Homework ~~# 24~~ # 27

26.17 bc, 21, 23 abde, 26, 28, 33, 35, 36

26.17 b

$$4a^2 y_{,\xi\eta} = H(\xi,\eta)$$

$$4a^2 y_{,\xi} = \int_{c(\xi)}^{\eta} H(\xi, v)\,dv + f_1(\xi)$$

$$4a^2 y = \int_{d(\eta)}^{\xi} \int_{c(u)}^{\eta} H(u,v)\,dv\,du + f(\xi) + g(\eta) \qquad f' = f_1$$

We have the flexibility to select $c(u)$ and $d(\eta)$!

$$4a^2 y(x,t) = \int_{d(x-at)}^{x+at} \int_{c(u)}^{x-at} H(u,v)\,dv\,du + f(x+at) + g(x-at)$$

$$4a^2 y(x,0) = \int_{d(x)}^{x} \int_{c(u)}^{x} H(u,v)\,dv\,du + f(x) + g(x)$$

Selecting $d(x) = x$ we reduce

1° the initial condition to $4a^2 y(x,0) = f(x) + g(x) = 4a^2 F(x)$

2° the general solution to :

$$4a^2 y(x,t) = \int_{x-at}^{x+at} \int_{c(u)}^{x-at} H(u,v)\,dv\,du + f(x+at) + g(x-at)$$

Differentiating in t, using the Leibnitz rule (page 18):

$$4a^2\frac{\partial y}{\partial t}(x,t) = \int_{x-at}^{x+at} \frac{d}{dt}\int_{c(u)}^{x-at} H(u,v)\,dv\,du$$

$$+\, a\int_{c(x+at)}^{x-at} H(x+at,v)\,dv + a\int_{c(x-at)}^{x-at} H(x-at,v)\,dv$$

$$+\, af'(x+at) - ag'(x-at)$$

$$= \int_{x-at}^{x+at} -a\,H(u, x-at)\,du + \dots$$

So:

$$4a^2\frac{\partial y}{\partial t}(x,0) = a\int_{c(x)}^{x} H(x,v)\,dv + a\int_{c(x)}^{x} H(x,v)\,dv$$

$$+\, af'(x) - ag'(x)$$

Selecting $c(x) = x$ we get

$$4a^2\frac{\partial y}{\partial t}(x,0) = af'(x) - ag'(x) = 4a^2G(x)$$

$$\therefore\ af(x) - ag(x) = 4a^2\int_0^x G(s)\,ds + 4a^2C$$

So the final system is

$$f(x) + g(x) = 4a^2F(x)$$
$$f(x) - g(x) = 4a\int_0^x G(s)\,ds + 4aC$$

$$\therefore\ f(x) = 2a\left(aF(x) + \int_0^x G(s)\,ds + C\right)$$
$$g(x) = 2a\left(aF(x) - \int_0^x G(s)\,ds - C\right)$$

Final formula

$$y(x,t) = \frac{1}{4a^2}\int_{x-at}^{x+at}\int_{u}^{x-at} H(u,v)\,dv\,du + \frac{1}{2a}\left(a\,F(x+at) + \int_0^{x+at} G(s)\,ds \cancel{+c}\right)$$

$$+ \frac{1}{2a}\left(a\,F(x-at) - \int_0^{x-at} G(s)\,ds \cancel{-c}\right)$$

$$= \frac{1}{4a^2}\int_{x-at}^{x+at}\int_{u}^{x-at} H(u,v)\,dv\,du + \frac{1}{2}\left(F(x+at) - F(x-at)\right)$$

$$+ \frac{1}{2a}\int_{x-at}^{x+at} G(s)\,ds$$

#

26.17c)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \quad /\cdot r$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{a^2}\frac{\partial}{\partial t^2}(ru)$$

and $$\frac{\partial^2}{\partial r^2}(ru) = \frac{\partial}{\partial r}\left(u + r\frac{\partial u}{\partial r}\right) = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}$$

$$= 2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}$$

whereas

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) = \frac{1}{r}\left(2r\frac{\partial u}{\partial r} + r^2\frac{\partial^2 u}{\partial r^2}\right) = 2\frac{\partial u}{\partial r} + r\frac{\partial^2 u}{\partial r^2}$$

#

26.21

$$a^2 y_{,xx} = y_{,tt} \qquad 0<x<l,\ t>0$$
$$y(l,t) = y(x,0) = y_{,t}(x,0) = 0$$
$$y(0,t) = \varepsilon \sin \omega(t)$$

