ASE 380P2 ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANIS II

Final Exam. Thursday, May 13, 2010

1. (a) State the Gauss divergence theorem (5 points).

   Let V be a bounded domain in $\mathbb{R}^3$, and $\boldsymbol{v}$ a vector-valued smooth function. Then,

$$\int_V \text{div}\boldsymbol{v}\, dV = \int_{\partial V} v_n\, dS$$

   where $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ denotes the (exterior) normal component of $\boldsymbol{v}$.

   (b) Evaluate $\int_S \boldsymbol{v} \cdot \boldsymbol{n}\, dS$, where S is the surface of the cylinder:

$$x^2 + y^2 \leq 4, \quad 0 \leq z \leq 3$$

   and

$$\boldsymbol{v} = (2, 0, 3z^5)$$

   (15 points).

   Apply the Gauss theorem. As $\text{div}\boldsymbol{v} = 15z^4$ is independent of x, y, the volume integral collapses to a single integral in z,

$$\int_V \text{div}\boldsymbol{v}\, dV = 4\pi \int_0^3 15z^4\, dz = 2916\pi$$

2. Consider the following initial-value problem.

$$\begin{cases} \ddot{x} + \dot{x} = F_0 \delta(t-1) \\ x(0) = \dot{x}(0) = 0 \, , \end{cases}$$

where δ denotes the Dirac's delta, $t_0 > 0$, and F_0 is a parameter.

(a) Explain how the presence of the delta functional translates into an interface condition, and solve the problem using elementary calculus (3 points).

Delta functional is the (distributional) derivative of the Heaviside step function. Consequently, in order to obtain $F_0\delta(t-1)$ on the right-hand side, the function must be continuous at $t = 1$, whereas the first derivative must experience a jump equal to F_0,

$$[x(1)] = 0, \; [\dot{x}(1)] = F_0$$

This leads to two initial-value problems. For $t \in (0,1)$,

$$\begin{cases} x(0) = 0, \; \dot{x}(0) = 1 \\ \ddot{x} + \dot{x} = 0 \end{cases}$$

implies $x(t) = 0$. For $t \in (1, \infty)$,

$$\begin{cases} x(1) = 0, \; \dot{x}(1) = F_0 \\ \ddot{x} + \dot{x} = 0 \end{cases}$$

implies $x(t) = F_0(1 - e^{1-t})$. Consequently,

$$x(t) = \begin{cases} 0 & t \in (0,1) \\ F_0(1 - e^{1-t}) & t \in (1, \infty) \end{cases}$$

(b) Define the Laplace transform for the delta distribution and compute it (2 points).

From the definition of the Laplace transform,

$$L(\delta(t-1))(s) = \int_0^\infty e^{-st} \delta(t-1) \, dt = e^{-s}$$

(c) Compute the Laplace transform of the solution to the initial-value problem (5 points).

Recalling the rule for transforming derivatives,

$$\widehat{(\dot{x})}(s) = s\hat{x}(s) - x(0) \, ,$$

we transform both sides of the differential equation to obtain,

$$(s^2 + s)\hat{x} - sx(0) - \dot{x}(0) - x(0) = F_0 e^{-s}$$

Taking into account the homogeneous initial conditions, we obtain

$$\hat{x}(s) = F_0 \frac{e^{-s}}{s(s+1)}$$

(d) Use the Residue Theorem to compute the inverse Laplace transform of the solution in the "Laplace domain" and compare it with the solution obtained using the elementary calculus (10 points).

We need to consider two cases.

- Case: $t \in (0,1)$. We use the integration contour shown in Fig. 1. The contour contains no

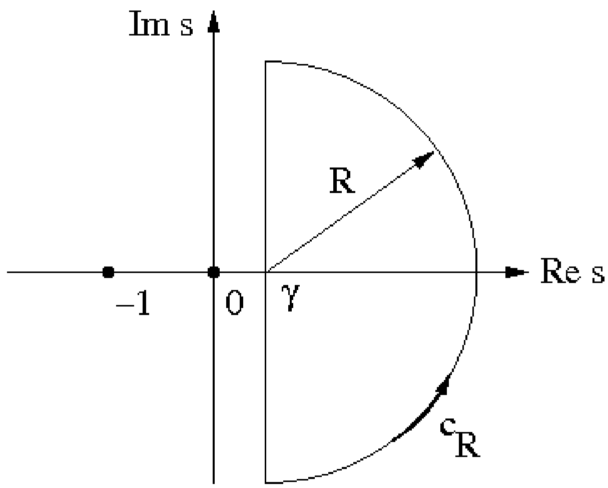

Figure 1: Case: $t \in (0,1)$. Integration contour.

singularities, so

$$\frac{1}{2\pi i}\lim_{R\to\infty}\int_{\gamma-iR}^{\gamma+iR} s^{st}\frac{e^{-s}}{s(s+1)}\,ds = \frac{1}{2\pi i}\lim_{R\to\infty}\int_{c_R}\frac{e^{s(t-1)}}{s(s+1)}\,ds = 0$$

The contour over the semicirle goes to zero since the numerator decreases exponentially and the denominator increases only algebraically.

