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ABSTRACT. We show that it is possible to apply the DPG methodology without reformulating a second order boundary value problem into a first order system, by considering the simple example of the Poisson equation. The result is a new weak formulation and a new DPG method for the Poisson equation, which has no numerical trace variable, but has a numerical flux approximation on the element interfaces, in addition to the primal interior variable.

1. INTRODUCTION

A typical approach to derive a discontinuous Petrov Galerkin (DPG) method for a second order boundary value problem, as presented in [5, 6], is to reformulate the problem as a first order system, multiply by test functions, and integrate every equation by parts. In this note, we show that it is not necessary to reformulate higher order equations into first order systems to apply the DPG methodology. This methodology applied to a simple transport equation was presented in [4] but the present paper uses the generalized methodology described in [5].

The DPG methodology relies on automatically computed test spaces. To describe it, suppose we want to approximate $x \in X$ satisfying

$$b(x, y) = l(y), \quad \forall y \in Y. \tag{1.1}$$

Here X and Y are real Hilbert spaces, $b(\cdot, \cdot) : X \times Y \to \mathbb{R}$ is a continuous bilinear form, and $l(\cdot) : Y \to \mathbb{R}$ is a continuous linear form. We use a closed subspace $X_h \subseteq X$ for approximation and a closed subspace $Y^r \subseteq Y$ to define the test spaces of the DPG method. Typically X_h and Y^r are finite-dimensional spaces and the indices h and r are related to their dimension. Let $T^r : X \to Y^r$ be defined by $(T^r w, y)_Y = b(w, y)$ for all $y \in Y^r$. Here and throughout, the inner product (and norm) on any Hilbert space W is denoted by $(\cdot, \cdot)_W$ (and $\| \cdot \|_W$, respectively). The DPG method uses the test space $Y_h^r = T^r(X_h)$ and computes an approximation $x_h \in X_h$ satisfying

$$b(x_h, y) = l(y) \quad \forall y \in Y_h^r. \tag{1.2}$$

This defines a Petrov-Galerkin method as $X_h \neq Y_h^r$ in general. It is clear from this abstract description of the method that the DPG methodology applies very generally to various

bilinear forms. In particular, there is no need to rewrite a second order boundary value problem as a first order system to formulate the method (1.2).

Nonetheless, for the method to be practical, the operator T^r must be local. This is usually ensured by finding a variational setting where Y^r is a DG space, i.e., it admits functions with no continuity constraints across mesh element interfaces. For the Dirichlet problem, we have shown in [6] that this can be achieved by rewriting it as a first order system. In this note, we show that it can also be accomplished without using the first order reformulation. In doing so, *we obtain perhaps the simplest example illustrating the ideas of the DPG method.*

By way of preliminaries, we recall one of the basic theorems used to analyze DPG methods, proved in [7]. It uses these three assumptions:

(1) Uniqueness:
$$\{w \in X : \ b(w, v) = 0 \quad \forall v \in Y\} = \{0\}. \tag{1.3a}$$

(2) Inf-sup and Continuity: There exist $C_1, C_2 > 0$ such that
$$C_1 \|v\|_Y \le \sup_{w \in X} \frac{|b(w, v)|}{\|w\|_X} \le C_2 \|v\|_Y. \tag{1.3b}$$

(3) There is a linear operator $\Pi : Y \to Y^r$ and a $C_\Pi > 0$ such that for all $w_h \in X_h$ and all $v \in Y$,
$$b(w_h, v - \Pi v) = 0, \qquad \|\Pi v\|_Y \le C_\Pi \|v\|_Y. \tag{1.3c}$$

Theorem 1.1 (see [7]). *Suppose Assumptions* (1.3a) *and* (1.3b) *hold. Then there is a unique* $x \in X$ *solving* (1.1), *i.e., Problem* (1.1) *is well posed. If, in addition, Assumption* (1.3c) *holds, then*
$$\|x - x_h\|_X \le \frac{C_2 C_\Pi}{C_1} \inf_{w \in X_h} \|x - w\|_X. \tag{1.4}$$

