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AN OVERVIEW OF THE DPG METHOD

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Abstract

We dicuss our current understanding of the Discontinuous Petrov Galerkin (DPG) Method with Optimal Test Functions and provide a literature review on the subject.

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1 Introduction

The adventure with the DPG method started in Spring 2009. Analyzing spectral methods for the simplest 1D convection problem, we realized that the choice of test function $v = u$ leading to the standard DG method was far from an optimal one in terms of implied stability properties [19]. The main breakthrough came with a realization that the use of ultraweak variational formulation and discontinuous test functions allowed for the computation of (approximate) optimal test functions [21]. After reporting the exciting results in a Mafelap plenary talk, in June 2009, we had learned that we owned neither the concept of the ultraweak formulation nor even the name - the Discontinuous Petrov Galerkin (DPG) method. Both were introduced several years earlier by the Italian colleagues [3, 4, 10, 11] [1].

But the concept of computing the optimal test functions on the fly was new, and we pursued a numerical implementation of hp-adaptivity quickly in [23] demonstrating the superior stability properties of the new method.

[1]To our credit, the ultraweak formulation was used at that point very formally, without a proper Functional Analysis setting which we established later in [20].

We devoted a considerable amount of our time and resources to the DPG research in the next three years. As it usually happens, our understanding did not grow in a systematic, "monotone" mode and, hence, attempting to follow the DPG work in a chronological order would rather be confusing. Instead we present a review of the main concepts behind the DPG methodology as we understand them today: minimization of residuals in dual norms in Section 2, use of discontinuous test functions in Section 3, ultraweak variational formulations in Section 4, selection of optimal test norm for singular perturbation problems in Section 5, and the important interpretation of the DPG method as a localization of the PG method with global optimal test functions in Section 6. We conclude with an outline of our current work in Section 7.

The work on the DPG methodology has barely begun and we hope that more colleagues will get interested in the subject and join us in this endeavor.

2 DPG is a Minimum Residual Method

DPG, like least squares, belongs to the class of *minimum residual methods*. We start with a (linear) variational problem,

$$\begin{cases} u \in U \\ b(u,v) = l(v) \quad \forall v \in V \, . \end{cases} \tag{2.1}$$

Here U is a *trial space* and V is a *test space*. We shall assume that both U and V are Hilbert spaces, b is a sesquilinear (bilinear in the real case) and continuous form on $U \times V$ and l is an antilinear (linear) continuous form on V, i.e. an element of the dual space V'. It is well known that every such form $b(u,v)$ generates two linear operators,

$$B : U \to V', \quad B' : V \to U', \quad b(u,v) = \langle Bu, v \rangle_{V' \times V}, \quad \overline{b(u,v)} = \langle B'v, u \rangle_{U' \times U} \quad u \in U, v \in V \, . \tag{2.2}$$

As every Hilbert spaces is reflexive, i.e. it is isomorphic and isometric with its bidual, the two maps are actually conjugates of each other. The abstract variational problem (2.1) is equivalent to the operator equation:

$$Bu = l \, . \tag{2.3}$$

One might argue that the nature of variational problems lies in the fact that the corresponding operator takes values in a dual space.

Banach Closed Range Theorem states that the following four conditions are equivalent to each other:

$$\begin{gathered} B \textit{ has closed range,} \\ B' \textit{ has closed range,} \\ B|_{\mathcal{N}(B)^\perp} \textit{ is bounded below,} \\ B'|_{\mathcal{N}(B')^\perp} \textit{ is bounded below.} \end{gathered}$$

Thus, at the expense of replacing U with the orthogonal complement of $\mathcal{N}(B)$, and V with the orthogonal complement of $\mathcal{N}(B')$, we can assume that both B and B' are bounded below,

$$\|Bu\|_{V'} \geq \gamma \|u\|_U \quad \forall u \in U, \qquad \|B'v\|_{U'} \geq \gamma \|v\|_V \quad \forall v \in V \, . \tag{2.4}$$

Notice that the constant $\gamma = \|B^{-1}\| = \|(B')^{-1}\|$ is the same for both operators.

Let $U_h \subset U$ be now a finite-dimensional approximate trial space. The minimum residual method seeks a solution $u_h \in U_h$ that minimizes the corresponding residual:

$$u_h = \text{arg min } J(w_h), \quad J(w_h) := \frac{1}{2}\|Bw_h - l\|^2_{V'} \, . \tag{2.5}$$

Of course, we have squared the norm of the residual and placed the half in front of it for elegance only. The use of the dual norm is a must, the operator takes values in the dual space V'. Problem (2.5) is equivalent to the minimization of the quadratic functional:

$$\frac{1}{2}(Bu_h, Bu_h)_{V'} - \text{Re}(Bu_h, l)_{V'} \tag{2.6}$$

where $(\cdot, \cdot)_{V'}$ denotes the inner product in the dual space V'. Indeed, boundedess below of B implies that the sesquilinear form:

$$(Bu_h, Bw_h)_{V'}$$

is U-coercive. Consequently, the minimum residual method can be classified as the classical Ritz method that experiences no preasymptotic behavior and delivers the best approximation error in the energy norm:

$$\|u\|_E := \|Bu\|_{V'} = \sup_{v \neq 0} \frac{|\langle Bu, v \rangle|}{\|v\|_V} = \sup_{v \neq 0} \frac{|b(u,v)|}{\|v\|_V} \, . \tag{2.7}$$

The dual norm, induced by the norm in test space[2],

$$\|l\|_{V'} = \sup_{v \neq 0} \frac{|l(v)|}{\|v\|_V} = \sup_{\|v\|_V \leq 1} |l(v)| = \sup_{\|v\|_V = 1} |l(v)| \, , \tag{2.8}$$

is not available analytically, unless we are dealing with the L^2-norm (possibly with a weight). We cannot thus compute directly with the dual norm. Coming to the rescue is the Riesz operator for the test space:

$$R_V \, : \, V \ni v \rightarrow (v, \cdot) \in V' \, , \tag{2.9}$$

which is an isometric isomorphism. At the expense of introducing the inverse of the Riesz operator, we can now trade the dual norm for the test norm, and reformulate the minimum residual method (2.5) in a new form,

$$u_h = \text{arg min } J(w_h), \quad J(w_h) = \frac{1}{2}\|R_V^{-1}(Bw_h - l)\|^2_V \, . \tag{2.10}$$

[2]For Hilbert space, the supremum is attained and can be replaced with maximum.

Computing the Gâteaux derivative of the quadratic functional,

$$\langle \delta J(u_h); \delta u_h \rangle = \text{Re}\, (R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u_h)_V \,, \tag{2.11}$$

we arrive at the linear problem equivalent[3] to minimization problem (2.10):

$$\begin{cases} u_h \in U_h \\ (R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u_h)_V = 0 \quad \forall \delta u_h \in U_h \,. \end{cases} \tag{2.12}$$

There are two ways to proceed now.

Petrov-Galerkin Method with Optimal Test Functions. We introduce the *trial-to-test operator*:

$$T \,: U_h \to V, \quad T := R_V^{-1} B \,, \tag{2.13}$$

with the corresponding range $V_h := U_h$ identified as the *optimal test functions*. The linear problem (2.12) reduces to:

$$(R_V^{-1}(Bu_h - l), v_h)_V = 0 \quad \forall v_h \in V_h := TU_h \,. \tag{2.14}$$

Recalling the definition of Riesz operator, we can rewrite it in the variational form:

$$\begin{cases} u_h \in U_h \\ b(u_h, v_h) = l(v_h) \quad \forall v_h \in V_h \end{cases} \tag{2.15}$$

The minimum residual method is thus *equivalent* to a Petrov-Galerkin method with the optimal test functions. Computation of the optimal test functions involves inverting the Riesz operator,

$$\begin{cases} v_{\delta u_h} \in V \\ (v_{\delta u_h}, \delta v)_V = b(\delta u_h, \delta v) \quad \forall \delta v \in V \,, \end{cases} \tag{2.16}$$

and, unfortunately, it is worth the solution of another boundary-value problem. Consequently, we have not got any practical method yet.

Being a minimum residual method[4], the PG method (2.15) delivers a hermitian, positive-definite stiffness matrix. Indeed, utilizing (2.16), we get:

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})} \,. \tag{2.17}$$

The energy norm of the Galerkin error equals the residual and can be computed without knowing the exact solution,

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \|R_V^{-1}(Bu_h - l)\|_V \,. \tag{2.18}$$

[3] Functional $I(\delta u_h) := (R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u_h)_V$ is antilinear. Real part of an antilinear functional vanishes if and only if the whole functional vanishes. This follows from the fact that, for any antilinear functional $I(v)$, $\text{Im}\, I(v) = \text{Re}\, I(iv)$.

[4] One might say, a generalized least squares method.

We shall call $\psi := R_V^{-1}(Bu_h - l)$ the *error representation function*. Computing ψ involves solving the same variational problem as for the optimal test functions but with the residual on the right-hand side,

$$\begin{cases} \psi \in V \\ (\psi, \delta v)_V = b(u_h, \delta v) - l(\delta v) \quad \forall \delta v \in V \ . \end{cases} \tag{2.19}$$

Thus, the method comes with a "built-in" a-posteriori error estimation or, more precisely, a-posteriori *error evaluation*. Of course, all of these nice properties will be available if we come up with a practical method of inverting the Riesz operator.

