}_{\tilde{r}}^-\Lambda^1(T;\mathbb{V})$ *as matrix-valued polynomial spaces whose rows stay in* $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$, $\mathcal{P}_{\tilde{r}}^-\Lambda^2(T)$, *and* $\mathcal{P}_{\tilde{r}}^-\Lambda^1(T)$ *respectively.*

REMARK 3.4. *Finite element spaces in definitions 3.1 and 3.3 are the same as those introduced in [17] for* $n = 3$. *In this paper, we just rewrite them in the language of standard calculus.*

3.2. Finite element spaces on a bounded polyhedral domain. Let $\mathcal{T}_h$ be a tetrahedral mesh. Here h represents the biggest diameter of tetrahedrons in $\mathcal{T}_h$. We extend the map $\tilde{r}$ to a mapping from $\triangle(\mathcal{T}_h)$ to $\mathbb{Z}_+$ such that if $e \subset f$, then $\tilde{r}(e) \leq \tilde{r}(f)$. For any $T \in \triangle_3(\mathcal{T}_h)$, the restriction of $\tilde{r}$ to $\triangle(T)$ is represented with the same symbol $\tilde{r}$. We denote by $\triangle_k(\mathcal{T}_h)$ the union of k-dimensional subsimplexes of $\mathcal{T}_h$, and by $\triangle(\mathcal{T}_h)$ the union of all subsimplexes of $\mathcal{T}_h$.

DEFINITION 3.5. *Let $\mathcal{T}_h$ be a tetrahedral mesh. We define $\mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h)$, $\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^2(\mathcal{T}_h)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h)$, $\mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V})$, $\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{V})$,$\mathcal{P}^-_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, and $\mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h;\mathbb{V})$ as spaces of piece-wisely smooth functions or vector fields on $\mathcal{T}_h$ whose restrictions on T are $\mathcal{P}_{\tilde{r}}\Lambda^3(T)$, $\mathcal{P}_{\tilde{r}}\Lambda^2(T)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^2(T)$, $\mathcal{P}^-_{\tilde{r}}\Lambda^1(T)$, $\mathcal{P}_{\tilde{r}}\Lambda^3(T;\mathbb{V})$, $\mathcal{P}_{\tilde{r}}\Lambda^2(T;\mathbb{V})$,$\mathcal{P}^-_{\tilde{r}}\Lambda^2(T;\mathbb{V})$, and $\mathcal{P}^-_{\tilde{r}}\Lambda^1(T;\mathbb{V})$ respectively, for any $T \in \triangle_3(\mathcal{T}_h)$.*

REMARK 3.6. *Obviously, we have*

$$\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h), \mathcal{P}^-_{\tilde{r}}\Lambda^2(\mathcal{T}_h) \subset H(div,\Omega), \qquad \mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h) \subset H(curl,\Omega),$$
$$\mathcal{P}_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{V}), \mathcal{P}^-_{\tilde{r}}\Lambda^2(\mathcal{T}_h;\mathbb{V}) \subset H(div,\Omega;\mathbb{M}), \qquad \mathcal{P}^-_{\tilde{r}}\Lambda^1(\mathcal{T}_h;\mathbb{V}) \subset H(curl,\Omega;\mathbb{M}).$$

Here Ω is an open subset in $\mathbb{R}^3$ with $\overline{\Omega} = \cup_{T\in\mathcal{T}_h} T$. The spaces defined in (3.5) have been introduced in [17] using the language of differential forms.

4. Algebraic operators and some auxiliary properties. In this section, we will introduce two algebraic operators, and prove some of their relevant properties.

DEFINITION 4.1. *We introduce a linear map S_2 defined as follows,*

$$S_2U = (u_{23} - u_{32}, u_{31} - u_{13}, u_{12} - u_{21})^\top.$$

Here U is an arbitrary matrix in $\mathbb{R}^{3\times 3}$.

REMARK 4.2. *It is easy to check that $S_2U = vec(U^\top - U)$.*

DEFINITION 4.3. *We define linear map S_1 as follows,*

$$S_1W = W^\top - tr(W)I.$$

Here W is an arbitrary matrix in $\mathbb{R}^{3\times 3}$.

LEMMA 4.4. *Operator S_1 is invertible. And $S_1^{-1}W = W^\top - \dfrac{1}{2}tr(W)I$.*

LEMMA 4.5. *$divS_1W + S_2curlW = 0, \forall W \in H^1(\Omega,\mathbb{M})$. Here, Ω is any open subset in $\mathbb{R}^3$.*

Proofs of Lemma 4.4 and Lemma 4.5 are straightforward.

LEMMA 4.6. *Let T be a tetrahedron in $\mathbb{R}^3$, and $W \in H^1(T,\mathbb{M})$. Let F be any face of T. If $W\cdot \boldsymbol{t}|_F = 0$ for all tangential vectors $\boldsymbol{t}$ on F, then $S_1W\cdot\boldsymbol{n}|_F = 0$ where $\boldsymbol{n}$ is an unit normal vector on F.*

Proof. According to definition 4.3, we have

$$\begin{aligned}
S_1W\cdot\boldsymbol{n} &= \begin{bmatrix} -w_{22}-w_{33} & w_{21} & w_{31} \\ w_{12} & -w_{11}-w_{33} & w_{32} \\ w_{13} & w_{23} & -w_{11}-w_{22} \end{bmatrix}\cdot\boldsymbol{n} \\
&= [n_1(-w_{22}-w_{33}) + n_2w_{21} + n_3w_{31}, n_1w_{12} + n_2(-w_{11}-w_{33}) + n_3w_{32}, \\
&\qquad n_1w_{13} + n_2w_{23} + n_3(-w_{11}-w_{22})]^\top.
\end{aligned}$$

In the following, we will show that $n_1(-w_{22}-w_{33}) + n_2w_{21} + n_3w_{31} = 0$ on F. The proof of the other two components of $S_1W\cdot\boldsymbol{n}$ being zero on F is similar.

Obviously, $(n_2,-n_1,0)^\top\cdot\boldsymbol{n} = 0$ and $(n_3,0,-n_1)^\top\cdot\boldsymbol{n} = 0$. This implies that $W\cdot(n_2,-n_1,0)^\top = 0$ and $W\cdot(n_3,0,-n_1)^\top = 0$ on F. So we have $-n_1w_{22} + n_2w_{21} = 0$ and $-n_1w_{33} + n_3w_{31} = 0$ on F. This shows that $n_1(-w_{22}-w_{33}) + n_2w_{21} + n_3w_{31} = 0$ on F. □

LEMMA 4.7. *For any $W,Q \in \mathbb{M}$, we have $S_1W : Q = W : S_1Q$.*

Proof.

$$
\begin{aligned}
S_1W : Q = &\begin{bmatrix} -w_{22}-w_{33} & w_{21} & w_{31} \\ w_{12} & -w_{11}-w_{33} & w_{32} \\ w_{13} & w_{23} & -w_{11}-w_{22} \end{bmatrix} : Q \\
=&(-w_{22}-w_{33})q_{11} + w_{21}q_{12} + w_{31}q_{13} \\
&+ w_{12}q_{21} + (-w_{11}-w_{33})q_{22} + w_{32}q_{23} \\
&+ w_{13}q_{31} + w_{23}q_{32} + (-w_{11}-w_{22})q_{33} \\
=&w_{11}(-q_{22}-q_{33}) + w_{12}q_{21} + w_{13}q_{31} \\
&+ w_{21}q_{12} + w_{22}(-q_{11}-q_{33}) + w_{23}q_{32} \\
&+ w_{31}q_{13} + w_{32}q_{23} + w_{33}(-q_{11}-q_{22}) \\
=&W : \begin{bmatrix} -q_{22}-q_{33} & q_{21} & q_{31} \\ q_{12} & -q_{11}-q_{33} & q_{32} \\ q_{13} & q_{23} & -q_{11}-q_{22} \end{bmatrix} = W : S_1Q.
\end{aligned}
$$

□

LEMMA 4.8. *Let T be a tetrahedron in $\mathbb{R}^3$. We take $W \in \mathring{\mathcal{P}}^-_{r+2}\Lambda^1(T;\mathbb{V})$. If*

$$\int_T S_1W : Q = 0, \quad Q \in \mathcal{P}_{r-1}(T;\mathbb{M}), \tag{4.1}$$

then $W = 0$ on T.

