A MAXIMUM-PRINCIPLE PRESERVING C^0 FINITE ELEMENT METHOD FOR SCALAR CONSERVATION EQUATIONS

Jean-Luc Guermond

Department of Mathematics
Texas A&M University

ICES
September 19th 2013,
University of Texas, Austin.

Acknowledgments

Collaborator: Murtazo Nazarov (Texas A&M)

Support:

Outline

1. **Introduction**

Outline

Outline

Outline

Outline

Introduction

Introduction

Formulation of the problem

The PDE

- Scalar conservation equation (u dependent variable)

$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \qquad (\mathbf{x}, t) \in \Omega \times \mathbb{R}_+.$$
$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega.$$

Formulation of the problem

The PDE

- Scalar conservation equation (u dependent variable)

$$\begin{aligned} &\partial_t u + \nabla\cdot \mathbf{f}(u) = 0, && (\mathbf{x}, t) \in \Omega\times\mathbb{R}_+. \\ &u(\mathbf{x}, 0) = u_0(\mathbf{x}), && \mathbf{x} \in \Omega. \end{aligned}$$

- Ω open polyhedral domain in $\mathbb{R}^d$.

Formulation of the problem

The PDE

- Scalar conservation equation (u dependent variable)

$$\begin{aligned} \partial_t u + \nabla\cdot\mathbf{f}(u) &= 0, \qquad (\mathbf{x}, t) \in \Omega\times\mathbb{R}_+. \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega. \end{aligned}$$

- Ω open polyhedral domain in $\mathbb{R}^d$.
- $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^d)$, the flux.

Formulation of the problem

The PDE

- Scalar conservation equation (u dependent variable)

$$\begin{aligned} &\partial_t u + \nabla\cdot\mathbf{f}(u) = 0, \qquad &&(\mathbf{x}, t) \in \Omega\times\mathbb{R}_+. \\ &u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad &&\mathbf{x} \in \Omega. \end{aligned}$$

- Ω open polyhedral domain in $\mathbb{R}^d$.
- $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^d)$, the flux.
- $u_0 \in L^\infty(\Omega)$, initial data.

Formulation of the problem

The PDE

- Scalar conservation equation (u dependent variable)

$$\partial_t u + \nabla\cdot\mathbf{f}(u) = 0, \qquad (\mathbf{x}, t) \in \Omega\times\mathbb{R}_+.$$
$$u(\mathbf{x}, 0) = u_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega.$$

- Ω open polyhedral domain in $\mathbb{R}^d$.
- $\mathbf{f} \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^d)$, the flux.
- $u_0 \in L^\infty(\Omega)$, initial data.
- Periodic BCs or u_0 has compact support (to simply BCs)

Formulation of the problem

Proposition (Entropy condition)

- *The problem has a unique entropy solution $u(x,t)$ such that*

$$\partial_t E(u) + \nabla \cdot \mathbf{F}(u) \leq 0$$

  for all convex entropy $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ and associated entropy flux $\mathbf{F} \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ with $\mathbf{F}_i'(u) = \int_0^u E'(v)\mathbf{f}_i'(v)\,\mathrm{d}v$, $1 \leq i \leq d$.
- **Kruskov (1970)** *and* **Bardos-LeRoux-Nedelec (1979)**.

Maximum principle

Corollary (Maximum principle)

The entropy solution satisfies the maximum principle

$$\inf_{\xi\in\Omega} u_0(\xi) \leq u(x,t) \leq \max_{\xi\in\Omega} u_0(\xi), \qquad a.e. x \in \Omega, t \in \mathbb{R}_+$$

Maximum principle

Corollary (Maximum principle)

The entropy solution satisfies the maximum principle

$$\inf_{\xi\in\Omega} u_0(\xi) \leq u(x,t) \leq \max_{\xi\in\Omega} u_0(\xi), \qquad a.e. x \in \Omega, t \in \mathbb{R}_+$$

Examples

- Traffic flow equation: describes car density on the road (car density: $0 \leq \rho(x,t) \leq \rho_{\text{jam}}$.)

Maximum principle

Corollary (Maximum principle)

The entropy solution satisfies the maximum principle

$$\inf_{\xi\in\Omega} u_0(\xi) \leq u(x,t) \leq \max_{\xi\in\Omega} u_0(\xi), \qquad a.e. x \in \Omega, t \in \mathbb{R}_+$$

Examples

- Traffic flow equation: describes car density on the road (car density: $0 \leq \rho(x,t) \leq \rho_{\text{jam}}$.)
- Buckley-Leverett: describes two-phase flow in porous media (water saturation: $0 \leq s(\mathbf{x},t) \leq 1$)

Finite element approximation

Continuous finite elements

- Let $\{\mathcal{K}_h\}_{h>0}$ be a mesh family (geometrically conforming).

$$X_h = \{v \in \mathcal{C}^0(\Omega;\mathbb{R});\ v|_K \circ \Phi_K \in Q,\ \forall K \in \mathcal{K}_h\},$$

- $\Phi_K : \widehat{K} \longrightarrow K \in \mathcal{K}_h,\quad (\widehat{K}$ reference element).

Finite element approximation

Continuous finite elements

- Let $\{\mathcal{K}_h\}_{h>0}$ be a mesh family (geometrically conforming).

$$X_h = \{v \in \mathcal{C}^0(\Omega;\mathbb{R});\ v|_K \circ \Phi_K \in Q,\ \forall K \in \mathcal{K}_h\},$$

- $\Phi_K : \widehat{K} \longrightarrow K \in \mathcal{K}_h, \quad (\widehat{K}$ reference element).
- Q is a polynomial space that has the property that

$$\min_{\ell \in \mathcal{I}(K)} v(\mathbf{a}_\ell) \le v(\mathbf{x}) \le \max_{\ell \in \mathcal{I}(K)} v(\mathbf{a}_\ell), \quad \forall v \in X_h, \forall \mathbf{x} \in K, \forall K \in \mathcal{K}_h.$$

- Vertices of the mesh $\mathcal{K}_h$: $\{\mathbf{a}_1, \ldots, \mathbf{a}_N\}$.