A Mixed Formulation. Another way to proceed was proposed by Dahmen at al. in [16, 17]. Instead of identifying the second argument in (2.12) as the optimal test function, we identify the first argument as the error representation function,

$$(\psi, R_V^{-1} B \delta u_h)_V = 0 \quad \forall \delta u_h \ . \tag{2.20}$$

Taking out the Riesz operator and combining it with the definition of ψ, we obtain a saddle-point problem:

$$\begin{cases} \psi \in V, \ u_h \in U_h & & \\ (\psi, \delta v)_V - b(u_h, \delta v) & = -l(\delta v) & \forall \delta v \in V \\ b(\delta u_h, \psi) & = 0 & \forall \delta u_h \in U_h \ . \end{cases} \tag{2.21}$$

In this unconventional saddle-saint problem, the approximate solution u_h comes from a finite-dimensional trial space and plays the role of the Lagrange multiplier for the error representation function.

Remark 1 The PG scheme with optimal test functions was proposed in [19, 21]. It is perhaps interesting that we had arrived at the concept of optimal test functions from a completely different angle. Babuška's theorem [1] assures that the *discrete stability and approximability imply convergence*. More precisely, if $M := \|b\|$ is the continuity constant for the form $b(u, v)$ and the form satisfies the discrete inf-sup condition with constant γ_h,

$$\sup_{v_h \in V_h} \frac{|b(u_h, v_h)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U \ , \tag{2.22}$$

then the Galerkin error satisfies the estimate,

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U \ . \tag{2.23}$$

The idea of optimal testing relies on employing test functions that *realize* the supremum (maximum) in the discrete inf-sup condition (2.22). For a class of simple convection problems discussed in [19], such optimal test functions can be determined analytically. With such optimal test functions, the Petrov-Galerkin method inherits automatically the stability from the continuous level, i.e.

$$\gamma_h \geq \gamma \ , \tag{2.24}$$

where γ is the infinite-dimensional inf-sup constant. This holds for any possible trial norm U. If we use the energy norm (2.7) in place of original norm $\|\cdot\|_U$, both the corresponding continuity and inf-sup constants are equal one, $M = \gamma = 1$. Babuška's estimate (2.23) implies then,

$$\|u_h - u\|_E \leq \frac{M}{\gamma_h}\|w_h - u\|_E \leq \frac{M}{\gamma}\|w_h - u\|_E = \|w_h - u\|_E \quad \forall w_h \in U_h \,. \tag{2.25}$$

Thus we have arrived at the minimum residual method. The moral of the story is that the minimum residual method *is the most stable Petrov Galerkin method* we can come up with.

∎

3 Broken Test Spaces

As we have learned in Section 2, the Petrov Galerkin scheme with Optimal Test Functions requires inversion of the Riesz operator. With test norms involving standard exact sequence energy spaces, i.e. the use of $H^1, H(\text{curl})$ and $H(\text{div})$ inner products, inversion of the Riesz operator is equivalent to the solution of a separate boundary value problem. This would make the PG scheme unfeasible. Critical for the practicality of the method is the use of *broken* energy spaces typical for Discontinuous Galerkin (DG) methods. Given a mesh $\mathcal{T}_h$ consisting of elements K, the corresponding broken energy spaces are defined as follows,

$$\begin{aligned} H^1(\Omega_h) &:= \{u \in L^2(\Omega) \,:\, u|_K \in H^1(K) \quad \forall K \in \mathcal{T}_h\} \,, \\ H(\text{curl},\Omega_h) &:= \{E \in (L^2(\Omega)^n \,:\, E|_K \in H(\text{curl},K) \quad \forall K \in \mathcal{T}_h\} \,, \\ H(\text{div},\Omega_h) &:= \{v \in (L^2(\Omega)^n \,:\, v|_K \in H(\text{div},K) \quad \forall K \in \mathcal{T}_h\} \,. \end{aligned} \tag{3.26}$$

The corresponding (standard) inner products are defined elementwise,

$$\begin{aligned} (u,\delta u)_{H^1(\Omega_h)} &:= \sum_K (u|_K, \delta u|_K)_{H^1(K)} \,, \\ (E,\delta E)_{H(\text{curl},\Omega_h)} &:= \sum_K (E|_K, \delta E|_K)_{H(\text{curl},K)} \,, \\ (v,\delta v)_{H(\text{div},\Omega_h)} &:= \sum_K (v|_K, \delta v|_K)_{H(\text{div},K)} \,. \end{aligned} \tag{3.27}$$

While the definition of the broken energy spaces is unique, we use frequently other (equivalent) inner products defined on them. Inner products (3.27) are examples of *localizable inner products*, i.e. each element contribution defines an inner product on the corresponding element energy space. Not every standard energy inner product is localizable. For instance, the standard $H^1_0(\Omega)$ product:

$$(u,\delta u)_{H^1_0(\Omega)} = \int_\Omega \nabla u \, \overline{\nabla \delta u} = \sum_K \int_K \nabla u \, \overline{\nabla \delta u} \,, \tag{3.28}$$

is an inner product on $H^1_0(\Omega)$ but it is not longer definite on the corresponding broken energy space. Indeed, for an element K that is not adjacent to the boundary of Ω, $\int_K |u|^2 = 0$ implies only that u is a constant on K but not necessarily zero.

The main point in using the broken energy test spaces is that the corresponding inversion of the Riesz operator *localizes* i.e. it is done element-wise. Problem (2.16) decouples into independent element problems:

$$\begin{cases} v_{\delta u_h} \in V(K) \\ (v_{\delta u_h}, \delta v)_{V(K)} = b_K(\delta u_h, \delta v) \quad \forall \delta v \in V(K) \, . \end{cases} \tag{3.29}$$

Here b_K is the element contribution to the global sesquilinear form, and the left-hand side of the equation is the element contribution to the global test inner product. Function $\delta u_h = (\delta u_h)|_K$ denotes the restriction of a trial basis function to element K (element trial shape function), and $V(K)$ stands for the element test space.

Problem (3.29) is still infinite dimensional and it is equivalent to a boundary-value problem with Neumann (natural) boundary conditions. Except for simple problems (like advection with constant velocity vector [19]), we can only solve it approximately. As the inner product is hermitian and positive definite, the standard Bubnov-Galerkin method is a natural choice. We introduce an approximate element test space $\widetilde{V}(K) \subset V(K)$ and seek *approximate optimal test functions*,

$$\begin{cases} \widetilde{v}_{\delta u_h} \in \widetilde{V}(K) \\ (\widetilde{v}_{\delta u_h}, \widetilde{\delta v})_{V(K)} = b_K(\delta u_h, \widetilde{\delta v}) \quad \forall \widetilde{\delta v} \in \widetilde{V}(K) \, . \end{cases} \tag{3.30}$$

This leads to the *approximate trial-to-test operator*:

$$\widetilde{T} \, : \, U_h \ni \delta u_h \to \widetilde{v}_{\delta u_h} \in \widetilde{V}(K) \, , \tag{3.31}$$

and the corresponding *approximate optimal test space*:

$$\widetilde{V}_h := \widetilde{T} U_h \, . \tag{3.32}$$

The ultimate, *practical DPG method*, is obtained by replacing in (2.15) the optimal test functions with their approximate counterparts,

$$\begin{cases} u_h \in U_h \\ b(u_h, \widetilde{v}_h) = l(v_h) \quad \forall \widetilde{v}_h \in \widetilde{V}_h \, . \end{cases} \tag{3.33}$$

In practice, the approximate test space $\widetilde{V}_h$ is obtained by raising locally the polynomial order of approximation. Roughly speaking, if the trial space involves polynomials of order p, we use polynomials of order $p + \Delta p$ for approximating the optimal test functions. Typically, $\Delta p = 2$. We might say that we are using the p-method for approximating element problem (3.30).

Example: Poisson problem. We shall use the simplest example of the Poisson equation with Dirichlet boundary condition to illustrate the main points made so far. We seek $u \in H^1(\Omega)$ that satisfies the boundary-value problem:

$$\begin{cases} -\Delta u &= f & \text{in } \Omega \\ u &= u_0 & \text{on } \Gamma := \partial\Omega \end{cases} \tag{3.34}$$

where $f \in L^2(\Omega), u_0 \in H^{1/2}(\Omega)$ are given data.

Let $\mathcal{T}_h$ be a FE mesh. Take an element K, multiply both sides of equation $(3.34)_1$ with a test function v, integrate over element K, and integrate the left-hand side by parts, to obtain:

$$\int_K \nabla u \, \nabla v - \int_{\partial K} \frac{\partial u}{\partial n} \, v = \int_K f \, v \, . \tag{3.35}$$

Summing up over all elements K, we get:

$$\sum_K \int_K \nabla u \, \nabla v - \sum_K \int_{\partial K} \frac{\partial u}{\partial n} \, v = \sum_K \int_K f \, v \, . \tag{3.36}$$

The second term on the left-hand side represents jump terms and, for regular solution u, can be rewritten by summing up over all edges (faces) e in the mesh,

$$\sum_K \int_{\partial K} \frac{\partial u}{\partial n} \, v = \sum_e \frac{\partial u}{\partial n_e} [v] \, . \tag{3.37}$$

Here n_e is a pre-defined unit normal for edge e and $[v]$ represents the jump term:

$$[v](x) := \begin{cases} v & \text{if } e \subset \Gamma \\ \lim\limits_{\epsilon \to 0} (v(x + \epsilon n_e) - v(x - \epsilon n_e)) & \text{otherwise} \, . \end{cases} \tag{3.38}$$

We have several choices now.