Proof. According to Lemma 4.7, we have

$$\int_T W : S_1Q = 0, \quad Q \in \mathcal{P}_{r-1}(T;\mathbb{M}).$$

By the definition of S_1, it is easy to see that $S_1\mathcal{P}_{r-1}(T;\mathbb{M}) \subset \mathcal{P}_{r-1}(T;\mathbb{M})$. According to Lemma 4.4, we conclude that $S_1\mathcal{P}_{r-1}(T;\mathbb{M}) = \mathcal{P}_{r-1}(T;\mathbb{M})$. According to Lemma 4.11 in [4], we have that $W = 0$ on T. □

5. Mixed formulation for the elasticity equations with weakly imposed symmetry. We begin by rewriting formulation (1.1) using operator S_2. The elasticity problem becomes: Find $(\sigma, u, p) \in H(\mathrm{div},\Omega;\mathbb{M}) \times L^2(\Omega;\mathbb{V}) \times L^2(\Omega;\mathbb{V})$ such that

$$
\begin{aligned}
\langle A\sigma,\tau\rangle + \langle \mathrm{div}\tau, u\rangle - \langle S_2\tau, p\rangle = 0,&\ \ \tau \in H(\mathrm{div},\Omega;\mathbb{M}), \\
\langle \mathrm{div}\sigma, v\rangle = \langle f, v\rangle\,,&\ \ v \in L^2(\Omega;\mathbb{V}), \\
\langle S_2\sigma, q\rangle = 0,&\ \ q \in L^2(\Omega;\mathbb{V}).
\end{aligned} \tag{5.1}
$$

Here $\langle\cdot,\cdot\rangle$ is the standard L^2 inner product on Ω. This problem is well-posed in the sense that, for each $f \in L^2(\Omega;\mathbb{V})$, there exists a unique solution $(\sigma, u, p) \in H(\mathrm{div},\Omega;\mathbb{M}) \times L^2(\Omega;\mathbb{V}) \times L^2(\Omega;\mathbb{V})$, and the solution operator is a bounded operator

$$L^2(\Omega;\mathbb{V}) \longrightarrow H(\mathrm{div},\Omega;\mathbb{M}) \times L^2(\Omega;\mathbb{V}) \times L^2(\Omega;\mathbb{V}).$$

See [4] and [12] for the proof.

Next, we consider a finite element discretization of (5.1). For this, we choose families of finite-dimensional subspaces

$$\Lambda_h^2(\mathbb{M}) \subset H(\mathrm{div},\Omega;\mathbb{M}), \Lambda_h^3(\mathbb{V}) \subset L^2(\Omega;\mathbb{V}), \overline{\Lambda}_h^3(\mathbb{V}) \subset L^2(\Omega;\mathbb{V}),$$

indexed by h, and seek the discrete solution $(\sigma_h, u_h, p_h) \in \Lambda_h^2(\mathbb{M}) \times \Lambda_h^3(\mathbb{V}) \times \overline{\Lambda}_h^3(\mathbb{V})$ such that

$$
\begin{aligned}
\langle A\sigma_h,\tau\rangle + \langle \mathrm{div}\tau, u_h\rangle - \langle S_2\tau, p_h\rangle = 0,&\ \ \tau \in \Lambda_h^2(\mathbb{M}), \\
\langle \mathrm{div}\sigma_h, v\rangle = \langle f, v\rangle\,,&\ \ v \in \Lambda_h^3(\mathbb{V}), \\
\langle S_2\sigma_h, q\rangle = 0,&\ \ q \in \overline{\Lambda}_h^3(\mathbb{V}).
\end{aligned} \tag{5.2}
$$

The stability of (5.2) will be ensured by the Brezzi stability conditions:

$$\text{(S1) } \|\tau\|^2_{H(\mathrm{div},\Omega;\mathbb{M})} \le c_1\langle A\tau,\tau\rangle \text{ whenever } \tau\in\Lambda^2_h(\mathbb{M}) \text{ satisfies } \langle \mathrm{div}\tau, v\rangle = 0 \quad (5.3)$$
$$\forall v\in\Lambda^3_h(\mathbb{V}) \text{ and } \langle S_2\tau, q\rangle = 0\ \forall q\in\overline{\Lambda}^3_h(\mathbb{V}),$$

$$\text{(S2) for all nonzero } (v,q)\in\Lambda^3_h(\mathbb{V})\times\overline{\Lambda}^3_h(\mathbb{V}), \text{ there exists nonzero} \quad (5.4)$$
$$\tau\in\Lambda^2_h(\mathbb{M}) \text{ with } \langle \mathrm{div}\tau, v\rangle - \langle S_2\tau, q\rangle \ge c_2\|\tau\|_{H(\mathrm{div},\Omega;\mathbb{M})}(\|v\|_{L^2(\Omega;\mathbb{V})}+\|q\|_{L^2(\Omega;\mathbb{V})}),$$

where constants c_1 and c_2 are independent of h.

For meshes of arbitrary but uniform order, conditions (5.3) and (5.4) have been proved in [4] and [12]. In what follows, we will demonstrate that they are also satisfied for meshes with elements of variable (but limited) order. In this paper, we define $\Lambda^2_h(\mathbb{M}) = \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, and $\Lambda^3_h(\mathbb{V}) = \overline{\Lambda}^3_h(\mathbb{V}) = \mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V})$. We assume that there is $r_{\max}\in\mathbb{N}$ such that for any $h>0$ and $f\in\Delta(\mathcal{T}_h)$,$\tilde{r}(f)\le r_{\max}$.

6. Preliminaries for the proof of stability. From now on, we assume that Ω is a bounded polyhedral domain in $\mathbb{R}^3$. We also use the standard assumptions for shape regular meshes, which means that the ratio between outer diameter and inner diameter of any tetrahedron in any mesh has an uniform upper bound.

In the proof of stability, the following three commuting diagrams are essential.

$$\begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{\mathrm{div}} & L^2(\Omega;\mathbb{V}) \\ \Pi^2_{\tilde{r},h}\downarrow & & \Pi^3_{\tilde{r},h}\downarrow \\ \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\mathrm{div}} & \mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V}) \end{array} \quad (6.1)$$

$$\begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{\mathrm{div}} & L^2(\Omega;\mathbb{V}) \\ \Pi^{2,-}_{\tilde{r},h}\downarrow & & \Pi^3_{\tilde{r},h}\downarrow \\ \mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\Pi^3_{\tilde{r},h}\circ\mathrm{div}} & \mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V}) \end{array} \quad (6.2)$$

$$\begin{array}{ccc} H^1(\Omega;\mathbb{M}) & \xrightarrow{S_1} & H^1(\Omega;\mathbb{M}) \\ \overline{\Pi}^{1,-}_{\tilde{r},h}\downarrow & & \Pi^{2,-}_{\tilde{r},h}\downarrow \\ \mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V}) & \xrightarrow{\Pi^{2,-}_{\tilde{r},h}\circ S_1} & \mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V}) \end{array} \quad (6.3)$$

Here $\Pi^3_{\tilde{r},h}$ is the L^2 orthogonal projection operator onto $\mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V})$. $\Pi^2_{\tilde{r},h}$, $\Pi^{2,-}_{\tilde{r},h}$, and $\overline{\Pi}^{1,-}_{\tilde{r},h}$ are projection operators into $\mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, $\mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$, and $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$ respectively.