- Case: $t \in (1,\infty)$. Use the integration contour shown in Fig. 2. The Residue Theorem implies that

$$\frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR}\frac{e^{s(t-1)}}{s(s+1)}\,ds + \frac{1}{2\pi i}\int_{c_R}\frac{e^{s(t-1)}}{s(s+1)}\,ds$$
$$= Res_0\left[\frac{e^{s(t-1)}}{s(s+1)}\right] + Res_{-1}\left[\frac{e^{s(t-1)}}{s(s+1)}\right]$$
$$= 1 - e^{1-t}$$

By the same argument as above, the integral over the semicircle goes to zero.

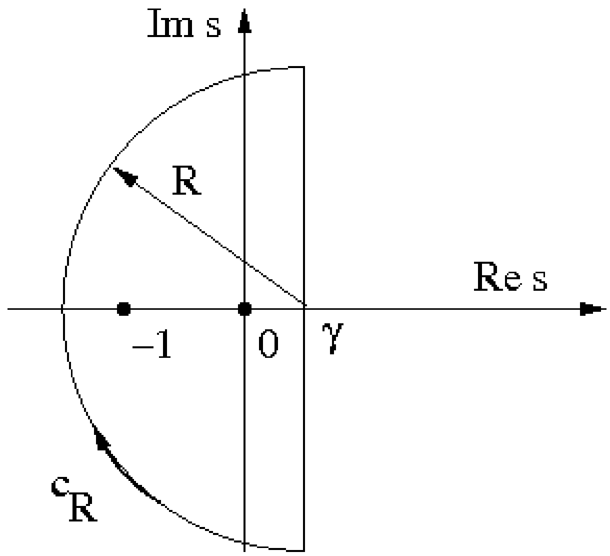

Figure 2: Case: $t \in (1, \infty)$. Integration contour.

3. Use the method of images and separation of variables to determine the solution of the Laplace equation in the triangle shown in Fig. 3 (20 points).

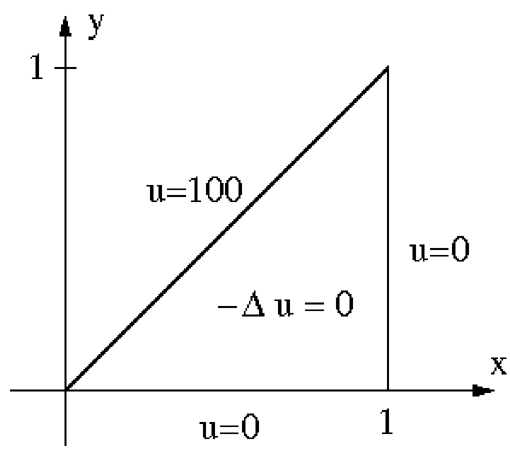

Figure 3: Laplace equation in a triangular domain

By superposition, solution can be obtained by considering first the same BVP with constant Dirichlet BCs ($= 100$) and then solution to the BVP shown in Fig. 4. The first solution can be guessed, it is

