For $f : X \to Y$, $\mathbf{sup}\, \boldsymbol{f} := \mathbf{sup}_X\, \mathcal{R}(f)$ and $\mathbf{inf}\, \boldsymbol{f} := \mathbf{inf}_X\, \mathcal{R}(f)$.

$\{X, +\}$ is an **Abelian group** iff

$x + (y + z) = (x + y) + z$	associative
$x + y = y + x$	commutative
$\exists 0 \in X : x + 0 = 0 + x = x$	identity
$\forall x \, \exists - x \in X : x + (-x) = (-x) + x = 0$	inverse

$\{\mathbb{F}, +, \times\}$ is a **commutative field** iff $\{\mathbb{F}, +\}$ is Abelian and

$x(yz) = (xy)z$	associative
$xy = yx$	commutative
$a(b + c) = (ab) + (ac)$	left distributive
$(a + b)c = (ac) + (bc)$	right distributive
$\exists 1 \in \mathbb{F} : x1 = 1x = x$	identity
$\forall x \, \exists x^{-1} \in \mathbb{F} : xx^{-1} = x^{-1}x = 1$	inverse

In an **order complete** ordering every nonempty subset with an upper bound ($x : a \le x \; \forall a \in A$) also has a least upper bound (sup), and vice versa for lower bounds ($x : x \le a \; \forall a \in A$) and greatest lower bounds (inf).

Properties of the **real numbers** $\{\mathbb{R}, \cdot, +, \le\}$:

(i) $\{\mathbb{R}, \cdot, +\}$ is a commutative field.
(ii) $\le$ is a total ordering on $\mathbb{R}$ which is order complete.
(iii) $x \le y \implies x + z \le y + z \; \forall z$
(iv) $0 \le x \wedge 0 \le y \implies 0 \le xy$

The **extended real numbers** are $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$.

Balls and the **Euclidean metric**:

$$\mathrm{d}(x, y) := \sqrt{\sum (x_i - y_i)^2}$$
$$B(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^n : \mathrm{d}(\mathbf{x}, \mathbf{y}) < r\}$$
$$\bar{B}(\mathbf{x}, r) := \{\mathbf{y} \in \mathbb{R}^n : \mathrm{d}(\mathbf{x}, \mathbf{y}) \le r\}$$

The set A is **bounded** iff $\exists x, r : A \subset B(x, r)$.

$A \in \mathbb{R}^n$ is a **neighborhood** of x iff $\exists \varepsilon > 0 : B(x, \varepsilon) \subset A$.

x is an **interior point** of A, denoted $\boldsymbol{x \in}$ int $\boldsymbol{A}$ iff one of

(a) $\exists N$, a neighborhood of $x : N \subset A$
(b) $\exists \varepsilon > 0 : B(x, \varepsilon) \subset A$
(c) A is a neighborhood of x.

The set A is **open** iff $A = \operatorname{int} A$. Set interiors and openness have the following properties:

$\operatorname{int}(\operatorname{int} A) = \operatorname{int} A$	idempotence
$\operatorname{int}(A \cup B) \supset \operatorname{int} A \cup \operatorname{int} B$	union is a superset of unions
$\operatorname{int}(A \cap B) = \operatorname{int} A \cap \operatorname{int} B$	intersection
$A \subset B \implies \operatorname{int} A \subset \operatorname{int} B$	subset relation
$\forall \iota \in I\ A_i$ open $\implies \bigcup_{\iota \in I} A_i$ open	denumerable union of opens is open
$A_1, \ldots, A_n$ open $\implies A_1 \cap \cdots \cap A_n$ open	finite intersection of opens is open
$\emptyset, \mathbb{R}^n$ open	odd examples

x is an **accumulation point** of the set A, sometimes denoted $x \in \hat{A}$, iff one of

(a) $\forall N\ N \cap A - \{x\} \neq \emptyset$
(b) $\forall B(x, \varepsilon)\ B(x, \varepsilon) \cap A - \{x\} \neq \emptyset$

The **closure** of a set A is $\overline{A} = \hat{A} \cup A$.

