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LEAST-SQUARES FINITE ELEMENT METHOD FOR THE STOKES PROBLEM WITH ZERO RESIDUAL OF MASS CONSERVATION*

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Abstract. In this paper the simulation of incompressible flow in two dimensions by the least-squares finite element method (LSFEM) in the vorticity-velocity-pressure version is studied. In the LSFEM, the equations for continuity of mass and momentum and a vorticity equation are minimized on a discretization of the domain of interest. A problem is these equations are minimized in a global sense. Thus this method may not enforce that $\mathrm{div}\underline{u} = 0$ at every point of the discretization. In this paper a modified LSFEM is developed which enforces near zero residual of mass conservation, i.e., $\mathrm{div}\underline{u}$ is nearly zero at every point of the discretization. This is accomplished by adding an extra restriction in the divergence-free equation through the Lagrange multiplier strategy. In this numerical method the inf-sup or say LBB condition is not necessary, and the matrix resulting from applying the method on a discretization is symmetric; the uniqueness of the solution and the application of the conjugate gradient method are also valid. Numerical experience is given in simulating the flow of a cylinder with diameter 1 moving in a narrow channel of width 1.5. Results obtained by the LSFEM show that mass is created or destroyed at different points in the interior of discretization. The results obtained by the modified LSFEM show the mass is nearly conserved everywhere.

Key words. least-squares finite element method (LSFEM), Stokes flow, mass conservation, conjugate gradient

AMS subject classifications. 65N30, 65N12, 76M10

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1. Introduction. During the last decade, many mathematicians and engineers have studied the least-squares finite element methods (LSFEM) for the incompressible Navier–Stokes equations, e.g., [2], [3], [4], [6], [7], [13], [14], [16], [17]. In these methods, a functional is defined which measures the error between any solution that may exist in the defined space and the continuous solution to governing equations of motion. For example, the functional defined for the general LSFEM in the vorticity-velocity-pressure formulation for Stokes flow is

$$J^h(\underline{U}) = \| \omega_y + p_x - f_1 \|_0^2 + \| -\omega_x + p_y - f_2 \|_0^2 + \| \mathrm{curl}\underline{u} - \omega \|_0^2 + \| \mathrm{div}\underline{u} \|_0^2, \tag{1.1}$$

where ω is the vorticity and is defined as in section 2. The member of the space which minimizes this functional in the space gives the approximated solution to the governing equations [8], [11]. These methods release the divergence-free restriction. Therefore equal-order finite element spaces can be applied for the test and trial function spaces. The advantages of the LSFEM are that the continuous piecewise polynomials can be used for test and trial functions without being subjected to the saddle point condition; and the corresponding matrix of the linear systems is symmetric and positive definite. This allows the use of efficient schemes to solve large systems.

In many cases the application of velocity boundary conditions plays the crucial role in the successful simulation of incompressible flows. Recently Bochev and Gunz-

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burger [4] presented a mesh-dependent LSFEM. In this method they formulate a weighted least-squares functional:

$$J^h(\underline{U}) = \| \omega_y + p_x - f_1 \|_0^2 + \| -\omega_x + p_y - f_2 \|_0^2 + h^{-2}(\| \text{curl}\underline{u} - \omega \|_0^2 + \| \text{div}\underline{u} \|_0^2). \tag{1.2}$$

A theoretical analysis by the theory of Agmon, Douglis, and Nirenberg [1] shows the mesh-dependent LSFEM is optimum for the simulation of flows with velocity boundary conditions. For example, after one defines a finite element space of piecewise quadratic polynomial functions for the velocity $\underline{u}$ and piecewise linear polynomial functions for ω and p, the approximated solution $\underline{U}^h$ which minimizes (1.2) is the approximated solution of the Stokes problem that is of optimal order.