In the next section, we present the new weak formulation of the Dirichlet problem. In Section 3, we verify the first two assumptions of the above theorem, thus proving that the new formulation is well-posed. In this proof we use that a piecewise harmonic function can be controlled by its jumps and the jumps of its normal derivatives, a fact established in [6]. The novelty lies in the bounds of gradients of piecewise H^1-functions using the orthogonal projection into $\nabla H_0^1(\Omega)$. In Section 4, we give a convergence result for the DPG method applied to the new formulation. We conclude in Section 5 with numerical reports.

2. The formulation

Let Ω be an open subset of $\mathbb{R}^N$ ($N = 2$ or 3) with Lipschitz connected boundary. The boundary value problem under consideration is
$$-\Delta u = f \qquad\qquad\qquad \text{on } \Omega, \tag{2.1a}$$
$$u = 0 \qquad\qquad\qquad \text{on } \partial\Omega. \tag{2.1b}$$

The weak formulation of this problem that we intend to study, is defined using a mesh Ω_h of Ω. We assume that Ω_h is a disjoint partitioning of Ω into open elements K such that the union of their closures is the closure of Ω. The element boundaries ∂K (for all the elements $K \in \Omega_h$) form the collection $\partial\Omega_h$. We assume that ∂K is Lipschitz for all $K \in \Omega_h$, so

that we may use trace theorems on each element, but the shape of the elements is otherwise arbitrary.

The new Petrov-Galerkin weak formulation of this problem is as follows: Find $(u, \hat{q}_n) \in X$ satisfying

$$(\nabla u, \nabla v)_{\Omega_h} - \langle \hat{q}_n, v \rangle_{\partial\Omega_h} = (f, v)_\Omega \qquad \forall v \in Y \tag{2.2}$$

where the spaces and the notations used are defined below.

First, define a broken Sobolev space (where as usual we use the adjective "broken" for spaces admitting functions with interelement discontinuities) by $H^1(\Omega_h) = \{v : \ v|_K \in H^1(K), \ \forall K \in \Omega_h\}$, and normed naturally by

$$\|v\|_{H^1(\Omega_h)}^2 = (v, v)_{\Omega_h} + (\nabla v, \nabla v)_{\Omega_h}. \tag{2.3}$$

The derivatives above, in (2.2), and in such notations throughout, are always calculated element by element, and

$$(r, s)_{\Omega_h} = \sum_{K \in \Omega_h} (r, s)_K, \qquad \langle \ell, w \rangle_{\partial\Omega_h} = \sum_{K \in \Omega_h} \langle \ell, w \rangle_{1/2, \partial K}.$$

where $(\cdot, \cdot)_K$ denotes the $L^2(K)$-inner product and $\langle \ell, \cdot \rangle_{1/2, \partial K}$ denotes the action of a functional ℓ in $H^{-1/2}(\partial K)$. We will also use $\|r\|_{\Omega_h}$ to denote the norm $(r, r)_{\Omega_h}^{1/2}$. Next, define the space of numerical fluxes by

$$H^{-1/2}(\partial\Omega_h) = \{\eta \in \prod_K H^{-1/2}(\partial K) : \exists q \in H(\mathrm{div}, \Omega) \text{ such that}$$
$$\eta|_{\partial K} = q \cdot n|_{\partial K}, \quad \forall K \in \Omega_h\},$$

where n generically denotes the unit outward normal of any domain under consideration – for instance, above it denotes the unit outward normal on the boundary of a mesh element K. This space is normed by

$$\|\hat{r}_n\|_{H^{-1/2}(\partial\Omega_h)} = \inf \{\|q\|_{H(\mathrm{div}, \Omega)} : \ \forall q \in H(\mathrm{div}, \Omega) \text{ such that } \hat{r}_n|_{\partial K} = q \cdot n|_{\partial K}\}. \tag{2.4}$$

