Chapter 15 17 15.1 bdefhj , 3, 5, 7, 11, 14, 16

15.1

b) $$\int_0^{2\pi} \cos^2\theta \, d\theta = \pi$$

$$\cos\theta = \frac{z + z^{-1}}{2} \quad , \quad \cos^2\theta = \frac{1}{4}\left(z^2 + \frac{1}{z^2} + 2\right)$$

where $z = e^{i\theta}$

$$\int_0^{2\pi} \cos^2\theta \, d\theta = \int_C \frac{1}{4}\left(z^2 + \frac{1}{z^2} + 2\right)\frac{dz}{iz}$$

$$= \operatorname{res}_0 \left(\frac{1}{4}\left(z^2 + \frac{1}{z^2} + 2\right)\frac{1}{iz} \right)$$

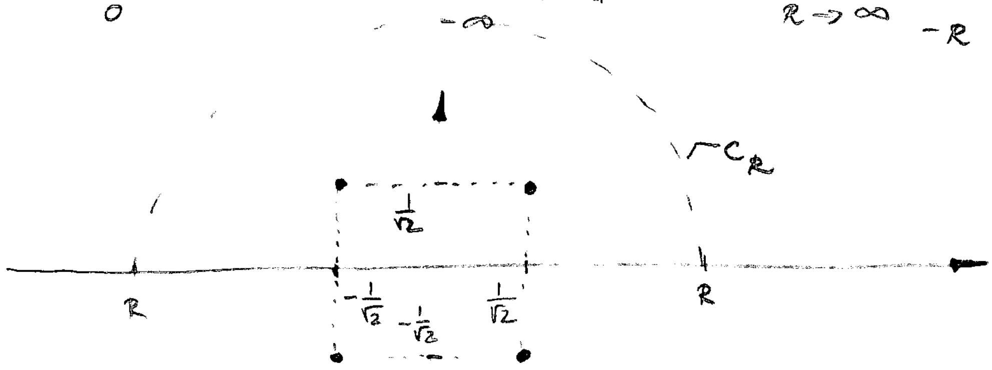
$$= 2\pi i\left(\operatorname{res}_0 \frac{1}{4iz^3} + \operatorname{res}_0 \frac{1}{2iz}\right) = \frac{2\pi i}{2i} = \pi$$

$$\operatorname{res}_0 \frac{1}{z^3} = \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}(1) = 0$$

$$\operatorname{res}_0 \frac{1}{2iz} = \frac{1}{2i}$$

15.1 d)

$$\int_0^\infty \frac{dx}{x^4+a^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R} \frac{dx}{x^4+a^4}$$



$$x^4+a^4=0 \qquad x^4=-a^4$$

$$x = \sqrt[4]{-1}\, a = \mp\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}} i$$

$$\int_{-R}^{R} \frac{dx}{x^4+a^4} + \int_{C_R} \frac{dz}{z^4+a^4} = 2\pi i \left(\operatorname{res}_{\left(-\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)} \frac{1}{z^4+a^4} \right.$$

$$\left. + \operatorname{res}_{\left(\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)} \frac{1}{z^4+a^4} \right)$$

$$\operatorname{res}_{\underbrace{\left(-\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)}_{b}} \frac{1}{z^4+a^4} = \lim_{z\to b} \frac{z-b}{z^4+a^4}$$

$$= \lim_{z\to b} \frac{1}{\left(z-\left(-\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i\right)\right)\left(z-\left(\frac{a}{\sqrt{2}} \text{[illegible]}\right.\right.}$$

$$= \frac{1}{\sqrt{2}i\,(-\sqrt{2}+\sqrt{2}i)(-\sqrt{2})\,a^3} = -\frac{(-\sqrt{2} \text{[illegible]})}{\text{[illegible]}\, i a^3}$$

$$= \left(\frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{8} i\right)\frac{1}{ia^3}$$

$$\operatorname{res}_{\underbrace{\left(\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)}_{b}} \frac{1}{z^4+a^4} = \lim_{z\to b} \frac{z-b}{z^4+a^4}$$

$$= \frac{1}{\left[\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i-\left(-\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i\right)\right]\left[\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i-\left(-\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i\right)\right]\left[\frac{a}{\sqrt{2}}+\frac{a}{\sqrt{2}}i-\left(\frac{a}{\sqrt{2}}-\frac{a}{\sqrt{2}}i\right)\right]}$$

$$= \frac{1}{a^3(\sqrt{2})(\sqrt{2}+\sqrt{2}i)(\sqrt{2}i)} \quad \frac{(\sqrt{2}-\sqrt{2}i)}{(\sqrt{2}-\sqrt{2}i)} = \frac{\sqrt{2}-\sqrt{2}i}{a^3\, 8i}$$

The sum of residues $= 2\,\frac{\sqrt{2}}{8_4}\,\frac{1}{ia^3} = \frac{\sqrt{2}}{4}\,\frac{1}{a^3 i}$

$$2\pi i \cdot \frac{\sqrt{2}}{4_2}\,\frac{1}{a^3 i} = \underline{\frac{\pi}{\sqrt{2}\,a^3}}$$

For $|z|=R$ large enough

$$|z^4+a^4| = \left|z-\left(-\tfrac{\sqrt{2}}{2}-\tfrac{\sqrt{2}}{2}i\right)\right| \cdots \left|z-\left(\tfrac{\sqrt{2}}{2}+\tfrac{\sqrt{2}}{2}i\right)\right| \geq \left(\frac{R}{2}\right)^4$$

So: $\left|\int_{C_R} \frac{dz}{z^4+a^4}\right| \leq \int_{C_R} \frac{ds}{|z^4+a^4|} \leq \frac{16}{R^4}\,\pi R = \frac{16\pi}{R^3} \xrightarrow[R\to\infty]{} 0$