In order to test the practical applicability of the LSFEM, several authors have used this method to simulate the flow in a driven cavity [3], [13], [14]. All calculations present reasonably good results. In this research, we are going to test the general and mesh-dependent LSFEM in the velocity-vorticity-pressure formulation [3], [6], [9] by simulating a cylinder of diameter 1 moving along the centerline of a narrow channel of width 1.5. The centerline of the channel is along the x-coordinate axis. If the cylinder is moving with speed 1, by mass conservation the average value of u_1 (x-component of velocity) along a vertical ($x =$ constant) line connecting the top of the cylinder and the nearest wall should be 3. Our calculations using the above methods give an average value of about 0.8, i.e., neither of the above methods ensures that mass is conserved in each element in our calculation. The cause of this problem is felt to be in the LSFEM. The error is minimized on a global scale, allowing errors of significant size to remain on a local scale, especially in areas in which the gradients of the variables are of significant size. These areas are the places of most interest. In this paper we modify the general LSFEM for the Stokes problem so that the method nearly conserves mass at every point. This is done by adding an extra restriction to this method (a restricted LSFEM) which ensures that the equation for conservation of mass is satisfied in every element.

2. The least-squares method in the vorticity-velocity-pressure formulation. In this section we present an overview of the LSFEM in the vorticity-velocity-pressure formulation. We assume Ω is a bounded and connected domain in two dimensions with a polygon boundary Γ. Let $\underline{f} \in [L^2(\Omega)]^2$ be a given function of body force. The Navier–Stokes problem can be presented as

$$\begin{cases} -\nu\Delta\underline{u} + \underline{u}\cdot\text{grad}\underline{u} + \text{grad}p = \underline{f} & \text{in } \Omega, \\ \text{div}\underline{u} = 0 & \text{in } \Omega, \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $\underline{u}$, p with $\int_\Omega p = 0$ are velocity and pressure, all of which are assumed to be nondimensionalized, and $\underline{u}_0$ is a given function on Γ. In the rest of this paper, u_1 denotes the x-component of the velocity and u_2 denotes the y-component of velocity. The parameter ν is the inverse of the Reynolds number $\Re$. The velocity-vorticity-pressure version has the following form if we introduce the vorticity $\omega = \text{curl}\ \underline{u}$:

$$\begin{cases} \nu\text{curl}\omega + \underline{u}\cdot\text{grad}\underline{u} + \text{grad}p = \underline{f} & \text{in } \Omega, \\ \text{curl}\underline{u} - \omega = 0 & \text{in } \Omega, \\ \text{div}\underline{u} = 0 & \text{in } \Omega, \\ \underline{u} = \underline{u}_0 & \text{on } \Gamma. \end{cases} \tag{2.2}$$

In this paper we restrict our attention to the Stokes problem with velocity boundary conditions. Without loss of generality we assume we have a homogeneous boundary condition as

$$\left\{\begin{array}{ll} \mathrm{curl}\omega + \mathrm{grad}p = \underline{f} & \text{in } \Omega, \\ \mathrm{curl}\underline{u} - \omega = 0 & \text{in } \Omega, \\ \mathrm{div}\underline{u} = 0 & \text{in } \Omega, \\ \underline{u} = 0 & \text{on } \Gamma. \end{array}\right. \tag{2.3}$$

This system can be written in a matrix form as

$$L\underline{U} = A\underline{U}_x + B\underline{U}_y + C\underline{U} = F, \tag{2.4}$$

where $\underline{U} = (\underline{u}, \omega, p)^T, \underline{F} = (f_1, f_2, 0, 0)^T$, and

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.5}$$

We calculate $\det(\xi A + \eta B) = (\xi^2 + \eta^2)^2$, which is always positive for any non-vanished pair (ξ, η), so that the linear system is elliptic. In [4] the authors prove that if we apply piecewise quadratic polynomials for $\underline{u}$ and piecewise linear polynomials for p and ω, the mesh-dependent, weighted LSFEM achieves optimal rates of convergence for all of the four unknowns in H_1 *and* L_2 norms.

