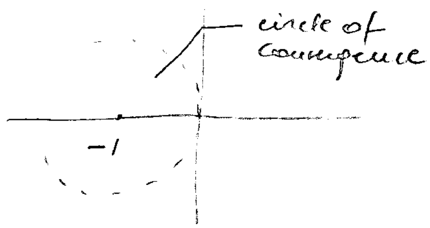
Chapter 14 : 14.3, 6, 7, 8, 15, 18, 19, 17

14.3

$$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$$



$$f(z) = \frac{1}{z^2} = z^{-2}$$
$$f'(z) = -2z^{-3}$$
$$f''(z) = (-2)(-3)z^{-4}$$
$$\vdots$$
$$f^{(n)}(z) = (-1)^n (n+1)!\, z^{-(n+2)}$$

$$\therefore \quad f^{(n)}(-1) = (-1)^n (n+1)!\, (-1)^{-(n+2)}$$
$$= (n+1)!$$

The Taylor series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(-1)}{n!} (z+1)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} (z+1)^n$$
$$= 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$$

valid for $|z+1| < 1$

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14.6 $\qquad f(z) = \dfrac{1}{z-1} \qquad |z-0| > 1$

$$= \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

Formula (14.21) $\qquad c_n = \dfrac{1}{2\pi i} \displaystyle\int_C \dfrac{f(z)}{z^{n+1}}\, dz$

$$c_0 = \frac{1}{2\pi i} \int_C \frac{dz}{z(z-1)} = *$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1) + Bz}{z(z-1)} = \frac{-A + (A+B)z}{z(z-1)}$$

$$\Rightarrow \quad A = -1 \qquad B = 1$$

$$* = \frac{1}{2\pi i} \int_C \left(\frac{1}{z-1} - \frac{1}{z} \right) dz$$

$$= \frac{1}{2\pi i} \int_{C_1} \frac{dz}{z-1} - \frac{1}{2\pi i} \int_{C_2} \frac{dz}{z}$$

$$= 1 - 1 = 0$$

where c_1 and c_2 sufficiently small circles around 1 and 0, respectively.

$$c_1 = \frac{1}{2\pi i}\int_C \frac{dz}{z^2(z-1)} = *$$

$$\frac{1}{z^2(z-1)} = \frac{Az+B}{z^2} + \frac{C}{z-1}$$

$$= \frac{(Az+B)(z-1)+Cz^2}{z^2(z-1)}$$

$$1 = (A+C)z^2 + (B-A)z - B$$

$B = -1$
$A = -1$
$C = 1$

$$= \frac{1}{2\pi i}\int_C \left(\frac{-z-1}{z^2} + \frac{1}{z-1}\right)dz$$

$$= -\frac{1}{2\pi i}\int_{C_1} \frac{z+1}{z^2}\,dz + \frac{1}{2\pi i}\int_{C_2}\frac{dz}{z-1}$$

$$= -(z+1)'(0) + 1 = -1+1 = 0$$

$$c_{-1} = \frac{1}{2\pi i}\int_C \frac{1}{z-1}\,dz = \frac{1}{2\pi i}\,2\pi i = 1$$

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14.7

a) $\dfrac{1}{z^2-3z+2}$ $\qquad 1 < |z-0| < 2$

$$z^2-3z+2 = (z-1)(z-2)$$

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\frac{1}{z-2} = -\frac{1}{2}\left(\frac{1}{1-\underbrace{\frac{z}{2}}_{q}}\right) = -\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\dots\right)$$

$$= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \qquad \left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$$

$$\frac{1}{z-1} = \frac{1}{z}\left(\frac{1}{1-\underbrace{\frac{1}{z}}_{q}}\right) = \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\dots\right)$$

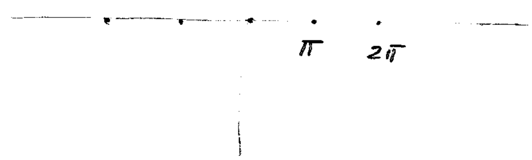
$$= \sum_{n=1}^{\infty}\frac{1}{z^n} \qquad \left|\frac{1}{z}\right| < 1 \Rightarrow |z| > 1$$

The final Laurent expansion :

$$f(z) = \sum_{n=0}^{\infty} -\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n}$$

14.7b)

$$\frac{1}{\sin z} \qquad \pi < |z| < 2\pi$$



$$\frac{1}{\sin z} = \underbrace{\frac{z(z-\pi)(z+\pi)}{\sin z}}_{\text{analytic in } |z|<2\pi} \quad \frac{1}{z\,(z^2-\pi^2)}$$

