16.3

Equation of a straight line

$$Ax + By + c = 0 \qquad A, B, c \in \mathbb{R}$$

$C = 0$ if the line is passing through the origin.

$$x = \frac{z + \bar{z}}{2} \qquad y = \frac{z - \bar{z}}{2i}$$

$$A\left(\frac{z+\bar{z}}{2}\right) + B\left(\frac{z-\bar{z}}{2i}\right) + c = 0$$

$$\underbrace{\frac{A - iB}{2}}_{\overset{\text{def}}{=} a} z + \underbrace{\frac{A + iB}{2}}_{= \bar{a}} \bar{z} + c = 0$$

$$az + \bar{a}\bar{z} + c = 0$$

Substituting $z = \frac{1}{w}$

$$\frac{a}{w} + \frac{\bar{a}}{\bar{w}} + c = 0$$

Case: $c = 0$ $\quad a\bar{w} + \bar{a}w = 0$ $\quad$ straight line equation

Case: $c \neq 0$ $\quad \left(\frac{a}{c}\right)\bar{w} + \left(\frac{\bar{a}}{c}\right)w + w\bar{w} = 0$ $\quad$ eq. of circle

16.8

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{\bar z}}\right) f(\zeta)\,d\zeta$$

(Since $\left|\frac{R^2}{\bar z}\right| = \frac{R^2}{|\bar z|} = \frac{R^2}{|z|} > \frac{R^2}{R} = R$, i.e.

point $\frac{R^2}{\bar z}$ is outside of the circle $|z| = R$

and, therefore, function $\frac{f(\zeta)}{\zeta - \frac{R^2}{\bar z}}$ is holomorfic

in ζ inside of $|z| = R$.)

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\zeta}{\zeta - z} + \frac{\bar z}{\bar\zeta - \bar z}\right) f(\zeta)\,d\varphi$$

(**algebra** : $|\zeta| = R \Rightarrow$

$$\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{\bar z}} = \frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{\zeta\bar\zeta}{\bar z}}$$

$$= \frac{1}{\zeta - z} - \frac{\bar z}{\zeta(\bar z - \bar\zeta)} = \frac{1}{\zeta}\left(\frac{\zeta}{\zeta - z} + \frac{\bar z}{\bar\zeta - \bar z}\right)$$

$$\zeta = Re^{i\varphi} \Rightarrow d\zeta = iRe^{i\varphi}d\varphi = i\zeta\,d\varphi$$)

$$\frac{\zeta}{\zeta - z} + \frac{\bar z}{\bar\zeta - \bar z} = \frac{\zeta\bar\zeta - \cancel{\zeta\bar z} + \cancel{\zeta\bar z} - z\bar z}{(\zeta - z)(\bar\zeta - \bar z)} = \frac{R^2 - r^2}{|\zeta - z|^2}$$

$$|\zeta - z|^2 = (Re^{i\varphi} - re^{i\theta})(Re^{-i\varphi} - re^{-i\theta})$$

$$= R^2 - Rre^{-i(\varphi-\theta)} - Rre^{i(\varphi-\theta)} + r^2$$

$$= R^2 - 2Rr\cos(\varphi - \theta) + r^2$$

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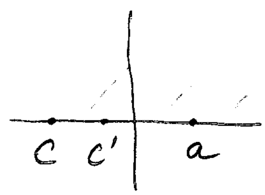
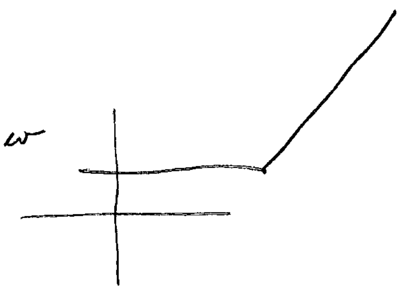
16.10