We require some spaces on Ω and Γ. The standard notations of the Sobolev spaces and their associated norms will be employed throughout this paper [10], [12], [15], [18]. We let $H^m(\Omega)$ denote the Sobolev space of functions having square integrable derivatives of order up to m over Ω. We define the norms by $\|u\|_m^2 = (u, u)_m$, and the inner product in $H^m(\Omega)$ is defined as

$$(u, v)_m = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \cdot \partial^\alpha v\ . \tag{2.6}$$

We also define the space for our problem:

$$S = \{\underline{V} \in [H^1(\Omega)]^4; \quad u_1, u_2 = 0 \ \text{ on } \Gamma \text{ and } \int_\Omega p = 0\}, \tag{2.7}$$

where $\underline{V} = (u_1, u_2, \omega, p)^T = (u_1, u_2, u_3, u_4)^T$. We will use finite-dimensional subspace $S^h \subset S$ of functions to approximate our solutions. The parameter h, which represents a mesh spacing, is used to indicate the approximation property of S^h. For example, if we define S^h to be the space consisting of continuous piecewise quadratic functions in Ω, the approximation property shows that for every $\underline{V} \in S \cap [H^2(\Omega)]^4$ there exists $\underline{V}^h \in S^h$ such that

$$h\|\underline{V} - \underline{V}^h\|_1 + \|\underline{V} - \underline{V}^h\|_0 \leq Ch^3\|\underline{V}\|_2, \tag{2.8}$$

where the positive constant C is independent of $\underline{V}$ and h.

We construct the least-squares quadratic functional

$$J(\underline{V}) = \int_\Omega (L\underline{V} - \underline{F}) \cdot (L\underline{V} - \underline{F}) \qquad \text{for } \underline{V} \in S. \tag{2.9}$$

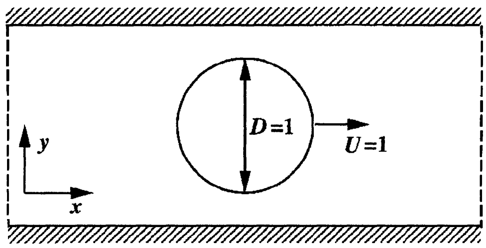

FIG. 1. *Problem setup.*

The least-squares method reads find $\underline{U} \in S$ such that

$$J(\underline{U}) \le J(\underline{V}) \qquad \text{for any } \underline{V} \in S. \tag{2.10}$$

If there is $\underline{U} \in S$ which minimizes $J(\underline{V})$ for any $\underline{V} \in S$, or say $J(\underline{U} + \epsilon\underline{V}) \ge J(\underline{U})$ for any $\underline{V} \in S$, we can easily have

$$\int_\Omega L\underline{U} \cdot L\underline{V} = \int_\Omega \underline{F} \cdot L\underline{V} \qquad \text{for any } \underline{V} \in S. \tag{2.11}$$

Similar to (2.11), if $\underline{U}^h$ minimizes (2.9) in the space S^h, we have the corresponding finite algebraic equations

$$\int_\Omega L\underline{U}^h \cdot L\underline{V}^h = \int_\Omega \underline{F} \cdot L\underline{V}^h \qquad \text{for any } \underline{V}^h \in S^h. \tag{2.12}$$

If the basis for S^h is chosen to be the piecewise quadratic polynomial, we can see that (2.12) is equivalent to a symmetric and positive definite linear algebraic system.

3. Numerical test case for general LSFEM. In order to test the practical applicability of the LSFEM, we ran a numerical test case using the functional (1.1) with piecewise quadratic elements. The same test case was run using the mesh-dependent, weighted LSFEM of the functional (1.2) using piecewise quadratic elements for u_1 and u_2 and piecewise linear elements for ω and p. For our test case we chose the phenomenon of a solid cylinder with diameter 1 moving with constant speed 1 parallel to the wall of a channel with width 1.5 (Fig.1). The domain is defined as a rectangle with corners (3, 0.75),(-1.5, 0.75),(-1.5, -0.75), and (3, -0.75). The center of the cylinder is on the origin.

We chose triangles as our elements. The triangle vertices and the midpoints of the triangle faces were chosen as the nodes for the quadratic basis functions. In our test case the domain was subdivided into 2,262 triangles with 3,485 faces and 4,708 nodes. There are four unknowns at each point except at the boundary points and the points on the cylinder, where there are two unknowns since u_1 and u_2 are given. Instead of setting $\int_\Omega p = 0$, we set $p = 0$ at the point (3,0). Figure 2 shows the grid which was used for the test case.

