ME 381P5

Applications of Incompressible Flow

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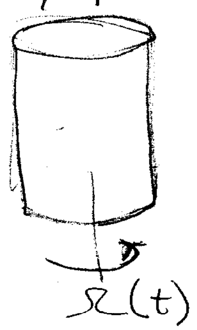
7.142D ETC

What is a viscous fluid? — How does a viscous fluid differ from an ideal fluid?

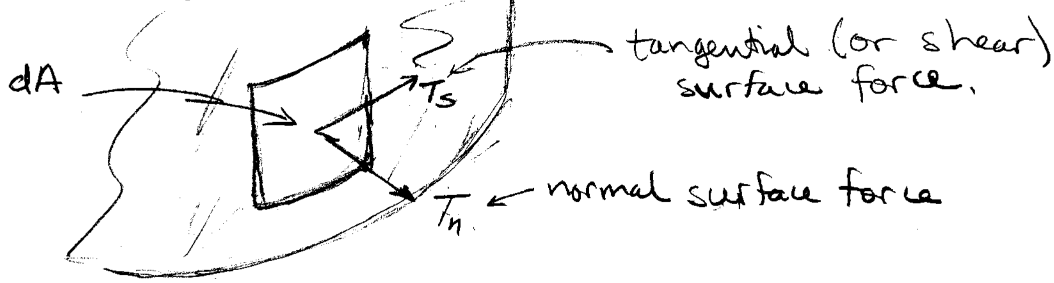
Consider a cylindrical can full of a viscous fluid, and another full of an ideal fluid.

Imagine translating (accelerating) the cans on some arbitrary path — what happens to the fluid?



Imagine rotating the can about the axis of the can. Suppose the fluid is at rest at $t=0$, what happens to the fluid?

The ideal fluid does not support shear forces. On a surface in the fluid (e.g. a surface parallel to the wall of the container)



without shear forces, the spinning can cannot accelerate the fluid.

Definition: a viscous fluid is one in which internal shear forces are possible.

Most fluids are viscous. Can you think of an example of a non-viscous (ideal) fluid?

Internal forces are expressed in terms of the stress tensor $\underline{\underline{\sigma}}$. Recall that stress is the 2nd rank tensor (i.e. linear operator) that maps a unit surface normal to the force per unit area on that surface. If $\underline{n}$ is the unit normal, then

$$\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n}\, dA$$

For $\underline{T}$ to be normal to the surface for all surface orientations, σ must be an <u>isotropic</u> tensor $\underline{\underline{\sigma}} = -p\,\underline{\underline{I}}$ ← identity tensor

so for an ideal fluid

$$\underline{T} = -p\underline{n}\, dA$$

Note that surface force normal to the surface for all orientations implies force/area being independent of direction. Why?

We call the isotropic stress the <u>Pressure</u>. In an ideal fluid pressure is the only stress. In a viscous fluid, we can write

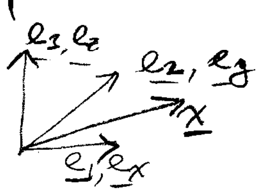
$$\underline{\underline{\sigma}} = -p\underline{\underline{I}} + \underline{\underline{\tau}} \quad \leftarrow \text{viscous stress}$$

We know several properties of stress tensors from general continuum mechanics. Particularly, the stress tensor is symmetric ($\underline{\underline{\sigma}}^T = \underline{\underline{\sigma}}$), and the stress must be independent of certain transformations (e.g. Galelean invariance implies that $\underline{\underline{\tau}}$ cannot depend on velocities, but can depend on velocity gradients). More about this later.

We will need a constituitive relation to express $\underline{\underline{\tau}}$ in terms of other quantities.

<u>Mathematical Notation</u> and Review. (<u>Chapter 3-Read IT</u>)

Language of fluid mechanics is vector and tensor analysis. In a euclidean 3-space ($\mathbb{R}^3$), we have position vector $\underline{x}$. If we define an (orthogonal) basis, $\underline{x} = x_1\underline{e}_1 + x_2\underline{e}_2 + x_3\underline{e}_3 = \underbrace{x_i\underline{e}_i}$



or! $\underline{x} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$ summation assumed! – Einstein Notation

we also write the vector $\underline{x}$ in cartesian tensor notation as x_i

The velocity of a fluid particle is just

$$\underline{v} = \frac{d\underline{x}}{dt}, \quad \text{where } \underline{x} \text{ is the position of the particle.}$$

$$\underline{v} = v_1 \underline{e_1} + v_2 \underline{e_2} + v_3 \underline{e_3} = v_i \underline{e_i}$$

or $\underline{v} = u\underline{e_x} + v\underline{e_y} + w\underline{e_z}$

A second ranked tensor (e.g. $\underline{\underline{\sigma}}$) is a linear operator mapping a vector to a vector (e.g. normal vector to a force vector). Expressing in terms of an orthogonal basis, yields a matrix (3x3 in 3-space).

$$\underline{\underline{\sigma}} \Rightarrow \sigma_{ij}$$

sum on repeated index – Einstein or summation convention

e.g. $\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n} \; dA \Rightarrow T_i = \sigma_{ij} n_j \, dA$

The trace of a 2nd rank tensor is just

$$\text{tr}\, \underline{\underline{\sigma}} = \sigma_{ii}$$

Now $\underline{v}$ or $\underline{\underline{\sigma}}$ (or other vectors or tensors) can be vector or tensor <u>fields</u>, that is they are functions of $\underline{x}$.

Derivative operators:

Gradient operator: $\nabla = \underline{e_1}\frac{\partial}{\partial x_1} + \underline{e_2}\frac{\partial}{\partial x_2} + \underline{e_3}\frac{\partial}{\partial x_3}$

Gradient of a vector $\nabla \underline{V}$ is a 2nd rank tensor

$$\nabla \underline{V} \Rightarrow \frac{\partial v_i}{\partial x_j} \quad \text{velocity gradient tensor } (\underline{\underline{D}})$$

Divergence $\nabla \cdot \underline{V} = \frac{\partial v_i}{\partial x_i} \quad (\text{tr}\, \underline{\underline{D}})$

Curl $\nabla \times \underline{V} = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$

alternating tensor

$$\varepsilon_{ijk} = \begin{cases} 1 & i,j,k = \text{cyclic permutation of } (1,2,3) \\ 0 & i=j,\ j=k \text{ or } i=k \\ -1 & (ijk) = \text{cyclic permutation of } (1,3,2) \end{cases}$$

ε_{ijk} is a pseudo-tensor
$\nabla \times v$ is a pseudo-vector
$\Big\}$ sign depends on handedness of coordinates

<u>Vorticity</u> $\underline{\omega} = \nabla \times \underline{v}$

Digression – 3 vector (tensor) products.

① Diadic Product (outer product)

$\underline{a}\,\underline{b}$ = a second rank tensor $\quad (\underline{a}\,\underline{b})_{ij} = a_i b_j$

② Inner Product

$$a \cdot b = tr(\underline{a}\,\underline{b}) = a_i b_i$$

③ Cross Product or vector product

$(a \times b)_i = \varepsilon_{ijk} a_k b_j$ (a pseudo-vector)

This is the <u>dual vector</u> of the tensor $\underline{a}\,\underline{b}$ (page 44)

Laplacian Operator

$\nabla^2 = \nabla \cdot \nabla$ (divergence of Gradient).

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i} = \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$$

scalar

$$(\nabla^2 \underline{v})_i = \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

Note the index notation we are using (Cartesian index notation) is valid in Cartesian coordinates only!