Finite element approximation

Continuous finite elements

- Let $\{\mathcal{K}_h\}_{h>0}$ be a mesh family (geometrically conforming).

$$X_h = \{v \in \mathcal{C}^0(\Omega;\mathbb{R});\ v|_K \circ \Phi_K \in Q,\ \forall K \in \mathcal{K}_h\},$$

- $\Phi_K : \widehat{K} \longrightarrow K \in \mathcal{K}_h, \quad (\widehat{K}$ reference element).
- Q is a polynomial space that has the property that

$$\min_{\ell \in \mathcal{I}(K)} v(\mathbf{a}_\ell) \le v(\mathbf{x}) \le \max_{\ell \in \mathcal{I}(K)} v(\mathbf{a}_\ell), \quad \forall v \in X_h, \forall \mathbf{x} \in K, \forall K \in \mathcal{K}_h.$$

- Vertices of the mesh $\mathcal{K}_h$: $\{\mathbf{a}_1, \dots, \mathbf{a}_N\}$.
- Ex: $\mathbb{P}_1$, $\mathbb{Q}_1$ (and prismatic $\mathbb{P}_1$, $\mathbb{Q}_1$ elements).

Approximation (time and space)

Objectives

- Approximate conservation equation in time and space.

Approximation (time and space)

Objectives

- Approximate conservation equation in time and space.
- Satisfy maximum principle at every time step

Approximation (time and space)

Objectives

- Approximate conservation equation in time and space.
- Satisfy maximum principle at every time step

Theorem (Godunov)

No linear method solving the linear transport equation can be second-order in space and monotone. **Godunov (1959)**

Approximation (time and space)

First-order viscosity

- Use Galerkin + $\mathcal{C}^0$ finite elements.

Approximation (time and space)

First-order viscosity

- Use Galerkin + $\mathcal{C}^0$ finite elements.
- First-order viscosity + Explicit Euler.

Approximation (time and space)

First-order viscosity

- Use Galerkin + $\mathcal{C}^0$ finite elements.
- First-order viscosity + Explicit Euler.

Am I wasting your time?

Approximation (time and space)

Higher-order in time

- Strong Stability Preserving methods (SSP) **Gottlieb-Shu-Tadmor (2001)**: convex combination of Euler sub-steps.

Approximation (time and space)

Higher-order in time

- Strong Stability Preserving methods (SSP) **Gottlieb-Shu-Tadmor (2001)**: convex combination of Euler sub-steps.
- High-order SSP method preserve the maximum principle.

Approximation (time and space)

Higher-order in time

- Strong Stability Preserving methods (SSP) **Gottlieb-Shu-Tadmor (2001)**: convex combination of Euler sub-steps.
- High-order SSP method preserve the maximum principle.
- Use SSP method to get higher-order in time.

Approximation (time and space)

Higher-order in space

- Use flux correction technique to get higher-order in space (FCT, **Boris-Book (1973)**, **Zalesak (1979)**. Purely algebraic.

Approximation (time and space)

Higher-order in space

- Use flux correction technique to get higher-order in space (FCT, **Boris-Book (1973)**, **Zalesak (1979)**. Purely algebraic.
- Assume u^n satisfies maximum principle.

$$\left.\begin{array}{l} u^n \to u_L^{n+1} \text{satisfies maximum principle} \\ u^n \to u_H^{n+1} \text{high-order method} \end{array}\right\} \overset{FCT}{\Rightarrow} u^n \to u^{n+1} \text{ high-order and max. principle}$$

Approximation (time and space)

Higher-order in space

- Use flux correction technique to get higher-order in space (FCT, **Boris-Book (1973)**, **Zalesak (1979)**. Purely algebraic.
- Assume u^n satisfies maximum principle.

$$\left.\begin{array}{l} u^n \to u_L^{n+1} \text{satisfies maximum principle} \\ u^n \to u_H^{n+1} \text{high-order method} \end{array}\right\} \overset{FCT}{\Rightarrow} u^n \to u^{n+1} \text{ high-order and max. principle}$$

- Similar limiting methods used in FV and DG literature (**Sanders (1988)**, **Jiang-Tadmor (1998)**, **Zhang-Shu (2010), (2011), (2012), (2013)**.

Approximation (time and space)

Galerkin + First-order viscosity + Explicit Euler.

- $\{\varphi_1, \ldots, \varphi_N\}$ nodal Lagrange basis

$$\int_\Omega \frac{u^{n+1} - u^n}{\Delta t} \varphi_i \, \mathrm{d}\mathbf{x} + \int_\Omega \nabla \cdot (\mathbf{f}(u^n)) \varphi_i \, \mathrm{d}\mathbf{x} + \sum_{K \in \mathcal{K}_h} \nu_K \int_K \nabla u^n \cdot \nabla \varphi_i \, \mathrm{d}\mathbf{x} = 0.$$

- ν_K: artificial viscosity. Piece-wise constant.

Approximation (time and space)

Question

What is the optimal value of ν_K?

Shortcomings of the traditional viewpoint

Shortcomings

Mass matrix

Proposition (Mass matrix, JLG + Yong Yang (2013) (and others ...))

The maximum principle can be violated for all values of ν_K if the mass matrix is not lumped.

Mass matrix

Proposition (Mass matrix, JLG + Yong Yang (2013) (and others ...))

The maximum principle can be violated for all values of ν_K if the mass matrix is not lumped.

Mass lumping

The mass matrix must be lumped and dealt with at the flux correction step.

Taking inspiration from DG0

Finite volume/DG0 in 1D

- U_{i-1}^k, U_i^k, U_{i+1}^k approximate u over cells $[x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}]$, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $[x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}]$ at time t^k.

Taking inspiration from DG0

Finite volume/DG0 in 1D

- U_{i-1}^k, U_i^k, U_{i+1}^k approximate u over cells $[x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}]$, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $[x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}]$ at time t^k.
- The Lax-Friedrichs scheme, **Lax (1954)**

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = -(\mathbf{n}_{\Gamma_{i-\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i-1}) + \mathbf{n}_{\Gamma_{i+\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i+1}))$$

Taking inspiration from DG0

Finite volume/DG0 in 1D

- U_{i-1}^k, U_i^k, U_{i+1}^k approximate u over cells $[x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}]$, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $[x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}]$ at time t^k.
- The Lax-Friedrichs scheme, **Lax (1954)**

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = -(\mathbf{n}_{\Gamma_{i-\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i-1}) + \mathbf{n}_{\Gamma_{i+\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i+1}))$$

- Lax-Friedrichs flux (V^i and V^e are the interior and exterior values)

$$\widehat{\mathbf{f}}(V^i, V^e) = \frac{1}{2}(\mathbf{f}(V^i) + \mathbf{f}(V^e)) + \frac{1}{2}\beta(V^i - V^e)\mathbf{n}_\Gamma,$$

- $|\beta| := \|f'\|_{L^\infty(\mathbb{R})}$: maximum wave speed.