In [4, 5], the canonical projection operators introduced by Arnold, Falk and Winther can make (6.1,6.2, 6.3) commute for meshes with uniform order. But for meshes with variable order, the natural generalization of the projection operators fails to make commute both (6.1) and (6.2), see a counter-example presented in the appendix of [17]. To overcome the difficulty, we recalled Projection Based (PB) interpolation operators from [17]. According to Lemma 19 and Lemma 20 in [17], there exists projection based interpolation operator $\Pi^2_{\tilde{r},h}$, which satisfies the following properties.

$$\mathrm{div}\Pi^2_{\tilde{r},h}\tau = \Pi^3_{\tilde{r},h}\mathrm{div}\tau, \quad \tau\in H^1(\Omega;\mathbb{M}). \quad (6.4)$$

$$\|\Pi^2_{\tilde{r},h}\tau\| \le C\|\tau\|_{H^1(\Omega;\mathbb{M})}, \quad \tau\in H^1(\Omega;\mathbb{M}). \quad (6.5)$$

Here C is independent of τ and h. Though Lemma 20 in [17] has been proved for quasi-uniform meshes only, it is straightforward to extend it to get (6.5) for shape regular meshes as well. Please refer to [17] for the details on the PB interpolation operators.

With the PB operators in place, the difficulty shifted to to defining a special projection operator $\overline{\Pi}^{1,-}_{\tilde{r},h}$, denoted by W_h in [17], that makes now (6.3) commute. The commutativity property followed directly from the construction of W_h but proving that it is well-defined, turned out to be difficult. We managed to show only that $\overline{\Pi}^{1,-}_{\tilde{r},h}$ is well-defined for $0 \le \tilde{r} \le 3$ with $n = 2$. In the following, we will use a different reasoning to demonstrate that *there exist* projection operators $\Pi^{2,-}_{\tilde{r},h}$ and $\overline{\Pi}^{1,-}_{\tilde{r},h}$, both well-defined, that make (6.2),(6.3) commute for arbitrary 3D meshes of arbitrary order. Note that the operators will not be constructed explicitly.

6.1. Projection operators on a reference tetrahedron. Let $\hat{T}$ be a fixed tetrahedron in $\mathbb{R}^3$. We are going to design projection operators $\Pi^{2,-}_{\tilde{r},\hat{T}}$ and $\Pi^{1,-}_{\tilde{r},\hat{T}}$ into $\mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\hat{T};\mathbb{V})$, and $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\hat{T};\mathbb{V})$ respectively.

DEFINITION 6.1. *We take $\tilde{r}$ to be a mapping from $\triangle(\hat{T})$ to $\mathbb{Z}_+$ such that if $\hat{e},\hat{f} \in \triangle(\hat{T})$ and $\hat{e} \subset \hat{f}$, then $\tilde{r}(\hat{e}) \le \tilde{r}(\hat{f})$. We put $k = \dim curl_{\hat{\boldsymbol{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}\Lambda^1(\hat{T};\mathbb{V})$.*

We define $\{\hat{\boldsymbol{f}}_{\tilde{r},1},\cdots,\hat{\boldsymbol{f}}_{\tilde{r},k}\}$ as a basis of $curl_{\hat{\boldsymbol{x}}}\mathring{\mathcal{P}}_{\tilde{r}(\hat{T})+1}\Lambda^1(\hat{T};\mathbb{V})$. We define $\{\hat{\boldsymbol{g}}_{\tilde{r},1},\cdots,\hat{\boldsymbol{g}}_{\tilde{r},k}\}$ as a linearly independent subset of $\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{M})$ such that $\{\hat{\boldsymbol{g}}_{\tilde{r},1},\cdots,\hat{\boldsymbol{g}}_{\tilde{r},k}\}\oplus\nabla_{\hat{\boldsymbol{x}}}\mathcal{P}_{\tilde{r}}\Lambda^3(\hat{T};\mathbb{V}) = \mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{M})$. We define $\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t) = (1-t)\hat{\boldsymbol{f}}_{\tilde{r}}(\hat{\boldsymbol{x}}) + t\hat{\boldsymbol{g}}_{\tilde{r}}(\hat{\boldsymbol{x}})$ for any $1 \le i \le k$.

REMARK 6.2. *It is easy to check that $k = \dim\mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{M}) - \dim\nabla_{\hat{\boldsymbol{x}}}\mathcal{P}_{\tilde{r}}\Lambda^3(\hat{T};\mathbb{V})$. Take $r = \tilde{r}(\hat{T})$. Then $\dim curl_{\hat{\boldsymbol{x}}}\mathring{\mathcal{P}}_{r+1}\Lambda^1(\hat{T};\mathbb{V}) = 3(\dim\mathring{\mathcal{P}}_{r+1}\Lambda^1(\hat{T}) - \dim\mathring{\mathcal{P}}_{r+2}\Lambda^0(\hat{T})) = \dfrac{1}{2}(2r+5)r(r-1)$. $\dim\mathcal{P}_{r-1}(\hat{T};\mathbb{M}) - \dim\nabla_{\hat{\boldsymbol{x}}}\mathcal{P}_r\Lambda^3(\hat{T};\mathbb{V}) = 3(\dim\mathcal{P}_{r-1}(\hat{T};\mathbb{V}) - \dim\mathcal{P}_r(\hat{T})/\mathbb{R}) = \dfrac{1}{2}(2r+5)r(r-1)$. For the dimensions of finite element spaces mentioned above, please refer to formula (3.1) in [4] and page 51 in [4].*

DEFINITION 6.3. *For any $t \in [0,1]$, we define the linear operator $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ mapping $H^1(\hat{T};\mathbb{M})$ onto $\mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\hat{T};\mathbb{V})$ by the following conditions.*

$$\int_{\hat{T}} div_{\hat{\boldsymbol{x}}}(\Pi^{2,-}_{\tilde{r},\hat{T},t}\hat{U} - \hat{U})\cdot\hat{\eta}d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V})/\mathbb{R}. \tag{6.6}$$

$$\int_{\hat{T}} (\Pi^{2,-}_{\tilde{r},\hat{T},t}\hat{U} - \hat{U}) : \hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t)d\hat{\boldsymbol{x}} = 0, \quad 1 \le i \le k. \tag{6.7}$$

$$\int_{\hat{F}} [(\Pi^{2,-}_{\tilde{r},\hat{T},t}\hat{U} - \hat{U})\cdot\hat{\boldsymbol{n}}]\cdot\hat{\mu}d\hat{s} = 0, \quad \hat{F} \in \triangle_2(\hat{T}), \quad \hat{\mu} \in \mathcal{P}_{\tilde{r}(\hat{F})}(\hat{F};\mathbb{V}). \tag{6.8}$$

