$$\left(\underline{\underline{A}}u\right)_{k(i,j)} = \frac{1}{h^2}\left\{4u_{k(i,j)} - \underbrace{u_{k(i+1,j)}}_{i<m} - \underbrace{u_{k(i-1,j)}}_{i>1} - \underbrace{u_{k(i,j+1)}}_{j<m} - \underbrace{u_{k(i,j-1)}}_{j>1}\right\} = f_{k(i,j)} + \frac{1}{h^2}\left\{\underbrace{g_{k(m+1,j}}_{i=m} + \underbrace{g_{k(0,j)}}_{i=0} + \underbrace{g_{k(i,m+1)}}_{j=m+1} + \underbrace{g_{k(i,0)}}_{j=0}\right\}$$

$$A_{k(i,j),k(i',j')} = \begin{cases} \frac{4}{h^2} & \text{if } k = k' \\ -\frac{1}{h^2} & \text{if } (i = i' \pm 1\,, j = j') \text{ or } (i = i'\,, j = j' \pm 1) \end{cases}$$

$$\lambda^{\alpha,\beta} = \frac{4}{h^2}\left\{\sin^2\left(\frac{\alpha\pi}{2(m+1)}\right) + \sin^2\left(\frac{\beta\pi}{2(m+1)}\right)\right\}$$

$$\mathbf{v}^{\alpha,\beta}_{k(i,j)} = \sin\left(\frac{\alpha\pi i}{m+1}\right)\sin\left(\frac{\beta\pi j}{m+1}\right)\,, \quad 1 \le \alpha\,, \beta \le m$$

$$\lambda_{\alpha,\beta} = (\alpha^2 + \beta^2)\pi^2$$

$$\mathbf{v}_{\alpha,\beta}(x,y) = \sin(\alpha\pi x)\sin(\beta\pi y)$$

Theorem If $u \in C^4(\bar{\Omega})$ then $\exists C > 0$ (independent of h) such that, for h sufficiently small,

$$\|u_{i,j} - u(x_i, y_j)\| \le \left\|\underline{\underline{A}}^{-1}\right\| \|\delta_h\| = C_u h^2$$

Problem Statement Given $\Omega \subset \mathbb{R}^d$, $\underline{\underline{a}} : \bar{\Omega} \to \mathbb{R}^{d\times d}$, $b : \bar{\Omega} \to \mathbb{R}^d$, and $\{c, f, \tilde{g}_d, \tilde{g}_N\} : \bar{\Omega} \to \mathbb{R}$
Find $u : \bar{\Omega} \to \mathbb{R}$ s.t.

$$-\nabla \cdot (\underline{\underline{a}}\nabla u + bu) + cu = f\,, \quad \Omega$$
$$u = g_d\,, \quad \Gamma_D$$
$$-(a\nabla u + bu)\cdot \nu = g_N\,, \quad \Gamma_N$$

Galerkin Method Find $u_h \in V_h + g$ s.t.

$$a(u_h, v) = f(v)\,, \quad \forall v \in V_h$$

Ritz Method $J(v) = \frac{1}{2}a(v,v) - f(v)$ Find $u_h \in V_h + g$ s.t.

$$J(u_h - g) < J(v)\,, \quad \forall v \in V_h\,, \quad v \ne u_h - g$$

Continuous (bounded)

$$\exists \eta^* > 0 \text{ s.t. } |a(u,v)| \le \eta^* \|u\|_H \|v\|_H\,, \forall u, v \in H$$

Coercive

$$\exists \eta_* > 0 \text{ s.t. } a(u,u) \ge \eta_* \|u\|_H^2\,, \forall u \in H$$

Energy Norm If $a(u,v)$ is bilinear, continuous, coercive, and symmetric it defines a norm

$$\|u\|_E = \sqrt{a(u,u)}$$

$$\sqrt{\eta_*}\,\|u\|_H \le \|u\|_E \le \sqrt{\eta^*}\,\|u\|_H$$

Lax-Milgram If H is Hilbert, $a : H\times H \to \mathbb{R}$ is bilinear, bounded, coercive and $F : H \to \mathbb{R}$ is linear, bounded, then $\exists! w \in H$ s.t.

$$a(w,v) = F(v)\,, \quad \forall v \in H$$

Poincarè Inequality If $\Omega \subset \mathbb{R}^d$ bounded then $\exists \quad C(\Omega)\,, C'(\Omega) > 0$ s.t.

