Chapter 9

Vector Field Theory

In this chapter we introduce the so-called gradient, divergence, and curl vector differential operators and then develop the several transformation theorems of Gauss, Green, and Stokes that relate volume, surface, and line integrals. These results, which are central to applied analysis, play an especially important role in the development of electrodynamics, thermodynamics, and the continuum mechanics. In fact, our illustrations include derivations of some of the governing equations of fluid mechanics and heat conduction, as well as a discussion of the gravitational potential. That is, here we see how the physical laws (e.g., conservation of mass, Fourier's law of heat conduction, and Newton's law of gravitational attraction) plus the vector integral theorems lead to the governing partial differential equations; later on we discuss their solution.

9.1. *LINE INTEGRALS*

To illustrate, suppose that we have defined throughout some region R of x, y, z space a vector force field $\mathbf{F}(x, y, z)$ per unit mass at x, y, z. If we denote by $\mathbf{r}$ the position vector from some fixed point O to a unit mass that moves along some space curve C (Fig. 9.1), then the work done by the force field on the particle in traversing C is defined by the **line integral**

$$W = \int_C \mathbf{F} \cdot d\mathbf{r}. \tag{9.1}$$

By the right-hand side we mean (as might be guessed from our discussion of the

Riemann integral in Section 1.4) the limit

$$\lim_{|P|\to 0} \sum_{k=1}^{n} \mathbf{F}(P_{k-1}) \cdot [\mathbf{r}(P_k) - \mathbf{r}(P_{k-1})], \tag{9.2}$$

where P_k is shorthand for the point x_k, y_k, z_k. The points $P_0(= A), P_1, \ldots, P_{n-1}, P_n(= B)$ define a "partition" P of C, and $|P| \equiv \max_{1\le k\le n} \|\mathbf{r}(P_k) - \mathbf{r}(P_{k-1})\|$ is called the "norm" of this partition. This definition is exactly analogous to the Riemann definition (1.14), where in this case we have decided to choose "ξ_k" = "x_{k-1}"—that is, at the left endpoint of each segment.

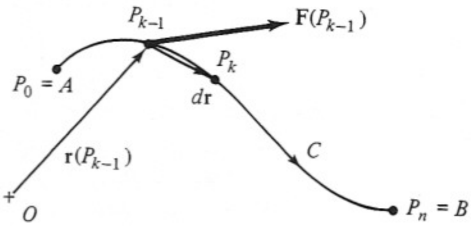

Figure 9.1. The line integral (9.1).

Note that there is an *orientation* to C, as indicated in the figure by the arrowhead. If we denote by $-C$ the same curve but with opposite orientation, then

$$\int_{-C} \mathbf{F} \cdot d\mathbf{r} = -\int_C \mathbf{F} \cdot d\mathbf{r}, \tag{9.3}$$

since the $d\mathbf{r}$'s are oppositely directed.

Line integrals also come in forms other than (9.1); for instance,

$$\int_C f\,ds, \quad \int_C f\,dx, \quad \int_C f\,dy, \quad \text{and} \quad \int_C f\,dz,$$

where $f = f(x, y, z)$, the Cs are given space curves, and s is arc length along C, are all line integrals and are defined as limits of Riemann-type sums. Practically speaking, however, they are simply special cases of (9.1). For instance, if $\mathbf{r} = \mathbf{r}(s)$, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \int_C f\,ds,$$

where
$$f \equiv \mathbf{F} \cdot \frac{d\mathbf{r}}{ds}.$$

Example 9.1. Consider the two-dimensional case where $\mathbf{F} = 2xy\hat{\mathbf{i}} + (x^2 - 1)\hat{\mathbf{j}}$ and C is the spiral $r = 2\theta$, from $\theta = 0$ to $\theta = 5\pi/2$, as shown in Fig. 9.2. In this case, $\mathbf{F}$ is given in terms of cartesian coordinates and C in terms of polar coordinates. Anticipating that it may be more convenient to work this problem in polar coordinates, let us take a step in that direction by expressing $d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta$. Then

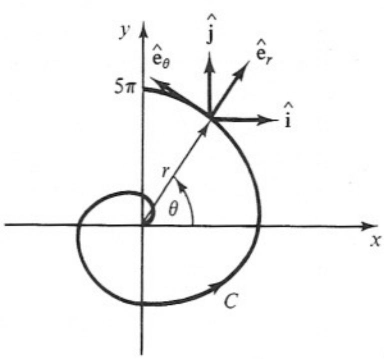

Figure 9.2. The spiral $r = 2\theta$.

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_C [2xy\hat{\mathbf{i}} + (x^2 - 1)\hat{\mathbf{j}}] \cdot (dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta) \\
&= \int_C [2xy(\hat{\mathbf{i}} \cdot \hat{\mathbf{e}}_r)\,dr + 2xy(\hat{\mathbf{i}} \cdot \hat{\mathbf{e}}_\theta)r\,d\theta \\
&\quad + (x^2 - 1)(\hat{\mathbf{j}} \cdot \hat{\mathbf{e}}_r)\,dr + (x^2 - 1)(\hat{\mathbf{j}} \cdot \hat{\mathbf{e}}_\theta)r\,d\theta] \\
&= \int_C [2r^2\,cs(c)\,dr + 2r^2cs(-s)r\,d\theta \\
&\quad + (r^2c^2 - 1)(s)\,dr + (r^2c^2 - 1)(c)r\,d\theta],
\end{aligned}$$

where $c \equiv \cos\theta$ and $s \equiv \sin\theta$ for short. Finally, on C $r = 2\theta$, so that

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{5\pi/2} (16\theta^2c^2s - 16\theta^3cs^2 + 8\theta^2c^2s - 2s + 8\theta^3c^3 - 2\theta c)\, d\theta$$

$$= \text{etc.} = -5\pi. \tag{9.4}$$

As in Chapter 8, we emphasize that $d\mathbf{r}$ in polar coordinates is $dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta$, *not* $dr\,\hat{\mathbf{e}}_r + d\theta\,\hat{\mathbf{e}}_\theta$! This situation results from expressing $\mathbf{r} = r\hat{\mathbf{e}}_r$, so that

$$d\mathbf{r} = dr\,\hat{\mathbf{e}}_r + r\,d\hat{\mathbf{e}}_r = dr\,\hat{\mathbf{e}}_r + r\left[\left(\frac{\partial \hat{\mathbf{e}}_r}{\partial r}\right) dr + \left(\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\right) d\theta\right] = dr\,\hat{\mathbf{e}}_r + r\,d\theta\,\hat{\mathbf{e}}_\theta,$$

as can also be seen graphically by sketching $\mathbf{r}$ and $\mathbf{r} + d\mathbf{r}$, identifying $d\mathbf{r}$, and breaking it into its $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ components. ∎

9.2. *THE CURVES, SURFACES, AND REGIONS UNDER CONSIDERATION*

Throughout this chapter we assume that our curves and surfaces are "suitably decent." In particular, we assume that they are **piecewise smooth**.

A curve C, given by $r = r(s)$, is said to be **smooth** if the unit tangent vector $\hat{\mathbf{t}} = d\mathbf{r}/ds$ is a continuous function of s over C, or, equivalently, if dx/ds, dy/ds, and dz/ds are continuous over C. If it is smooth except at a finite number of points, such that "left-" and "right-hand derivatives" $d\mathbf{r}/ds$ exist at these points, then we say that it is **sectionally** or **piecewise smooth**. Certainly the curves in Examples 9.1 and 9.2 are piecewise smooth. It can be shown that every piecewise smooth curve between endpoints is *rectifiable*—that is, has finite length. Finally, if the initial and final points of C (A and B in Fig. 9.1) coincide, we say that C is a *closed* curve.

Turning to surfaces, we say that a surface S is **smooth** if its unit normal $\hat{\mathbf{n}}$ is a continuous function of position over S. If S can be divided by smooth curves into a finite number of segments, each of which is smooth, then we say that S is **piecewise smooth**. For instance, a cube is not smooth, but it is piecewise smooth.

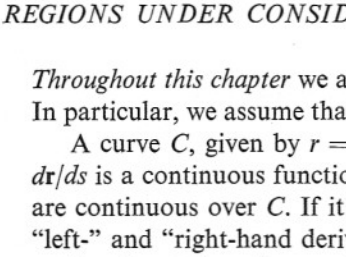

Figure 9.3. Möbius band.

Actually, we will also require our surfaces to be **orientable**—that is, having two sides—so that a skink could live on one side and a small klipspringer on the other without ever meeting. The classic example of a *non*orientable surface is the Möbius band, which can be constructed by taking a rectangular strip of paper and gluing the upper surface of one end to the lower surface of the other end (Fig. 9.3). If we start at any point P and follow the arrows, we eventually visit the entire surface, so that the Möbius band has only one side.

Our three-dimensional regions are to be bounded (by one or more piecewise smooth surfaces) and connected. By **connected** we mean that every pair of points in the region R can be joined by a piecewise smooth curve lying entirely in R. Furthermore, a connected region R is said to be **simply connected** if every *simple closed curve* (i.e., a piecewise smooth, closed curve) in R can be shrunk down to a point without leaving R; otherwise R is **multiply connected**.

To illustrate, consider the regions shown in Fig. 9.4; R_1 is the interior of a sphere, R_2 is the region *between* concentric spherical surfaces, and R_3 is the interior of a bagel. R_1 and R_2 are both simply connected, since every simple closed curve C (which we might think of as made of string) can be shrunk to a point; this fact is less obvious for R_2, since it appears that the inner spherical boundary gets in the way, but observe that the string can be slipped over this sphere and shrunk to the point P, for example. R_3 on the other hand, is multiply connected, since curves such as the one shown can*not* be shrunk to a point.

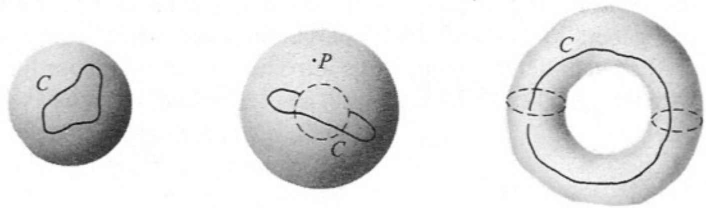

Figure 9.4. Simple and multiple connectedness.

9.3. DIVERGENCE, GRADIENT, AND CURL

In the following discussion we consider both scalar and vector fields, say $u(x, y, z)$ and $\mathbf{v}(x, y, z)$, or $u(P)$ and $\mathbf{v}(P)$ for short, defined throughout various three-dimensional regions R. Physically, u might be a temperature field and $\mathbf{v}$ a fluid velocity field, for example. They may vary with time as well, but the present discussion involves spatial considerations, and so we will simply ignore any t dependence except in applications where relevant.

We will assume of u and the three components of $\mathbf{v}$ that their first partial derivatives with respect to x, y and z all exist and are continuous throughout R.

Divergence. Associated with the vector field $\mathbf{v}$, we define a scalar field called div $\mathbf{v}$—that is, *the divergence of* $\mathbf{v}$—by

$$\operatorname{div} \mathbf{v}(P) \equiv \lim_{\tau \to 0} \left\{ \frac{\int_\sigma \hat{\mathbf{n}} \cdot \mathbf{v} \, d\sigma}{\tau} \right\}, \qquad (9.5)$$

where τ and σ are the volume and surface of a little "control volume," having an outward unit normal vector $\hat{\mathbf{n}}$ defined over σ[1] and containing the point P (see Fig. 9.5).

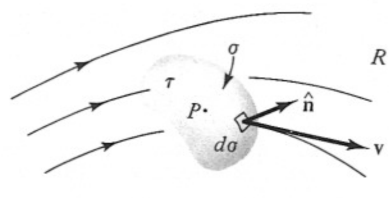

Figure 9.5. The control volume at P.

[1] That is, $\hat{\mathbf{n}}$ is defined over *most* of σ, since σ is only required to be piecewise smooth; for instance, for a cube, $\hat{\mathbf{n}}$ is defined over the six faces, although not along their edges.

For definiteness, let us think of $\mathbf{v}$ as a *fluid velocity field*, as indicated by the streamlines in Fig. 9.5. Note that the control volume τ is only a "mathematical" surface; it is completely permeable, and its presence in no way affects the flow.

Let us now interpret the right-hand side of (9.5). The outflow through the surface element $d\sigma$ is the normal velocity component $\hat{\mathbf{n}} \cdot \mathbf{v}$ (meters per second, say) times $d\sigma$ (meters squared)—that is, $\hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma$ m³/sec. (Note that the component of $\mathbf{v}$ that is *tangential* to $d\sigma$ produces no outflow.) Integrating over σ, $\int_\sigma \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma$ is the *net* outflow, in cubic meters per second, say, across σ. And dividing by the volume τ, we have outflow per unit volume. Letting τ tend to 0, we see that div $\mathbf{v}(P)$ is the *outflow per unit volume at P*.

An important asset of (9.5) is that it provides a so-called *intrinsic* definition of div $\mathbf{v}$; that is, it contains no reference to any particular coordinate system. The price of this generality is that it is not particularly convenient computationally.

For definiteness, then, let us introduce a cartesian coordinate system, for instance, with the usual $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ base vectors, and let us attempt to carry out the limit indicated in (9.5). The rectangular coordinate system suggests that we choose τ to be a prism, as sketched in Fig. 9.6. With x, y, z as our cartesian variables, let P be the point x_0, y_0, z_0 at the center of the prism.

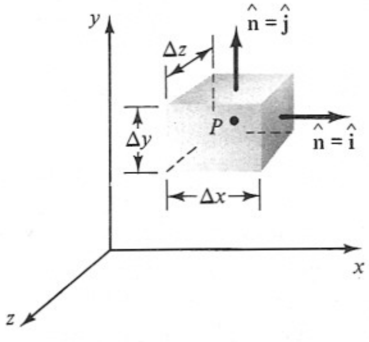

Figure 9.6. Cartesian coordinates.

Consider first the contribution from the right- and left-hand faces, on which $\hat{\mathbf{n}} = +\hat{\mathbf{i}}$ and $-\hat{\mathbf{i}}$, respectively. Since $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, we have, with some help from the mean value theorem of the integral calculus,[2,3]

$$\int_{\text{right face}} \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma = \int v_x\left(x_0 + \frac{\Delta x}{2}, y, z\right) d\sigma$$
$$= v_x\left(x_0 + \frac{\Delta x}{2}, y_1, z_1\right) \Delta y\, \Delta z$$

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[2]If $f(x)$ has minimum and maximum values of m and M over $a \le x \le b$, then clearly

$$m(b - a) \le \int_a^b f\, dx \le M(b - a),$$

so that the integral equals $c(b - a)$, where $m \le c \le M$. But if f is continuous over $[a, b]$, then there is some point, say ξ, such that $f(\xi) = c$. Then

$$\int_a^b f\, dx = f(\xi)(b - a) \tag{9.6}$$

for some ξ in $[a, b]$, and this is the **mean value theorem of the integral calculus**. Similarly, if $f(x, y)$ is continuous over a bounded area A, then there is some point ξ, η in A such that

$$\iint_A f\, dx\, dy = f(\xi, \eta)A. \tag{9.7}$$

[3]Often we use only a single integral sign for multiple integrals.

and
$$\int_{\text{left face}} \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma = \int -v_x\left(x_0 - \frac{\Delta x}{2}, y, z\right) d\sigma$$
$$= -v_x\left(x_0 - \frac{\Delta x}{2}, y_2, z_2\right) \Delta y\, \Delta z,$$

where $x_0 + (\Delta x/2), y_1, z_1$ is some point on the right-hand face, and $x_0 - (\Delta x/2), y_2, z_2$ is some point on the left hand face. Since $\tau = \Delta x\, \Delta y\, \Delta z$,

$$\lim_{\tau\to 0}\left\{\frac{\int_{\text{left+right faces}} \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma}{\tau}\right\}$$
$$= \lim_{\Delta x, \Delta y, \Delta z\to 0}\left\{\frac{v_x[x_0 + (\Delta x/2), y_1, z_1] - v_x[x_0 - (\Delta x/2), y_2, z_2]}{\Delta x}\right\}$$
$$= \lim_{\Delta x\to 0} \frac{v_x[x_0 + (\Delta x/2), y, z] - v_x[x_0 - (\Delta x/2), y, z]}{\Delta x} = \frac{\partial v_x}{\partial x}. \tag{9.8}$$

Similarly, the top and bottom faces contribute $\partial v_y/\partial y$, and the front and back faces contribute $\partial v_z/\partial z$, and so

$$\operatorname{div} \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \tag{9.9}$$

Thus if $\mathbf{v}$ is given in cartesian form—that is, as $v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, then its divergence is obtained easily according to (9.9). Noncartesian systems will be treated in Section 9.7.

Rewriting (9.9) as

$$\operatorname{div} \mathbf{v} = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot (v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}),$$

let us define the vector differential operator

$$\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} \equiv \nabla \tag{9.10}$$

called *del* or *nabla*, so that

$$\operatorname{div} \mathbf{v} = \nabla \cdot \mathbf{v}. \tag{9.11}$$

Gradient. Starting instead with a *scalar* field $u(P)$, it will be fruitful to define an associated vector field called grad u, that is, *the gradient of u*—by

$$\operatorname{grad} u(P) \equiv \lim_{\tau\to 0}\left\{\frac{\int_\sigma \hat{\mathbf{n}} u\, d\sigma}{\tau}\right\}. \tag{9.12}$$

If we introduce cartesian coordinates, take τ to be a prism, and carry out the limit as before, we find (Exercise 9.5) that

$$\operatorname{grad} u = \frac{\partial u}{\partial x}\hat{\mathbf{i}} + \frac{\partial u}{\partial y}\hat{\mathbf{j}} + \frac{\partial u}{\partial z}\hat{\mathbf{k}}. \tag{9.13}$$

Recalling (9.10), we see that this is none other than ∇ acting on u:

$$\operatorname{grad} u = \nabla u. \tag{9.14}$$

Physical interpretation of grad u does not seem to fall out of the definition (9.12) easily. Instead let us consider a field point P and the $u =$ constant surface (e.g., an *isothermal surface* if u is temperature) that passes through P (Fig. 9.7). To facilitate our story, let us introduce a space curve C through P according to $x = x(s)$, $y = y(s)$, $z = z(s)$ and examine the rate of change du/ds along C, in particular, at the point P on C.

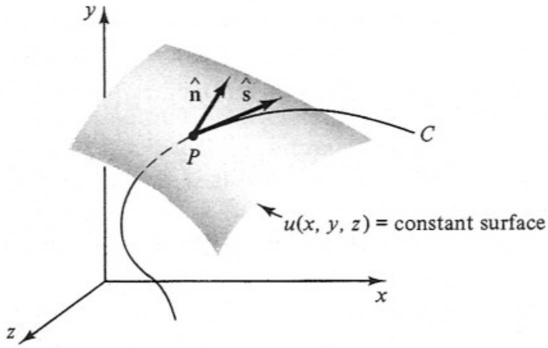

Figure 9.7. Interpretation of gradient.