In other coordinates, more complicated expressions arise for the same differential operators, to account for the derivation of the coordinate directions with spatial location (See Appendix B).

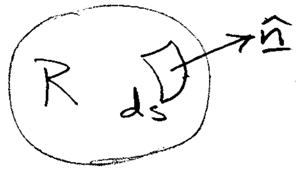
Some more Nomenclature – shorthand

$$\frac{\partial V_i}{\partial x_j} = \partial_j V_i = V_{i,j}$$

vector indices   derivative indices.

Theorems:

Gauss' Theorem



volume differential

$$\int_R \nabla \underline{\underline{T}}\, d\mathbb{V} = \int_S \hat{\underline{n}}\, \underline{T}\, ds$$

was an error

tensor of any rank

R a closed domain with boundary $S = \partial R$

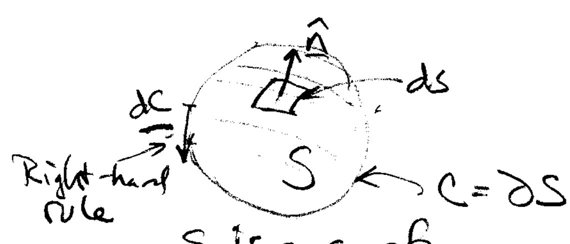
in Cartesian index notation:

$$\int_R \partial_i T_{\{jke--\}}\, d\mathbb{V} = \int_S \hat{\underline{n}}_i\, T_{\{jke..\}}\, dS \quad \leftarrow \text{was an error}$$

Special cases – divergence theorem

$$\int_R \nabla \cdot \underline{V}\, d\mathbb{V} = \int_S \underline{V} \cdot \hat{\underline{n}}\, ds \quad (\text{also } \hat{\underline{n}}\, ds = \underline{\underline{ds}})$$

Stokes Theorem



$$\int_S \hat{\underline{n}} \cdot \nabla \times \underline{V}\, ds = \oint_C V \cdot dc$$

S is a surface with boundary C

Or in index notation

$$\int_S \hat{n}_i\, \varepsilon_{ijk}\, \partial_j v_k\, ds = \oint_C V_k\, dC_k$$

Using Stokes and Gauss' theorem, how would you show that:

$$\nabla \cdot (\nabla \times v) = 0 \qquad \nabla \times (\nabla \phi) = 0$$

Fluid representations: What is the difference between the Eulerian and Lagrangian descriptions of a fluid system?

Reynolds Transport Theorem:

Leibnitz theorem of integral calculus:

Consider a region of space R with boundaries S, where R and its boundaries are changing in time. Let T be any field quantity (scalar, vector, ~~tensor~~).

Leibnitz theorem states

$$\frac{d}{dt}\int_{R(t)} T(\underline{x},t)\, d\underline{V} = \int_R \frac{\partial T}{\partial t}(\underline{x},t)\, dV + \int_S \hat{\underline{n}}\cdot\underline{w}\, T\, ds$$

where $\bar{\underline{n}}$ is the outward unit normal, and $\underline{w}$ is the velocity of of the boundary of R. This is just the multi-dimensional version of the well-known 1-D Leibnitz theorem. (freshman calculus).

$$\frac{d}{dt}\int_a^b f\, dx = \int_a^b \frac{df}{dt}\, dx + f(b)\frac{db}{dt} - f(a)\frac{da}{dt}$$

Now consider a special region R that moves with the fluid, so it always contains the same fluid particles. $\underline{w}$ then becomes the fluid velocity $\underline{V}$ and we have the relationship between the evolution of an integral quantity in a material region, and the evolution of the same quantity in the coincident fixed region.

$$\frac{d}{dt}\int_{R(t)} T \, d\mathcal{V} = \underbrace{\int_{R} \frac{\partial T}{\partial t} \, d\mathcal{V}}_{\text{storage term}} + \underbrace{\int_{S} T \, \underline{V} \cdot \hat{\underline{n}} \, ds}_{\text{flux term.}}$$

This allows us to relate the Lagrangian representation to the Eulerian. Note - it is generally easier to write the laws of mechanics in Lagrangian form! Why?

We can now easily write down the equations of mechanics applied to a "blob" of fluid occupying region $R(t)$.

Mass Conservation: $M_R = \text{const.} \Rightarrow \frac{dM_R}{dt} = 0$

Momentum Conservation $\frac{d\underline{m}_R}{dt} = \Sigma F$

Energy Conservation $\frac{dE}{dt} = \underbrace{\Sigma W}_{\text{work rate}} + \underbrace{\Sigma Q}_{\text{Heat addition rate.}}$

Writing these in integral form and applying RTT

Mass Conservation:

$$M_R = \int_R \rho \, d\forall \qquad (\rho \text{ is fluid density, or mass/unit volume})$$

$$\frac{dM_R}{dt} = \frac{d}{dt}\int_R \rho \, d\forall = \int_R \frac{\partial \rho}{\partial t} d\forall + \int_S \rho \, \underline{V}\cdot\underline{\hat{n}} \, ds = 0$$

Finally the surface integral can be rewritten using the divergence theorem.

$$\int_S \rho \underline{V}\cdot\underline{\hat{n}} \, dS = \int_S (\rho \underline{V})\cdot\underline{\hat{n}} \, ds = \int_R \nabla\cdot(\rho \underline{V}) \, d\underline{\forall}$$

$$\therefore \quad \int_R \left(\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{V})\right) d\underline{\forall} = 0$$

This must be true for any volume R, which can only be so if the integrand is zero! Thus we have derived the differential form of the mass conservation equation!

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{V}) = 0}$$

Momentum Equation:

Similarly, we start with the calculation of $\underline{m}_R$

$$\underline{m}_R = \int_R \rho \underline{V} \, d\forall \,, \quad \text{where } \underline{V} \text{ is the fluid velocity.}$$

or momentum/unit mass

The forces are of two types – volumetric (body) and surface forces.

$$\Sigma F = \int_R F_b \, d\forall + \int_S F_s \, ds$$

where f_b is the force/unit volume (e.g. $\rho \underline{g}$ for gravity).
and f_s is the force/unit area of the surface. But we know
that

stress tensor!

$$f_s = \underline{\underline{\sigma}} \cdot \underline{\hat{n}}$$

Apply the same procedure used in the mass equation:

to get

$$\boxed{\frac{\partial \rho \underline{V}}{\partial t} + \nabla \cdot (\underline{V}\, \rho \underline{V}) - \underline{f_b} - \nabla \cdot \underline{\underline{\sigma}} = 0}$$

Energy Equation:

internal energy/unit mass

$$E = \int_R \rho\left(e + \tfrac{1}{2}\underline{V} \cdot \underline{V}\right) d\forall$$

$$\Sigma W =$$

The heat addition Q is (let us assume) through the surface so $\quad \Sigma Q = \int_S \underset{\sim}{q} \cdot \underline{\hat{n}} \, ds \qquad$ ($\underset{\sim}{q}$ is the heat flux vector)

Again applying the theorems

we get

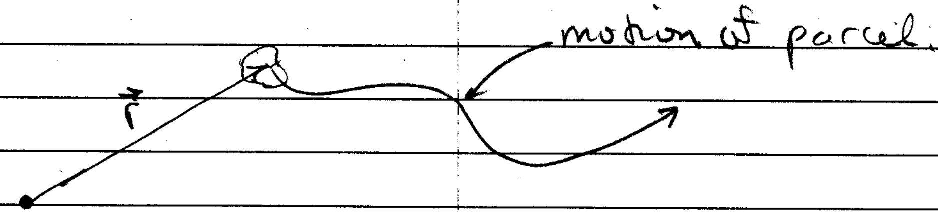
$$\boxed{\frac{\partial \rho (e + \frac{1}{2}\underline{v}\cdot\underline{v})}{\partial t} + \nabla\cdot(\underline{v}\,\rho\,(e+\tfrac{1}{2}\underline{v}\cdot\underline{v})) - \nabla\cdot q - \nabla\cdot(\underline{\underline{\sigma}}\cdot\underline{v}) + \underline{v}\cdot\underline{f_b}}$$

What approximations and assumptions have we made so far?