1. Assume that test functions are globally conforming and vanish on Γ, $v \in H_0^1(\Omega)$. All boundary terms vanish and we arrive at the classical variational formulation:

$$\begin{cases} u \in H^1(\Omega), u = u_0 \text{ on } \Gamma \\ \displaystyle\int_\Omega \nabla u \, \nabla v = \int_\Omega f \, v \quad \forall v \in H_0^1(\Omega) \, . \end{cases} \tag{3.39}$$

2. We may assume that the test functions are globally conforming but do not necessarily vanish on Γ (we "do test" on Γ). The jump terms vanish but we are left with the normal derivative term on the domain boundary Γ:

$$\int_\Omega \nabla u \, \nabla v - \int_\Gamma \frac{\partial u}{\partial n} \, v = \int_\Omega f \, v \, . \tag{3.40}$$

   For $u \in H^1(\Omega)$, the normal derivative (the *flux*) is not well-defined and we identify it as a new, separate unknown (placing the hat symbol over the normal derivative). The right energy setting is as follows:

$$\begin{cases} u \in H^1(\Omega), u = u_0 \text{ on } \Gamma, \ \widehat{\frac{\partial u}{\partial n}} \in H^{-1/2}(\Gamma) \\ \displaystyle\int_\Omega \nabla u \, \nabla v - \int_\Gamma \widehat{\frac{\partial u}{\partial n}} \, v = \int_\Omega f \, v \quad \forall v \in H^1(\Omega) \, . \end{cases} \tag{3.41}$$

3. We test with *discontinuous test functions*, i.e. $v \in H^1(\Omega_h)$. The jump terms remain. We introduce a new unknown, the *flux*

$$\widehat{\frac{\partial u}{\partial n}} \in H^{-1/2}(\Gamma_h) \,, \tag{3.42}$$

that lives not only on Γ but the whole *mesh skeleton:*

$$\Gamma_h = \bigcup_K \partial K \tag{3.43}$$

and, for regular solutions, coincides with the normal derivative of u. Space $H^{-1/2}(\Gamma_h)$, introduced in [20], is identified as the space of traces of functions from $H(\text{div},\Omega)$ to skeleton Γ_h, equipped with the minimum energy extension norm. The ultimate variational formulation looks as follows:

$$\left\{ \begin{array}{l} u \in H^1(\Omega), u = u_0 \text{ on } \Gamma, \ \widehat{\frac{\partial u}{\partial n}} \in H^{-1/2}(\Gamma_h) \\ \displaystyle \int_\Omega \nabla u \, \nabla v - \sum_K \int_{\partial K} \widehat{\frac{\partial u}{\partial n}} \, v = \int_\Omega f \, v \quad \forall v \in H^1(\Omega_h) \,. \end{array} \right. \tag{3.44}$$

One can show that the last two problems are well-posed [24].

With the discontinuous test functions, we can pursue now the idea of optimal testing. The price paid for the localization is the introduction of new unknown: the flux. The unknown solution is a group variable consisting of the original unknown u, and the flux $\widehat{\frac{\partial u}{\partial n}}$. A typical discretization uses standard H^1-conforming elements for u, and traces of Raviart-Thomas $H(\text{div},\Omega)$-conforming elements on mesh skeleton Γ_h for the flux. For instance, for 2D quadrilateral elements, the standard choice would be the isoparametric $\mathcal{P}^p \otimes \mathcal{P}^p$ element for u, and the discontinuous $\mathcal{P}^{p-1}$ element (with the Piola pull back map) for the flux. The lowest order elements use bilinear vertex shape functions for u and piece-wise constant functions for the flux.

Let u be an H^1-conforming trial basis function. For each element K in the support of u, we solve for the optimal test function v_u,

$$\left\{ \begin{array}{l} v_u \in H^1(K) \\ \displaystyle \int_K \nabla v_u \, \nabla \delta v + v_u \, \delta v = \int_K \nabla u \, \nabla \delta v \quad \forall \delta v \in H^1(K) \,. \end{array} \right. \tag{3.45}$$

If u is discretized with $\mathcal{P}^p \otimes \mathcal{P}^p$ element, the corresponding optimal test function is approximated with $\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p}$ element (in practice, $\Delta p = 2$).

The support of v_u coincides with the support of u. If u is a vertex shape function, the corresponding optimal test function v_u will span over all elements sharing the vertex, if u represents an edge basis function, the support of v_u will include all elements sharing the edge etc. In particular, if u is an element bubble, so is the corresponding test function v_u. Note though that, contrary to the continuous basis function u, the corresponding optimal test function (exact or approximate) v_u is *discontinuous*.

Similarly, let $g = g_e$ be a discontinuous flux function that lives on an edge (face in 3D) e. For each element K *adjacent* to the edge e, we solve for the optimal test function v_g,

$$\left\{ \begin{array}{l} v_g \in H^1(K) \\ \displaystyle\int_K \nabla v_g \, \nabla \delta v + v_g \, \delta v = - \int_{\partial K} g \text{sgn}_K \, \delta v \quad \forall \delta v \in H^1(K) \, , \end{array} \right. \tag{3.46}$$

with

$$\text{sgn}_K = \left\{ \begin{array}{rl} 1 & \text{if } n_K = n_e \\ -1 & \text{if } n_K = -n_e \end{array} \right. \tag{3.47}$$

where n_e is the predefined edge normal and n_K is the outward normal for element K.

The support of v_g will span over all elements sharing the edge. We use the same $\mathcal{P}^{p+\Delta p} \otimes \mathcal{P}^{p+\Delta p}$ element to compute the approximate optimal test function.

Remark 2 It is important to notice that the DPG method does not destroy the classical logical flow of finite elements. In a classical PG FE method, we enter an element with sesquilinear (stiffness) and antilinear (load) forms, and two sets of approximating functions: trial and test shape functions. We integrate then for the element stiffness matrix and load vector that are returned to a global direct or iterative solver. In the DPG method, we enter an element with trial shape functions and *test inner product*. We compute then the approximate optimal test (shape) functions and proceed with the computation of corresponding element matrices. The "on the fly" computation of approximate optimal test functions takes place in the element routine and it does not affect the rest of the code.

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Remark 3 With the use of the *enriched* approximate test space, the logic is actually much simpler. We enter element K with the following forms: sesquilinear stiffness form $b_K(u,v)$, antilinear load $l_K(v)$ form, and sesquilinear $(v, \delta v)_K$ test inner product form. We have two sets of shape functions: trial shape functions e_i and enriched space basis functions $\hat{e}_j$. We compute the following matrices:

$$\begin{array}{rl} K_{ki} := b_K(e_i, \hat{e}_k) & \text{"extended" element stiffness matrix,} \\ l_k := l_K(\hat{e}_k) & \text{"extended" element load vector,} \\ G_{kl} := (\hat{e}_k, \hat{e}_l)_V & \text{Gram test matrix .} \end{array} \tag{3.48}$$

We invert (factorize) the Gram matrix, and compute the ultimate element stiffness matrix and load vector using the simple formulas:

$$G^{-1}_{kl} K_{ki} \bar{K}_{lj}, \quad G^{-1}_{kl} l_k \bar{K}_{lj} \, . \tag{3.49}$$

For the L^2 test inner product, we obtain the standard least squares method. The DPG method can thus be viewed as a *preconditioned least squares method.* ∎

Convergence of the DPG method with exact optimal test functions is analyzed in [24]. We will discuss convergence analysis in context of general ultraweak variational formulations in more detail in Section 4. The effect of using the approximate optimal test functions was studied in [29].

The development of the DPG method was not motivated with the solution of simple elliptic problems like the Poisson equation for which the standard Bubnov Galerkin method works just fine, and we do not necessarily advocate the use of DPG method for such problems. The price paid for the localization is high. If we neglect the cost of all local degrees-of-freedom (static condensation), in the standard H^1-conforming FE method we solve for *traces* of u on the mesh skeleton Γ_h. In the DPG method, we solve for *both* traces and fluxes, so the number of unknowns essentially doubles.

On the positive side, remember that the DPG method comes with a "built-in" a-posteriori error evaluator. Once the solution has been determined, a calc copy of the element routine is used to evaluate the (approximate) error representation function, and the corresponding element contribution to the global residual. With the use of adaptivity, the additional cost of solving for fluxes becomes less significant. The idea of using broken spaces and approximate inverse Riesz operators was used a long time ago in context of implicit a-posteriori error estimation, see e.g. [35].

4 Ultraweak Variational Formulations

Whereas the localization requires only the test functions to be discontinuous, it is also desirable to work with a variational setting in which the trial functions are discontinuous as well. We shall first present such a formulation and only then discuss its advantages. As in the theory of Schwartz's distributions, the idea behind the *ultraweak variational formulation* is to move *all derivatives* to test functions.

We return to our model Poisson problem and rewrite it in terms of a system of first order equations:

$$\left\{ \begin{array}{rl} \sigma - \nabla u &= 0 \\ \quad \mathrm{div}\sigma &= -f \, . \end{array} \right. \tag{4.50}$$

We will discuss first a *global formulation.* We multiply the two equations with *globally conforming* test functions $\tau \in H(\mathrm{div},\Omega), v \in H^1(\Omega)$, integrate over domain Ω, and integrate by parts to obtain:

$$\left\{ \begin{array}{rl} \int_\Omega \sigma \cdot \tau + \int_\Omega u \, \mathrm{div}\tau - \int_\Gamma u \, v &= 0 \\ - \int_\Omega \sigma \, \nabla v + \int_\Gamma \sigma n \, v &= - \int_\Omega f \, v \, . \end{array} \right. \tag{4.51}$$

With no derivatives left on solution u, σ, the natural energy space for both components of the solution is the L^2 space: $\sigma \in (L^2(\Omega)^2, u \in L^2(\Omega)$. With such functional setting for the solution, the *trace* u and *flux* $t := \sigma n$ are not well defined and we declare them to be independent, additional unknowns $\hat{u} \in H^{1/2}(\Gamma)$ and $\hat{t} = \widehat{\sigma n} \in H^{-1/2}(\Gamma)$. Additionally, with Dirichlet boundary condition imposed on the whole boundary,

the trace $\hat{u}$ is known from the boundary condition. We can substitute u_0 for $\hat{u}$ and move it to the right-hand side. This leads to the final formulation in the form:

$$\begin{cases} (u,\sigma,\hat{t}) \in L^2(\Omega) \times (L^2(\Omega))^2 \times H^{-1/2}(\Gamma) \\ (u,\text{div}\tau) + (\sigma,\tau - \nabla v) + \langle \hat{t}, v\rangle = -(f,v) + \langle u_0, v\rangle \quad \forall v \in H^1(\Omega), \tau \in H(\text{div},\Omega) \end{cases} \tag{4.52}$$

where $(\cdot,\cdot)$ stands for the $L^2(\Omega)$-product and $\langle \cdot,\cdot\rangle$ for the duality pairing in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$.