By chain differentiation,

$$\frac{d}{ds}u[x(s), y(s), z(s)] = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} + \frac{\partial u}{\partial z}\frac{dz}{ds}. \tag{9.15}$$

Recalling from Exercise 8.20 that the unit tangent vector to C is[4] $\hat{\mathbf{s}} = (dx/ds)\hat{\mathbf{i}} + (dy/ds)\hat{\mathbf{j}} + (dz/ds)\hat{\mathbf{k}}$, we observe that the right-hand side of (9.15) can be expressed more compactly as $\nabla u \cdot \hat{\mathbf{s}}$:

$$\frac{du}{ds} = \nabla u \cdot \hat{\mathbf{s}}. \tag{9.16}$$

This is an important little result. It says that the "**directional derivative**" du/ds at any point P is computed as the gradient of u at P dotted into a unit vector in the desired direction.

Equation (9.16) is very informative. Suppose, for instance, that we choose C to lie *in* the $u = c$ surface, so that $\hat{\mathbf{s}}$ is *tangent* to the surface at P. Since u is thereby constant on C, $du/ds = 0$. Thus $0 = \nabla u \cdot \hat{\mathbf{s}}$ for *all* $\hat{\mathbf{s}}$'s that are tangent to the surface at P, and it follows that *the ∇u vector at P must be perpendicular to the $u =$ constant surface at P.*

So much for the *direction* of ∇u. What about its *magnitude*? To answer, consider the problem of choosing the orientation of $\hat{\mathbf{s}}$ so as to achieve the largest possible value of du/ds. From (9.16) $du/ds = \|\nabla u\|\,\|\hat{\mathbf{s}}\| \cos\theta = \|\nabla u\| \cos\theta$, where θ is the angle between $\hat{\mathbf{s}}$ and ∇u. Thus du/ds is a maximum, namely, $\|\nabla u\|$, when $\cos\theta = 1$—that is, $\theta = 0$; so *the magnitude $\|\nabla u\|$ equals the maximum possible du/ds, and this value occurs when $\hat{\mathbf{s}}$ is normal to the $u = c$ surface.*

[4]We will use the designation $\hat{\mathbf{s}}$ here rather than $\hat{\mathbf{t}}$.

Example 9.2. Find the directional derivative of $u = x^2y + 3z$ at the point (1, 2, 0) in the direction of the vector $\hat{\mathbf{i}} - 4\hat{\mathbf{k}}$.

Solution.

$$\nabla u = 2xy\hat{\mathbf{i}} + x^2\mathbf{j} + 3\hat{\mathbf{k}} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}} \quad \text{at } (1, 2, 0)$$

$$\hat{\mathbf{s}} = \frac{\hat{\mathbf{i}} - 4\hat{\mathbf{k}}}{\sqrt{17}}$$

$$\frac{du}{ds} = \nabla u \cdot \hat{\mathbf{s}} = (4\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot \frac{\hat{\mathbf{i}} - 4\hat{\mathbf{k}}}{\sqrt{17}} = -\frac{8}{\sqrt{17}}. \quad \blacksquare$$

Curl. Finally, we also define for a vector field $\mathbf{v}$ an associated vector field called curl $\mathbf{v}$, *the curl of* $\mathbf{v}$, according to

$$\operatorname{curl} \mathbf{v}(P) \equiv \lim_{\tau \to 0} \left\{ \frac{\int_\sigma \hat{\mathbf{n}} \times \mathbf{v}\, d\sigma}{\tau} \right\}. \tag{9.17}$$

Introducing cartesian coordinates and taking τ to be a prism (Fig. 9.6), we can carry out the limit, as we did for div $\mathbf{v}$. But it is shorter and neater to proceed as follows. Since $\mathbf{A} = (\mathbf{A} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\mathbf{A} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\mathbf{A} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}}$ for any $\mathbf{A}$, we can express

$$\begin{aligned}\hat{\mathbf{n}} \times \mathbf{v} &= (\hat{\mathbf{n}} \times \mathbf{v} \cdot \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{n}} \times \mathbf{v} \cdot \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{n}} \times \mathbf{v} \cdot \hat{\mathbf{k}})\hat{\mathbf{k}} \\ &= (\hat{\mathbf{n}} \cdot \mathbf{v} \times \hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{n}} \cdot \mathbf{v} \times \hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{n}} \cdot \mathbf{v} \times \hat{\mathbf{k}})\hat{\mathbf{k}}.\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{curl} \mathbf{v}(P) &= \lim_{\tau \to 0} \left\{ \frac{\int_\sigma \hat{\mathbf{n}} \cdot (\mathbf{v} \times \hat{\mathbf{i}})\, d\sigma}{\tau} \right\} \hat{\mathbf{i}} + \lim_{\tau \to 0} \left\{ \frac{\int_\sigma \hat{\mathbf{n}} \cdot (\mathbf{v} \times \hat{\mathbf{j}})\, d\sigma}{\tau} \right\} \hat{\mathbf{j}} \\ &\quad + \lim_{\tau \to 0} \left\{ \frac{\int_\sigma \hat{\mathbf{n}} \cdot (\mathbf{v} \times \hat{\mathbf{k}})\, d\sigma}{\tau} \right\} \hat{\mathbf{k}} \\ &= [\nabla \cdot (\mathbf{v} \times \hat{\mathbf{i}})]\hat{\mathbf{i}} + [\nabla \cdot (\mathbf{v} \times \hat{\mathbf{j}})]\hat{\mathbf{j}} + [\nabla \cdot (\mathbf{v} \times \hat{\mathbf{k}})]\hat{\mathbf{k}} \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{k}}.\end{aligned} \tag{9.18}$$

We can also express curl $\mathbf{v}$ in terms of ∇ as

$$\operatorname{curl} \mathbf{v} = \nabla \times \mathbf{v}, \tag{9.19}$$

since, from (8.34),

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right) \hat{\mathbf{i}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right) \hat{\mathbf{j}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right) \hat{\mathbf{k}},\end{aligned}$$

in agreement with the last line of (9.18). Or backing up to the next to last line of (9.18), we could have noted that

$$\begin{aligned}&[\nabla \cdot (\mathbf{v} \times \hat{\mathbf{i}})]\hat{\mathbf{i}} + [\nabla \cdot (\mathbf{v} \times \hat{\mathbf{j}})]\hat{\mathbf{j}} + [\nabla \cdot (\mathbf{v} \times \hat{\mathbf{k}})]\hat{\mathbf{k}} \\ &\qquad = [(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{i}}]\hat{\mathbf{i}} + [(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{j}}]\hat{\mathbf{j}} + [(\nabla \times \mathbf{v}) \cdot \hat{\mathbf{k}}]\hat{\mathbf{k}} = \nabla \times \mathbf{v}\end{aligned}$$

again.

Example 9.3. For instance, if $\mathbf{v} = y^2\hat{\mathbf{i}} + 3xy\hat{\mathbf{j}} - x^2z\hat{\mathbf{k}}$, then

$$\text{curl } \mathbf{v} = (0 - 0)\hat{\mathbf{i}} + (0 + 2xz)\hat{\mathbf{j}} + (3y - 2y)\hat{\mathbf{k}} = 2xz\hat{\mathbf{j}} + y\hat{\mathbf{k}}. \quad \blacksquare$$

To interpret curl $\mathbf{v}$ physically, consider the case of fluid flow, where $\mathbf{v}$ is chosen to be $\mathbf{q}$, the fluid velocity vector:

$$\mathbf{q} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}};$$

that is, u, v, w are the x, y, z velocity components. (With the letter v used for one of the velocity components, it would be confusing to use $\mathbf{v}$ for the velocity vector, and so we use $\mathbf{q}$ instead.) Let us examine curl $\mathbf{q}$, which is called the **vorticity** and denoted $\mathbf{\Omega}$. For simplicity, let $\mathbf{q}$ be a *plane flow*, that is, where $w = 0$ and u, v are functions of x and y but not z; thus $\mathbf{q} = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$. Then

$$\text{curl } \mathbf{q} = \mathbf{\Omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}. \tag{9.20}$$

Consider the motion of the little rectangular element of fluid shown in Fig. 9.8. At time t it is in position 1. If the element were *rigid*, its motion would consist of a translation plus a rotation. At $t + \Delta t$ it might be in configuration 2, say, due to a translation OO' plus a rotation $\Delta\alpha$, and its angular velocity (taken as positive counterclockwise) would be $\omega \sim (\Delta\alpha/\Delta t)\hat{\mathbf{k}}$.

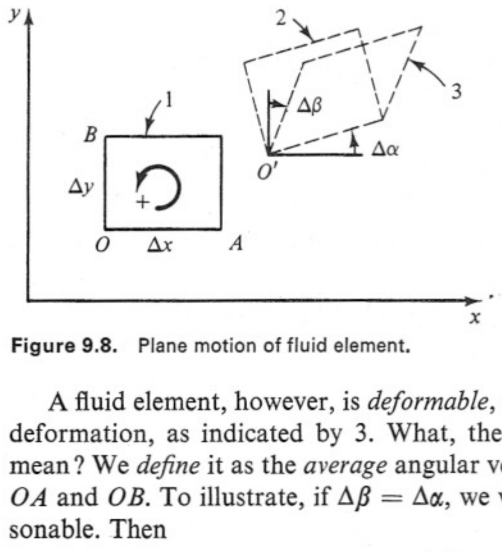

Figure 9.8. Plane motion of fluid element.

A fluid element, however, is *deformable*, not rigid, and it may also suffer a "shear" deformation, as indicated by 3. What, then, does the element's "angular velocity" mean? We *define* it as the *average* angular velocity of the initially perpendicular edges OA and OB. To illustrate, if $\Delta\beta = \Delta\alpha$, we would have $\boldsymbol{\omega} = \mathbf{0}$, as seems fair and reasonable. Then

$$\text{Angular velocity of } OA = \frac{v(A) - v(O)}{\Delta x}\hat{\mathbf{k}} \longrightarrow \frac{\partial v}{\partial x}\hat{\mathbf{k}} \quad \text{as } \Delta x \longrightarrow 0$$

$$\text{Angular velocity of } OB = \frac{u(O) - u(B)}{\Delta y}\hat{\mathbf{k}} \longrightarrow -\frac{\partial u}{\partial y}\hat{\mathbf{k}} \quad \text{as } \Delta y \longrightarrow 0$$

so that the

$$\text{Average of the two} \equiv \boldsymbol{\omega} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}. \tag{9.21}$$

Comparing (9.21) and (9.20), we see that $\mathbf{\Omega} = 2\boldsymbol{\omega}$; *the vorticity curl* $\mathbf{q}(P)$ *is twice the* *angular velocity of the fluid at P*. Although our discussion was for plane flows, it's not hard to show that the conclusion is the same for nonplanar flows as well.

The Laplace Operator. So far we've introduced the div, grad, and curl operators and found that all three can be expressed in terms of ∇. Specifically, div $= \nabla \cdot$, grad $= \nabla$, and curl $= \nabla\times$.

Finally, we also introduce the Laplacian operator "∇^2" as

$$\nabla^2 \equiv \nabla \cdot \nabla = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{9.22}$$

Read as *del square*, we emphasize that it is not really ∇ times ∇ but rather ∇ *dot* ∇, as given above for cartesian coordinates. It can act on either scalar or vector fields:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{9.23a}$$

$$\nabla^2 \mathbf{v} = \hat{\mathbf{i}}\,\nabla^2 v_x + \hat{\mathbf{j}}\,\nabla^2 v_y + \hat{\mathbf{k}}\,\nabla^2 v_z \tag{9.23b}$$

$$= \hat{\mathbf{i}}\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) + \hat{\mathbf{j}}(\text{etc.}) + \hat{\mathbf{k}}(\text{etc.}).$$

The importance of the div, grad, curl, and ∇^2 operators will become apparent shortly.

Some Important Vector Identities. The following properties and formulas will prove quite useful.

For α, β arbitrary scalars,

$$\text{grad}\,(\alpha u + \beta v) = \nabla(\alpha u + \beta v) = \alpha\,\nabla u + \beta\,\nabla v \tag{9.24}$$

$$\text{div}\,(\alpha\mathbf{u} + \beta\mathbf{v}) = \nabla \cdot (\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\,\nabla \cdot \mathbf{u} + \beta\,\nabla \cdot \mathbf{v} \tag{9.25}$$

$$\text{curl}\,(\alpha\mathbf{u} + \beta\mathbf{v}) = \nabla \times (\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha\,\nabla \times \mathbf{u} + \beta\,\nabla \times \mathbf{v}; \tag{9.26}$$

that is, grad, div, and curl are *linear* operators (Exercise 9.9). Furthermore,

$$\text{div curl}\,\mathbf{v} = \nabla \cdot \nabla \times \mathbf{v} = 0 \tag{9.27}$$

$$\text{curl grad}\,u = \nabla \times \nabla u = 0 \tag{9.28}$$

$$\text{div}\,(u\mathbf{v}) = \nabla \cdot (u\mathbf{v}) = \nabla u \cdot \mathbf{v} + u\,\nabla \cdot \mathbf{v} \tag{9.29}$$

$$\text{curl}\,(u\mathbf{v}) = \nabla \times (u\mathbf{v}) = u\,\nabla \times \mathbf{v} + \nabla u \times \mathbf{v} \tag{9.30}$$

$$\text{div}\,(\mathbf{u} \times \mathbf{v}) = \nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v} \tag{9.31}$$

$$\text{curl}\,(\mathbf{u} \times \mathbf{v}) = \nabla \times (\mathbf{u} \times \mathbf{v}) = \mathbf{u}\,\nabla \cdot \mathbf{v} - \mathbf{v}\,\nabla \cdot \mathbf{u} + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} \tag{9.32}$$

$$\text{curl}^2\,\mathbf{v} = \nabla \times (\nabla \times \mathbf{v}) = \nabla\,\text{div}\,\mathbf{v} - \nabla^2\mathbf{v}, \tag{9.33}$$

where $(\mathbf{u} \cdot \nabla)\mathbf{v}$ in (9.32) means

$$\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)(v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}})$$

$$= \left(u_x\frac{\partial v_x}{\partial x} + u_y\frac{\partial v_x}{\partial y} + u_z\frac{\partial v_x}{\partial z}\right)\hat{\mathbf{i}} + (\text{etc.})\hat{\mathbf{j}} + (\text{etc.})\hat{\mathbf{k}},$$

and similarly for $(\mathbf{v} \cdot \nabla)\mathbf{u}$.

Let us prove just one of these formulas, say (9.29), and leave the others for Exercise 9.10.

Example 9.4. To verify (9.29), we simply expand[5]

$$\begin{aligned}\nabla \cdot (u\mathbf{v}) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot (uv_x\hat{\mathbf{i}} + uv_y\hat{\mathbf{j}} + uv_z\hat{\mathbf{k}}) \\ &= \frac{\partial}{\partial x}(uv_x) + \frac{\partial}{\partial y}(uv_y) + \frac{\partial}{\partial z}(uv_z) \\ &= v_x\frac{\partial u}{\partial x} + v_y\frac{\partial u}{\partial y} + v_z\frac{\partial u}{\partial z} + u\frac{\partial v_x}{\partial x} + u\frac{\partial v_y}{\partial y} + u\frac{\partial v_z}{\partial z} \\ &= \nabla u \cdot \mathbf{v} + u\nabla \cdot \mathbf{v}. \quad \blacksquare\end{aligned}$$

Note carefully that all scalars, vectors, and operators in (9.25) through (9.32) are defined intrinsically—that is, without reference to any special coordinate system. Thus verification in any one coordinate system (e.g., cartesian) is equivalent to verification of *all* coordinate systems!

9.4. *THE DIVERGENCE THEOREM*

Consider a vector field $\mathbf{v}$ defined throughout some region V. Let us partition V into a large number of tiny subregions, say N of them, as suggested in Fig. 9.9(a), and define a point P_i in each of them. Then from definition (9.5) we have

$$\text{div}\,\mathbf{v}(P_i)(d\tau)_i \approx \int_{\sigma_i} \hat{\mathbf{n}}_i \cdot \mathbf{v}\, d\sigma \tag{9.34}$$

for each little subregion.[6] Letting $i = 1, 2, \ldots, N$, we have N equations. Adding them,

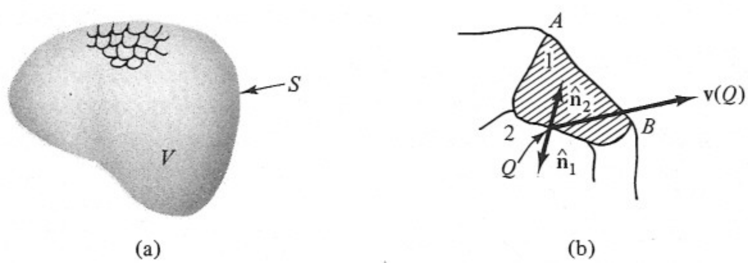

Figure 9.9. Gauss divergence theorem.

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[5]Actually, such manipulations are substantially more concise, using cartesian tensor notation. See, for example, Harold Jeffreys, *Cartesian Tensors*, Cambridge University Press, Cambridge, 1952.

[6]We've replaced the infinitesimal τ by $d\tau$. Incidentally, if you are worried about having an infinitesimal left-hand side of (9.34) and a finite integral on the right, note that σ_i is itself infinitesimal, and so the integral is, too.

the left-hand side is

$$\sum_{i=1}^{N} \operatorname{div} \mathbf{v}(P_i)(d\tau)_i,$$

which $\longrightarrow \int_V \operatorname{div} \mathbf{v}\, d\tau$ as the partition becomes infinitely fine.

The sum of the right-hand sides, however, is less obvious. The idea is that contributions from all abutting surfaces cancel. Consider, for instance, subregions 1 and 2 in Fig. 9.9(b), which have part of their boundaries in common. At any point Q of their interface note that their outward unit normals are negatives of each other, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$, whereas $\mathbf{v}(Q)$ is uniquely defined, so that $\hat{\mathbf{n}}_1 \cdot \mathbf{v}(Q)\, d\sigma = -\hat{\mathbf{n}}_2 \cdot \mathbf{v}(Q)\, d\sigma$. Thus we have cancellation from all abutting surfaces, and the only contributions that survive are those from the surface S of V (e.g., the portion AB of subregion 1), since they suffer no such cancellation. So

$$\sum_{i=1}^{n} \int_{\sigma_i} \hat{\mathbf{n}}_i \cdot \mathbf{v}\, d\sigma = \int_S \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma.$$

Equating these two results, we have

$$\int_V \nabla \cdot \mathbf{v}\, d\tau = \int_S \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma, \tag{9.35}$$

which is the **Gauss divergence theorem**. Of course, our derivation has been quite formal.[7]

Example 9.5. In order to illustrate the manipulations involved in (9.35), consider the case where V is the right circular cone shown in Fig. 9.10, and $\mathbf{v} = z^2\hat{\mathbf{k}}$. Then $\nabla \cdot \mathbf{v} = 2z$, and therefore the volume integral in (9.35) is

$$\int_V \nabla \cdot \mathbf{v}\, d\tau = \int_V 2z\, d\tau = \int_0^1 (2z)(\pi r^2\, dz)$$
$$= 2\pi \int_0^1 z(1-z)^2\, dz = \text{etc.} = \frac{\pi}{6}, \tag{9.36}$$

where r is the radius of a coin-shaped element of volume.