Step 1 Expand $\frac{1}{z(z^2-\pi^2)}$ for $\pi < |z| < 2\pi$

$$\frac{1}{z^2-\pi^2} = -\frac{1}{\pi^2-z^2} = \frac{-1}{\pi^2}\,\frac{1}{1-\left(\frac{z}{\pi}\right)^2}$$

$$= -\frac{1}{\pi^2}\left(1 + \frac{z^2}{\pi^2} + \frac{z^4}{\pi^4} + \frac{z^6}{\pi^6} + \dots\right)$$

$$\frac{1}{z\,(z^2-\pi^2)} = -\frac{1}{\pi^2}\left(\frac{1}{z} + \frac{z}{\pi^2} + \frac{z^3}{\pi^4} + \frac{z^5}{\pi^6} + \dots\right)$$

Step 2 Expand $\frac{z(z-\pi)(z+\pi)}{\sin z}$ for $|z| < 2\pi$

$$z(z^2-\pi^2) = -\pi^2 z + z^3$$

$$-\pi^2 + \left(-\frac{\pi^2}{6}+1\right)z^2 + \dots$$

$$-\pi^2 z + z^3 \;:\; z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \dots$$

$$+\pi^2 z - \frac{\pi^2}{6}z^3 -$$

$$= \left(-\frac{\pi^2}{6}+1\right)z^3$$

Step 3 Multiply $\left(-\pi^2 + \left(-\frac{\pi^2}{6}+1\right)z^2 + \dots\right)\left(-\frac{1}{\pi^2}\right)\left(\frac{1}{z} + \frac{z}{\pi^2} + \frac{z^3}{\pi^4} + \dots\right)$

Question: Can you do it simpler ?

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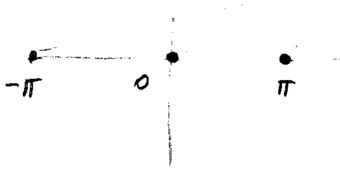
14.8

(a) $\csc z$ in $0 < |z| < \pi$

$$\csc z = \frac{1}{\sin z}$$

simple poles



$$\frac{1}{\sin z} = \frac{z}{\sin z} \quad \frac{1}{z}$$

$$\sin z = z - \frac{1}{3!} z^3 + \frac{1}{5!} z^5 - \frac{1}{7!} z^7 + \dots$$

$$\frac{\sin z}{z} = 1 - \frac{1}{3!} z^2 + \frac{1}{5!} z^4 - \frac{1}{7!} z^6 + \dots$$

analytic, $\neq 0$ for $|z| < \pi$

$\therefore$ $\frac{z}{\sin z}$ is analytic in $|z| < \pi$

$\therefore$ $\frac{z}{\sin z}$ is expandable into its Taylor series (no singular part) at 0 for $|z| < \pi$

At this point one can evaluate the Taylor series directly or through the division of series

$$1 + \frac{1}{6} z^2 + \frac{7}{360} z^4 + \dots$$

$$1 : \quad 1 - \frac{1}{6} z^2 + \frac{1}{120} z^4 -$$

$$-1 + \frac{1}{6} z^2 - \frac{1}{120} z^4 + \dots$$

$$= \quad \frac{1}{6} z^2 - \frac{1}{120} z^4 + \dots$$

$$- \frac{1}{6} z^2 + \frac{1}{36} z^4 - \dots$$

$$= \quad \frac{7}{360} z^4$$

So finally,

$$\frac{1}{\sin z} = \frac{z}{\sin z}\ \frac{1}{z} = \left(1 + \frac{1}{6}z^2 + \frac{7}{360}z^4 + \dots\right)\frac{1}{z}$$

$$= \underbrace{\frac{1}{z}}_{\text{singular part}} + \underbrace{\frac{1}{6}z + \frac{7}{360}z^3 + \dots}_{\text{reg. part}}$$

b) $\sec z = \dfrac{1}{\cos z}$ in $|z| < \dfrac{\pi}{2}$

$$\cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 -$$
$$= 1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 -$$

$$1 + \frac{1}{2}z^2 + \frac{5}{24}z^4$$

$$1 \quad : \left(1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 - \dots\right)$$
$$-1 + \frac{1}{2}z^2 - \frac{1}{24}z^4 + \dots$$
$$= \frac{1}{2}z^2 - \frac{1}{24}z^4 +$$
$$-\frac{1}{2}z^2 + \frac{1}{4}z^4 - \dots$$
$$= \frac{5}{24}z^4 + \dots$$

so: $\sec z = 1 + \frac{1}{2}z^2 + \frac{5}{24}z^4 + \dots \qquad |z| < \frac{\pi}{2}$

c) $\csc z = \dfrac{1}{\sin z} = \dfrac{1}{\sin(z-\pi)}$ in $0 < |z-\pi| < \pi$

Compare with a, just a shift in z.