Taking inspiration from DG0

Finite volume/DG0 in 1D

- U_{i-1}^k, U_i^k, U_{i+1}^k approximate u over cells $[x_{i-\frac{3}{2}}, x_{i-\frac{1}{2}}]$, $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, $[x_{i+\frac{1}{2}}, x_{i+\frac{3}{2}}]$ at time t^k.
- The Lax-Friedrichs scheme, **Lax (1954)**

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = -(\mathbf{n}_{\Gamma_{i-\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i-1}) + \mathbf{n}_{\Gamma_{i+\frac{1}{2}}} \cdot \widehat{\mathbf{f}}(U_i, U_{i+1}))$$

- Lax-Friedrichs flux (V^i and V^e are the interior and exterior values)

$$\widehat{\mathbf{f}}(V^i, V^e) = \frac{1}{2}(\mathbf{f}(V^i) + \mathbf{f}(V^e)) + \frac{1}{2}\beta(V^i - V^e)\mathbf{n}_\Gamma,$$

- $|\beta| := \|f'\|_{L^\infty(\mathbb{R})}$: maximum wave speed.
- Finally, after re-arrangement:

$$\frac{U_i^{k+1} - U_i^k}{\Delta t} = -\frac{(f(U_{i+1}^k) - f(U_{i-1}^k))}{2h} + \frac{1}{2}|\beta| h \frac{(U_{i+1}^k - 2U_i^k + U_{i-1}^k)}{h^2}.$$

Taking inspiration from DG0

Upwinding

- Lax-Friedrichs flux = upwind flux.

Taking inspiration from DG0

Upwinding

- Lax-Friedrichs flux = upwind flux.
- Ex: transport $\mathbf{f}(u) = \beta e_x$,

$$\widehat{\mathbf{f}}(V^i, V^e) = \begin{cases} V^e \beta \mathbf{e}_x = \mathbf{f}(V^e) & \text{if } \beta \mathbf{e}_x \cdot \mathbf{n}_\Gamma < 0 \\ V_i \beta \mathbf{e}_x = \mathbf{f}(V^i) & \text{otherwise.} \end{cases}$$

Taking inspiration from DG0

Theorem (Maximum principle, Lax (1954))

Assume that $f \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$

$$-\infty < u_{\min} := \min_{x\in\mathbb{R}} u_0(x) \le \min_i U_i^0 \le \max_i U_i^0 \le \max_{x\in\mathbb{R}} u_0(x) := u_{\max} < \infty.$$

Assume that $|\beta|\Delta t \le h$*, then the local discrete maximum principle holds:*

$$u_{\min} \le \min(U_{i-1}^k, U_i^k, U_{i+1}^k) \le U_i^{k+1} \le \max(U_{i-1}^k, U_i^k, U_{i+1}^k) \le u_{\max}.$$

Taking inspiration from DG0

Finite difference interpretation

- Finite volume/DG0 $\Leftrightarrow$ centered finite difference + artificial viscosity $\nu_K = \frac{1}{2}|\beta|\boldsymbol{h}$

Shortcomings of scalar viscosities

Scalar viscosities for finite elements

- Finite volume/DG0 $\Rightarrow \nu_K = c_M|\beta|h$.

Shortcomings of scalar viscosities

Scalar viscosities for finite elements

- Finite volume/DG0 $\Rightarrow \nu_K = c_M|\beta|h$.

What is h?

- What is h on nonuniform meshes? Anisotropic meshes? (Many clever and elegant ideas but no provable maximum principle; **Tezduyar-Osawa (2000)**, **Tezduyar-Sathe (2003)**, **Campbell-Shashkov (2001)**, **Dobrev-Kolev-Rieben (2012)**)

Shortcomings of scalar viscosities

What is c_M?

- What should be the value of c_M? Is $\frac{1}{2}$ universal?

Shortcomings of scalar viscosities

What is c_M?

- What should be the value of c_M? Is $\frac{1}{2}$ universal?
- Same question for stabilized method, what is τ? (one of **T.J.R. Hughes**' research program)

Shortcomings of scalar viscosities

The killing argument

- The key argument for maximum principle (with scalar-valued viscosity) is that

$$\int_{S_{ij}} \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}\mathbf{x} < 0$$

for all pairs of shape functions, φ_i, φ_j, with common support of nonzero measure.

Shortcomings of scalar viscosities

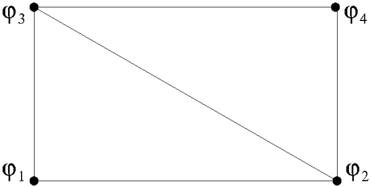
The killing argument

- The key argument for maximum principle (with scalar-valued viscosity) is that

$$\int_{S_{ij}} \nabla\varphi_i \cdot \nabla\varphi_j \,\mathrm{d}\mathbf{x} < 0$$

for all pairs of shape functions, φ_i, φ_j, with common support of nonzero measure.

- Nice Cartesian meshes violate the maximum principle



$$\int_{S_{23}} \nabla\varphi_2 \cdot \nabla\varphi_3 \,\mathrm{d}\mathbf{x} = 0$$

Simplicial meshes

Simplices

Some geometry

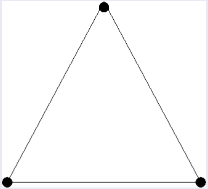
Regular simplices

- Assume $\widehat{K}$ is the regular simplex whose edges all have length 1.

Some geometry

Regular simplices

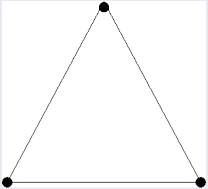
- Assume $\widehat{K}$ is the regular simplex whose edges all have length 1.
- Ex: $\widehat{K}$ is the equilateral triangle of side 1 in two space dimension, and $\widehat{K}$ is the regular tetrahedron (all four faces are equilateral triangles) in three space dimensions.



Some geometry

Regular simplices

- Assume $\widehat{K}$ is the regular simplex whose edges all have length 1.
- Ex: $\widehat{K}$ is the equilateral triangle of side 1 in two space dimension, and $\widehat{K}$ is the regular tetrahedron (all four faces are equilateral triangles) in three space dimensions.



- Assume $\mathcal{K}_h$ composed of affine simplices.