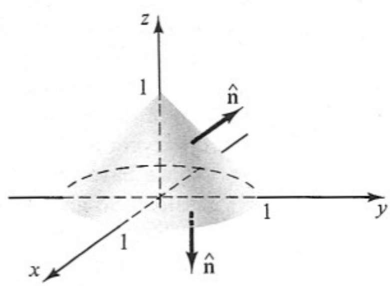


Figure 9.10. Application of divergence theorem.

In evaluating the surface integral, we first need expressions for $\hat{\mathbf{n}}$. On the bottom face ($x^2 + y^2 < 1$ and $z = 0$) $\hat{\mathbf{n}} = -\hat{\mathbf{k}}$, and so $\hat{\mathbf{n}} \cdot \mathbf{v} = -\hat{\mathbf{k}} \cdot (z^2\hat{\mathbf{k}}) = -z^2 = 0$ on $z = 0$ and there is no contribution from that face. Determining $\hat{\mathbf{n}}$ on the "side" of the cone is less obvious. Imagine a one-parameter family of surfaces, of which our cone is one member. If the equation of this family is

$$F(x, y, z; c) = 0,$$

where c_0 is the value of c corresponding to our particular cone, then

$$\hat{\mathbf{n}} = \pm \frac{\nabla F(x, y, z; c)}{\|\nabla F(x, y, z; c)\|}\bigg|_{c=c_0} = \pm \frac{\nabla F(x, y, z; c_0)}{\|\nabla F(x, y, z; c_0)\|}, \tag{9.37}$$

[7]For details, see, for example, O. D. Kellogg's *Foundations of Potential Theory*, Springer-Verlag, Berlin, 1929, p. 84.

where the sign is chosen so that $\hat{\mathbf{n}}$ is the *out*ward normal, as desired. In the present example,

$$F(x, y, z; c_0) = x^2 + y^2 - (1 - z)^2 = 0;$$

hence

$$\hat{\mathbf{n}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (1 - z)\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + (1 - z)^2}} = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (1 - z)\hat{\mathbf{k}}}{\sqrt{2}\,(1 - z)}, \tag{9.38}$$

where the last step follows from the fact that $x^2 + y^2 = (1 - z)^2$ on our particular cone. Then $\hat{\mathbf{n}} \cdot \mathbf{v} = z^2/\sqrt{2}$; therefore (see Fig. 9.11)

$$\int_S \hat{\mathbf{n}} \cdot \mathbf{v}\, d\sigma = \int_0^1 \frac{z^2}{\sqrt{2}} 2\pi(1 - z)(\sqrt{2}\, dz)$$

$$= \text{etc.} = \frac{\pi}{6},$$

in agreement with (9.36).

Actually, you may have noticed that we've made the problem too hard; that is, $\hat{\mathbf{n}} \cdot \mathbf{v} = z^2(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}) = z^2/\sqrt{2}$, since $\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = \text{constant} = 1/\sqrt{2}$ over the whole side of the cone. But this simplification is fortuitous, and we chose to follow a more general approach. ■

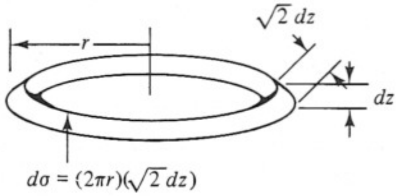


Figure 9.11. Area element $d\sigma$.

Example 9.6. *The Continuity Equation of Fluid Mechanics.* Let us *make up* a fluid flow. To illustrate, let the velocity field be $\mathbf{q} = -x\hat{\mathbf{i}}$ and the mass density $\rho = \text{constant} = \rho_0$. Is such a flow *possible*? Consider a control volume consisting of the slab $-1 < x < 1$. Fluid flows in through the left face ($x = -1$) at the rate 1 m/sec and in through the right face ($x = 1$) at the same rate. Since the amount of fluid in the control volume is constantly increasing, it follows from the principle of conservation of mass that the average density therein must be increasing with time. Yet we have prescribed that $\rho = \text{constant}$, which means that such a flow is not possible! So instead of $\mathbf{q}$ and ρ being independent, there is apparently some relationship between them that is implied by the fundamental requirement of conservation of mass. Let us find this relationship.

Consider a stationary control volume V of arbitrary shape and location within the flow. The amount of mass inside V at any time t is

$$M = \int_V \rho\, d\tau, \tag{9.39}$$

where $\rho = \rho(x, y, z, t)$ is the fluid mass density. Thus the rate of increase of M is (Exercise 9.13)

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \rho(x, y, z, t)\, d\tau = \int_V \frac{\partial \rho}{\partial t}\, d\tau. \tag{9.40}$$

But if mass is to be conserved, it must be true that dM/dt also equals the rate at which mass enters V through its surface S; that is,

$$\frac{dM}{dt} = -\int_S \rho\mathbf{q} \cdot \hat{\mathbf{n}}\, d\sigma, \tag{9.41}$$

where the minus sign is needed, since $\hat{\mathbf{n}}$ is the *out*ward normal, and so $\rho\mathbf{q} \cdot \hat{\mathbf{n}}\, d\sigma$ is *out*flow, whereas we want the *in*flow. Consequently, we must have

$$\int_V \frac{\partial \rho}{\partial t}\, d\tau = -\int_S \rho \mathbf{q} \cdot \hat{\mathbf{n}}\, d\sigma. \tag{9.42}$$

In order to combine the two integrals, let us convert the surface integral to a volume integral by means of the divergence theorem:

$$\int_S \rho \mathbf{q} \cdot \hat{\mathbf{n}}\, d\sigma = \int_V \nabla \cdot (\rho \mathbf{q})\, d\tau;$$

then (9.42) becomes.

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q})\right] d\tau = 0. \tag{9.43}$$

The only way that this result can be true for *all* choices of V within the flow (recall that V is arbitrary) is if the integrand is identically zero throughout the flow—for all t, since (9.43) holds at every instant. Suppose the integrand were *positive*, say, at some point P and thus (by continuity) throughout some sufficiently small sphere N centered at P. Then choosing V to be N, the integral would be positive (i.e., nonzero), contradicting (9.43). Therefore

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) = 0 \tag{9.44}$$

throughout the flow and for all time t; (9.44) is a partial differential equation and is called the **continuity equation** of fluid mechanics.

In particular, if $\rho =$ constant (i.e., the fluid is **incompressible**[8]), it reduces to

$$\nabla \cdot \mathbf{q} = 0, \tag{9.45}$$

or in terms of a cartesian reference frame

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{9.46}$$

where u, v, w are the x, y, z velocity components.

COMMENT. Here we have elected to have V stationary, in which case dM/dt equals the rate of mass flux across the surface S of V. Alternatively, suppose that we define a control volume V at some instant t_0 and then require that it continue to enclose the same set of fluid particles for all $t > t_0$. In other words, V is to "drift" with the fluid. In this case, the statement of conservation of mass is simply

$$\frac{dM}{dt} = 0. \tag{9.47}$$

But this time the Leibnitz-type differentiation of

$$M = \int_{V(t)} \rho(x, y, z, t)\, d\tau$$

[8]Just as the *Reynolds number* turns out to be a measure of the effects of *viscosity*, the nondimensional parameter that provides a measure of the effects of *compressibility* is the so-called *mach number* M, defined as the ratio of the fluid velocity to the local speed of sound. If $M \ll 1$, then we can, to good accuracy, make the simplifying assumption that $\rho =$ constant. For example, the speed of sound in air, at 20°C and sea level, is 344 m/sec, so that if $|\mathbf{q}| < 30$ m/sec, say, the flow may be considered as incompressible. Do you think we could take $\rho =$ constant in the theory of water waves on the ocean?

is more difficult because the region V varies with t. Proceeding formally,[9] along the same lines as in our derivation of the Leibnitz rule (1.41), we have

$$\begin{aligned}\frac{M(t+\Delta t)-M(t)}{\Delta t} &= \frac{1}{\Delta t}\left[\int_{V(t+\Delta t)} \rho(P, t+\Delta t)\, d\tau - \int_{V(t)} \rho(P, t)\, d\tau\right] \\ &= \int_{V(t)} \frac{\rho(P, t+\Delta t)-\rho(P, t)}{\Delta t}\, d\tau \\ &\quad + \frac{1}{\Delta t}\left\{\int_{V(t+\Delta t)} \rho(P, t+\Delta t)\, d\tau - \int_{V(t)} \rho(P, t+\Delta t)\, d\tau\right\},\end{aligned}$$

where P is short for x, y, z. From the sketch in Fig. 9.12 we see that the last term, in braces, is

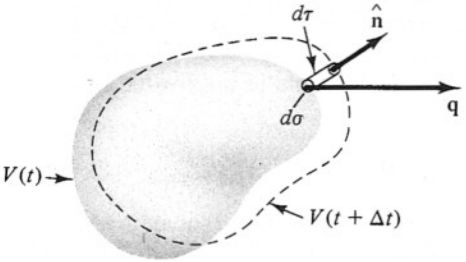


Figure 9.12. Leibnitz differentiation of $M(t)$.

$$\frac{1}{\Delta t}\{\quad\} \sim \frac{1}{\Delta t}\int_{S(t)} \rho(P, t+\Delta t)[\mathbf{q}(P, t)\,\Delta t] \cdot \hat{\mathbf{n}}\, d\sigma,$$

where $S(t)$ is the surface of $V(t)$. Letting $\Delta t \longrightarrow 0$,

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t}\, d\tau + \int_S \rho \mathbf{q} \cdot \hat{\mathbf{n}}\, d\sigma = 0,$$

which coincides with our previous result (9.42), and we end up with the same continuity equation (9.44) (as we had better!), independent of whether our derivation considers V to be fixed or drifting with the fluid. ∎

Example 9.7. *Unsteady Heat Conduction.* Consider the unsteady conduction of heat within some region R. Our derivation of a differential equation governing the temperature field $u(x, y, z, t)$ will closely parallel our derivation of the continuity equation in the previous example.

First, we choose a fixed arbitrary control volume V within R. Instead of a "mass balance," this time we carry out a heat balance; that is, we equate the rate at which heat enters V through its surface S to the rate at which it is stored.

The **Fourier law of heat conduction** states that the heat flux q (calories per unit time, say) across a plane face is proportional to its area and the temperature gradient $\partial u/\partial n$ (i.e., $\nabla u \cdot \hat{\mathbf{n}}$) normal to the face, with a constant of proportionality k, called the *thermal conductivity* of the material.

Thus the flux through S is

$$q = \int_S k\frac{\partial u}{\partial n}\, d\sigma, \tag{9.48}$$

where $\hat{\mathbf{n}}$ is the outward normal on S. Note that the sign is correct because a positive $\partial u/\partial n$ causes a flux *into* V, as assumed.

Alternatively, since the amount of heat (in calories, say) in a mass m at tempera-

[9]For a rigorous derivation, see W. Kaplan, *Advanced Calculus*, Addison-Wesley, Reading, Mass., 1953, pp. 223, 299, and 300; also see the detailed discussion by H. Flanders, Differentiation under the Integral Sign, *American Mathematical Monthly*, Vol. 80, June–July, 1973, pp. 615–627.

ture u is mcu, where c is the *specific heat* of the material, we can express

$$q = \frac{d}{dt} \int_V cu\rho \, d\tau, \tag{9.49}$$

where ρ is the mass density.

Assuming, for simplicity, that the material is homogeneous so that k, c, ρ are all constant, we therefore have

$$\int_V \frac{\partial u}{\partial t} d\tau = \alpha^2 \int_S \frac{\partial u}{\partial n} d\sigma \qquad \left(\alpha^2 \equiv \frac{k}{c\rho}\right)$$

$$= \alpha^2 \int_S \nabla u \cdot \hat{\mathbf{n}} \, d\sigma = \alpha^2 \int_V \nabla \cdot \nabla u \, d\tau.$$

Thus

$$\int_V \left(\frac{\partial u}{\partial t} - \alpha^2 \nabla^2 u\right) d\tau = 0; \tag{9.50}$$

and since V is arbitrary, we must have

$$\alpha^2 \nabla^2 u = \frac{\partial u}{\partial t} \tag{9.51}$$

throughout the region R and for all t.

The differential equation (9.51) is called the equation of heat conduction or sometimes simply the **heat equation**, and α is the *coefficient of diffusivity*; (9.51) is also known as the **diffusion equation** because it applies not only to heat conduction but also to diffusion processes in general, such as material diffusion due to variation in concentration and the diffusion of vorticity in viscous flows.

Of special importance is the case of *steady-state* heat conduction—that is, where $\partial u/\partial t = 0$—since then (9.51) becomes the **Laplace equation**

$$\nabla^2 u = 0. \tag{9.52}$$

The importance of the diffusion equation, Laplace's equation, and the wave equation (not yet discussed) cannot be overstated. Most of our discussion of these equations is contained in Part V.

COMMENT. We have assumed that there is no generation or disappearance of heat in the material. Such is often the case, but not always. For instance, a chemical reaction might be taking place throughout the material so that heat is generated at the rate $Q(x, y, z, t)$ calories per unit volume per unit time ($Q > 0$ if the reaction is exothermic; $Q < 0$ if it is endothermic), or we might have a mechanical specimen in fatigue being heated by internal hysteresis losses. If this effect is included, it is not hard to see that the result is $c\rho u_t = k \nabla^2 u + Q$ in place of (9.51). ▮

Example 9.8. *Green's Identities.* If u and v are scalar fields, then $u \nabla v$ is a vector field. Applying the divergence theorem to the field $u \nabla v$ gives

$$\int_V \nabla \cdot (u \nabla v) \, d\tau = \int_S \hat{\mathbf{n}} \cdot (u \nabla v) \, d\sigma.$$

But [recall (9.29)] $\nabla \cdot (u \nabla v) = \nabla u \cdot \nabla v + u \nabla^2 v$ and $\hat{\mathbf{n}} \cdot (u \nabla v) = u\hat{\mathbf{n}} \cdot \nabla v = u \, \partial v/\partial n$, and therefore

$$\int_V (\nabla u \cdot \nabla v + u \nabla^2 v) \, d\tau = \int_S u \frac{\partial v}{\partial n} d\sigma. \tag{9.53}$$

This useful formula is known as **Green's first identity**. Interchanging u and v,

$$\int_V (\nabla v \cdot \nabla u + v\,\nabla^2 u)\, d\tau = \int_S v \frac{\partial u}{\partial n}\, d\sigma,$$

and expunging this equation from (9.53), we arrive at **Green's second identity**,

$$\int_V (u\,\nabla^2 v - v\,\nabla^2 u)\, d\tau = \int_S \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) d\sigma. \tag{9.54}$$

Recall that we required the three components of $\mathbf{v}$ in the divergence theorem to have continuous first-order partials with respect to x, y, and z. In deriving Green's identities, we set "$\mathbf{v}$" $= u\,\nabla v = uv_x\hat{\mathbf{i}} + uv_y\hat{\mathbf{j}} + uv_z\hat{\mathbf{k}}$ and then "$\mathbf{v}$" $= v\,\nabla u = vu_x\hat{\mathbf{i}} + vu_y\hat{\mathbf{j}} + vu_z\hat{\mathbf{k}}$. Thus we require $uv_x, uv_y, uv_z, vu_x, vu_y, vu_z$ to have continuous first-order partials with respect to x, y, z. It follows that we require the continuity of u, v and all their first- and second-order partials throughout V in Green's identities (9.53) and (9.54). ▮

9.5. STOKES' THEOREM

Consider a caplike surface S bounded by a simple, closed, oriented curve C (Fig. 9.13) and with a unit normal $\hat{\mathbf{n}}$ defined at each point at which S is smooth. We will not say that $\hat{\mathbf{n}}$ is to be directed "outward" or "inward," since S is only a cap, not the surface of a closed volume, and such terms are not meaningful. By *convention*, we will take the direction of the unit normals $\hat{\mathbf{n}}$ to be dictated by the "right-hand rule"; that is, curling the fingers of our right hand around $\hat{\mathbf{n}}$ in the direction indicated by the orientation of C, our thumb points in the direction of $\hat{\mathbf{n}}$.

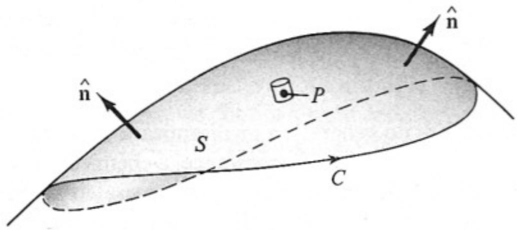


Figure 9.13. Caplike surface S.

At any point P on S let us construct a tiny cylindrical house (not necessarily of circular cross section), sitting on S as shown in Fig. 9.13. If S is immersed in some vector field $\mathbf{v}$, then from the definition (9.17) of the curl

$$\text{curl}\,\mathbf{v}(P) \approx \frac{1}{\tau} \int_\sigma \hat{\mathbf{N}} \times \mathbf{v}\, d\sigma, \tag{9.55}$$

with equality holding in the limit as the volume τ of the house tends to zero. $\hat{\mathbf{N}}$ is the outward unit normal over the surface σ of τ and should not be confused with $\hat{\mathbf{n}}$, which is the normal to S; in fact, $\hat{\mathbf{N}} = \hat{\mathbf{n}}$ on the top of σ, $\hat{\mathbf{N}} = -\hat{\mathbf{n}}$ on the bottom, and $\hat{\mathbf{N}} = \hat{\mathbf{e}}_n$, say, on the side, as shown in the blowup of the house in Fig. 9.14. Dotting $\hat{\mathbf{n}}$ into (9.55),

$$\hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v} \approx \frac{1}{\tau}\left\{\int_{\text{top}} \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \mathbf{v})\, d\sigma + \int_{\text{bottom}} \hat{\mathbf{n}} \cdot (-\hat{\mathbf{n}} \times \mathbf{v})\, d\sigma + \int_{\text{side}} \hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_n \times \mathbf{v})\, d\sigma\right\}.$$

But $\hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \mathbf{v}) = \hat{\mathbf{n}} \cdot (-\hat{\mathbf{n}} \times \mathbf{v}) = 0$, since $\hat{\mathbf{n}} \times \mathbf{v}$ is orthogonal to both $\hat{\mathbf{n}}$ and $\mathbf{v}$, and [recalling (8.11)] $\hat{\mathbf{n}} \cdot (\hat{\mathbf{e}}_n \times \mathbf{v}) = (\hat{\mathbf{n}} \times \hat{\mathbf{e}}_n) \cdot \mathbf{v} = \hat{\mathbf{e}}_t \cdot \mathbf{v}$; therefore

$$\hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v} \approx \frac{1}{\tau} \int_{\text{side}} \hat{\mathbf{e}}_t \cdot \mathbf{v}\, d\sigma.$$

Expressing $d\sigma = h\, ds$ and $\tau = h\sigma_B$, where σ_B is the (cross-hatched) base area of the house,

$$\hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v} \approx \frac{1}{h\sigma_B} \int_c \hat{\mathbf{e}}_t \cdot \mathbf{v}\, h\, ds = \frac{1}{\sigma_B} \int_c \mathbf{v} \cdot d\mathbf{r}, \tag{9.56}$$

where the tiny closed curve c is the edge of σ_B, oriented in accordance with $\hat{\mathbf{n}}(P)$ and the right-hand rule, and $d\mathbf{r}$ has entered through the combination $ds\hat{\mathbf{e}}_t$, as shown in Fig. 9.14.[10]

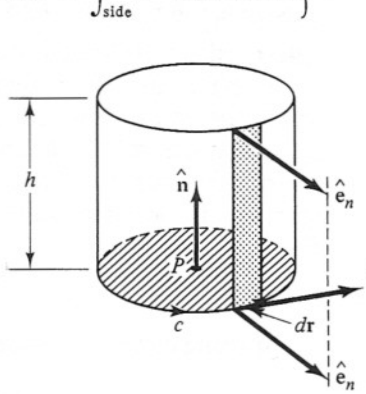


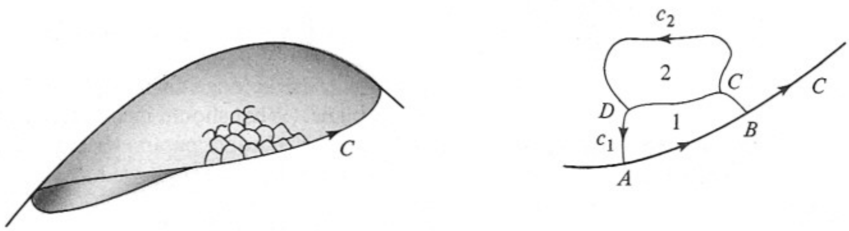
Figure 9.14. Blowup of the house at P.