Eulerian and Lagrangian time derivatives

(Read sections 4.1 - 4.3).

Consider a small parcel of fluid located a position $\underline{r}$ at time t.



the position of the parcel is given by $\underline{r}(t)$ and it velocity by $V_p = \frac{dr}{dt}$. We can label all parcels by their position at $t=0$ (say). Then the velocity field is given by

$$V(\underline{r_0}, t) = \frac{\partial}{\partial t} r(\underline{r_0}, t)\Big|_{r_0}$$

what is the acceleration?

This is the Lagrangian description of velocity, acceleration etc. In contrast, for Eulerian, we consider properties at a fixed point in space ($\underline{x}$ say).

The time derivative of the velocity at $\underline{x}$ is

$$\frac{\partial}{\partial t}\underline{v}(\underline{x},t)\Big|_{\underline{x}}$$

We want to relate the Lagrangian derivative to the Eulerian. Why?

For a general property f we have

Lagrangian $f_L(\underline{r_0},t)$

Eulerian $f_E(x,t)$

Obviously $f_L(\underline{r_0},t) = f_E(\underline{r}(r_0,t),t)$

So

$$\frac{\partial f_L}{\partial t}\Big|_{r_0} = \frac{\partial f_E}{\partial t}\Big|_{x} + \frac{\partial r_i(r_0,t)}{\partial t}\Big|_{r_0}\frac{\partial f_E}{\partial x_i} \quad \text{chain rule}$$

But $\frac{\partial r}{\partial t}\Big|_{r_0} = \underline{v}$

so $$\frac{\partial f_L}{\partial t}\Big|_{r_0} = \frac{\partial f_E}{\partial t}\Big|_{x} + \underline{v}\cdot\nabla f_E$$

$$= \frac{\partial f_E}{\partial t} + v_i\frac{\partial f_E}{\partial x_i} = \frac{Df_E}{Dt}$$

This is called the substantial derivative.

What is the acceleration of a fluid partical in Eulerian representation?

What is Newton's second law ~~to~~ for a fluid particle?
(Volume ΔV)

The force F on the particle will be given by

$$F = \int_{\Delta V} \underbrace{f_b}_{\text{body force/unit volume}} dV + \int_{\Delta S} \underbrace{f_s}_{\text{surface force}} ds = \int_{\Delta V} f_b \, dV + \int_{\Delta S} \hat{n} \cdot \underline{\underline{\sigma}} \, ds$$

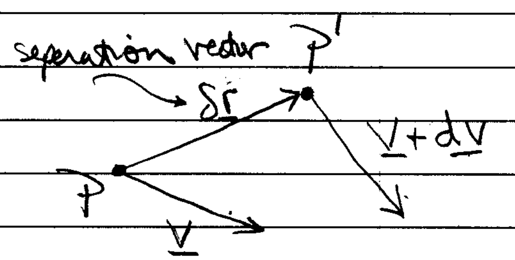
$$F = \int_{\Delta V} (f_b + \nabla \cdot \underline{\underline{\sigma}}) \, dV = \Delta V \, (f_b + \nabla \cdot \underline{\underline{\sigma}})$$

Yielding the final equation:

Kinematics of Fluid Deformation [Sections 4.4 – 4.7 Please Read!]

Why do we care about deformation?

Consider a fluid particle at a point P and and another fluid particle at a point P',



We are interested in how the particle at P' moves relative to that at P! (If the move together, i.e. $d\underline{V} = 0$, nothing interesting happens, why?).

Assuming $\delta \underline{r}$ is small:

$$d\underline{V} = \delta \underline{r} \cdot \underline{\underline{G}} \Rightarrow dV_i = \delta r_j G_{ji} = \delta r_j \frac{\partial v_i}{\partial x_j}$$

Where $\underline{\underline{G}}$ is the second-rank velocity gradient tensor

Symmetric / Anti-symmetric decomposition:

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \frac{1}{2}\left(G_{ji} + G_{ij}\right) \Rightarrow \underline{\underline{S}} = \frac{1}{2}\left(\underline{\underline{G}} + \underline{\underline{G}}^T\right)$$

$$\Omega_{ij} = \tfrac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) = \tfrac{1}{2}\left(G_{ij} - G_{ji}\right)$$

$$\Rightarrow \underline{\underline{\Omega}} = \tfrac{1}{2}\left(\underline{\underline{G}} - G^T\right)$$

$$\underline{\underline{G}} = \underline{\underline{S}} + \underline{\underline{\Omega}} \Rightarrow G_{ij} = S_{ij} + \Omega_{ij}$$

Some Special Cases –

$\underline{\underline{S}} = 0 \quad \Omega \neq 0$

$$dv_i = \Omega_{ji}\, dr_j$$

Proposition:

1) $d\underline{v} \perp d\underline{r}$: Proof $dv_i\, dr_i = \Omega_{ji}\, dr_j\, dr_i$

$$\Omega_{ji}\, dr_j\, dr_i = \Omega_{ji}\, dr_i\, dr_j$$
$$= \Omega_{ij}\, dr_j\, dr_i$$
$$= -\Omega_{ji}\, dr_j\, dr_i$$
$$= 0!$$

2) $\Omega_{ij} = \tfrac{1}{2}\varepsilon_{ijk}\omega_k$ where $\omega_k = \varepsilon_{k\ell m}\dfrac{\partial v_m}{\partial x_\ell}$ "Vorticity"

Proof: $\underline{\omega} = \nabla \times \underline{v}$

$$\tfrac{1}{2}\varepsilon_{ijk}\omega_k = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon_{k\ell m}\frac{\partial v_m}{\partial x_\ell} = \tfrac{1}{2}\left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right)\frac{\partial v_m}{\partial x_\ell}$$

$$= \tfrac{1}{2}\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right) = \Omega_{ij}$$

3) The motion is solid body rotation:

Proof

$$dv_i = \Omega_{ji}\, dr_j = \tfrac{1}{2}\varepsilon_{jik}\,\omega_k\,\delta r_j = \tfrac{1}{2}\varepsilon_{ikj}\,\delta r_j\,\omega_k$$

$$d\underline{v} = \delta \underline{r} \times \tfrac{1}{2}\underline{\omega}$$ which is the velocity distribution for solid body rotation.

Vorticity $\underline{\omega}$ is just ½ the rotation vector.

Ω is the "Rotation Rate" tensor.

Sort out sign!!

Second Special Case:

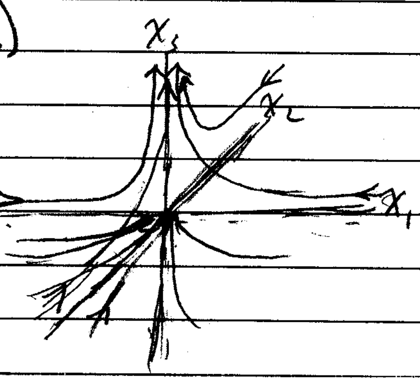
$$\Omega = 0. \quad S \neq 0$$

$$\delta v_i = S_{ji}\, dr_j$$

Suppose $S_{ji} = \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$ (for example)

what happens to 2 fluid particles separated in only the x_1 direction?