Some geometry

Re-orientation of the gradients

- $\Phi_K : \widehat{K} \longrightarrow K$ affine mapping that transforms $\widehat{K}$ to K.
- $\mathbb{J}_K$ the Jacobian matrix of Φ_K. Chain rule:

$$\nabla(u \circ \Phi_K) = \mathbb{J}_K^{\mathrm{T}}(\nabla u(\Phi_K)).$$

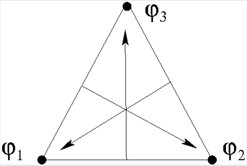
Some geometry

Re-orientation of the gradients

- $\Phi_K : \widehat{K} \longrightarrow K$ affine mapping that transforms $\widehat{K}$ to K.
- $\mathbb{J}_K$ the Jacobian matrix of Φ_K. Chain rule:

$$\nabla(u \circ \Phi_K) = \mathbb{J}_K^{\mathrm{T}}(\nabla u(\Phi_K)).$$

- Gradients of two reference shape functions $\nabla(\varphi_i \circ \Phi_K)$ and $\nabla(\varphi_j \circ \Phi_K)$ form a maximal angle.



Some geometry

Lemma (Re-orientation of the gradients)

$$\int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -\frac{1}{d\widehat{h}^2}|K|, \qquad \forall K \in \mathcal{K}_h, \quad \forall i, \forall j \in S(K), j \neq i.$$

$$\int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = \frac{1}{\widehat{h}^2}|K|, \qquad \forall K \in \mathcal{K}_h, \quad \forall i \in S(K).$$

Some geometry

Lemma (Re-orientation of the gradients)

$$\int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -\frac{1}{d\widehat{h}^2}|K|, \qquad \forall K \in \mathcal{K}_h, \quad \forall i, \forall j \in S(K), j \neq i.$$

$$\int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = \frac{1}{\widehat{h}^2}|K|, \qquad \forall K \in \mathcal{K}_h, \quad \forall i \in S(K).$$

Examples

$$\int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -|K|\begin{cases}\frac{2}{3} & \text{in 2D}\\ \frac{1}{2} & \text{in 3D.}\end{cases}$$

$$\int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = |K|\begin{cases}\frac{4}{3} & \text{in 2D}\\ \frac{3}{2} & \text{in 3D.}\end{cases}$$

Some geometry

Re-orientation of the gradients

$\mathbb{J}_K^{\mathrm{T}}(\nabla\varphi_i)\cdot\mathbb{J}_K^{\mathrm{T}}(\nabla\varphi_j)$ has exactly the right property we are looking for.

Tensor-valued viscosity

Re-orientation of the gradients

Define the artificial viscosity bilinear form:

$$b(u, \varphi_i) = \sum_{K \subset S_i} \int_K \nu_K^k (\mathbb{J}_K^{\mathrm{T}} \nabla u_h^k) \cdot (\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_i) \, \mathrm{d}\mathbf{x}$$

The scheme

The scheme

- Define $u^{k+1} = \sum_{i=1}^{N} U_i^{k+1} \varphi_i$.

The scheme

The scheme

- Define $u^{k+1} = \sum_{i=1}^{N} U_i^{k+1} \varphi_i$.
- Define $m_i := \int_{S_i} \varphi_i \,\mathrm{d}\mathbf{x}$ (lumped mass matrix coefficients)

The scheme

The scheme

- Define $u^{k+1} = \sum_{i=1}^{N} U_i^{k+1}\varphi_i$.
- Define $m_i := \int_{S_i} \varphi_i \,\mathrm{d}\mathbf{x}$ (lumped mass matrix coefficients)
- Define U_i^{k+1} so that

$$U_i^{k+1} = U_i^k - \Delta t^k m_i^{-1} \sum_{K\subset S_i} \int_K \left(\nu_K^k (\mathbb{J}_K^{\mathrm{T}} \nabla u_h^k)\cdot(\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_i) + \nabla\cdot(\mathbf{f}(u_h^k))\varphi_i \right) \mathrm{d}\mathbf{x}.$$

The viscosity

Observe the convex combination:

$$U_i^{k+1} = U_i^k \Big(1 - \Delta t^k m_i^{-1} \sum_{K\subset S_i} \int_K \Big(\nu_K^k \|\mathbb{J}_K^{\mathrm{T}} \nabla\varphi_i\|^2 + (\mathbf{f}'(u_h^k)\cdot\nabla\varphi_i)\varphi_i\Big)\,\mathrm{d}\mathbf{x}\Big)$$
$$- \Delta t^k m_i^{-1} \sum_{\mathcal{I}(S_i)\ni j\neq i} U_j^k \sum_{K\subset S_{ij}} \int_K \Big(\nu_K^k (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i) + (\mathbf{f}'(u_h^k)\cdot\nabla\varphi_j)\varphi_i\Big)\,\mathrm{d}\mathbf{x}.$$

Definition (Viscosity)

Define ν_K on each cell so that

$$\nu_K^k = \max_{i\neq j\in\mathcal{I}(K)} \frac{\left|\int_{S_{ij}} (\mathbf{f}'(u_h^k)\cdot\nabla\varphi_j)\varphi_i\,\mathrm{d}\mathbf{x}\right|}{-\int_{S_{ij}} (\mathbb{J}^{\mathrm{T}}\nabla\varphi_j)\cdot(\mathbb{J}^{\mathrm{T}}\nabla\varphi_i)\,\mathrm{d}\mathbf{x}}.$$

Maximum principle

Theorem (Maximum principle (JLG-Nazarov (2013)))

Assume that $\mathbf{f} \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$ *and let* $u_{\min} := \inf_{\mathbf{x}\in\mathbb{R}^d} u_0(\mathbf{x})$, $u_{\max} := \sup_{\mathbf{x}\in\mathbb{R}^d} u_0(\mathbf{x})$, *and* $\beta = \max_{v\in[u_{\min},u_{\max}]} \|\mathbf{f}'(v)\|$. *Assume that*

$$u_{\min} \le U_i^0 \le u_{\max}, \quad \forall i = 1,\dots,N, \quad \textit{and} \quad \beta\Delta t^k h^{-1} \le 1/(c_I(1+d)).$$

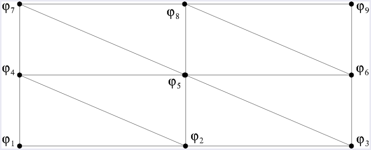
Then the local discrete maximum principle holds

$$u_{\min} \le \min_{j\in\mathcal{I}(S_i)} U_j^k \le U_i^{k+1} \le \max_{j\in\mathcal{I}(S_i)} U_j^k \le u_{\max}, \quad \forall k \ge 0.$$