Finally:

$$\int_{-R}^{R} \frac{dx}{x^4+a^4} \xrightarrow[R\to\infty]{} \frac{\pi}{\sqrt{2}\,a^3}$$

e) $$\int_0^{2\pi} \sin^6\theta \, d\theta = \frac{5\pi}{8}$$

$r = 1$ $\quad z = e^{i\theta} = \cos\theta + i\sin\theta$

$\quad z^{-1} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$\therefore \sin\theta = \frac{z - z^{-1}}{2i}$$

$$\sin^6\theta = \frac{1}{(2i)^6} \left(z - \frac{1}{z}\right)^6$$

$$= -\frac{1}{64}\left(\frac{z^2-1}{z}\right)^6 = -\frac{1}{64}\frac{(z^2-1)^6}{z^6}$$

$$\int_0^{2\pi} \sin^6\theta \, d\theta = -\frac{1}{64} \oint_{|z|=1} \frac{(z^2-1)^6}{z^6} \frac{dz}{iz}$$

$$= -\frac{2\pi \cancel{i}}{64 \cancel{i}} \frac{1}{6!}\frac{d^6}{dz^6}(z^2-1)^6 \Big|_{z=0}$$

$$(fg)' = f'g + fg'$$

By induction.

$$(fg)^{(n)} = \binom{n}{0} f^{(n)} g + \binom{n}{1} f^{(n-1)} g' + \dots + \binom{n}{n-1} f' g^{(n-1)} + \binom{n}{n} f g^{(n)}$$

$f(x) = (z-1)^6 \qquad f^{(l)}(0) = \dfrac{6!}{(6-l)!}(-1)^l$

$g(x) = (z+1)^6 \qquad g^{(l)}(0) = \dfrac{6!}{(6-l)!}$

$$(fg)^{(6)}(0) = \sum_{k=0}^{6} \binom{6}{k} f^{(6-k)}(0)\ g^{(k)}(0)$$

$$= \sum_{k=0}^{6} \binom{6}{k} \frac{6!}{k!} (-1)^{(6-k)} \frac{6!}{(6-k)!}$$

$$= 6! \sum_{k=0}^{6} \binom{6}{k}^2 (-1)^{(6-k)}$$

$$= 6! \left(1-6^2+15^2-20^2+15^2-6^2+1\right)$$

$$= 6! \left(1-36+225-400+225-36+1\right)$$

$$= -20\ 6!$$

Finally :

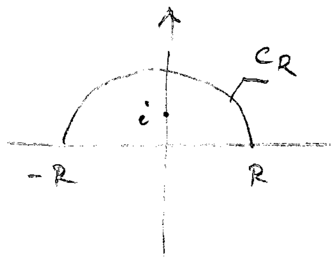
$$\int_0^{2\pi} \sin^6\theta\, d\theta = \frac{2\pi}{64}\, 20. = \frac{\pi}{8} 5 = \frac{5\pi}{8}$$

#

(f) $\displaystyle\int_0^\infty \frac{\cos x\,dx}{(x^2+1)^2} = \frac{\pi}{2e}$

$$e^{ix} = \cos x + i \sin x$$

$$\int_0^\infty \frac{\cos x\,dx}{(x^2+1)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+1)^2} = \frac{1}{2}\,\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{(x^2+1)^2}$$



$$\int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+1)^2} + \int_{C_R} \frac{e^{iz}\,dz}{(z^2+1)^2} = 2\pi i \;\mathrm{res}_i\; \frac{e^{iz}}{(z-i)^2(z+i)^2}$$

$$= 2\pi i \lim_{z \to i} \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2}\right) = 2\pi i \left.\frac{ie^{iz}(z+i)^2 - e^{iz}2(z+i)}{(z+i)^4}\right|_{z=i}$$

$$= 2\pi i \frac{e^{-1}(-4i-4i)}{16} = \frac{e^{-1}2\pi i \cdot -8i}{\cancel{16}} = \frac{\pi}{e}$$

It remains to show that

$$\int_{C_R} \frac{e^{iz}\,dz}{(z^2+1)^2} \xrightarrow[R \to \infty]{} 0$$

$$\left| \int_{C_R} \frac{e^{iz}\,dz}{(z^2+1)^2} \right| \leq \int_{C_R} \frac{|e^{iz}|\,ds}{|z^2+1|^2} = (*)$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = |e^{ix}|\,e^{-y} = e^{-y} \leq 1$$

$$|z^2+1| = |z-i||z+i| \geq \frac{R}{2}\,\frac{R}{2} \quad \text{for } R \text{ large enough}$$

So: $\quad (*) \leq \frac{4}{R\cancel{^2}}\,\pi \cancel{R} \xrightarrow[R\to\infty]{} 0$

h) $$\int_0^{\pi} \frac{d\Theta}{a-\cos\Theta} = \frac{\pi}{\sqrt{a^2-1}} \qquad a>1$$

$$z = e^{i\Theta} \Rightarrow \cos\Theta = \frac{z+z^{-1}}{2} \qquad d\Theta = \frac{dz}{iz}$$

$$\int_0^{\pi} \frac{d\Theta}{a-\cos\Theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\Theta}{a-\cos\Theta} = \frac{1}{2i}\int_C \frac{dz}{z\left(a-\frac{z+z^{-1}}{2}\right)}$$

$$= \frac{1}{2i}\int_C \left(-\frac{2}{z^2+1-2za}\right)dz$$

$$= i\int_C \frac{1}{(z-z_1)(z-z_2)}\,dz = -2\pi \operatorname{res}_{z_2} \frac{1}{(z-z_1)(z-z_2)} = (*)$$

where $z_1 = a+\sqrt{a^2-1}$, $z_2 = a-\sqrt{a^2-1}$ $\qquad a>1$!