DEFINITION 6.4. *For any $t \in [0,1]$, we define the linear operator $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ mapping $H^1(\hat{T};\mathbb{M})$ into $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\hat{T};\mathbb{V})$ by the following conditions.*

$$\int_{\hat{T}} div_{\hat{\boldsymbol{x}}}S_1(\Pi^{1,-}_{\tilde{r},\hat{T},t}\hat{W} - \hat{W})\cdot\hat{\eta}d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V})/\mathbb{R}. \tag{6.9}$$

$$\int_{\hat{T}} S_1(\Pi^{1,-}_{\tilde{r},\hat{T},t}\hat{W} - \hat{W}) : \hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t)d\hat{\boldsymbol{x}} = 0, \quad 1 \le i \le k. \tag{6.10}$$

$$\int_{\hat{F}} [(\Pi^{1,-}_{\tilde{r},\hat{T},t}\hat{W} - \hat{W})\cdot\hat{\boldsymbol{t}}]\cdot\hat{\mu}d\hat{s} = 0, \quad \hat{F} \in \triangle_2(\hat{T}), \quad \hat{\mu} \in \mathcal{P}_{\tilde{r}(\hat{F})}(\hat{F};\mathbb{V}). \tag{6.11}$$

$$\Pi^{1,-}_{\tilde{r},\hat{T},t}\hat{W}\cdot\hat{\boldsymbol{t}}|_{\hat{e}} = 0, \quad \hat{e} \in \triangle_1(\hat{T}). \tag{6.12}$$

In (6.8), $\hat{\boldsymbol{n}}$ is a unit normal vector on $\hat{F}$. In (6.11), $\hat{\boldsymbol{t}}$ is any tangential vector on $\hat{F}$. Notice that the dimension of tangential vector space on $\hat{F}$ is two. In (6.12), $\hat{\boldsymbol{t}}$ is a tangential vector along $\hat{e}$.

LEMMA 6.5. *For any $\tilde{r}(\hat{T}) \in \mathbb{Z}_+$, operator $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ is a linear projection, and a well-defined operator for all but finitely many values of $t \in [0,1]$.*

Proof. It is easy to see that the conditions (6.6,6.7,6.8) are well-defined for any $\hat{U} \in H^1(\hat{T};\mathbb{M})$. Obviously, if $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ is well-defined, then it is linear and a projection. It is sufficient to show that for any $\hat{U} \in \mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\hat{T};\mathbb{V})$, $\hat{U} = 0$ if $\Pi^{2,-}_{\tilde{r},\hat{T},t}\hat{U} = 0$.

By Theorem 4.12 in [4], $\hat{U} \in \mathring{\mathcal{P}}^-_{\tilde{r}(\hat{T})+1}\Lambda^2(\hat{T};\mathbb{V})$ because

$$\int_{\hat{F}} [\hat{U}\cdot\hat{\boldsymbol{n}}]\cdot\hat{\mu}d\hat{s} = 0, \quad \hat{F} \in \triangle_2(\hat{T}), \quad \hat{\mu} \in \mathcal{P}_{\tilde{r}(\hat{F})}(\hat{F};\mathbb{V}).$$

So it is sufficient to show that, for any $\hat{U} \in \mathring{\mathcal{P}}^-_{\tilde{r}(\hat{T})+1}\Lambda^2(\hat{T};\mathbb{V})$, $\hat{U} = 0$, provided,

$$\int_{\hat{T}} \mathrm{div}_{\hat{\boldsymbol{x}}}\hat{U}\cdot\hat{\eta}d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V})/\mathbb{R}; \tag{6.13}$$

$$\int_{\hat{T}} \hat{U} : \hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t)d\hat{\boldsymbol{x}} = 0, \quad 1 \le i \le k. \tag{6.14}$$

Since $\hat{U} \in \mathring{\mathcal{P}}^-_{\tilde{r}(\hat{T})+1}\Lambda^2(\hat{T};\mathbb{V})$, (6.13) can be integrated by parts to yield

$$\int_{\hat{T}} \hat{U} : \nabla_{\hat{\boldsymbol{x}}}\hat{\eta}d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V}). \tag{6.15}$$

According to the definition of $\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t)$ and the fact that $\mathcal{P}_{\tilde{r}}\Lambda^3(\hat{T};\mathbb{V}) = \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V})$, the assertion is true for $t = 0$. Indeed, when $t = 0$, conditions (6.15,6.14) can be rewritten as

$$\int_{\hat{T}} \hat{U} : \hat{Q}d\hat{\boldsymbol{x}} = 0, \quad \hat{Q} \in \mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{M}).$$

By Lemma 4.11 in [4], we have $\hat{U} = 0$. This implies that $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ is well-defined for $t = 0$.

We denote by $C(t)$ the matrix associated with the left hand side of conditions (6.6,6.7,6.8). Then $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ is well-defined if and only if $C(t)$ is a square nonsingular matrix. We have already known that $C(0)$ is a square nonsingular matrix. So $C(t)$ is a square matrix for any $t \in [0,1]$. Notice that $\det(C(t))$ is a polynomial of a single variable t. Since $\det(C(0)) \neq 0$, then there are at most finitely many $t \in [0,1]$ which make $\det(C(t)) = 0$. This implies that $\Pi^{2,-}_{\tilde{r},\hat{T},t}$ is well-defined for all but finitely many values of $t \in [0,1]$. ∎

LEMMA 6.6. *For any $\tilde{r}(\hat{T}) \in \mathbb{Z}_+$, operator $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ is a well-defined, linear projection operator for all but finitely many values of $t \in [0,1]$.*

Proof. It is easy to see that the conditions (6.9,6.10,6.11, 6.12) are well-defined for any $\hat{W} \in H^1(\hat{T};\mathbb{M})$. Obviously, if $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ is well-defined, then it is linear and a projection. It is sufficient to show that for any $\hat{W} \in \mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\hat{T};\mathbb{V})$ with $\hat{W}\cdot\hat{\boldsymbol{t}}|_{\hat{e}} = 0$, then $\hat{W} = 0$ if $\Pi^{1,-}_{\tilde{r},\hat{T},t}\hat{W} = 0$. Here $\hat{e}$ is any edge of $\hat{T}$, and $\hat{\boldsymbol{t}}$ is a tangential vector along $\hat{e}$.

By Theorem 4.12 in [4], $\hat{W} \in \mathring{\mathcal{P}}^-_{\tilde{r}(\hat{T})+2}\Lambda^1(\hat{T};\mathbb{V})$ because

$$\int_{\hat{F}} [\hat{W}\cdot\hat{\boldsymbol{t}}]\cdot\hat{\mu}d\hat{s} = 0, \quad \hat{F} \in \triangle_2(\hat{T}), \quad \hat{\mu} \in \mathcal{P}_{\tilde{r}(\hat{F})}(\hat{F};\mathbb{V});$$

$$\hat{W}\cdot\hat{\boldsymbol{t}}|_{\hat{e}} = 0, \quad \hat{e} \in \triangle_1(\hat{T}).$$

So it is sufficient to show that, for any $\hat{W} \in \mathring{\mathcal{P}}^{-}_{\tilde{r}(\hat{T})+2}\Lambda^1(\hat{T};\mathbb{V})$, $\hat{W} = 0$, provided,

$$\int_{\hat{T}} \mathrm{div}_{\hat{\boldsymbol{x}}} S_1 \hat{W} \cdot \hat{\eta} d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V})/\mathbb{R}; \tag{6.16}$$

$$\int_{\hat{T}} S_1 \hat{W} : \hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}}, t) d\hat{\boldsymbol{x}} = 0, \quad 1 \le i \le k. \tag{6.17}$$

Notice that $\hat{W} \in \mathring{\mathcal{P}}^{-}_{\tilde{r}(\hat{T})+2}\Lambda^1(\hat{T};\mathbb{V})$. By Lemma 4.6, we have that $S_1\hat{W}\cdot\hat{\boldsymbol{n}}|_{\hat{F}} = 0$ for any $\hat{F} \in \triangle_2(\hat{T})$. So we can integrate thus (6.16) by parts without obtaining any boundary term. Condition (6.16) can be rewritten as follows.

$$\int_{\hat{T}} S_1 \hat{W} : \nabla_{\hat{\boldsymbol{x}}} \hat{\eta} d\hat{\boldsymbol{x}} = 0, \quad \hat{\eta} \in \mathcal{P}_{\tilde{r}(\hat{T})}(\hat{T};\mathbb{V}). \tag{6.18}$$

By the definition of $\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}}, t)$, for $t = 1$, conditions (6.18) and (6.17) can be rewritten as

$$\int_{\hat{T}} S_1 \hat{W} : \hat{Q} d\hat{\boldsymbol{x}} = 0, \quad \hat{Q} \in \mathcal{P}_{\tilde{r}(\hat{T})-1}(\hat{T};\mathbb{M}).$$

Lemma 4.8 implies then that $\hat{W} = 0$. This shows that $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ is well-defined for $t = 1$.

