ASE 380P2 ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANIS II

Exam 2. Monday, Apr 5, 2010

1. (a) Define a complex differentiable function and state the Cauchy-Riemann condition (5 points)

   (b) Check if the following function is complex-differentiable (15 points)

$$f(z) = |z| \sin z$$

2. (a) Define branch cuts, and select a specific single-value function for

$$f(z) = \sqrt{1 + \sqrt{z}}$$

3. (a) State the Laurent Expansion Theorem (5 points).

   (b) Expand the following function into its Laurent series in $1 < |z| < 2$,

$$\frac{1}{z^2 - 3z + 2}$$

   (15 points).

4. (a) State the Residue Theorem (5 points).

   (b) Show that

$$\int_{-\infty}^{\infty} \frac{x\,dx}{x^3 + 1} = \frac{\pi}{\sqrt{3}}$$

   (15 points).

5. (a) Solve the following problem. Use first elementary means, and then Laplace Transform, and Residue Theorem to compute the inverse Laplace transform. Compare the results (20 points).

$$\ddot{x} - \dot{x} = H(t-1), \quad x(0) = 0, \dot{x}(0) = 1$$

Solutions

1a) $f: \mathbb{C} \to \mathbb{C}$ is complex-differentiable in $\Omega \subset \mathbb{C}$, if

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists for every $z \in \Omega$. The key point in the definition is the fact that the division by Δz is understood in the sense of complex numbers.

Function f is complex-differentiable (holomorphic) iff it satisfies Cauchy – Riemann conditions:

$$z = x + iy \;, \quad f(z) = u(z) + i\,v(z) \;, \quad x, y, u, v \in \mathbb{R}$$

$$u_{,x} = v_{,y} \quad \text{and} \quad u_{,y} = -v_{,x}$$ 5

1b) It is not, because it involves $\bar{z}$ ($|z| = z\bar{z}$, and so $f(z) = \sqrt{z\bar{z}}\,\sin z$) which is not holomorphic

$$\bar{z} = x - iy \qquad u = x \;, \quad v = -y$$

$$u_{,x} = 1 \;, \quad v_{,y} = -1 \qquad 1 \neq -1$$

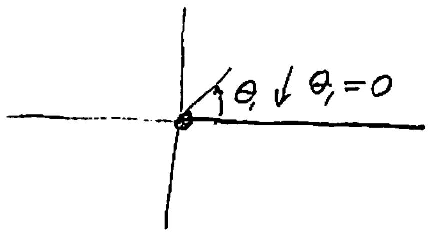
15

2) $\quad f(z) = \sqrt{1+\sqrt{z}}$ ⓶

f is a composition of two functions: $\sqrt{z}$ and $\sqrt{1+z}$.
Both functions are double-valued and require branch cuts.

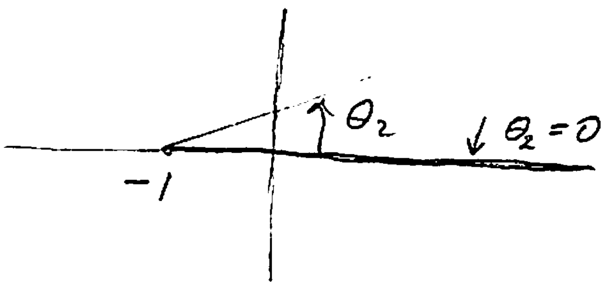
$\sqrt{z}$:

$$z = r_1 e^{i\theta_1}$$
$$\sqrt{z} = \sqrt{r_1}\, e^{i\frac{\theta_1}{2}}$$



$\sqrt{z+1}$

$$z = -1 + r_2 e^{i\theta_2}$$
$$\sqrt{z+1} = \sqrt{r_2}\, e^{i\frac{\theta_2}{2}}$$



The technical part is to make sure that the range of $\sqrt{z}$ does not interfere with the domain of $\sqrt{z+1}$. For $0 < \theta < 2\pi$, range of the selected branch of $\sqrt{z}$ is the upper half of the complex plane in the second picture. Consequently, one must not cut through the upper half. Any other cut is OK, including the cut shown in the picture.

