Chapter 7

Functions of Several Variables

Here we extend concepts already established for functions of a single variable to functions of two or more variables. As before, both the variables and the functions are assumed to be real valued.

7.1. CHAIN DIFFERENTIATION

We will consider only two independent variables, since the extension to the case of three or more should be transparent.

If $x = x(t)$ and $y = y(t)$, then $u(x, y) = u[x(t), y(t)]$ can be regarded as a function of the single variable t; we say that u is a **composite function** of t. Recalling the "chain rule" from calculus, we expect that

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}, \tag{7.1}$$

assuming, of course, that the four derivatives on the right-hand side all exist. Let us have a closer look, however.

Example 7.1.

$$u(x, y) = \begin{cases} x & \text{on } y = x^2 \\ 0 & \text{for } y \neq x^2. \end{cases} \tag{7.2}$$

Introduce a curve C in the x, y plane according to parametric equations $x = x(t)$, $y = y(t)$, as sketched in Fig. 7.1. Then du/dt is the rate of change of u with respect to t along C. For instance, let us choose $x = 3t$ and $y = 9t^2$, so that C happens to coincide with the curve $y = x^2$, and let us compute du/dt at the origin. The simplest way to do so is to note that on C we have $u(x, y) = x = 3t$; therefore du/dt is clearly 3. Alternatively, (7.1) gives

$$\frac{du}{dt} = (0)(3) + (0)(0) = 0,$$

which is incorrect! ▮

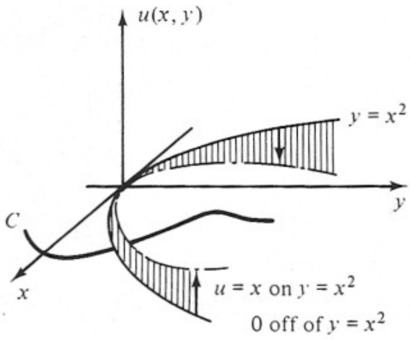

Figure 7.1. Chain differentiation, Example 7.1.

THEOREM 7.1. (The Chain Rule) If $u(x, y)$ has continuous partials u_x and u_y and x and y are differentiable functions of t, then[1]

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}. \tag{7.3}$$

The difficulty within Example 7.1 is that u_x and u_y are *not* continuous at the point in question, $x = y = 0$, since they don't even *exist* throughout an arbitrarily small disk centered at the origin; in particular, u_x and u_y do not exist at any points on the curve $y = x^2$ (except for the origin) due to the jump discontinuity in u along that curve.

Why is the requirement of continuity on u_x and u_y in Theorem 7.1 entirely reasonable? It's reasonable because, after all, (7.3) is basically an *interpolation* formula, where we are interpolating between the x and y directions, and interpolation is fundamentally dependent on continuity.

Suppose that we parameterize, in Example 7.1, according to $x(t) = t^2$, $y(t) = t^4$ instead; C again coincides with $y = x^2$. On C we have $u(x, y) = x = t^2$, so that $du/dt = 2t = 0$ at the origin. This time, however, (7.3) gives $du/dt = (0)(0) + (0)(0) = 0$, which *is* correct, even though u_x and u_y are still discontinuous at the point in question. We conclude that the conditions of the theorem are *sufficient* but not necessary.

Do you agree that $u_x = 0$ at $x = y = 0$, as we've been claiming, or, looking at (7.2), do you think it should be 1? Well,

$$u_x(0, 0) = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0, 0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} = 0!$$

Example 7.2. Express the quantity

$$u_{xx} + u_{yy} \tag{7.4}$$

[1]By "continuous" and "differentiable" we mean at the point, or points, of application of the formula, not necessarily everywhere. Incidentally, exactly what do we mean by these terms for functions of *several* variables, say $f(x_1, \ldots, x_n)$? Extending the basic idea from Chapter 1, we say that $f(x_1, \ldots, x_n)$ is *continuous* at X_0 if $\lim_{X \to X_0} f(X) = f(X_0)$; that is, if to each $\epsilon > 0$ there corresponds a $\delta(\epsilon, X_0)$ such that $|f(X) - f(X_0)| < \epsilon$ for all X such that $d(X, X_0) < \delta$, where X is shorthand for the "point" $(x_1, \ldots, x_n)$ and d is some distance function, for example the *n-dimensional Euclidean distance* $d(X, Y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$. Similarly for differentiability.

in terms of **polar coordinates** r, θ, which are related to the original cartesian x, y variables according to