Seek $\quad y(x,t) = X(x)\,T(t) \qquad y(0,t) = \overbrace{X(0)}^{\text{a number}}\,T(t)$

$$a^2 X'' T = X T''$$

$$\frac{X''}{X} = \frac{1}{a^2}\frac{T''}{T} = c$$

So $\quad T'' - ca^2 T = 0$

$$y(0,t) = \varepsilon \sin \omega t \quad \Rightarrow \quad T(t) = \varepsilon \sin \omega t$$

$$\therefore\ -ca^2 = \omega^2 \quad \Rightarrow \quad T(t) = \overset{0}{\cancel{C}} \cos \omega t + \overset{\varepsilon}{\cancel{D}} \sin \omega t$$

$$\therefore\ c = -\left(\frac{\omega}{a}\right)^2 \Rightarrow \quad X'' + \left(\frac{\omega}{a}\right)^2 X = 0$$

$$\Rightarrow \quad X(x) = A \cos \frac{\omega}{a} x + B \sin \frac{\omega}{a} x$$

$$X(0)\,\varepsilon \sin \omega t = \varepsilon \sin \omega t \quad \Rightarrow \quad A = 1$$

$$X(l) = 0 \quad \Rightarrow \quad \cos \frac{\omega l}{a} + B \sin \frac{\omega l}{a} = 0$$

$$\therefore\ B = -\cot \frac{\omega l}{a}$$

Finally : $\quad y(x,t) = \left(\cos \frac{\omega x}{a} - \cot \frac{\omega l}{a} \sin \frac{\omega x}{a}\right) \varepsilon \sin \omega t$

Auxiliary problem:

$$\begin{cases} a^2 y_{,xx} = y_{,tt} \\ y(0,t) = y(l,t) = 0 \\ y(x,0) = 0 \\ y_{,t}(x,0) = G(x) \end{cases}$$

Using separation of variables we end up with the solution

$$y(x,t) = \sum_{n=1}^{\infty} G_n \sin\frac{n\pi x}{l} \sin\frac{n\pi a t}{l}$$

so: $$y_{,t} = \sum_{n=1}^{\infty} \frac{n\pi a}{l} G_n \sin\frac{n\pi x}{l} \cos\frac{n\pi a t}{l}$$

so: $$G(x) = y_{,t}(x,0) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} G_n \sin\frac{n\pi x}{l}$$

As $$\int_0^l \sin^2\frac{n\pi x}{l}\,dx = \frac{l}{n\pi}\int_0^{n\pi} \sin^2 t\,dt = \frac{l}{n\pi}\,\frac{t}{2}\Big|_0^{n\pi} = \frac{l}{2}$$

$$\frac{n\pi x}{l} = t$$

$$\cos 2t = 1 - 2\sin^2 t \Rightarrow \sin^2 t = \frac{1-\cos 2t}{2}$$

we get

$$\frac{n\pi a}{\cancel{l}} G_n \underbrace{\cancel{\int_0^l \sin^2\frac{n\pi x}{l}\,dx}}_{\frac{l}{2}} = \int_0^l G(x)\sin\frac{n\pi x}{l}\,dx$$

and consequently $$G_n = \frac{2}{n\pi a}\int_0^l G(x)\sin\frac{n\pi x}{l}\,dx$$

$$\int_0^l \cos\frac{\omega x}{a}\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_0^l (\sin\alpha + \sin\beta)\,dx$$

$$\left.\begin{aligned} 2\frac{n\pi x}{l} &= \alpha+\beta \\ 2\frac{\omega x}{a} &= \alpha-\beta \end{aligned}\right\} \Rightarrow \begin{aligned} \alpha &= \left(\frac{n\pi}{l}+\frac{\omega}{a}\right)x \\ \beta &= \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)x \end{aligned}$$

$$= -\frac{1}{2}\left\{\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1}\cos\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)x\Big|_0^l + \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\cos\left(\frac{n\pi}{l}-\frac{\omega}{a}\right)x\Big|_0^l\right\}$$

$$= -\frac{1}{2}\left\{\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1}\left[\cos\left(n\pi+\frac{\omega l}{a}\right)-1\right] + \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\left[\cos\left(n\pi-\frac{\omega l}{a}\right)-1\right]\right\}$$

$$= -\frac{1}{2}\left[\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1} + \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\right]\left[\cos\left(n\pi+\frac{\omega l}{a}\right)-1\right] =: A_n$$

$$\int_0^l \sin\frac{\omega x}{a}\sin\frac{n\pi x}{l}\,dx = -\frac{1}{2}\int_0^l \cos\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)x - \cos\left(\frac{n\pi}{l}-\frac{\omega}{a}\right)x\,dx$$

$$= -\frac{1}{2}\left\{\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1}\sin\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)x\Big|_0^l - \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\sin\left(\frac{n\pi}{l}-\frac{\omega}{a}\right)x\Big|_0^l\right\}$$

$$= -\frac{1}{2}\left\{\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1}\sin\left(n\pi+\frac{\omega l}{a}\right) - \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\sin\left(n\pi-\frac{\omega l}{a}\right)\right\}$$

$$= -\frac{1}{2}\left[\left(\frac{n\pi}{l}+\frac{\omega}{a}\right)^{-1} + \left(\frac{n\pi}{l}-\frac{\omega}{a}\right)^{-1}\right]\sin\left(n\pi+\frac{\omega l}{a}\right) =: B_N$$