Example

Cartesian mesh

- Consider linear transport equation: $\mathbf{f}(u) = \boldsymbol{\beta} u$ on following grid



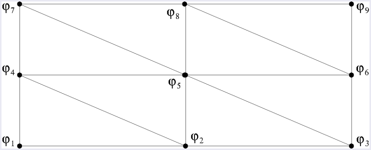
- Viscosity given by

$$\nu_K^k = \max_{i \neq j \in \mathcal{I}(K)} \frac{\left| \int_{S_{ij}} (\boldsymbol{\beta} \cdot \nabla \phi_j) \varphi_i \, \mathrm{d}\mathbf{x} \right|}{\frac{4}{3}|K|} = \frac{1}{4} \max\left(2\frac{|\beta_x|}{h_x} + \frac{|\beta_y|}{h_y}, \frac{|\beta_x|}{h_x} + 2\frac{|\beta_y|}{h_y}\right),$$

Example

Cartesian mesh

- Consider linear transport equation: $\mathbf{f}(u) = \boldsymbol{\beta} u$ on following grid

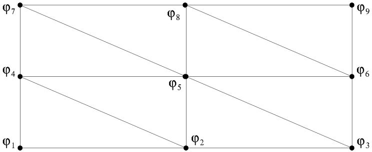


- Viscosity given by

$$\nu_K^k = \max_{i \neq j \in \mathcal{I}(K)} \frac{\left| \int_{S_{ij}} (\boldsymbol{\beta} \cdot \nabla \phi_j) \varphi_i \, \mathrm{d}\mathbf{x} \right|}{\frac{4}{3}|K|} = \frac{1}{4} \max(2\frac{|\beta_x|}{h_x} + \frac{|\beta_y|}{h_y}, \frac{|\beta_x|}{h_x} + 2\frac{|\beta_y|}{h_y}),$$

- Observe that $\nu_K \sim$ speed/meshsize

Example



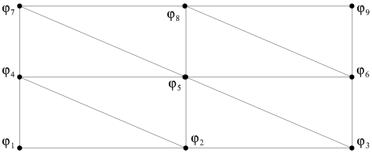
Cartesian mesh

- The scheme:

$$\frac{U_5^{k+1}-U_5^k}{\Delta t^k}+\frac{1}{3}(2\beta_x+\beta_y\frac{h_x}{h_y})\frac{U_6^k-U_4^k}{2h_x}+\frac{1}{3}(2\beta_y+\beta_x\frac{h_y}{h_x})\frac{U_8^k-U_2^k}{2h_y}$$
$$+\frac{1}{3}(\frac{\beta_x}{h_x}-\frac{\beta_y}{h_y})\sqrt{h_x^2+h_y^2})\frac{U_3^k-U_7^k}{2\sqrt{h_x^2+h_y^2}}-\frac{Bh_x^2}{6}\frac{U_6^k-2U_5^k+U_4^k}{h_x^2}$$
$$-\frac{Bh_y^2}{6}\frac{U_8^k-2U_5^k+U_2^k}{h_x^2}-\frac{B(h_x^2+h_y^2)}{6}\frac{U_7^k-2U_5^k+U_3^k}{h_x^2+h_y^2}=0.$$

where $B=\max(2\frac{|\beta_x|}{h_x}+\frac{|\beta_y|}{h_y},\frac{|\beta_x|}{h_x}+2\frac{|\beta_y|}{h_y})$.

Example



Cartesian mesh

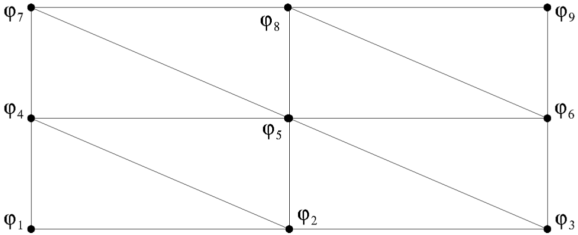
- The scheme:

$$\frac{U_5^{k+1}-U_5^k}{\Delta t^k} + \frac{1}{3}(2\beta_x + \beta_y\frac{h_x}{h_y})\frac{U_6^k-U_4^k}{2h_x} + \frac{1}{3}(2\beta_y + \beta_x\frac{h_y}{h_x})\frac{U_8^k-U_2^k}{2h_y}$$
$$+ \frac{1}{3}(\frac{\beta_x}{h_x} - \frac{\beta_y}{h_y})\sqrt{h_x^2+h_y^2})\frac{U_3^k-U_7^k}{2\sqrt{h_x^2+h_y^2}} - \frac{Bh_x^2}{6}\frac{U_6^k-2U_5^k+U_4^k}{h_x^2}$$
$$- \frac{Bh_y^2}{6}\frac{U_8^k-2U_5^k+U_2^k}{h_x^2} - \frac{B(h_x^2+h_y^2)}{6}\frac{U_7^k-2U_5^k+U_3^k}{h_x^2+h_y^2} = 0.$$

where $B = \max(2\frac{|\beta_x|}{h_x} + \frac{|\beta_y|}{h_y}, \frac{|\beta_x|}{h_x} + 2\frac{|\beta_y|}{h_y})$.

- Right amount of viscosity put in each direction.

Example: Cartesian mesh



Dir.	h	speed	$\frac{1}{2}\lvert\text{speed}\rvert \times h$	viscosity
$\mathbf{a}_4\mathbf{a}_6$	h_x	$\frac{1}{3}(2\beta_x + \beta_y \frac{h_x}{h_y})$	$\frac{1}{6}(2\frac{\lvert\beta_x\rvert}{h_x} + \frac{\lvert\beta_y\rvert}{h_y})h_x^2$	$\frac{Bh_x^2}{6}$
$\mathbf{a}_2\mathbf{a}_8$	h_y	$\frac{1}{3}(2\beta_y + \beta_x \frac{h_y}{h_x})$	$\frac{1}{6}(2\frac{\lvert\beta_y\rvert}{h_y} + \frac{\lvert\beta_x\rvert}{h_x})h_y^2$	$\frac{Bh_y^2}{6}$
$\mathbf{a}_7\mathbf{a}_3$	$\sqrt{h_x^2 + h_y^2}$	$\frac{1}{3}(\frac{\beta_x}{h_x} - \frac{\beta_y}{h_y})\sqrt{h_x^2 + h_y^2}$	$\frac{1}{6}(\frac{\lvert\beta_x\rvert}{h_x} + \frac{\lvert\beta_y\rvert}{h_y})(h_x^2 + h_y^2)$	$\frac{B(h_x^2+h_y^2)}{6}$

$$B = \max(2\frac{|\beta_x|}{h_x} + \frac{|\beta_y|}{h_y}, \frac{|\beta_x|}{h_x} + 2\frac{|\beta_y|}{h_y}).$$

Generalization

Generalization

The cubic obstruction

The cubic obstruction

- Consider a cubic uniform mesh, $\mathbb{J}_K = h\mathbb{I}$.