We denote by $C(t)$ the matrix associated with the left hand side of conditions (6.9,6.10,6.11,6.12). Then $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ is well-defined if and only if $C(t)$ is a square non-singular matrix. We have already known that $C(1)$ is a square non-singular matrix. So $C(t)$ is a square matrix for any $t \in [0,1]$. Notice that $\det(C(t))$ is a polynomial of a single variable t. Since $\det(C(1)) \neq 0$, then there are at most finitely many $t \in [0,1]$ which make $\det(C(t)) = 0$. This implies that $\Pi^{1,-}_{\tilde{r},\hat{T},t}$ is well-defined for all but finitely many values of $t \in [0,1]$. □

According to Lemma 6.5 and Lemma 6.6, we can choose $t_r \in [0,1]$ for any $\tilde{r}(\hat{T})$ such that both $\Pi^{2,-}_{\tilde{r},\hat{T},t_r}$ and $\Pi^{1,-}_{\tilde{r},\hat{T},t_r}$ are well-defined. Here t_r depends only on $\tilde{r}(\hat{T})$.

DEFINITION 6.7. *We define operators* $\Pi^{2,-}_{\tilde{r},\hat{T}} := \Pi^{2,-}_{\tilde{r},\hat{T},t_r}$ *and* $\Pi^{1,-}_{\tilde{r},\hat{T}} := \Pi^{1,-}_{\tilde{r},\hat{T},t_r}$.

6.2. Projection operators on a physical tetrahedron. Let T be an arbitrary tetrahedron in $\mathbb{R}^3$. Then there exists an affine mapping from the reference tetrahedron $\hat{T}$ to T, defined by

$$\boldsymbol{x} = A\hat{\boldsymbol{x}} + \boldsymbol{b}. \tag{6.19}$$

Here A is a 3×3 real non-singular matrix, and $\boldsymbol{b}$ is a vector in $\mathbb{R}^3$. In the following, we always relate $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ by (6.19). We take $\tilde{r}$ to be a mapping from $\triangle(T)$ to $\mathbb{Z}_+$ such that if $e, f \in \triangle(T)$ and $e \subset f$, then $\tilde{r}(e) \le \tilde{r}(f)$. In the following, we denote by $\hat{\boldsymbol{x}}(\boldsymbol{x})$ the inverse of the affine mapping described above.

DEFINITION 6.8. *We define the linear operator* $\Pi^{2,-}_{\tilde{r},T}$ *mapping* $H^1(T;\mathbb{M})$ *onto* $\mathcal{P}^{-}_{\tilde{r}+1}\Lambda^2(T;\mathbb{V})$ *by the following conditions.*

$$\int_T div(\Pi^{2,-}_{\tilde{r},T}U - U) \cdot \eta d\boldsymbol{x} = 0, \quad \eta \in \mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})/\mathbb{R}. \tag{6.20}$$

$$\int_T (\Pi^{2,-}_{\tilde{r},T}U - U)(\boldsymbol{x}) : [A\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}}(\boldsymbol{x}), t_r)A^{-1}] d\boldsymbol{x} = 0, \quad 1 \le i \le k. \tag{6.21}$$

$$\int_F [(\Pi^{2,-}_{\tilde{r},T}U - U) \cdot \boldsymbol{n}] \cdot \mu ds = 0, \quad F \in \triangle_2(T), \quad \mu \in \mathcal{P}_{\tilde{r}(F)}(F;\mathbb{V}). \tag{6.22}$$

DEFINITION 6.9. *We define the linear operator* $\Pi^{1,-}_{\tilde{r},T}$ *mapping* $H^1(T;\mathbb{M})$ *into* $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(T;\mathbb{V})$ *by the following conditions.*

$$\int_T divS_1(\Pi^{1,-}_{\tilde{r},T}W - W)\cdot \eta d\boldsymbol{x} = 0, \quad \eta \in \mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})/\mathbb{R}. \tag{6.23}$$

$$\int_T S_1(\Pi^{1,-}_{\tilde{r},T}W - W) : [A\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}}(\boldsymbol{x}), t_r)A^{-1}]d\boldsymbol{x} = 0, \quad 1 \le i \le k. \tag{6.24}$$

$$\int_F [(\Pi^{1,-}_{\tilde{r},T}W - W)\cdot \boldsymbol{t}]\cdot \mu ds = 0, \quad F \in \triangle_2(T), \quad \mu \in \mathcal{P}_{\tilde{r}(F)}(F;\mathbb{V}). \tag{6.25}$$

$$\Pi^{1,-}_{\tilde{r},T}W \cdot \boldsymbol{t}|_e = 0, \quad e \in \triangle_1(T). \tag{6.26}$$

We want to "pull back" $\tilde{r}$ from $\triangle(T)$ to $\triangle(\hat{T})$. We put $\tilde{r}(\hat{e}) = \tilde{r}(e)$ for any $\hat{e} \in \triangle(\hat{T})$. Here $e := A\hat{e} + \boldsymbol{b}$. Then we have the following lemma.

LEMMA 6.10. *For any* $U, W \in H^1(T;\mathbb{M})$*, we define* $\hat{U}, \hat{W} \in H^1(\hat{T};\mathbb{M})$ *by*

$$U(\boldsymbol{x}) = A^{-\top}\hat{U}(\hat{\boldsymbol{x}})A^{\top}, \quad W(\boldsymbol{x}) = A\hat{W}(\hat{\boldsymbol{x}})A^{-1}.$$

Then we have

$$\Pi^{2,-}_{\tilde{r},T}U(\boldsymbol{x}) = A^{-\top}\Pi^{2,-}_{\tilde{r},\hat{T}}\hat{U}(\hat{\boldsymbol{x}})A^{\top}, \quad \Pi^{1,-}_{\tilde{r},T}W(\boldsymbol{x}) = A\Pi^{1,-}_{\tilde{r},\hat{T}}\hat{W}(\hat{\boldsymbol{x}})A^{-1}, \quad \boldsymbol{x} \in T. \tag{6.27}$$

So operators $\Pi^{2,-}_{\tilde{r},T}$ *and* $\Pi^{1,-}_{\tilde{r},T}$ *are well-defined.*

Proof. For the result of $\Pi^{2,-}_{\tilde{r},T}$, the proof is straightforward. For the result of $\Pi^{1,-}_{\tilde{r},T}$, we need utilize the definition of S_1. Notice that

$$S_1W(\boldsymbol{x}) = W(\boldsymbol{x})^{\top} - \mathrm{tr}(W(\boldsymbol{x}))I = A^{-\top}[\hat{W}(\hat{\boldsymbol{x}})^{\top} - \mathrm{tr}(\hat{W}(\hat{\boldsymbol{x}}))I]A^{\top} = A^{-\top}S_1\hat{W}(\hat{\boldsymbol{x}})A^{\top}. \tag{6.28}$$

Using (6.28), it is now straightforward to prove the result for $\Pi^{1,-}_{\tilde{r},T}$. □

LEMMA 6.11. *For any* $U \in H^1(T;\mathbb{M})$*, we have*

$$\Pi^3_{\tilde{r},T} div \Pi^{2,-}_{\tilde{r},T}U = \Pi^3_{\tilde{r},T} divU.$$

Here $\Pi^3_{\tilde{r},T}$ *is the orthogonal projection operator from* $L^2(T;\mathbb{V})$ *onto* $\mathcal{P}_{\tilde{r}}\Lambda^3(T;\mathbb{V})$.