$$(*) = -2\pi \lim_{z\to z_2} \frac{1}{z-z_1} = -2\pi \frac{1}{z_2-z_1} = -2\pi \frac{1}{-2\sqrt{a^2-1}}$$

$$= \frac{\pi}{\sqrt{a^2-1}}$$

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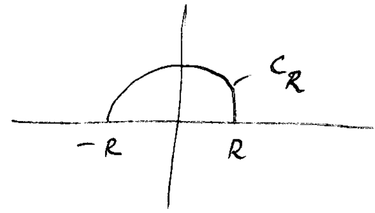
j) $$\int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1} = \frac{\pi}{\sqrt{3}}$$



$$4x^2+2x+1 = 4\Big(x-\big(\underbrace{-\tfrac{1}{4}-i\tfrac{\sqrt{3}}{4}}_{z_1}\big)\Big)\Big(x-\big(\underbrace{-\tfrac{1}{4}+i\tfrac{\sqrt{3}}{4}}_{z_2}\big)\Big)$$

$$\Delta = 4-16 = -12$$

$$x_{1,2} = \frac{-2 \mp i2\sqrt{3}}{8} = -\frac{1}{4} \mp i\frac{\sqrt{3}}{4}$$

$$\int_{-R}^{R} \frac{dx}{4x^2+2x+1} + \int_{C_R} \frac{dz}{4(z-z_1)(z-z_2)} = 2\pi i \operatorname{res}_{z_2} \frac{1}{4(z-z_1)(z-z_2)}$$

$$= 2\pi i \lim_{z\to z_2} \frac{1}{4(z-z_1)} = \cancel{2}\pi i \frac{1}{\cancel{4}_2(z_2-z_1)} = \frac{\pi i}{\cancel{2}\left(2i\frac{\sqrt{3}}{\cancel{4}}\right)} = \frac{\pi}{\sqrt{3}}$$

It remains to show that $\displaystyle\int_{C_R} \frac{dz}{4(z-z_1)(z-z_2)} \xrightarrow[R\to\infty]{} 0$

But, for R large enough, $\left|4(z-z_1)(z-z_2)\right| \geq 4\,\frac{R}{2}\,\frac{R}{2} = R^2$

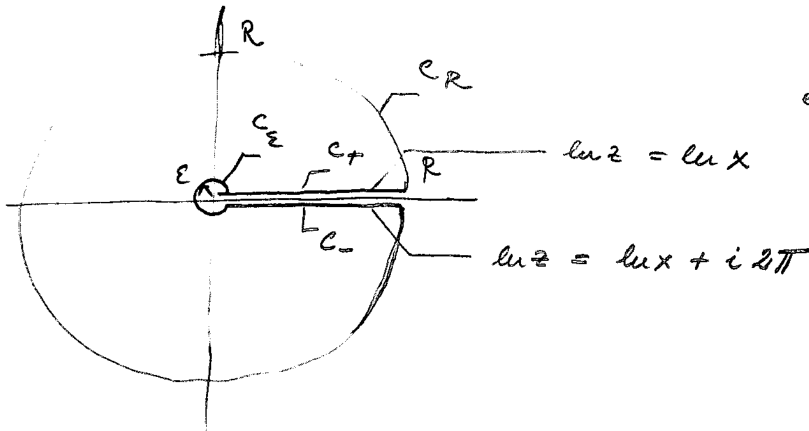
so: $$\left|\int_{C_R} \frac{dz}{4(z-z_1)(z-z_2)}\right| \leq \int_{C_R} \frac{ds}{4|z-z_1||z-z_2|} \leq \frac{2\pi R}{R^2} \xrightarrow[R\to\infty]{} 0$$

#

15.3

$$\int_0^\infty \frac{dx}{x^2+x+1}$$

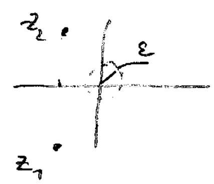
$$u+iv = w = \ln z$$
$$e^w = z$$
$$e^u e^{iv} = r e^{i\theta}$$
$$u = \ln r$$
$$v = \theta + 2k\pi$$



$$\int_{c_+} - \int_{c_-} = \int_\varepsilon^R \frac{\ln x - \ln x - 2\pi i}{x^2+x+1}\,dx = -2\pi i \int_\varepsilon^R \frac{dx}{x^2+x+1}$$

$$\int_{c_\varepsilon} \frac{\ln z}{z^2+z+1}\,dz$$

$$z = \varepsilon e^{i\theta} \qquad z^2+z+1 = 0$$
$$dz = \varepsilon i e^{i\theta}\,d\theta \qquad \Delta = 1-4 = -3$$
$$\ln z = \ln\varepsilon + i\theta \qquad z_{1,2} = \frac{-1 \mp i\sqrt{3}}{2}$$



$|z-z_1||z-z_2| > \frac{1}{16}$ for ε small enough

$$\left| \int_{c_\varepsilon} \frac{\ln z}{z^2+z+1}\,dz \right| \le \int_{c_\varepsilon} \frac{|\ln z|}{|z^2+z+1|}\,ds$$

$$\le 16\,(\ln\varepsilon + 2\pi)\,\varepsilon \cdot 2\pi \xrightarrow[\varepsilon \to 0]{} 0$$

$$\left| \int_{C_R} \frac{\ln z}{z^2+z+1}\, dz \right| \leq \int_{C_R} \frac{|\ln z|}{|z^2+z+1|}\, ds$$

$$\leq \frac{\ln R + 2\pi}{\frac{R^2}{4}} \; R \; 2\pi \xrightarrow[R \to \infty]{} 0$$

Now:

$$\int_{C_+} + \int_{C_R} - \int_{C_-} - \int_{C_\varepsilon} = 2\pi i \left(Res_{z_1} + Res_{z_2} \right)$$

$$Res_{z_1} = \frac{\ln z_1}{z_1 - z_2} = \frac{\frac{4}{3}\pi \cancel{i}}{-\cancel{i}\sqrt{3}} = -\frac{4}{3\sqrt{3}}\pi$$

$$z_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \qquad |z_1| = \frac{1}{4} + \frac{3}{4} = 1$$

$$Arg\, \theta = \frac{4}{3}\pi$$

$$\ln z_1 = \ln 1 + i\frac{4}{3}\pi = \frac{4}{3}\pi i$$

$$Res_{z_2} = \frac{\ln z_2}{z_2 - z_1} = \frac{\frac{2}{3}\pi \cancel{i}}{\cancel{i}\sqrt{3}} = \frac{2}{3\sqrt{3}}\pi$$