The cubic obstruction

The cubic obstruction

- Consider a cubic uniform mesh, $\mathbb{J}_K = h\mathbb{I}$.
- Use $\mathbb{Q}_1$ finite elements

The cubic obstruction

The cubic obstruction

- Consider a cubic uniform mesh, $\mathbb{J}_K = h\mathbb{I}$.
- Use $\mathbb{Q}_1$ finite elements
- Consider two shape functions, φ_i, φ_j, on the same edge of an element K,

$$\int_K (\mathbb{J}_K^{\mathrm{T}} \nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}} \nabla\varphi_j)\,\mathrm{d}\mathbf{x} = h^3 \int_K \nabla\varphi_i\cdot\nabla\varphi_j\,\mathrm{d}\mathbf{x} = 0.$$

The cubic obstruction

The cubic obstruction

- Consider a cubic uniform mesh, $\mathbb{J}_K = h\mathbb{I}$.
- Use $\mathbb{Q}_1$ finite elements
- Consider two shape functions, φ_i, φ_j, on the same edge of an element K,

$$\int_K (\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_i) \cdot (\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_j) \, \mathrm{d}\mathbf{x} = h^3 \int_K \nabla \varphi_i \cdot \nabla \varphi_j \, \mathrm{d}\mathbf{x} = 0.$$

$\Rightarrow$ Above definition of ν_K cannot work!

Back to basics

Key property (for simplices)

$$b_K(\varphi_i,\varphi_j) = \int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -|K|\begin{cases}\frac{2}{3} & \text{in 2D}\\ \frac{1}{2} & \text{in 3D.}\end{cases}$$

$$b_K(\varphi_i,\varphi_i) = \int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = |K|\begin{cases}\frac{4}{3} & \text{in 2D}\\ \frac{3}{2} & \text{in 3D.}\end{cases}$$

Back to basics

Key property (for simplices)

$$b_K(\varphi_i, \varphi_j) = \int_K (\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_i) \cdot (\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_j) \, \mathrm{d}\mathbf{x} = -|K| \begin{cases} \frac{2}{3} & \text{in 2D} \\ \frac{1}{2} & \text{in 3D.} \end{cases}$$

$$b_K(\varphi_i, \varphi_i) = \int_K \|\mathbb{J}_K^{\mathrm{T}} \nabla \varphi_i\|^2 \, \mathrm{d}\mathbf{x} = |K| \begin{cases} \frac{4}{3} & \text{in 2D} \\ \frac{3}{2} & \text{in 3D.} \end{cases}$$

- $b_K(\varphi_i, \varphi_j) = 0 \quad$ if $i \notin \mathcal{I}(K)$ or $j \notin \mathcal{I}(K)$.

Back to basics

Key property (for simplices)

$$b_K(\varphi_i,\varphi_j) = \int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -|K| \begin{cases} \frac{2}{3} & \text{in 2D} \\ \frac{1}{2} & \text{in 3D.} \end{cases}$$

$$b_K(\varphi_i,\varphi_i) = \int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = |K| \begin{cases} \frac{4}{3} & \text{in 2D} \\ \frac{3}{2} & \text{in 3D.} \end{cases}$$

- $b_K(\varphi_i,\varphi_j) = 0 \quad$ if $i \notin \mathcal{I}(K)$ or $j \notin \mathcal{I}(K)$.
- $b_K(\varphi_i,\varphi_j) = c|K|$, $i \neq j$.

Back to basics

Key property (for simplices)

$$b_K(\varphi_i,\varphi_j) = \int_K (\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i)\cdot(\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_j)\,\mathrm{d}\mathbf{x} = -|K|\begin{cases}\frac{2}{3} & \text{in 2D}\\ \frac{1}{2} & \text{in 3D.}\end{cases}$$

$$b_K(\varphi_i,\varphi_i) = \int_K \|\mathbb{J}_K^{\mathrm{T}}\nabla\varphi_i\|^2\,\mathrm{d}\mathbf{x} = |K|\begin{cases}\frac{4}{3} & \text{in 2D}\\ \frac{3}{2} & \text{in 3D.}\end{cases}$$

- $b_K(\varphi_i,\varphi_j) = 0$ if $i \notin \mathcal{I}(K)$ or $j \notin \mathcal{I}(K)$.
- $b_K(\varphi_i,\varphi_j) = c|K|$, $i \neq j$.
- $\sum_{j\neq i} b_K(\varphi_i,\varphi_j) = -b_K(\varphi_i,\varphi_i)$ (conservation).

Generalization

The viscosity

- Let K be a cell in $\mathcal{K}$. Let n_K be the number of vertices in K.

Generalization

The viscosity

- Let K be a cell in $\mathcal{K}$. Let n_K be the number of vertices in K.
- Define

$$b_K(\varphi_j,\varphi_i) = \begin{cases} -\frac{1}{n_K-1}|K| & \text{if } i \neq j, \quad i,j \in \mathcal{I}(K), \\ |K| & \text{if } i = j, \quad i,j \in \mathcal{I}(K), \\ 0 & \text{if } i \notin \mathcal{I}(K) \text{ or } j \notin \mathcal{I}(K). \end{cases}$$

Generalization

The viscosity

- Let K be a cell in $\mathcal{K}$. Let n_K be the number of vertices in K.
- Define

$$b_K(\varphi_j,\varphi_i) = \begin{cases} -\frac{1}{n_K-1}|K| & \text{if } i\neq j, \quad i,j\in\mathcal{I}(K),\\ |K| & \text{if } i=j, \quad i,j\in\mathcal{I}(K),\\ 0 & \text{if } i\notin\mathcal{I}(K) \text{ or } j\notin\mathcal{I}(K).\end{cases}$$

- Define ν_K

$$\nu_K^k = \max_{i\neq j\in\mathcal{I}(K)} \frac{\left|\int_{S_{ij}}(\mathbf{f}'(u_h^k)\cdot\nabla\varphi_j)\varphi_i\,\mathrm{d}\mathbf{x}\right|}{-\sum_{T\subset S_{ij}} b_T(\varphi_j,\varphi_i)}.$$

Generalization

The viscosity

- Observe the automatic rescaling of ν_K (i.e. global scaling of b_K does not matter)

Generalization

The viscosity

- Observe the automatic rescaling of ν_K (i.e. global scaling of b_K does not matter)
- New definition coincides with definition for simplicial meshes.