A set is **closed** iff $A = \overline{A}$. Set closures have the following properties:

$\overline{\overline{A}} = \overline{A}$	idempotence
$\overline{(A \cup B)} = \overline{A} \cup \overline{B}$	union
$\overline{(A \cap B)} \subset \overline{A} \cap \overline{B}$	intersection is a subset of intersections
$A \subset B \implies \overline{A} \subset \overline{B}$	subset relation
$\forall \iota \in I\ A_i$ closed $\implies \bigcap_{\iota \in I} A_i$ closed	denumerable intersection of closeds is closed
$A_1, \ldots, A_n$ closed $\implies A_1 \cup \cdots \cup A_n$ closed	finite union of closeds is closed
$\emptyset, \mathbb{R}^n$ closed	odd examples

x is a **cluster point** of the set A iff one of

(a) $x \in \overline{A}$
(b) $\forall N\ N \cap A \neq \emptyset$
(c) $\forall B(x, \varepsilon)\ B(x, \varepsilon) \cap A \neq \emptyset$

Interior/closure complements: $\operatorname{int} A = \left(\overline{(A')}\right)'$

Open closed duality: A open $\iff A'$ closed

Bolzano-Weierstrass theorem for sets: $A \subset \mathbb{R}$ infinite and bounded $\implies \exists x \in \mathbb{R}$, an accumulation point.

A **sequence**, denoted x_n, is a function $\mathbb{N} \to \mathbb{R}$.

A sequence x_n **converges** to $x \in \mathbb{R}$, denoted $\mathbf{x}_n \to \mathbf{x}$ iff $\forall \varepsilon > 0\, \exists N : n \geq N \implies \mathrm{d}(\mathbf{x}_n, \mathbf{x}) < \varepsilon$.

(i) $x_n \to +\infty \in \bar{\mathbb{R}} := \forall c\ \exists N : n \geq N \implies x_n > c$
(ii) $x_n \to -\infty \in \bar{\mathbb{R}} := \forall c\ \exists N : n \geq N \implies x_n < c$

x is an **accumulation point** of A, iff $\exists x_n \in A : x_n \to x$.

$A \subset \mathbb{R}^n$ is **sequentially closed** iff $x_n \in A, x_n \to x \implies x \in A$.

A is sequentially closed iff it is closed.

$t : \mathbb{N} \to \mathbb{R}$ is a **subsequence** of $s : \mathbb{N} \to \mathbb{R}$ iff $\exists r : \mathbb{N} \to \mathbb{N}$ injective and $t = sr$.

x is a **cluster point** of the sequence x_n iff $\exists x_{n_k}$, a subsequence, such that $x_{n_k} \to x$.

$a_n \in \mathbb{R}$ is a **bounded sequence** iff $\exists B(x, r) : \{a_n\} \subset B(x, r)$.

(i) a_n is **bounded above** if $\exists b \in \mathbb{R} : a_n \leq b \; \forall n$.
(ii) a_n is **bounded below** if $\exists b \in \mathbb{R} : a_n \geq b \; \forall n$.

Every monotone, bounded sequence converges.

(i) a_n is **monotone increasing** if $a_{n+1} \geq a_n$.
(ii) a_n is **monotone decreasing** if $a_{n+1} \leq a_n$.

Squeeze convergence: Given $x_n, y_n, z_n \in \mathbb{R}$ where $x_n \leq y_n \leq z_n \; \forall n$. $x_n \to c \wedge z_n \to c \implies y_n \to c$.

Bolzano-Weierstrass theorem for sequences: Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Let $a_n \in \mathbb{R}$ bounded and $\hat{A}$ be the cluster points of a_n. $\hat{A}$ is not empty by Bolzano-Weierstrass. Define:

(i) $\boldsymbol{\lim_{n\to\infty} \sup a_n := \sup \hat{A}}$
(ii) $\boldsymbol{\lim_{n\to\infty} \inf a_n := \inf \hat{A}}$
(iii) Either value may be $\pm\infty$ if using $\bar{\mathbb{R}}$. Every set is bounded in $\bar{\mathbb{R}}$.

Characterization of the **limit inferior** and **limit superior** for a sequence a_n and its accumulation points $\hat{A}$:

$$\lim_{N\to\infty} \inf_{n\geq N} \{a_n\} := \inf \hat{A} = \min \hat{A} = \sup_N \inf_{n\geq N} \{a_n\}$$
$$\lim_{N\to\infty} \sup_{n\geq N} \{a_n\} := \sup \hat{A} = \max \hat{A} = \inf_N \sup_{n\geq N} \{a_n\}$$

(i) $a_n \leq b_n \; \forall n \implies \liminf a_n \leq \liminf b_n$
(ii) $a_n \leq b_n \; \forall n \implies \limsup a_n \leq \limsup b_n$
(iii) $\liminf \{a_n\} + \liminf \{b_n\} \leq \liminf \{a_n + b_n\}$
(iv) $\liminf \{a_n + b_n\} \leq \liminf \{a_n\} + \liminf \{b_n\}$

f has a **function limit** a at x_0, denoted $\lim_{x\to x_0} f(x) = a$, iff $\forall \varepsilon > 0 \; \exists \delta > 0 : \mathrm{d}(x, x_0) < \delta \implies \mathrm{d}(f(x), a) < \varepsilon$.