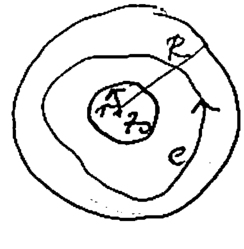
20

3

3 a) <u>Laurent Expansion Thm</u>

$f : D \to \mathbb{C}$ holomorphic in an annulus

$$D := \{ z \in \mathbb{C} : r < |z - z_0| < R \}$$



$\Longrightarrow$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad \text{where} \quad a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

5

3b)
$$\frac{1}{z^2 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\Delta = 9 - 4 \cdot 2 = 1 \qquad z_1 = \frac{3+1}{2} = 2, \quad z_2 = \frac{3-1}{2} = 1$$

$$z^2 - 3z + 2 = (z-1)(z-2)$$

$$\frac{1}{z-2} = \frac{-1}{2-z} = -\frac{1}{2} \underbrace{\frac{1}{1 - \frac{z}{2}}}_{q} = -\frac{1}{2} \left\{ 1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots \right\}$$

$$= - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

$$\frac{1}{z-1} = \frac{1}{z} \left(\frac{1}{\underbrace{1 - \frac{1}{z}}_{q}} \right) = \frac{1}{z} \left\{ 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \right\}$$

$$= \sum_{n=1}^{\infty} \frac{1}{z^n} \qquad \left|\frac{1}{z}\right| < 1 \iff |z| > 1$$

So:
$$\frac{1}{z^2 - 3z + 2} = - \sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

15

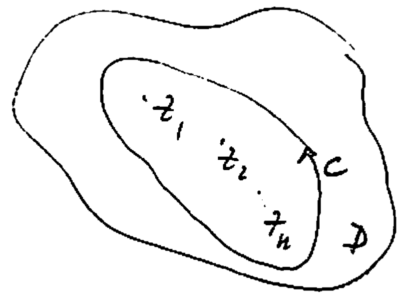
4a) Residue Thm

$D \subset \mathbb{C}$ simply connected

$z_1, \dots, z_n \in D$

$f : D \setminus \{z_1, \dots, z_n\} \longrightarrow \mathbb{C}$ holo

$c \subset D$, ccw, surrounding $z_1, \dots z_n$



$\Longrightarrow$

$$\oint_c f(z)\, dz = 2\pi i \sum_{l=1}^{n} \mathrm{res}_{z_l} f(z)$$

5

4b) $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{x^3+1}$

$$z^3+1 = (z+1)(z^2-z+1)$$

$z^2 - z + 1 = 0$

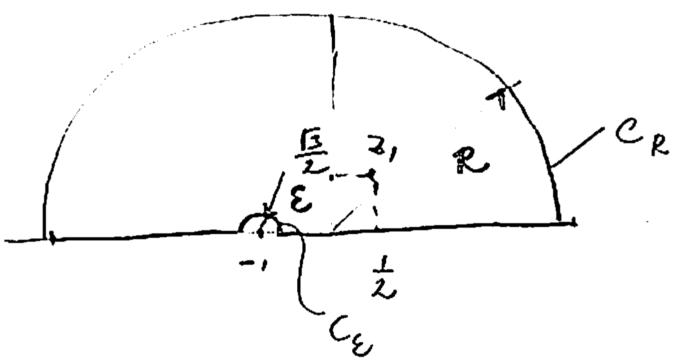
$\Delta = 1 - 4 = -3$

$z_{1,2} = \dfrac{1 \pm i\sqrt{3}}{2}$

$f(x) = \dfrac{x}{x^3+1}$ is singular at $x = -1$

$= O\left(\frac{1}{x^2}\right)$ at $\pm\infty$

So, the integral is to be understood in the CPV sense at -1 but it can be understood in Lebesgue sense at $x = \pm\infty$.



5

$$\operatorname{Res}_{z_1} \frac{z}{z^3+1} = \lim_{z \to \frac{1+i\sqrt{3}}{2}} \frac{z}{(z+1)\left(z - \frac{1-i\sqrt{3}}{2}\right)}$$

$$= \frac{\frac{1+i\sqrt{3}}{\cancel{2}}}{\frac{3+i\sqrt{3}}{\cancel{2}} \; i\sqrt{3}} = \frac{1+i\sqrt{3}}{-3+3i\sqrt{3}}$$

$$= \frac{(1+i\sqrt{3})(-3-3i\sqrt{3})}{9+27} = \frac{-3+9+i(-6\sqrt{3})}{36}$$

$$= +\frac{1}{6} - \frac{\sqrt{3}}{6} i = \frac{1}{6}(1-\sqrt{3}\, i)$$

$$2\pi i \operatorname{Res}_{z_1} = \frac{\pi i}{3}(1-\sqrt{3}\, i) = +\frac{\sqrt{3}}{3}\pi + \frac{1}{3}\pi i$$

