Chapter 9

9.6

$$f(x,y,z) = xyz$$

$$\underset{\sim}{a} = (2, 0, -1)$$

$$D^{a} f\,(1,3,2) = \varphi'(0)$$

where $\varphi(t) = f(1+2t,\ 3,\ 2-t) = (1+2t)\,3\,(2-t)$

so: $\varphi'(t) = 2\cdot 3(2-t) + (1+2t)\,3\,(-1)$

$\varphi'(0) = 12 - 3 = 9$

Equivalently: $D^{a} f = \frac{\partial f}{\partial x} a_x + \frac{\partial f}{\partial y} a_y + \frac{\partial f}{\partial z} a_z$

$$= 6\cdot 2 + 3(-1) = 9$$

Find the <u>maximum</u> possible directional derivative at $(1,3,2)$

As $D^{a} f = \frac{\partial f}{\partial x} a_x + \frac{\partial f}{\partial y} a_y + \frac{\partial f}{\partial z} a_z$,

the problem reduces to

$$\max_{a_x^2+a_y^2+a_z^2=1} \left\{ \frac{\partial f}{\partial x} a_x + \frac{\partial f}{\partial y} a_y + \frac{\partial f}{\partial z} a_z \right\}$$

with the obvious solution $\underset{\sim}{a} = \dfrac{\underset{\sim}{\nabla} f}{|\underset{\sim}{\nabla} f|}$

So $\underset{\sim}{a} = \frac{(6,2,3)}{\sqrt{49}} = \left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

The equation of the tangent plane:

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \dots + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

$$6(x-1) + 2(y-3) + 3(z-2) = 0$$

9.10 a) $\operatorname{div} \operatorname{curl} \underset{\sim}{v} = 0$ $\qquad \operatorname{curl} \underset{\sim}{v} = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right) \times \left(v_x, v_y, v_z\right)$

$$\frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial v_x}\right)$$

$$= \left(\frac{\partial v_z}{\partial x \partial y} - \frac{\partial v_z}{\partial y \partial x}\right) + \dots = 0$$

$$\underset{\sim}{\operatorname{curl}}\, \underset{\sim}{\operatorname{grad}}\, \varphi = \left(\partial/\partial x, \partial/\partial y, \partial/\partial z\right) \times \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$$

$$\left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y}, \dots\right) = \underset{\sim}{0}$$

b) $\underset{\sim}{\operatorname{curl}}(\varphi \underset{\sim}{v}) = \underset{\sim}{\nabla} \times (\varphi \underset{\sim}{v}) = \varphi\, \underset{\sim}{\operatorname{curl}}\, \underset{\sim}{v} - \underset{\sim}{\nabla}\varphi \times \underset{\sim}{v}$

straightforward computations

9.20 Use the Gauss theorem

a) $$\int_S \underset{\sim}{v} \circ \underset{\sim}{n}\, d\sigma = \int_V 1+1+1\, dV = 3 \cdot \int_V dV$$

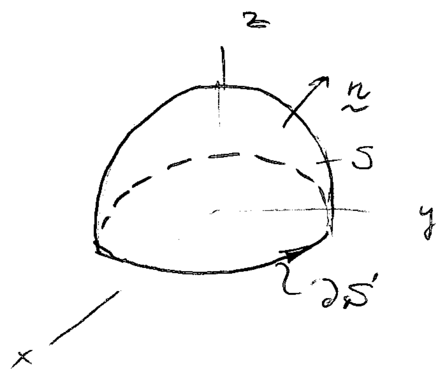
$$= 3\ \pi 4\, 3 = 36\pi$$

b) $$\int_S \underset{\sim}{v} \circ \underset{\sim}{n}\, d\sigma = \int_V \left(0+0+15z^4\right) dV$$

$$= \pi 4 \cdot \int_0^3 15z^4\, dz = 60\pi \left.\frac{z^5}{5}\right|_0^3$$

$$= 12\pi \cdot 3^5 = 2916\pi$$

9.23 a) $\underset{\sim}{v} = (y, -x, 2z+3)$



$$\int_{\partial S} \underset{\sim}{v} \cdot d\underset{\sim}{r} = \int_S \operatorname{curl} \underset{\sim}{v} \circ \underset{\sim}{n}$$