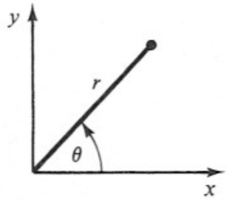

Figure 7.2. Polar variables.

$$x = r\cos\theta \tag{7.5a}$$

$$y = r\sin\theta, \tag{7.5b}$$

as shown in Fig. 7.2. First, let us transform $\partial/\partial x$ and $\partial/\partial y$. Applying the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}, \tag{7.6}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial y}. \tag{7.7}$$

[It is important to keep the bookkeeping straight. There are two alternative sets of independent variables, x, y and r, θ. By $\partial(\ \ )/\partial x$, for example, we mean the derivative of ( ), considered as a function of the x, y variables, with respect to x, with all the other variables in the x group held fixed—namely, y. Similarly, $\partial/\partial\theta$ means r is held fixed and so on.]

We need to know $r_x, \theta_x, r_y, \theta_y$ in (7.6) and (7.7).

Although $x_r = \cos\theta$, $x_\theta = -r\sin\theta$, $y_r = \sin\theta$, $y_\theta = r\cos\theta$ are easily obtained directly from (7.5), the "reverse" derivatives $r_x, \ldots, \theta_y$ are a little more elusive. To obtain r_x and θ_x, let (7.6) act on the functions x and y, say, in turn:

$$\frac{\partial x}{\partial x} = 1 = \frac{\partial x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial x} = cr_x - rs\theta_x \tag{7.8a}$$

$$\frac{\partial y}{\partial x} = 0 = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial x} = sr_x + rc\theta_x, \tag{7.8b}$$

where c and s are shorthand for $\cos\theta$ and $\sin\theta$, respectively. Equations (7.8) are two simultaneous equations in the desired quantities r_x and θ_x. Solving,

$$r_x = \cos\theta \tag{7.9}$$

$$\theta_x = -\frac{\sin\theta}{r}. \tag{7.10}$$

Thus[2]

$$\begin{aligned} u_{xx} &= \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right)u = \left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial\theta}\right)\left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial\theta}\right)u \\ &= \left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial\theta}\right)\left(cu_r - \frac{s}{r}u_\theta\right) \end{aligned} \tag{7.11}$$

$$u_{xx} = c^2u_{rr} + \frac{cs}{r^2}u_\theta - \frac{cs}{r}u_{\theta r} + \frac{s^2}{r}u_r - \frac{sc}{r}u_{r\theta} + \frac{sc}{r^2}u_\theta + \frac{s^2}{r^2}u_{\theta\theta}. \tag{7.12}$$

To obtain r_y and θ_y, we let (7.7) act on the functions x and y,

$$\frac{\partial x}{\partial y} = 0 = \frac{\partial x}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial x}{\partial \theta}\frac{\partial \theta}{\partial y} = cr_y - rs\theta_y \tag{7.13a}$$

$$\frac{\partial y}{\partial y} = 1 = \frac{\partial y}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial y}{\partial \theta}\frac{\partial \theta}{\partial y} = sr_y + rc\theta_y. \tag{7.13b}$$

[2] Note that (7.6) really means $\partial(\ \ )/\partial x = [\partial(\ \ )/\partial r]r_x + [\partial(\ \ )/\partial\theta]\theta_x$; that is, the $\partial/\partial r$ and $\partial/\partial\theta$ act only on ( ) and *not* on their respective neighbors r_x and θ_x. We sometimes emphasize this point by writing (7.6) as $\partial/\partial x = r_x\partial/\partial r + \theta_x\partial/\partial\theta$ instead, as in (7.11). Similarly for (7.7).

Solving, we have

$$r_y = \sin\theta \tag{7.14}$$

$$\theta_y = \frac{\cos\theta}{r}. \tag{7.15}$$

Then

$$u_{yy} = \left(\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y}\right)u = \text{etc.}$$

$$= s^2 u_{rr} - \frac{sc}{r^2}u_\theta + \frac{sc}{r}u_{\theta r} + \frac{c^2}{r}u_r + \frac{cs}{r}u_{r\theta} - \frac{cs}{r^2}u_\theta + \frac{c^2}{r^2}u_{\theta\theta}. \tag{7.16}$$

Adding (7.12) and (7.16) gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{7.17}$$

or in terms of the differential operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}, \tag{7.18}$$

which is the well-known **Laplace operator**, or **Laplacian**, expressed in both cartesian and polar coordinates; we'll be seeing much more of it later on.