So:

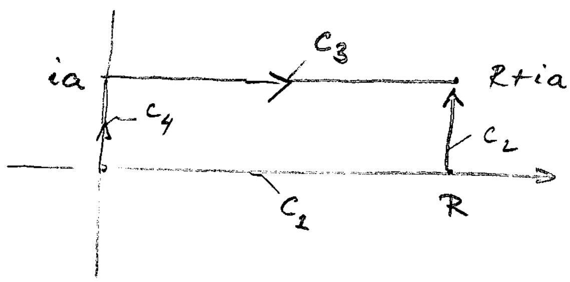
$$-\cancel{2\pi i} \int_0^\infty \frac{dx}{x^2+x+1} = \cancel{2\pi i} \left(- \frac{2}{3\sqrt{3}}\pi \right)$$

and

$$\int_0^\infty \frac{dx}{x^2+x+1} = \frac{2\pi}{3\sqrt{3}}$$

$\#$

15.5 $$\int_0^\infty e^{-x^2}\cos 2ax\,dx = \frac{\sqrt{\pi}\,e^{-a^2}}{2}$$



$$\underbrace{\int_0^R e^{-x^2}dx}_{=\frac{\sqrt{\pi}}{2}} + \int_{C_2} e^{-z^2}dz - \int_{C_3} - \int_{C_4} = 0$$

C_2: $z = R + iat$, $t \in (0,1)$

$dz = ia\,dt$

$$e^{-z^2} = e^{-(R^2 - a^2t^2 + i\,2Rat)}$$
$$= e^{-(R^2-a^2t^2)}\,e^{-i2Rat}$$

$$|e^{-z^2}| = e^{-(R^2-a^2t^2)}$$

$$\left|\int_{C_2} e^{-z^2}dz\right| \le \int_0^1 e^{-(R^2-a^2t^2)}dt = e^{-R^2}\int_0^1 e^{a^2t^2}dt$$

$$\to 0 \text{ as } R \to \infty$$

C_3: $z = t + ia$, $t \in (0,R)$

$$e^{-z^2} = e^{-(t+ia)^2} = e^{-(t^2-a^2+2ati)}$$

$$= e^{-t^2}e^{a^2}e^{-2ati} = e^{a^2}e^{-t^2}(\cos 2at - \sin 2at)$$

$$\int_{C_3} e^{-z^2}dz = e^{a^2}\int_0^R e^{-x^2}(\cos 2ax - \sin 2ax)\,dx$$

C_4 : $\quad z = iat \quad t \in (0,1) \quad dz = ia\,dt$

$$e^{-z^2} = e^{a^2t^2}$$

$$\int_{C_4} e^{-z^2} dz = ia \int_0^1 e^{a^2t^2} dt$$

Summing up and passing with $R \to \infty$, we get

$$\frac{\sqrt{\pi}}{2} - ia \int_0^1 e^{a^2t^2} dt = e^{a^2} \int_0^\infty e^{-x^2}(\cos 2ax - \sin 2ax)\,dx$$

Comparing the real parts of both sides we get the result required.

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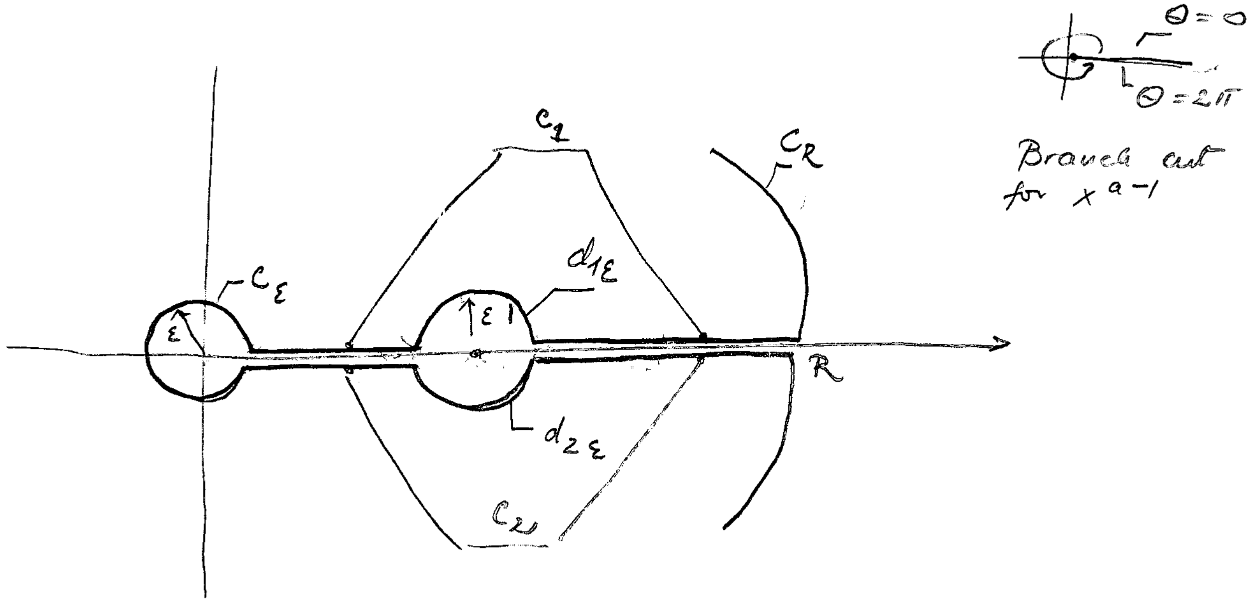
15.7 a) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1-x}\,dx = \pi \cot (a-1)\pi \qquad (0<a<1)$



$$-\oint_{c_\varepsilon} + \int_{c_1} - \int_{c_2} - \oint_{d_{1\varepsilon}} - \oint_{d_{2\varepsilon}} + \oint_{C_R} = 0$$