With these notations, we can now complete the definition of the weak formulation (2.2) by setting the trial and test spaces in (2.2) by

$$X = H_0^1(\Omega) \times H^{-1/2}(\partial\Omega_h)$$
$$Y = H^1(\Omega_h).$$

The norm in Y is defined by (2.3) and the norm in X is defined by

$$\|(w, \hat{r}_n)\|_X^2 = \|\nabla w\|_{L^2(\Omega)}^2 + \|\hat{r}_n\|_{H^{-1/2}(\partial\Omega_h)}^2.$$

This completes the definition of all the notations that appeared in (2.2).

At this point, we note that a variational problem very similar to (2.2) can be found in [2, p.141], by the name of a *primal hybrid method*. There, it is used as motivation to introduce hybrid and non-conforming methods. An important difference between their formulation and (2.2) is that while theirs is standard Galerkin (or Bubnov-Galerkin), ours is a Petrov-Galerkin formulation since $X \neq Y$. We now proceed to prove that (2.2) is well-posed in the norms introduced above.

3. Wellposedness

Let us denote the bilinear form in (2.2) by

$$b(\,(w,\hat{r}_n),\,v) = (\nabla w, \nabla v)_{\Omega_h} - \langle \hat{r}_n, v \rangle_{\partial \Omega_h}$$

for any $(w, \hat{r}_n) \in X$ and any $v \in Y$. The wellposedness of (2.2) follows from Theorem 1.1, once we verify (1.3a) and (1.3b).

Lemma 3.1 (Uniqueness). *Any $(w, \hat{r}_n) \in X$ satisfying*

$$b(\,(w,\hat{r}_n),\,v) = 0 \qquad \forall v \in Y \tag{3.1}$$

vanishes.

Proof. Choosing $v \in \mathcal{D}(K)$ on any fixed mesh element $K \in \Omega_h$, equation (3.1) implies that $-\Delta(w|_K) = 0$. Then $r = -\nabla w|_K$ is in $H(\mathrm{div}, K)$. Therefore, considering a general $v \in Y$, we may integrate (3.1) by parts element by element, and use $\nabla \cdot (r|_K) = 0$ to obtain

$$\langle r \cdot n - \hat{r}_n, v \rangle_{\partial \Omega_h} = 0 \qquad \forall v \in H^1(\Omega_h).$$

Choosing v supported only on K, this shows that $r \cdot n = \hat{r}_n$ on ∂K. Furthermore, for any $\phi \in \mathcal{D}(\Omega)$, integrating by parts element by element,

$$-(r, \nabla \phi)_\Omega = (\nabla \cdot r, \phi)_{\Omega_h} + \langle r \cdot n, \phi \rangle_{\partial \Omega_h} = (\nabla \cdot r, \phi)_{\Omega_h} + \langle \hat{r}_n, \phi \rangle_{\partial \Omega_h}$$

Now, by definition of $H^{-1/2}(\partial \Omega_h)$, there is a $q \in H(\mathrm{div}, \Omega)$ such that $q \cdot n = \hat{r}_n$ on each ∂K. Hence integrating by parts, first element by element, and then over all Ω, we have $\langle \hat{r}_n, \phi \rangle_{\partial \Omega_h} = \langle q \cdot n, \phi \rangle_{\partial \Omega_h} = (\nabla \cdot q, \phi)_\Omega + (q, \nabla \phi)_\Omega = \langle q \cdot n, \phi \rangle_{\partial \Omega} = 0$. In particular, this proves that the distributional divergence of r satisfies $(\nabla \cdot r)(\phi) = -(r, \nabla \phi)_\Omega = (\nabla \cdot r, \phi)_{\Omega_h}$, so $r \equiv -\nabla w \in H(\mathrm{div}, \Omega)$. Therefore, $\Delta w = 0$ on Ω and since $w \in H_0^1(\Omega)$, we conclude that $w \equiv 0$. Moreover on the boundary of each element, $\hat{r}_n = -\partial w / \partial n = 0$. $\square$