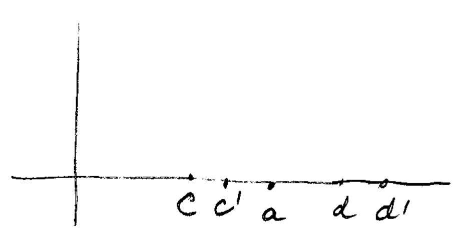
$$\frac{dw}{dz} = A(z-a)^{-\alpha}\quad , \quad \alpha \in \mathbb{R},\ A \in \mathbb{C}$$

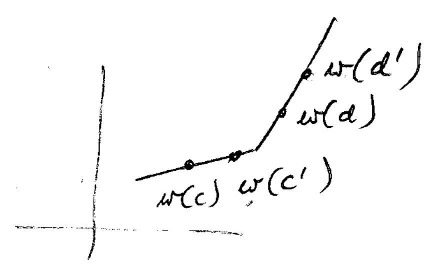
$$w(c') = w(c) + \int_c^{c'} \frac{dw}{dz}\,dz$$

$$= w(c) + A\underbrace{\int_c^{c'} (z-a)^{-\alpha}\,dz}_{\text{a real parameter, if } z = x \text{ is real (some function of } x)}$$

So, indeed, the x axis to the left of a will be transformed into a straight line

a)





$$\frac{w(c') - w(c)}{c' - c} \to \frac{dw}{dz}(c) = A(c-a)^{-\alpha}$$

$$\therefore \quad \mathrm{Arg}\,(w(c') - w(c)) - \mathrm{Arg}\,(c'-c) = \mathrm{Arg}\,A - \alpha \underbrace{\mathrm{Arg}\,(c-a)}_{=\pi}$$

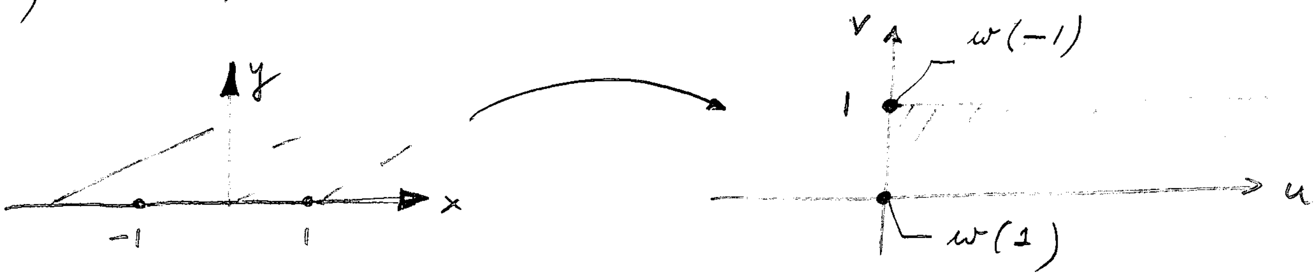
Similarly

$$\mathrm{Arg}\,(w(d') - w(d)) - \mathrm{Arg}\,(d'-d) = \mathrm{Arg}\,A - \alpha\,\mathrm{Arg}\,(d-a)$$

Subtracting :

$$\mathrm{Arg}\left(w(d') - w(d)\right) - \mathrm{Arg}\left(w(c') - w(c)\right) = -\alpha(0 - \pi)$$

c) $w = A \sin^{-1} z + B$



$$i = w(-1) = A \sin^{-1}(-1) + B = A\left(-\frac{\pi}{2}\right) + B$$
$$0 = w(1) = A \sin^{-1}(1) + B = A\left(\frac{\pi}{2}\right) + B$$

$$A(-\pi) = i \Rightarrow A = -\frac{i}{\pi}$$

$$\frac{i}{\cancel{\pi}} \frac{\cancel{\pi}}{2} = B \Rightarrow B = \frac{i}{2}$$

The mapping we need is : $w = -\frac{i}{\pi} \sin^{-1} z + \frac{i}{2}$

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