Actually, σ_B constitutes an element of area "$d\sigma$" of S, and so (9.56) yields

$$\hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v}\, d\sigma \approx \int_c \mathbf{v} \cdot d\mathbf{r} \tag{9.57}$$

with equality holding in the limit as $d\sigma \longrightarrow 0$. Thus if we partition S into a large number of tiny subregions, say N of them, as suggested in Fig. 9.15(a), and define a point P_i in each, then from (9.57)

$$\hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v}(P_i)(d\sigma)_i \approx \int_{c_i} \mathbf{v} \cdot d\mathbf{r} \quad \text{for } i = 1, 2, \ldots, N. \tag{9.58}$$



(a) (b)

Figure 9.15. Tortoise-shell partition of S.

[10]Note that our derivation is rather formal. For instance, taking $d\sigma$ to be the shaded sliver $h\, ds$ really assumes that the integrand $\hat{\mathbf{e}}_t \cdot \mathbf{v}$ is constant over the sliver. Well, $\hat{\mathbf{e}}_t$ is, but $\mathbf{v}$ varies throughout space and thus over the height of the sliver. But we are letting h and σ_B tend to zero. Letting $h \longrightarrow 0$ first, the variation in $\mathbf{v}$ over the height of the sliver has no effect.

Adding these N equations, the left-hand side is

$$\sum_{i=1}^{N} \hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v}(P_i)(d\sigma)_i,$$

which $\longrightarrow \int_S \hat{\mathbf{n}} \cdot \operatorname{curl} \mathbf{v}\, d\sigma$ as the partition becomes infinitely fine. The sum of the right-hand sides is less obvious, however. Consider, for instance, subregions 1 and 2 in Fig. 9.15(b), which have the portion DC of their boundaries in common. At any point on the segment DC note that the $d\mathbf{r}$'s for c_1 and c_2 are oppositely oriented, whereas $\mathbf{v}$ is uniquely defined. So the contribution from the DC portion of c_2 exactly cancels the contribution from the CD portion of c_1. Similarly, we have cancellation from *all* the c_i's except for segments along the boundary curve C, such as AB, which are not shared. Therefore

$$\sum_{i=1}^{N} \int_{c_i} \mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot d\mathbf{r}.$$

Equating these two results, we have the well-known and important **Stokes theorem,**

$$\int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, d\sigma = \int_C \mathbf{v} \cdot d\mathbf{r}. \tag{9.59}$$

Notice how our derivation parallels that of the divergence theorem, starting with equation (9.57).

Example 9.9. Evaluate

$$I = \int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, d\sigma,$$

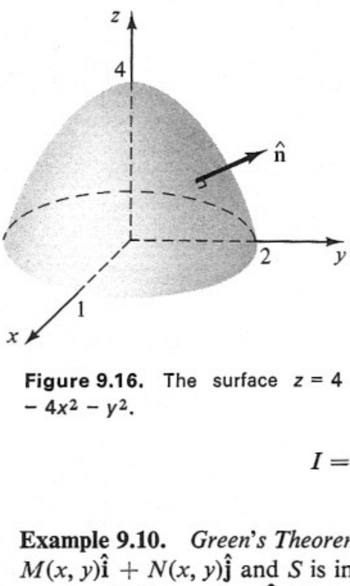


Figure 9.16. The surface $z = 4 - 4x^2 - y^2$.

where $\mathbf{v} = x^3\hat{\mathbf{j}} - (z+1)\hat{\mathbf{k}}$, S is the surface $z = 4 - 4x^2 - y^2$ between $z = 0$ and $z = 4$, and $\hat{\mathbf{n}}$ is directed as shown in Fig. 9.16.

Solution. Rather than evaluate I directly, it is easier to apply Stokes' theorem and deal with the line integral

$$I = \int_C \mathbf{v} \cdot d\mathbf{r} = \int_C [x^3\, dy - (z+1)\, dz],$$

where C is the edge curve $4x^2 + y^2 = 4$ in the x, y plane. (What should its orientation be?) Then $dz = 0$ on C, and parameterizing C according to $x = \cos t$, $y = 2\sin t$, we have

$$I = 2\int_0^{2\pi} \cos^4 t\, dt = \frac{3\pi}{2}. \quad ∎$$

Example 9.10. *Green's Theorem.* Consider the two-dimensional case, where $\mathbf{v} = M(x, y)\hat{\mathbf{i}} + N(x, y)\hat{\mathbf{j}}$ and S is in the x, y plane. Let C be oriented *counterclockwise*, say, in which case $\hat{n} = \hat{\mathbf{k}}$. Then Stokes' theorem (9.59) becomes

$$\int_S \hat{\mathbf{k}} \cdot \left[0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{\mathbf{k}}\right] d\sigma = \int_C (M\hat{\mathbf{i}} + N\hat{\mathbf{j}}) \cdot (dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}})$$

or

$$\int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\sigma = \int_C (M\, dx + N\, dy), \tag{9.60}$$

which is known as **Green's theorem** and should not be confused with "Green's identities," (9.53) and (9.54). Recall, from the first sentence of this section, that C is to be a *simple closed curve* or, equivalently, that S is to be a *simply connected region.*

COMMENT. Actually, it is also possible to bypass Stokes' theorem, in deriving Green's theorem, and to proceed instead along elementary lines. For instance, consider the N terms in (9.60). Referring to Fig. 9.17, we see that

$$\begin{aligned}\int_S \frac{\partial N}{\partial x}\,d\sigma &= \int_{y_1}^{y_2}\int_{x_L(y)}^{x_R(y)} \frac{\partial N}{\partial x}\,dx\,dy\\ &= \int_{y_1}^{y_2} N(x_R, y)\,dy - \int_{y_1}^{y_2} N(x_L, y)\,dy\\ &= \int_{C_R} N(x, y)\,dy + \int_{C_L} N(x, y)\,dy = \int_C N(x, y)\,dy,\end{aligned}$$

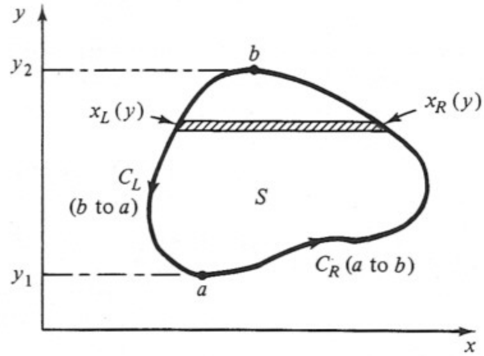


Figure 9.17. Green's theorem.

where C_R and C_L are the right and left halves of C, as shown in the figure.[11] We leave it for the reader to show, in similar fashion, that (Exercise 9.24)

$$\int_S \frac{\partial M}{\partial y}\,d\sigma = -\int_C M\,dx, \tag{9.61}$$

completing our alternative derivation of (9.60). ∎

Example 9.11. *A Multiply Connected Region.* Consider

$$I = \int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\sigma, \tag{9.62}$$

where S is the annulus $1 < r < 2$ in the x, y plane, $N = x/(x^2 + y^2)$, and $M = -y/(x^2 + y^2)$. Differentiating, we find that $\partial N/\partial x - \partial M/\partial y = 0$ in S, so that $I = 0$ and we're done. But the reason that we pose this problem is to discuss the conversion of (9.62) to a line integral by means of Green's theorem (9.60). The catch is that the boundary C in Green's theorem is to be a simple closed curve, whereas that's not the case here, since S is multiply connected.

[11]Note that we've tacitly assumed that the region is *convex* in the x direction; that is, each cross-hatched "sliver" running from $x_L(y)$ to $x_R(y)$ lies entirely within S. Extension to the nonconvex case is not hard, and the final result (9.60) is the same.

To fix things, let us "cut" the annulus S, say along the x axis from $x = 1$ to $x = 2$. The cut annulus, S' say, is simply connected and *is* bounded by a simple closed curve C', as shown in Fig. 9.18(a). The two straight portions of C' are shown as separated for clarity; actually, they abut each other on $y = 0$.

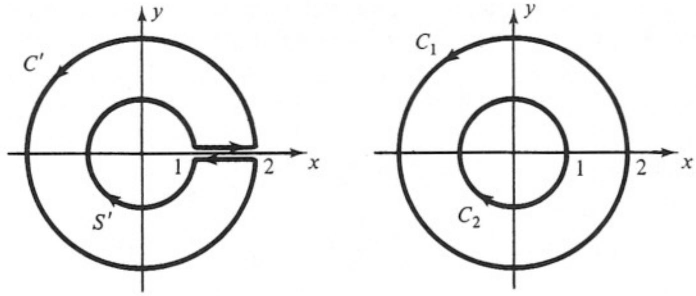


Figure 9.18. Extension of Green's theorem to a multiply connected region.

Surely

$$\int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\sigma = \int_{S'} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\sigma,$$

since S and S' differ only by an infinitely thin cut. *Now* applying Green's theorem,

$$I = \int_{S'} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) d\sigma = \int_{C'} (M\,dx + N\,dy)$$

$$= \int_{C_1} (M\,dx + N\,dy) + \int_{C_2} (M\,dx + N\,dy),$$

since the two straight segments cancel. We leave it for the reader to show that the two integrals on the right are 2π and -2π, respectively, so that $I = 2\pi - 2\pi = 0$ as before. ∎

9.6. IRROTATIONAL AND SOLENOIDAL FIELDS

Irrotational Fields. If $\nabla \times \mathbf{v} = 0$ throughout R, then we say that $\mathbf{v}$ is ***irrotational*** in R.[12]

THEOREM 9.1. $\nabla \times \mathbf{v} = 0$ *throughout a simply connected region R if and only if there exists a scalar field u(P) such that* $\mathbf{v} = \nabla u$ *in R.* (Note that u is to have continuous *second* partials, $u_{xx}, u_{yy}, u_{zz}, u_{xy}, u_{xz}$, and u_{yz}, since the components of $\mathbf{v}$ are to have continuous first partials and $\mathbf{v} = \nabla u$.)

[12] Recall (Section 9.3) that for all scalar fields $u(P)$ and vector fields $\mathbf{v}(P)$ we assume continuity of the first partials throughout R. And from section 9.2 R is understood always to be bounded and connected; our surfaces are to be piecewise smooth and orientable, and our curves are to be piecewise smooth. These conditions will be assumed, without reiteration, throughout this chapter. Only when *more* is required, will these conditions be stated—for example, in Theorem 9.1, where R is required to be *simply* connected.

Proof. First, assume that there *is* a u such that $\mathbf{v} = \nabla u$. Then $\nabla \times \mathbf{v} = \nabla \times \nabla u = (u_{zy} - u_{yz})\hat{\mathbf{i}} + (u_{xz} - u_{zx})\hat{\mathbf{j}} + (u_{yx} - u_{xy})\hat{\mathbf{k}} = 0$. Next, we must show that $\nabla \times \mathbf{v} = 0$ implies the existence of a u such that $\mathbf{v} = \nabla u$.

Consider $\int \mathbf{v} \cdot d\mathbf{r}$ from any fixed point x_0, y_0, z_0 to a field point x, y, z along any two paths C_1 and C_2 that lie within R (Fig. 9.19). Let us denote by $C = C_1 + (-C_2)$ the closed path from P_0 to P along C_1 and then back to P_0 along $-C_2$. Next, let us stretch a caplike surface S across from C_1 to C_2. Then the edge of S is C. (Note that if R were not simply connected, S might need to have holes in it, so that we could not say that the edge of S is C.) Applying Stokes' theorem,

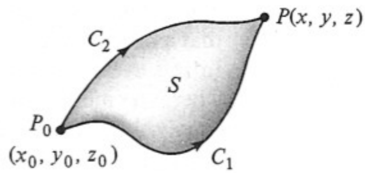


Figure 9.19. Getting ready to apply Stokes' theorem.

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_S \hat{\mathbf{n}} \cdot \nabla \times \mathbf{v}\, d\sigma = 0,$$

since $\nabla \times \mathbf{v}$ is zero in R, and hence on S. But

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_{C_1} \mathbf{v} \cdot d\mathbf{r} + \int_{-C_2} \mathbf{v} \cdot d\mathbf{r} = \int_{C_1} \mathbf{v} \cdot d\mathbf{r} - \int_{C_2} \mathbf{v} \cdot d\mathbf{r},$$

and therefore

$$\int_{C_1} \mathbf{v} \cdot d\mathbf{r} = \int_{C_2} \mathbf{v} \cdot d\mathbf{r}; \tag{9.63}$$

that is, the integral is *independent of the path* and depends only on the endpoints P_0 and P. Thus we can express[13]

$$\int_{x_0, y_0, z_0}^{x, y, z} \mathbf{v} \cdot d\mathbf{r} \equiv f(x, y, z), \tag{9.64}$$

where f is uniquely determined by the point x, y, z[14]; if the integral *did* depend on path, then it would not define a single-valued function of x, y, z. Then if we hold y and z fixed and express (see Fig. 9.20)

$$f(x, y, z) = \int_{x_0, y_0, z_0}^{x_1, y, z} (v_x\, dx + v_y\, dy + v_z\, dz) + \int_{x_1, y, z}^{x, y, z} (v_x\, dx + v_y\, dy + v_z\, dz), \tag{9.65}$$

we see that the first integral is constant and that the second reduces to

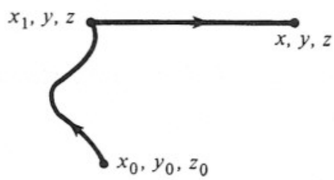


Figure 9.20. Getting ready for the Fundamental Theorem of the Integral Calculus.

$$\int_{x_1}^{x} v_x(x, y, z)\, dx \quad \text{or} \quad \int_{x_1}^{x} v_x(\xi, y, z)\, d\xi$$

and so, by the Fundamental Theorem of the Integral Calculus (1.19), we have $\partial f/\partial x = v_x(x, y, z)$. Similarly, we have $\partial f/\partial y = v_y(x, y, z)$ and $\partial f/\partial z = v_z(x, y, z)$; consequently, $\mathbf{v} = \nabla f$, and f is the scalar field u that we've been looking for. This completes the proof.

---

[13]When a line integral is independent of path, we will denote this fact by dropping the subscripted C and indicating the endpoints as upper and lower limits.

[14]We will regard P_0 as fixed and P as variable, so that f is a function of the "active" variables x, y, z.

Example 9.12. *Conservative Force Fields.* A force field is said to be **conservative** if the work

$$W = \int_{P_0}^{P} \mathbf{F} \cdot d\mathbf{r},$$

is independent of the path for any points P_0 and P in the region. We've just seen that a necessary and sufficient condition for $\mathbf{F}$ to be conservative is that $\nabla \times \mathbf{F} = 0$. In this event, we can associate with $\mathbf{F}$ a *scalar* field, ϕ say, such that $\mathbf{F} = -\nabla\phi$, where the minus sign is inserted for convenience; ϕ is called the scalar potential or simply the **potential** of $\mathbf{F}$.

As a simple illustration, consider the uniform gravitational field

$$\mathbf{F} = -g\hat{\mathbf{k}} \tag{9.66}$$

per unit mass, where g is the acceleration of gravity. Clearly, $\nabla \times \mathbf{F} = 0$, since $\mathbf{F}$ is a constant. To find ϕ, note from $\mathbf{F} = -g\hat{\mathbf{k}} = -\nabla\phi$ that

$$\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial \phi}{\partial y} = 0, \qquad \frac{\partial \phi}{\partial z} = g,$$

and so

$$\phi = gz + C, \tag{9.67}$$

where C is an arbitrary constant of integration; (9.67) is the familiar "gravitational potential energy" from elementary physics. There is *always* an arbitrary additive constant in ϕ that has no physical significance because it drops out when we compute the "physical" quantity $\mathbf{F} = -\nabla\phi$; we can take C to be zero or any other convenient value.

As a final example, let us reconsider the two-dimensional case discussed in Example 9.1. Noting that $\nabla \times \mathbf{F}$ happens to be zero, we have $\mathbf{F} = -\nabla\phi$; as a result,

$$\begin{aligned} W &= \int \mathbf{F} \cdot d\mathbf{r} = \int (F_x\, dx + F_y\, dy) \\ &= -\int \left(\frac{\partial \phi}{\partial x}\, dx + \frac{\partial \phi}{\partial y}\, dy\right) = -\int d\phi = -\phi(x, y)\bigg|_{0,0}^{0,5\pi}. \end{aligned}$$

To find ϕ, let us integrate $\mathbf{F} = -\nabla\phi$—that is, $2xy\hat{\mathbf{i}} + (x^2 - 1)\hat{\mathbf{j}} = -\dfrac{\partial \phi}{\partial x}\hat{\mathbf{i}} - \dfrac{\partial \phi}{\partial y}\hat{\mathbf{j}}$—or

$$\frac{\partial \phi}{\partial x} = -2xy \tag{9.68a}$$

$$\frac{\partial \phi}{\partial y} = 1 - x^2. \tag{9.68b}$$

Integrating (9.68a),[15]

$$\phi = \int (-2xy)\, \partial x = -x^2 y + A(y), \tag{9.69}$$

and inserting this expression in (9.68b) gives

$$-x^2 + A'(y) = 1 - x^2;$$

[15] By $\int f(x, y)\, \partial x$ we mean the integral of f with respect to x *with y held fixed.* So in order to be as general as possible, the constant of integration A must be permitted to depend on y, since y has been regarded as constant. If unhappy about $A(y)$ work backward: taking $\partial/\partial x$ of (9.69), we have $\partial\phi/\partial x = -2xy$, in agreement with (9.68a).

hence $A'(y) = 1$ and $A = y + C$. Thus $\phi = -x^2y + y + C$ and

$$W = -(-x^2y + y + C)\Big|_{0,0}^{0,5\pi} = -5\pi$$

in agreement with our result in Example 9.1.