3.1. Test case simulation using general LSFEM. For our actual computation, we simulated the flow in a reference frame moving with the cylinder. We set $u_1 = 1$ and $u_2 = 0$ at each point on the outer boundary and set $u_1 = u_2 = 0$ on the surface of the cylinder. We also set $p = 0$ at the point (3,0) to ensure there is a unique solution for the pressure (equivalent to $\int_\Omega p = 0$). Piecewise quadratic functions were

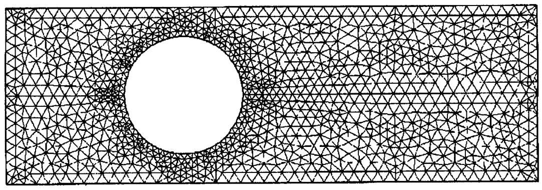

FIG. 2. *Grid used in test case simulation.*

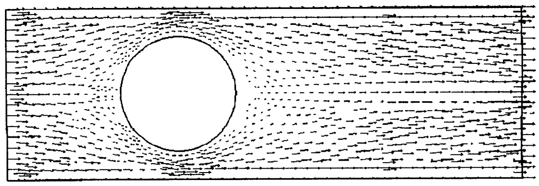

FIG. 3. *Velocity vectors for solution of test case using LSFEM.*

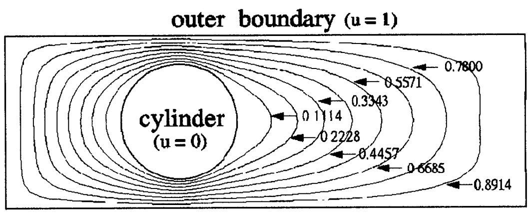

FIG. 4. *Level curves of u_1 for solution of test case using LSFEM.*

applied for all four unknowns at each node. Equation (1.1) was minimized in the finite element space S^h, which had 18,095 elements.

We assembled the linear system by the formulation (2.11). A sparse square matrix with dimension 18,095 $\times$ 18,095 was generated. Only the nontrivial entries of this matrix (with tolerance of 10^{-5}) were stored using upper storage by row. A double precision, smoothly converging variant of the conjugate gradient squared method was used to solve the system. Figures 3 and 4 show velocity vectors and level contours of u_1 for the calculated solution using the LSFEM.

3.2. Test case simulation using mesh-dependent LSFEM. In our test case simulation with the mesh-dependent LSFEM, we used the space defined in [4]; i.e., S^h was defined to be $u_1, u_2 \in H^1$ with piecewise quadratic polynomials in each element, and $\omega, p \in H^1$ were represented by piecewise linear polynomials in each element. The same boundary conditions and grid that were used in the simulation using the general LSFEM were used for this simulation. There were a total of 11,125 elements in the finite element space S^h.

After minimizing (1.2) with $h = 0.1$ in the space S^h, the resulting linear system was solved using the conjugate gradient method. After 4,000 iterations, the relative error $\| A\underline{x} - \underline{b} \|_0 \,/\, \| \underline{x} \|_0$ was less that 10^{-8}. Figures 5 and 6 show velocity vectors and level contours of u_1 for the calculated solution for our test case using the mesh-dependent LSFEM.

Upon inspection the numerical results for both simulations look fine. All of the dynamic equations, the vorticity relation equation, and the mass conservative equation

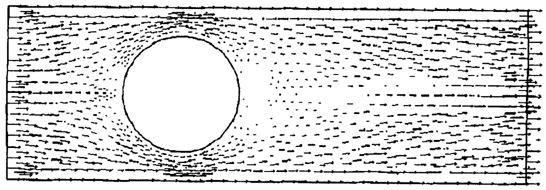

FIG. 5. *Velocity vectors for solution of test case using the mesh-dependent LSFEM.*