On c_1 : $z^{a-1} = x^{(a-1)}$

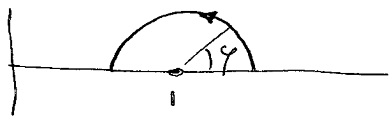
On c_2 : $z^{a-1} = (xe^{i2\pi})^{a-1} = x^{(a-1)} e^{i2\pi(a-1)}$

$$\int_{c_1} - \int_{c_2} = \int_\varepsilon^{1-\varepsilon} \sim + \int_{1+\varepsilon}^{R} \frac{x^{(a-1)}}{1-x}\left(1-e^{i2\pi(a-1)}\right)$$

$$\longrightarrow \left(1-e^{i2\pi(a-1)}\right) \int_0^\infty \frac{x^{(a-1)}}{1-x}\,dx$$

$$\left| \int_{C_R} \frac{z^{a-1}}{1-z}\,dz \right| \le \frac{R^{a-1}}{\frac{R}{2}} R \cdot 2\pi \xrightarrow[R\to\infty]{} 0$$

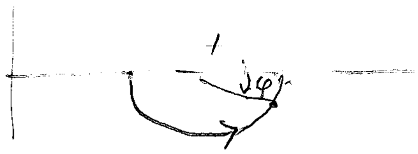
$$\left| \int_{c_\varepsilon} \frac{z^{a-1}}{1-z}\,dz \right| \le \frac{\varepsilon^{a-1}}{\frac{1}{2}} \varepsilon 2\pi \xrightarrow[\varepsilon\to 0]{} 0$$



$$z = 1 + \varepsilon e^{i\varphi} \qquad dz = \varepsilon i e^{i\varphi} d\varphi$$

$$\oint_{d_{1\varepsilon}} \frac{z^{a-1}}{1-z} dz = \int_0^{\pi} \frac{(1+\varepsilon e^{i\varphi})^{a-1}}{-\cancel{\varepsilon e^{i\varphi}}} \cancel{\varepsilon} i \cancel{e^{i\varphi}} d\varphi = -i \int_0^{\pi} (1+\varepsilon e^{i\varphi})^{a-1} d\varphi$$

$$\xrightarrow[\varepsilon \to 0]{} -i \int_0^{\pi} 1^{a-1} d\varphi = -\pi i$$



$$z = 1 + \varepsilon e^{-i\varphi} \qquad dz = -\varepsilon i e^{-i\varphi} d\varphi$$

$$\oint_{d_{2\varepsilon}} \frac{z^{a-1}}{1-z} dz = -\int_0^{\pi} \frac{(1+\varepsilon e^{-i\varphi})^{a-1}}{-\cancel{\varepsilon e^{-i\varphi}}} (-\cancel{\varepsilon} i \cancel{e^{-i\varphi}}) d\varphi = -i \int_0^{\pi} (1+\varepsilon e^{-i\varphi})^{a-1} d\varphi$$

$$\longrightarrow -i \int_0^{\pi} 1^{a-1} d\varphi = -i \left(1 e^{i2\pi}\right)^{a-1} \pi$$

$$= -\pi i \, e^{i2\pi(a-1)}$$

Summing up :

$$\left(1 - e^{i2\pi(a-1)}\right) \int_0^{\infty} \frac{x^{(a-1)}}{1-x} dx = -\pi i \left(1 + e^{i2\pi(a-1)}\right)$$

$$\gamma = 2\pi(a-1)$$

$$\frac{1+e^{i\gamma}}{1-e^{i\gamma}} = \frac{(1+e^{i\gamma})(1+e^{-i\gamma})}{(1-e^{i\gamma})(1+e^{-i\gamma})} = \frac{1+e^{i\gamma}+e^{-i\gamma}+1}{\cancel{1}-e^{i\gamma}+e^{-i\gamma}\cancel{-1}}$$

$$= \frac{2+2\cos\gamma}{-2i\sin\gamma}$$

$$\frac{2+2\cos x}{-2\cancel{x}\sin x}\,(-\pi\cancel{x}) = \frac{1+\cos x}{\sin x}\,\pi$$

$$= \frac{\cancel{1}+2\cos^2\frac{x}{2}\cancel{-1}}{\cancel{2}\sin\frac{x}{2}\cos\frac{x}{2}} = \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} = \cot\frac{x}{2} = \underline{\underline{\cot\pi(a-1)}}$$

Passing to the limits we get the result required.

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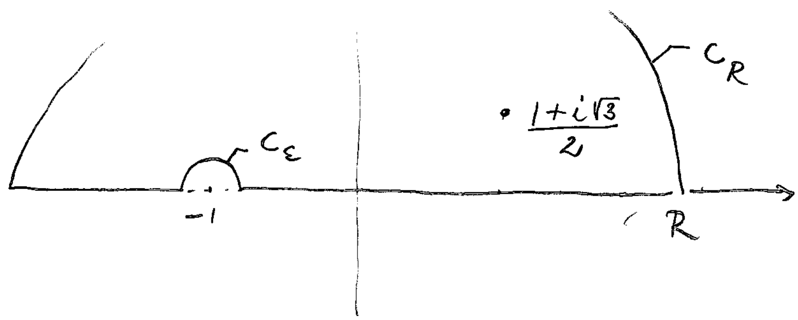
15.7 b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{x^3+1} = \frac{\pi}{\sqrt{3}}$