COMMENT 1. In fact, r_x, θ_x, r_y, and θ_y could have been determined more easily by solving (7.5) for $r(x, y)$ and $\theta(x, y)$: squaring and adding give $r = \sqrt{x^2 + y^2}$, and dividing gives $\theta = \tan^{-1}(y/x)$. Thus $r_x = x/\sqrt{x^2 + y^2} = (r\cos\theta)/r = \cos\theta$, and so on. Had the $x = x(r, \theta)$, $y = y(r, \theta)$ relations been more complicated, however, we might not have been able to solve for $r(x, y)$ and $\theta(x, y)$, and we might then have been glad to fall back on the more systematic procedure demonstrated above.

COMMENT 2. Observe that $\partial x/\partial r$ and $\partial r/\partial x$, for instance, are *not* algebraic reciprocals of each other the way that ordinary derivatives are; that is, $\partial x/\partial r = \cos\theta$ and $\partial r/\partial x = \cos\theta$ (identical by coincidence), *not* $1/\cos\theta$. This situation occurs because $\partial x/\partial r$ is computed with θ fixed (i.e., the other member of the r, θ group), whereas $\partial r/\partial x$ is computed with y fixed (i.e., the other member of the x, y group), as displayed in Fig. 7.3. Since different triangles are involved, there's no reason why $\Delta x/\Delta r$ and $\Delta r/\Delta x$ should be reciprocals of each other (see Exercise 7.18, however).

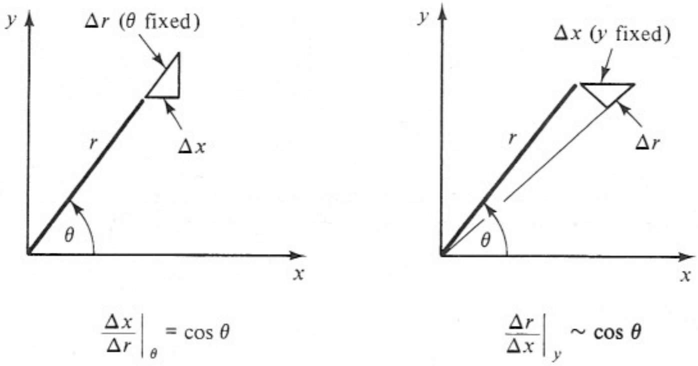

Figure 7.3. Graphical interpretation of $\frac{\partial x}{\partial r}$ and $\frac{\partial r}{\partial x}$.

On the other hand, consider the equation of state of an ideal gas, $pv = RT$, where p, v, T are the pressure, volume, and absolute temperature and R is a physical constant. In this case,

$$\frac{\partial p}{\partial T} = \frac{R}{v} \quad \text{and} \quad \frac{\partial T}{\partial p} = \frac{v}{R},$$

for instance, *are* reciprocals. The reason that they are is that both partials are computed with the *same* variable (v) held constant:

$$\frac{\partial p}{\partial T} = \frac{1}{\partial T/\partial p}.$$

(As soon as we write down $\partial p/\partial T$, say, it's implied that we are regarding p as the dependent variable and T, v as independent. Similarly, $\partial T/\partial p$ implies that T is to be regarded as a function of p and v.) ▮

Generalizing a bit, equations (7.5) are of the form

$$F(x, y, u, v) = 0 \tag{7.19a}$$

$$G(x, y, u, v) = 0 \tag{7.19b}$$

For example, (7.5a) is equivalent to $F = x - u \cos v = 0$. In (7.19) x, y, u, v are all independent variables. However, we may wish to change our point of view and regard u and v as functions of x and y, say, as defined *implicitly* by (7.19). How might we compute the partials u_x, u_y, v_x, v_y? Well,

$$\frac{\partial}{\partial x} F[x, y, u(x, y), v(x, y)] = F_x + F_u u_x + F_v v_x = 0 \tag{7.20a}$$

$$\frac{\partial}{\partial x} G[x, y, u(x, y), v(x, y)] = G_x + G_u u_x + G_v v_x = 0 \tag{7.20b}$$

and solving in terms of *determinants*,[3]

$$u_x = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{J(x, v)}{J(u, v)}, \tag{7.21a}$$

$$v_x = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{J(u, x)}{J(u, v)}, \tag{7.21b}$$

[3]Determinants and simultaneous linear algebraic equations will be discussed in depth in Part III. For the present, let us merely define the so-called 2×2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \equiv ad - bc$$

and the 3×3 determinant

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \equiv a(ei - fh) - b(di - fg) + c(dh - eg)$$

where the J determinants are known as **Jacobians.** For instance, $J(x, v)$ is *the Jacobian of F and G with respect to x and v*. The alternative notations

$$\frac{\partial(F, G)}{\partial(x, v)} \quad \text{and} \quad J\left(\frac{F, G}{x, v}\right)$$

contain reference to F and G and are sometimes preferred.