Proof. According to the definition of $\Pi^{2,-}_{\tilde{r},T}$, we have $(I - \Pi^{2,-}_{\tilde{r},T})\Pi^{2,-}_{\tilde{r},T}U = 0$ for any $U \in H^1(T;\mathbb{M})$. So it is sufficient to show that $\Pi^3_{\tilde{r},T}\mathrm{div}U = 0$ for any $U \in H^1(T;\mathbb{M})$ with $\Pi^{2,-}_{\tilde{r},T}U = 0$.

Now, we choose $U \in H^1(T;\mathbb{M})$ with $\Pi^{2,-}_{\tilde{r},T}U = 0$. We only need to show that $\int_T \mathrm{div}U \cdot \overline{\eta} = 0$ for any $\overline{\eta} \in \mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})$. Obviously, we can choose $\boldsymbol{c} \in \mathbb{R}^3$ such that $\overline{\eta} = \eta + \boldsymbol{c}$, where $\eta \in \mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})/\mathbb{R}$. Then we have

$$\int_T \mathrm{div}U \cdot \overline{\eta} d\boldsymbol{x} = \int_T \mathrm{div}U \cdot \eta d\boldsymbol{x} + \int_T \mathrm{div}U \cdot \boldsymbol{c} d\boldsymbol{x} = \int_T \mathrm{div}U \cdot \eta d\boldsymbol{x} + \int_{\partial T}(U\cdot \boldsymbol{n})\cdot \boldsymbol{c} ds.$$

By (6.20),(6.22) and the face that $\Pi^{2,-}_{\tilde{r},T}U = 0$, we have $\int_T \mathrm{div}U\cdot\overline{\eta}d\boldsymbol{x} = 0$. This implies that $\Pi^3_{\tilde{r},T}\mathrm{div}\Pi^{2,-}_{\tilde{r},T}U = \Pi^3_{\tilde{r},T}\mathrm{div}U$ for any $U \in H^1(T;\mathbb{M})$. □

LEMMA 6.12. *For any* $W \in H^1(T;\mathbb{M})$*, we have*

$$\Pi^{2,-}_{\tilde{r},T}S_1\Pi^{1,-}_{\tilde{r},T}W = \Pi^{2,-}_{\tilde{r},T}S_1W.$$

Proof. According to the definition of $\Pi^{1,-}_{\tilde{r},T}$, we have

$$(I-\Pi^{1,-}_{\tilde{r},T})\Pi^{1,-}_{\tilde{r},T}W=0, \quad W\in H^1(T;\mathbb{M}).$$

So it is sufficient to show that $\Pi^{2,-}_{\tilde{r},T}S_1W=0$ for any $W\in H^1(T;\mathbb{M})$ with $\Pi^{1,-}_{\tilde{r},T}W=0$.

Now, we choose $W\in H^1(T;\mathbb{M})$ with $\Pi^{1,-}_{\tilde{r},T}W=0$. By (6.23) and (6.24), we have,

$$\int_T \mathrm{div}S_1W\cdot\eta d\boldsymbol{x}=0, \quad \eta\in\mathcal{P}_{\tilde{r}(T)}(T;\mathbb{V})/\mathbb{R};$$

$$\int_T S_1W:[A\hat{\boldsymbol{h}}_{\tilde{r},i}(\hat{\boldsymbol{x}},t_r)A^{-1}]d\boldsymbol{x}=0, \quad 1\le i\le k.$$

In order to demonstrate that $\Pi^{2,-}_{\tilde{r},T}S_1W=0$, we only need to show that

$$\int_F[S_1W\cdot\boldsymbol{n}]\cdot\mu ds=0, \quad F\in\triangle_2(T), \quad \mu\in\mathcal{P}_{\tilde{r}(F)}(F;\mathbb{V}).$$

According to the definition of S_1, we have,

$$\begin{aligned}S_1W\cdot\boldsymbol{n}&=\begin{bmatrix}-w_{22}-w_{33} & w_{21} & w_{31}\\ w_{12} & -w_{11}-w_{33} & w_{32}\\ w_{13} & w_{23} & -w_{11}-w_{22}\end{bmatrix}\cdot\boldsymbol{n}\\ &=\begin{bmatrix}(n_2w_{21}-n_1w_{22})+(n_3w_{31}-n_1w_{33})\\ -(n_2w_{11}-n_1w_{12})+(n_3w_{32}-n_2w_{33})\\ -(n_3w_{11}-n_1w_{13})-(n_3w_{22}-n_2w_{23})\end{bmatrix}.\end{aligned}$$

Consequently, for any $\mu\in\mathcal{P}_{\tilde{r}(F)}(F;\mathbb{V})$, we have,

$$\begin{aligned}[S_1W\cdot\boldsymbol{n}]\cdot\mu=&\left(W\cdot\begin{bmatrix}n_2\\ -n_1\\ 0\end{bmatrix}\right)\cdot\begin{bmatrix}-\mu_2\\ \mu_1\\ 0\end{bmatrix}\\ &+\left(W\cdot\begin{bmatrix}n_3\\ 0\\ -n_1\end{bmatrix}\right)\cdot\begin{bmatrix}-\mu_3\\ 0\\ \mu_1\end{bmatrix}+\left(W\cdot\begin{bmatrix}0\\ n_3\\ -n_2\end{bmatrix}\right)\cdot\begin{bmatrix}0\\ -\mu_3\\ \mu_2\end{bmatrix}.\end{aligned}$$

By (6.25) and the fact that $\Pi^{1,-}_{\tilde{r},T}W=0$, we conclude that

$$\int_F[S_1W\cdot\boldsymbol{n}]\cdot\mu ds=0, \quad F\in\triangle_2(T), \quad \mu\in\mathcal{P}_{\tilde{r}(F)}(F;\mathbb{V}).$$

Consequently, $\Pi^{2,-}_{\tilde{r},T}S_1W=0$. □

LEMMA 6.13. *There exists $c>0$ such that, for any $U,W\in H^1(T;\mathbb{M})$,*

$$\|\Pi^{2,-}_{\tilde{r},T}U\|_{L^2(T;\mathbb{M})}\le c\|U\|_{H^1(T;\mathbb{M})}; \tag{6.29}$$

$$\|curl\Pi^{1,-}_{\tilde{r},T}W\|_{L^2(T;\mathbb{V})}\le c(h_T^{-1}\|W\|_{L^2(T;\mathbb{M})}+\|W\|_{H^1(T;\mathbb{M})}). \tag{6.30}$$

Here h_T is the outer diameter of T, and c is independent of T. The constant c may depend upon the ratio of outer and inner diameters of T.