IF $s_1 < s_2 < 0 < s_3$ (for example)



A particle eventually leaves asymptotically in the x_3-direction (in this case).

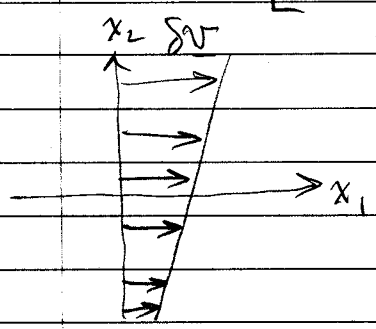
There is always a coordinate system in which $\underline{\underline{S}}$ is diagonal. Why?

These are the principle coordinates (principle axis) of $\underline{\underline{S}}$. Can in general have s_1, s_2, s_3 (the principle strains). $\underline{\underline{S}}$ is the strain-rate or deformation rate of the fluid!

If we started with a spherical blob of fluid, what would it look like after a time.

Third special case - A pure shear

$$\frac{\partial v_j}{\partial x_i} = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & s/2 & 0 \\ s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \Omega = \begin{bmatrix} 0 & -s/2 & 0 \\ s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

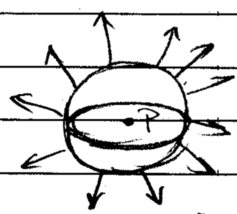


Sheer is a combination of rotation and strain. What is the vorticity? What are the principle axis of the strain? What is the rotation rate?

Final Specicle case:

$\Omega = 0, \quad S_{ij} = s\,\delta_{ij}$ - isotropic strain.

Particles travel radially away from (or toward) P.



Note that $\nabla \cdot \underline{V} = \frac{\partial v_i}{\partial x_i} = 3s$ in this case

$\nabla \cdot \underline{V}$ is the dillation!

If $\mathcal{V}$ is the volume of a fluid parcel, then

$$\nabla \cdot \underline{V} = \frac{1}{\mathcal{V}} \frac{d\mathcal{V}}{dt}$$

When $\nabla \cdot \underline{V} = 0$ (No volume change) the flow is "incompressible". In this case $S_{ii} = 0$

Properties of the Stress tensor.

1) - It's a second rank tensor - this is a general result from continuum mechanics

(i.e. the force on an internal surface element is given by $\hat{n} \cdot \underline{\underline{\sigma}}\, dS$ where σ is the stress tensor).

This is a consequence of momentum conservation on a volume of material as the volume $\to 0$ (see section 5.4)

This is the "Cauchy Stress Principle"

2) The stress tensor is symmetric - Well not in general, but in most fluids - (Section 5.8 - this is somewhat confused in the book).

Angular Momentum Conservation

Angular momentum in region R call it $\hat{M}$ (about O say)

$$\underline{\hat{M}} = \int_{R(t)} \rho(\underline{\theta} + \underline{x} \times \underline{V})\, dV$$

"internal angular momentum/unit mass

What internal angular momentum represent? How does it arise? Why don't we worry about "internal linear momentum"?

Write the Conservation of angular momentum

$$\frac{d\hat{M}}{dt} = \underbrace{\int_S (\underline{x} \times \hat{n} \cdot \underline{\underline{\sigma}})\, ds}_{\text{moment arising from surface forces}} + \underbrace{\int_V (\underline{x} \times \underline{f}_b)\, dV}_{\text{moment due to body forces}}$$

$$+ \int_S \underline{n} \cdot \underline{\underline{C}}\, ds$$

"surface couples" - a second rank tensor

What microscopic process does $\underline{\underline{C}}$ represent?

- We can apply Divergence theorem to first integral on right:

$$\int_S \underline{x} \times \underline{n} \cdot \underline{\underline{\sigma}}\, ds = \int_S (\varepsilon_{ijk}\, x_j \sigma_{pk})\, \hat{n}_p\, ds$$

$$= \int_V \partial_p (\varepsilon_{ijk}\, x_j\, \sigma_{pk})\, dV$$

but $\partial_p(\varepsilon_{ijk}\, x_j\, \sigma_{pk}) = \varepsilon_{ijk}\left(x_j\, \partial_p \sigma_{pk} + (\partial_p x_j)\, \sigma_{pk}\right)$

and $\partial_p x_j = \delta_{pj}$ (why?)

so

$$\partial_p(\varepsilon_{ijk}\, x_j \sigma_{pk}) = \varepsilon_{ijk}\, x_j\, \partial_p \sigma_{pk} + \varepsilon_{ijk}\, \sigma_{jk}$$

$$= \underline{x} \times \nabla \cdot \underline{\underline{\sigma}} + \underline{\sigma_x}$$

$\underline{\sigma_x}$ is the dual vector of $\underline{\underline{\sigma}}$

Applying transport theorem, and rearranging

$$\int_R \underline{x} \times \left(\cancel{\frac{\partial \rho \underline{v}}{\partial t} + \nabla \cdot \rho \underline{v}\underline{v}} \xrightarrow{} 0 - \nabla \cdot \underline{\underline{\sigma}} - f_b \right) dV$$

$$+ \int_R \left(\frac{\partial \rho \underline{\theta}}{\partial t} + \nabla \cdot (\rho \underline{v} \underline{\theta}) - \nabla \cdot \underline{\underline{C}} - \underline{\sigma_x} \right) dV = 0$$

We get an equation for internal angular momentum

$$\frac{\partial \rho \underline{\theta}}{\partial t} + \nabla \cdot (\rho \underline{v} \underline{\theta}) = \nabla \cdot \underline{\underline{C}} + \underline{\sigma_x}$$

In most fluids $\underline{\theta} = 0$ why? In this case, $\underline{\underline{C}} = 0$, why?

Fluids for which $\underline{\theta} \neq 0$ are called polar -

in non polar fluids the internal angular momentum equation reduces to $\underline{\sigma_x} = 0$.

Recall that (as with the gradient tensor)

$$\tfrac{1}{2}(\sigma_{ij} - \sigma_{ji}) \equiv \tfrac{1}{2}\varepsilon_{ijk}\, \sigma_{x\,k} = 0$$

$\Rightarrow$ $\boxed{\underline{\underline{\sigma}} \text{ is symmetric}}$

Constitutive Relation for stress: Assumptions:
(Read 6.1, 6.2)

1) $\underline{\underline{\sigma}}$ is symmetric (no internal angular momentum)
2) $\underline{\underline{\sigma}}$ depends continuously on $\rho, e, \underline{\underline{S}}$ where
$S_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$

Note: Not dependent on $\underline{v}$ → Galilean invariance
Not dependent on Ω → No shear stress in solid body rotation.

3) Fluid is homogeneous → $\underline{\underline{\sigma}}$ dependance on $\underline{x}$ due only to variation of $\rho, e, \underline{\underline{S}}$ with $\underline{x}$

4) Fluid is isotropic → principle directions of $\underline{\underline{\sigma}}$ same as those of $\underline{\underline{S}}$

What are principle directions?, what does this imply about the relation between $\underline{\underline{\sigma}}$ and $\underline{\underline{S}}$?

5) When $\underline{\underline{S}} = 0$ $\sigma_{ij} = -p\,\delta_{ij}$, where p is the thermodynamic pressure.

Note: Earlier we just defined the pressure to be the trace of the stress tensor $p = -\frac{1}{3}\sigma_{ii}$, we might consider this the "mechanical pressure" p_m. Pressure is also a thermodynamic variable p_t, does $p_m = p_t$?

Assumption 5 says that when $\underline{\underline{S}}=0$
$p_m = p_t$.