$$= \frac{\pi}{\sqrt{3}} + \frac{1}{3}\pi i$$

10

Integral over C_ε: $\quad z = -1 + \varepsilon e^{i\theta} \qquad 0 < \theta < \pi$

$$\int_{C_\varepsilon} f(z)\, dz = \int_0^\pi \frac{-1+\varepsilon e^{i\theta}}{\cancel{\varepsilon e^{i\theta}}\left(\varepsilon^2 e^{i2\theta} - \varepsilon e^{i\theta} - 1\right)} \cancel{\varepsilon}\, i \cancel{e^{i\theta}}\, d\theta$$

$$\xrightarrow[\varepsilon \to 0]{} -i \int_0^\pi \frac{1}{3}\, d\theta = -\frac{1}{3}\pi i$$

5

Integral over C_R vanishes in the limit as

$$\left| \int_{C_R} \frac{z}{z^3+1}\, dz \right| \leq \int_{C_R} \frac{|z|}{|z^3+1|}\, ds$$

5

$z = e^{iR\theta}$

$dz = iRe^{iR\theta} d\theta$

$ds = |dz| = R\, d\theta$

$$\leq C \cdot \int_{C_R} \frac{R}{R^3} R\, d\theta \sim \frac{1}{R} \xrightarrow[R \to \infty]{} 0$$

Consequently, by the Residue Thm,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\,dx}{x^3+1} = \frac{\pi}{\sqrt{3}} + \frac{1}{3}\pi i - \frac{1}{3}\pi i = \underline{\underline{\frac{\pi}{\sqrt{3}}}}$$

---

5. Elementary solution:

$0<t<1$

$\ddot{x} - \dot{x} = 0 \qquad x(t) = A + Be^{t} \qquad \dot{x}(t) = Be^{t}$

$x(0) = 0 \qquad \Rightarrow \quad A = -1$

$\dot{x}(0) = 1 \qquad \Rightarrow B = 1$

So: $\boxed{x(t) = -1 + e^{t}}$, $\quad \dot{x}(t) = e^{t}$

$1<t<\infty$

$\ddot{x} - \dot{x} = 1 \qquad x(t) = A + Be^{(t-1)} - (t-1), \quad \dot{x}(t) = Be^{t-1} - 1$

$x(1) = A + B = -1 + e \qquad \Rightarrow A = -1 + e - 1 - \cancel{e} = -2$

$\dot{x}(1) = B - 1 = e \quad \Rightarrow \quad B = 1 + e$

So: $\boxed{x(t) = -2 + (1+e)e^{t-1} - (t-1)}$

$\dot{x}(t) = (1+e)e^{t-1} - 1$

Check: $\quad x(1) = -2 + 1 + e = -1 + e$

$\dot{x}(1) = 1 + e - 1 = e$

5

$$\ddot{x} - \dot{x} = H(t-1) \Big/ \mathcal{L}$$ ⑦

$$s^2\bar{x} - s\cancel{x(0)}^{0} - \dot{x}(0) - (s\bar{x} - \cancel{x(0)}^{0}) = \frac{e^{-s}}{s}$$

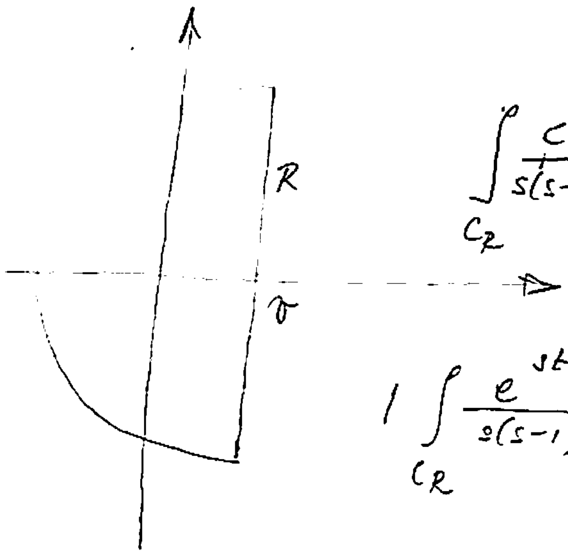
$$\dot{x}(0) = 1$$

$$\int_0^\infty H(t-1)\, e^{-ts}\, dt = \int_1^\infty e^{-ts}\, dt = -\frac{1}{s} e^{-ts}\Big/_1^\infty = \frac{e^{-s}}{s} \quad \mathrm{Re}\, s > 0$$

