CAM 389C Exercise Set I.6

Truman Ellis

October 4, 2010

Problem 1

Consider the small deformations and heating of a thermo-elastic solid constructed of a material characterized by the following constitutive equations:

$$\begin{aligned}
&\text{Free energy: } \rho_0\psi_0 = \frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{e})^2 + \mu\boldsymbol{e}:\boldsymbol{e} + c(\mathrm{tr}\,\boldsymbol{e})\theta + \frac{c_0}{2}\theta^2\,,\\
&\text{Heat Flux: } \quad \mathbf{q}_0 = k\nabla\theta\,,
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{e} =& \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T) = \text{the "infinitesimal" strain tensor } (\approx \mathbf{E})\,,\\
\boldsymbol{u} =& \text{ the dispacement field}\,,\\
\theta =& \text{ the temperature field}\,,\\
\lambda,\mu,c,c_0,k =& \text{ material constants.}
\end{aligned}$$

A body $\mathbb{B}$ is constructed of such a material and is subjected to body forces $\mathbf{f}_0$ and to surface contact forces $\boldsymbol{g}$ on a portion Γ_g of its boundary $\Gamma_g \subset \partial\Omega_0$. On the remainder of its boundary, $\Gamma_u = \partial\Omega_0 \setminus \Gamma_g$, the displacements $\boldsymbol{u}$ are prescribed as zero ($\boldsymbol{u} = \mathbf{0}$ on Γ_u). The mass density of the body is ρ_0, and, when in its reference configuration at time $t = 0$, $\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x})$, $\partial\boldsymbol{u}(\boldsymbol{x}, 0)/\partial t = \boldsymbol{v}_0(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega_0$, where $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ are given functions. A portion Γ_q of the boundary is heated, resulting in a prescribed heat flux $h = \mathbf{q}\cdot\mathbf{n}$, and the complementary boundary, $\Gamma_\theta = \partial\Omega_0 \setminus \Gamma_q$ is subjected to a prescribed temperature $\theta(\boldsymbol{x}, t) = \tau(\boldsymbol{x}, t)$, $\boldsymbol{x} \in \Gamma_\theta$.

Develop a mathematical model of this physical phenomena (a set of partial differential equations, boundary and initial conditions): the dynamic, thermo-mechanical behavior of a thermoelastic solid.

Solution

From the given formula for the Helmholtz free energy, we can see that

$$\psi_0 = \Psi(\boldsymbol{e}, \theta)\,.$$

According to the Coleman-Noll Method, we can rewrite our constraint on the second law of thermodynamics as

$$-\rho_0\dot{\psi}_0 - \rho_0\eta_0\dot{\theta} + \mathbf{S} : \dot{\mathbf{E}} - \frac{1}{\theta}\mathbf{q}_0 \cdot \nabla\theta \geq 0\,.$$

Since we are dealing with small deformations, we can assume from here on that $\mathbf{E} \approx \boldsymbol{e}$, and the second law constraint becomes

$$-\rho_0\dot{\psi}_0 - \rho_0\eta_0\dot{\theta} + \mathbf{S} : \dot{\boldsymbol{e}} - \frac{1}{\theta}\mathbf{q}_0 \cdot \nabla\theta \geq 0\,.$$

When we substitute

$$\dot{\psi}_0 = \frac{\partial\Psi}{\partial\boldsymbol{e}} : \dot{\boldsymbol{e}} + \frac{\partial\Psi}{\partial\theta} : \dot{\theta}$$

into the constraint, we get

$$\left(\mathbf{S} - \rho_0\frac{\partial\Psi}{\partial\boldsymbol{e}}\right) : \dot{\boldsymbol{e}} - \rho_0\left(\frac{\partial\Psi}{\partial\theta} + \eta_0\right)\dot{\theta} - \frac{1}{\theta}\mathbf{q}_0 \cdot \nabla\theta \geq 0\,.$$

It is sufficient to satisfy this condition that the coefficients of the rates be zero. Thus

$$\mathbf{S} = \rho_0\frac{\partial\Psi}{\partial\boldsymbol{e}}\,,\ \text{an} \qquad \eta_0 = -\frac{\partial\Psi}{\partial\theta}\,.$$

Now,

$$\begin{aligned}\rho_0\frac{\partial\Psi}{\partial\boldsymbol{e}} &= \frac{\partial}{\partial\boldsymbol{e}}\left(\frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{e})^2 + \mu\boldsymbol{e} : \boldsymbol{e} + c(\mathrm{tr}\,\boldsymbol{e})\theta + \frac{c_0}{2}\theta^2\right)\\ &= \frac{1}{2}\lambda\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}(\mathrm{tr}\,\boldsymbol{e})^2 + \mu\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}\,\mathrm{tr}(\boldsymbol{e}^T\boldsymbol{e}) + c\theta\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}(\mathrm{tr}\,\boldsymbol{e})\end{aligned}$$