$$\csc z = \frac{1}{z-\pi} + \frac{1}{6}(z-\pi) + \frac{7}{360}(z-\pi)^3 + \dots$$

d) $\dfrac{1}{e^z - 1}$ in $0 < |z| < 2\pi$

$$e^z = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots$$

$$e^z - 1 = z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + \dots$$
$$= z\left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \dots\right)$$

$$\underline{1 - \frac{1}{2}z + \frac{1}{12}z^2}$$
$$1 \quad : \left(1 + \frac{1}{2}z + \frac{1}{6}z^2 + \frac{1}{24}z^3 + \dots\right)$$
$$\underline{- \; 1 - \frac{1}{2}z - \frac{1}{6}z^2 - \frac{1}{24}z^3}$$
$$= -\frac{1}{2}z - \frac{1}{6}z^2 - \frac{1}{24}z^3 -$$
$$\underline{+\frac{1}{2}z + \frac{1}{4}z^2 + \frac{1}{12}z^3 +}$$
$$= \quad \frac{1}{12}z^2 + \frac{1}{24}z^3$$
$$\underline{-\frac{1}{12}z^2 - \frac{1}{24}z^3 +}$$
$$= \quad = \quad +$$

So
$$\frac{1}{e^z - 1} = \frac{1}{z}\left(1 - \frac{1}{2}z + \frac{1}{12}z^2 + \dots\right)$$

$$= \underbrace{\frac{1}{z}}_{\text{sing.}} \underbrace{- \frac{1}{2} + \frac{1}{12}z + \dots}_{\text{reg. part}}$$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

$\therefore$ For $x = 0,\ y = n \cdot 2\pi$ $\quad e^z - 1 = 0$ and $\dfrac{1}{e^z - 1}$

is singular (simple poles) at those points. Consequently the region of convergence for the Laurent expansion is $0 < |z| < 2\pi$

↑

distance from 0 to the nearest pole!

e) $\frac{1}{e^z+1}$ in $0 < |z - \pi i| < 2\pi$

$e^z = e^x(\cos y + i \sin y)$

$e^{\pi i} = \cos \pi = -1$

$\therefore \; e^z+1 = 0$ at πi

Same at $(\pi + 2k\pi)i$ which explains the region of convergence.

Expansion:

$$e^z = e^x(\cos y + i \sin y) = e^x(-\cos(y-\pi) - i \sin(y-\pi))$$
$$= -e^{(z-\pi i)}$$

$$\frac{1}{e^z+1} = \frac{-1}{e^{(z-\pi i)}-1} \quad \ldots \quad \text{Proceed like in d)}$$

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14.15

$$\sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^n = (z-1) \sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^{n-1}$$

Focus on: $\sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^{n-1} =: g(z)$

Integrate term by term: $G(z) = \sum_{n=1}^{\infty} (-1)^{n+1} (z-1)^n$

$$G' = g$$

Now

$$G(z) = (z-1) - (z-1)^2 + (z-1)^3 \qquad q = -(z-1)$$

is a geometric series. Its analytic extension is equal to the sum of the series (use the same symbol)

$$G(z) = \frac{z-1}{1+z-1} = \frac{z-1}{z} = 1 - \frac{1}{z}$$

Differentiating G, we get

$$g(z) = \frac{1}{z^2}$$

Consequently, the analytic extension of the original function is

$$\boxed{f(z) = \frac{z-1}{z^2}}$$

Check:

$$f(z) = \frac{z-1}{z^2} = \frac{1}{z} - \frac{1}{z^2} = z^{-1} - z^{-2}$$

$$f'(z) = -z^{-2} + 2z^{-3} \qquad f'(1) = -1+2 = 1$$

$$f''(z) = 2z^{-3} - 6z^{-4} \qquad f''(1) = 2-6 = -4$$

$$\frac{f''(1)}{2!} = -2$$

$\vdots$

d) $\sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^{-n}$

Same idea as for c)

$$\sum_{n=1}^{\infty} (-1)^{n+1} n (z-1)^{-n} = \sum_{n=1}^{\infty} (-1)^{n} (-n)(z-1)^{-n}$$

$$= (z-1) \underbrace{\sum_{n=1}^{\infty} (-1)^{n} (-n)(z-1)^{-(n+1)}}_{g(z)} = (*)$$

$$G(z) = \sum_{n=1}^{\infty} (-1)^{n} (z-1)^{-n} \quad , \quad G'(z) = g(z)$$

$$= -\frac{(z-1)^{-1}}{1 + (z-1)^{-1}} = -\frac{1}{z\cancel{-1}\cancel{+1}} = -\frac{1}{z}$$