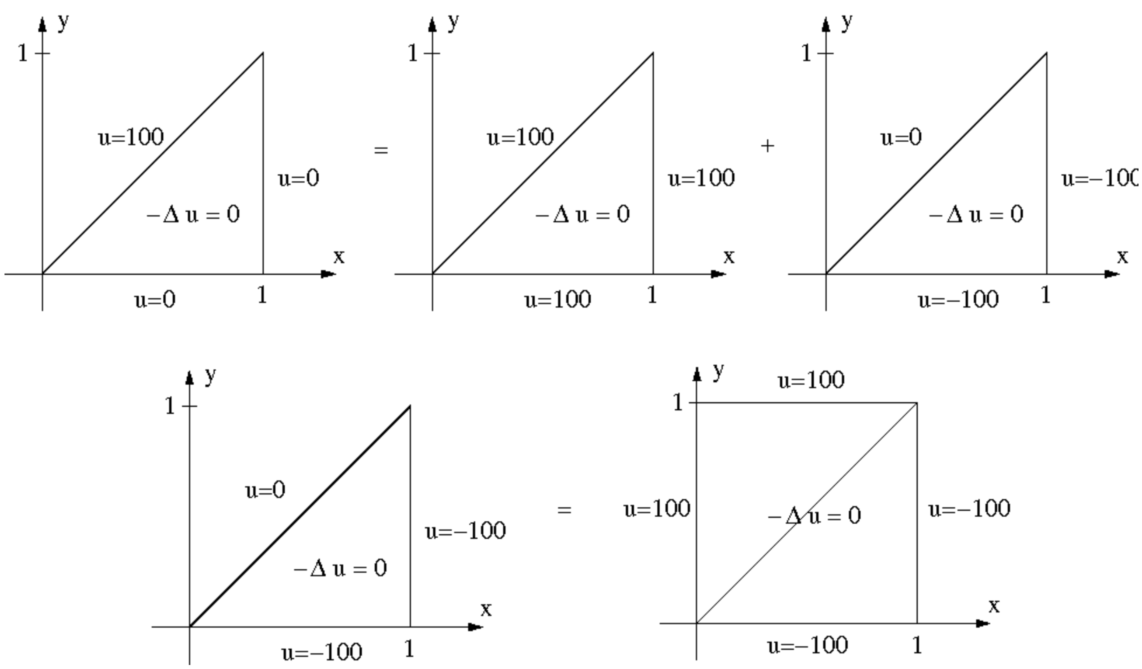

Figure 4: Use of superposition and method of images.

a constant solution $u_1 = 100$. It does satisfy the BCs and the Laplace equation so, by uniqueness, it must be *the* solution. In turn, by the method of images, the second solution can be obtained from the corresponding solution in square domain, shown again in Fig. 4. This solution can be obtained by using again the superposition principle.

For the case of BCs $u(1,y) = 100$ and all remaining BC's equal zero, separation of variables gives,

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sinh n\pi x \sin n\pi y$$

where constants a_n are given by

$$a_n = \frac{\int_0^1 100 \sin n\pi y \, dy}{\sinh n\pi \int_0^1 \sin^2 n\pi y \, dy}$$

The remaining three solutions are given by analogous formulas. The final solution looks as follows,

$$\begin{aligned} u(x,y) = 100 \quad &+ \sum_{n=1}^{\infty} a_n \sinh n\pi x \sin n\pi y + \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi y \\ &- \sum_{n=1}^{\infty} a_n \sinh n\pi(1-x) \sin n\pi y - \sum_{n=1}^{\infty} a_n \sin n\pi x \sinh n\pi(1-y) \end{aligned}$$

(20 points)

Remark. A shorter but more tricky solution is to extend the constant ($= 100$) solution to the whole square and reduce the problem to the solution on the square with BCs shown in Fig. 5

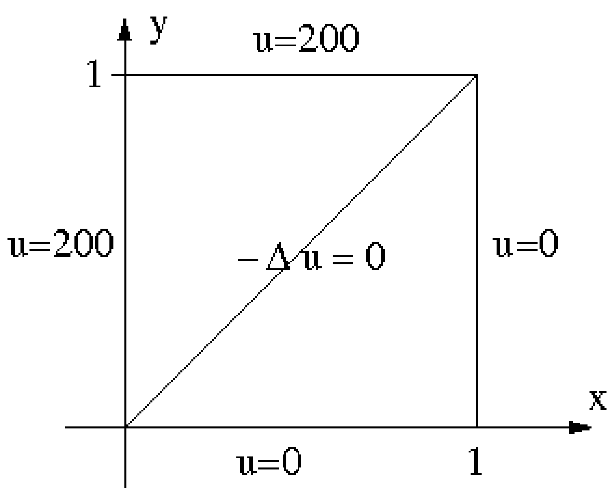

Figure 5: An alternative solution.

4. Solve the following eigenvalue problem.

   (a) Derive the formula for the gradient in polar system of coordinates,

$$\boldsymbol{\nabla} u = \frac{\partial u}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\boldsymbol{e}_\theta$$

   (3 points).

   Use the general formula for the gradient in a curvilinear system of coordinates $u^j,\ j = 1, 2$,

$$\boldsymbol{\nabla} f = \sum_{j=1}^{2} \frac{\partial f}{\partial u_j}\boldsymbol{a}^j$$

   where $\boldsymbol{a}^j$ are the co-basis vectors. For polar coordinates r, θ, basis vectors,

$$\boldsymbol{a}_r = (\cos\theta, \sin\theta) = \boldsymbol{e}_r, \quad \boldsymbol{a}_\theta = (-r\sin\theta, r\cos\theta) = r\boldsymbol{e}_\theta$$

   are orthogonal, so the determination of the co-basis reduces to the rescaling,

$$\boldsymbol{a}^r = \boldsymbol{e}_r, \quad \boldsymbol{a}^\theta = \frac{1}{r}\boldsymbol{e}_\theta$$

   (b) Use the formula for the gradient, and the integration by parts to derive the formula for the Laplacian in polar coordinates,

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

   (3 points).