$f : \mathbb{R}^n \supset A \to \mathbb{R}^m$ is **continuous** at point $x_0 \in A$ iff one of

(a) $f(\mathbf{x}_0)$ exists and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$.
(b) $\forall \varepsilon > 0 \; \exists \delta > 0 : \mathrm{d}(\mathbf{x}_0, \mathbf{x}) < \delta \implies \mathrm{d}(f(\mathbf{x}_0), f(\mathbf{x})) < \varepsilon$
(c) $\forall N$ of $f(\mathbf{x}_0) \; \exists M$ of $\mathbf{x} : f(M) \subset N$

$f : \mathbb{R}^n \supset A \to \mathbb{R}^m$ is **sequentially continuous** at $\mathbf{x}_0 \in A$ iff $\forall \mathbf{x}_n \in A \; \mathbf{x}_n \to \mathbf{x}_0 \implies f(\mathbf{x}_n) \to f(\mathbf{x}_0)$.

$f : \mathbb{R}^n \supset A \to \mathbb{R}^m$ is continuous at x_0 iff it is sequentially continuous at x_0.

$f : A \to \mathbb{R}^m$ is **globally continuous** if one of

(a) f is continuous at every point in A
(b) $\forall G \subset \mathbb{R}^M$ open, $f^{-1}(G)$ open $\in \mathbb{R}^n$.
(c) $\forall H \subset \mathbb{R}^M$ closed, $f^{-1}(H)$ closed $\in \mathbb{R}^n$.

A set $K \in \mathbb{R}^n$ is **compact** iff it is bounded and closed.

A set $K \in \mathbb{R}^n$ is **sequentially compact** iff $\forall a_n \in A\ \exists a_{n_k} : a_{n_k} \to x_0 \in A$.

A set $K \in \mathbb{R}^n$ is sequentially compact iff it is compact.

Weierstrass Theorem: If $f : \mathbb{R}^n \supset K \to \mathbb{R}$ continuous and K compact, then

$$\exists \mathbf{x}_{\min}, \mathbf{x}_{\max} \in K : f(\mathbf{x}_{\min}) = \inf_K f \wedge f(\mathbf{x}_{\max}) = \sup_K f$$

$\{X, +, \mathbb{F}, +, \times, *\}$ is a **vector space** iff $\{X, +\}$ is Abelian, $\{\mathbb{F}, +, \times\}$ is a field, and $* : \mathbb{F} \times X \to X$ satisfies

$\alpha(\beta x) = (\alpha\beta)x$	associative
$\alpha(x + y) = \alpha x + \alpha y$	left distributive
$(\alpha + \beta)x = \alpha x + \beta x$	right distributive
$1x = x$	identity
$0x = 0$	*implied*
$-1x = -x$	*implied*

V^E is a **function vector space** given

$$(f + g)(x) := f(x) + g(x)$$
$$(\alpha f)(x) := \alpha f(x)$$

$C^k(\Omega) :=$ **space of all continuous functions** on Ω with k^{th} order derivatives.

$C^\omega(\Omega) :=$ **space of all analytic functions**.

$f \in C^k(\bar{\Omega})$ iff $\bar{\Omega} \in \Omega_1$, $f_1 \in C^k(\Omega_1)$, $f_1\big|_\Omega = f$.

$W \subset V$, a **subspace** iff

$u, v \in W \implies u + v \in W$	closed wrt vector sum
$u \in W \implies \alpha u \in W$	close wrt scalar product
$0 \in W$	*implied*

If X, Y are subspaces of V then

$X \cup Y$	is not generally a subspace
$X \cap Y \neq \emptyset$	since $0 \in X, Y$
$X \cap Y$	is a subspace
$X + Y := \{x + y, x \in X, y \in Y\}$	is an **algebraic sum**, a subspace.
$X \oplus Y$	is a **direct sum**, a subspace, if $X \cap Y = \{0\}$

If $V = X \oplus Y$ then Y is a **complement** of X.