Similarly,

$$u_y = -\frac{J(y, v)}{J(u, v)} \quad \text{and} \quad v_y = -\frac{J(u, y)}{J(u, v)}. \tag{7.22}$$

7.2. *TAYLOR SERIES IN TWO OR MORE VARIABLES*

Suppose that we know "all about" some function $f(x, y)$ at a point (a, b) of the x, y plane; that is, we know all the values $f(a, b)$, $f_x(a, b)$, $f_y(a, b)$, $f_{xx}(a, b)$, $f_{xy}(a, b)$, $f_{yy}(a, b)$, and so on. Based on these data, can we predict the value of f at some other point, say x_0, y_0? That is, can we extrapolate away from (a, b) throughout some or all the x, y plane?

Having already explored this question for functions $f(x)$ of a *single* variable, it is natural to try to reduce the two-dimensional case to one dimension; if we can do so, we're "home free." We can accomplish it by running a curve C from (a, b) to (x_0, y_0). Defining C parametrically by $x = x(t)$, $y = y(t)$, we then have $f(x, y) = f[x(t), y(t)] \equiv F(t)$, say, a function of the single variable t. This line of approach is left for Exercise 7.15.

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and note that the solution of

$$\begin{aligned} ax + by &= p \\ cx + dy &= q \end{aligned}$$

is given by

$$x = \frac{\begin{vmatrix} p & b \\ q & d \end{vmatrix}}{D}, \qquad y = \frac{\begin{vmatrix} a & p \\ c & q \end{vmatrix}}{D},$$

where

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

provided that $D \neq 0$, and that the solution of

$$\begin{aligned} ax + by + cz &= p \\ dx + ey + fz &= q \\ gx + hy + iz &= r \end{aligned}$$

is

$$x = \frac{\begin{vmatrix} p & b & c \\ q & e & f \\ r & h & i \end{vmatrix}}{D}, \qquad y = \frac{\begin{vmatrix} a & p & c \\ d & q & f \\ g & r & i \end{vmatrix}}{D}, \qquad z = \frac{\begin{vmatrix} a & b & p \\ d & e & q \\ g & h & r \end{vmatrix}}{D},$$

where

$$D = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix},$$

provided again that $D \neq 0$.

Instead note that we can reduce the problem to one dimension, alternatively, simply by expanding in one variable at a time. Expanding in x first (holding y fixed),

$$f(x, y) = f(a, y) + f_x(a, y)(x - a) + \frac{f_{xx}(a, y)}{2!}(x - a)^2 + \cdots. \tag{7.23}$$

Next, we expand the coefficients $f(a, y), f_x(a, y), \ldots$ in terms of y,

$$\begin{aligned} f(x, y) = &\left[f(a, b) + f_y(a, b)(y - b) + \frac{f_{yy}(a, b)}{2!}(y - b)^2 + \cdots \right] \\ &+ [f_x(a, b) + f_{xy}(a, b)(y - b) + \cdots](x - a) \\ &+ [f_{xx}(a, b) + \cdots]\frac{(x - a)^2}{2!} + \cdots. \end{aligned} \tag{7.24}$$

Finally, we arrange the terms according to ascending degree,[4]

$$\begin{aligned} f(x, y) = &f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &+ \frac{f_{xx}(a, b)}{2!}(x - a)^2 + \frac{2f_{xy}(a, b)}{2!}(x - a)(y - b) \\ &+ \frac{f_{yy}(a, b)}{2!}(y - b)^2 + \cdots, \end{aligned} \tag{7.25}$$

which, it turns out, is expressible as

$$f(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m!\, n!} \frac{\partial^{m+n} f}{\partial x^m\, \partial y^n}(a, b)(x - a)^m (y - b)^n. \tag{7.26}$$