Alternatively, recall from our proof of Theorem 9.1 that $\nabla \times \mathbf{F} = 0$ implies that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C and depends only on the endpoints (which is consistent with the fact that $\int_C \mathbf{F} \cdot d\mathbf{r} = -\phi\Big|_{\text{initial point}}^{\text{final point}}$). Thus we can *simplify* C from the given spiral to a straight-line path $x = 0$, from $y = 0$ to $y = 5\pi$, so that

$$W = \int \mathbf{F} \cdot d\mathbf{r} = \int [2xy\,dx + (x^2 - 1)\,dy] = -\int_0^{5\pi} dy = -5\pi$$

again.

Of these two approaches, the first has the advantage that it yields ϕ, and once we have ϕ, we are able to compute W between *any* two points P_0 and P simply as $-\phi(x, y)\Big|_{P_0}^{P}$ with no further calculation. ∎

Solenoidal Fields. If $\nabla \cdot \mathbf{v} = 0$ throughout R, we say that $\mathbf{v}$ is ***solenoidal*** in R.

THEOREM 9.2. $\nabla \cdot \mathbf{v} = 0$ *throughout a simply connected region R if and only if there exists a vector field* $\mathbf{w}$ *such that* $\mathbf{v} = \nabla \times \mathbf{w}$ *in* R. (Note that $\mathbf{w}$ is to have continuous *second* partials in R, since $\mathbf{v} = \nabla \times \mathbf{w}$, and $\mathbf{v}$ is to have continuous first partials.)

Proof. First, suppose that there *is* a $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}$. Then

$$\begin{aligned}\nabla \cdot \mathbf{v} &= \frac{\partial}{\partial x}\left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)\\ &= \text{etc.} = 0.\end{aligned}$$

It remains to prove the converse—namely, that if $\mathbf{v}$ is given such that $\nabla \cdot \mathbf{v} = 0$, then there is a solution $\mathbf{w}$ of $\nabla \times \mathbf{w} = \mathbf{v}$. But let us skip the proof, which is actually a generalization of the method outlined in the following example.

Example 9.13. If $\mathbf{v} = (xy - 1)\hat{\mathbf{i}} - xz\hat{\mathbf{j}} + (2 - yz)\hat{\mathbf{k}}$, then $\nabla \cdot \mathbf{v} = y - y = 0$, and there exists a $\mathbf{w}$ such that $\mathbf{v} = \nabla \times \mathbf{w}$. Equating components,

$$\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z} = xy - 1 \tag{9.70a}$$

$$\frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x} = -xz \tag{9.70b}$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = 2 - yz. \tag{9.70c}$$

To find $\mathbf{w}$, we are faced with the three coupled, first-order, partial differential equations (9.70) in the unknowns w_x, w_y, and w_z. As a start, let us (tentatively) try setting

$w_x \equiv 0$. Then from (9.70b)

$$\frac{\partial w_z}{\partial x} = xz$$

$$w_z = \int xz\, \partial x = \frac{x^2 z}{2} + f(y, z) \tag{9.71}$$

and from (9.70c)

$$\frac{\partial w_y}{\partial x} = 2 - yz$$

$$w_y = \int (2 - yz)\, \partial x = 2x - xyz + g(y, z). \tag{9.72}$$

Inserting (9.71) and (9.72) into (9.70a), we need merely find an f and a g such that

$$\frac{\partial f}{\partial y} + xy - \frac{\partial g}{\partial z} = xy - 1.$$

Doing so is easy; for instance, we can set $g \equiv 0$ and $f = -y$; then

$$\mathbf{w} = 0\hat{\mathbf{i}} + (2x - xyz)\hat{\mathbf{j}} + \left(\frac{x^2 z}{2} - y\right)\hat{\mathbf{k}} \equiv \mathbf{w}_0.$$

In fact, $\mathbf{w}_0$ is not the *only* possible $\mathbf{w}$. We can add to it any gradient, ∇u, since the curl of a gradient is zero (per Theorem 9.1); that is,

$$\nabla \times \mathbf{w} = \nabla \times (\mathbf{w}_0 + \nabla u) = \nabla \times \mathbf{w}_0 + \nabla \times \nabla u = \mathbf{v} + \mathbf{0} = \mathbf{v}.$$

For example, with $u = yz^3$ we have $\mathbf{w} = \mathbf{w}_0 + \nabla u = \mathbf{w}_0 + z^3\hat{\mathbf{j}} + 3yz^2\hat{\mathbf{k}}$.

COMMENT 1. A nice way to remember the important formulas $\nabla \times \nabla u = 0$ for all u and $\nabla \cdot \nabla \times \mathbf{v} = 0$ for all $\mathbf{v}$ contained in Theorems 9.1 and 9.2 is to think of the letters d, g, and c (for divergence, gradient, and curl). The only combinations that come to mind (for me at least) are **cg** (for center of gravity) and **dc** (for direct current); that is, the **c**url of a **g**radient is zero and the **d**ivergence of a **c**url is zero.

COMMENT 2. Observe that the *scalar* potential ϕ of an irrotational field is unique to within an additive arbitrary *constant*, whereas the *vector* potential $\mathbf{w}$ of a solenoidal field is unique to within an additive arbitrary *gradient*.

COMMENT 3. It is interesting to look at some of the formulas we've developed as generalizations of the Fundamental Theorem of the Integral Calculus, which states that if $f(x)$ is continuous over $a \leq x \leq b$, we can express

$$\int_a^x f(\xi)\, d\xi = F(x) - F(a) \tag{9.73}$$

for $a \leq x \leq b$, where $F'(x) = f(x)$. [This statement is equivalent to our earlier statement (1.19); "$F(x)$" in (1.19) is our $F(x) - F(a)$ in (9.73).] To illustrate, in the course of proving Theorem 9.1, we found that

$$\int_{a,b,c}^{x,y,z} [v_x(\xi, \eta, \zeta)\, d\xi + v_y(\xi, \eta, \zeta)\, d\eta + v_z(\xi, \eta, \zeta)\, d\zeta]$$

$$= F(x, y, z) - F(a, b, c), \tag{9.74}$$

where $\nabla F = \mathbf{v}$; that is $\partial F/\partial x = v_x$, $\partial F/\partial y = v_y$, and $\partial F/\partial z = v_z$. This is the generalization of (9.73) to line integrals. But whereas we merely assumed f to be continuous in (9.73), here we require v_x, v_y, and v_z to be not only continuous but also related so that $\nabla \times \mathbf{v} = 0$; that is,

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0, \qquad \frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0, \qquad \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} = 0.$$

Just as (9.74) relates a *line* integral to certain *end* values, Stokes' theorem relates a *surface* integral to an integral around the *edge* of the surface, and the divergence theorem relates a *volume* integral to an integral over the *surface* of the volume.[16] ▮

9.7. NONCARTESIAN SYSTEMS

Having introduced the gradient, divergence, and curl operators through intrinsic definitions, we've worked them out thus far only with respect to cartesian coordinates. Many situations, however, are more naturally referred to curvilinear systems, and it is therefore important to obtain more general representations of these operators.

In order to derive such expressions, we can start with definitions (9.5), (9.12), (9.17) and introduce a curvilinear u, v, w system, as defined in Section 8.5. The derivation would follow essentially the same lines as our derivation of (9.9), for example, so let us simply state the final results.

First, note (as in Sections 8.4 and 8.5) that the "line element" $d\mathbf{s}$ is

$$\begin{aligned} d\mathbf{s} &= dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}} \\ &= (x_u\,du + x_v\,dv + x_w\,dw)\hat{\mathbf{i}} + (y_u\,du + y_v\,dv + y_w\,dw)\hat{\mathbf{j}} \\ &\quad + (z_u\,du + z_v\,dv + z_w\,dw)\hat{\mathbf{k}} \\ &= (x_u\hat{\mathbf{i}} + y_u\hat{\mathbf{j}} + z_u\hat{\mathbf{k}})\,du + (x_v\hat{\mathbf{i}} + y_v\hat{\mathbf{j}} + z_v\hat{\mathbf{k}})\,dv \\ &\quad + (x_w\hat{\mathbf{i}} + y_w\hat{\mathbf{j}} + z_w\hat{\mathbf{k}})\,dw \\ &\equiv \mathbf{U}\,du + \mathbf{V}\,dv + \mathbf{W}\,dw. \end{aligned} \tag{9.75}$$

Limiting ourselves to the case of *orthogonal* curvilinear coordinates—that is, where $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ are mutually orthogonal at each point in the space—it follows that

$$\begin{aligned} (ds)^2 = d\mathbf{s}\cdot d\mathbf{s} &= (\mathbf{U}\cdot\mathbf{U})(du)^2 + (\mathbf{V}\cdot\mathbf{V})(dv)^2 + (\mathbf{W}\cdot\mathbf{W})(dw)^2 \\ &\equiv h_1^2(du)^2 + h_2^2(dv)^2 + h_3^2(dw)^2, \end{aligned} \tag{9.76}$$

where[17]

$$\begin{aligned} h_1 &= \sqrt{\mathbf{U}\cdot\mathbf{U}} = \sqrt{x_u^2 + y_u^2 + z_u^2} \\ h_2 &= \sqrt{\mathbf{V}\cdot\mathbf{V}} = \sqrt{x_v^2 + y_v^2 + z_v^2} \\ h_3 &= \sqrt{\mathbf{W}\cdot\mathbf{W}} = \sqrt{x_w^2 + y_w^2 + z_w^2}. \end{aligned} \tag{9.77}$$

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[16] For a generalized development, see, for example, R. C. Buck, *Advanced Calculus*, 2nd ed., McGraw-Hill, New York, 1965, Section 7.4.

[17] Many authors use the symbols g_{11}, g_{22}, g_{33} in place of our h_1^2, h_2^2, h_3^2; they are called **metric coefficients**.

Then we state without derivation

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u}\hat{\mathbf{i}}_1 + \frac{1}{h_2}\frac{\partial f}{\partial v}\hat{\mathbf{i}}_2 + \frac{1}{h_3}\frac{\partial f}{\partial w}\hat{\mathbf{i}}_3 \tag{9.78}$$

$$\nabla \cdot \mathbf{Q} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}(h_2 h_3 Q_1) + \frac{\partial}{\partial v}(h_3 h_1 Q_2) + \frac{\partial}{\partial w}(h_1 h_2 Q_3)\right] \tag{9.79}$$

$$\begin{aligned}\nabla \times \mathbf{Q} &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{i}}_1 & h_2\hat{\mathbf{i}}_2 & h_3\hat{\mathbf{i}}_3 \\ \partial/\partial u & \partial/\partial v & \partial/\partial w \\ h_1 Q_1 & h_2 Q_2 & h_3 Q_3 \end{vmatrix} \\ &= \frac{1}{h_2 h_3}\left[\frac{\partial(h_3 Q_3)}{\partial v} - \frac{\partial(h_2 Q_2)}{\partial w}\right]\hat{\mathbf{i}}_1 + \frac{1}{h_1 h_3}\left[\frac{\partial(h_1 Q_1)}{\partial w} - \frac{\partial(h_3 Q_3)}{\partial u}\right]\hat{\mathbf{i}}_2 \\ &\quad + \frac{1}{h_1 h_2}\left[\frac{\partial(h_2 Q_2)}{\partial u} - \frac{\partial(h_1 Q_1)}{\partial v}\right]\hat{\mathbf{i}}_3,\end{aligned} \tag{9.80}$$

where $\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3$ are the normalized $\mathbf{U}, \mathbf{V}, \mathbf{W}$ base vectors:

$$\begin{aligned}\hat{\mathbf{i}}_1 &= \frac{\mathbf{U}}{\|\mathbf{U}\|} = \frac{\mathbf{U}}{\sqrt{\mathbf{U}\cdot\mathbf{U}}} = \frac{\mathbf{U}}{h_1} \\ \hat{\mathbf{i}}_2 &= \text{etc.} = \frac{\mathbf{V}}{h_2} \\ \hat{\mathbf{i}}_3 &= \text{etc.} = \frac{\mathbf{W}}{h_3}.\end{aligned} \tag{9.81}$$

Combining (9.79) and (9.78), we have for the important *Laplacian operator* (Exercise 9.41)

$$\nabla^2 f = \nabla \cdot \nabla f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial w}\right)\right]. \tag{9.82}$$

Naturally, for the simple cartesian case—that is, where $u = x$, $v = y$, and $w = z$—these expressions should reduce to our previous results. In this case, $h_1 = \sqrt{x_x^2 + y_x^2 + z_x^2} = \sqrt{1 + 0 + 0} = 1$, $h_2 = \text{etc.} = 1$, and $h_3 = \text{etc.} = 1$, so that (9.82), for example, becomes

$$\nabla^2 f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} + \frac{\partial^2 f}{\partial w^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

as before.

For **cylindrical coordinates**, with $u = r, v = \theta, w = z$ say,[18] $\hat{\mathbf{i}}_1 = \hat{\mathbf{e}}_r$, $\hat{\mathbf{i}}_2 = \hat{\mathbf{e}}_\theta$, $\hat{\mathbf{i}}_3 = \hat{\mathbf{e}}_z$, and according to (9.77),

$$\begin{aligned}h_1 &= \sqrt{x_r^2 + y_r^2 + z_r^2} = \sqrt{\cos^2\theta + \sin^2\theta + 0} = 1 \\ h_2 &= \sqrt{x_\theta^2 + y_\theta^2 + z_\theta^2} = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta + 0} = r \\ h_3 &= \sqrt{x_z^2 + y_z^2 + z_z^2} = \sqrt{0 + 0 + 1} = 1.\end{aligned}$$

[18] We could just as well have taken $u = z$, $v = r$, and $w = \theta$, for example; it doesn't matter.

It follows that

$$\nabla f = \frac{\partial f}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{\partial f}{\partial z}\hat{\mathbf{e}}_z, \tag{9.83}$$

$$\nabla \cdot \mathbf{Q} = \frac{1}{r}\frac{\partial}{\partial r}(rQ_r) + \frac{1}{r}\frac{\partial}{\partial \theta}Q_\theta + \frac{\partial}{\partial z}Q_z, \tag{9.84}$$

$$\nabla \times \mathbf{Q} = \left[\frac{1}{r}\frac{\partial Q_z}{\partial \theta} - \frac{\partial Q_\theta}{\partial z}\right]\hat{\mathbf{e}}_r + \left[\frac{\partial Q_r}{\partial z} - \frac{\partial Q_z}{\partial r}\right]\hat{\mathbf{e}}_\theta + \frac{1}{r}\left[\frac{\partial (rQ_\theta)}{\partial r} - \frac{\partial Q_r}{\partial \theta}\right]\hat{\mathbf{e}}_z, \tag{9.85}$$

and

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}, \tag{9.86}$$

where $\mathbf{Q} = Q_r\hat{\mathbf{e}}_r + Q_\theta\hat{\mathbf{e}}_\theta + Q_z\hat{\mathbf{e}}_z$.

For **spherical polars** ρ, θ, ϕ (defined in Exercise 7.11) we leave it for you to show (Exercise 9.38) that

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\hat{\mathbf{e}}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial f}{\partial \phi}\hat{\mathbf{e}}_\phi, \tag{9.87}$$

$$\nabla \cdot \mathbf{Q} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 Q_\rho) + \frac{1}{\rho\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, Q_\theta) + \frac{1}{\rho\sin\theta}\frac{\partial Q_\phi}{\partial \phi}, \tag{9.88}$$

$$\begin{aligned}\nabla \times \mathbf{Q} = {} & \frac{1}{\rho\sin\theta}\left[\frac{\partial Q_\theta}{\partial \phi} - \frac{\partial}{\partial \theta}(\sin\theta\, Q_\phi)\right]\hat{\mathbf{e}}_\rho \\ & + \frac{1}{\rho}\left[\frac{\partial(\rho Q_\phi)}{\partial \rho} - \frac{1}{\sin\theta}\frac{\partial Q_\rho}{\partial \phi}\right]\hat{\mathbf{e}}_\theta + \frac{1}{\rho}\left[\frac{\partial Q_\rho}{\partial \theta} - \frac{\partial(\rho Q_\theta)}{\partial \rho}\right]\hat{\mathbf{e}}_\phi,\end{aligned} \tag{9.89}$$

and

$$\nabla^2 f = \frac{1}{\rho^2}\left[\frac{\partial}{\partial \rho}\left(\rho^2\frac{\partial f}{\partial \rho}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}\right], \tag{9.90}$$

where $\mathbf{Q} = Q_\rho\hat{\mathbf{e}}_\rho + Q_\theta\hat{\mathbf{e}}_\theta + Q_\phi\hat{\mathbf{e}}_\phi$.

Of all the formulas (9.83) through (9.90), two that should definitely be memorized are (9.83) and (9.86).

9.8. FLUID MECHANICS; IRROTATIONAL FLOW

Some flows are irrotational—that is the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{q}$ is identically zero—and some are not; here we consider only the class of flows that are. To understand which flows are irrotational and which are not, we state the following crucial result, which goes back to Helmholtz and Lord Kelvin (Sir William Thomson).

If the fluid is barotropic and inviscid, then the vorticity vector $\mathbf{\Omega}$ *of any fluid particle* (which we saw in Section 9.3 is actually twice the particle's angular velocity) *is constant in time*. By *barotropic* we mean that the fluid mass density ρ depends on the pressure p only, that is, $\rho = \rho(p)$. For instance, if the fluid is a perfect gas (i.e., $p = \rho RT$, where R is a constant and T is the temperature) and heat is transferred so that the flow is isothermal, then RT is a constant and ρ is proportional to p. Or if the fluid can be considered as incompressible (recall footnote 8), then, of course, $\rho =$ constant. And by *inviscid* we mean that the fluid has no viscosity.

It follows from the Kelvin–Helmholtz statement that if, for such a fluid, a particle once has zero vorticity, then it will *always* have zero vorticity. Physically, this situation makes sense, since the way to impart some spin (and hence vorticity) to the particle is by applying a torque through *shear stresses* at its surface; but an inviscid fluid, by definition, cannot support any shear stresses.

If, in fact, $\mathbf{\Omega} = 0$ throughout the flow, it follows from Theorem 9.1 (since $\nabla \times \mathbf{q} = 0$) that there exists a fluid **velocity potential** ϕ such that[19]

$$\mathbf{q} = \nabla\phi. \tag{9.91}$$

If, *in addition*, the fluid is incompressible, the continuity equation (9.45) holds:

$$\nabla \cdot \mathbf{q} = 0. \tag{9.92}$$

Inserting $\nabla\phi$ for $\mathbf{q}$, (9.92) becomes $\nabla \cdot \nabla\phi = 0$ or

$$\nabla^2\phi = 0, \tag{9.93}$$

which is the well-known Laplace equation. This partial differential equation and its solutions constitute the rich and well-developed subject of **potential theory**. Most of our discussion of potential theory is contained in Part V on partial differential equations.