$V = X \oplus Y \iff \forall v\ \exists! x, y : v = x + y$

If M is a subspace of V, $X \subset V$ then $x + M := \{x + m, y \in M, x \in V\}$ is an **affine subspace**.

Any subspace M of V generates an equivalence relation R_M and corresponding **quotient vector space** V/M

$$\begin{aligned} x \, R_M \, y &:= x - y \in M \\ [x] &= \{v \in V : v - x \in M\} \\ &= x + M \\ [x] + [y] &:= [x + y] \\ \alpha\,[x] &:= [\alpha x] \end{aligned}$$

$\sum_{i=1}^{k} \alpha_i \mathbf{x}_i$ is a **linear combination**.

If $\exists \alpha_i : \mathbf{x} = \sum_{i=1}^{k} \alpha_i \mathbf{x}_i$ then x is **linearly dependent** (**LD**) on $\mathbf{x}_i$. Otherwise $\mathbf{x}$ is **linearly independent** (**LI**).

(i) $\{\mathbf{x}_i\}$ LI iff none of the $\mathbf{x}_i$ is LD on the remaining elements.
(ii) $\{\mathbf{x}_i\}$ LI iff $\sum \alpha_i \mathbf{x}_i = 0 \implies \alpha_i = 0$.
(iii) $\{\mathbf{x}_i\}$ LI $\implies \mathbf{x}_i \neq 0$
(iv) $B \subset$ LI $\implies B$ LI.
(v) Infinite P LI iff every finite subset of P is LI.

$\{\mathbf{x}_i\} \subset V$ **spans** V iff $\forall \mathbf{v} \in V \; \exists \alpha_i : \mathbf{v} = \alpha_i \mathbf{x}_i$.

If $X \subset V$, X LI, and X is maximal wrt set inclusion then X is a **Hamel basis**. Basis are not unique.

(i) X is a basis of V iff $\forall \mathbf{v} \in V \; \exists! \alpha_i : \mathbf{v} = \sum \alpha_i \mathbf{x}_i$.
(ii) X is a basis of V iff X LI and X spans V.
(iii) Every LI $A \subset V$ can be extended to a basis.
(iv) Every nontrivial V possesses a basis.
(v) If B a basis of V, $P \subset V$ LI then $\#P \leq \#B$.
(vi) B_1, B_2 basis of V implies $\#B_1 = \#B_2$.

The **dimension** of V is the cardinality of any basis B: $\dim V := \#B$.

Construction of a complement: $X \subset V$, a subspace. $\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ a basis for X. $\{\mathbf{e}_{k+1}, \ldots, \mathbf{e}_m\}$ basis extended to V. $Y :=$ linear combinations of $\mathbf{e}_{k+1} \ldots \mathbf{e}_m$. Then $V = X \oplus V \wedge X \cap Y = \{0\}$.

$T : X \to Y$ is a **linear transform** iff

$$\begin{aligned} T(x + y) &= T(x) + T(y) && \text{additive} \\ T(\alpha x) &= \alpha T(x) && \text{homogeneous} \\ T(0) &= 0 && \textit{implied} \end{aligned}$$

For a linear transform $T : V \to W$

(i) $\boldsymbol{\mathcal{N}(T)} := \ker T := \{\mathbf{v} \in V : T\mathbf{v} = 0\}$
(ii) $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are subspaces of V and W respectively.
(iii) T monomorphism (injective) iff $\mathcal{N}(T) = \{0\}$
(iv) T epimorphism (surjective) iff $\mathcal{R}(T) = W$
(v) **rank $\boldsymbol{T}$** $:= \dim \mathcal{R}(T)$
(vi) **nullity $\boldsymbol{T}$** $:= \dim \mathcal{N}(T)$

Rank and Nullity Theorem: If $\dim V < \infty$ then $\dim V = \operatorname{nullity} T + \operatorname{rank} T$.

(i) T is nonsingular (injective) iff $\mathcal{N}(T) = \{0\}$
(ii) T is a monomorphism (injective) iff $\operatorname{rank} T = \dim V$.
(iii) T is an epimorphism (surjective) iff $\operatorname{rank} T = \dim W$.
(iv) T is an isomorphism (bijective) iff $\dim V = \dim W = \operatorname{rank} T$.

X and Y are **isomorphic vector spaces** iff $\exists \iota : X \to Y$, a bijection.