A fluid satisfying these assumptions is a "Stokesian fluid".

Emperical observation: Many fluids (liquids and gases) are stokesian.

The most general constitutive relation for a stokesian fluid is given by:

$$\sigma_{ij} = \left(A(\rho, e, I_s, II_s, III_s) - p(\rho, e)\right)\delta_{ij}$$

$$B(\rho, e, I_s, II_s, III_s) S_{ij} +$$

$$C(\rho, e, I_s, II_s, III_s) S_{ik} S_{kj}$$

Where I_s, II_s, III_s are scalar invariants of S.

$$I_s = S_{ii}$$
$$II_s = S_{ij} S_{ji}$$
$$III_s = S_{ij} S_{jk} S_{ki} \qquad (\text{Note } A(\rho, e, 0, 0, 0) = 0)$$

They are called invariants because they do not depend on the coordinate system in which they are writen.

Theorem: (by Caley-Hamilton theorem) any other invariant scalar function of S can be determined from I, II, III

A Newtonian fluid is a Stokesian fluid in which $\underline{\underline{\sigma}}$ depends linearly on $\underline{\underline{S}}$.

What are the implications of linearity on A, B, C?

The most general Newtonian Stress Law is then

$$\sigma_{ij} = (\lambda(\rho,e)\, S_{kk} - p(\rho,e))\,\delta_{ij} + 2\mu(\rho,e)\, S_{ij}$$

where p is the thermodynamic pressure, μ and λ are the first and second ~~velocity~~ viscosity coeficient.

What is the mechanical pressure?

$P_m = P_t$ requires $\lambda = -\frac{2}{3}\mu$. Let $K = \lambda + \frac{2}{3}\mu$

K is the "Bulk Viscosity".

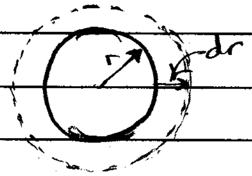
$$\sigma_{ij} = -p\,\delta_{ij} + 2\mu\underbrace{\left(S_{ij} - \tfrac{1}{3} S_{kk}\,\delta_{ij}\right)}_{\text{Deviatoric Strain Rate}} + K\, S_{kk}\,\delta_{ij}$$

What is another expression for $\mathrm{tr}\,S = S_{ii}$?

Note that S_{ii} can also be writen $-\frac{1}{\rho}\frac{D\rho}{Dt}$

Understandiy K:

Consider a spherical region of fluid undergoing uniform expansion + contractions



$S_{ij} = \frac{1}{r}\frac{dr}{dt}\,\delta_{ij}$ div v or

$\sigma_{ij} = \left(-p + \cancel{3}K\frac{1}{r}\frac{dr}{dt}\right)\delta_{ij}$

Work done by/on fluid

$dW_+ = 4\pi r^2 \underline{\underline{\sigma}}\cdot n\, dr =$

If the expansion is reversed at the same rate

$dW_- =$

So the net work is $dW = dW_+ + dW_- = \overset{8}{\cancel{24}}\pi r^2 K \left|\frac{1}{r}\frac{dr}{dt}\right| dr$

This is an irreversibility. From thermodynamics, we expect a volume expansion to be reversible (true if $K=0$!) But if $K \neq 0$, in what limit is the process reversible? What does this mean?

Notes: $K=0$ for monatomic gases.
$K \geq 0$ required by second law of thermo.
($\mu > 0$ too!).

Bulk viscosity generally important only for rapid dilatation - e.g. dissipation of sound waves, shocks, detonations etc. - Also, this linear relation may not be a particularly good model of the relaxation process.

For our purposes - we will generally be able to ignore $K \neq 0$.

Rewriting the momentum equation ($K=0$)

$$\frac{\partial \rho V_i}{\partial t} + \partial_j \rho V_i V_j = \partial_j \left[-P\delta_{ij} + \mu\left(\partial_j V_i + \partial_i V_j - \tfrac{2}{3}\partial_k V_k \delta_{ij}\right)\right] + F_b$$

Fourier Heat Conduction (Read 6.5).

By a similar analysis we assume

$q(\rho, e, \nabla T)$ where T is temperature

linearity in ∇T requires:

$$q = -\underbrace{k(\rho,e)}_{\text{thermal conductivity}}\nabla T \leftarrow \text{heat flow from high to low temp.}$$

Rewriting the energy equation:

Let $E = e + \frac{1}{2}\underline{V}\cdot\underline{V}$ be the total energy/unit volume

$$\frac{\partial \rho E}{\partial t} + \partial_j(V_j \rho E) = \partial_j(\sigma_{ji} V_i - k\partial_j T) + V_j f_{b_j}$$

where

$$\sigma_{ji} = \mu\left(\partial_j V_i + \partial_i V_j - \tfrac{2}{3}\partial_k V_k \delta_{ij}\right) - p\delta_{ij}$$

Let $\mathring{S}_{ij} = \left(\partial_j V_i + \partial_i V_j - \tfrac{2}{3}\partial_k V_k \delta_{ij}\right)$

Then $\sigma_{ji} = -p\,\delta_{ji} + \mu \mathring{S}_{ij}$

The Navier Stokes Equations are Then:

$$\frac{\partial \rho}{\partial t} + \partial_j \rho V_j = 0$$

$$\frac{\partial \rho V_i}{\partial t} + \partial_j \rho \underline{V_i}\underline{V_j} = -\partial_i p + \mu \mathring{S}_{ij} + f_b$$

$$\frac{\partial \rho E}{\partial t} + \partial_j(V_j \rho E) = -\partial_j(pV_j) + \partial_j(\mu \mathring{S}_{ji} V_i) - \partial_j k \partial_j T + V_j f_{b_j}$$

With auxilier definitions:

$E = e + \frac{1}{2}V_i V_i$ $\quad \mathring{S}_{ij}$ as above

Need Equation of state (Review Thermo in Chap. 2)

$p = p(\rho, e) \quad T = T(\rho, e)$

Or more commonly:

$p = p(\rho, T) \quad e = e(\rho, T)$

Example: Ideal Gas

Specific Heat (constant volume).

$$P = \rho R T \qquad e = e_0 + \int_{T_0}^{T} C_v(T)\, dT$$

Need Viscosity Relation. In general, $\mu = \mu(T)$ (no pressure dependence).

Example Sutherlands Law for Gases

$$\mu = \mu_0 \left(\frac{T}{T_0}\right)^{3/2} \frac{C + T_0}{C + T}$$

This is essentially a curve fit

For air $T_0 = 273.2\,K$ $C = 111\,K$

Thermal Conductivity:

Typically $Pr = \dfrac{C_p \mu}{k}$ is a constant (Prandtl #).

C_p is specific heat at constant pressure

For example $Pr = 0.71$ in air.

<u>Boundary Conditions</u> (read 6.4)

At a solid wall there are several conditions

No flow through: fluid doesn't cross the wall

$$\underline{V}_f \cdot \hat{n} = V_s \cdot \hat{n} \quad \text{at the wall}$$

No Slip condition: This is a consequence of viscosity

$$\underline{V}_f = \underline{V}_s \quad \text{at the wall}$$

What are mechanisms causing this? – No slip is best thought of as an emperical observation.

A temperature boundary condition. Commonly

$$T = T_w \quad \text{at the wall}$$

or $\quad q \cdot n = q_w \cdot n \quad$ at the wall

We have been working with the compressible Navier-Stokes equations (Newtonian Viscosity – Fourier Heat Conduction).