   Use the fact that, in any system of coordinates,

$$\int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v = -\int_\Omega \Delta u\, v + \text{boundary terms}$$

   Integrating by parts,

$$\int \left(\frac{\partial u}{\partial r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial v}{\partial \theta}\right) r dr d\theta = -\int \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) v\, r dr d\theta$$

   (c) Consider the eigenvalue problem,

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

   where Ω is the segment of the unit circle,

$$\Omega = \{(r, \theta)\ :\ 0 < r < 1,\ 0 < \theta < \frac{3}{2}\pi\}$$

Prove that the eigenvalues λ must be real and positive (4 points).

Laplacian with homogeneous Dirichlet boundary conditions is self-adjoint. This implies that the eigenvalues must be real. The operator is also positive-definite. Indeed, integrating by parts, we get,

$$\int_\Omega (-\Delta u)u = \int_\Omega |\boldsymbol{\nabla} u|^2 = 0 \quad \text{implies} \quad \boldsymbol{\nabla} u = 0$$

which in turn (due to the homogeneous BC) implies that $u = 0$. Positive-definitness implies now that all eigenvalues must be positive. Indeed, let (λ, u) be an eigenpair. Then,

$$\int_\Omega (-\Delta u)u = \lambda \int_\Omega |u|^2 > 0 \quad \text{implies} \quad \lambda > 0$$

(d) Use separation of variables to solve the eigenvalue problem (10 points).

Starting with $u = R(r)\Theta(\theta)$, we arrive at

$$-\left[\frac{1}{r}(rR')'\Theta + \frac{1}{r^2}R\Theta\right] = \lambda R\Theta$$

or, equivalently,

$$\frac{r(rR')'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = k^2$$

Note that the separation constant must be real and positive. This is a consequence that operator $-\Theta''$ in $(0, \frac{3}{2}\pi)$, with homogeneous Dirichlet conditions, is self-adjoint and positive-definite. Solving for Θ, we get,

$$\Theta = A\sin k\theta, \quad k = k_n = \frac{2}{3}n$$

This leads to the equation in r,

$$r^2R'' + rR' + (\lambda r^2 - \frac{4}{9}n^2)R = 0$$

Scaling the independent variable $\sqrt{\lambda}r = x$, we obtain the Bessel equation,

$$x^2R'' + xR' + (x^2 - \nu^2)R = 0$$

where $\nu = \frac{2}{3}n$. The solution must be finite at $x = 0$, so the general solution of the equation is,

$$R(r) = J_\nu(\sqrt{\lambda}r)$$

The function has to vanish at $r = 1$,

$$J_\nu(\sqrt{\lambda}) = 0$$

so the eigenvalues

$$\lambda = \lambda_m = r_m^2$$

where $r_m, m = 1, 2, \ldots$ represent roots of J_ν. We obtain a two-parameter family of eigenvalues with the corresponding eigenvectors,

$$u = J_\nu(r_m r)\sin(\frac{2}{3}n\theta)$$

5. Consider the Poisson problem in $\mathbb{R}^2$,

$$\Delta u = f(x, y)$$

(a) With $z = x + iy,\ \bar{z} = x - iy$, demonstrate that,

$$\Delta u = 4\frac{\partial^2 u}{\partial z \partial \bar{z}}.$$

(5 points).

This is an elementary change of variables exercise.

(b) Use the result to derive the general solution of the Poisson equation in terms of the complex variables (10 points).

Defining

$$g(z, \bar{z}) := f(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2})$$

we integrate with respect to $\bar{z}$, and then in z, to obtain,

$$u = \frac{1}{4}\left(\int\int g(z, \bar{z})\, d\bar{z}dz + A(z) + B(\bar{z})\right)$$

where $A(z), B(\bar{z})$ are arbitrary functions, and the integrals are indefinite.

(c) Work out and verify a particular solution for $f(x, y) = xy$ (5 points).

Just algebra.

$$xy = \frac{z+\bar{z}}{2}\frac{z-\bar{z}}{2i} = \frac{1}{4i}(z^2 - \bar{z}^2)$$

Integrating and setting $A = B = 0$, we get

$$\frac{1}{4i}\frac{z^3}{3}\bar{z} - \frac{1}{4i}z\frac{\bar{z}^3}{3} = \frac{1}{3}z\bar{z}\frac{1}{4i}(z^2 - \bar{z}^2) = \frac{1}{3}(x^2 + y^2)xy = \frac{1}{3}x^3y + \frac{1}{3}xy^3$$

which gives

$$u = \frac{1}{12}(x^3y + xy^3)$$