Next, to prove (1.3b), we start by defining

$$\big|[\tau \cdot n]\big|_{\partial \Omega_h} \overset{\text{def}}{=} \sup_{\phi \in H_0^1(\Omega)} \frac{\langle \tau \cdot n, \phi \rangle_{\partial \Omega_h}}{\|\phi\|_{H^1(\Omega)}}, \tag{3.2a}$$

$$\big|[vn]\big|_{\partial \Omega_h} \overset{\text{def}}{=} \sup_{\hat{r}_n \in H^{-1/2}(\partial \Omega_h)} \frac{\langle \hat{r}_n, v \rangle_{\partial \Omega_h}}{\|\hat{r}_n\|_{H^{-1/2}(\partial \Omega_h)}} = \sup_{r \in H(\mathrm{div}, \Omega)} \frac{\langle r \cdot n, v \rangle_{\partial \Omega_h}}{\|r\|_{H(\mathrm{div}, \Omega)}}. \tag{3.2b}$$

The last equality is a consequence of the definition of the quotient norm in (2.4).

An important ingredient in our analysis is $L^2(\Omega)$-orthogonal projection into $\nabla H_0^1(\Omega)$, which we denote by P:

$$(Pq, \nabla \phi)_\Omega = (q, \nabla \phi)_\Omega \qquad \forall \phi \in H_0^1(\Omega). \tag{3.3}$$

Lemma 3.2. *There is a positive constant C independent of Ω_h such that for all v in $H^1(\Omega_h)$,*

$$\|v\|_{\Omega_h} \leq C \left(\|P\nabla v\|_{\Omega_h} + \big|[vn]\big|_{\partial \Omega_h} \right).$$

Proof. This can be proved along the same lines as [6, Lemma 4.2]. $\square$

Lemma 3.3. *There is a positive constant C independent of Ω_h such that for all v in $H^1(\Omega_h)$,*

$$\|\nabla v\|_{\Omega_h} \leq \|P\nabla v\|_{\Omega_h} + C\big|[vn]\big|_{\partial \Omega_h}.$$

Proof. By the definition of P, there is $z \in H_0^1(\Omega)$ such that $P\nabla v = \nabla z$. Let $\varepsilon = v - z$ and define $r|_K = -\nabla\varepsilon|_K$ on all $K \in \Omega_h$. Then, (3.3) implies

$$(r, \nabla\phi)_K = 0 \qquad \forall \phi \in H_0^1(\Omega). \tag{3.4}$$

Choosing $\phi \in \mathcal{D}(K)$, we immediately find that $\nabla \cdot (r|_K) = 0$. In other words,

$$r + \nabla\varepsilon = 0 \qquad\qquad \text{on } K,$$
$$\nabla \cdot r = 0 \qquad\qquad \text{on } K,$$

Applying a known result [6, Lemma 4.3] for locally harmonic functions, we thus obtain

$$\|r\|_{\Omega_h} = \|\nabla\varepsilon\|_{\Omega_h} \leq C\left(\left|[r \cdot n]\right|_{\partial\Omega_h} + \left|[\varepsilon n]\right|_{\partial\Omega_h}\right). \tag{3.5}$$

Now we claim that $\left|[r \cdot n]\right|_{\partial\Omega_h} = 0$. This is because we may integrate by parts element by element to conclude from (3.4) that

$$-(\nabla \cdot r, \phi)_{\Omega_h} + \langle r \cdot n, \phi \rangle_{\partial\Omega_h} = \langle r \cdot n, \phi \rangle_{\partial\Omega_h} = 0.$$