Example 9.14. To illustrate, consider the steady two-dimensional flow of an inviscid, incompressible fluid over a rigid, stationary, circular cylinder of radius a, as sketched in Fig. 9.21. Except for the presence of the cylinder, the flow would simply be the uniform "free stream" $\mathbf{q} = U\hat{\mathbf{i}}$; the deviation from this free stream, due to the cylinder, is called the "disturbance" velocity field.

Now as $x \longrightarrow -\infty$, we have $\mathbf{q} \sim U\hat{\mathbf{i}}$. (In fact, $\mathbf{q} \sim U\hat{\mathbf{i}}$ as $r \longrightarrow \infty$ in *any* direction.) Moreover, since $\nabla \times U\hat{\mathbf{i}} = 0$, we conclude that $\nabla \times \mathbf{q} = 0$ initially (i.e., far upstream) and hence $\nabla \times \mathbf{q} = 0$ *throughout* the flow. Thus $\mathbf{q} = \nabla\phi$, and putting this expression into $\nabla \cdot \mathbf{q} = 0$ yields the governing partial differential equation

$$\nabla^2\phi = 0. \tag{9.94}$$

In addition, we have *boundary conditions* at infinity and on the cylinder.

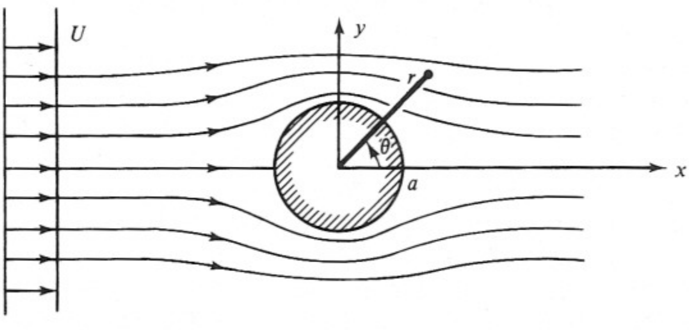

Figure 9.21. Potential flow over a cylinder.

[19]Some authors use $-\nabla\phi$ instead of $\nabla\phi$; it doesn't matter as long as we are consistent.

At infinity, $\mathbf{q} = \nabla\phi \sim U\hat{\mathbf{i}}$; that is, $\phi_x \sim U$ and $\phi_y \longrightarrow 0$. Integrating, $\phi \sim Ux$ (plus an arbitrary constant, which we might as well set equal to zero).

On the cylinder $r = a$, we must have the normal velocity equal to zero, since the cylinder is rigid and stationary; that is, $\mathbf{q} \cdot \hat{\mathbf{e}}_r = \nabla\phi \cdot \hat{\mathbf{e}}_r = \partial\phi/\partial r = 0$ on $r = a$.

The complete boundary value problem is therefore as follows.

Partial differential equation:

$$\nabla^2\phi = 0 \qquad \text{in } r > a$$

Boundary conditions:

$$\begin{aligned} \phi &\sim Ux \quad \text{as } r \longrightarrow \infty \\ \frac{\partial\phi}{\partial r} &= 0 \qquad \text{on } r = a. \end{aligned} \tag{9.95}$$

We should decide whether we want to use cartesian or polar coordinates. The free stream $\mathbf{q} \sim U\hat{\mathbf{i}}$ seems to suggest a cartesian system, whereas the circular shape of the cylinder strongly suggests polar coordinates. We will find later on that it is best to use the r, θ variables shown in the figure. Then (9.95) becomes

$$\begin{aligned} \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} &= 0 \quad \text{in } r > a \\ \phi &\sim Ur\cos\theta \quad \text{as } r \longrightarrow \infty \\ \frac{\partial\phi}{\partial r} &= 0 \quad \text{on } r = a. \end{aligned} \tag{9.96}$$

The *solution* of (9.96) is the subject of Exercise 26.39 in Part V. Here we are concerned only with the problem formulation.

COMMENT 1. Sometimes people prefer to split $\mathbf{q}$ into two parts, the free stream plus the disturbance velocity $\mathbf{q}'$, say: $\mathbf{q} = U\hat{\mathbf{i}} + \mathbf{q}'$. Then $\mathbf{q}' = \mathbf{q} - U\hat{\mathbf{i}} = \nabla\phi - \nabla(Ux) = \nabla(\phi - Ux) \equiv \nabla\phi'$, where ϕ' is the "disturbance potential." To formulate the problem in terms of ϕ', take the divergence of the equation $\mathbf{q} = U\hat{\mathbf{i}} + \mathbf{q}'$: $\nabla \cdot \mathbf{q} = \nabla \cdot (U\hat{\mathbf{i}}) + \nabla \cdot \mathbf{q}'$ or $0 = 0 + \nabla \cdot \mathbf{q}'$. Thus $\nabla \cdot \nabla\phi' = 0$ or

$$\nabla^2\phi' = \frac{\partial^2\phi'}{\partial r^2} + \frac{1}{r}\frac{\partial\phi'}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi'}{\partial\theta^2} = 0, \tag{9.97a}$$

that is, ϕ' satisfies the same (Laplace) equation as ϕ. We leave it for the reader to show (Exercise 9.38) that the associated boundary conditions are

$$\phi' \longrightarrow 0 \quad \text{as } r \longrightarrow \infty \tag{9.97b}$$

$$\frac{\partial\phi'}{\partial r} = -Ua\cos\theta \quad \text{on } r = a. \tag{9.97c}$$

The boundary value problem (9.97) is actually no easier to solve than (9.96); the choice is primarily a matter of personal preference.

COMMENT 2. Observe that in this problem we had both $\nabla \times \mathbf{q} = 0$ and $\nabla \cdot \mathbf{q} = 0$. We used the first to imply the existence of a scalar potential ϕ, so that $\mathbf{q} = \nabla\phi$, and the second to yield the differential equation (9.94) on ϕ. Alternatively, we could have reversed the procedure and used the continuity equation $\nabla \cdot \mathbf{q} = 0$ to imply the existence of a *vector potential* $\mathbf{A}$ (from Theorem 9.2), so that

$$\mathbf{q} = \nabla \times \mathbf{A}. \tag{9.98}$$

Inserting (9.98) into $\nabla \times \mathbf{q} = 0$ then yields the governing partial differential equation

$$\nabla \times \nabla \times \mathbf{A} = 0 \tag{9.99}$$

on **A**. Nevertheless, it is standard to use the scalar potential approach instead. Why? Well, it's certainly preferable to deal with the single unknown ϕ than a vector **A** with two or three components. Furthermore, whereas the Laplace equation on ϕ is well known and susceptible to standard methods of solution, equation (9.99) is less tractable (Exercise 9.40). ∎

Example 9.15. Next, consider the plane axisymmetric, irrotational, incompressible flow in the annulus $a < r < b$ between stationary, rigid, concentric, circular cylinders, defined by the potential $\phi = \kappa\theta$, where κ is any given constant. First checking to see that such a flow is *possible*, note that $\kappa\theta$ does satisfy the Laplace equation $\nabla^2\phi = 0$. Also, the velocity

$$\mathbf{q} = \nabla\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta = 0\hat{\mathbf{e}}_r + \frac{\kappa}{r}\hat{\mathbf{e}}_\theta \tag{9.100}$$

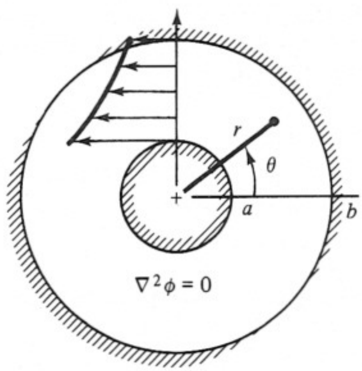

Figure 9.22. Axisymmetric potential flow between cylinders.

does satisfy the boundary conditions $\mathbf{q} \cdot \hat{\mathbf{e}}_r = 0$ at $r = a$ and $r = b$. In fact, the radial velocity component $\mathbf{q} \cdot \hat{\mathbf{e}}_r$ is zero for *all* $a < r < b$; the velocity is purely tangential and independent of θ (hence the adjective "axisymmetric"). The variation in the tangential component $\mathbf{q} \cdot \hat{\mathbf{e}}_\theta = \kappa/r$ is sketched in Fig. 9.22. (Anyone who has studied fluid mechanics probably recognizes this flow as the flow induced by a fictitious "vortex" of strength $\Gamma = 2\pi\kappa$ at $r = 0$.)

Let us calculate the kinetic energy of the fluid per unit depth (i.e., perpendicular to the paper). Surely,

$$\begin{aligned}\mathrm{KE} &= \int \frac{1}{2}q^2\, dm = \frac{\rho}{2}\int_V (\mathbf{q}\cdot\mathbf{q})\, d\tau \\ &= \frac{\rho}{2}\int_0^1\int_0^{2\pi}\int_a^b \frac{\kappa^2}{r^2}\, r\, dr\, d\theta\, dz = \pi\kappa^2\rho \ln\frac{b}{a}.\end{aligned} \tag{9.101}$$

But suppose instead that we convert to a surface integral by Green's first identity (9.53) with "u" = "v" = ϕ:

$$\mathrm{KE} = \frac{\rho}{2}\int_V \nabla\phi\cdot\nabla\phi\, d\tau = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n}\, d\sigma - \frac{\rho}{2}\int_V \phi\,\nabla^2\phi\, d\tau = \frac{\rho}{2}\int_S \phi\frac{\partial\phi}{\partial n}\, d\sigma, \tag{9.102}$$

since $\nabla^2\phi = 0$ throughout V. Now V is the "washer"-shaped region $a \le r \le b$, $0 \le z \le 1$, and so its surface S has four parts: an outer edge $r = b$, an inner edge $r = a$, a front face $z = 1$, and a back face $z = 0$. On each of these four parts $\partial\phi/\partial n = \nabla\phi \cdot \hat{\mathbf{n}} = \mathbf{q} \cdot \hat{\mathbf{n}}$ is zero. (On $z = 0$ and $z = 1$ it is zero because there is no velocity in the z direction.) Thus the right side of (9.102) is zero, which disagrees with (9.101) and which is obviously incorrect. What's wrong? The difficulty is that ϕ and its first-and second-order partials are not continuous in V as required in Green's identity (9.53). That is, if we consider the θ domain to be $0 \leq \theta < 2\pi$, say, then ϕ is not even *continuous* in V, since $\phi \longrightarrow 0$ as we approach the line $\theta = 0$ from above and $\phi \longrightarrow 2\pi\kappa$ as we approach it from below!

We overcome this difficulty by redefining our region. Specifically, we take a hacksaw (with an infinitely thin blade) and cut the washer as in Fig. 9.23. Inside the *cut* washer ϕ *is* continuous, as are its partials of all orders, which means that (9.102) should now be correct. But note carefully that S now has two additional faces, the top of the cut ($\theta = 0$, say) and the bottom of the cut ($\theta = 2\pi$). The top part gives no contribution because $\phi = \kappa\theta = 0$ there. On the bottom, however,

$$\phi = \kappa\theta = 2\pi\kappa,$$

and
$$\frac{\partial \phi}{\partial n} = \nabla\phi \cdot \hat{\mathbf{n}} = \mathbf{q} \cdot \hat{\mathbf{e}}_\theta = \frac{\kappa}{r}$$

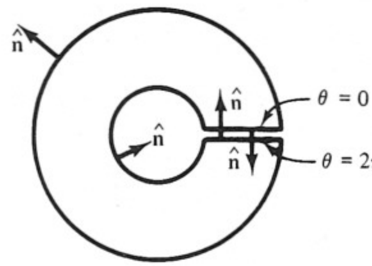

Figure 9.23. The cut washer.

from (9.100). So from the six parts of S we obtain

$$\text{KE} = 0 + 0 + 0 + 0 + 0 + \frac{\rho}{2}\int_0^1 \int_a^b (2\pi\kappa)\left(\frac{\kappa}{r}\right) dr\, dz = \pi\rho\kappa^2 \ln\frac{b}{a},$$

which is now in agreement with (9.101).

Note that the same result is obtained if we decide instead to let $\theta = 6\pi$, for example, on the top of the cut; then it is 8π on the bottom (Exercise 9.40).

COMMENT. Recalling that the vorticity $\mathbf{\Omega} = \nabla \times \mathbf{q}$ is twice the angular velocity of the fluid particles, the reader may well wonder how the flow around an annulus can possibly be irrotational, since it certainly *looks* as though the fluid particles must have some rotation. Of course, we're certain that it *is* irrotational because $\mathbf{q} = (\kappa/r)\hat{\mathbf{e}}_\theta$ is derived from the scalar potential $\phi = \kappa\theta$ and so, by Theorem 9.1, $\nabla \times \mathbf{q}$ must be zero. Or we can work out $\nabla \times [(\kappa/r)\hat{\mathbf{e}}_\theta]$ directly from (9.85) if you like, and again we obtain zero.

The trick is that the fluid particles simply move about like seats on a ferris wheel as shown in Fig. 9.24(a)—that is, without rotating! (Does this situation seem to fit together with the way that the tangential velocity varies with r?)

In contrast, consider the flow $\mathbf{q} = \omega r\hat{\mathbf{e}}_\theta$, where ω is any given constant. Using

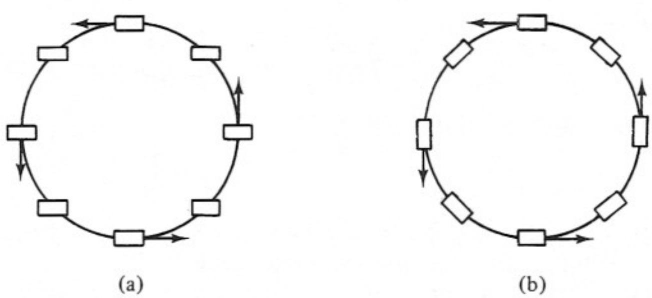

Figure 9.24. Irrotational and rotational flow in an annulus.

(9.85), we find that $\mathbf{\Omega} = \nabla \times \mathbf{q} = 2\omega\hat{\mathbf{e}}_z \neq 0$; therefore the flow is *not* irrotational; it is rotational. In fact, since $\mathbf{\Omega}$ is a constant vector, it follows that *all* fluid particles have the same angular velocity—namely, $\omega\hat{\mathbf{e}}_z$ (i.e., half of $\mathbf{\Omega}$)—so that $\mathbf{q} = \omega r\hat{\mathbf{e}}_\theta$ happens to correspond to a state of "solid-body rotation" as shown in Fig. 9.24(b). ▮

9.9. *THE GRAVITATIONAL POTENTIAL*

According to *Newton's law of gravitation*, the attractive force F exerted by a point mass m on another mass at a distance ρ from m, per unit mass at that location, is proportional to m and the inverse square of ρ (Fig. 9.25):

$$\mathbf{F} = -\gamma \frac{m}{\rho^2}\hat{\mathbf{e}}_\rho, \tag{9.103}$$

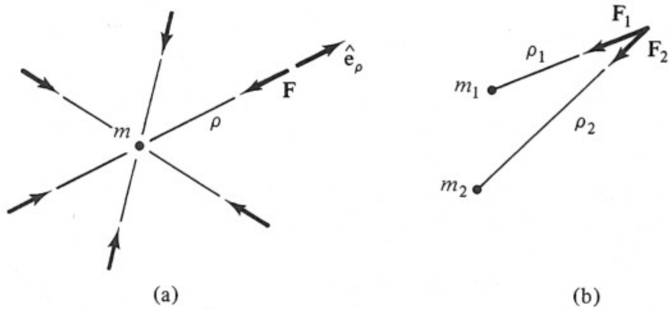

Figure 9.25. Gravitational force field.

where γ is a suitable constant of proportionality that can (and will) be set equal to unity by an appropriate choice of the units of mass. Note that we've adopted a spherical polar coordinate system, rather than cylindrical say, in order to take advantage of the spherical symmetry; that is, $F_\theta = F_\phi = 0$, and $F_\rho = -m/\rho^2$ is a function of ρ only.

From (9.89) we see that $\nabla \times \mathbf{F} = 0$; thus $\mathbf{F}$ is conservative and can be expressed as $\nabla\Phi$, where Φ is the so-called gravitational or Newtonian potential. To determine Φ, we equate the components of $\mathbf{F}$ to the corresponding components of $\nabla\Phi$. Recalling (9.87),

$$-\frac{m}{\rho^2} = \frac{\partial \Phi}{\partial \rho}, \qquad 0 = \frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}, \qquad 0 = \frac{1}{\rho \sin\theta}\frac{\partial \Phi}{\partial \phi},$$

from which it follows that

$$\Phi = \frac{m}{\rho} \tag{9.104}$$

(plus an arbitrary constant, which we will choose to be zero).

If there are *two* point masses, m_1 and m_2 [Fig. 9.25(b)], the force at any point P is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \nabla\Phi_1 + \nabla\Phi_2 = \nabla(\Phi_1 + \Phi_2) = \nabla\Phi$, where

$$\Phi = \Phi_1 + \Phi_2 = \frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}$$

is the total potential at P. (Note how vector addition of the $\mathbf{F}_j$'s is equivalent to the scalar addition of their potentials.) Generalizing to a *distribution* of mass, of density $\sigma(P)$, we can say that

$$\Phi = \int_M \frac{dm}{\rho} = \int_V \frac{\sigma\, d\tau}{\rho}, \tag{9.105}$$

where M is the total mass and V is the region that it occupies. Or in terms of some cartesian reference frame (Fig. 9.26)

$$\Phi(x, y, z) = \int_V \frac{\sigma(x', y', z')\, dx'\, dy'\, dz'}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}. \tag{9.106}$$

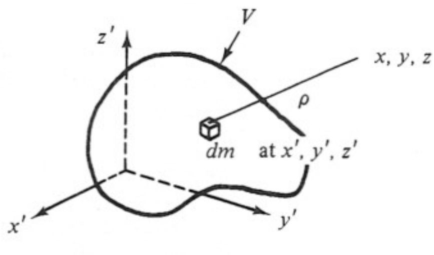

Figure 9.26. Distribution of mass.

Note carefully that although the "field point" x, y, z is arbitrary, it is regarded as *fixed* in the calculation (9.106).

Example 9.16. *A Hollow Sphere.* Suppose that V is a spherical shell with inner radius a, outer radius b, its center at $x' = y' = z' = 0$, and constant density σ. Because of the spherical symmetry, we can consider our field point to be on the z axis, say, with no loss of generality; in other words, set $x = y = 0$ in (9.106). Switching to the more convenient spherical polars,

$$x' = \rho \sin\theta \cos\phi, \qquad y' = \rho \sin\theta \sin\phi, \qquad z' = \rho\cos\theta,$$

and $$dx'\, dy'\, dz' \longrightarrow |J|\, d\rho\, d\theta\, d\phi = |\rho^2 \sin\theta|\, d\rho\, d\theta\, d\phi.$$

Therefore

$$\begin{aligned}\Phi(0, 0, z) &= \sigma \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{\rho^2 \sin\theta\, d\rho\, d\theta\, d\phi}{\sqrt{\rho^2 + z^2 - 2\rho z \cos\theta}} \\ &= 2\pi\sigma \int_a^b \rho^2\, d\rho \int_0^{\pi} \frac{\sin\theta\, d\theta}{\sqrt{\rho^2 + z^2 - 2\rho z\cos\theta}}.\end{aligned}$$

Letting the terms inside the radical be u, say,

$$\Phi(0, 0, z) = \frac{\pi\sigma}{z}\int_a^b \rho\, d\rho \int_{(\rho - z)^2}^{(\rho+z)^2} u^{-1/2}\, du = \frac{2\pi\sigma}{z}\int_a^b \rho\, d\rho \left\{\sqrt{u}\,\Big|_{(\rho-z)^2}^{(\rho+z)^2}\right\}. \tag{9.107}$$

Next comes a key point. Tracing the square root back to (9.106), we see that it is actually a distance, and so the *positive* root is to be used. Thus

$$\sqrt{(\rho - z)^2} = |\rho - z| = \begin{cases}\rho - z & \text{if } z < \rho \\ z - \rho & \text{if } z > \rho.\end{cases} \tag{9.108}$$

Case 1: $z > b$.