So, the find solution is

$$y(x,t) = \varepsilon\left(\cos\frac{\omega x}{a} - \cot\frac{\omega l}{a}\sin\frac{\omega x}{a}\right)\sin\omega t - \sum_{n=1}^{\infty} G_n \sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l},$$

where $G_n = \frac{2\varepsilon\omega}{n\pi a}\left(A_n - \cot\frac{\omega l}{a}B_n\right)$

\#

26.23

a)

$$\alpha^2 \Delta w = w_{,tt} \qquad \Delta w = w_{,xx} + w_{,yy}$$

$$w = \varphi(x,y)\, T(t)$$

$$\alpha^2 \Delta\varphi T = \varphi T''$$

$$\frac{\Delta\varphi}{\varphi} = \frac{T''}{\alpha^2 T} = c$$

It must be $c = -k^2$, so the following equation would have a solution:

$$\begin{cases} \Delta\varphi + k^2\varphi = 0 & \text{in } D \\ \varphi \equiv 0 & \text{on } \partial D \end{cases}$$

where $D = (0,a)\times(0,a)$

Separating further variables: $\varphi(x,y) = X(x)Y(y)$

$$X''Y + XY'' + k^2 XY = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} + k^2 = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} - k^2 = -l^2$$

$$X'' + l^2 X = 0 \quad X(0) = X(a) = 0 \Rightarrow l = l_n = \frac{n\pi}{a}, \quad n = 1, 2, \ldots$$

$$X_n(x) = C_n \sin\frac{n\pi}{a}x$$

$$\frac{Y''}{Y} = l_n^2 - k^2 \quad \Rightarrow \quad Y'' + (k^2 - l_n^2) Y = 0$$

$$\therefore \quad k = k_{mn}, \text{ where } \quad k_{mn}^2 - l_n^2 = \left(\frac{m\pi}{a}\right)^2, \quad m = 1, 2, \ldots$$

and $Y_m(y) = D_m \sin \frac{m\pi}{a} y$

Consequently $\quad \varphi = \varphi_{mn} = C_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$

$$\text{and} \quad k_{nm}^2 = \frac{(n^2+m^2)\pi^2}{a^2}$$

Finally $\quad T'' + \alpha^2 k_{nm}^2 T = 0$

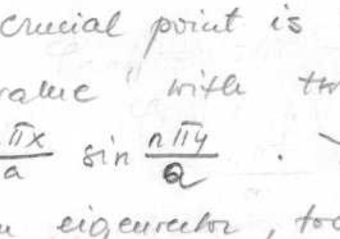
implies that $\quad T(t) = A_{nm} \cos \alpha k_{nm} t + B_{nm} \sin \alpha k_{nm} t$

and therefore the final solution is

$$w(x,y,t) = \sum_{n,m=1}^{\infty} \left(A_{nm} \cos \alpha k_{nm} t + B_{nm} \sin \alpha k_{nm} t \right) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$$

#

b)



The crucial point is that $k_{nm} = \frac{\sqrt{n^2+m^2}\,\pi}{a}$ is a double eigenvalue with two eigenvectors: $\sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a}$ and $\sin\frac{m\pi x}{a} \sin\frac{n\pi y}{a}$. Thus $\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a} - \sin\frac{n\pi y}{a}\sin\frac{m\pi x}{a}$ is an eigenvector, too, and it vanishes along $y=x$! The minimum eigenvalue obtained that way is for $n=1, m=2$ $\left(\frac{\sqrt{5}\,\pi}{a}\right)$.

##

d) $$\begin{cases} \Delta\varphi + k^2\varphi = 0 & \text{in } D \\ \varphi \equiv 0 & \text{on } D \end{cases}$$

$$D = \{(x,y) : x^2+y^2 < a^2\}$$

Separating variables $\varphi = R(r)\,\Theta(\theta)$

$$\Delta\varphi = \varphi_{,rr} + \frac{1}{r}\varphi_{,r} + \frac{1}{r^2}\varphi_{,\theta\theta}$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + k^2R\Theta = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + k^2 = -\frac{1}{r^2}\frac{\Theta''}{\Theta} \quad / \cdot r^2$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} + k^2r^2 = -\frac{\Theta'}{\Theta} = C$$

We seek periodic solutions of $\Theta'' + C\Theta = 0$
which requires $C = k^2 = k_n^2$, $k_n = n$,
Consequently

$$\Theta(\theta) = C_n \cos n\theta + D_n \sin n\theta \qquad n = 0, 1, 2, \ldots$$