The bilinear form:

$$b((u,\sigma,\hat{t}),(v,\tau)) := (u,\text{div}\tau) + (\sigma,\tau - \nabla v) + \langle \hat{t}, v\rangle \tag{4.53}$$

generates two operators: the ultraweak operator B defining the problem and its conjugate, *strong version* of the same operator (the problem is self-adjoint):

$$H^1(\Omega) \times H(\text{div},\Omega) \ni (v,\tau) \to (\tau - \nabla v, \text{div}\tau, v|_\Gamma) \in (L^2(\Omega))^2 \times L^2(\Omega) \times H^{1/2}(\Gamma)\,. \tag{4.54}$$

The ultraweak variational formulation can now be repeated in the DPG setting. Given a mesh $\mathcal{T}_h$ consisting of elements K, we repeat step (4.51) over every element K,

$$\begin{cases} \displaystyle\int_K \sigma\cdot\tau + \int_K u\,\text{div}\tau - \int_\Gamma u\,v &= 0 & \tau \in H(\text{div},K) \\ \displaystyle -\int_K \sigma\,\nabla v + \int_\Gamma \sigma n\,v &= -\int_K f\,v & v \in H^1(K)\,. \end{cases} \tag{4.55}$$

Next we sum up over all elements to obtain:

$$\sum_K \left\{ \int_K u\,\text{div}\tau + \int_K \sigma\,(\tau - \nabla v) - \int_{\partial K} u\,v + \int_{\partial K} \underbrace{\sigma n}_{=:t}\, v \right\} = -\sum_K \int_K f v\,. \tag{4.56}$$

As in the global ultraweak formulation, we introduce additional unknowns: trace $\hat{u} \in H^{1/2}(\Gamma_h)$ and flux $\hat{t} \in H^{-1/2}(\Gamma_h)$. Both are defined not only on domain boundary Γ but *the whole mesh skeleton* Γ_h. Spaces $H^{1/2}(\Gamma_h), H^{-1/2}(\Gamma_h)$ are defined as traces of functions from $H^1(\Omega)$ and $H(\text{div},\Omega)$ (see [20, 36] for a detailed discussion) and equipped with minimum energy extension norms:

$$\begin{aligned} \|\hat{u}\|_{H^{1/2}(\Gamma_h)} &:= \inf\left\{ \|u\|_{H^1(\Omega)} \,:\, u \in H^1(\Omega),\, u|_{\Gamma_h} = \hat{u} \right\}\,, \\ \|\hat{t}\|_{H^{-1/2}(\Gamma_h)} &:= \inf\left\{ \|\sigma\|_{H(\text{div},\Omega)} \,:\, \sigma \in H(\text{div},\Omega),\, (\sigma n)|_{\Gamma_h} = \hat{t} \right\}\,. \end{aligned} \tag{4.57}$$

The unknown trace $\hat{u}$ is represented as a sum of a lift of the known boundary data $\widetilde{u}_0$ to the mesh skeleton Γ_h, and unknown component $\hat{u}$ that vanishes on Γ (watch for the overloaded symbol),

$$\hat{u} := \widetilde{u}_0 + \hat{u}\,. \tag{4.58}$$

More precisely, the unknown component comes from the space of traces of $H^1_0(\Omega)$, denoted $\widetilde{H}^{1/2}(\Gamma_h^0)$, where $\Gamma_h^0 := \Gamma_h - \Gamma$ stands for the *interior mesh skeleton*.

The ultimate *DPG ultraweak variational formulation* reads as follows:

$$\left\{\begin{array}{l} (u,\sigma,\hat{u},\hat{t}) \in L^2(\Omega) \times (L^2(\Omega))^2 \times \widetilde{H}^{1/2}(\Gamma_h^0) \times H^{-1/2}(\Gamma_h) \\ \underbrace{(u, \mathrm{div}_h \tau) + (\sigma, \tau - \nabla_h v) - \langle \hat{u}, [\tau n] \rangle + \langle \hat{t}, [v] \rangle}_{=: b((u,\sigma,\hat{u},\hat{t}),(v,\tau))} = -(f,v) + \langle \widetilde{u}_0, v \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H^1(\Omega_h), \tau \in H(\mathrm{div}, \Omega_h) \, . \end{array}\right. \tag{4.59}$$

Above, the test functions come from the broken test spaces $H^1(\Omega_h), \tau \in H(\mathrm{div}, \Omega_h)$, the duality pairings extend over Γ_h^0 and Γ_h, respectively, and the grad and div operators in the first two terms are *understood elementwise* as indicated by index h. Similarly to the global ultraweak formulation, the conjugate operator generated by the bilinear form b corresponds to a strong version of the operator applied elementwise and accompanied by interface conditions across interelement boundaries expressing continuity of v and τn,

$$\begin{array}{l} H^1(\Omega_h) \times H(\mathrm{div}, \Omega_h) \ni (v,\tau) \to \\ \qquad (\mathrm{div}_h, \tau - \nabla_h v, [v], [\tau n]) \in L^2(\Omega) \times (L^2(\Omega))^2 \times \Pi_K H^{1/2}(\partial K) \times \Pi_K H^{-1/2}(\partial K) \, . \end{array} \tag{4.60}$$

Note that the jump terms have to be understood globally, i.e.,

$$\begin{array}{ll} \langle \hat{u}, [\tau n] \rangle & := \sum_K \langle \hat{u}, \tau n \rangle_{\partial K} \\ \langle \hat{t}, [v] \rangle & := \sum_K \langle \hat{t}, v \rangle_{\partial K} \, . \end{array} \tag{4.61}$$

The DPG ultraweak formulation (4.59) provides a natural setting for the PG method with optimal test functions. The test functions are discontinuous enabling local computation of approximate optimal test functions. Solution consists of several components: the original unknown u, the (continuous) flux σ, traces $\hat{u}$ and fluxes $\hat{t}$. At a first glance, it looks like the formulation based on the first order system is much more expensive than the one based on the second order equation discussed in Section 3. Actually, it is not. The L^2-variables u, σ are discretized with discontinuous elements and can be statically condensed out. After the condensation, we solve a global problem for traces $\hat{u}$ and fluxes $\hat{t}$ whose cost is identical to the one discussed in the previous section. Thus, if we disregard the cost of local computations (that are trivially parallelizable), the two methods are essentially equally expensive.

Abstract setting. The DPG ultraweak variational formulation discussed above has been applied to and analyzed for a number of different problems: convection-diffusion [20], linear elasticity [5] and linear acoustics [22]. In the latter, we began to see the emerging general abstract setting. We further formulated it and applied to Stokes problem in [36]. We shall attempt now to outline the main points of the study in [36].

The starting point is an operator representing a system of first order differential equations:

$$(L^2(\Omega))^N \supset H_A(\Omega) \ni u \to Au \in (L^2(\Omega))^N \, . \tag{4.62}$$

Here N denotes the total number of scalar unknowns, and the domain of the operator is the graph space equipped with the graph norm:

$$\|u\|^2_{H_A} = \|u\|^2 + \|Au\|^2 \tag{4.63}$$

where, as usual, $\|\cdot\|$ denotes the L^2-norm.

Integration by parts leads to the introduction of the formal L^2-adjoint and a sesquilinear form representing boundary terms:

$$(Au, v) = (u, A^*v) + c(\mathrm{tr}_A u, \mathrm{tr}_{A^*} v) \tag{4.64}$$

Here v comes from the graph space for the formal adjoint:

$$H_{A^*}(\Omega) := \{v \in (L^2(\Omega))^N \ : \ A^*v \in (L^2(\Omega))^N\}\,, \tag{4.65}$$

equipped with a graph norm. We assume that we have at our disposal trace operators along with trace spaces for both energy spaces:

$$\begin{aligned} \mathrm{tr}_A \ : \ H_A(\Omega) \ni u \quad &\to \mathrm{tr}_A u = \hat{u} \in \hat{H}_A(\Gamma) \\ \mathrm{tr}_{A^*} \ : \ H_{A^*}(\Omega) \ni v \quad &\to \mathrm{tr}_{A^*} v = \hat{v} \in \hat{H}_{A^*}(\Gamma)\,, \end{aligned} \tag{4.66}$$

and that $c(\hat{u}, \hat{v})$ is a definite sesquilinear form. For the Poisson problem, we have (watch for overloaded symbols):

$$\begin{aligned} &u := (u, \sigma), \quad v = (v, \tau)\,, \\ &Au = A(u, \sigma) = (\sigma - \nabla u, \mathrm{div}\sigma), \quad A^* = A\,, \\ &H_A(\Omega) = H^1(\Omega) \times H(\mathrm{div}, \Omega), \quad H_{A^*}(\Omega) = H_A(\Omega)\,, \\ &\hat{H}_A(\Gamma) = H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma), \quad \mathrm{tr}_A u = (u, \sigma n), \quad \hat{H}_{A^*}(\Gamma) = \hat{H}_A(\Gamma), \quad \mathrm{tr}_{A^*} = \mathrm{tr}_A\,, \\ &c(\hat{u}, \hat{v}) = c((u, \sigma n), (v, \tau n)) = \langle u, \tau n\rangle_{H^{1/2}(\Gamma)\times H^{-1/2}(\Gamma)} + \langle \sigma n, v\rangle_{H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)}\,. \end{aligned} \tag{4.67}$$