That is, the field point lies outside the sphere. Thus $z > \rho$ for all $a < \rho < b$, and (9.107) becomes

$$\Phi(0, 0, z) = \frac{2\pi\sigma}{z}\int_a^b \rho[(\rho + z) - (z - \rho)]\, d\rho = \frac{4\pi\sigma}{3z}(b^3 - a^3). \tag{9.109}$$

But since $M = V\sigma = [4\pi(b^3 - a^3)/3]\sigma$, (9.109) is simply

$$\Phi(0, 0, z) = \frac{M}{z}, \tag{9.110}$$

and the force [per unit mass at $(0, 0, z)$] is

$$F = \frac{\partial \Phi}{\partial z} = -\frac{M}{z^2}. \tag{9.111}$$

It is remarkable that (9.111) *is precisely the same as if all the mass were concentrated at the origin*! The "averaging" process that leads to this result is not at all transparent.

Case 2: $z < a$.

That is, the field point lies within the cavity. Thus $z < \rho$ for all $a < \rho < b$, and (9.107) becomes

$$\Phi(0, 0, z) = \frac{2\pi\sigma}{z}\int_a^b \rho[(\rho + z) - (\rho - z)]\, d\rho = 2\pi\sigma(b^2 - a^2), \tag{9.112}$$

which is a *constant*, so that the force in the cavity is

$$F = \frac{\partial \Phi}{\partial z} = 0, \tag{9.113}$$

which is no less remarkable than (9.111).

Case 3: $a < z < b$.

That is, the field point is somewhere in the material. Then

$$\begin{aligned} \Phi(0, 0, z) &= \frac{2\pi\sigma}{z}\left\{\int_a^z \rho[(\rho + z) - (z - \rho)]\, d\rho + \int_z^b \rho[(\rho + z) - (\rho - z)]\, d\rho\right\} \\ &= \text{etc.}, \end{aligned} \tag{9.114}$$

but it is easier simply to apply what we have already learned from Cases 1 and 2: there should be no force at z due to the $z < \rho < b$ portion of the shell, and the $a < \rho < z$ portion should be felt exactly the same as if it were lumped as a point mass at the origin. Therefore

$$F = -\frac{\text{mass}}{z^2} = -\frac{(4/3)\pi(z^3 - a^3)\sigma}{z^2}, \tag{9.115}$$

which, of course, agrees with (9.113) as $z \longrightarrow a$ and with (9.111) as $z \longrightarrow b$.

COMMENT 1. Let us return to (9.106) and pursue an entirely different line of approach.

Recall from the Leibnitz rule (1.41) that if

$$I(\alpha) = \int_a^b f(x, \alpha)\, dx,$$

where a and b are independent of α, then

$$\frac{dI}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x, \alpha)\, dx$$

if $\partial f/\partial \alpha$ is a continuous function of both x and α over the pertinent x and α intervals. Similarly, it is a fact that if V is independent of α, then

$$\frac{d}{d\alpha}\int_V f(P, \alpha)\, d\tau = \int_V \frac{\partial f}{\partial \alpha}(P, \alpha)\, d\tau \tag{9.116}$$

if $\partial f/\partial \alpha$ is continuous for all Ps (i.e., x, y, z) in V and α's in the given α interval.

Using this result to obtain $\partial/\partial x$ of (9.106), we have

$$\frac{\partial \Phi}{\partial x}(x, y, z) = \int_V \frac{(-1/2)(2)(x - x')\sigma(x', y', z')\, dx'\, dy'\, dz'}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}}.$$

But the integrand is *not* continuous, as desired, if x, y, z is inside V because it "blows up" at that point—that is, when $x' = x$, $y' = y$, and $z' = z$. So we can't be sure that the differentiation under the integral sign is legitimate, and from here on we will restrict our x, y, z field point to be *outside* V.

Differentiating again with respect to x yields an expression for $\partial^2\Phi/\partial x^2$; computing $\partial^2\Phi/\partial y^2$ and $\partial^2\Phi/\partial z^2$ in similar fashion and adding, we have much cancellation and find (Exercise 9.42) that

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0. \tag{9.117}$$

That is,

$$\nabla^2\Phi = 0 \tag{9.118}$$

for all x, y, z outside V![20] Writing (9.118) in spherical polars per (9.90) and noting that Φ is a function of ρ only, for the spherical shell case,

$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{d}{d\rho}\left(\rho\,\frac{2d\Phi}{d\rho}\right) = 0,$$

so that

$$\rho^2\frac{d\Phi}{d\rho} = A, \qquad \frac{d\Phi}{d\rho} = A\rho^{-2},$$

and

$$\Phi = -\frac{A}{\rho} + B, \qquad F = \frac{d\Phi}{d\rho} = \frac{A}{\rho^2}, \tag{9.119}$$

where A and B are the constants of integration.

Note carefully that we are solving (9.118) in two distinct regions, the exterior $\rho > b$ and the cavity $\rho < a$, and that there is no reason why the constants A and B for the exterior region need be the same as for the cavity. In $\rho < a$, (9.119) says that F is unbounded; that is, it tends to infinity as $\rho \longrightarrow 0$. Certainly this is ridiculous, and so we must choose $A = 0$. Thus we obtain $F = 0$ in the cavity as before.

---

[20] We cannot claim that (9.118) holds for points *within* V because the Leibnitz differentiations are then apparently incorrect. In fact, it turns out that within the material Φ satisfies the so-called **Poisson equation** $\nabla^2\Phi = -4\pi\sigma(x, y, z)$; of course, this reduces to (9.118) outside V, where $\sigma = 0$. Although we will not derive the Poisson equation (see, for example, Philip Franklin's *Methods of Advanced Calculus*, McGraw-Hill, New York, 1944, pp. 319–322), let us at least check it for the present example. Recalling the results of case 3,

$$F = -\frac{4\pi\sigma(\rho^3 - a^3)}{3\rho^2} = \frac{\partial\Phi}{\partial\rho},$$

so that

$$\nabla^2\Phi = \frac{1}{\rho^2}\frac{\partial(\rho^2\partial\Phi/\partial\rho)}{\partial\rho} = \frac{1}{\rho^2}\frac{-4\pi\sigma}{3}(3\rho^2) = -4\pi\sigma,$$

as claimed.

Having applied a "boundedness" condition in order to evaluate A for the region $\rho < a$, what boundary condition can we come up with so as to evaluate A in the exterior region $\rho > b$? Well, we must have

$$F \sim -\frac{M}{\rho^2} \quad \text{as } \rho \longrightarrow \infty, \tag{9.120}$$

because as $\rho \longrightarrow \infty$, the shell will look more and more like a *point* mass. Comparing (9.120) and (9.119), we see that we must choose $A = -M$ for the exterior region. Consequently, (9.119) becomes $F = -M/\rho^2$ for all $\rho > b$ (not just as $\rho \longrightarrow \infty$), in agreement with our previous result (9.111).

COMMENT 2. In Section 9.8 we noted that irrotationality ($\nabla \times \mathbf{q} = 0$) implied that $\mathbf{q} = \nabla\phi$. The *additional* assumption of incompressibility ($\nabla \cdot \mathbf{q} = 0$) then led to the Laplace equation $\nabla \cdot \nabla\phi = \nabla^2\phi = 0$. In the present section we verified directly that $\nabla \times \mathbf{F} = 0$ for the gravitational force field (9.103), so that $\mathbf{F} = \nabla\Phi$. Finally, in Comment 1, we showed by Leibnitz differentiation that $\nabla^2\Phi = 0$ (at points not in the material). More directly, however, this result follows on taking the divergence of (9.103). Using (9.88), we have

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 F_\rho) = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(-m) = 0$$

for all points $\rho \neq 0$. So $\nabla \cdot \mathbf{F} = \nabla \cdot \nabla\Phi = 0$. It is also interesting to observe that whereas $\nabla \times \mathbf{F} = 0$ for *any* central force field $\mathbf{F} = F_\rho(\rho)\hat{\mathbf{e}}_\rho$, it is only for an *inverse-square law* $F_\rho(\rho) = \text{constant}/\rho^2$ that $\nabla \cdot \mathbf{F} = 0$ as well and hence that $\nabla^2\Phi = 0$.

*

COMMENT 3. Recall from Chapter 3 that if $f(x) \longrightarrow \infty$ as $x \longrightarrow x_0$, where x_0 is inside an interval I ($a \leq x \leq b$), then the improper integral $\int_a^b f\,dx$ is defined as the limit of the integral with an arbitrary neighborhood N_0 of x_0 deleted [Fig. 9.27(a)] as the length of N_0 shrinks to zero in any manner; symbolically,

$$\int_I f\,dx \equiv \lim_{N_0 \to 0} \int_{I-N_0} f\,dx. \tag{9.121}$$

In particular, we saw that a singularity $f(x) = O[1/(x - x_0)^p]$ causes divergence if and only if $p \geq 1$.

Improper *multiple* integrals are defined essentially the same way. For instance, if $f(x, y, z) \longrightarrow \infty$ as $x, y, z \longrightarrow x_0, y_0, z_0$, where x_0, y_0, z_0 is inside a volume V,

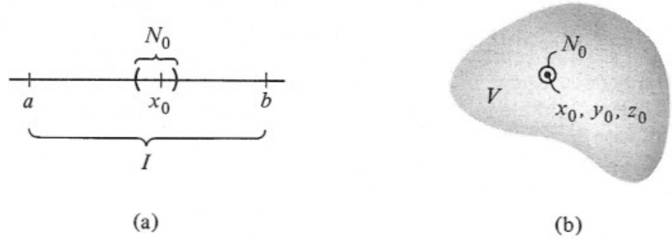

Figure 9.27. Deleting a neighborhood of singular point.

then [Fig. 9.27(b)]

$$\int_V f\,d\tau \equiv \lim_{N_0 \to 0} \int_{V-N_0} f\,d\tau, \tag{9.122}$$

where the N_0s are a sequence of neighborhoods of x_0, y_0, z_0 of arbitrary shape that shrink to zero; that is, their largest dimension tends to zero.

We mention this definition because a number of integrals in the "shell example" were, in fact, singular; for instance, assuming that σ is nonzero and continuous at the field point x, y, z in (9.106), observe that x, y, z is a singular point of the integral.

More generally, consider the integral

$$I = \int_V \frac{g(x, y, z)\,dx\,dy\,dz}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^p}, \tag{9.123}$$

where P_0 (i.e., x_0, y_0, z_0) is inside V and $\lim_{P \to P_0} g(P) = g(P_0) \neq 0$. Switching to a spherical polar system ρ, θ, ϕ with its origin at P_0, in order to expose the nature of the singularity, we have

$$dx\,dy\,dz \longrightarrow \rho^2 \sin\theta\,d\rho\,d\theta\,d\phi$$

and

$$[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^p = \rho^{2p}.$$

Note how the ρ^2, which enters through the Jacobian, helps out. Instead of having an $O(1/\rho^{2p})$ behavior as $\rho \longrightarrow 0$, we have an $O(\rho^2/\rho^{2p})$ behavior, and so we have convergence of I if $2p - 2 < 1$—that is, if $p < \frac{3}{2}$.

The reader should, I think, be reasonably comfortable with this claim. Anyone who wishes to see a careful proof, plus some further discussion of the gravitational potential, should refer to the book by Tychonov and Samarski.[21] ▌

EXERCISES

9.1. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a straight line from (0, 1, 0) to (1, 2, 3) and

(a) $\mathbf{F} = xz\hat{\mathbf{i}} - y^2z\hat{\mathbf{k}}$.

(b) $\mathbf{F} = x^2y\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - x\hat{\mathbf{k}}$.

(c) $\mathbf{F} = xyz\hat{\mathbf{k}}$.

9.2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the cardioid $r = a(1 + \cos\theta)$ from $\theta = 0$ to 2π and

(a) $\mathbf{F} = y\hat{\mathbf{i}}$, (b) $\mathbf{F} = 6\hat{\mathbf{e}}_r + r\hat{\mathbf{e}}_\theta$.

9.3. Evaluate $\int_C (xy^2\,dx - y\,dz)$, where C is given parametrically by $x = a\cos\zeta$, $y = a\sin\zeta$, $z = \zeta$, as ζ goes from 0 to 5π.

9.4. Evaluate the following line integrals, where C is the ellipse $(x/a)^2 + (y/b)^2 = 1$ clockwise. *Hint*: The parameterization $x = a\cos t$, $y = b\sin t$ may be of help in some of them.

[21] A. N. Tychonov and A. A. Samarski, *Partial Differential Equations of Mathematical Physics*, Vol. I, Holden-Day, San Francisco, 1964, pp. 292–303.

(a) $\int_C (y\,dx - x\,dy)$ (b) $\int_C (y\,dx + x\,dy)$

(c) $\int_C (x^6 \sin x\,dx + dy)$ (d) $\int_C x^3 y\,dy$

(e) $\int_C x^7\,dx$ (f) $\int_C (x + y)\,dy$

9.5. Taking τ to be a prism, as in Fig. 9.6, derive the expression (9.13) for grad u from definition (9.12).

9.6. Find the derivative of $f = xyz$ at the point (1, 3, 2) in the direction of the vector $2\hat{\mathbf{i}} - \hat{\mathbf{k}}$. What is the *maximum* possible directional derivative of f at that point, and what is its direction? What is the equation of the tangent plane to the surface $f =$ constant at the point (1, 3, 2)?

9.7. Find the derivative of $u = xy^2 - 3z^3$ at the point $(1, -2, 4)$ in the direction of the normal to the surface $xy + xz + yz = -6$.

9.8. Sketch the velocity profile for the one-dimensional flow $\mathbf{q} = (100 + 5y)\hat{\mathbf{i}}$. Compute $\nabla \times \mathbf{q}$ and explain qualitatively why the result "makes sense" in light of our physical interpretation of the vorticity vector.

9.9. Why are the linearity relations (9.24) to (9.26) true?

9.10. Verify the formulas (a) (9.27) and (9.28), (b) (9.30) and (9.31), (c) (9.32) and (9.33).

9.11. If $\mathbf{A}$ is a constant vector and $r = \sqrt{x^2 + y^2 + z^2}$, show that

$$\mathbf{A} \cdot \nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{A} \cdot \mathbf{r}}{r^3} \quad \text{and} \quad \nabla(\mathbf{A} \cdot \mathbf{r}) = \mathbf{A}.$$

9.12. Just as we derived the divergence theorem (9.35) from definition (9.5), obtain the companion results

$$\int_V \nabla u\,d\tau = \int_S \hat{\mathbf{n}} u\,d\sigma \quad \text{and} \quad \int_V \nabla \times \mathbf{v}\,d\tau = \int_S \hat{\mathbf{n}} \times \mathbf{v}\,d\sigma. \tag{9.124}$$

9.13. Previously we applied Leibnitz differentiation only to single integrals. Can you explain why the Leibnitz differentiation of the *triple* integral in (9.40) is valid?

9.14. According to the fluid mechanics version of the **Biot–Savart law**, the velocity $\mathbf{q}$ induced in an incompressible fluid at a point P by a "vortex filament" of strength Γ is given by the line integral

$$\mathbf{q}(P) = -\frac{\Gamma}{4\pi} \int_C \frac{\mathbf{r} \times d\mathbf{s}}{r^3}, \tag{9.125}$$

where Γ is a constant, $\mathbf{r}$ is the vector from the $d\mathbf{s}$ element to the point P, as shown in Fig. 9.28, and the orientation of Γ and $d\mathbf{s}$ are related by the right-hand rule. For the case where C is an infinitely long helix, defined parametrically by $x = a\cos\theta$, $y =$

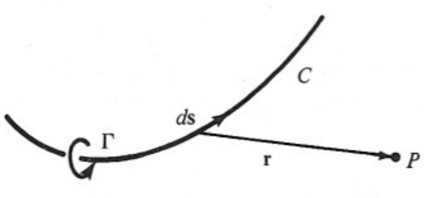

Figure 9.28. Vortex filament.

$a \sin\theta$, and $z = \kappa\theta$, for $-\infty < \theta < \infty$, use this integral to show that the z component of $\mathbf{q}$ at any point on the z axis is equal to $\Gamma/(2\pi\kappa)$. As a partial check, does this result make sense for the limiting cases $\kappa \longrightarrow \infty$ and $\kappa \longrightarrow 0$? How about the fact that it's independent of the helix radius a? (Note that our formula is called the Biot–Savart law by analogy with the expression for the magnetic flux induced by a current-carrying wire.)

9.15. (*Convective derivative*) Suppose that we have a fluid velocity field $\mathbf{q}(x, y, z, t)$. Consider some property of the flow, such as the temperature $T(x, y, z, t)$. If we swim along any desired path according to $x = x(t)$, $y = y(t)$, $z = z(t)$, then the rate of change observed is

$$\frac{d}{dt} T[x(t), y(t), z(t), t] = \frac{\partial T}{\partial t} + \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt}.$$

Note carefully that if we simply choose to *drift* with the fluid, then dx/dt, dy/dt, and dz/dt are the fluid velocity components u, v, and w. In this case, show that

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \mathbf{q} \cdot \nabla T.$$

This is called the **convective derivative** because it corresponds to the case where we drift, or convect, with the fluid. The special notation D/Dt is often used:

$$\frac{D(\quad)}{Dt} = \frac{\partial(\quad)}{\partial t} + \mathbf{q} \cdot \nabla(\quad). \tag{9.126}$$

9.16. (*Equations of motion of fluid mechanics*) Considering a fluid velocity field $\mathbf{q}$, let us apply Newton's second law to an arbitrary control volume V with surface S that drifts with the fluid. If ρ and p are the fluid mass density and pressure, $\mathbf{F}$ is a body force per unit mass (e.g., a gravitational field), and $\mathfrak{M}$ is the total momentum of the fluid in V, then

$$\mathfrak{M} = \int_V \rho\mathbf{q}\, d\tau$$

and according to Newton's law,

$$\frac{d}{dt}\int_V \rho\mathbf{q}\, d\tau = -\int_S p\hat{\mathbf{n}}\, d\sigma + \int_V \rho\mathbf{F}\, d\tau \tag{9.127}$$

if we assume the fluid to be inviscid, so that there is no viscous shear force on S to include. Noting that both the integrand and V depend on t, show (with the help of Fig. 9.12 and the continuity equation) that the Leibnitz differentiation on the left of (9.127) yields

$$\begin{aligned}\frac{d}{dt}\int_V \rho\mathbf{q}\, d\tau &= \int_V \frac{\partial}{\partial t}(\rho\mathbf{q})\, d\tau + \int_S (\hat{\mathbf{n}} \cdot \mathbf{q})(\rho\mathbf{q})\, d\sigma \\ &= \text{etc.} = \int_V \left(\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla\mathbf{q}\right)\rho\, d\tau.\end{aligned}$$

Finally, converting the pressure term in (9.127) to a volume integral with the help of (9.124), show that

$$\frac{\partial\mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla\mathbf{q} = -\frac{1}{\rho}\nabla p + \mathbf{F} \tag{9.128}$$

or using (9.126),

$$\frac{D\mathbf{q}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{F}. \tag{9.129}$$

Here is the important vector equation of motion for an inviscid fluid. Whereas the continuity equation was a statement of conservation of mass (i.e., *kinematics*), the equation of motion (9.129) is a statement of Newton's second law (i.e., *dynamics*).