$$\|v\|_{L_2} \le C(\Omega)\,\|\nabla v\|_{L_2}\,, \quad \forall v \in H^1_0(\Omega)$$
$$\|v\|_{H^1} \le C'(\Omega)\,\|\nabla v\|_{L_2}\,, \quad \forall v \in H^1_0(\Omega)$$

BVP Solvability Given $\Omega \subset \mathbb{R}^d$ bounded with Lipshitz boundary, $\Gamma_D = \Gamma$, $\tilde{g}_D \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$. Consider the problem of finding $u = \tilde{u} + \tilde{h}_d$, $\tilde{u} \in H^1_0(\Omega)$ s.t.

$$\left(\underline{\underline{a}}\nabla u\,, \nabla v\right) + (bu\,, \nabla v) + (cu\,, v) = \langle f\,, v\rangle\,, \quad \forall v \in H^1_0(\Omega)$$

or

$$\underbrace{\left(\underline{\underline{a}}\nabla \tilde{u}\,, \nabla v\right) + (b\tilde{u}\,, \nabla v) + (c\tilde{u}\,, v)}_{a(\tilde{u},v)} = \underbrace{\langle f\,, v\rangle - a(\tilde{g}_D, v)}_{F(v)}$$

1. If $\underline{\underline{a}} \in L_\infty(\Omega)^{d\times d}$, $b \in L_\infty(\Omega)^d$ and $c \in L_\infty(\Omega)$ then $a(\tilde{u}, v)$ and $F(v)$ are bounded.
2. Additionally, if $\underline{\underline{a}}$ is symmetric and uniformly positive definite, $c \ge 0$ and $\|b\|_\infty < \frac{a_*}{C(\Omega)}$ then $a(\tilde{u}, v)$ is coercive.
3. Additionally, if $b \equiv 0$, then $a(\tilde{u}, v)$ is symmetric and there is an energy $J(\cdot)$, inner product $a(\cdot,\cdot)$ and norm $\|\cdot\|_E$ associated with the problem

By Lax-Milgram: 1. + 2. $\Rightarrow \exists!$ solution $u = \tilde{u} + \tilde{g}_D$
1. + 2. + 3. $\Rightarrow$ $\tilde{u}$ is a strict minimum of J

Proof of 1.

$$\begin{aligned}
|a(\tilde{u}, v)| &\le \left|\left(\underline{\underline{a}}\nabla\tilde{u}, \nabla v\right)\right| + |(b\tilde{u}, \nabla v)| + |(c\tilde{u}, v)| \\
&\le \int_\Omega \left|\underline{\underline{a}}\nabla\tilde{u}\cdot v\right| \mathrm{d}\Omega + \int_\Omega |b\tilde{u}\cdot\nabla v|\,\mathrm{d}\Omega + \int_\Omega |c\tilde{u}v|\,\mathrm{d}\Omega \\
&\le \int_\Omega \left|\underline{\underline{a}}\nabla\tilde{u}\right| |\nabla v|\,\mathrm{d}\Omega + \int_\Omega |b\tilde{u}|\,|\nabla v|\,\mathrm{d}\Omega + \int_\Omega |c\tilde{u}|\,|v|\,\mathrm{d}\Omega \\
&\le \|\underline{\underline{a}}\|_\infty (|\nabla\tilde{u}|\,, |\nabla v|) + \|b\|_\infty (|\tilde{u}|\,, |\nabla v|) + \|c\|_\infty (|\tilde{u}|\,, |v|) \\
&\le \|\underline{\underline{a}}\|_\infty \|\nabla\tilde{u}\|_{L_2} \|\nabla v\|_{L_2} + \|b\|_\infty \|\tilde{u}\|\,\|\nabla v\| + \|c\|_\infty \|\tilde{u}\|\,\|v\| \\
&\le \underbrace{(\|\underline{\underline{a}}\|_\infty + \|b\|_\infty + \|c\|_\infty)}_{\eta^* > 0} \|\tilde{u}\|_{H^1} \|v\|_{H^1}
\end{aligned}$$

$$\begin{aligned}
|F(v)| &\le |\langle f, v\rangle| + |a(\tilde{g}, v)| \\
&\le C\,\|v\|_{H^1} + \eta^*\,\|\tilde{g}_D\|_{H^1}\,\|v\|_{H^1}\,, \qquad f \in H^{-1} \\
&\le \hat{C}\,\|v\|_{H^1}
\end{aligned}$$