Let C be a boundary operator generated by the boundary sesquilinear form:

$$C \ : \ \hat{H}_A(\Gamma) \to (\hat{H}_{A^*}(\Gamma))', \quad \langle C\hat{u}, \hat{v}\rangle = c(\hat{u}, \hat{v})\,. \tag{4.68}$$

We assume that C can be split into two operators,

$$C = C_1 + C_2\,, \tag{4.69}$$

in such a way that operators A and A^* restricted to spaces corresponding to homogeneous boundary conditions:

$$\begin{aligned} U_0 &:= \{u \in H_A(\Omega) \ : \ C_1 u = 0\}\,, \\ V_0 &:= \{v \in H_{A^*}(\Omega) \ : \ C_2' v = 0\}\,, \end{aligned} \tag{4.70}$$

are L^2-adjoint. According to Banach Closed Range Theorem, the following four conditions are equivalent to each other:

$$\begin{gathered} A|_{U_0} \text{ has a closed range,} \\ A^*|_{V_0} \text{ has a closed range,} \\ A|_{U_0} \text{ is bounded below in the orthogonal component of its null space } \mathcal{N}(A|_{U_0}), \\ A^*|_{V_0} \text{ is bounded below in the orthogonal component of its null space } \mathcal{N}(A^*|_{V_0}). \end{gathered}$$

For simplicity, we assume that both $A|_{U_0}$ and $A^*|_{V_0}$ are injective. Consequently, both operators are bounded below with the same constant γ,

$$\|Au\| \geq \gamma \|u\| \quad \forall u \in U_0 \quad \text{and} \quad \|A^* v\| \geq \gamma \|v\| \quad \forall v \in V_0 \,. \tag{4.71}$$

Operator split (4.69) implies a split of the trace space $\hat{H}_A$,

$$\hat{H}_A = X_1 \oplus X_2, \quad X_1 = \mathcal{N}(C_2), \quad X_2 = \mathcal{N}(C_1) \,. \tag{4.72}$$

For the Poisson problem,

$$C_1(u, \sigma n) = u, \quad C_2(u, \sigma n) = \sigma n, \quad C_2'(v, \tau n) = v, \quad U_0 = V_0 \,, \tag{4.73}$$

i.e. the operator $A|_{U_0}$ is self-adjoint. The trace space split is very simple:

$$X_1 = H^{1/2}(\Gamma) \times \{0\} \sim H^{1/2}(\Gamma), \quad X_2 = \{0\} \times H^{-1/2}(\Gamma) \sim H^{-1/2}(\Gamma) \,. \tag{4.74}$$

We are finally ready to write down the *abstract ultraweak variational formulation* for the boundary-value problem:

$$Au = f, \quad C_1 u = g \tag{4.75}$$

Assuming that $g \in \mathcal{R}(C_1)$, we have,

$$\begin{cases} u \in L^2(\Omega), \quad \hat{u}_2 \in X_2 \\ \underbrace{(u, A^* v) + < \hat{u}_2, C_2' \text{tr}_{A^*} v >}_{=: b((u, \hat{u}_2), v)} = (f, v) - < g, \text{tr}_{A^*} v > \quad \forall v \in H_{A^*}(\Omega) \,. \end{cases} \tag{4.76}$$

It has been proved in [20, 36] that problem (4.76) is well posed with an inf-sup constant of order[5] γ. The conjugate operator corresponding to form $b((u, \hat{u}_2), v)$ is the strong operator:

$$H_{A^*}(\Omega) \ni v \to (A^* v, C_2' \text{tr}_{A^*} v) \in (L^2(\Omega))^N \times \hat{H}_{A^*}(\Gamma) \,. \tag{4.77}$$

We refer also to [36] for the abstract DPG ultraweak variational formulation. As for the Poisson problem, the unknowns include additionally the (abstract) trace variable $\hat{u}$ defined on the whole interior mesh skeleton. Notice that the abstract notion of "trace" includes equivalents of both trace and flux for the Poisson problem.

There are two main points about the results presented in [36], generalizing the earlier results for particular problems in [20, 5, 22]. First of all, we prove that, under the assumptions outlined above, the DPG ultraweak variational formulation is well-posed with a *mesh independent* inf-sup constant of order γ. Mesh independence is critical for h-convergence and it is not obvious at all as, with h mesh refinements, the skeleton grows and there is "more" traces (and fluxes). The second important observation is that the inf-sup constant is of order γ. If, for a singular perturbation problem involving a parameter, γ is independent of

[5] Under the assumption that the traces spaces are equipped with minimum energy extension norms.

the parameter, then this uniform stability *automatically carries over* to the DPG ultraweak variational formulation. As the PG method with optimal test functions inherits the inf-sup constant from the continuous level[6], our DPG method is automatically *uniformly stable* i.e. the approximation error is bounded by the best approximation error times a stability constant independent of the perturbation parameter. Using the abstract notation we have:

$$\left(\|u-u_h\|^2+\|\hat{u}-\hat{u}_h\|^2\right)^{1/2} \leq C \inf_{(w_h,\hat{w}_h)} \left(\|u-w_h\|^2+\|\hat{u}-\hat{w}_h\|^2\right)^{1/2} . \tag{4.78}$$

The abstract trace $\hat{u}$ incorporates both all (abstract) traces on the interior skeleton and the unknown traces on boundary Γ. The minimum extension energy norm used to measure traces is mesh dependent but the *field variables* u are measured in mesh-independent L^2-norm. Two particular cases are of interest: convection-dominated diffusion and linear acoustics. For linear acoustics, under appropriate regularity assumptions on the domain Ω, constant γ is independent of wave number k [22]. For convection-dominated diffusion with specific boundary conditions and assumptions on the advection vector, constant γ is independent of diffusion parameter ϵ. For both classes of problems, the uniform stability result (4.78) should be approached with care (for different reasons). The issue of *robustness* for singular perturbation problems will be discussed in Section 5.

For additional studies of DPG method based on the test graph norm, see [34, 33] (convection-dominated diffusion) and [32, 9] (thin walled structures), [26] (2D cloaking problems). A relation between various versions of DPG and DG methods was studied in [8].

For a related study on well-posedness of DPG formulations for general Friedrichs' systems, see [7].

Accounting for approximation of optimal test functions. A general theory for taking into account the approximation of optimal test functions was put forth in [29]. Let $\widetilde{V}$ be the *enriched* approximate test space in which the optimal test functions are approximated. Suppose we can identify a Fortin-like operator,

$$\Pi : V \to \widetilde{V} \quad \|\Pi v\| \leq C\|v\| , \tag{4.79}$$

that satisfies the orthogonality property:

$$b((u_h,\hat{u}_h), v-\Pi v) = (u_h, A^*(v-\Pi v)) + \langle \hat{u}_h, (v-\Pi v)\rangle = 0 \quad \forall u_h, \hat{u}_h . \tag{4.80}$$

We have then:

$$\begin{aligned} \sup_{v\in V} \frac{|b((u_h,\hat{u}_h),v)|}{\|v\|} &= \sup_{v\in V} \left[\frac{|b((u_h,\hat{u}_h),v-\Pi v)|}{\|v\|} + \frac{|b((u_h,\hat{u}_h),\Pi v)|}{\|v\|} \right] \\ &= \sup_{v\in V} \frac{|b((u_h,\hat{u}_h),Piv)|}{\|v\|} \frac{\|\Pi v\|}{\|v\|} \leq C \sup_{v\in\widetilde{V}} \frac{|b((u_h,\hat{u}_h),v)|}{\|v\|} \\ &= C \sup_{v_h\in V_h} \frac{|b((u_h,\hat{u}_h),v_h)|}{\|v_h\|} \end{aligned} \tag{4.81}$$

[6] Neglecting the error due to the approximation of optimal test functions.

where the last equality follows from the fact that the approximate optimal test functions realize the supremum in the enriched space. Thus, at the expense of the additional C stability factor, the practical DPG method preserves the optimal stability. For examples of such Fortin operators for Poisson and elasticity problems, see [29]. In practice, the orthogonality condition **??**4.80) serves as a defining property for such operators. The Fortin operators constructed in [29] are defined for arbitrary polynomial order p, but the estimates of constant C are p-dependent. The results provide thus a basis for h-convergence analysis only.

5 Robustness

By now, the reader should realize that the DPG method is more a methodology than a single method. One can combine the method with different variational formulations and, most importantly, one can compute with *different test norms*. For each test norm, we get a different version of the method. For "standard" problems and formulation based on the first order systems, the adjoint operator graph norm is a natural choice. For singular perturbation problems, where we strive for uniform (with respect to the perturbation parameter) approximation properties, the so-called *robustness*, the optimal choice of the test norm is much more difficult.