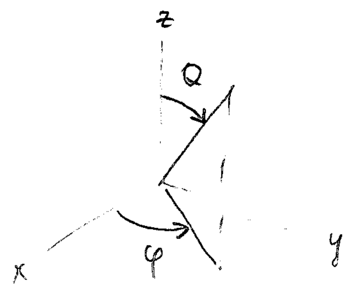
$x^2+y^2+z^2 = a^2$

Parametrization: ∂S . $\begin{cases} x = a\cos\theta \\ y = a\sin\theta \\ z = 0 \end{cases}$ $\theta \in [0, 2\pi]$

$$d\underset{\sim}{r} = (-a\sin\theta,\ a\cos\theta,\ 0)\, d\theta$$

$$\int_{\partial S} \underset{\sim}{v} \circ d\underset{\sim}{r} = \int_0^{2\pi} \left(a\sin\theta(-a\sin\theta) - a\cos\theta\, a\cos\theta\right) d\theta = -2\pi a^2$$

Parametrization: $\quad S: \begin{cases} x = r\sin\theta\cos\varphi & 0<\varphi<2\pi \\ y = r\sin\theta\sin\varphi & 0<\theta<\frac{\pi}{2} \\ z = r\cos\theta \end{cases}$



$$\begin{array}{l} \frac{\partial \underset{\sim}{r}}{\partial\theta}\ (r\cos\theta\cos\varphi,\ r\cos\theta\sin\varphi,\ -r\sin\theta) \\ \times\ \frac{\partial \underset{\sim}{r}}{\partial\varphi}\ (-r\sin\theta\sin\varphi,\ r\sin\theta\cos\varphi,\ 0) \\ \hline d\underset{\sim}{S} = (r^2\sin^2\theta\cos\varphi,\ r^2\sin^2\theta\sin\varphi,\ r^2\cos\theta\sin\theta) \end{array}$$

$$\text{curl}\ \underset{\sim}{v} = \begin{array}{l} (\partial/\partial x\quad \partial/\partial y\quad \partial/\partial z) \\ \times\ (y\quad -x\quad 2z+3) \\ \hline (0,\ 0,\ 1+1) \end{array} = (0,0,2)$$

$$\int_{S'} \text{curl}\ \underset{\sim}{v}\cdot\underset{\sim}{n}\ dS' = \int_0^{2\pi}\int_0^{\pi/2} 2a^2\cos\theta\sin\theta\ d\theta\, d\varphi$$

$$= 2\pi a^2 - \frac{1}{2}\cos 2\theta \Big|_0^{\frac{\pi}{2}} = -2\pi a^2$$

b), c) – same as a)

9.30

a) Check whether the field is conservative

$$\frac{\partial}{\partial x}\left(-2x^3 \sin 2y\right) - \frac{\partial}{\partial y}\left(3x^2 \cos 2y\right) = -6x^2 \sin 2y + 6x^2 \sin 2y = 0$$

So the integral is independent of the path.

Change the path to a segment of a straight line from

$\theta = 0 \Rightarrow r = 0 \Rightarrow x = y = 0$

$\theta = 6\pi \Rightarrow r = 432 \Rightarrow x = 432 \cos 6\pi = 432$

$y = 432 \sin 6\pi = 0$

New path: $\begin{cases} x = t \\ y = 0 \end{cases}$ $\quad t \in [0, 432]$

$$\int_C \dots = \int_0^{432} 3t^2 \cos 0 \cdot 1 \, dt = t^3 \Big|_0^{432} = (432)^3$$

b), c) – same technique as in a)

9.31

a) $\underset{\sim}{v} = (2x^2, -2yz, -y^2-3)$

$$\underset{\sim}{rot}\, \underset{\sim}{v} = \frac{\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \times \begin{pmatrix} 2x^2 & -2yz & -y^2-3 \end{pmatrix}}{(-2y+2y,\ 0-0,\ 0-0)} = \underset{\sim}{0}$$