The equation for $R(r)$ then becomes:

$$r^2R'' + rR' + (k^2r^2 - n^2)R = 0$$

Introducing a new variable $\bar{r} = kr$ we have

$$\frac{dR}{dr} = \frac{dR}{d\bar{r}}\frac{d\bar{r}}{dr} = k\frac{dR}{d\bar{r}}$$

and $$\frac{d^2R}{dr^2} = k^2\frac{d^2R}{d\bar{r}^2}$$

Consequently, we get the Bessel equation

$$\bar{r}^2 \frac{d^2R}{d\bar{r}^2} + \bar{r}\frac{dR}{d\bar{r}} + (\bar{r}^2 - n^2)R = 0$$

with the general solution:

$$R(\bar{r}) = A_n J_n(\bar{r}) + B_n Y_n(\bar{r})$$

$R(0)$ must be finite which implies that $B_n = 0$

For a nontrivial solution it must be

$$J_n(ka) = 0$$

which means that $ka = z_{nm}$ (m-th root of Bessel functions $J_n(x)$), $m = 1, 2, \ldots$

The modes therefore are of the folowing form

$$\varphi_{nm}(r,\theta) = J_n\left(\frac{z_{nm}}{a} r\right)\left(C_{nm}\cos n\theta + D_{nm}\sin n\theta\right)$$

$$n = 0, 1, 2, \ldots, \quad m = 1, 2, \ldots$$

and the corresponding eigenfrequencies $k_{nm} = \frac{z_{nm}}{a}$

For $n = 0$ we get the axisymmetric modes (the only ones listed in the book!)

Solution of the transient problem with axisymmetric data will be

$$w(r,\Theta,t) = \sum_{m=1}^{\infty} J_o\left(\frac{z_{om}}{a} r\right)\left(C_m \cos \alpha k_{om} t + D_m \sin \alpha k_{om} t\right)$$

$w(r,\Theta,0) = 0$ implies $C_m = 0 \quad m = 1, 2, \ldots$ and therefore

$$w_{,t}(r,\Theta,t) = \sum_{m=1}^{\infty} J_o\left(\frac{z_{om}}{a} r\right) \alpha k_{om} D_m \cos \alpha k_{om} t$$

which implies

$$f(r) = w_{,t}(r,\Theta,0) = \sum_{m=1}^{\infty} J_o\left(\frac{z_{om}}{a} r\right) \alpha k_{om} D_m$$

Finally:

$$D_m = \frac{\int_0^a J_o\left(\frac{z_{om}}{a} r\right) f(r) r\,dr}{\alpha k_{om} \int_0^a J_o^2\left(\frac{z_{om} r}{a}\right) r\,dr}$$

as the Bessel equ falls into the Sturm-Liouville category (see discussion on page 539).

e)

$$k_{nm} = \frac{\sqrt{n^2+m^2}\,\pi}{a} \qquad \text{for the square membrane}$$

$$k_{nm} = \frac{z_{nm}}{a} \qquad \text{for the circular membrane}$$

these are lower (easier to excite?).

26.26

$$\sum_1^\infty \left(\frac{r}{a}\right)^n \cos n(\varphi-\theta) \stackrel{?}{=} \frac{a^2 - ar\cos(\varphi-\theta)}{a^2 - 2ar\cos(\varphi-\theta) + r^2} - 1$$

$$\sum_1^\infty \left(\frac{r}{a}\right)^n \cos n(\varphi-\theta) = Re \sum_1^\infty \left(\frac{r}{a}\right)^n e^{in(\varphi-\theta)}$$

Denoting $z = \frac{r}{a} e^{i(\varphi-\theta)}$

$$= Re \sum_1^\infty z^n = Re \frac{z}{1-z} \qquad \text{for } |z| < 1$$

$$= Re \frac{\frac{r}{a} e^{i(\varphi-\theta)}}{1 - \frac{r}{a} e^{i(\varphi-\theta)}} = Re \frac{r e^{i(\varphi-\theta)}}{a - r\cos(\varphi-\theta) - ir\sin(\varphi-\theta)}$$

$$= Re \frac{r\big(\cos(\varphi-\theta) + i\sin(\varphi-\theta)\big)\big(a - r\cos(\varphi-\theta) + ir\sin(\varphi-\theta)\big)}{[a - r\cos(\varphi-\theta)]^2 + r^2\sin^2(\varphi-\theta)}$$

$$= \frac{ra\cos(\varphi-\theta) - r^2\cos^2(\varphi-\theta) - r^2\sin^2(\varphi-\theta)}{a^2 + r^2 - 2ar\cos(\varphi-\theta)}$$

$$= \frac{ra\cos(\varphi-\theta) - r^2}{a^2 + r^2 - 2ar\cos(\varphi-\theta)}$$

$$= \frac{a^2 - ar\cos(\varphi-\theta)}{a^2 + r^2 - 2ar\cos(\varphi-\theta)} - 1$$

#

26.28

a) $u_{,rr} + \frac{1}{r} u_{,r} + u_{,zz} = 0 \qquad r < a, \quad 0 < z < \infty$

$$u(r,z) = R(r) Z(z)$$

$$R'' Z + \frac{1}{r} R' Z + R Z'' = 0$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} = -\frac{Z''}{Z} = c$$

$$R'' + \frac{1}{r} R' - cR = 0$$

$$r^2 R'' + r R' - c r^2 R = 0$$

For $-c = k^2$ we get

$$r^2 R'' + r R' + k^2 r^2 R = 0$$

or introducing a new variable $\bar{r} = kr$

$$\bar{r}^2 \frac{d^2 R}{d\bar{r}^2} + \bar{r}\frac{dR}{d\bar{r}} + \bar{r}^2 R = 0 \qquad \text{(Bessel eqn. of 0 order)}$$