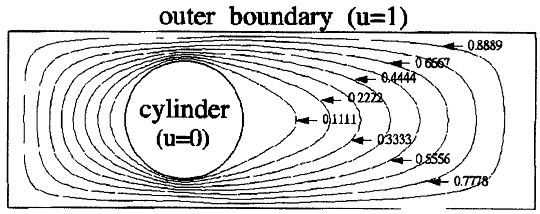

FIG. 6. *Level curves of u_1 for solution of test case using the mesh-dependent LSFEM.*

are minimized globally. But if one pays attention to some special points, for example, the points between the solid wall and the top point of the cylinder, one finds that the value of u_1 is 1 at the solid wall and reduces steadily to 0 at the cylinder. Since the flow at the entrance $x = -1.5$ and the outlet $x = 3.0$ has speed 1, the average speed between (0, 0.5) and (0, 0.75) should be 3, but the numerical results from both simulations show that the maximum value of u_1 is 1.0, and the average value of u_1 along the above-mentioned line is about 0.8 (see Fig. 9). At this region the mass conservative law has broken down totally, and this region could be a very interesting part of our application. The result shows that the simple LSFEM cannot be applied to similar problem directly.

4. Zero residual for mass conservative law. In [5] and [6], boundary conditions were considered as constraints and Lagrange multipliers were used to introduce the boundary conditions into the formulation of the problems they were studying. For the LSFEM, the mass conservation property is critical, so that we wish to enforce $\int_{\Omega_i}(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}) = 0$ in each triangle. We will view the mass continuity equation as a constraint and use the Lagrange multiplier method as a natural means of incorporating the constraint within the LSFEM statement of the problem without any special weighting. The restricted LSFEM is then as follows: find $\underline{U}_e^h \in S_h$, such that

$$J(\underline{U}_e^h) \leq J(\underline{V}^h) \qquad \text{for any } \underline{V}^h \in S^h \tag{4.1}$$

subject to the condition

$$\int_{\Omega_i} \left(\frac{\partial u_1^h}{\partial x} + \frac{\partial u_2^h}{\partial y} \right) = 0 \qquad \text{for } i = 1, \ldots, \ell, \tag{4.2}$$

where ℓ is the total number of elements, and for our test case $\ell = 2,262$ (the number of triangles). Here S_h is defined the same as in section 3.2, i.e., $u_1, u_2, \omega, p \in H^1(\Omega)$ with piecewise quadratic polynomial representations for u_1 and u_2 and piecewise linear polynomial representations for ω and p.

An equivalent formulation of this problem is to find the vector function $\underline{U}_e^h \in S^h$ which minimizes the expression

$$J_e(\underline{V}^h, \underline{\mu}) = \frac{1}{2}\int_\Omega (L\underline{V}^h - \underline{F}) \cdot (L\underline{V}^h - \underline{F}) \; + \; \underline{\mu}^T \cdot \Lambda \underline{V}^h \tag{4.3}$$

for all $\underline{V}^h \in S^h$ and $\underline{\mu}$. Here $\underline{\mu} = (\mu_1, \mu_2, \dots, \mu_\ell)^T$, and $\Lambda\underline{V}^h$ is a linear vector functional defined in S^h with ℓ elements, the ith element of which represents the numerical integration of $\int_{\Omega_i}(\frac{du_1^h}{dx} + \frac{du_2^h}{dy})$ in the ith triangle.

Taking the first-order variation of J_e with respect to $\underline{V}^h$ and $\underline{\mu}$, respectively, and setting $\delta J_e = 0$ leads to the weak statement: find $\underline{U}_e^h \in S^h$ and $\underline{\lambda}$ such that

$$\int_\Omega L\underline{U}_e^h \cdot L\underline{V}^h + \underline{\lambda}^T \cdot \Lambda \underline{V}^h + \underline{\mu}^T \cdot \Lambda \underline{U}_e^h = \int_\Omega L\underline{V}^h \cdot \underline{F} \qquad \text{for any } \underline{V}^h \in S^h \text{ and any } \underline{\mu}, \tag{4.4}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_\ell)^T$. The restricted LSFEM then has ℓ more unknowns than the standard LSFEM for the same discretization, which for our test case means there are 11,125 + 2,262 = 13,387 unknowns using the restricted method if we let $\underline{U}_e^h$ and $\underline{V}^h$ be represented by piecewise quadratic polynomials for u_1 and u_2 and piecewise linear polynomials for ω and p.