Since $\boldsymbol{e}$ is square and symmetric, $\boldsymbol{e}^T\boldsymbol{e} = \boldsymbol{e}^2$. Therefore,

$$\begin{aligned}\rho_0\frac{\partial\Psi}{\partial\boldsymbol{e}} &= \lambda(\mathrm{tr}\,\boldsymbol{e})\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}\,\mathrm{tr}(\boldsymbol{e}) + \mu\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}\,\mathrm{tr}(\boldsymbol{e}^2) + c\theta\frac{\mathrm{d}}{\mathrm{d}\boldsymbol{e}}\,\mathrm{tr}(\boldsymbol{e})\\ &= \left(\lambda(\mathrm{tr}\,\boldsymbol{e}) + 2\mu\,\mathrm{tr}(\boldsymbol{e}) + c\theta\right)\mathbf{I}\,.\end{aligned}$$

Next,

$$\begin{aligned}\rho_0\frac{\partial\Psi}{\partial\theta} &= \frac{\partial}{\partial\theta}\left(\frac{1}{2}\lambda(\mathrm{tr}\,\boldsymbol{e})^2 + \mu\boldsymbol{e} : \boldsymbol{e} + c(\mathrm{tr}\,\boldsymbol{e})\theta + \frac{c_0}{2}\theta^2\right)\\ &= c\,\mathrm{tr}\,\boldsymbol{e} + c_0\theta\,.\end{aligned}$$

Therefore, we can automatically satisfy the second law if we use

$$\mathbf{S} = \left(\lambda(\mathrm{tr}\,\boldsymbol{e}) + 2\mu\,\mathrm{tr}(\boldsymbol{e}) + c\theta\right)\mathbf{I}\,,$$

and

$$\eta_0 = -\frac{1}{\rho_0}\left(c\,\mathrm{tr}\,\boldsymbol{e} + c_0\theta\right)\,.$$

Please note that $\mathbf{F} = \mathbf{I} + \nabla\boldsymbol{u}$.

Therfore, our system of partial differential equations describing this thermoelastic solid are

Conservation of Mass

$$\begin{aligned}\rho_0 &= \rho \det \mathbf{F} \\ &= \rho \det(\mathbf{I} + \nabla \boldsymbol{u}) \\ &= \text{const}\end{aligned}$$

Conservation of Linear Momentum

$$\begin{aligned}\rho_0 \ddot{\boldsymbol{u}} &= \operatorname{Div} \mathbf{F}\mathbf{S} + \mathbf{f}_0 \\ &= \operatorname{Div}((\mathbf{I} + \nabla \boldsymbol{u})\mathbf{S}) + \mathbf{f}_0 \\ &= \operatorname{Div}(\mathbf{S} + \nabla \boldsymbol{u}\,\mathbf{S}) + \mathbf{f}_0\end{aligned}$$

Conservation of Angular Momentum

$$\mathbf{S} = \mathbf{S}^T$$

Conservation of Energy

$$\begin{aligned}\mathbf{S} : \dot{\boldsymbol{e}} - \operatorname{Div} \mathbf{q}_0 + r_0 &= \rho_0 \dot{e_0} \\ \mathbf{S} : \dot{\boldsymbol{e}} - \operatorname{Div} \operatorname{Grad} k\theta + r_0 &= \rho_0(\dot{\psi} + \theta\dot{\eta} + \eta\dot{\theta})\end{aligned}$$

Second Law of Thermodynamics

$$\rho_0 \dot{\eta_0} + \operatorname{Div} \frac{k \operatorname{Grad} \theta}{\theta} - \frac{r_0}{\theta} \geq 0$$

This should be satisfied automatically by our choice of $\mathbf{S}$ and η_0. Also, according to Fourier's law, k should be negative.

Where

$$\mathbf{S} = (\lambda(\operatorname{tr} \boldsymbol{e}) + 2\mu \operatorname{tr}(\boldsymbol{e}) + c\theta)\, \mathbf{I}\,,$$

$$\eta_0 = -\frac{1}{\rho_0}\left(c \operatorname{tr} \boldsymbol{e} + c_0\theta\right)\,,$$

and

$$\rho_0 \dot{\psi}_0 = (\lambda(\operatorname{tr} \boldsymbol{e}) + 2\mu \operatorname{tr}(\boldsymbol{e}) + c\theta)\, \mathbf{I} : \dot{\boldsymbol{e}} + (c \operatorname{tr} \boldsymbol{e} + c_0\theta)\, \dot{\theta}\,.$$

The kinematic boundary conditions are

$$\begin{aligned}\rho_0 \ddot{\boldsymbol{u}} &= \boldsymbol{g} \quad \text{on } \Gamma_g \\ \boldsymbol{u} &= 0 \quad \text{on } \Gamma_u\,,\end{aligned}$$

while the thermodynamic boundary conditions are

$$\begin{aligned}\rho_0 \dot{e} &= h && \text{on} \quad \Gamma_q \\ \theta &= \tau(\boldsymbol{x}, t) && \text{on} \quad \Gamma_\theta.\end{aligned}$$

The initial kinematic conditions are

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0(\boldsymbol{x}) \\ \frac{\partial \boldsymbol{u}(\boldsymbol{x},0)}{\partial t} &= \boldsymbol{v}_0(\boldsymbol{x})\,. \end{aligned}$$

Additionally, the initial temperature and entropy must be chosen so that they satisfy the governing equations.