Proof. (6.29,6.30) are obtained by standard scaling techniques. The proof for (6.29) is the same as that for Lemma 20 in [17]. The proof for (6.30) is the same as that for Lemma 29 in [17]. □

6.3. Projection operators on tetrahedral meshes. As we stated at the beginning of this section, we use standard assumptions for regular meshes. This means that the ratio between outer diameter and inner diameter of any tetrahedron in any mesh has a uniform upper bound. We are going to extend operators $\Pi^{2,-}_{\tilde{r},T}$ and $\Pi^{1,-}_{\tilde{r},T}$ now to the whole mesh $\mathcal{T}_h$ in such a way that they make (6.2,6.3) commute.

DEFINITION 6.14. *We define mappings* $\Pi^{2,-}_{\tilde{r},h} : H^1(\Omega;\mathbb{M}) \to \mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$ *and* $\Pi^{1,-}_{\tilde{r},h} : H^1(\Omega;\mathbb{M}) \to \mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$ *by*

$$(\Pi^{2,-}_{\tilde{r},h}U)|_T = \Pi^{2,-}_{\tilde{r},T}(U|_T); \quad (\Pi^{1,-}_{\tilde{r},h}W)|_T = \Pi^{1,-}_{\tilde{r},T}(W|_T).$$

Here $T \in \triangle_3(\mathcal{T}_h)$*, and* $U, W \in H^1(\Omega;\mathbb{M})$.

LEMMA 6.15. *For any* $U, W \in H^1(\Omega;\mathbb{M})$*,* $\Pi^{2,-}_{\tilde{r},h}U \in \mathcal{P}^-_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$ *and* $\Pi^{1,-}_{\tilde{r},h}W \in \mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$. *And we have*

$$\Pi^3_{\tilde{r},h} div \Pi^{2,-}_{\tilde{r},h}U = \Pi^3_{\tilde{r},h} div U; \quad \Pi^{2,-}_{\tilde{r},h} S_1 \Pi^{1,-}_{\tilde{r},h} W = \Pi^{2,-}_{\tilde{r},h} S_1 W. \tag{6.31}$$

And there exists a constant $c > 0$*, which is independent of* $\mathcal{T}_h, U, W$ *, so that*

$$\|\Pi^{2,-}_{\tilde{r},h}U\|_{L^2(\Omega;\mathbb{M})} \le c\|U\|_{H^1(\Omega;\mathbb{M})}; \tag{6.32}$$

$$\|curl \Pi^{1,-}_{\tilde{r},h}W|_T\|_{L^2(T;\mathbb{V})} \le c(h_T^{-1}\|W|_T\|_{L^2(T;\mathbb{M})} + \|W|_T\|_{H^1(T;\mathbb{M})}). \tag{6.33}$$

Here $T \in \triangle_3(\mathcal{T}_h)$*, and* h_T *is the outer diameter of* T.

Proof. This is by definitions of $\Pi^{2,-}_{\tilde{r},h}$ and $\Pi^{1,-}_{\tilde{r},h}$, Lemma 6.11, Lemma 6.12, and Lemma 6.13. □

To remove the h_T^{-1} factor in (6.33), we introduce a Clement-type interpolant R_h mapping $H^1(\Omega;\mathbb{M})$ into continuous piece-wise linear M-valued function on $\mathcal{T}_h$ (The operator Π^0_h in Theorem 5.1 of [7], using example 1 in [7]). Then there exists a constant $c > 0$ such that $\forall W \in H^1(\Omega;\mathbb{M}), T \in \mathcal{T}_h$, we have that

$$\|W - R_h W\|_{L^2(T;\mathbb{M})} \le c h_T \|W\|_{H^1(T;\mathbb{M})}; \|W - R_h W\|_{H^1(T;\mathbb{M})} \le c\|W\|_{H^1(\Sigma_T;\mathbb{M})}.$$

Here $\Sigma_T := \bigcup_{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset} T'$. Then we follow [6] and define $\overline{\Pi}^{1,-}_{\tilde{r},h} := \Pi^{1,-}_{\tilde{r},h}(I - R_h) + R_h$.

LEMMA 6.16. $\overline{\Pi}^{1,-}_{\tilde{r},h}$ *maps from* $H^1(\Omega;\mathbb{M})$ *into* $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$.

$$\Pi^{2,-}_{\tilde{r},h} S_1 \overline{\Pi}^{1,-}_{\tilde{r},h} W = \Pi^{2,-}_{\tilde{r},h} S_1 W, \quad W \in H^1(\Omega;\mathbb{M}). \tag{6.34}$$

And there exists a constant $c > 0$ *such that for any* $W \in H^1(\Omega;\mathbb{M})$,

$$\|curl \overline{\Pi}^{1,-}_{\tilde{r},h} W\|_{L^2(\Omega;\mathbb{V})} \le c\|W\|_{H^1(\Omega;\mathbb{M})}. \tag{6.35}$$

Proof. Since R_h maps $H^1(\Omega;\mathbb{M})$ into continuous piece-wise linear M-valued function on $\mathcal{T}_h$, we have $\overline{\Pi}^{1,-}_{\tilde{r},h}$ maps from $H^1(\Omega;\mathbb{M})$ into $\mathcal{P}^-_{\tilde{r}+2}\Lambda^1(\mathcal{T}_h;\mathbb{V})$. The proof for (6.34,6.35) is straightforward. □

7. Stability of the finite element discretization. We will use the following well-known result from partial differential equations, see [14].

LEMMA 7.1. *Let* Ω *be a bounded domain in* $\mathbb{R}^3$ *with a Lipschitz boundary. Then, for all* $\mu \in L^2(\Omega)$*, there exists* $\eta \in H^1(\Omega;\mathbb{V})$ *satisfying* $div\eta = \mu$*. If, in addition,* $\int_\Omega \mu d\boldsymbol{x} = 0$*, then we can choose* $\eta \in \mathring{H}^1(\Omega;\mathbb{V})$.

REMARK 7.2. *The domain* Ω *need* not *be contractible.*

The main result of this paper is the following theorem. In the proof we follow the lines of proof of Theorem 9.1 in [12], Theorem 7.1 in [5] and Theorem 11.4 in [4]. The main difference is in the use of our operator $\overline{\Pi}^{1,-}_{\tilde{r},h}$ in place of the operator $\tilde{\Pi}^{n-2}_h$ from [12].

THEOREM 7.3. *Let Ω be a bounded polyhedral domain in $\mathbb{R}^3$ with a Lipschitz boundary. We assume that the meshes are regular. Then for any $(\omega,\mu) \in \mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V}) \times \mathcal{P}_{\tilde{r}}\Lambda^3(\mathcal{T}_h;\mathbb{V})$, there exists $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$ such that $div\sigma = \mu$, $-\Pi^3_{\tilde{r},\mathcal{T}_{\tilde{r},h}} S_2\sigma = \omega$. And we have*

$$\|\sigma\|_{H(div,\Omega;\mathbb{M})} \le c(\|\omega\|_{L^2(\Omega;\mathbb{V})} + \|\mu\|_{L^2(\Omega;\mathbb{V})}), \tag{7.1}$$

where the constant c is independent of ω, μ and h, but it may depend upon $\max_{T\in\triangle_3(\mathcal{T}_h)} \tilde{r}(T)$.

Proof. We want to show that Brezzi stability conditions (5.3),(5.4) are satisfied. The condition (5.3) is obviously satisfied since, by construction, $\mathrm{div}\mathcal{P}_{\tilde{r}+1}\Lambda^2(\Omega;\mathbb{V}) \subset \mathcal{P}_{\tilde{r}}\Lambda^3(\Omega;\mathbb{V})$ and the fact that A is coercive.

Now we only need to prove that the condition (5.4) is satisfied as well.