One can check that the linear algebraic system resulting from the restricted method is symmetric, as is the case when the standard LSFEM method is used. Furthermore we can prove that this linear algebraic system is so-called pseudopositive definite, which we explain in the following. The restricted LSFEM generates an extended linear algebraic system as

$$A_e \begin{bmatrix} \underline{x} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ \underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{b} \\ \underline{b}_\ell \end{bmatrix}, \tag{4.5}$$

where A is a symmetric positive definite matrix with dimension of $n \times n$; B is a matrix with dimension of $n \times \ell$; $\underline{b}$, $\underline{\lambda}$, $\underline{b}_\ell$ are vectors with dimensions of n, ℓ and ℓ, respectively; and $\underline{b}_\ell$ is formed by the velocity boundary conditions $u_1 = 1$ at the outer boundary when the matrix B^T is assembled.

From the first n equations in (4.5) we have

$$A\underline{x} + B\underline{\lambda} = \underline{b}, \tag{4.6}$$

so that

$$\underline{x} = A^{-1}(\underline{b} - B\underline{\lambda}). \tag{4.7}$$

Also from the last ℓ equations we have

$$B^T\underline{x} = \underline{b}_\ell. \tag{4.8}$$

Combining the above two expressions, we obtain

$$B^T A^{-1} B \underline{\lambda} = B^T A^{-1} \underline{b} - \underline{b}_\ell. \tag{4.9}$$

We can prove the matrix of $B^T A^{-1} B$ is also symmetric and positive definite.

LEMMA 1. *If A is positive definite of dimension $n \times n$ and B is a matrix with dimension of $n \times \ell$ and if $B\underline{y}$ is nontrivial for any nontrivial vector $\underline{y}$ with dimension ℓ, then $B^T A^{-1} B$ is positive definite.*

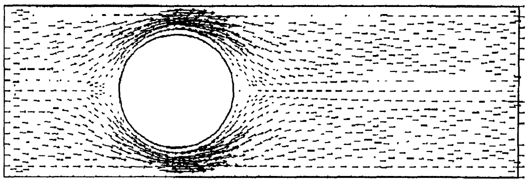

FIG. 7. *Velocity vectors for solution of test case using restricted LSFEM.*

Proof. For any equation $A\underline{x} = \underline{b}$, if $\underline{b}$ is a nontrivial vector, then $\underline{x}$ is also nontrivial, and vice versa. Since A is positive definite, $\underline{x}^T A \underline{x} = \underline{x}^T \underline{b} > 0$ for any nontrivial $\underline{x}$. We then have $\underline{b}^T A^{-1} \underline{b} = \underline{b}^T \underline{x} > 0$, so that A^{-1} is also positive definite.

For any given nontrivial vector $\underline{y}$ with dimension ℓ,

$$\underline{y}^T (B^T A^{-1} B) \underline{y} = (B\underline{y})^T A^{-1} (B\underline{y}) > 0. \tag{4.10}$$

Therefore the linear algebraic system of (4.5) has a unique solution if the matrix A is positive definite and if B is defined as in (4.2). □

THEOREM 1. *The linear algebraic system* (4.5) *in which* A *and* B *are defined by* (4.4) *and* (4.2) *has a unique solution.*

Proof. From the definition of matrix B in (4.4), each column represents the numerical integration of $\int_{\Omega_i} \operatorname{div} \underline{u}$ in each triangle Ω_i, so the column vectors are linearly independent. Therefore, the rank of B is ℓ, since $\ell < n$. Then for any nontrivial vector $\underline{y}$, $B\underline{y}$ is nontrivial. Combining Theorem 1 with Lemma 1, the proof follows. □