$z^3+1 = (z+1)(z^2-z+1)$

$z^2-z+1=0$

$z_{1,2} = \frac{1 \pm i\sqrt{3}}{2}$



$$\int_{-R}^{-1-\varepsilon} + \int_{C_\varepsilon} + \int_{-1+\varepsilon}^{R} + \int_{C_R} = 2\pi i \operatorname{res}_{\frac{1+i\sqrt{3}}{2}}$$

$$\left| \int_{C_R} \frac{z\,dz}{z^3+1} \right| \le \frac{R}{\left(\frac{R}{2}\right)^3} R\pi \xrightarrow[R\to\infty]{} 0$$

on C_ε: $\quad z = -1 + \varepsilon e^{i\varphi} \qquad \varphi \in (0,\pi)$

$dz = \varepsilon i e^{i\varphi}\, d\varphi$

$z^3+1 = (-1+\varepsilon e^{i\varphi})^3 + 1 = -1 + 3\varepsilon e^{i\varphi} - 3\varepsilon^2 e^{2i\varphi} + \varepsilon^3 e^{3i\varphi} + 1$

$$\int_0^{\pi} \frac{(-1+\varepsilon e^{i\varphi})}{3\varepsilon e^{i\varphi} - 3\varepsilon^2 e^{i2\varphi} + \varepsilon^3 e^{i3\varphi}} \varepsilon i e^{i\varphi}\, d\varphi$$

$$= \int_0^{\pi} \frac{(-1+\varepsilon e^{i\varphi})\, i}{3 - 3\varepsilon e^{i\varphi} + \varepsilon^2 e^{i2\varphi}}\, d\varphi \longrightarrow \int_0^{\pi} \frac{-i}{3}\, d\varphi = -\frac{\pi i}{3}$$

$$\operatorname{res}_{\frac{1+i\sqrt{3}}{2}} = \left.\frac{z}{(z+1)\left(z - \frac{1-i\sqrt{3}}{2}\right)}\right|_{z=\frac{1+i\sqrt{3}}{2}}$$

$$= \frac{\frac{1+i\sqrt{3}}{2}}{\frac{3+i\sqrt{3}}{\cancel{2}}\; i\sqrt{3}} = \frac{1+i\sqrt{3}}{-3+3i\sqrt{3}} =$$

$$= \frac{1}{3} \frac{(1+i\sqrt{3})(-1-i\sqrt{3})}{1+3} = -\frac{1}{12}\left(1-3+2i\sqrt{3}\right)$$

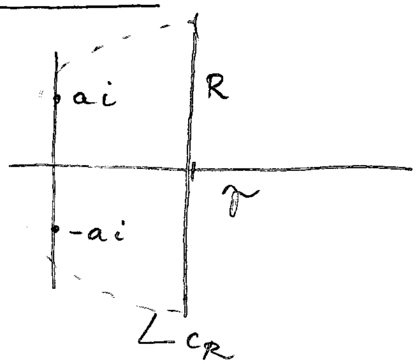
$$= -\frac{1}{6} - \frac{1}{6} i\sqrt{3}$$

$$2\pi i \cdot \operatorname{res}_{\frac{1+i\sqrt{3}}{2}} = -\frac{\pi i}{3} + \underline{\underline{\frac{\pi}{3}\sqrt{3}}}$$

Summing up $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{x^3+1} = \frac{\pi}{\sqrt{3}}$

15.11 a) $$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{st}}{s^2+a^2}\,ds$$

Case $t>0$



$$|e^{st}| = |e^{(x+iy)t}| = |e^{xt}| \leq e^{\gamma t}$$

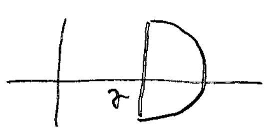
$$\left|\int_{C_R}\frac{e^{st}}{s^2+a^2}\,ds\right| \leq \frac{e^{\gamma t}}{\frac{R^2}{4}}\,2\pi R \xrightarrow[R\to\infty]{} 0$$

$$res_{ai} = \frac{e^{iat}}{2ai} \qquad res_{-ai} = \frac{e^{-iat}}{-2ai}$$

$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}\frac{e^{st}}{s^2+a^2}\,ds + \underbrace{\int_{C_R}\sim dz}_{\to 0} = \frac{e^{iat}-e^{-iat}}{2ai}$$

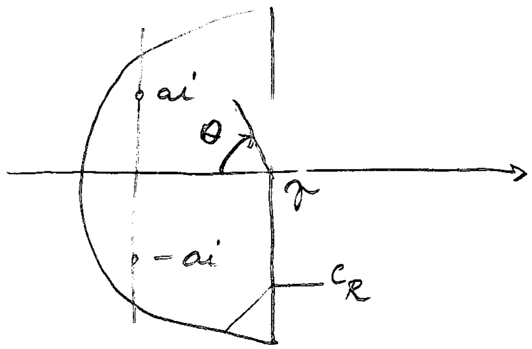
$$= \frac{1}{a}\sin(at)$$

Case $t<0$

use  to show that $\int ds = 0$

15.11 b)
$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{s\,e^{st}}{s^2+a^2}\,ds$$

Case $t>0$



$$C_R:\begin{cases} x=\gamma-R\cos\theta \\ y=R\sin\theta\end{cases}$$

$$|e^{st}|=e^{xt}=e^{(\gamma-R\cos\theta)t}=e^{\gamma t}\,e^{-Rt\cos\theta}$$

$$\left|\int_{C_R}\frac{s\,e^{st}}{s^2+a^2}\,dz\right|\le\int_{C_R}\frac{|s|\,|e^{st}|}{|s^2+a^2|}\,ds$$

$$\le\frac{2R}{\frac{R^2}{4}}\,Re^{\gamma t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}e^{-Rt\cos\theta}\,d\theta\xrightarrow[R\to\infty]{}0$$

$$\text{res}_{ai}=\frac{\cancel{ai}\,e^{ait}}{2\cancel{ai}}\qquad\text{res}_{-ai}=\frac{-ai\,e^{-iat}}{-2ai}$$