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V_j}{\partial x_j} = 0$$

$$\frac{\partial \rho V_i}{\partial t} + \frac{\partial \rho V_i V_j}{\partial x_j} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

$$\frac{\partial \rho(e+\frac{1}{2}V^2)}{\partial t} + \frac{\partial \rho V_j(e+\frac{1}{2}V^2)}{\partial x_j} + \frac{\partial V_j P}{\partial x_j} = \frac{\partial}{\partial x_j}(\tau_{ij}V_i) - \frac{\partial q_j}{\partial x_j} + V_i f_{b_i}$$

$$\tau_{ij} = 2\mu \overset{\circ}{S}_{ij} \qquad q_j = -k\frac{\partial T}{\partial x_j} \qquad P = P(\rho, T) \quad e = e(\rho, T)$$

$$\mu = \mu(T) \qquad k = k(T)$$

Interested in other forms and simplifications of these equations. Want something easier to analyse. To this end we will work through several transformations.

Continuity (by chain rule):

$$\frac{D\rho}{Dt} = -\rho\frac{\partial V_j}{\partial x_j}$$

Momentum (subtracting $V_i \cdot$ continuity):

$$\rho\frac{DV_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

Energy: Subtract kinetic energy equation (eq. 5.10.1 in Panton)

$$\frac{\partial \rho e}{\partial t} + \frac{\partial \rho e}{\partial x_j} + P\frac{\partial V_j}{\partial x_j} = \tau_{ij} G_{ij} - \frac{\partial q_j}{\partial x_j}$$

Subtract $e \cdot$ continuity

$$\rho\frac{De}{Dt} + P\frac{\partial V_j}{\partial x_j} = \tau_{ij}S_{ij} - \frac{\partial q_j}{\partial x_j}$$

Substitute for $\frac{\partial V_j}{\partial x_j}$ from continuity

$$\frac{De}{Dt} - \frac{P}{\rho^2}\frac{D\rho}{Dt} = \frac{1}{\rho}\left(\tau_{ij}S_{ij} - \frac{\partial q_j}{\partial x_j}\right).$$

Now from thermodynamics we have

$$T\,ds = de + p\,d(1/\rho)$$

where s is the entropy/unit mass.

$$d(1/\rho) = -\frac{1}{\rho^2}d\rho \Rightarrow T\,ds = de - \frac{p}{\rho^2}d\rho$$

What are the restrictions on the validity of this expression?

This is thus an entropy equation:

① $$\frac{Ds}{Dt} = \frac{1}{\rho T}\left(\tau_{ij}S_{ij} - \frac{\partial q_j}{\partial x_j}\right)$$

what does this imply about the role of viscous stresses and conductive heat flux?

The equations of state can be solved for p in terms of s, ρ so $p = p(s,\rho)$. Thus:

$$\frac{Dp}{Dt} = \frac{Ds}{Dt}\frac{\partial p}{\partial s}\Big|_\rho + \frac{D\rho}{Dt}\frac{\partial p}{\partial \rho}\Big|_s$$

Note that $\frac{\partial p}{\partial \rho}\Big|_s = a^2$ where a is the sound speed.

Also, by Maxwell's relations: $\frac{\partial p}{\partial s}\Big|_\rho = -\frac{\partial T}{\partial(1/\rho)}\Big|_s = \rho^2\frac{\partial T}{\partial \rho}\Big|_s$

substituting in ①

$$\frac{Dp}{Dt} - a^2\frac{D\rho}{Dt} = \frac{\rho}{T}\frac{\partial T}{\partial \rho}\Big|_s\left(\tau_{ij}S_{ij} - \frac{\partial q_j}{\partial x_j}\right)$$

or

$$\frac{Dp}{Dt} + a^2\rho\frac{\partial v_j}{\partial x_j} = \frac{\rho}{T}\frac{\partial T}{\partial \rho}\Big|_s\left(\tau_{ij}S_{ij} - \frac{\partial q_i}{\partial x_j}\right)$$

To make further simplifications, we will need to nondimensionalize the equations. Why?

The trick is to figure out what the scaling quantities are. Assume there is a length scale δ and a velocity scale U associated with the flow being analyzed. Also that there is a reference thermodynamic state. What might these represent.

We therefore define the nondimensional quantities

$$x_j^* = x_j/\delta \qquad v_i^* = v_i/U \qquad \rho^* = \rho/\hat{\rho} \qquad \cancel{p^* = p/\hat{p}} \qquad f_{bi}^* = \frac{\delta f_{bi}}{\hat{\rho} U^2}$$

$$T^* = T/\hat{T} \qquad \mu^* = \mu/\hat{\mu} \qquad k^* = k/\hat{k} \qquad a^* = a/\hat{a} \qquad t^* = \frac{tU}{\delta}$$

Note that $\hat{p}, \hat{\rho}, \hat{T}, \hat{\mu}, \hat{k}, \hat{a}$ are values at the reference thermodynamic state, and $\hat{p}, \hat{\rho}, \hat{T}, \hat{a}$ are related through the equation of state. Also

$$\tau_{ij}^* = 2\,\mu/\hat{\mu}\; \mathring{S}_{ij}\, \delta/U = 2\mu^* \mathring{S}_{ij}^* = \frac{\delta}{\hat{\mu} U}\tau_{ij}$$

$$q_j^* = -\,k/\hat{k}\, \frac{\partial T}{\partial x_j}\, \delta/\hat{T} = \frac{\delta}{\hat{k}\hat{T}} q_j$$

In particular $p/\rho a^2$ is nondimensional and generally of order 1. For example for an ideal gas $p/\rho a^2 = 1/\gamma = c_v/c_p$.

So we can redefine $\underline{\underline{p^* = p/\hat{\rho}\hat{a}^2}}$.

Aside: Notice that

$$\alpha_1 = \frac{\rho a^2}{P} = \frac{\rho}{P}\frac{\partial P}{\partial \rho}\Big|_s$$ is order 1 (γ for ideal gas, $\gamma = c_p/c_v$)

Similarly:

$$\alpha_2 = \frac{\rho}{T}\frac{\partial T}{\partial \rho}\Big|_s$$ is order 1 ($\gamma - 1$ for an ideal gas).

In any case, these quantities are thermodynamic, and are determined by the equation of state.

Non dimensionalizing equations:

Continuity: (multiply by $\delta / U\hat{\rho}$).

$$\frac{D\rho^*}{Dt^*} = -\rho^* \frac{\partial V_i^*}{\partial x_i^*}$$

Momentum: (multiply by $\frac{\delta}{\hat{\rho} U^2}$)

$$\rho^* \frac{DV_i^*}{Dt^*} = -\frac{\hat{a}^2}{U^2}\frac{\partial p^*}{\partial x_i^*} + \frac{\hat{\mu}}{\hat{\rho}\delta U}\frac{\partial T_{ij}^*}{\partial x_j^*} + f_{bi}^*$$

Energy / Pressure (multiply by $\frac{\delta}{\hat{\rho}\hat{a}^2 U}$):

$$\frac{DP^*}{Dt^*} + a^{*2}\rho^*\frac{\partial V_j^*}{\partial x_j^*} = \frac{\rho^*\partial T^*}{T^*\partial \rho^*}\Big|_s \left(\frac{\hat{\mu}}{\hat{\rho}U\delta}\frac{U^2}{\hat{a}^2} T_{ij}^* S_{ij}^* - \frac{\hat{K}\hat{T}}{\hat{\rho}U\delta\hat{a}^2}\frac{\partial q_i^*}{\partial x_j^*}\right)$$

There are several Non-dimensional #s of relevance

$$\frac{\hat{\rho} U \delta}{\hat{\mu}} = Re \quad \text{Reynolds \#}$$

$$\frac{U}{\hat{a}} = M_a \quad \text{Mach \#}$$

$$\frac{\hat{\mu}\hat{C}_p}{\hat{K}} = P_r \quad \text{Prandtl \#}$$

$C_p = \left(\frac{\partial h}{\partial T}\right)_P$ = Specific heat at constant P.

enthalpy

Note $$\frac{\hat{K}\hat{T}}{\hat{\rho}U\delta\hat{a}^2} = \frac{\hat{\mu}}{\hat{\rho}U\delta}\frac{\hat{K}}{\hat{\mu}\hat{C}_p}\frac{\hat{C}_p\hat{T}}{\hat{a}^2} = \frac{1}{Re}\frac{1}{P_r}\frac{\hat{C}_p\hat{T}}{\hat{a}^2}$$

$\alpha_3 = \frac{C_p T}{a^2}$ is a nondimensional thermodynamic quantity of order 1 ($\frac{1}{\gamma - 1}$ for an ideal gas).