Thus the solution is:

$$R(r) = A J_0(kr) + B Y_0(kr)$$

$R(0)$ finite $\Rightarrow B = 0$

$R(a) = 0 \Rightarrow ka = z_n$ the n-th root of $J_0(r)$

Thus $k = k_n = \frac{z_n}{a}$

Consequently,

$$\frac{Z''}{Z} = k_n^2 \quad \Rightarrow \quad Z'' - k_n^2 Z = 0$$

$$Z(z) = C e^{k_n z} + D e^{-k_n z}$$

Requesting $Z(\infty) = 0$ we get $C = 0$.

Finally

$$u(r,z) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{z_n r}{a}\right) e^{-\frac{z_n z}{a}}$$

Implementing the BC

$$u(r,0) = \sum_{n=1}^{\infty} c_n J_0\left(\frac{z_n r}{a}\right) = 100$$

we get

$$c_n = \frac{100 \int_0^a J_0\left(\frac{z_n r}{a}\right) r \, dr}{\int_0^a J_0^2\left(\frac{z_n r}{a}\right) r \, dr}$$

\#

b)
$$\begin{cases} u_{,rr} + \frac{1}{r} u_{,r} = \frac{1}{\alpha^2} u_{,t} \\ u(r,0) = 0 \\ u(a,t) = 50 \end{cases}$$

Step 1 Look for a steady-state solution satisfying the BC.

By inspection: $u(r,t) \equiv 50$

Step 2 Look for a solution to the transient problem with homogeneous BC and an IC matching the steady-state solution

$$\begin{cases} u_{,rr} + \frac{1}{r} u_{,r} = \frac{1}{\alpha^2} u_{,t} \\ u(r,0) = 50 \\ u(a,t) = 0 \end{cases}$$

$$u(r,t) = \sum_1^{\infty} D_n J_0(k_n r) e^{-\alpha^2 k_n^2 t}$$

where $D_n = \frac{2}{a^2 [J_1(k_n a)]^2} \int_0^a 50 J_0(k_n r) r \, dr$

(compare page 559)

By superposition, the final solution is

$$u(r,t) = 50 - \sum_1^{\infty} D_n J_0(k_n r) e^{-\alpha^2 k_n^2 t}$$

#

26.28C

$$\alpha^2 \Delta u = u_{,t} \qquad r < a, \quad t > 0$$
$$u(r,\Theta,0) = u_0$$
$$u(a,\Theta,t) = u_0 + u_0 \cos\Theta$$

Step 1 Look for steady-state solution satisfying the nonhomogeneous BC

$$\begin{cases} \alpha^2 \Delta u = 0 \\ u(a,\Theta,t) = u_0 + u_0 \cos\Theta \end{cases}$$

$$\Delta u = u_{,rr} + \frac{1}{r} u_{,r} + \frac{1}{r^2} u_{,\Theta\Theta} = 0$$

$$u(r,\Theta) = R(r)\,\Theta(\Theta)$$

$$R''\Theta + \frac{1}{r} R'\Theta + \frac{1}{r^2} R\Theta'' = 0 \qquad \Big/ \frac{r^2}{R\Theta}$$
$$r^2 \frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = c$$

$$\Theta'' + c\Theta = 0$$

For periodic solutions $\quad c = n^2 \implies \Theta = C\cos n\Theta + D \sin n\Theta$

Matching BC, we can restrict ourselves to two modes only

$\Theta = c_0 \qquad (n=0)$

$\Theta = c_1 \cos\Theta \qquad (n=1)$

For $n=0$ $\qquad r^2 R'' + r R' = 0$

$\qquad R(r) = r^\alpha \qquad \alpha(\alpha-1) + \alpha = 0$

$\qquad\qquad \alpha^2 = 0 \implies \alpha = 0$

$\qquad R(r) = \text{const}$

For $n=1$ $\qquad r^2R'' + rR' - R = 0$

$$R(r) = r^{\alpha} \Rightarrow \quad \alpha(\alpha-1) + \alpha - 1 = 0$$

$$\alpha^2 = 1$$

$$\alpha = \pm 1$$

$$R(r) = Cr + D\frac{1}{r}$$

$$R(0) \text{ finite} \Rightarrow D = 0$$

So the final solution is

$$u(r,\theta) = C_0 + C_1 r\cos\theta$$

$$u(a,\theta) = C_0 + C_1 a\cos\theta = u_0 + u_0\cos\theta \Rightarrow$$

$$\underline{u(r,\theta) = u_0 + \frac{u_0}{a} r\cos\theta}$$

<u>Step 2</u> Look for transient solution satisfying the homogeneous BC and nonhomogeneous IC, matching the steady-state solution

$$\begin{cases} \alpha^2 \Delta u = u_{,t} & r<a,\ t>0 \\ u(r,\theta,0) = \frac{u_0}{a} r\cos\theta \\ u(a,\theta,t) = 0 \end{cases}$$

$$u_{,rr} + \frac{1}{r}u_{,r} + \frac{1}{r^2}u_{,\theta\theta} = \frac{1}{\alpha^2}u_t$$