The very definition of what we mean by a *robust method* for a singular perturbation problem is shaky. Rather than attempting to develop a general theory, we will focus in this section on an important model problem: convection-dominated diffusion. The "confusion" problem, as we call it, is an important stepping stone for compressible and incompressible fluid dynamics. To fix ideas, we shall consider a model problem illustrated in Fig. 1. We shall start right away with the first order system setting. The problem of interest is:

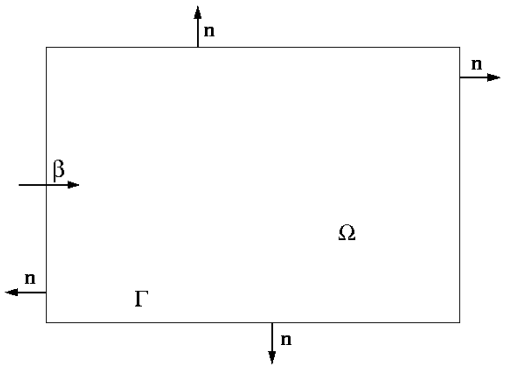

Figure 1: A model convection-dominated diffusion problem.

$$\left\{ \begin{array}{rl} \frac{1}{\epsilon}\sigma - \nabla u & = 0 \\ \text{div}(\sigma - \beta u) & = -f \end{array} \right. \tag{5.82}$$

where ϵ is the diffusion parameter and $\beta = \beta(x) \approx O(1)$ is a prescribed advection field. We shall consider

two types of boundary conditions:

$$\begin{aligned} \sigma_n - \beta_n u &= -\beta_n u_0 \qquad \text{on } \Gamma_{in} \quad := \{x \in \Gamma \ : \ \beta_n(x) < 0\} \\ u &= 0 \qquad\qquad\ \ \text{on } \Gamma_{out} \quad := \{x \in \Gamma \ : \ \beta_n(x) \geq 0\}\,. \end{aligned} \tag{5.83}$$

Here $\sigma_n = \sigma n$ and $\beta_n = \beta n$ represent normal components of flux σ and advection field β. The first boundary condition (BC) imposed on Γ represents an attempt at bringing from infinity condition $u = u_0$ where u_0 is a prescribed value. In presence of small diffusion ϵ, flux σ is expected to be small and the BC should approximate well the desired condition. The second BC is the main BC of interest as it produces a strong boundary layer on the outflow part of Γ.

Proceeding along lines discussed in Section 4, we obtain the following DPG ultraweak variational formulation for the problem:

$$\begin{cases} u \in L^2(\Omega),\ \sigma \in (L^2(\Omega))^2,\ \hat{u} \in \widetilde{H}^{1/2}(\Gamma_h),\ \hat{t} \in \widetilde{H}^{-1/2}(\Gamma_h) \\ \underbrace{(u, \mathrm{div}_h \tau + \beta \nabla_h v) + (\sigma, \frac{1}{\epsilon}\tau - \nabla_h v) - \langle \hat{u}, \tau_n \rangle + \langle \hat{t}, v \rangle}_{=: b((u,\sigma,\hat{u},\hat{t}),(v,\tau))} = -(f, v) + \langle \widetilde{\beta_n u_0}, v \rangle \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H^1(\Omega_h),\ \tau \in H(\mathrm{div}, \Omega_h) \end{cases} \tag{5.84}$$

where $t = \sigma_n - \beta_n u$. Remember that all skeleton terms are global. Thus $\widetilde{H}^{1/2}(\Gamma_h)$ are traces of functions from $H^1(\Omega)$, vanishing on Γ_{out}, and $\widetilde{H}^{-1/2}(\Gamma_h)$ are traces of functions from $H(\mathrm{div}, \Omega)$ with (normal) trace vanishing on Γ_{in}. Finally, $\widetilde{\beta_n u_0}$ is an extension of $\beta_n u_0$ to the whole skeleton Γ_h.

If possible, we would like to have a robust behavior of L^2-error of the original unknown u. In other words, for a given mesh, we would like $\|u - u_h\|$ to be of the same order *uniformly* in $\epsilon \to 0$. The request is not so unreasonable, one can show that the L^2-norm of the solution u is bounded by the data f, u_0 *uniformly* in ϵ. This happens despite the fact that the solution develops a boundary layer on Γ_{out} which steepens up with $\epsilon \to 0$, the L^2-norm is simply insensitive to the developing boundary layer.

The critical question is:

How to define the test norm ?

The *ideal* DPG method delivers then the best approximation error in the energy norm (2.7). So one might want to solve an inverse problem: determine a test norm for which the corresponding energy norm coincides with the original norm used for the solution. This question can actually be answered at the abstract level. If both operators B and B' corresponding to the original sesquilinear form are injective, the ideal test norm is obtained by switching the role of spaces in the inf-sup condition(see [40] and Remark 5 below):

$$\|v\|_V := \sup_{u \neq 0} \frac{|b(u, v)|}{\|u\|_U}\,. \tag{5.85}$$

The particular advantage of the ultraweak formulation (4.76) is that we can compute the ideal test norm explicitly:

$$\|v\|_V^2 = \|A^* v\|^2 + \left(\sup_{\hat{u}_2 \in X_2} \frac{|\langle \hat{u}_2, C_2' \mathrm{tr}_{A^*} v \rangle|}{\|\hat{u}_2\|_{\hat{H}_A}} \right)^2 . \tag{5.86}$$

For the model Poisson problem, we simply get:

$$\|(v,\tau)\|_V^2 = \|\tau - \nabla v\|^2 + \|\mathrm{div}\tau\|^2 + \|v|_\Gamma\|^2_{H^{1/2}(\Gamma)} . \tag{5.87}$$

The norm is very close to the adjoint operator graph norm used in our well-posedness analysis which prompted us to call in [40, 22] the graph norm a *quasi-optimal test norm*.

Remark 4 A sesquilinear form $b(u,v),\ u \in U, v \in V$ is called a *duality pairing* if

$$\|u\|_U = \sup_{v \neq 0} \frac{|b(u,v)|}{\|v\|_V} \quad \text{and} \quad \|v\|_V = \sup_{u \neq 0} \frac{|b(u,v)|}{\|u\|_U} . \tag{5.88}$$

A class of problems with explicitly known norms $\|\cdot\|_U,\ \|\cdot\|_v$ for which the corresponding sesquilinear form is a duality pairing, was studied in [6]. ∎

We shall discuss now shortly a more general approach to the problem of determining an optimal test norm for singular perturbation problems that was proposed in [25]. Let us assume that we have found an optimal test norm $\|\cdot\|_V$. Consider a very special test function (v,τ) which, when substituted into the bilinear form (5.84) delivers the L^2-norm of solution u. This is obtained by requesting the conditions:

$$\begin{aligned} \mathrm{div}_h \tau + \beta \nabla_h v &= u \\ \tfrac{1}{\epsilon}\tau - \nabla_h v &= 0 \\ \langle \hat{u}, \tau_n \rangle &= 0 \quad \forall \hat{u} \\ \langle \hat{t}, v \rangle &= 0 \quad \forall \hat{t} . \end{aligned} \tag{5.89}$$

The last two conditions imply that the test function (v,τ) must be globally conforming, $v \in H^1(\Omega), \tau \in H(\mathrm{div},\Omega)$, and satisfy the homogeneous boundary conditions for the adjoint operator. This implies that the differential operators in the first two conditions can be understood globally. Simply, the test function (v,τ) solves the continuous adjoint problem with homogeneous BCs:

$$\begin{cases} v \in H^1(\Omega),\ \tau \in H(\mathrm{div},\Omega) \\ \tfrac{1}{\epsilon}\tau - \nabla v = 0, \quad \mathrm{div}\tau + \beta \nabla v = u \\ \tau_n = 0 \text{ on } \Gamma_{in}, \quad v = 0 \text{ on } \Gamma_{out} . \end{cases} \tag{5.90}$$

Notice that, contrary to the actual confusion problem, solution (v,τ) does not develop [7] a strong boundary layer on Γ_{in}, the outflow boundary for the adjoint problem.

[7] Actually, BC $\tau_n = 0$ *does* produce a very weak boundary layer, hard to observe even with very accurate adaptive simulations, see [38].

We get:

$$\|u\|^2 = b((u,\sigma,\hat{u},\hat{t}),(v,\tau)) \leq \underbrace{\sup_{(v,\tau)} \frac{|b((u,\sigma,\hat{u},\hat{t}),(v,\tau))|}{\|(v,\tau)\|_V}}_{\|(u,\sigma,\hat{u},\hat{t})\|_E} \|(v,\tau)\|_V \,. \tag{5.91}$$

Thus, if the test norm has been selected in such a way that the solution of the adjoint problem (5.90) can be bounded by the data u *robustly* , i.e. *uniformly* in ϵ :

$$\|(v,\tau)\| \lesssim \|u\| \,, \tag{5.92}$$

the L^2-norm of the solution u is bounded robustly by the energy norm,

$$\|u\| \lesssim \|(u,\sigma,\hat{u},\hat{t})\|_E \,. \tag{5.93}$$

As the DPG method delivers the best approximation error in the energy norm, we obtain:

$$\|u-u_h\| \lesssim \|(u-u_h,\sigma-\sigma_h,\hat{u}-\hat{u}_h,\hat{t}-\hat{t}_h)\|_E \leq \underbrace{\inf_{(u_h,\sigma_h,\tau u_h,\hat{t}_h)} \|(u-u_h,\sigma-\sigma_h,\hat{u}-\hat{u}_h,\hat{t}-\hat{t}_h)\|_E}_{\text{best approximation error in energy norm}} \,. \tag{5.94}$$

The design of an optimal test norm leads thus to the stability analysis of the adjoint problem. Condition (5.93) is only necessary for the robustness. If the solution of the adjoint problem cannot be bounded in the test norm robustly by $\|u\|$, the robust estimate above is gone and the whole game is lost.

The following stability estimates for the adjoint problem (5.90) have been proved in [13] (under some assumptions on advection β):

$$\|\beta\cdot\nabla v\|,\ \epsilon^{1/2}\|\nabla v\|,\ \|v\|,\ \frac{1}{\epsilon}\|\beta\cdot\boldsymbol{\tau}\|,\ \frac{1}{\epsilon^{1/2}}\|\boldsymbol{\tau}\|,\ \|\mathrm{div}\tau\|^2 \lesssim \|u\| \,. \tag{5.95}$$

The terms on the left are our "Lego blocks" to construct a test norm. In particular, the use of the graph norm:

$$\|(v,\tau)\|_V^2 = \|\frac{1}{\epsilon}\tau - \nabla v\|^2 + \|\mathrm{div}v + \beta\cdot\nabla v\|^2 + \|v\|^2 + \|\tau\|^2 \tag{5.96}$$

is admissible. The main trouble with the graph norm and other possible test norms is that they inherit the main trouble of the original problem - optimal test functions may develop a boundary layer and, therefore, one may not be able to resolve them using the simple enrichment strategy. With unresolved optimal test functions, we cannot claim anymore the robust lower bound (5.95). In other words, our "Lego play" has to take into account another factor:

The optimal test functions should be easy to resolve.