Since this holds for all $\phi \in H_0^1(\Omega)$, the definition (3.2a) immediately proves the claim. It also follows from (3.2b) that $\left|[\varepsilon n]\right|_{\partial\Omega_h} = \left|[vn]\right|_{\partial\Omega_h}$. With these facts in mind, returning to (3.5), we conclude that

$$\|\nabla v\|_{\Omega_h} \leq \|P\nabla v\|_{\Omega_h} + \|\nabla\varepsilon\|_{\Omega_h} \leq \|P\nabla v\|_{\Omega_h} + C\left|[vn]\right|_{\partial\Omega_h},$$

which proves the lemma. $\qquad\square$

Lemma 3.4 (Inf-sup condition). *There exists a mesh-independent $C_1 > 0$ such that*

$$C_1\|v\|_Y \leq \sup_{(w,\hat{r}_n)\in X} \frac{b((w,\hat{r}_n), v)}{\|(w,\hat{r}_n)\|_X}, \qquad \forall v \in Y. \tag{3.6}$$

Proof. Let s denote supremum in (3.6). Choosing $\hat{r}_n = 0$ and taking the supremum over nontrivial $w \in H_0^1(\Omega)$ we obtain

$$s \geq \sup_{w\in H_0^1(\Omega)} \frac{(\nabla w, \nabla v)_{\Omega_h}}{\|\nabla w\|_{L^2(\Omega)}} = \|P\nabla v\|_{L^2(\Omega)}.$$

Next, setting $w = 0$ and maximizing over $\hat{r}_n$, we also obtain

$$s \geq \sup_{\hat{r}_n\in H^{-1/2}(\partial\Omega_h)} \frac{-\langle \hat{r}_n, v \rangle_{\partial\Omega_h}}{\|\hat{r}_n\|_{H^{-1/2}(\partial\Omega_h)}} = \left|[vn]\right|_{\partial\Omega_h}$$

where we have used the definition (3.2b). Thus, we have shown that

$$2s \geq \|P\nabla v\|_{L^2(\Omega)} + \left|[vn]\right|_{\partial\Omega_h}$$

for all $v \in Y$. The required inf-sup condition now follows from Lemmas 3.2 and 3.3. $\qquad\square$

Lemma 3.5 (Continuity). *There is a mesh-independent $C_2 > 0$ such that*

$$b((w,\hat{r}_n), v) \leq C_2 \|(w,\hat{r}_n)\|_X \|v\|_Y, \qquad \forall (w,\hat{r}_n) \in X, \, v \in Y. \tag{3.7}$$

holds.

Proof. Since it is immediate from the definition (3.2b) that

$$\langle \hat{r}_n, v \rangle_{\partial \Omega_h} \leq \|\hat{r}_n\|_{H^{-1/2}(\partial \Omega_h)} |[vn]|_{\partial \Omega_h}$$

we have

$$b(\,(w, \hat{r}_n),\, v) \leq \|(w, \hat{r}_n)\|_X \left(\|\nabla v\|^2_{\Omega_h} + |[vn]|_{\partial \Omega_h} \right)^{1/2}.$$

Hence, it suffices to prove that

$$|[vn]|_{\partial \Omega_h} \leq C\|v\|_Y \qquad \forall v \in Y. \tag{3.8}$$

But using the definition (3.2b) again,

$$|[vn]|_{\partial \Omega_h} = \sup_{r \in H(\mathrm{div},\Omega)} \frac{\langle r \cdot n, v \rangle_{\partial \Omega_h}}{\|r\|_{H(\mathrm{div},\Omega)}} = \sup_{r \in H(\mathrm{div},\Omega)} \frac{(r, \nabla v)_{\Omega_h} + (\nabla \cdot r, v)_{\Omega_h}}{\|r\|_{H(\mathrm{div},\Omega)}},$$