Let $\underset{\sim}{\nabla}\varphi = \underset{\sim}{v}$

$$\frac{\partial \varphi}{\partial x} = 2x^2 \Rightarrow \varphi(x,y,z) = \frac{2x^3}{3} + C_1(y,z)$$

$$\frac{\partial \varphi}{\partial y} = -2yz \Rightarrow \frac{\partial C_1}{\partial y} = -2yz$$

$$C_1(y,z) = -y^2 z + C_2(z)$$

$$\frac{\partial \varphi}{\partial z} = -y^2-3 \Rightarrow \frac{\partial C_1}{\partial z} = -y^2-3 \Rightarrow \cancel{-y^2} + \frac{\partial C_2}{\partial z} = \cancel{-y^2} - 3$$

$$C_2 = -3z + C$$

$$\varphi(x,y,z) = \frac{2x^3}{3} - y^2 z - 3z + C$$

b) $\underset{\sim}{v} = (1, -4, 0)$

same technique as in a) ; $\varphi(x,y,z) = x - 4y + C$

c) $\underset{\sim}{v} = (3x^2\cos 2y\ ,\ -2x^3\sin 2y\ ,\ 0)$

same technique as in a) $\varphi(x,y,z) = x^3\cos 2y + C$

d) $\underset{\sim}{v} = (0,\ 0,\ z^3)$

same technique as in a) $\varphi(x,y,z) = \frac{z^4}{4} + C$

9.32

a) $\underset{\sim}{v} = (1, -4, 0)$

$\operatorname{div} \underset{\sim}{v} = 0+0+0 = 0$

$$\underset{\sim}{\operatorname{curl}}\ \underset{\sim}{w} = \begin{matrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ w_x & w_y & w_z \end{matrix}$$

$$\left(\frac{\partial w_z}{\partial y} - \frac{\partial w_y}{\partial z},\ \frac{\partial w_x}{\partial z} - \frac{\partial w_z}{\partial x},\ \frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y}\right)$$

Set $w_z \equiv 0$

$$-\frac{\partial w_y}{\partial z} = 1 \quad \Rightarrow \quad w_y(x,y,z) = -z + c_1(x,y)$$

$$\frac{\partial w_x}{\partial z} = -4 \quad \Rightarrow \quad w_x(x,y,z) = -4z + c_2(x,y)$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = 0 \quad \Rightarrow \quad \frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y} = 0$$

Set $c_1 = c_2 \equiv 0$

$$\begin{cases} w_x = -4z + \dfrac{\partial \varphi}{\partial x} \\ w_y = -z + \dfrac{\partial \varphi}{\partial y} \\ w_z = \dfrac{\partial \varphi}{\partial z} \end{cases}$$

$\varphi = \varphi(x,y,z)$ arbitrary !

b) $\underset{\sim}{v} = (z, x, y)$ $\qquad\qquad$ $\operatorname{div} \underset{\sim}{v} = 0$

Set $w_z = 0$

$$-\frac{\partial w_y}{\partial z} = z \Rightarrow w_y(x,y,z) = -\frac{z^2}{2} + c_1(x,y)$$

$$\frac{\partial w_x}{\partial z} = x \Rightarrow w_x(x,y,z) = -xz + c_2(x,y)$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = y \Rightarrow \frac{\partial c_1}{\partial x} - \frac{\partial c_2}{\partial y} = y$$

Set $c_1 \equiv 0$, $c_2(y) = -\frac{y^2}{2}$

$$\begin{cases} w_x = -xz - \frac{y^2}{2} + \frac{\partial \varphi}{\partial x} \\ w_y = -\frac{z^2}{2} + \frac{\partial \varphi}{\partial y} \\ w_z = \frac{\partial \varphi}{\partial z} \end{cases}$$

$\varphi(x,y,z)$ arbitrary

c) $\underset{\sim}{v} = (2y, x^2y^2, -2x^2yz)$ $\qquad\qquad$ $\operatorname{div} \underset{\sim}{v} = 0$