Proof of 2.

$$\begin{aligned}
a(v,v) &\ge (\underline{\underline{a}}\nabla v, \nabla v) + (bv, \nabla v) + (cv, v) \\
&\ge \underline{\underline{a}}_*\,\|\nabla v\|^2_{L_2} - |(bv, \nabla v)| + (cv, v)\,, \text{ Cauchy-Schwarz} \\
&\ge \underline{\underline{a}}_*\,\|\nabla v\|^2_{L_2} - \|b\|_\infty \|v\|_{L_2} \|\nabla v\|_{L_2}\,, \text{ Poincare} \\
&\ge \underline{\underline{a}}_*\,\|\nabla v\|^2_{L_2} - \|b\|_\infty C(\Omega)\,\|\nabla v\|_{L_2} \\
&\ge \left(\frac{\underline{\underline{a}}_*}{C(\Omega} - \|b\|_\infty\right) C(\Omega)\,\|\nabla v\|_{L_2} \\
&\ge \left(\frac{\underline{\underline{a}}_*}{C(\Omega} - \|b\|_\infty\right) \frac{C(\Omega)}{(C'(\Omega))^2}\,\|v\|_{H^1} \\
&\ge \eta_*\,\|v\|^2_{H^1}\,, \quad \forall v \in H^1_0(\Omega)
\end{aligned}$$

Stability Choose $v = u_h - g \in V_h$

$$\begin{aligned}
a(u_h, u_h - g) &= f(u_h - g) \\
a(u_h - g, u_h - g) &= f(u_h - g) - a(g, u_h - g) \\
a_*\,\|u_h - g\|^2_V &\le \|f\|_V\,\|u_h - g\|_V + a^*\,\|g\|_V\,\|u_h - g\|_V \\
\|u_h - g\|_V &\le \frac{\|f\|_V + a^*\,\|g\|_V}{a_*}
\end{aligned}$$

Error Estimation

$$a(u - u_h, v) = 0\,, \quad \forall v \in V_h$$

Take $v \mapsto u - u_h - (u - v)\,, \quad v \in V_h$

$$\begin{aligned}
a(u - u_h, u - u_h) &= a(u - u_h, u - v)\,, \quad \forall v \in V_h \\
a_*\,\|u - u_h\|^2_V &\le a^*\,\|u - u_h\|_V\,\|u - v\|_V\,, \quad \forall v \in V_h \\
\|u - u_h\|_V &\le \frac{a^*}{a_*}\,\|u - v\|_V\,, \quad \forall v \in V_h
\end{aligned}$$

Cea's Lemma

$$\|u - u_h\|_V \le \left(\frac{a^*}{a_*}\right) \inf_{v\in V_h} \|u - v\|_V$$

Galerkin Orthogonality

$$a(u - u_h, v) = 0\,, \quad \forall v \in V_h$$

Theorem

$$\|u-u_h\|_a = \inf_{v\in V_h} \|u-v\|_a$$

Proof

$$\begin{aligned}\|u-u_h\|_a^2 &= a(u-u_h, u-uh)\\ &= a(u-u_h, u-v) + \cancel{a(u-u_h, v-u_h)}\\ &\le \|u-u_h\|_a \|u-v\|_a\end{aligned}$$

Ciarlet FEM Definition Let

1. $E \subseteq \mathbb{R}^d$ domain with piecewise smooth boundary
2. $\mathcal{P}$ is a finite dimensional vector space of functions on E (shape functions)
3. $\mathcal{N} = \{N_1, \dots, N_k\}$ is a basis for $\mathcal{P}'$, a set of linear functionals on $\mathcal{P}$ (nodal variables or DOFs)

Then $(E, \mathcal{P}, \mathcal{N})$ is a finite element

Definition Let $\{\phi_1, \dots, \phi_k\}$ be a basis for $\mathcal{P}$ dual to $\mathcal{N}$, $(N_i(\phi_j) = \delta_{ij})$

Unisolvence If dim$\mathcal{P} = k$ and $\{N_1, \dots, N_k\} \subseteq \mathcal{P}'$