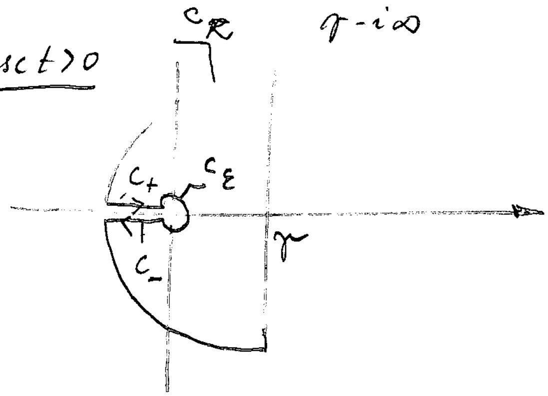
$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}\frac{s\,e^{st}}{s^2+a^2}\,ds+\underbrace{\int_{C_R}\sim dz}_{\to 0}=\frac{e^{ait}+e^{-ait}}{2}=\cos at$$

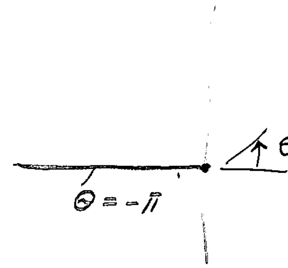
Case $t<0$ same in a) $\equiv 0$.

\#

15.11 c) $\qquad \dfrac{1}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{\sqrt{s}}\, ds$

<u>case $t>0$</u>





cut for $\sqrt{z}$

$$\int_{C_R} \frac{e^{st}}{\sqrt{s}}\, ds \xrightarrow{\;0\;} \quad \text{by Exercise 15.14}$$

On C_ε : $\qquad s = \varepsilon e^{i\theta} \qquad \theta \in (-\pi, \pi)$

$$\sqrt{s} = \sqrt{\varepsilon}\, e^{i\frac{\theta}{2}}$$

$$|e^{st}| = |e^{(x+iy)t}| = |e^{xt}||e^{iyt}| \le e^{\varepsilon t}$$

So: $\quad \left| \displaystyle\int_{C_\varepsilon} \frac{e^{st}}{\sqrt{s}}\, ds \right| \le \dfrac{e^{\varepsilon t}}{\sqrt{\varepsilon}}\, 2\pi\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$

$$\int_{C_+} \frac{e^{st}}{\sqrt{s}}\, ds = \int_{\gamma - R}^{-\varepsilon} \frac{e^{xt}}{\sqrt{-x}\, e^{+i\frac{\pi}{2}}}\, dx = \int_{\gamma - R}^{-\varepsilon} \frac{e^{xt} e^{-i\frac{\pi}{2}}}{\sqrt{-x}}\, dx$$

$$\int_{C_-} \frac{e^{st}}{\sqrt{s}}\, ds = \int_{\gamma - R}^{-\varepsilon} \frac{e^{xt}}{\sqrt{-x}\, e^{-i\frac{\pi}{2}}}\, dx = \int_{\gamma - R}^{-\varepsilon} \frac{e^{xt} e^{+i\frac{\pi}{2}}}{\sqrt{-x}}\, dx$$

$$\int_{C_+} - \int_{C_-} = \int_{\gamma-R}^{-\varepsilon} \frac{e^{xt}\left(e^{-i\frac{\pi}{2}} - e^{+i\frac{\pi}{2}}\right)}{\sqrt{-x}}\,dx$$

$$= -2i\int_{\gamma-R}^{-\varepsilon} \frac{e^{xt}}{\sqrt{-x}}\,dx \xrightarrow[R\to\infty]{\varepsilon\to 0} -2i\int_{-\infty}^{0} \frac{e^{xt}}{\sqrt{-x}}\,dx$$

$$= -2i\int_{0}^{\infty} \frac{e^{-xt}}{\sqrt{x}}\,dx = -2i\int_{0}^{\infty} \frac{\sqrt{t}\,e^{-\xi}}{\sqrt{\xi}}\,\frac{d\xi}{t}$$

$$xt = \xi \qquad\qquad = -2i\,t^{-\frac{1}{2}}\int_{0}^{\infty} \frac{e^{-\xi}}{\sqrt{\xi}}\,d\xi$$

$$x = \frac{\xi}{t} \qquad\qquad = -2i\,t^{-\frac{1}{2}}\,\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{=\sqrt{\pi}}$$

$$dx = \frac{1}{t}\,d\xi$$

$$\sqrt{x} = \frac{1}{\sqrt{t}}\sqrt{\xi}$$

So:

$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{\sqrt{s}}\,ds + \cancelto{0}{\frac{1}{2\pi i}\int_{C_R}} - \cancelto{0}{\frac{1}{2\pi i}\int_{C_\varepsilon}} + \underbrace{\frac{1}{2\pi i}\left(\int_{C_+} - \int_{C_-}\right)}_{\downarrow} = 0$$

$$- \frac{t^{-\frac{1}{2}}}{\pi}\,\underbrace{\Gamma\left(\tfrac{1}{2}\right)}_{=\sqrt{\pi}} = - \frac{t^{-\frac{1}{2}}}{\sqrt{\pi}}$$

So, the inverse transform of $\dfrac{1}{\sqrt{s}}$ is $\dfrac{t^{-\frac{1}{2}}}{\sqrt{\pi}}$ for $t>0$. For $t<0$ we get 0, as usual.

Remark. Notice that the value of $\int_{C_+} - \int_{C_-}$ is independent of the branch cut!