(1) By Lemma 7.1, we can find $\eta \in H^1(\Omega;\mathbb{M})$ with $\mathrm{div}\eta = \mu$ and $\|\eta\|_{H^1(\Omega;\mathbb{M})} \le c\|\mu\|_{L^2(\Omega;\mathbb{V})}$.

(2) Since $\omega + \Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta \in L^2(\Omega;\mathbb{V})$, we can apply Lemma 7.1 again to find $\tau \in H^1(\Omega;\mathbb{M})$ with $\mathrm{div}\tau = \omega + \Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta$ and

$$\|\tau\|_{H^1(\Omega;\mathbb{M})} \le c(\|\omega\|_{L^2(\Omega;\mathbb{V})} + \|\Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta\|_{L^2(\Omega;\mathbb{V})}).$$

(3) Since S_1 is an isomorphism from $H^1(\Omega;\mathbb{M})$ to $H^1(\Omega;\mathbb{M})$, we have $\varrho \in H^1(\Omega;\mathbb{M})$ with $S_1\varrho = \tau$, and $\|\varrho\|_{H^1(\Omega;\mathbb{M})} \le c\|\tau\|_{H^1(\Omega;\mathbb{M})}$.

(4) Define $\sigma = \mathrm{curl}\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho + \Pi^2_{\tilde{r},h}\eta \in \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$. According to Lemma 14 in [17], $\mathrm{curl}\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho \in \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$. So we have $\sigma \in \mathcal{P}_{\tilde{r}+1}\Lambda^2(\mathcal{T}_h;\mathbb{V})$.

(5) From step (4), (6.4), step (1), and the fact that $\Pi^3_{\tilde{r},h}$ is a projection, we have

$$\mathrm{div}\sigma = \mathrm{div}\Pi^2_{\tilde{r},h}\eta = \Pi^3_{\tilde{r},h}\mathrm{div}\eta = \Pi^3_{\tilde{r},h}\mu = \mu.$$

(6) Also from step (4),

$$-\Pi^3_{\tilde{r},h}S_2\sigma = -\Pi^3_{\tilde{r},h}S_2\mathrm{curl}\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho - \Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta.$$

Applying, in order, Lemma 4.5, (6.31), (6.34), step (3), (6.31), step (2), and the fact that $\Pi^3_{\tilde{r},h}$ is a projection, we obtain

$$\begin{aligned}-\Pi^3_{\tilde{r},h}S_2\mathrm{curl}\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho &= \Pi^3_{\tilde{r},h}\mathrm{div}S_1\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho = \Pi^3_{\tilde{r},h}\mathrm{div}\Pi^{2,-}_{\tilde{r},h}S_1\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho\\ &= \Pi^3_{\tilde{r},h}\mathrm{div}\Pi^{2,-}_{\tilde{r},h}S_1\varrho = \Pi^3_{\tilde{r},h}\mathrm{div}\Pi^{2,-}_{\tilde{r},h}\tau = \Pi^3_{\tilde{r},h}\mathrm{div}\tau\\ &= \Pi^3_{\tilde{r},h}(\omega + \Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta) = \omega + \Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta.\end{aligned}$$

Combining, we have $-\Pi^3_{\tilde{r},h}S_2\sigma = \omega$.

(7) Finally, we prove the norm bound. From the boundedness of S_2 in L^2, (6.5), and step (1),

$$\|\Pi^3_{\tilde{r},h}S_2\Pi^2_{\tilde{r},h}\eta\|_{L^2(\Omega;\mathbb{V})} \le c\|S_2\Pi^2_{\tilde{r},h}\eta\|_{L^2(\Omega;\mathbb{V})} \le c\|\Pi^2_{\tilde{r},h}\eta\|_{L^2(\Omega;\mathbb{M})} \le c\|\eta\|_{H^1(\Omega;\mathbb{M})} \le c\|\mu\|_{L^2(\Omega;\mathbb{V})}.$$

Combining with the bounds in step (3) and (2), this gives $\|\varrho\|_{H^1(\Omega;\mathbb{M})} \le c(\|\omega\|_{L^2(\Omega;\mathbb{V})} + \|\mu\|_{L^2(\Omega;\mathbb{V})})$. From (6.34), we then have $\|\mathrm{curl}\overline{\Pi}^{1,-}_{\tilde{r},h}\varrho\|_{L^2(\Omega;\mathbb{V})} \le c\|\varrho\|_{H^1(\Omega;\mathbb{M})} \le c(\|\omega\|_{L^2(\Omega;\mathbb{V})} + \|\mu\|_{L^2(\Omega;\mathbb{V})})$. From (6.5) and the bound in Step (1), $\|\Pi^2_{\tilde{r},h}\eta\|_{L^2(\Omega;\mathbb{M})} \le c\|\eta\|_{H^1(\Omega;\mathbb{M})} \le c\|\mu\|_{L^2(\Omega;\mathbb{V})}$. In view of the definition of σ, these two last bounds imply that $\|\sigma\|_{L^2(\Omega;\mathbb{M})} \le c(\|\omega\|_{L^2(\Omega;\mathbb{V})} + \|\mu\|_{L^2(\Omega;\mathbb{V})})$, while $\|\mathrm{div}\sigma\|_{L^2(\Omega;\mathbb{V})} = \|\mu\|_{L^2(\Omega;\mathbb{V})}$ by Step (5), and thus we have the desired bound (7.1). □

We have thus verified the stability conditions (5.3) and (5.4), and so obtain the following quasi-optimal error estimate.

THEOREM 7.4. *Suppose (σ,u,p) is the solution of the elasticity system (5.1) and (σ_h,u_h,p_h) is the solution of discrete system (5.2), where the finite element spaces satisfy the hypotheses of Theorem 7.3. We also assume that there is $r_{\max} \in \mathbb{N}$ such that for any $h > 0$ and $f \in \Delta(\mathcal{T}_h)$,$\tilde{r}(f) \le r_{\max}$. Then there is a constant C, independent of h, such that*

$$\begin{aligned}&\|\sigma-\sigma_h\|_{H(div,\Omega;\mathbb{M})} + \|u-u_h\|_{L^2(\Omega;\mathbb{V})} + \|p-p_h\|_{L^2(\Omega;\mathbb{V})}\\ \le& C\inf(\|\sigma-\tau\|_{H(div,\Omega;\mathbb{M})} + \|u-v\|_{L^2(\Omega;\mathbb{V})} + \|p-q\|_{L^2(\Omega;\mathbb{V})}),\end{aligned}$$

where the infimum is taken over all $\tau \in \mathcal{P}^2_{\tilde{r}+1}(\mathcal{T}_h;\mathbb{V}), v \in \mathcal{P}^3_{\tilde{r}}(\mathcal{T}_h;\mathbb{V})$, and $q \in \mathcal{P}^3_{\tilde{r}}(\mathcal{T}_h;\mathbb{V})$.

8. Conclusions and future work. In the paper, we have presented a generalization of Arnold-Falk-Winther (AFW) elements to the case of elements of variable order for a three dimensional domain. The proof of stability is based on the use of some variant of projection based interpolation operators, and a specially designed operator $\overline{\Pi}^{1,-}_{\tilde{r},h}$ discussed in the text. We have proved the h-stability for meshes with variable order under the assumption that there is an uniform upper bound on the highest polynomial order used.

We plan to continue the research on several fronts. On the numerical side, we intend to implement and test the hp-adaptive algorithm based on the coarse/fine grid paradigm. The code will be applied to a detailed study of problems with large material contrast including the streamer problem, discussed in [17]. The results obtained using the AFW elements will be compared with results obtained using the classical H^1-conforming elements, in terms of memory use and CPU time.

On the theoretical side, we will attempt to prove p-stability and, ultimately, the hp-stability of the method.

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