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

$$R''\Theta T + \frac{1}{r} R'\Theta T + \frac{1}{r^2} R\Theta'' T = \frac{1}{\alpha^2} R\Theta T' \Big/ R\Theta T$$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \frac{1}{\alpha^2}\frac{T'}{T} = c$$

$$r^2 \frac{R''}{R} + r\frac{R'}{R} - cr^2 = -\frac{\Theta''}{\Theta} = d$$

$$\therefore \quad \Theta'' + d\Theta = 0$$

For periodic solutions $d = n^2$, $n = 0, 1, 2, \ldots$

$$\Theta(\theta) = C\cos n\theta + D\sin n\theta$$

For $c = -k^2$

$$\therefore \quad r^2 R'' + rR' + (k^2 r^2 - n^2)R = 0$$

$$R(r) = A J_n(kr) + B Y_n(kr)$$

$$R(0) < \infty \Rightarrow B = 0$$

$R(a) = 0 \Rightarrow ka = z_{mn}$, the m-th rooth of J_n

$$\therefore \quad k = k_{mn} = \frac{z_{mn}}{a}$$

Finaly $\quad \frac{1}{\alpha^2}\frac{T'}{T} = -k^2$

$$T' + \alpha^2 k^2 T = 0 \Rightarrow T(t) = Ce^{-\alpha^2 k^2 t}$$

Final solution

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_n(k_{nm} r)\left(C_{mn}\cos n\theta + D_{mn}\sin n\theta\right) e^{-\alpha^2 k_{nm}^2 t}$$

Matching IC

$$u(r,\theta,0) = \sum_{m=1}^{\infty} C_m J_1(k_{1m} r)\cos\theta = \frac{u_0}{a} r\cos\theta$$

where

$$C_m = \frac{\int_0^a \frac{u_0}{a} r\, J_1(k_{1m} r)\, r\, dr}{\int_0^a J_1^2(k_{1m} r)\, r\, dr}$$

and the final solution is:

$$u(r,\theta,t) = u_0 + \frac{u_0}{a} r\cos\theta - \sum_{m=1}^{\infty} C_m J_1(k_{1m} r)\cos\theta$$

#

26.33

a) $\Delta u = 0 \qquad u = u(r,\theta) = R(r)\Theta(\theta)$

$$\Delta u = u_{,rr} + \frac{1}{r}u_{,r} + \frac{1}{r^2}u_{,\theta\theta} = 0$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \;\Big/ \frac{r^2}{R\Theta}$$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = c$$

$$\Theta'' + c\Theta = 0 \qquad \Theta(0) = \Theta(\alpha) = 0$$

For a nontrivial solution to exist it must be $c = k^2$

then $\Theta = A\sin k\theta + B\cos k\theta$

$\Theta(0) = 0 \Rightarrow B = 0 \qquad \Theta(\alpha) = 0 \Rightarrow k\alpha = n\pi, \; n = 1, 2$

$\therefore \; k = k_n = n\frac{\pi}{\alpha}, \quad n = 1, 2, \ldots$

So:

$$r^2 R'' + rR' - k_n^2 R = 0$$

$R(r) = r^\alpha \qquad \alpha(\alpha-1) + \alpha - k_n^2 = 0 \qquad \alpha = \pm k_n$

$R(r) = C r^{-k_n} + D r^{k_n}$

$R(0) < \infty \Rightarrow C = 0$

Finally: $u(r,\theta) = \sum_{n=1}^{\infty} c_n \, r^{k_n} \sin k_n \theta$

Imposing BC: $u(a,\theta) = \sum_{n=1}^{\infty} c_n \, a^{k_n} \sin k_n \theta = 100$

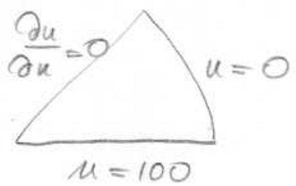
$$\therefore \quad c_n = \frac{\int_0^\alpha 100 \sin k_n\theta \, d\theta}{a^{k_n} \int_0^\alpha \sin^2 k_n\theta \, d\theta}$$

b) same as a) except that BC on $\Theta(\theta)$ are $\Theta(0) = \Theta'(\alpha) = 0$

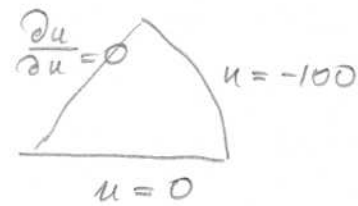
$\Theta'(\alpha) = 0 \Rightarrow k \cos k\alpha = 0 \Rightarrow k = 0$ or $k\alpha = \frac{\pi}{2} + n\pi, \; n = 1, 2, \ldots$