(i) Finite dimensional spaces are iso to $\mathbb{R}^{\dim V}$, called the **model space**.
(ii) For $V = X \oplus Y$, X's complement and quotient space are iso given $\iota : Y \ni y \to [y] = y + X \in V/X$.
(iii) $X^\Omega \times Y^\Omega$ is iso to $(X \times Y)^\Omega$.

Inverses of isomorphic linear transforms are linear.

$P : V \to V$ is a **projection** iff

(i) $P^2 = PP = P$
(ii) $\exists X, Y : V = X \oplus Y, Tv = x$ where $v = x + y$

If X is a subspace of V then $\exists P : X = \mathcal{R}(P)$.

Given linear $T : V \to W$ and M, a subspace of $\mathcal{N}(T)$, then $\bar{T} : V/M \to W$ where $\bar{T}([v]) := T(v)$. If $M = \mathcal{N}(T)$ then $\bar{T}$ is monomorphic.

$\boldsymbol{L(X, Y)} := \{f : f : X \to Y, \text{linear}\}$ is a linear subspace of Y^X.

$\{X, +, \mathbb{F}, +, \times, *, \circ\}$ is a **linear algebra** iff $\{X, +, \mathbb{F}, +, \times, *, \}$ is a vector space, and $\circ : V \times V \to V$ satisfies

$(x \circ y) \circ z = x \circ (y \circ z)$	associative
$(\alpha x) \circ y = \alpha(x \circ y) = x \circ (\alpha y)$	commutative wrt scalars
$z \circ (x + y) = z \circ x + z \circ y$	left distributive
$(x + y) \circ z = x \circ z + y \circ z$	right distributive

Composition of linear transforms is linear.

$\boldsymbol{L(X)} := L(X, X)$ is a linear algebra.

Given a bijection $\iota : X \to Y$ where X is a vector space and Y is an arbitrary set, ι **induces** or **transfers** a vector space structure onto Y using

$$y_1 + y_2 := \iota\left(\iota^{-1}(y_1) + \iota^{-1}(y_2)\right)$$
$$\alpha y := \iota\left(\alpha\iota^{-1}(y)\right)$$

If $\mathbf{v} = \sum v_i \mathbf{e}_i$, $\mathbf{w} = \sum w_i \mathbf{g}_i$, and $T : V \ni \mathbf{v} =\to \mathbf{w} \in W$ then

$$T\mathbf{v} = T\left(\sum v_j \mathbf{e}_j\right) = \sum v_j T\mathbf{e}_j = \sum v_j \sum T_{ij}\mathbf{g}_i$$

Matrix-scalar multiplication, matrix addition, and matrix multiplication representations follow.

Matrix rank is the rank of the corresponding linear transformation. Matrix rank is equivalent to the number of LI column vectors.

Elements of $L(V, \mathbb{F})$, e.g. $f : V \to \mathbb{F}$ are called **linear functionals**.

$V^* := L(V, \mathbb{F})$ is the **algebraic dual** of $L(V, V)$.

Given V, a space, $\dim V < \infty$, $\{\mathbf{e}_i\}$, a basis then $\forall f \in V^*$

$$f(v) = f\left(\sum v_i \mathbf{e}_i\right) = \sum v_i f(\mathbf{e}_i) = \sum v_i l_i \text{ where } l_i := f(\mathbf{e}_i)$$

Given a finite basis $\{\mathbf{e}_i\} \in V$, $\left\{\mathbf{e}_j^*\right\}$ forms the **dual basis** in V^* where $\mathbf{e}_j^*(\mathbf{e}_i) := \delta_{ij}$. Corollary $\mathbf{\dim V = \dim V^*}$.

$l : V \times W \to \mathbb{F}$ is **bilinear** iff l is linear wrt each argument. For some basis, $l(v, w) = \sum\sum l_{ij} v_i w_j$.

$\boldsymbol{M(X, Y)}$ denotes the space of bilinear functionals.

A **duality pairing** $\langle v^*, v\rangle$ is a definite bilinear functional: $V^* \times V \ni (v^*, v) \to \boldsymbol{\langle v^*, v\rangle := v^*(v)} \in \mathbb{R}$

(i) $\langle v^*, v\rangle = 0 \ \forall v \implies v^* = 0$
(ii) $\langle v^*, v\rangle = 0 \ \forall v^* \implies v = 0$

$U^\perp := \{v^* \in V : \langle v^*, v\rangle = 0 \ \forall v \in U\}$ is called the **orthogonal complement** of U.