Example 7.3. Expand $f(x, y) = y^x$ about (1, 1), up to and including second-degree terms. Instead of simply plugging into (7.26), it may be more informative to proceed by expanding in one variable at a time, as mentioned above. Since y^x is a simpler function of y than of x, let us expand in y first, about $y = 1$, with x considered as *fixed.*

$$\begin{aligned} f(x, y) &= f(x, 1) + f_y(x, 1)(y - 1) + \frac{f_{yy}(x, 1)}{2!}(y - 1)^2 + \cdots \\ y^x &= [1]^x + [x \cdot 1^{x-1}](y - 1) + [x(x - 1) \cdot 1^{x-2}]\frac{(y - 1)^2}{2} + \cdots \\ &= [1] + [x](y - 1) + [x(x - 1)]\frac{(y - 1)^2}{2} + \cdots. \end{aligned} \tag{7.27}$$

The coefficients of this expansion (in the square brackets) are functions of x; we now expand each of them about $x = 1$. Since we plan to go only as far as second degree and $(y - 1)^2$ is already second degree, we need $[x(x - 1)]$ only through *zeroth* degree; similarly, we need $[x]$ through first degree and $[1]$ through second degree. Thus

$$x(x - 1) = 0 + \cdots, \qquad x = 1 + (x - 1), \qquad 1 = 1,$$

so that (7.27) becomes

$$y^x = 1 + [1 + (x - 1)](y - 1) + \cdots.$$

Finally, we rearrange so that the terms are in ascending degree:

$$y^x = 1 + (y - 1) + (x - 1)(y - 1) + \cdots.$$

[4] $(x - a)^m (y - b)^n$ is said to be of degree (or "order") $m + n$.

We do *not* "simplify" per

$$y^x = 1 + y - 1 + xy - x - y + 1 + \cdots = 1 - x + xy + \cdots;$$

in expanding about (a, b), we leave the quantities $(x - a)$ and $(y - b)$ intact! ▌

You may have noticed that in going from (7.24) to (7.25), we tacitly assumed that $f_{yx}(a, b) = f_{xy}(a, b)$, and similarly for higher-order partials, such as f_{xxy} and f_{xyx}. That this assumption is not necessarily true should come as no surprise in view of our discussion of limit interchange in Chapter 4. It can be shown, for instance, that *if f_x, f_y, and f_{xy} are continuous in some neighborhood of a, b, then we do have $f_{yx} = f_{xy}$ there*, and similar statements can be made about mixed partials of higher order as well.[5] In engineering applications, however, our functions turn out to be sufficiently well behaved with such regularity that we generally don't worry about the order of differentiation—although you may worry if you wish.

EXERCISES

7.1. Defining $u(x, y) = xy/(x^2 + y^2)$ except at $x = y = 0$, where it is defined to be zero, show that $u(x, y)$ is discontinuous at $x = y = 0$. Nevertheless, $u(x, y)$ is a continuous function of x for all x and all y, and a continuous function of y for all x and all y. Explain how this fact is compatible with the fact that u is discontinuous at the origin.

7.2. Defining $u(x, y) = (x^3 - y^3)/(x^2 + y^2)$ except at $x = y = 0$, where it is defined to be zero, show that $u_x(x, y)$ exists at $x = y = 0$ (Evaluate it.) but is not continuous there.

7.3. (a) Verify that $u_{xy} = u_{yx}$ for all values of x and y, for the cases $u = e^{x-2\sin y}$ and $u = 4x \sin x^3y$. *Note:* By u_{xy} we mean $(u_x)_y$.

(b) Given that $u(x, y) = xy^3/(x^2 + y^2)$ for $x^2 + y^2 \neq 0$, and $u(0, 0) = 0$, compute $u_{xy}(0, 0)$ and $u_{yx}(0, 0)$.

7.4. The *second law of thermodynamics* may be expressed in differential form as $T\,ds = dh - v\,dp$. Since $s = s(p, T)$ and $h = h(p, T)$, we have $ds = s_T\,dT + s_p\,dp$ and $dh = h_T\,dT + h_p\,dp$. Thus show that $s_T = h_T/T$ and $s_p = (h_p - v)/T$. Eliminating s between these two equations, show that $h_{pT} = T[h_p - v)/T]_T$. So for the case of a *perfect gas* (i.e., where $pv = RT$ with R a constant), show that h is, in fact, a function of T alone; that is, $h = h(T)$.

7.5. Show that $u = f(x + at) + g(x - at)$ satisfies the partial differential ("wave") equation $a^2u_{xx} = u_{tt}$, where a is a constant and f and g are arbitrary twice-differentiable functions. With $a = \sqrt{-1} = i$, it follows that $u = f(x + iy) + g(x - iy)$ satisfies the partial differential ("Laplace") equation $u_{xx} + u_{yy} = 0$.