Generalization

The viscosity

- Observe the automatic rescaling of ν_K (i.e. global scaling of b_K does not matter)
- New definition coincides with definition for simplicial meshes.
- No restriction on the mesh geometry.

Generalization

The viscosity

- Observe the automatic rescaling of ν_K (i.e. global scaling of b_K does not matter)
- New definition coincides with definition for simplicial meshes.
- No restriction on the mesh geometry.
- Note similarity with graph Laplacian: $\sum_{j\in\mathcal{I}(S_i)} w_{ij}(U_i^k - U_j^k)$ with weights $\{w_{ij}\}_{i,j=1\ldots,N}$.

$$b(u_h^k,\varphi_i) = \sum_{K\subset S_i} \nu_K^k \sum_{i\neq j\in\mathcal{I}(K)} (U_i^k - U_j^k)\frac{|K|}{n_K - 1},$$

The scheme

The scheme

- Advance in time as follows:

$$U_i^{k+1} = U_i^k - \Delta t^k m_i^{-1} \sum_{K \subset S_i} \left(\nu_K^k b_K(u_h^k, \varphi_i) + \int_K \nabla \cdot (\mathbf{f}(u_h^k)) \varphi_i \, \mathrm{d}\mathbf{x} \right).$$

Maximum principle

Theorem (Maximum principle (JLG-Nazarov (2013)))

Assume that $\mathbf{f} \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^d)$ *and let* $u_{\min} := \inf_{\mathbf{x}\in\mathbb{R}^d} u_0(\mathbf{x})$, $u_{\max} := \sup_{\mathbf{x}\in\mathbb{R}^d} u_0(\mathbf{x})$, *and* $\beta = \max_{v\in[u_{\min},u_{\max}]} \|\mathbf{f}'(v)\|$. *Let*

$$\rho := \min_{K\in\mathcal{K}_h} \frac{1}{n_K - 1}.$$

Assume that

$$u_{\min} \le U_i^0 \le u_{\max}, \quad \textit{for all } i = 1,\ldots,N, \textit{ and } \quad \beta \Delta t^k h^{-1} \le 1/(c_I(1+\rho^{-1})).$$

Then the local discrete maximum principle holds,

$$u_{\min} \le \min_{j\in\mathcal{I}(S_i)} U_j^k \le U_i^{k+1} \le \max_{j\in\mathcal{I}(S_i)} U_j^k \le u_{\max} \quad \forall k \ge 0.$$

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.
- Define entropy residual over $K \in \mathcal{K}_h$ as follows:

$$R_K^k(u_h) = \| \frac{1}{\Delta t^{k-1}} (E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k) \cdot \nabla E(u_h^k) \|_{L^\infty(K)}. \tag{1}$$

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.
- Define entropy residual over $K \in \mathcal{K}_h$ as follows:

$$R_K^k(u_h) = \|\frac{1}{\Delta t^{k-1}}(E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k)\cdot\nabla E(u_h^k)\|_{L^\infty(K)}. \tag{1}$$

- Define entropy jump across cell interfaces

$$J_F^k(u_h) = \|\mathbf{f}'(u_h^k)\cdot\mathbf{n}[\![\partial_n E(u_h^k)]\!]\|_{L^\infty(F)}. \tag{2}$$

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.
- Define entropy residual over $K \in \mathcal{K}_h$ as follows:

$$R_K^k(u_h) = \|\tfrac{1}{\Delta t^{k-1}}(E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k)\cdot\nabla E(u_h^k)\|_{L^\infty(K)}. \tag{1}$$

- Define entropy jump across cell interfaces

$$J_F^k(u_h) = \|\mathbf{f}'(u_h^k)\cdot\mathbf{n}[\![\partial_n E(u_h^k)]\!]\|_{L^\infty(F)}. \tag{2}$$

- Let $\nu_K^{v,k}$ be the first-order viscosity.

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.
- Define entropy residual over $K \in \mathcal{K}_h$ as follows:

$$R_K^k(u_h) = \|\tfrac{1}{\Delta t^{k-1}}(E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k)\cdot\nabla E(u_h^k)\|_{L^\infty(K)}. \qquad (1)$$

- Define entropy jump across cell interfaces

$$J_F^k(u_h) = \|\mathbf{f}'(u_h^k)\cdot\mathbf{n}[\![\partial_n E(u_h^k)]\!]\|_{L^\infty(F)}. \qquad (2)$$

- Let $\nu_K^{v,k}$ be the first-order viscosity.
- Define entropy viscosity

$$\nu_K^k := \min(\nu_K^{v,k}, \frac{c_E R_K^k + c_J \max_{F\in\partial K} J_F^k}{\|E(u_h^k) - \overline{E}(u_h^k)\|_{L^\infty(\Omega)}})$$

$c_E \sim 1$ and $c_J \sim 1$ parameters.

High-order extension (Entropy viscosity)

Entropy viscosity

- Let $E \in \mathcal{C}^2(\mathbb{R};\mathbb{R})$ be a convex entropy with associated flux $\mathbf{F}$.
- Define entropy residual over $K \in \mathcal{K}_h$ as follows:

$$R_K^k(u_h) = \|\frac{1}{\Delta t^{k-1}}(E(u_h^k) - E(u_h^{k-1})) + \mathbf{f}'(u_h^k)\cdot\nabla E(u_h^k)\|_{L^\infty(K)}. \tag{1}$$

- Define entropy jump across cell interfaces

$$J_F^k(u_h) = \|\mathbf{f}'(u_h^k)\cdot\mathbf{n}[\![\partial_n E(u_h^k)]\!]\|_{L^\infty(F)}. \tag{2}$$

- Let $\nu_K^{v,k}$ be the first-order viscosity.
- Define entropy viscosity

$$\nu_K^k := \min(\nu_K^{v,k}, \frac{c_E R_K^k + c_J \max_{F\in\partial K} J_F^k}{\|E(u_h^k) - \overline{E}(u_h^k)\|_{L^\infty(\Omega)}})$$

  $c_E \sim 1$ and $c_J \sim 1$ parameters.
- Observe that $\nu_K^{v,k} \sim$ speed/meshsize $\sim R_K^k \sim J_F^k$,
  (dimensions are correct now! no need for h!).