we find that an application of Cauchy-Schwarz inequality proves (3.8). □

4. THE DPG APPROXIMATION

In this section, we assume that Ω_h consists of a conforming shape regular finite element mesh of simplices and $h = \max_{K \in \Omega_h}(\mathrm{diam}\, K)$. We turn to the DPG approximation of $(u, \hat{q}_n) \in X$, denoted by $(u_h, \hat{q}_{n,h})$. It lies in X_h, which is set to

$$X_h = \{(w, \hat{r}_n) \in X : w|_K \in P_{k+1}(K),\ \hat{r}_n|_{\partial K} \in P_k(\partial K)\}, \tag{4.1}$$

where $k \geq 0$ is an integer, $P_k(K)$ denotes the space of polynomials of degree at most k restricted to K and $P_k(\partial K)$ denotes the set of functions on ∂K whose values on each $(N-1)$ dimensional subsimplex F (face) of K is a polynomial of degree at most k on F. Next, we set

$$Y^r = \{v \in Y : v|_K \in P_r(K)\}.$$

With these settings, we consider the DPG method (1.2) and its solution $(u_h, \hat{q}_{n,h})$, and prove the following result.

Theorem 4.1. *Suppose $r \geq k + N$. Then, there is a constant $C > 0$ independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} + \|\hat{q} - \hat{q}_{n,h}\|_{H^{-1/2}(\partial \Omega_h)} \leq C \inf_{(w_h, \hat{r}_{n,h}) \in X_h} \left(\|u - w_h\|_{H^1(\Omega)} + \|\hat{q} - \hat{r}_{n,h}\|_{H^{-1/2}(\partial \Omega_h)} \right).$$

Proof. We apply Theorem 1.1. We only need to verify (1.3c). This can be verified element by element. Let $K \in \Omega_h$. It suffices to show that there exists a bounded linear map $\Pi : H^1(K) \to P_r(K)$ such that

$$(\nabla w_h, \nabla(v - \Pi v))_{\Omega_h} - \langle \hat{q}_{n,h}, v - \Pi v \rangle_{\partial \Omega_h} = 0 \tag{4.2}$$

for all $v \in H^1(K)$, $w_h \in P_k(K)$ and $\hat{q}_{n,h} \in P_k(\partial K)$.

We now recall the result of [7, Lemma 3.2], which asserts that, on any simplex K, there exists a C independent of $h_K = \mathrm{diam}(K)$ and a bounded linear operator $\Pi_{p+N} : H^1(K) \to$

$P_{p+N}(K)$ such that

$$(\Pi_{p+N}v - v, q_{p-1})_K = 0, \qquad \forall q_{p-1} \in P_{p-1}(K), \qquad (4.3)$$

$$\langle \Pi_{p+N}v - v, \mu_p \rangle_{\partial K} = 0, \qquad \forall \mu_p \in P_p(\partial K), \qquad (4.4)$$

$$\|\Pi_{p+N}v\|_{H^1(K)} \leq C\|v\|_{H^1(K)}, \qquad \forall v \in H^1(K).$$

Since $r \geq k + N$, we may set $\Pi = \Pi_{k+N}$. Then,

$$(\nabla w_h, \nabla(v - \Pi v))_{\Omega_h} - \langle \hat{q}_{n,h}, v - \Pi v \rangle_{\partial \Omega_h}$$
$$= -(\Delta w_h, v - \Pi v)_{\Omega_h} - \langle \hat{q}_{n,h} - n \cdot \nabla w_h, v - \Pi v \rangle_{\partial \Omega_h} = 0,$$

which proves (4.2), and concludes the proof. $\qquad \square$

Remark 4.1. The above analysis can be easily generalized to a mesh of hypercubes. For example, consider the case of squares ($N = 2$) and let $Q_{p,p}$ denote the (tensor product) space of polynomials of degree at most p in each variable. Then, a generalization of the construction in [7, Lemma 3.2] shows that on any square K, there is a bounded linear operator $\Pi_{k+2}^{\text{sqr}} : H^1(K) \to Q_{k+2,k+2}(K)$ such that

$$(\Pi_{k+2}^{\text{sqr}}v - v, q_k)_K = 0, \qquad \forall q_k \in Q_{k,k}(K),$$

$$\langle \Pi_{k+2}^{\text{sqr}}v - v, \mu_k \rangle_{\partial K} = 0, \qquad \forall \mu_k \in P_k(\partial K).$$