The equations again are (dropping the *'s).

$$\frac{D\rho}{Dt} = -\rho \frac{\partial V_i}{\partial x_i}$$

$$\rho \frac{DV_i}{Dt} = -\frac{1}{M_a^2}\frac{\partial P}{\partial x_i} + \frac{1}{Re}\frac{\partial \tau_{ij}}{\partial x_j} + f_{bi}$$

$$\frac{DP}{Dt} + \alpha_1 P \frac{\partial V_j}{\partial x_j} = \frac{\alpha_2 M_a^2}{Re}\tau_{ij}S_{ij} - \frac{\alpha_2 \hat{\alpha}_3}{Re Pr}\frac{\partial q_j}{\partial x_j}$$

For an ideal gas, the Pressure equation is

$$\frac{DP}{Dt} + \gamma P \frac{\partial V_j}{\partial x_j} = \frac{(\gamma-1) M_a^2}{Re}\tau_{ij}S_{ij} - \frac{1}{Re Pr}\frac{\partial q_j}{\partial x_j}$$

Now, what happens in the limit $Ma \to 0$, or $Re \to \infty$?

Examine $Ma \to 0$ first. To answer this, we must sort out the scaling of velocity, pressure, density and temperature fluctuations with Ma. To keep pressure term in momentum equation finite, P fluctuation must scale $\sim Ma^2$ so

$$P = P^0 + Ma^2 P^2 + \ldots \qquad P^0 = \frac{\hat{P}}{\hat{\rho}\hat{a}^2} = 1/\hat{\alpha}_1$$

velocity fluctuations are order 1 so

$$V_i = V_i^0 + Ma V_i^1 + Ma^2 V_i^2 + \ldots$$

From equation of state Ma^2 pressure fluctuations suggest that T and ρ fluctuations $\sim Ma^2$ so

$$T = 1 + Ma^2 T^2 + \ldots$$
$$\rho = 1 + Ma^2 \rho^2 + \ldots$$

Is the Ma^2 scaling of density and Temp. fluctuations necessary?

- Substituting in the equations and collecting terms of equal order we get

$$\frac{\partial V_i^0}{\partial x_i} = -Ma\frac{\partial V_i^1}{\partial x_i} + O(Ma^2)$$

$$\frac{DV_i^0}{Dt} = -\frac{\partial P_i^2}{\partial x_i} + \frac{1}{Re}\frac{\partial \tau_{ij}^0}{\partial x_j} + f_{bi} - Ma\left(\frac{DV_i^1}{Dt} + \frac{\partial P_i^3}{\partial x_i} - \frac{1}{Re}\frac{\partial \tau_{ij}^1}{\partial x_j}\right) + O(Ma^2)$$

$\mu = 1 + Ma^2\mu^2 + \cdots \quad \tau_{ij}^0 = 2\overset{\circ}{S}_{ij} \quad \tau_{ij}^1 = 2\overset{\circ}{S}{}^1_{ij}$

↑ why?

$$\frac{\alpha_i^0}{\hat{\alpha}_i}\frac{\partial V_j^0}{\partial x_j} = -\frac{\alpha_i^0}{\hat{\alpha}_1}Ma\frac{\partial V_j^1}{\partial x_j} + O(Ma^2)$$

In limit as $Ma \to 0$

$$\frac{\partial V_i^0}{\partial x_i} = 0 \qquad \frac{DV_i^0}{Dt} = -\frac{\partial P_i^2}{\partial x_i} + \frac{1}{Re}\frac{\partial \tau_{ij}^0}{\partial x_j} + f_{bi}$$

What equation is this? What is τ_{ij}^0? What do we make of P_i^2 in the momentum equation?

Suppose T and ρ fluctuations go like Ma, how can this be if P fluctuations $\sim Ma^2$?

$$T = 1 + Ma\,T^1 + Ma^2 T^2 + \cdots$$
$$\rho = 1 + Ma\,T^1 + Ma^2 T^2 + \cdots$$

then we get

$$\frac{\partial V_i^0}{\partial x_i} = -Ma\left(\frac{D\rho^1}{Dt} + \frac{\partial V_i^1}{\partial x_i} - \rho_1 \frac{\partial V_i^0}{\partial x_i}\right)$$

$$\frac{DV_i^0}{Dt} = -\frac{\partial P_i^2}{\partial x_i} + \frac{1}{Re}\frac{\partial \tau_{ij}^0}{\partial x_j} + f_{bi} - Ma\left(\frac{DV_i^1}{Dt} + \frac{\partial P_i^3}{\partial x_i} - \frac{1}{Re}\frac{\partial \tau_{ij}^1}{\partial x_j}\right) + O(Ma^2)$$

$$\mu = 1 + Ma\,\mu^1 + Ma^2\mu^2 + \cdots \quad \tau_{ij}^0 = 2S_{ij}^0 \quad \tau_{ij}^1 = 2\mathring{S}_{ij}^1 + 2\mu^1 \mathring{S}_{ij}^0$$

$$\frac{\alpha_1^0}{\hat\alpha_1}\frac{\partial V_i^0}{\partial x_i} = -Ma\left(\frac{\alpha_1^0}{\hat\alpha_1}\frac{\partial V_i^1}{\partial x_i} + \frac{\alpha_1^1}{\hat\alpha_1}\frac{\partial V_i^0}{\partial x_i} + \frac{\alpha_2^0\hat\alpha_3}{Re Pr}\frac{\partial q_j^1}{\partial x_j}\right) + O(Ma^2)$$

Note that as $Ma \to 0$ we again get the incompressible equations.
But look at the order Ma equations for P and ρ.

$$P: \quad \frac{\alpha_1^0}{\hat\alpha_1}\frac{\partial V_i^1}{\partial x_i} = -\frac{\hat\alpha_2\hat\alpha_3}{Re Pr}\frac{\partial q_j^1}{\partial x_j}$$

$$\rho: \quad \frac{D\rho^1}{Dt} = -\frac{\partial V_i^1}{\partial x_i}$$

(2) so $$\frac{D\rho^1}{Dt} = \frac{\hat\alpha_2\hat\alpha_3}{Re P_r}\frac{\partial q_j^1}{\partial x_j}$$

Note $\underbrace{\alpha_1 = \alpha_1^0}_{\hat\alpha_1} + Ma\,\alpha_1^1(T) + \cdots \qquad \underbrace{\alpha_2 = \alpha_2^0}_{\hat\alpha_2} + Ma\,\alpha_2^1(T) + \cdots$

$$q_j = -Ma\,\underbrace{\frac{\partial T^1}{\partial x_j}}_{q_j^1} + \cdots \qquad \text{(why?)}$$