$$(s^2 - s)\bar{x} = \frac{e^{-s}}{s} + 1$$

$$\bar{x} = \frac{e^{-s}}{s^2(s-1)} + \frac{1}{s(s-1)}$$

<u>Calculation of the inverse transforms</u>



$$\int_{C_R} \frac{e^{st}}{s(s-1)}\, ds + \int_{\gamma - iR}^{\gamma + iR} \frac{e^{st}}{s(s-1)}\, ds = 2\pi i\, [\mathrm{Res}_0 + \mathrm{Res}_1]$$

$$\left| \int_{C_R} \frac{e^{st}}{s(s-1)}\, ds \right| \leq \int \frac{e^{\gamma t}}{|s||s-1|}\, |ds| \sim \frac{e^{\gamma t}}{R} \xrightarrow[R \to \infty]{} 0$$

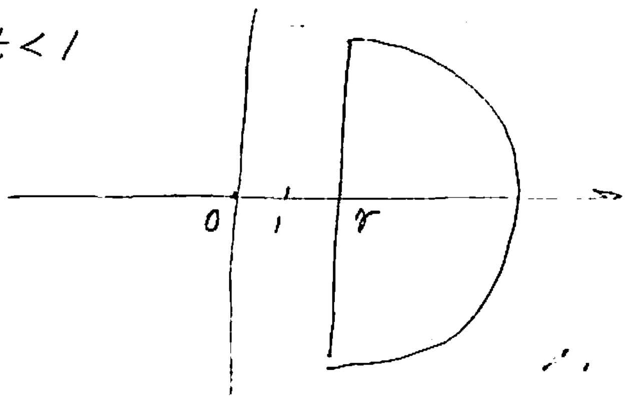
$$\mathrm{Res}_0 = \lim_{s \to 0} \frac{e^{st}}{s-1} = -1$$

$$\mathrm{Res}_1 = \lim_{s \to 1} \frac{e^{st}}{s} = e^t$$

So, $\mathcal{L}^{-1}\left(\frac{1}{s(s-1)}\right) = -1 + e^t$ for <u>$t > 0$</u>

$$\cdot \quad \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2(s-1)}\right) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{s(t-1)}}{s^2(s-1)}\,ds$$

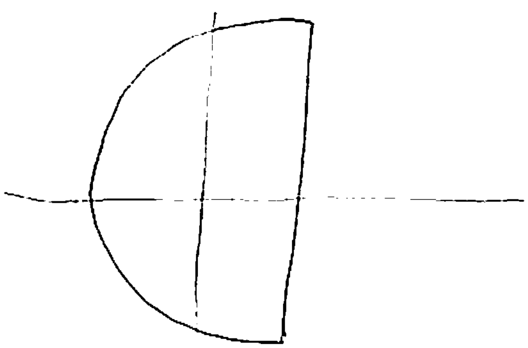
$t<1$



$$\int_{\gamma-iR}^{\gamma+iR} + \cancel{\int_{C_R}} = 0 \qquad R\to\infty, \ \int_{C_R} \to 0$$

$$\therefore \quad \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2(s-1)}\right) = 0 \quad \text{for } t<1$$

$t>1$



$$\mathrm{Res}_1 = \lim_{s\to 1} \frac{e^{s(t-1)}}{s^2} = e^{(t-1)}$$

$$\mathrm{Res}\, s_0 = \lim_{s\to 0} \left[\frac{d}{ds}\frac{e^{s(t-1)}}{s-1}\right]$$

$$= \lim_{s\to 0} \frac{(t-1)e^{s(t-1)}(s-1) - e^{s(t-1)}}{(s-1)^2}$$

$$= -(t-1) - 1 = -t$$

$$\int_{C_R} \to 0$$

10

$$\text{So} \quad \mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2(s-1)}\right) = -t + e^{(t-1)}$$

<u>Check:</u> $-t + e^{(t-1)} - 1 + e^{t} \overset{?}{=} -2 + (1+e)e^{t-1} - (t-1)$

$$= -1 + e^{t-1} + e^{t} - t$$

✓ <u>ok.</u>