(Details of this construction on squares are explicitly written down in [3].) Therefore, proceeding as in the proof of Theorem 4.1, we find that for any integer $k \geq 1$, the error estimate of the theorem continues to hold on square meshes with

$$X_h = \{(w, \hat{r}_n) \in X : w|_K \in Q_{k,k}(K), \ \hat{r}_n|_{\partial K} \in P_k(\partial K)\}, \qquad (4.5)$$

$$Y^r = \{v \in Y : v|_K \in Q_{k+2,k+2}(K)\}. \qquad (4.6)$$

Note the difference in the degree of w with that in (4.1).

Remark 4.2. It is easy to see that the condition number κ of the resulting symmetric positive definite DPG matrix system is $O(h^{-2})$ on quasiuniform meshes. It is proved abstractly in [7, Remark 2.3] that any DPG method has

$$\kappa \leq \frac{\lambda_1}{\lambda_0} \frac{C_2^2 C_\Pi^2}{C_1^2},$$

where λ_0 and λ_1 are positive constants for which the vector of coefficients $\vec{x}$ in the basis expansion of any $x \in X_h$ satisfies $\lambda_0 \|\vec{x}\|_{\ell^2}^2 \leq \|x\|_X^2 \leq \lambda_1 \|\vec{x}\|_{\ell^2}^2$. Using the norm equivalences in the proof of [7, Theorem 3.7], or the ones in [1], we can easily see that $\lambda_1/\lambda_0 \leq Ch^{-2}$ for our method. The remaining constants C_0, C_1, C_Π have already been proved to be independent of h for the formulation under consideration.

5. NUMERICAL ILLUSTRATION

We now report a few numerical results. We consider square meshes of two dimensional domains and set, for integers $k \geq 1$,

$$X_h = \{(w, \hat{r}_n) \in X : w|_K \in Q_{k,k}(K), \ \hat{r}_n|_{\partial K} \in P_{k-1}(\partial K)\},$$

$$Y^r = \{v \in Y : v|_K \in Q_{k+2,k+2}(K)\}.$$

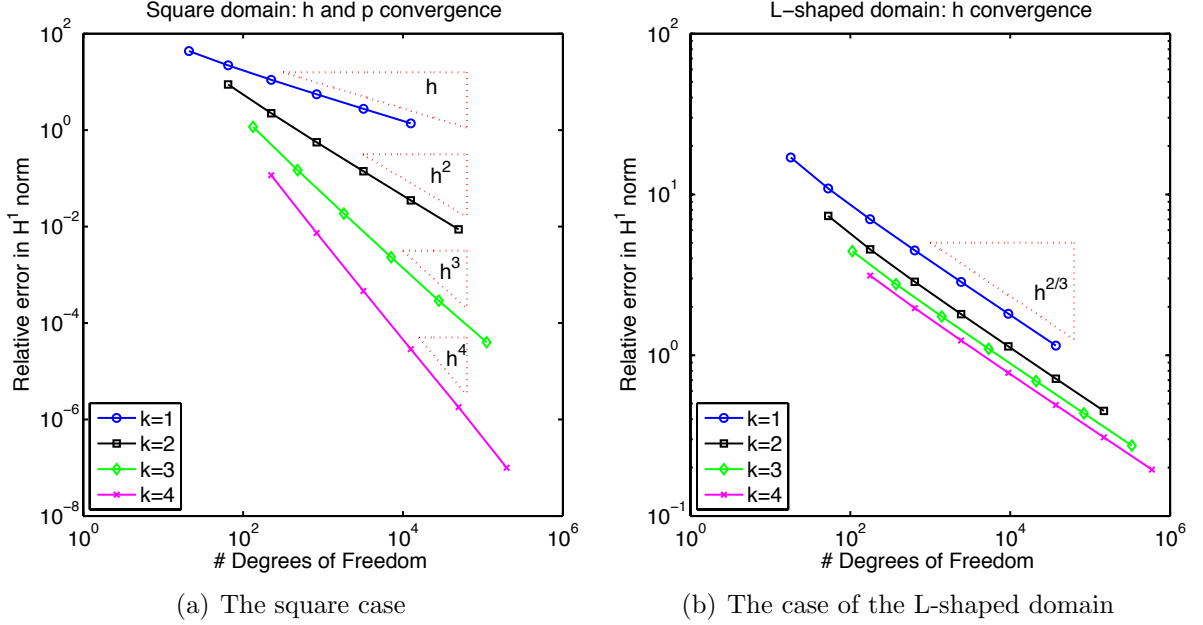(a) The square case   (b) The case of the L-shaped domain