Now from the equation of state: $\quad P = P(\rho, T) = \frac{\hat P}{\rho \hat a^2} + Ma^2 P^2$

so at order Ma, P is constant $\Rightarrow \rho^1 = \left.\frac{\partial \rho}{\partial T}\right|_P T^1$

let $\alpha_4 = \left.\frac{\rho}{T}\frac{\partial T}{\partial \rho}\right|_P$

Then 2 is written.

$$\frac{DT'}{Dt} = -\frac{\hat{\alpha}_2 \hat{\alpha}_3 \hat{\alpha}_4}{Re\,Pr} \frac{\partial^2 T'}{\partial x_i \partial x_i}$$

$$\alpha_2 \alpha_3 \alpha_4 = \frac{\rho}{T}\frac{\partial T}{\partial \rho}\Big|_s \; c_p T \frac{\partial \rho}{\partial p}\Big|_s \; \frac{\rho}{T}\frac{\partial T}{\partial \rho}\Big|_p$$

$$= \frac{\rho^2 c_p}{T}\frac{\partial T}{\partial p}\Big|_s \frac{\partial T}{\partial \rho}\Big|_p$$

Thermodynamic Relations:

$$c_p = \frac{\partial h}{\partial T}\Big|_p \qquad \frac{\partial T}{\partial p}\Big|_s = -\frac{1}{\rho^2}\frac{\partial \rho}{\partial s}\Big|_p \qquad \frac{\partial h}{\partial s}\Big|_p = T$$

$$\Rightarrow \quad \alpha_2 \alpha_3 \alpha_4 = -1$$

$$\frac{DT'}{Dt} = \frac{1}{Re\,Pr}\frac{\partial^2 T'}{\partial x_j \partial x_j}$$

So the order Ma temperature fluctuations satisfy the passive scalar equation.

Finally, let us consider the case of finite Temperature and density fluctuations:

$$T = T^0 + Ma\,T' + \cdots$$
$$\rho = \rho^0 + Ma\,T' + \cdots$$

What does the equation of state reduce to for T_0, ρ_0?
What about μ and K?

Again substituting in the equations:

$$\frac{D\rho^0}{Dt} = -\rho^0 \frac{\partial V_i^0}{\partial x_i} + O(Ma)$$

$$\rho^\circ \frac{DV_i^\circ}{Dt} = -\frac{\partial P^2}{\partial x_i} + \frac{1}{Re}\frac{\partial \tau_{ij}^\circ}{\partial x_j} + f_{bi} + O(Ma)$$

$$\tau_{ij}^\circ = 2\mu^\circ \mathring{S}_{ij}^\circ$$

$$\frac{\alpha_1^\circ}{\hat\alpha_1}\frac{\partial V_j^\circ}{\partial x_j} = -\frac{\alpha_2^\circ\hat\alpha_3}{RePr}\frac{\partial q_j^\circ}{\partial x_j} + O(Ma)$$

$$\alpha_1^\circ = \alpha_1(T^\circ) \qquad \alpha_2^\circ = \alpha_2(T^\circ) \qquad q_j^\circ = -k^\circ \frac{\partial T^\circ}{\partial x_j}$$

Again, to order Ma, we can write

$$\frac{D\rho^\circ}{Dt} = \rho^\circ \frac{\hat\alpha_1\alpha_2^\circ\hat\alpha_3}{\alpha_1^\circ RePr}\frac{\partial q_j^\circ}{\partial x_j}$$

and $\frac{D\rho^\circ}{Dt} = \frac{\partial\rho}{\partial T}\Big|_p \frac{DT^\circ}{Dt}$ or $\frac{D\rho^\circ}{Dt} = \frac{\rho^\circ}{T^\circ\alpha_4^\circ}\frac{DT^\circ}{Dt}$

$$\frac{DT^\circ}{Dt} = \frac{\alpha_2^\circ\alpha_3^\circ\alpha_4^\circ}{RePr}\cdot\frac{T^\circ\hat\alpha_1\hat\alpha_3}{\alpha_1^\circ\alpha_3^\circ}\frac{\partial q_j^\circ}{\partial x_j}$$

$$\frac{T^\circ\hat\alpha_1\hat\alpha_3}{\alpha_1^\circ\alpha_3^\circ} = \frac{\cancel{T^\circ}\,\cancel{\hat p}\,\hat C_p\,\cancel{\alpha^{\circ 2}}}{\cancel{\hat p}\,\rho^\circ\cancel{\alpha^{\circ 2}}\,C_p^\circ\,\cancel{T^\circ}} = \frac{\hat C_p}{\rho^\circ C_p^\circ}$$

$$\boxed{\rho^\circ\frac{C_p^\circ}{\hat C_p}\frac{DT^\circ}{Dt} = \frac{1}{RePr}\frac{\partial}{\partial x_j}k^\circ\frac{\partial T^\circ}{\partial x_j}}$$

Similarly, we have an equation for the divergence

$$\frac{\partial V_j^\circ}{\partial x_j} = -\frac{\alpha_2^\circ\alpha_3^\circ\alpha_4^\circ}{RePr}\cdot\frac{\hat\alpha_1\hat\alpha_1}{\alpha_1^\circ\alpha_3^\circ\alpha_4^\circ}\frac{\partial q_j}{\partial x_j} = -\frac{\hat C_p}{\rho^\circ C_p}\frac{1}{\rho^\circ}\frac{\partial\rho}{\partial T}\Big|_p\frac{\partial}{\partial x_j}k^\circ\frac{\partial T^\circ}{\partial x_j}$$

or $\boxed{\frac{\partial V_j^\circ}{\partial x_j} = \beta(T^\circ)\frac{DT^\circ}{Dt}}$ where $\beta(T^\circ) = -\frac{1}{\rho}\frac{\partial\rho}{\partial T}\Big|_p$ is thermal expansion coefficient

Along with the momentum equation

$$\boxed{\rho^\circ\frac{DV_i^\circ}{Dt} = -\frac{\partial P^2}{\partial x_j} + \frac{1}{Re}\frac{\partial}{\partial x_j}\mu^\circ \mathring{S}_{ij}^\circ + f_{bi}}$$

this is the anelastic approximation!

We will be concerned from now on with (isothermal) incompressible flows, (i.e. $Ma \to 0$). This is generally a good approximation even for $Ma \sim 0.1$ (or higher depending on the flow). The equations (non dimensionalized w.r.t ρ, U, δ)

$$\frac{DV_i}{Dt} = -\frac{\partial P}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 V_i}{\partial x_j \partial x_j} + \frac{1}{F_r^2}\hat{g}_i \qquad \frac{\partial u_i}{\partial x_i} = 0$$

gravity body force

or in dimensional form

$$\frac{DV_i}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 V_i}{\partial x_j \partial x_j} + g_i \qquad \frac{\partial V_i}{\partial x_i} = 0 \qquad \nu = \frac{\mu}{\rho} \text{ kinematic viscosity.}$$

ρ = constant.

where g_i is the gravity acceleration vector, $\hat{g}_i$ is $\frac{g_i}{|g|}$ the unit vector pointing in the direction of gravity acceleration.

$F_r = \frac{U}{\sqrt{g\delta}}$ is the Froude Number

$Re = \frac{\rho U \delta}{\mu} = \frac{U\delta}{\nu}$ is the Reynolds Number.

What does $F_r \to \infty$ mean? $F_r \to 0$?
$Re \to \infty$? $Re \to 0$

Recall that the pressure arose as the $O(Ma^2)$ fluctuations from the background, or reference. For the incompressible equations, the reference pressure is of no consequence. The fluctuations in p scale with ρU^2 and the nondimensional pressure is thus $P/\rho U^2$.