If $V = U \oplus W$ and $\dim V = n < \infty$ then

(i) $\dim U^\perp = \dim V - \dim U$
(ii) $V^* = U^* \oplus W^*$

Any vector space can be identified with a subspace of its bidual.

If $\dim V < \infty$ then V **and** V^{**} **are isomorphic** by the map

$$\iota : V \ni v \to \{V^* \ni v^* \to \langle v^*, v\rangle \in \mathbb{R}\} \in V^{**}$$

$T^{\mathrm{T}} : W^* \to V$ is the **transpose** of $T : V \to W$:

$$T^{\mathrm{T}}(w^*) := w^* T$$
$$\langle T^{\mathrm{T}} w^*, v\rangle = \langle w^*, Tv\rangle$$

Transpose properties:

(i) A linear $\implies A^{\mathrm{T}}$ linear
(ii) $(ST)^{\mathrm{T}} = T^{\mathrm{T}} S^{\mathrm{T}}$
(iii) $\mathrm{id}_V^{\mathrm{T}} = \mathrm{id}_{V*}$
(iv) $\left(T^{\mathrm{T}}\right)^{-1} = \left(T^{-1}\right)^{\mathrm{T}}$
(v) $\operatorname{rank} T = \operatorname{rank} T^{\mathrm{T}}$

$\{X, \mathrm{d}\}$ is a **metric space** given a **metric** $d : X \times X \to [0, \infty)$ obeying

$\mathrm{d}(x, y) = 0 \implies x = y$	positive definite
$\mathrm{d}(x, y) = \mathrm{d}(y, x)$	symmetric
$\mathrm{d}(x, z) \leq \mathrm{d}(x, y) + \mathrm{d}(y, z)$	triangle inequality

$\{V, \|\cdot\|\}$ is a **normed vector space** given a **norm** $\|\cdot\| : V \ni v \to \|v\| \in [0, \infty)$ obeying

$\|v\| = 0 \implies v = 0$	positive definite
$\|\alpha v\| = \|\alpha\| \, \|v\|$	homogenaity
$\|u + v\| \le \|u\| + \|v\|$	triangle inequality

Every normed vector space is a metric space given the **induced metric** $\mathrm{d}(x, y) := \|x - y\|$.

$\{V, (\cdot, \cdot)\}$ is an **inner product space** given an **inner product** $(\cdot, \cdot)_V : V \times V \ni (u, v) \to (u, v)_V \in \mathbb{R}$ obeying

$(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 (u_1, v) + \alpha_2 (u_2, v)$	linear in the first argument
$(u, v) = \overline{(v, u)}$	Hermitian $\implies$ antilinear in second argument
$(v, v) \ge 0 \wedge (v, v) = 0 \implies v = 0$	positive definite

Every inner product space is a normed vector space given the **induced Euclidean norm** $\|v\| := \sqrt{(v, v)}$.

Cauchy Inequality: $|(u, v)| \le \sqrt{(u, u)}\,\sqrt{(v, v)}$

The **Riesz map** is a linear and injective function.

$$R : V \ni u \to Ru := \{V \ni v \to (u, v)_V \in \mathbb{R}\} \in V^*$$

If V finite, R is a cannonical isomorphism between V and V^*. V^* constructs can be "brought back" to V:

(i) Dual basis brought back to a **cobasis**: $\mathbf{e}^j := R^{-1} \mathbf{e}_j^*$ yielding $\left(\mathbf{e}_i, \mathbf{e}^j\right) = \delta_{ij}$.
(ii) **Orthogonal complement** brought back: $X^\perp := R^{-1} X^\perp = \{y \in V : (y, x) = 0 \ \forall x \in X\}$
(iii) Given $A : X \to Y$, the *antilinear* **adjoint transformation** is $A^* := R_X^{-1} \circ A^T \circ R_Y$

A basis $\{\mathbf{e}_i\}$ is **orthonormal** iff $\left(\mathbf{e}_i, \mathbf{e}_j\right) = \delta_{ij}$. An orthonormal basis coincides with the cobasis.

For $A \in L(V, W)$, the adjoint transformation $A^* \in L(W, V)$ is unique and satisfies $(v, A^* w)_V = (Tv, w)_W$.

Adjoint properties:

(i) $(ST)^* = T^* S^*$
(ii) $\mathrm{id}_V^* = \mathrm{id}_V$
(iii) $(T^*)^{-1} = \left(T^{-1}\right)^*$
(iv) $\operatorname{rank} T = \operatorname{rank} T^*$