7.6. The differential equation $xy'' + y' + xy = 0$, known as **Bessel's equation of order zero,** has the general solution $y = AJ_0(x) + BY_0(x)$, where J_0 is the **Bessel function of the first kind and order zero** and Y_0 is the **Bessel function of the second kind and order zero.**

(a) Solve $xy'' + y' + k^2xy = 0$ in terms of Bessel functions. *Hint:* Set $x = \alpha t$ and choose α so that the new differential equation is the Bessel equation.

(b) Solve $xy'' + y' + k^2y = 0$ in terms of Bessel functions. *Hint:* Set $x = \alpha t^\beta$.

7.7. $f(x_1, \ldots, x_n)$ is said to be **homogeneous of degree k** if

$$f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n). \tag{7.28}$$

[5] T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, Mass., 1957, pp. 120–123.

(a) Are the functions $f = x^2 + 3xy$, $f = \ln(x^2 + y^2)$, $f = (x^2 - xy)/(2x + y)$, $f = x^2 e^{x/2y}$ homogeneous? Of what degree?

(b) Show that if $f(x, y, z)$ is homogeneous of degree k, then $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$ are homogeneous of degree $k - 1$.

(c) (*Euler's theorem on homogeneous functions*) Show that if $f(x_1, \ldots, x_n)$ is homogeneous of degree k, then

$$x_1 \frac{\partial f}{\partial x_1} + \cdots + x_n \frac{\partial f}{\partial x_n} = kf.$$

Hint: Differentiate (7.28) with respect to λ and then set $\lambda = 1$. *Note:* There's an important point of notation to straighten out. By $f_x(\lambda x, \lambda y)$, say, we mean $f_x(x, y)$ with x and y *then* replaced by λx and λy, respectively. For instance, if $f(x, y) = x^2 y + 3y$, then $f_x(\lambda x, \lambda y) = 2(\lambda x)(\lambda y) = 2\lambda^2 xy$.

(d) Verify Euler's theorem for the case

$$f = \sqrt{x^4 + y^4}\, \sin^{-1}\frac{y}{2x}.$$

7.8. **(Similarity transformation)** The one-dimensional diffusion in an isotropic medium, where the diffusion coefficient is a function of the concentration C, is governed by the (nonlinear) partial differential equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[D(C)\frac{\partial C}{\partial x}\right]. \tag{7.29}$$

Suppose that the x, t dependence of a solution $C(x, t)$ is such that C is actually a function of some combination of x and t, say $x/\sqrt{t}$; then with $\eta \equiv x/\sqrt{t}$, we have $C = C(\eta)$, which is fortunate because of the reduction from two independent variables to one. Are there such combinations? Let us try, for example, the form $\eta = xt^\alpha$, where α is at our disposal. Then

$$\frac{\partial C}{\partial x} = \frac{\partial C[\eta(x, t)]}{\partial x} = C'(\eta)\frac{\partial \eta}{\partial x} = t^\alpha C',$$

and similarly for C_t and $[D(C)C_x]_x$ in (7.29). Thus show that (7.29) becomes $\alpha\eta C' = t^{2\alpha+1}(D'C'^2 + DC'')$, so that x and t can be suppressed, in favor of η, if we choose $2\alpha + 1 = 0$, $\alpha = -\frac{1}{2}$.[6] Then (7.29) reduces to the *ordinary* differential equation $(D'C'^2 + DC'') = -\eta C'/2$, where $\eta = x/\sqrt{t}$. (Of course, there may also be boundary and initial conditions attached to (7.29), which we've ignored. More about this in Part V.)

7.9. **(Von Mises transformation)** With ν the constant kinematic viscosity and the dp/dx term omitted, the equations governing the boundary layer on a semi-infinite flat plate (Fig. 7.4) are

$$uu_x + vu_y = \nu u_{yy} \qquad (x \text{ momentum})$$

$$u_x + v_y = 0 \qquad (\text{continuity})$$

With the "stream function" ψ defined by $u = \psi_y$, $v = -\psi_x$, so that the continuity equation is automatically satisfied, the Von Mises transformation consists of replacing the independent variable y by ψ. Thus $u(x, y) = U(x, \psi)$, say, so that $u_x = U_x + U_\psi \psi_x = U_x - vU_\psi$, and similarly for u_y and u_{yy}. Thus show that the x-mo-

[6] D' means dD/dC. By a prime, we will *always* mean differentiation with respect to the argument, whatever the argument happens to be. In the present case, D is a function of C, and so D' must be dD/dC.