$\{N_1, \dots, N_k\}$ is a basis iff

$N_i(v) = 0\,, \quad \forall i\,,$ then $v = 0$

Peano Kernel Theorem If L is a continuous linear functional on $C^{k+1}(a,b)$ s.t. $L(p) = 0\,, \forall p \in \mathbb{P}$. Then

$$L(f) = \int_a^b f^{(k+1)}(\xi) K(\xi) \mathrm{d}\xi$$

$$K(\xi) = \frac{1}{k!} L\left((\cdot - \xi)_+^k\right)$$

Sobolev Spaces

$$W^{m,r(\Omega)} = \{f : D^\alpha f \in L^r(\Omega)\,, \forall \alpha \le m\}$$

Norm and Semi-norm

$$\|f\|_{m,r,\Omega} = \left(\sum_{|\alpha|\le m} \|D^\alpha f\|_{0,r,\Omega}^r\right)^{\frac{1}{r}}$$

$$|f|_{m,r,\Omega} = \left(\sum_{|\alpha| = m} \|D^\alpha f\|_{0,r,\Omega}^r\right)^{\frac{1}{r}}$$

Theorem If E is connected with Lipshitz boundary, then $\exists$ constant $C = C(E)$ s.t. $\forall v \in W^{k+1,r}(E)$

$$\inf_{p\in\mathbb{P}_k(E)} \|v-p\|_{k+1,r,E} \le C(E)\, |v|_{k+1,r,E}$$

Theorem $F : \hat{E} \to E$ affine, $F(\hat{x}) = \underline{\underline{A}}\hat{x} + \underline{b}$

$$\begin{aligned}\hat{v}(\hat{x}) &= v(F(\hat{x})) \in W^{m,r}(\hat{E})\\ v(x) &= \hat{v}(F^{-1}(x))\end{aligned}$$

$\exists C = C(m,d)$ s.t. $\forall v \in W^{m,r}(E)$

$$|\hat{v}|_{m,r,\hat{E}} \le C \left\|\underline{\underline{A}}\right\|^m \left|\det \underline{\underline{A}}\right|^{-\frac{1}{r}} |v|_{m,r,E}$$

$$|v|_{m,r,E} \le C \left\|\underline{\underline{A}}^{-1}\right\|^m \left|\det \underline{\underline{A}}\right|^{\frac{1}{r}} |\hat{v}|_{m,r,\hat{E}}$$

Theorem

$$\inf_{\hat{p}\in\mathbb{P}+k} \|\hat{v}-\hat{p}\|_{k+1,r,\hat{E}} \le C(\hat{E})\, |\hat{v}|_{k+1,r,E}$$

Theorem

$$\begin{aligned}\|A\| &\le \frac{h}{\hat{\rho}}\\ \left\|A^{-1}\right\| &\le \frac{\hat{h}}{\rho}\\ \det A &= \frac{|E|}{|\hat{E}|}\end{aligned}$$

Corollary $\forall v \in W^{k+1,p}(E)$, $m \le k+1$

$$\inf_{p\in\mathbb{P}_k} |v-p|_{m,r,E} \le C(\hat{E}) \frac{h^{k+1}}{\rho^m} |v|_{k+1,r,E}$$

Proof

$$\begin{aligned}\inf_{p\in\mathbb{P}_k} |v-p|_{m,r,E} &\le \left(\inf_{\hat{p}\in\mathbb{P}_k} (\hat{v}-\hat{p})_{m,r,\hat{E}}\right) \left\|A^{-1}\right\|^m |\det A|^{\frac{1}{r}}\\ &\le C(\hat{E}) \left\|A^{-1}\right\|^m |\det A|^{\frac{1}{r}} |\hat{v}|_{k+1,r,\hat{E}}\\ &\le C(\hat{E}) \left\|A^{-1}\right\|^m \|A\|^{k+1} |\hat{v}|_{k+1,r,E}\\ &\le C(\hat{E}) \frac{\hat{h}^m h^{k+1}}{\rho^m \hat{\rho}^{k+1}} |\hat{v}|_{k+1,r,E}\end{aligned}$$

Corollary If the mesh is quasi-regular ($\rho \ge \alpha h$ for some $\alpha > 0$) then

$$\inf_{p\in\mathbb{P}_k} (E)\, |v-p|_{m,r,E} \le C(\hat{E}, \alpha) h^{k+1-m} |v|_{k+1,r,E}$$