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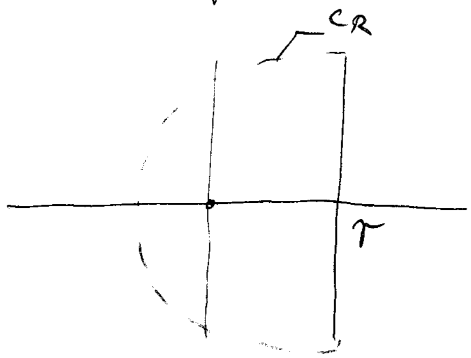
15.11 d) $\displaystyle \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{s^3}\, ds$



Case $t > 0$

$$\left| \int_{C_R} \frac{e^{zt}}{z^3}\, dz \right| \leq \frac{e^{\gamma t}}{\left(\frac{R}{2}\right)^3} R\pi \xrightarrow[R \to \infty]{} 0$$

$$\operatorname{res}_0 \frac{e^{zt}}{z^3} = \lim_{z \to 0} \frac{1}{2} \frac{d^2}{dz^2} \left(e^{zt} \right) = \frac{t^2}{2}$$

So the inverse transform is $\frac{t^2}{2}$

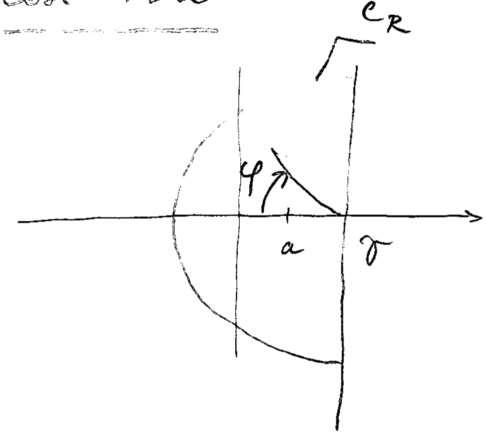
Case $t < 0$ as usual gives 0.

#

15.11 e) $$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}e^{-as}}{s}\,ds$$

$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-a)}}{s}\,ds$$

<u>Case $t > a$</u>



$$\left|\int_{C_R} \frac{e^{z(t-a)}}{z}\,dz\right| \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{|e^{z(t-a)}|}{\frac{\cancel{R}}{2}}\,\cancel{R}\,d\varphi = (*)$$

$$|e^{z(t-a)}| = |e^{(x+iy)(t-a)}| = |e^{x(t-a)}|\,\cancel{|e^{iy(t-a)}|}^{\,1}$$

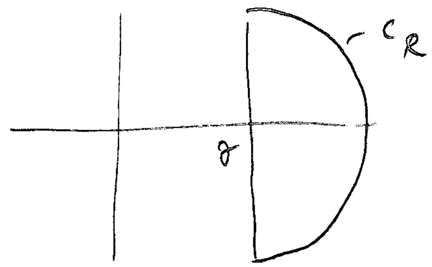
$$= e^{(\gamma - R\cos\varphi)(t-a)} = e^{\gamma(t-a)}\,e^{-R(t-a)\cos\varphi}$$

$$(*) = \frac{1}{2}e^{\gamma(t-a)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R(t-a)\cos\varphi}\,d\varphi \to 0$$

$$\operatorname{res}_0 \frac{e^{s(t-a)}}{s} = e^{0(t-a)} = 1$$

Case $t<a$

use



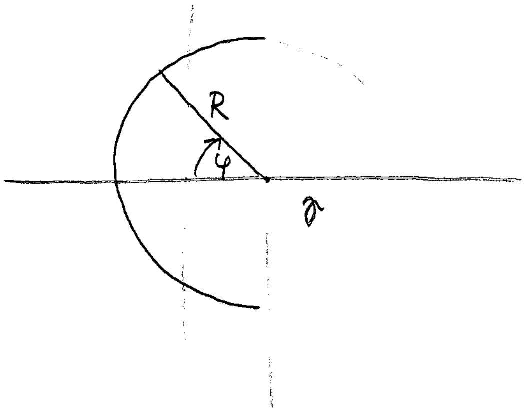
to show that $\int_{C_R} \longrightarrow 0$

Consequently the inverse transform of $\frac{e^{-as}}{s}$ is

$$\begin{cases} 1 & \text{for } t>a \\ 0 & \text{for } t<a \end{cases} = H(t-a)$$

#

15.14

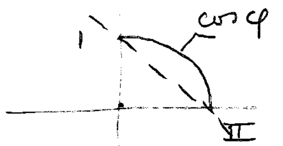


<u>Case $t>0$</u>

$$\left|\int_C e^{st}\bar{f}(z)\,dz\right| \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |e^{st}|\,\overbrace{|\bar{f}(z)|}^{\le\, MR^{-\alpha}}\, R\,d\varphi = (*) \qquad (M>0)$$

$$|e^{st}| = \left|e^{(\gamma - R\cos\varphi + iy)t}\right| = e^{(\gamma - R\cos\varphi)t}$$

$$(*) \le Me^{\gamma t}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-Rt\cos\varphi} R^{1-\alpha}\,d\varphi = 2Me^{\gamma}\int_0^{\frac{\pi}{2}} e^{-Rt\cos\varphi} R^{1-\alpha}\,d\varphi$$



$$\cos\varphi \ge -\frac{2}{\pi}\left(\varphi - \frac{\pi}{2}\right)$$

$$e^{-Rt\cos\varphi} \le e^{+Rt\frac{2}{\pi}\left(\varphi - \frac{\pi}{2}\right)}$$

$$\leq 2Me^{\gamma} R^{1-\alpha} \int_0^{\frac{\pi}{2}} e^{Rt\frac{2}{\pi}(\varphi - \frac{\pi}{2})} d\varphi$$

$$Rt\frac{2}{\pi}\left(\varphi - \frac{\pi}{2}\right) = -x$$

φ	0	$\frac{\pi}{2}$
x	$-Rt$	0

$$Rt\frac{2}{\pi}\, d\varphi = -dx$$

$$= 2Me^{\gamma} R^{1-\alpha} \frac{\pi}{2Rt} \int_0^{Rt} e^{-x} dx$$

$$\leq Me^{\gamma} R^{-\alpha} \frac{\pi}{t} \underbrace{\int_0^{\infty} e^{-x} dx}_{\text{finite}} \xrightarrow[R\to\infty]{} 0$$

<u>Case $t<0$</u> analogous ..

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15,16

We cannot speak about the residue of this function at 0. To speak about the residue at 0, the function must be holomorphic in the annulus $|z| > 0$. whereas the given geometric series converges only for $|z| > 1$ and for $|z| \leq 1$ function is simply not defined !

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