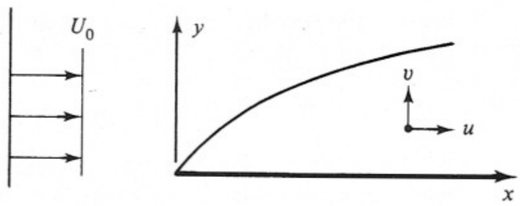

Figure 7.4. Boundary layer on flat plate.

mentum equation becomes

$$\frac{\partial U}{\partial x} = \frac{\partial}{\partial \psi}\left[\nu U \frac{\partial U}{\partial \psi}\right],$$

which is of the type (7.29) with $D(U) = \nu U$. The transformations referred to Exercises 7.8, 7.9, and 7.23, as well as many other techniques of this type, are discussed in the book by W. F. Ames, *Nonlinear Partial Differential Equations in Engineering*, Academic, New York, 1965.

7.10. If $u = x^2 + y^2$ and $v = x + y$, evaluate u_x, x_u, x_v and x_{uu}.

7.11. If $u = u(x, y, z)$, and x, y, z are given in terms of **spherical polar coordinates** ρ, θ, ϕ (Fig. 7.5), according to

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho\cos\theta, \tag{7.30}$$

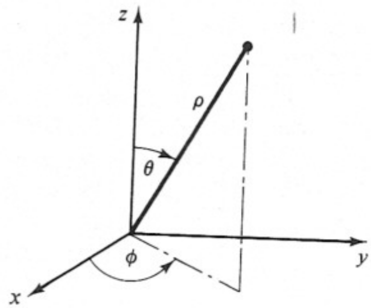

Figure 7.5. Spherical polar coordinates.

(a) show that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{\rho^2 \sin\theta}\left[\frac{\partial}{\partial \rho}\left(\rho^2 \sin\theta \frac{\partial u}{\partial \rho}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial \phi}\right)\right] \tag{7.31}$$

(b) show that the Jacobian

$$\frac{\partial(x, y, z)}{\partial(\rho, \theta, \phi)} = \begin{vmatrix} x_\rho & x_\theta & x_\phi \\ y_\rho & y_\theta & y_\phi \\ z_\rho & z_\theta & z_\phi \end{vmatrix} \text{ is } = \rho^2 \sin\theta.$$

7.12. Consider the change of variables $u = u(x, y)$, $v = v(x, y)$, subject to the restrictions $u_x = v_y$ and $u_y = -v_x$ (the so-called **Cauchy-Riemann conditions**). Show that under this change of variables

$$f_{xx} + f_{yy} = (u_x^2 + u_y^2)(f_{uu} + f_{vv}).$$

(This fact is actually the key to the *conformal mapping* technique of complex variable theory, as we will see in Part II.) *Hint:* Note that

$$f_{xx} = \left(u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}\right)\left(u_x \frac{\partial}{\partial u} + v_x \frac{\partial}{\partial v}\right) f,$$

which expands to eight terms, and similarly for f_{yy}. Quantities like $(\partial/\partial u)(u_x)$ that arise may well be puzzling. It is *not* true that $(u_x)_u = (u_u)_x = (1)_x = 0$ [e.g., if $u = x^{1/2}$, then $u_x = 1/(2x^{1/2}) = 1/(2u)$, so that $(u_x)_u = -1/(2u^2) \neq 0$.] Instead note that

$$\frac{\partial}{\partial u}(u_x)u_x + \frac{\partial}{\partial v}(u_x)v_x = u_{xx},$$

and similarly for various other combinations of such terms.

7.13. If $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$f_{xx} + f_{yy} = e^{2x}(f_{uu} + f_{vv}).$$

Hint: Recall Exercise 7.12.

7.14. If $f(x, y, z) = 0$, show that

$$\frac{\partial y}{\partial x} = -\frac{f_x}{f_y}, \qquad \frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1, \qquad \frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = -1.$$

7.15. Derive the Taylor series (7.25), following the line of approach suggested in the second paragraph of Section 7.2. *Hint:* Setting $x = a + (x_0 - a)t$, $y = b + (y_0 - b)t$, expand per $f(x, y) = f[x(t), y(t)] = F(t) = F(0) + F'(0)t + \cdots$ or, with $t = 1$, $f(x_0, y_0) = F(0) + F'(0) + \cdots$. Finally, change "$x_0, y_0$" to x, y.