FIGURE 1. Convergence rates on a square and L-shaped domain

The above defined X_h is contained in (although not equal to) the space defined in (4.5), so by Remark 4.1, the error estimate of Theorem 4.1 continues to hold. We choose this X_h so that the best approximation rates for both the variables are comparable. Convergence rates in terms of h can be immediately obtained (the $H^{-1/2}(\partial\Omega_h)$ best approximation error can be bounded using known $H(\mathrm{div})$-projections of the flux $q = -\nabla u$ as in [6, Corollary 4.1]), namely,

$$\|u - u_h\|_{H^1(\Omega)} + \|\hat{q} - \hat{q}_{n,h}\|_{H^{-1/2}(\partial\Omega_h)} \leq Ch^s \left(|u|_{H^{s+1}(\Omega)} + |\mathrm{div}\, q|_{H^s(\Omega)}\right) \qquad (5.1)$$

for all $s \leq k$, whenever the right hand side norms are finite.

We illustrate the practical manifestation of this error bound by solving the Dirichlet problem (2.1) on the unit square with f set such that the exact solution is $u = \sin(\pi x)\sin(\pi y)$. Due to the difficulties in computing the $H^{-1/2}(\partial\Omega_h)$-norm, we only report the error in the $H^1(\Omega)$-norm of the primal variable:

$$e_h = \frac{\|u - u_h\|_{H^1(\Omega)}}{\|u\|_{H^1(\Omega)}}.$$

The computations were done on a sequence of meshes with $h = 1/2^n$, from $n = 1, \ldots, 6$. A plot of e_h versus the number of degrees of freedom (which is $O(h^{-2})$) is shown in Figure 1(a). Clearly, e_h decreases at the rates predicted by (5.1) for each k. By cross connecting the multiple curves, one can also see that the method exhibits exponential decrease of error with k (i.e., p-convergence), even though that is not covered by Theorem 4.1.

Since the solution in above experiment was infinitely smooth, the maximal rate of convergence with $s = k$ obtainable in (5.1) was observed in Figure 1(a). In contrast, if we approximate the singular solution $u(r, \theta) = r^{2/3}\sin\left(\frac{2}{3}(\theta + \frac{\pi}{2})\right)$ on an L-shaped domain using the DPG method, then as seen from Figure 1(b), due to the regularity limit, we only

observe convergence at $O(h^{2/3})$. Moreover, the convergence rate does not improve even if k is increased. Since the solution is in $H^{s+1}(\Omega)$ for $s < 2/3$, these observations are also in accordance with (5.1). Note, however, that by cross connecting the multiple curves, one can observe that the p-refinements deliver also an algebraic convergence with a doubled rate compared with h-refinements. This is consistent with a similar behavior of standard conforming elements.

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