The rest of the procedure the same.

c) Use the superposition



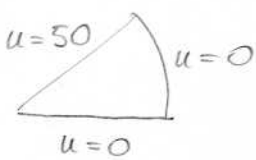
$=$ 100 $+$



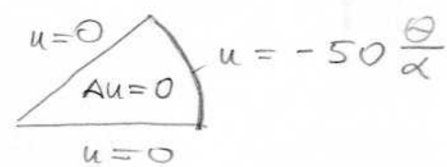
d) use a particular solution

$$u(r, \theta) = 50 \frac{\theta}{\alpha} \qquad (\Delta u = 0 \, !)$$

and the superposition principle :



$=$ $50 \frac{\theta}{\alpha}$ $+$



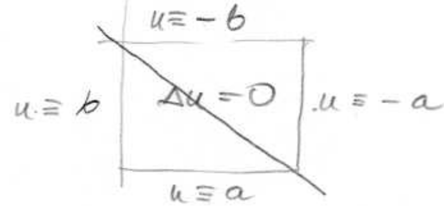
e) Use a particular solution

$$u(r, \theta) = 50\, \theta$$

and the superposition principle as in d).

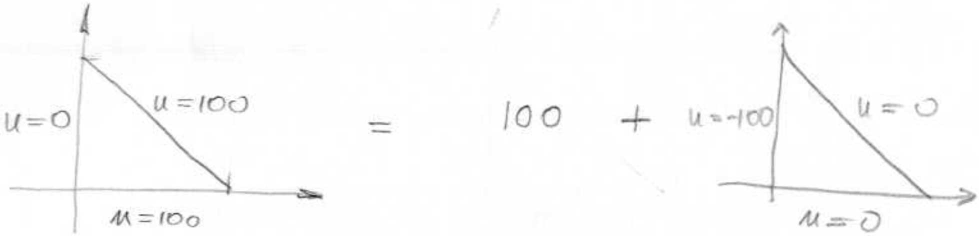
26.35

Step 1: Notice that the solution on the square domain

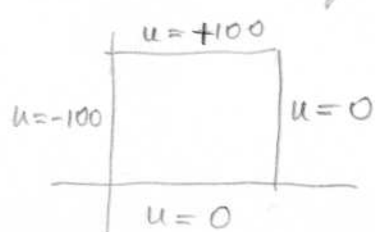


vanishes along the diagonal

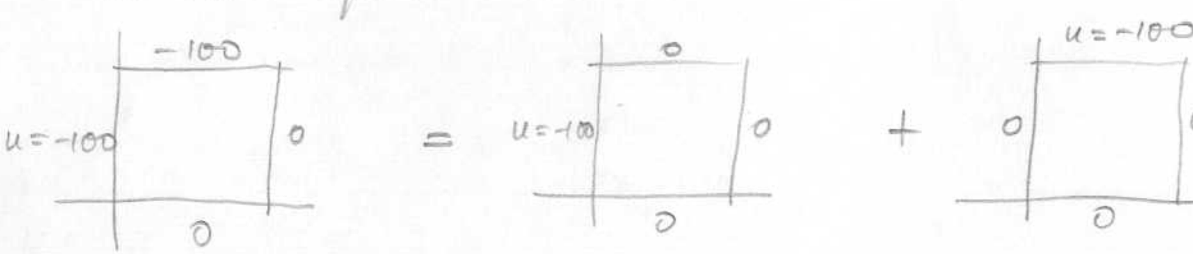
Step 2: use the superposition principle



where the solution to the second problem is constructed via the square domain problem



Step 3 Use the superposition principle to solve the problem on the square domain



26.36

a) $u_{,rr} + \frac{1}{r} u_{,r} + \frac{1}{r^2} u_{,\theta\theta} = 0 \qquad u(r,\theta) = R(r)\Theta(\theta)$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Theta''}{\Theta} = k^2$$

Periodicity in $\theta \Rightarrow \quad k = 0, 1, 2,$

$$\Theta = A\cos k\theta + B\sin k\theta$$

$$r^2 R'' + r R' - k^2 R = 0$$

$$R(r) = C r^k + D r^{-k}$$

$R(a) = C a^k + D a^{-k} = 1$

$R(b) = C b^k + D b^{-k} = 0$

$$\therefore \quad C = \frac{\begin{vmatrix} 1 & a^{-k} \\ 0 & b^{-k} \end{vmatrix}}{\begin{vmatrix} a^k & a^{-k} \\ b^k & b^{-k} \end{vmatrix}} = \frac{b^{-k}}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k}$$

$$D = \frac{\begin{vmatrix} a^k & 1 \\ b^k & 0 \end{vmatrix}}{\begin{vmatrix} & \end{vmatrix}} = -\frac{b^k}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k}$$