7.16. Carry out the Taylor expansions below.

(a) $\sin(x + y)$ about 0, 0 through third degree.
(b) $\sin(x + y + z^2)$ about 0, 0, 1 through second degree.
(c) x^2y about 1, 2 through 27th degree.
(d) $\ln(x + y)$ about 0, 1 through second degree.

7.17. (a) Recalling that if $x = x[u(s)]$, then

$$\frac{dx}{du}\frac{du}{ds} = \frac{dx}{ds},$$

generalize this result to

$$\frac{\partial(x, y)}{\partial(u, v)}\frac{\partial(u, v)}{\partial(r, s)} = \frac{\partial(x, y)}{\partial(r, s)} \quad \text{and} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)}\frac{\partial(u, v, w)}{\partial(r, s, t)} = \frac{\partial(x, y, z)}{\partial(r, s, t)}.$$

Note: The three-dimensional Jacobian notation is defined in Exercise 7.11.

(b) Verify for the case where $x = u \cos v$, $y = u \sin v$ and $u = r + s$, $v = r^2 + s^2$.

7.18. (a) Show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \left[\frac{\partial(u, v)}{\partial(x, y)}\right]^{-1} \quad \text{and} \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \left[\frac{\partial(u, v, w)}{\partial(x, y, z)}\right]^{-1} \tag{7.32}$$

that is

$$J(u, v) = [J(x, y)]^{-1} \quad \text{and} \quad J(u, v, w) = [J(x, y, z)]^{-1}.$$

Recall that in Comment 2 of Example 7.2 we noted that if $x = x(u)$, then dx/du and du/dx *are* inverses, i.e.,

$$\frac{dx}{du} = \left(\frac{du}{dx}\right)^{-1}, \tag{7.33}$$

whereas, if $x = x(u, v)$, and $y = y(u, v)$, say, then (in general) pairs such as $\partial x/\partial u$ and $\partial u/\partial x$ are *not* inverses. Nevertheless, we now see that there *is* an analog of (7.33), namely, the first of equations (7.32). Similarly, for the case where $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ the analog of (7.33) is provided by the second of equations (7.32).

Note: The three-dimensional Jacobian notation is defined in Exercise 7.11.

(b) Verify for the case $x = u \cos v$, $y = u \sin v$.

7.19. (a) Find z_x and z_y if $xy + \sin(x + z) - z^2 = 5$.
(b) Find y_x and y_z if $xe^y - y^2 - z^2 \sin z = 0$.

7.20. Expand $y(x)$ about a point x of your choosing, through second order, if $(y - 1)e^y = \sqrt{x} - 1$.

7.21. Expand $z(x, y)$ about $x = y = 0$, through second order, if $x^2 + y^2 + z^2 = 4$ and $z \geq 0$. Calling the second-order approximation $z_{\text{approx}}(x, y)$, sketch the original and approximating surfaces, $z(x, y)$ and $z_{\text{approx}}(x, y)$, respectively.

7.22. Expand $t(q, r, s)$ about the point 1, 1, 1 through second order if $q^2 + r^2 + s^2 + t^2 - 2t = 3$; choose the root $t(1, 1, 1) = 0$.

7.23. **(Hodograph transformation)** A variety of problems in fluid mechanics, plasticity, and vibration theory can be modeled by the partial differential equations

$$\left.\begin{aligned} Au_x + Bu_y + Cv_x + Dv_y &= P \\ Eu_x + Fu_y + Gv_x + Hv_y &= Q. \end{aligned}\right\} \tag{7.34}$$

If $P = Q = 0$ and A through H are functions only of u, v, then we say that equations (7.33) are *reducible*; suppose that such is the case. Let us switch the dependent and independent variables; that is, we now regard x and y as functions of u and v:

$$x = x(u, v) \quad \text{and} \quad y = y(u, v). \tag{7.35}$$

This is called the hodograph transformation. Show that for any region in which the Jacobian $J = \partial(u, v)/\partial(x, y) \neq 0$, (7.34) converts the original *nonlinear* system to the *linear* system

$$\begin{aligned} Ay_v - Bx_v - Cy_u + Dx_u &= 0 \\ Ey_v - Fx_v - Gy_u + Hx_u &= 0. \end{aligned}$$

Again, we recommend the book by Ames mentioned in Exercise 7.9.

7.24. Make up two cogent exercises on Chapter 7.