$$G'(z) = \frac{1}{z^2}$$

$$(*) = \frac{z-1}{z^2}$$

14.18

a) $\sin\left(\frac{1}{t}\right) = \frac{1}{t} - \frac{1}{3!}\frac{1}{t^3} + \frac{1}{5!}\frac{1}{t^5} - \dots$

essential singularity at $t=0$

b) $e^{\frac{1}{t}} = 1 + \frac{1}{t} + \frac{1}{2}\frac{1}{t^2} + \frac{1}{6}\frac{1}{t^3} + \dots$

essential singularity at $t=0$

c) $e^t = 1 + t + \frac{1}{2}t^2 + \dots$ analytic at $t=0$

d) e^{-t} analytic at $t=0$

e) $\dfrac{\frac{1}{t}}{\sin\left(\frac{1}{t}\right)} = \dfrac{1}{t\sin\left(\frac{1}{t}\right)}$ does not have an isolated singularity at $t=0$

f) $\dfrac{\frac{1}{t}}{\frac{1}{t^3}+2} = \dfrac{t^3}{t(2t^3+1)} = \dfrac{t^2}{2t^3+1}$ analytic at $t=0$

g) $\frac{1}{t} + 3t$ simple pole at $t=0$

14.19

The left-hand side is discontinuous at $x=0$ and the expansion simply is valid for $x>0$ only.

14.17

a) $\csc z = \dfrac{1}{\sin z}$

$\sin z = 0 \iff z = k\pi \quad k \in \mathbb{Z}$

Case $k = 2n$

$$\frac{1}{\sin z} = \frac{(z - 2n\pi)}{\sin(z - 2n\pi)} \quad \frac{1}{(z - 2n\pi)}$$

Part $\dfrac{z - 2n\pi}{\sin(z - 2n\pi)}$ can be extended by 1 to an analytic function*, so $z = 2n\pi$ is a simple pole

Case $k = 2n-1$

$$\frac{1}{\sin z} = \frac{-1}{\sin(z - 2n\pi - \pi)} = -\frac{z - (2n+1)\pi}{\sin(z - (2n+1)\pi)} \; \frac{1}{z - (2n+1)\pi}$$

Same conclusion as above

* the easiest way to see it is to look at the inverse $\dfrac{\sin(z - 2n\pi)}{z - 2n\pi}$ which is an analytic function with value $= 1$ at $2n\pi$

Thus $\csc z$ has simple poles at $z = k\pi$, $k \in \mathbb{Z}$

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b) $\sec z = \dfrac{1}{\cos z}$

Same reasoning as in a), simple poles at $z = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

c) $\dfrac{1}{e^z - 1}$

$$e^z - 1 = z + \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \dots$$

$$e^z - 1 = 0 \iff z = 0$$

$$\frac{e^z - 1}{z} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$

can be extended by 1 to an analytic function in the whole $\mathbb{C}$

Thus $\dfrac{1}{e^z - 1} = \underbrace{\dfrac{z}{e^z - 1}}_{\text{analytic}} \dfrac{1}{z}$

$\therefore$ The function has a simple pole at $z = 0$

d) $\dfrac{1}{e^z + 1}$

$$e^z + 1 = 0 \iff e^z = -1 \iff z = -\pi i$$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

$$\left.\begin{aligned} e^x \cos y &= -1 \\ e^x \sin y &= 0 \end{aligned}\right\} \Rightarrow x = 0,\ y = -\pi$$

$f(z) = e^z$

$$f(z) = f(-\pi i) + f'(-\pi i)(z+\pi i) + \frac{f''(-\pi i)}{2!}(z+\pi i)^2 + \dots$$

$$e^z = -1 - (z+\pi i) - \frac{1}{2!}(z+\pi i)^2 - \dots$$

$$e^z + 1 = -(z+\pi i) - \frac{1}{2!}(z+\pi i)^2 - \dots$$

$$\frac{e^z+1}{z+\pi i} = 1 - \frac{1}{2!}(z+\pi i) - \dots$$ is analytic in a neighborhood of $z = -\pi i$

Thus

$$f(z) = \frac{1}{e^z+1} = \underbrace{\frac{z+\pi i}{e^z+1}}_{\text{analytic}} \frac{1}{z+\pi i}$$

has a simple pole at $z = -\pi i$

e) $$\frac{e^{-z}}{z(z^2+1)} = \frac{e^{-z}}{z(z+i)(z-i)}$$

Simple poles at $z = 0, i, -i$

f) $z e^{-\frac{1}{z}}$

$$e^z = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots$$

$$e^{-\frac{1}{z}} = 1 - \frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3!z^3}$$

$$z e^{-\frac{1}{z}} = z - 1 + \frac{1}{2z} - \frac{1}{3!z^2} + \dots$$ essential singularity at $z=0$