Numerical illustration

Numerical illustration

2D Burgers

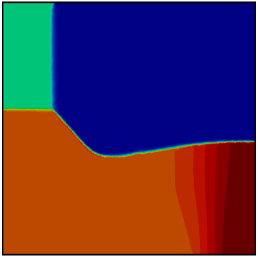
2D Burgers

$$\partial_t u + \nabla\cdot\left(\tfrac{1}{2}\boldsymbol{\beta} u^2\right) = 0, \qquad u(\mathbf{x},0) = u_0(\mathbf{x}),$$

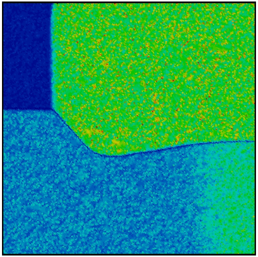
where $\boldsymbol{\beta} = (1,1)$ constant vector field, and initial condition is

$$u_0 = \begin{cases} -0.2, & \text{if } x < 0.5 \text{ and } y > 0.5, \\ -1, & \text{if } x > 0.5 \text{ and } y > 0.5, \\ 0.5, & \text{if } x < 0.5 \text{ and } y < 0.5, \\ 0.8, & \text{if } x > 0.5 \text{ and } y < 0.5. \end{cases}$$

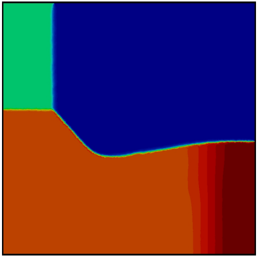
2D Burgers



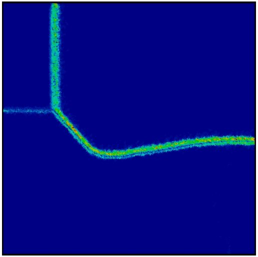
(a) Viscous solution, $t = 0.5$



(b) $\nu_K^{v,k}$



(c) Entropy solution, $t = 0.5$



(d) Entropy viscosity

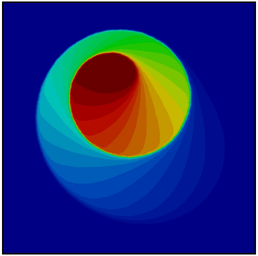
KPP rotating wave

KPP rotating wave, Kurganov-Petrova-Popov (2007)

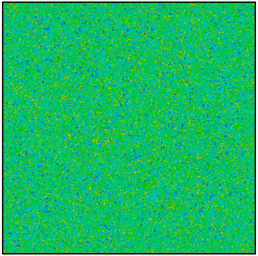
$$\partial_t u + \nabla \cdot \mathbf{f}(u) = 0, \quad u(\mathbf{x}, 0) = u_0(\mathbf{x}) = \begin{cases} \frac{14\pi}{4}, & \text{if } \sqrt{x^2 + y^2} \leq 1, \\ \frac{\pi}{4}, & \text{otherwise.} \end{cases},$$

where $\mathbf{f}(u) = (\sin u, \cos u)$.

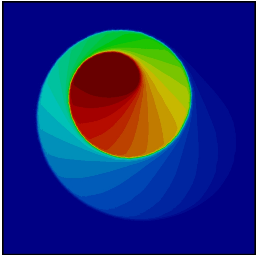
KPP rotating wave: First-order vs. Entropy viscosity



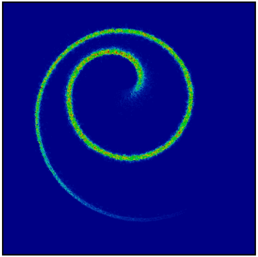
(a) Viscous solution, $t = 1$



(b) $\nu_K^{v,k}$

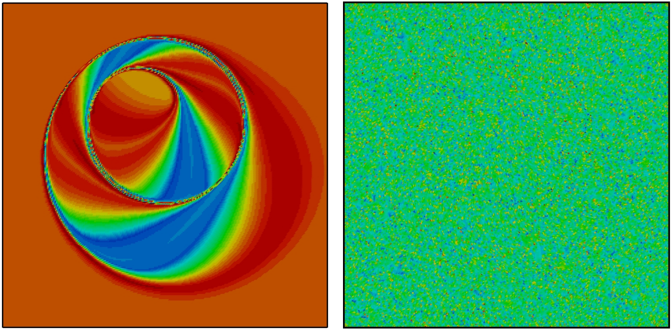


(c) Entropy solution, $t = 1$



(d) Entropy viscosity

KPP rotating wave: Uniform mesh vs. non-uniform



(a) $\nu_K^{v,k}$, uniform grid (b) $\nu_K^{v,k}$, non-uniform grid

Two wave problem

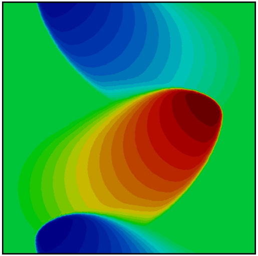
Two wave problem Holden (2010)

$$\partial_t u + \partial_x(\tfrac{1}{2}u^2) + \partial_y(\tfrac{1}{3}u^3) = 0, \quad u(\mathbf{x}, 0) = u_0(\mathbf{x}),$$

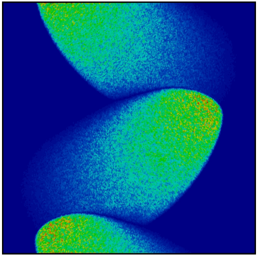
with periodic boundary conditions and initial condition:

$$u_0 = \begin{cases} -1, & \text{if } (x-0.5)^2 + (y-0.5)^2 < 0.16, \\ 1, & \text{if } (x+0.5)^2 + (y+0.5)^2 < 0.16, \\ 0, & \text{otherwise} . \end{cases}$$

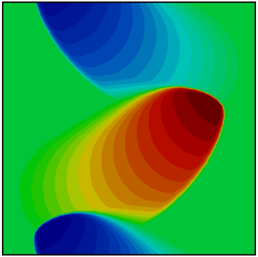
Two wave problem: First-order vs. Entropy viscosity



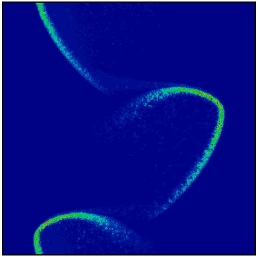
(a) Viscous solution, $t = 2$



(b) $\nu_K^{v,k}$



(c) Entropy solution, $t = 2$



(d) Entropy viscosity