Set $w_z = 0$

$$-\frac{\partial w_y}{\partial z} = 2y \Rightarrow w_y = -2yz + c_1(x,y)$$

$$\frac{\partial w_x}{\partial z} = x^2y^2 \Rightarrow w_x = x^2y^2z + c_2(x,y)$$

$$\frac{\partial w_y}{\partial x} - \frac{\partial w_x}{\partial y} = -2x^2yz \Rightarrow \frac{\partial c_1}{\partial x} - \cancel{2x^2yz} - \frac{\partial c_2}{\partial y} = -\cancel{2x^2yz}$$

Set $c_1 = c_2 = 0$

$$\begin{cases} w_x = x^2y^2z + \frac{\partial \varphi}{\partial x} \\ w_y = -2yz + \frac{\partial \varphi}{\partial y} \\ w_z = \frac{\partial \varphi}{\partial z} \end{cases}$$

$\varphi(x,y,z)$ arbitrary

9.33

$$\underset{\sim}{q} = \underset{\sim}{v}_1 + \underset{\sim}{v}_2 \qquad \underset{\sim}{\text{curl}}\, \underset{\sim}{v}_1 = 0\,, \quad \text{div}\, \underset{\sim}{v}_2 = 0$$

$$\underset{\sim}{v}_1 = \underset{\sim}{\nabla} \varphi, \quad \underset{\sim}{v}_2 = \underset{\sim}{\text{curl}}\, \underset{\sim}{w}$$

$$\underset{\sim}{q} = \underset{\sim}{\nabla} \varphi + \underset{\sim}{\text{curl}}\, \underset{\sim}{w} \quad / \text{div}$$

$$\underline{\text{div}\, q} = \text{div}\, \underset{\sim}{\nabla} \varphi + 0 = \underline{\Delta \varphi}$$

Applying curl on both sides we get

$$\text{curl}\, q = \cancel{\underset{\sim}{\text{curl}}\, \underset{\sim}{\nabla} \varphi}^{\underset{\sim}{0}} + \underset{\sim}{\text{curl}}\, \text{curl}\, \underset{\sim}{w}$$

$$= (\text{identity } 9.33)\ \underset{\sim}{\nabla}\, \text{div}\, \underset{\sim}{w} - \Delta \underset{\sim}{w}$$

Assuming $\text{div}\, \underset{\sim}{w} = 0$ we may look for $\underset{\sim}{w}$ such that

$$\Delta \underset{\sim}{w} = \text{curl}\, q$$

<u>Ex</u>: $q = (x, x^2, 0)$

$$\text{curl}\, q = \begin{array}{c} \begin{pmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & x^2 & 0 \end{pmatrix} \\ \hline (0, 0, 2x) \end{array}$$

$$\Delta w_x = 0 \quad \Rightarrow \quad \text{set } w_x \equiv 0$$
$$\Delta w_y = 0 \quad \Rightarrow \quad \text{set } w_y \equiv 0$$
$$\Delta w_z = 2x \quad \Rightarrow \quad \text{set } w_z \equiv \frac{x^3}{3}$$

check: $\text{div}\, \underset{\sim}{w} = 0$ <u>OK</u>.

So

$$\underset{\sim}{v}_2 = \underset{\sim}{\text{curl}}\ \underset{\sim}{w} = \begin{array}{l} \times \left(\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z\right) \\ \left(0,\ 0,\ \frac{x^3}{3}\right) \\ \hline \left(0, -x^2,\ 0\right) \end{array}$$

$$\underset{\sim}{v}_1 = \underset{\sim}{q} - \underset{\sim}{v}_2 = (x, 0, 0)$$

$$\text{curl}\ \underset{\sim}{v}_1 = \begin{array}{l} \times \left(\partial/\partial x\ ,\ \partial/\partial y\ ,\ \partial/\partial z\right) \\ \left(x \quad 0 \quad 0\right) \\ \hline (0,\ 0,\ 0) \end{array}$$

o.k.

Equivalently, one could start with determining $\underset{\sim}{v}_1$ first.

Note the fact that the decomposition is not unique!