The following mesh dependent test norm has been studied extensively in [13]:

$$\|(v,\tau)\|_V^2 = \|\beta\cdot\nabla v\|^2 + \epsilon\|\nabla\|^2 + \|C_v v\|^2 + \|\mathrm{div}\tau\|^2 + \|C_\tau\tau\|^2 \tag{5.97}$$

with

$$C_v|_K = \min\left\{\sqrt{\frac{\epsilon}{|K|}}, 1\right\} \quad \text{and} \quad C_\tau|_K = \min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K\|}}\right\} \tag{5.98}$$

where $|K|$ denotes the element area. The zero order terms have been selected in such a way that they do not dominate the diffusion terms. Contrary to the graph norm, components v and τ have been separated, so the inversion of the Riesz operator can be done componentwise. The use of mesh dependent zero order terms allows to employ the optimal powers of ϵ for small elements. Every time, we use suboptimal powers of ϵ in the construction of the test norm, we pay a price for that in the best approximation error estimate. With the mesh-dependent terms, we regain at least the optimality for small elements which appear in boundary layers and other places of high gradients. We refer once again to [13] for an extensive analysis and numerical experiments.

Finally, it is perhaps interesting to mention that, for general boundary conditions, it is impossible to avoid strong boundary layers in the solution of the adjoint problem. Robust estimates analogous to (5.97) are still possible but they employ *weighted* L^2-norms[8]. The weighted test norms have been discovered in a purely experimental way in [23] and rediscovered through the theoretical analysis in [25].

An alternative to the presented philosophy is to work with the graph norm but employ special means (Shishkin meshes) to resolve the optimal test functions with boundary layers, see [34, 33].

Remark 5 (Connecting old dots...) A general approach for controlling convergence in a desired norm through the choice of optimal test functions was proposed a long time ago in [27, 28]. The idea is very simple. Let $\|\cdot\|_U$ be a trial norm in which we want the PG method to be optimal. Define optimal test functions $v_h \in V$ as follows:

$$\begin{cases} v_h \in V \\ b(\Delta u, v_h) = (\Delta u, \delta u_h)_U \quad \forall \Delta u \in U \end{cases} \tag{5.99}$$

where $(\cdot,\cdot)_U$ is the inner product corresponding to norm in U. The Galerkin orthogonality condition,

$$b(u - u_h, v_h) = 0 \quad \forall v_h \,, \tag{5.100}$$

translates then (use $\Delta u := u - u_h$ in (5.99)) into the corresponding orthogonality condition in terms of the inner product,

$$(u - u_h, \delta u_h)_U = 0 \quad \forall \delta u_h \in U_h \,. \tag{5.101}$$

This implies that the PG scheme coincides with the orthogonal projection in the trial norm.

The optimal test space is given here by a different *trial-to-test operator*

$$U_h \ni \delta u_h \to (B')^{-1} R_U \delta u_h \in V \tag{5.102}$$

[8] Intuitively speaking, the weights are selected in such a way that they "kill" the effect of the boundary layers.

implied by the conjugate B' and Riesz operator R_U for trial space U.

For operators (5.102) and (2.13) to coincide, we need:

$$(B')^{-1}R_U = R_V^{-1}B \quad \Longleftrightarrow \quad R_V = BR_U^{-1}B' \,. \tag{5.103}$$

This implies the test norm:

$$\begin{aligned} \|v\|_V^2 &= (v,v)_V = \langle R_V v, v\rangle_{V'\times V} = \langle BR_U^{-1}v, v\rangle_{V'\times V} = \langle B'v, R_U^{-1}B'v\rangle_{U'\times U} \\ &= (R_U^{-1}B'v, R_U^{-1}B'v)_U = \|R_U^{-1}B'v\|_U^2 = \|B'\|_{U'}^2 \end{aligned} \tag{5.104}$$

which coincides with norm (5.85).

Note also that the ideal test norm is not readily available unless we can invert the Riesz operator R_U explicitly. This is the case of the L^2-norm used in the ultraweak formulations but it is not available if the U-norm includes derivatives. This may be considered to be an additional advantage of the ultraweak formulations.

∎

6 Global Optimal Test Functions

In this section we discuss an alternate interpretation of the DPG method based on the concept of global optimal test functions.

Suppose that we pursue the PG method with optimal test functions for the global ultraweak variational formulation (4.76). In other words, for each trial function $(u_h, \hat{u}_{2,h})$, we determine the corresponding *globally optimal test functions* by solving the global problem:

$$\begin{cases} v \in H_{A^*}(\Omega) \\ (v,\delta v)_V = (u_h, A^*\delta v) + \langle \hat{u}_{2,h}, C_2'\mathrm{tr}A^*\delta v\rangle \quad \forall \delta v \in H_{A^*}(\Omega)\,. \end{cases} \tag{6.105}$$

The relation between the DPG (local) *approximate* optimal test functions and the global optimal test functions is quite revealing.

In order to study a relation between the approximate local and global test functions, we will resort to the following abstract notation for the sesquilinear form:

$$b((w,\hat{w}),v) = b_0(w,v) + \langle\langle \hat{w}, v\rangle\rangle. \tag{6.106}$$

By w above we mean the group unknown corresponding to the ultra-weak formulation with globally conforming test functions (6.105). It consists of field variable $u \in L^2(\Omega)$ and the unknown part $\hat{u}_2$ of trace on Γ. The second variable - $\hat{w}$ denotes the unknown abstract trace defined on the *internal* skeleton only. For conforming test functions, the second term vanishes.

Let $W_p, \hat{W}_p$ be now appropriate discrete spaces for the two sets of variables, and let $V_r(\Omega_h)$ denote the enriched approximate broken test space ($r = p + \Delta p$) used to determine the *practical trial-to-test operator:*

$$\begin{cases} T^r(w, \hat{w}) \in V_r(\Omega_h) \\ (T^r(w, \hat{w}), \delta v)_V = b((w, \hat{w}), \delta v) \quad \forall \delta v \in V_r(\Omega_h) \, . \end{cases} \tag{6.107}$$

The *local optimal test space* of the practical DPG method is

$$V_r^p(\Omega_h) := T^r(W_p \times \hat{W}_p) \, . \tag{6.108}$$

To study its relation with a weakly conforming approximate test space, define

$$\widetilde{V}_r(\Omega_h) := \{ v \in V_r(\Omega_h) \, : \, \langle\langle \hat{w}_p, v \rangle\rangle = 0 \quad \forall \hat{w} \in \hat{W}_p \} \, . \tag{6.109}$$

Then, let $\widetilde{T}^r w$ in $\widetilde{V}_r(\Omega_h)$ be defined by

$$(\widetilde{T}^r w, \delta v)_V = b_0(w, \delta v) \quad \forall \delta v \in \widetilde{V}_r(\Omega_h) \, . \tag{6.110}$$

The *weakly conforming global optimal test space* is defined by

$$\widetilde{V}_r^p(\Omega_h) := \widetilde{T}^r(W_p) \, . \tag{6.111}$$

We have (comp. also [23]).

Proposition 1

$$\widetilde{V}_r^p(\Omega_h) \subset V_p^r(\Omega_h) \, . \tag{6.112}$$

∎

Proof: Since $V_r^p(\Omega_h) \subset V_r(\Omega_h)$, we can decompose $V_r(\Omega_h) = V_r^p(\Omega_h) + V_r^\perp(\Omega_h)$ where $V_r^\perp(\Omega_h)$ is the V-orthogonal component of $V_r^p(\Omega_h)$ in $V_r(\Omega_h)$.

Let $\widetilde{v} = \widetilde{T}^r w_p$. Since $\widetilde{v}$ is in $V_r(\Omega_h)$, we can apply the decomposition above to get

$$\widetilde{v} = v^p + \widetilde{v}^\perp, \quad v^p \in V_r^p(\Omega_h), \; \widetilde{v}^\perp \in V_r^\perp(\Omega_h) \, . \tag{6.113}$$

Since $\widetilde{v}^\perp$ is V-orthogonal to $V_r^p(\Omega_h)$, for every $\hat{w}_p \in \hat{W}_p$, we have

$$0 = (T^r(0, \hat{w}_p), \widetilde{v}^\perp)_V = (b((0, \hat{w}_p), \widetilde{v}^\perp) = \langle\langle \hat{w}_p, \widetilde{v}^\perp \rangle\rangle \, , \tag{6.114}$$

hence $\widetilde{v}^\perp \in \widetilde{V}_r(\Omega_h)$. Therefore, we may substitute $\widetilde{v}^\perp$ for v in (6.110) to get

$$(\widetilde{T}^r w_p, \widetilde{v}^\perp)_V = b_0(w_p, \widetilde{v}^\perp) \, . \tag{6.115}$$

Now, the right hand side above must vanish because

$$b_0(w_p, \widetilde{v}^{\perp}) = ((w_p, 0), \widetilde{v}^{\perp}) = (T^r(w_p, 0), \widetilde{v}^{\perp})_V = 0\,. \tag{6.116}$$

Therefore, $0 = (\widetilde{T}^r w_p, \widetilde{v}^{\perp})_V = (\widetilde{v}, \widetilde{v}^{\perp})_V = \|\widetilde{v}^{\perp}\|_V$. Returning to (6.113), we find that $\widetilde{v} = v^p \in V_r^p(\Omega_h)$, thus finishing the proof. ∎

Now consider two DPG methods corresponding to the two test spaces defined previously. The first is the standard practical DPG method that defines $w_p \in W_p$ and $\hat{w}_p \in \hat{W}_p$ satisfying

$$b((w_p, \hat{w}_p), v) = l(v) \quad \forall v \in V_r^p(\Omega_h)\,. \tag{6.117}$$

The second is the DPG method with the weakly conforming globally optimal test space, which finds $\widetilde{w}_p \in W_p$ satisfying

$$b_0(\widetilde{w}_p, \widetilde{v}) = l(\widetilde{v}) \quad \forall \widetilde{v} \in \widetilde{V}_r^p(\Omega_h) \tag{6.118}$$

Proposition 2
If both methods are uniquely solvable, then

$$w_p = \widetilde{w}_p\,. \tag{6.119}$$

∎

Proof: By virtue of Proposition 1, we can substitute any $\widetilde{v} \in \widetilde{V}_r^p(\Omega_h)$ into (6.117), which then immediately reduces to (6.118). ∎

The moral of the story is that the actual DPG method may be interpreted simply as a *localization* of the corresponding global PG methodology. This fact has a number of important implications.