It is easy to check that the extended matrix A_e defined by (4.5) is still symmetric but no longer positive definite. Since A is symmetric and positive definite, the solution of (4.5) is equivalent to locating the minimum of the quadratic problem

$$\min_{(\underline{\xi},\underline{\mu})} \left(\frac{1}{2} \underline{\xi}^T A \underline{\xi} + \frac{1}{2} \underline{\mu}^T B^T \underline{\xi} + \frac{1}{2} \underline{\xi}^T B \underline{\mu} - \underline{\xi}^T \underline{b} - \underline{\mu}^T \underline{b_\ell} \right), \tag{4.11}$$

where $\underline{\xi}$ and $\underline{\mu}$ are vectors of dimension n and ℓ, respectively. The above equation is equivalent to

$$\min \left(\frac{1}{2} \begin{bmatrix} \underline{\xi} \\ \underline{\mu} \end{bmatrix}^T A_e \begin{bmatrix} \underline{\xi} \\ \underline{\mu} \end{bmatrix} - \begin{bmatrix} \underline{\xi} \\ \underline{\mu} \end{bmatrix}^T \begin{bmatrix} \underline{b} \\ \underline{b_\ell} \end{bmatrix} \right). \tag{4.12}$$

Therefore the common conjugate gradient method can still be used to solve the system (4.5). Figures 7 and 8 show the solution of our test case simulation using the restricted LSFEM.

The numerical results after solving this linear system show that the divergence of the velocity $\int_{\Omega_i} (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y})$ is less than 10^{-4} in each triangle. In checking the mass conservation in the region we examined earlier, we draw a line from (0, 0.5) and (0, 0.75). The average velocity normal to the line is 2.96. Figure 9 shows profiles of u_1 calculated along the above-mentioned line using the mesh-dependent, weighted LSFEM and the restricted LSFEM.

The equation $\int_{\Omega_i} (\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}) = 0$ for $i = 1, \ldots, \ell$ does not mean the flow is divergence free at each point but at the center of each triangle. As we mentioned in section 3.1, there are 11,125 degrees of freedom totally for the general LSFEM. After

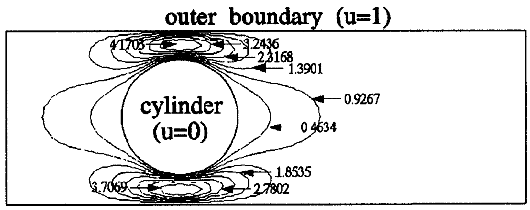

FIG. 8. *Level curves of u_1 for solution of test case using the restricted LSFEM.*

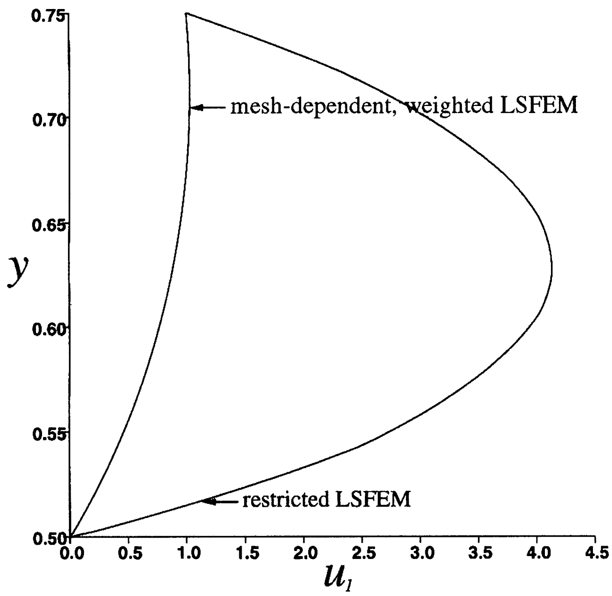

FIG. 9. *Profile of u_1 along a vertical line connecting the top of the cylinder and the nearest wall using the mesh-dependent, weighted LSFEM and the restricted LSFEM.*

adding the divergence-free restriction at each center of each triangle, the number of degrees of freedom will reduce, but the reduced freedom will not influence the global results since a divergence-free solution at each point is the result as $h \to 0$.

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