Similarly, requesting

$R(a) = 0$

$R(b) = 1$

we get $$C = \frac{\begin{vmatrix} 0 & a^{-k} \\ 1 & b^{-k} \end{vmatrix}}{\begin{vmatrix} & \end{vmatrix}} = -\frac{a^{-k}}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k}$$

$$D = \frac{\begin{vmatrix} a^k & 0 \\ b^k & 1 \end{vmatrix}}{\begin{vmatrix} & \end{vmatrix}} = \frac{a^k}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k}$$

Finally, we can represent the general solution in the form

$$R(r) = C\,\frac{\left(\frac{r}{b}\right)^k - \left(\frac{b}{r}\right)^k}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k} + D\,\frac{\left(\frac{a}{r}\right)^k - \left(\frac{r}{a}\right)^k}{\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k}$$

or

$$R(r) = C\left[\left(\frac{r}{b}\right)^k - \left(\frac{b}{r}\right)^k\right] + D\left[\left(\frac{a}{r}\right)^k - \left(\frac{r}{a}\right)^k\right]$$

(constants C and D redefined twice !)

By superposition, solution to the original problem is

$$u(r,\theta) = \sum_{k=0}^{\infty}\left(A_k^a \cos k\theta + B_k^a \sin k\theta\right)\left[\left(\frac{r}{b}\right)^k - \left(\frac{b}{r}\right)^k\right]$$
$$+ \sum_{k=0}^{\infty}\left(A_k^b \cos k\theta + B_k^b \sin k\theta\right)\left[\left(\frac{a}{r}\right)^k - \left(\frac{r}{a}\right)^k\right]$$

$$u(a,\theta) = \sum_{k=0}^{\infty}\left(A_k^a \cos k\theta + B_k^a \sin k\theta\right)\left[\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k\right] = f(\theta)$$

$$A_k^a = \frac{\int_0^{2\pi} \cos k\theta \, f(\theta)\, d\theta}{\left[\left(\frac{a}{b}\right)^k - \left(\frac{b}{a}\right)^k\right]^{-1} \int_0^{2\pi} \cos^2 k\theta \, d\theta}$$

$$B_k^a = \frac{\int_0^{2\pi} \sin k\theta \, f(\theta)\, d\theta}{\left[\qquad\qquad\right] \int_0^{2\pi} \sin^2 k\theta \, d\theta}$$

$$u(b,\theta) \equiv 0 \quad \Rightarrow \quad A_k^b = B_k^b \equiv 0$$

b) same as a), except that we have both A_k^a, B_k^a and A_k^b, B_k^b terms and A_k^b, B_k^b are expressed in terms of $g(\theta)$

c) We have to solve eigenvalue problems

$$u_{,rr} + \frac{1}{r} u_{,r} + \frac{1}{r^2} u_{,\theta\theta} = -\lambda^2 u \quad \text{(eigenvalues)}$$

$$u(r,\theta) = R(r)\,\Theta(\theta)$$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = -\lambda^2 R\Theta \quad \Big/ \frac{r^2}{R\Theta}$$

$$\frac{r^2R'' + rR'}{R} + \frac{\Theta''}{\Theta} + \lambda^2 r^2 = 0$$

so $\quad \dfrac{\Theta''}{\Theta} = \text{const} = c$

$$\Theta'' - c\Theta = 0$$

Periodicity $\Rightarrow \quad c = -k^2 \quad k = 0, 1, 2, \ldots$

$$\Theta(\theta) = A\cos k\theta + B\sin k\theta$$

$$\therefore \quad \frac{r^2R'' + rR'}{R} + \lambda^2 r^2 - k^2 = 0$$

$$r^2R'' + rR' + (\lambda^2 r^2 - k^2)R = 0 \qquad \text{Bessel eqn.}$$

$$R(r) = C\,J_k(\lambda r) + D\,Y_k(\lambda r)$$

$$R(0) < \infty \implies D = 0$$

$R(1) = 0 \implies \lambda = z_{km}$, m-th root of Bessel function of order k
$\quad \lambda = \lambda_{km}$

Expanding solution u into the series of eigenfunctions:

$$u(r,\theta) = \sum_{k=0}^{\infty} \sum_{m=1}^{\infty} J_k(\lambda_{km} r)\left(A_{km}\cos k\theta + B_{km}\sin k\theta\right)$$

(Notice that each of the eigenvalues, except for $k = 0$, is a double eigenvalue!). Finally, applying Laplace operator to both sides we get

$$\sum_{k=0}^{\infty} \sum_{m=1}^{\infty} \lambda_{km}\, J_k(\lambda_{km} r)\left(A_{km}\cos k\theta + B_{km}\sin k\theta\right) \equiv 1$$

As the right-hand side is θ-independent, we get

$$\sum_{m=1}^{\infty} \lambda_{om} A_{om} J_0(\lambda_{om} r) = 1$$

where

$$A_{om} = \frac{\int_0^1 1\, J_0(\lambda_{om} r)\, r\, dr}{\lambda_{om} \int_0^1 J_0^2(\lambda_{om} r)\, r\, dr}$$

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