The most important one is the fact that we do not have to resolve the element local (DPG) optimal test functions but only element restrictions of global optimal test functions. For instance, if we insist on using the graph norm for the confusion problem discussed in Section 5, the global optimal test functions *do not form (strong) boundary layers*. If we trade the flux BC for the BC on u, the global optimal test functions do develop a strong layer on the inflow boundary. Consequently, if we can resolve (by whatever means) optimal test functions adjacent to the inflow boundary (and use the standard enriched spaces with $\Delta p = 2$ elsewhere), we observe a robust behavior of the method. On the contrary, if we go after the DPG (local) optimal test functions, we need to resolve boundary layers within each element. For concrete examples illustrating the discussion, see [30].

From the analysis point of view, it looks like one can deemphasize convergence of traces (and fluxes) and focus on studying the convergence of the field variables only. In particular, a discretization of traces with discontinuous elements is non-conforming form the point of view of the DPG method (traces live in

$H^{1/2}$ space) but it is perfectly OK from the point of view of the global PG method and non-conforming discretization of optimal test functions.

As the approximation of optimal test functions in non-conforming, one has to account for both approximation and consistency errors. For 1D problems, weak conformity is equivalent to conformity and there is no consistency error. The DPG method delivers optimal convergence in the L^2-error. This explains, in particular, why the 1D DPG method for linear acoustics is pollution-free [40].

7 Generalizations and Conclusions

The DPG method is a minimum residual method minimizing the residual in a dual norm. It generalizes the classical least squares approach based on L^2-residuals. The general philosophy is straightforward - we minimize the residual in hope of the corresponding error (measured in a desired norm) converging to zero s well. It is thus clear from the very beginning that the choice of the norm used to measure the residual is critical. We need what Wolfgang Dahmen calls "the right mapping property". This leads to the task of selecting the right test norm. The issue is especially critical for singular perturbation problems.

In the paper, we attempted to review the main ideas behind the DPG method and review our and our collaborators' work in the last 3 1/2 years, since the inception of the main idea of computing (approximate) optimal test functions on the fly [19, 21]. Despite several success stories, the method is still in its infancy. We hope that this overview will help to propagate an interest in the DPG methodology.

We will finish the paper with a few additional comments on current work and open research problems.

Adaptivity. Once the resolution of optimal test functions is secured, the DPG methods guarantees stability for any well posed linear problem and *any discretization*, in particular hp elements. This promisses superior convergence rates for problems with singular solutions and boundary layers. The method comes with *a built in error evaluator*. We have made the point of using higher order elements, h- and hp-adaptive meshes in most of our papers to demonstrate that the method remains stable and delivers optimal approximation properties for arbitrary hp meshes. Whereas the element contributions to the global residual serve as perfect element error indicators, the subject of automatic hp-adaptivity remains completely open. In all our examples of hp-adaptivity, we have used so far simple marking strategies only.

DPG for nonlinear problems. The idea of minimizing the dual residual can be extended to nonlinear problems. This was first pursued in [31]. If operator B in (2.10) is nonlinear, the corresponding formula (2.11) for the first Gâteaux derivative must be modified:

$$\langle \delta J(u_h); \delta u_h \rangle = \text{Re } (R_V^{-1} \underbrace{(Bu_h - l)}_{=:r_h}, R_V^{-1} B'(u_h; \delta u_h))_V \, . \tag{7.120}$$

Above, $B'(u_h;\delta u_h)$ denotes the first derivative of operator B at u_h in the direction of δu_h, and r_h is the residual.

The second derivative (Hessian) consists of two terms:

$$\langle \delta^2 J(u_h);\delta u_h,\Delta u_h\rangle = \text{Re}\ \left[(R_V^{-1}(B'(u_h;\Delta u_h),R_V^{-1}B'(u_h;\delta u_h))_V + (R_V^{-1}r_h,R_V^{-1}B''(u_h;\delta u_h,\Delta u_h))_V\right] . \tag{7.121}$$

The trial-to-test operator depends now upon u_h:

$$v_{\delta u_h} = T(u_h)\delta u_h = R_V^{-1}B'(u_h;\delta u_h)\ . \tag{7.122}$$

Introducing the optimal test functions into the formulas, we obtain:

$$\begin{aligned} \langle \delta J(u_h);\delta u_h\rangle = &\quad \text{Re}\ \left[b(u_h,T(u_h)\delta u_h) - l(T(u_h)\delta u_h)\right] \\ \langle \delta^2 J(u_h);\delta u_h,\Delta u_h\rangle = &\quad \text{Re}\ \left[(T(u_h)\Delta u_h,T(u_h)\delta u_h)_V + \langle B''(u_h;\delta u_h,\Delta u_h),R_V^{-1}r_h\rangle\right]\ . \end{aligned} \tag{7.123}$$

With the elementwise inversion of the Riesz operator, we can compute not only the gradient but also the hessian of the residual and use it to solve the nonlinear minimum residual problem. Note that the formula for the hessian assumes that the test inner product is fixed. In practice, an optimal inner product depends upon the first derivative and it also evolves with u_h.

For preliminary attempts to apply the DPG method to nonlinear problems, see also [12].

Element conservation properties. In general, the DPG method does not assure element conservation properties. For problems involving conservation laws, enforcement of conservation properties is deemed desirable. For instance, in the convection-dominated diffusion, the second equation in (5.82) represents a conservation law. The element conservation property translates into the requirement that the test space includes functions $(v,\tau) = (1_K,0)$ where 1_K denotes indicator function for element K. There is no reason why, in general, the optimal test space should include such functions.

It was though also noticed by the MIT colleagues [31] that it is relatively easy to enforce the element conservation property. The main idea is to turn the residual minimization problem (2.10) into a *constrained minimization problem*. We form the Lagrangian:

$$L(u_h,\lambda_K) := \frac{1}{2}\|R_V^{-1}(Bu_h - l)\|_V^2 - \sum_K \text{Re}\ \left[b(u_h,\lambda_K(1_K,0)) - l(\lambda_K(1_K,0))\right] \tag{7.124}$$

and seek its stationary points. We arrive at the mixed problem:

$$\begin{cases} b(u_h,T\delta u_h) - \displaystyle\sum_K \lambda_K b(\delta u_h,(1_K,0)) &= l(T\delta u_h) \qquad \forall \delta u_h \\ b(u_h,(1_K,0)) &= l((1_K,0)) \quad \forall K\ . \end{cases} \tag{7.125}$$

Thus, at the expense of introducing an extra scalar unknown per element for each conserved quantity, the minimum residual approach allows naturally for enforcing conservation laws at the element level, a critical property in CFD simulations. We refer to [14] for additional details.

Preconditioning. Numerical experiments indicate that the DPG method based on the ultraweak formulation delivers stiffness matrices with same condition numbers as conforming finite elements. The work on preconditioners and iterative solvers for the DPG method is an its infancy, see [2] for the only results we are aware of at this point.

Maxwell problems. For an application of DPG methodology to a 2D Maxwell cloaking problem, see [26]. A theoretical groundwork for the 3D DPG Maxwell method has recently been laid down by Wieners and Wohlmut [39].

Implementation of DPG method. We conclude with a short discussion on implementational issues.

The DPG method is definitely more expensive than standard, conforming finite elements or hybridizible DG elements [15]. Even if we disregard element interior degrees-of-freedom (d.o.f.) ("bubbles"), the number of d.o.f. is of the same order as for mixed and standard DG methods, i.e. it is doubled. On top of it, the element computations are essentially more expensive. The extra computational cost is balanced with the extraordinary stability properties and the possibility of controling the norm in which we converge through the selection of test norm. This is a unique property that distinguishes DPG from other methods.

The method comes with a built-in a posteriori *error evaluator*. A routine evaluating the residual error for an element is a calc copy of the element routine. Anyone who had to implement even a simple, explicit a-posteriori error estimate, will appreciate this point. The methodology provides thus a very natural framework for adaptive methods including hp-adaptivity.

The main difficulty in implementing the DPG method comes from the fact that it is a hybrid FE method. As the field variables are discretized with L^2-conforming elements, trace variables with traces of H^1-conforming elements, and flux variables with traces of $H(\text{div})$-conforming elements, the DPG method is naturally implementable within any framework supporting the exact sequence elements. Solution of Maxwell problems in 3D will require traces of $H(\text{curl})$-conforming elements as well. Two implementations of this type have been built at ICES and are available for interested parties: a parallel implementation built on top of Sandia's Trilinos [37], and a workstation Fortran 90 version based on our earlier work on hp methods [18].

As we have tried to convince the reader in Remark 3, with the enriched spaces approach to the computation of optimal test functions, implementation of the element routine is rather straightforward. Otherwise, the rest of the code (assembling, interfacing with solvers, graphical postprocessing etc.) uses the standard FE technology.

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