9.17. To obtain a work-energy equation for the flow of an inviscid, incompressible fluid, it is reasonable to dot an arbitrary displacement vector $d\mathbf{s}$ into (9.128), since $\mathbf{F} \cdot d\mathbf{s}$, for example, is clearly a work term. Is the following derivation correct? If not, correct it.

Suppose that the flow $\mathbf{q}$ is irrotational and that the force field $\mathbf{F}$ is conservative. Then with $\mathbf{q} \equiv \nabla\phi$ and $\mathbf{F} \equiv -\nabla V$,

$$\frac{\partial}{\partial t}(\nabla\phi) \cdot d\mathbf{s} + (\mathbf{q} \cdot \nabla\mathbf{q}) \cdot d\mathbf{s} = -\frac{1}{\rho}\nabla p \cdot d\mathbf{s} - \nabla V \cdot d\mathbf{s},$$

$$\nabla\left(\frac{\partial\phi}{\partial t}\right) \cdot d\mathbf{s} + \mathbf{q} \cdot (\nabla\mathbf{q} \cdot d\mathbf{s}) = -\frac{1}{\rho}\frac{\partial p}{\partial s}\,ds - \frac{\partial V}{\partial s}\,ds,$$

$$\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial t}\right) + \mathbf{q} \cdot \frac{\partial\mathbf{q}}{\partial s} = -\frac{\partial}{\partial s}\left(\frac{p}{\rho}\right) - \frac{\partial}{\partial s}V,$$

$$\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\mathbf{q} \cdot \mathbf{q} + \frac{p}{\rho} + V\right) = 0,$$

and we obtain the *energy equation*

$$\frac{\partial\phi}{\partial t} + \frac{q^2}{2} + \frac{p}{\rho} + V = C(t) \tag{9.130}$$

throughout the flow, where the "constant" of integration $C(t)$ is often determined by considering the value of the terms on the left at "infinity" if the flow is unbounded. For instance, suppose that $V = 0$ and $\mathbf{q}$ and p approach the "free stream" values $U\hat{\mathbf{i}}$ and p_0 at infinity. Then $C(t)$ must equal $U^2/2 + p_0/\rho$.

9.18. The following scalar equations occur in fluid and solid mechanics. Reexpress them more concisely in terms of vector and vector differential operator notation.

(a)
$$\rho\ddot{u}_x = \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial\Theta}{\partial x} + F_x$$
$$\rho\ddot{u}_y = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial\Theta}{\partial y} + F_y$$
$$\rho\ddot{u}_z = \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + (\lambda + \mu)\frac{\partial\Theta}{\partial z} + F_z,$$

where the x, y, z subscripts denote the respective components of a given vector, the dots denote partial derivatives with respect to t, and ρ, μ, λ are constants.

(b)
$$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\,\partial y^2} + \frac{\partial^4\psi}{\partial y^4} = 0.$$

9.19. Evaluate $\int_V \nabla \cdot \mathbf{v}\, d\tau$ both directly and as a surface integral, where

(a) $\mathbf{v} = -z^2\hat{\mathbf{k}}$ and V is bounded by four flat faces with corners at (0, 0, 0), (0, 1, 1), (0, 0, 1), and (1, 0, 1).

(b) $\mathbf{v} = 3xy^2\hat{\mathbf{j}}$ and V is the region $0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3$.

9.20. Evaluate $\int_S \mathbf{v} \cdot \hat{\mathbf{n}}\, d\sigma$, where S is the surface of the cylinder $x^2 + y^2 \le 4$, $0 \le z \le 3$, and

(a) $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, (b) $\mathbf{v} = 3z^5\hat{\mathbf{k}} + 2\hat{\mathbf{i}}$.

9.21. Suppose that $\phi(x, y, z)$ has continuous first-order partials and satisfies the Poisson equation $\nabla^2\phi = f(x, y, z)$ throughout some simply connected region V, together with the boundary condition $\phi = g(x, y, z)$ on the surface S of V.

(a) Show that the solution (if it exists) is unique. *Hint:* Suppose that there are *two* solutions, say ϕ_1 and ϕ_2. With $\phi_1 - \phi_2 \equiv \Phi$, show that $\nabla^2\Phi = 0$ in V; $\Phi = 0$ on S. Setting $u = v = \Phi$ in (9.53), show that

$$\int_V (\Phi_x^2 + \Phi_y^2 + \Phi_z^2)\, d\tau = 0$$

and conclude that $\Phi_x = \Phi_y = \Phi_z = 0$ throughout V, so that $\Phi =$ constant $= 0$ and $\phi_1 = \phi_2$.

(b) Repeat part (a) with the boundary condition $\phi = g$ replaced by $\partial\phi/\partial n = g$.

(c) Repeat part (a) but with a "mixed" boundary condition, where $\phi = g$ over part of S and $\partial\phi/\partial n = h$ over the rest of S.

9.22. Evaluate the surface integral in Example 9.9 directly—that is, without using Stokes' theorem.

9.23. Verify Stokes' theorem if S is the hemispherical cap $z = +\sqrt{a^2 - x^2 - y^2}$ and
(a) $\mathbf{v} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + (2z + 3)\hat{\mathbf{k}}$.
(b) $\mathbf{v} = x^3y\hat{\mathbf{i}} + (y^2 + 4)\hat{\mathbf{j}}$.
(c) $\mathbf{v} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$.

9.24. Derive (9.61) by means of integration by parts.

9.25. Evaluate the line integrals of Exercise 9.4(b), (c), (d), and (e) with the help of Green's theorem.

9.26. If we set $M = 0$ and $N = x$ in Green's theorem, we obtain the line integral representation

$$A = \int_C x\, dy$$

for the area enclosed by C. Or with $N = 0$ and $M = -y$, we obtain an alternative representation

$$A = -\int_C y\, dx.$$

Interpret these two formulas graphically and apply them to the following areas.
(a) The triangle with vertices at (0, 0), (0, 2), and (2, 2) counterclockwise.
(b) The triangle with vertices at (0, 0), (0, 2) and (4, 2) counterclockwise.

9.27. Express

$$\int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

as one or more line integrals, where S is the shaded region in Fig. 9.29.

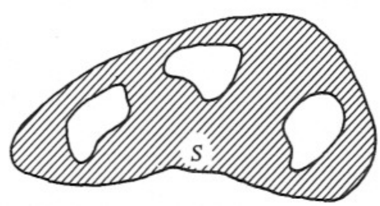

Figure 9.29. Green's theorem.

9.28. Show carefully that for the *two*-dimensional case [i.e., where $\mathbf{v} = \mathbf{v}(x, y)$, S is a region in the x, y plane, and C is its boundary], the *divergence theorem* (9.35) becomes

$$\int_S \nabla \cdot \mathbf{v}\, d\sigma = \int_C \hat{\mathbf{n}} \cdot \mathbf{v}\, |ds|,$$

where $|ds|$ emphasizes that all ds increments are to be positive. Similarly, write out the two-dimensional version of Green's first and second identities, (9.53) and (9.54).

9.29. Show that $\mathbf{F}$ in Example 9.1, is conservative. Use this information to evaluate the line integral much more simply.

9.30. Evaluate the line integrals shown.

(a) $\int_C (3x^2 \cos 2y\, dx - 2x^3 \sin 2y\, dy)$.

(b) $\int_C [5x^{3/2}e^{y^2}\, dx + (4yx^{5/2}e^{y^2} + 5y^2)\, dy]$,
where C is the curve $r = 2\theta^3$ as θ goes from 0 to 6π.

(c) $\int_C [3\, dx + 6y^2z^{5/2}\, dy + (5y^3z^{3/2} + 2)\, dz]$,
where C is the curve $x = \sin t$, $y = \cos 3t$, $z = 2t$ as t goes from 0 to 5π. *Hint:* They are *not* hard.

9.31. Show that the following vector fields are irrotational and in each case determine a scalar potential $u(x, y, z)$ such that $\mathbf{v} = \nabla u$.

(a) $\mathbf{v} = 2x^2\hat{\mathbf{i}} - 2yz\hat{\mathbf{j}} - (y^2 + 3)\hat{\mathbf{k}}$ (b) $\mathbf{v} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}}$
(c) $\mathbf{v} = 3x^2 \cos 2y\hat{\mathbf{i}} - 2x^3 \sin 2y\hat{\mathbf{j}}$ (d) $\mathbf{v} = z^3\hat{\mathbf{k}}$

9.32. Show that the vector fields below are solenoidal and in each case find a vector potential $\mathbf{w}(x, y, z)$ such that $\mathbf{v} = \nabla \times \mathbf{w}$:

(a) $\mathbf{v} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}}$ (b) $\mathbf{v} = z\hat{\mathbf{i}} + x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$
(c) $\mathbf{v} = 2y\hat{\mathbf{i}} + x^2y^2\hat{\mathbf{j}} - 2x^2yz\hat{\mathbf{k}}$

9.33. (a) Consider the decomposition, or splitting, of a given vector field $\mathbf{q}$ into the sum $\mathbf{q} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is irrotational (so that there exists a ϕ such that $\mathbf{v}_1 = \nabla\phi$) and $\mathbf{v}_2$ is solenoidal (so that there exists a $\mathbf{w}$ such that $\mathbf{v}_2 = \nabla \times \mathbf{w}$). The problem is as follows. Given $\mathbf{q}$, find $\mathbf{v}_1$ and $\mathbf{v}_2$ or, equivalently, their potentials ϕ and $\mathbf{w}$. Show that ϕ is found as any solution of the Poisson equation

$$\nabla^2\phi = \operatorname{div} \mathbf{q} \tag{9.131}$$

and that $\mathbf{w}$ is found as any solution of

$$\nabla \operatorname{div} \mathbf{w} - \nabla^2\mathbf{w} = \operatorname{curl} \mathbf{q}$$

(if indeed such solutions exist). Note that since the gradient of an arbitrary scalar function, say f, can be included in $\mathbf{w}$ (Why?), it may be possible to adjust f so that $\operatorname{div} \mathbf{w} = 0$, in which case $\mathbf{w}$, like ϕ, satisfies a Poisson equation,

$$\nabla^2\mathbf{w} = -\operatorname{curl} \mathbf{q}. \tag{9.132}$$

(b) Carry out the decomposition described in part (a) for $\mathbf{q} = x\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$. (Although solving the Poisson equation has yet to be discussed, you should be able to find solutions by inspection.)

9.34. (*The stream function for plane steady flow*) Consider the plane flow [i.e., $\mathbf{q} = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$] of an incompressible fluid. The volume flow rate per unit depth (in cubic meters per second, say) crossing the curve OA from left to right (Fig. 9.30) is

$$Q = \int_C (u\,dy - v\,dx) \equiv \int_C \mathbf{F} \cdot d\mathbf{r}$$

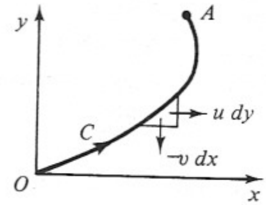

Figure 9.30. Stream function.

where $\mathbf{F} = -v\hat{\mathbf{i}} + u\hat{\mathbf{j}}$. Since $\nabla \times \mathbf{F} = (\partial u/\partial x + \partial v/\partial y)\hat{\mathbf{k}} = 0$ by the continuity equation, it follows that

$$Q = \int_{0,0}^{x,y} (u\,dy - v\,dx)$$

is a (single-valued) function of x, y, say $\psi(x, y)$, where

$$\frac{\partial \psi}{\partial x} = -v, \qquad \frac{\partial \psi}{\partial y} = u, \tag{9.133}$$

right? The $\psi(x, y) =$ constant lines are called **streamlines.**

(a) Show that

$$\nabla^2 \psi = -\Omega(x, y) \tag{9.134}$$

where $\mathbf{\Omega} = \Omega\hat{\mathbf{k}}$ is the vorticity. In particular, $\nabla^2\psi = 0$ if the flow is irrotational (i.e., if $\mathbf{\Omega} = 0$).

(b) Reformulate the problem of the flow over a circular cylinder (Example 9.14) in terms of ψ instead of ϕ. What is the resulting boundary value problem on ψ, analogous to the problem (9.96) on ϕ?

(c) Show that for plane, steady, *compressible* flow we can again introduce a stream function ψ according to

$$\frac{\partial \psi}{\partial x} = -\rho v, \qquad \frac{\partial \psi}{\partial y} = \rho u,$$

where this time ψ is the *mass* flow rate per unit depth across OA.

9.35. Show how (9.82) results from the combination of (9.79) and (9.78) as claimed in the text.

9.36. Write out the continuity equation (9.44) in cartesian, cylindrical, and spherical polar coordinates.

9.37. Write out the equations of motion (9.129) as three scalar equations in cartesian and cylindrical coordinates.

9.38. Derive equations (9.87) to (9.90) from (9.78) to (9.82).

9.39. Show how (9.97b) and (9.97c) were derived. Interpret (9.97c) in physical terms. (A picture would help.)

9.40. Write out the equation $\nabla \times \nabla \times \mathbf{A} = 0$ from Comment 2 of Example 9.14 as three scalar equations in terms of cartesian coordinates.

9.41. (a) In Example 9.15 show that if we set $\theta \equiv 6\pi$, say, on the top of the cut (Fig. 9.23), we still arrive at the same result, $\text{KE} = \pi\rho\kappa^2 \ln (b/a)$.

(b) Show also that the same answer results if we use a different cut, say along the y axis instead.

9.42. Verify that the Leibnitz differentiation of (9.106) does lead to (9.117). Is this result surprising in view of Exercise 9.44?

9.43. Show that *any* central force field $\mathbf{F} = f(\rho)\hat{\mathbf{e}}_\rho$ is conservative, where $f(\rho)$ is continuously differentiable [i.e., $f'(\rho)$ exists and is continuous]. What is the potential Φ such that $\mathbf{F} = -\nabla\Phi$?

9.44. (a) *Spherical Polars.* With $\rho = \sqrt{x^2 + y^2 + z^2}$, show that

$$\nabla^2(\rho^\alpha) = 0 \qquad (\text{for all } 0 < \rho < \infty)$$

only for $\alpha = -1$ (and, of course, $\alpha = 0$).

(b) *Cylindrical Coordinates.* With $r = \sqrt{x^2 + y^2}$, show that there are *no* α's (except, of course, $\alpha = 0$) such that

$$\nabla^2(r^\alpha) = 0$$

but that

$$\nabla^2(\ln r) = 0 \qquad (\text{for all } 0 < r < \infty).$$

(c) In order to attach a physical significance to the solutions $1/\rho$ and $\ln r$ of Laplace's equation, consider the context of the gravitational potential (Section 9.9). From (9.104) we see that $1/\rho$ may be regarded as the potential field induced by a unit mass at $\rho = 0$. Next, consider a straight, infinite wire with a uniform mass density of $\frac{1}{2}$ (for convenience) per unit length (Fig. 9.31). Integrating over $-\infty < z < \infty$,

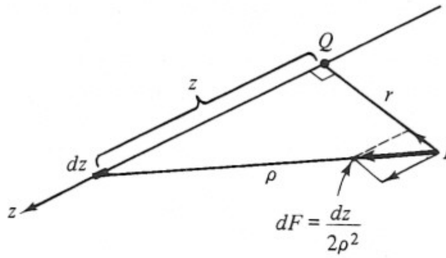

Figure 9.31. Potential due to an infinite wire.

show that the resulting force at P is $1/r$ and is normal to the wire—that is, toward Q. Thus deduce that the potential field induced by the wire is $\Phi = -\ln r$. This potential is called the two-dimensional or **logarithmic potential**. (These results will be important in our discussion of Green's functions in Part V.)

***9.45.** Consider the solution of $\nabla \times \mathbf{F} = \Phi(x, y, z)\mathbf{r}$ for $\mathbf{F}$, where

$$\Phi = \frac{1}{\pi} \frac{lx + my + nz}{(x^2 + y^2 + z^2)^2} \quad \text{and} \quad \mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}.$$

It will help to define $\mathbf{A} \equiv l\hat{\mathbf{i}} + m\hat{\mathbf{j}} + n\hat{\mathbf{k}}$ and introduce spherical polars r, θ, ϕ with $\mathbf{A}$ as polar axis (Fig. 9.32), for then $\mathbf{F}$ will be independent of ϕ because of the symmetry about the polar axis (Right?). Thus

$$\nabla \times \mathbf{F} = \frac{1}{\pi} \frac{\mathbf{A} \cdot \mathbf{r}}{r^4} \mathbf{r} = \frac{1}{\pi} \frac{A \cos\theta}{r^2} \hat{\mathbf{e}}_r.$$

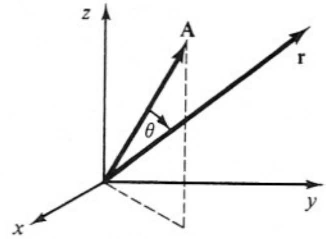

Figure 9.32. A as polar axis.

Integrating, obtain the solution

$$\mathbf{F} = \frac{(ny - mz)\hat{\mathbf{i}} + (lz - nx)\hat{\mathbf{j}} + (mx - ly)\hat{\mathbf{k}}}{2\pi r^2}.$$

(Note the similarity between this problem and Exercise 8.30.)

***9.46.** (*Improper multiple integrals*) In Comment 3 following Example 9.16 we briefly discussed improper triple integrals. For the *double* integral

$$I = \int_S \frac{g(x, y)\, dx\, dy}{[(x - x_0)^2 + (y - y_0)^2]^p}, \tag{9.135}$$

where x_0, y_0 is inside S, put forward a convergence definition, analogous to (9.122), and discuss the values of p for which (9.135) converges.

9.47. Consider the following claims: $\mathbf{F} = \theta\hat{\mathbf{e}}_r + \hat{\mathbf{e}}_\theta$ is conservative and $\mathbf{F} = -\nabla\Phi$, where $\Phi = -r\theta$ plus a constant, which can be taken to be zero. If C is the *ccw* unit circle, say, then

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = -\Phi\Big|_{\text{initial}}^{\text{final}} = 2\pi.$$

On the other hand, W should be zero by Stokes' theorem, since $\mathbf{F}$ is conservative and C is a closed curve. Explain the apparent contradiction.

9.48. Make up two examination-type problems on this chapter.