Chapter 8

8.5 $\quad a = i - k \qquad\qquad a = (1, 0, -1)$

$b = 2i + j + k \qquad\qquad b = (2, 1, 1)$

$c = -i + j \qquad\qquad c = (-1, 1, 0)$

- $2a - 3b = 2(i - k) - 3(2i + j + k)$
  $= -4i - 3j - 5k$

- $a \cdot b = b \cdot a = (i - k)\cdot(2i + j + k) = 1\cdot 2 + 0\cdot 1 - 1\cdot 1 = 1$

- $a \times b = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{vmatrix} = i - 3j + k$

- $b \times 3a = -3a \times b = -3(a \times b) = -3i + 9j - 3k$

$$b \times c = \times \begin{array}{c} (2 \quad 1 \quad 1) \\ (-1 \quad 1 \quad 0) \\ \hline (-1, -1, 3) \end{array}$$

- $a \times (b \times c) = \times \begin{array}{c} (1 \quad 0 \quad -1) \\ (-1, -1, 3) \\ \hline (-1, -2, -1) \end{array} = -i - 2j - k$

$\cos \measuredangle (a, b) = \dfrac{a \cdot b}{|a||b|} = \dfrac{1}{\sqrt{1+1}\,\sqrt{4+1+1}} = \dfrac{1}{\sqrt{12}}$

- $\measuredangle (a, b) = \underset{\cos^{-1}}{\arccos}\left(\dfrac{1}{\sqrt{12}}\right)$

- Projection of $\mathbf{b}$ on $\mathbf{a}$

$$\left(\mathbf{b}\cdot\frac{\mathbf{a}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \left[(2,1,1)\cdot\left(\tfrac{1}{\sqrt{2}},0,-\tfrac{1}{\sqrt{2}}\right)\right]\left(\tfrac{1}{\sqrt{2}},0,-\tfrac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(\tfrac{1}{\sqrt{2}},0,-\tfrac{1}{\sqrt{2}}\right) = \left(\tfrac{1}{2},0,-\tfrac{1}{2}\right)$$

$$= \tfrac{1}{2}\mathbf{i} - \tfrac{1}{2}\mathbf{k}$$

Projection of $\mathbf{a}$ on $\mathbf{b}$

$$\left(\mathbf{a}\cdot\frac{\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|} = \left[(1,0,-1)\cdot\left(\tfrac{2}{\sqrt{6}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}}\right)\right]\left(\tfrac{2}{\sqrt{6}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}}\right)$$

$$= \frac{1}{\sqrt{6}}\left(\tfrac{2}{\sqrt{6}},\tfrac{1}{\sqrt{6}},\tfrac{1}{\sqrt{6}}\right) = \left(\tfrac{1}{3},\tfrac{1}{6},\tfrac{1}{6}\right)$$

$$= \tfrac{1}{3}\mathbf{i} + \tfrac{1}{6}\mathbf{j} + \tfrac{1}{6}\mathbf{k}$$

- $(\mathbf{a}+\alpha\mathbf{b})\cdot\mathbf{a} = 0$

$$\mathbf{a}\cdot\mathbf{a} + \alpha(\mathbf{b}\cdot\mathbf{a}) = 0$$
$$2 + \alpha\, 1 = 0 \implies \alpha = -2$$

- $$(\mathbf{a}+\alpha\mathbf{b}+\beta\mathbf{c})\mathbf{a} = \mathbf{a}\cdot\mathbf{a} + \alpha(\mathbf{b}\cdot\mathbf{a}) + \beta(\mathbf{c}\cdot\mathbf{a}) = 0$$
$$(\mathbf{a}+\alpha\mathbf{b}+\beta\mathbf{c})\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \alpha(\mathbf{b}\cdot\mathbf{b}) + \beta(\mathbf{c}\cdot\mathbf{b}) = 0$$

$$\left.\begin{array}{l} 2 + \alpha\cdot 1 + \beta\cdot(-1) = 0 \\ 1 + \alpha\cdot 6 + \beta(-1) = 0 \end{array}\right\} \implies \underline{\alpha = \tfrac{1}{5},\ \beta = 11/5}$$

8.11

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \circ (\underset{\sim}{a} \times \underset{\sim}{b}) = \begin{vmatrix} \underset{\sim}{a} \circ \underset{\sim}{a} & \underset{\sim}{a} \circ \underset{\sim}{b} \\ \underset{\sim}{a} \circ \underset{\sim}{b} & \underset{\sim}{b} \circ \underset{\sim}{b} \end{vmatrix}$$

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \circ (\underset{\sim}{a} \times \underset{\sim}{b}) = [\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{a} \times \underset{\sim}{b}] \quad \text{(mixed product)}$$

$$= \underset{\sim}{a} \circ \underset{\sim}{b} \times (\underset{\sim}{a} \times \underset{\sim}{b})$$

$$= \underset{\sim}{a} \circ ((\underset{\sim}{b} \circ \underset{\sim}{b})\underset{\sim}{a} - (\underset{\sim}{a} \circ \underset{\sim}{b})\underset{\sim}{b})$$

$$= (\underset{\sim}{b} \circ \underset{\sim}{b})(\underset{\sim}{a} \circ \underset{\sim}{a}) - (\underset{\sim}{a} \circ \underset{\sim}{b})(\underset{\sim}{a} \circ \underset{\sim}{b})$$

#

8.20

- Show that $\frac{dt}{ds}$ is perpendicular to $\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{ds}$

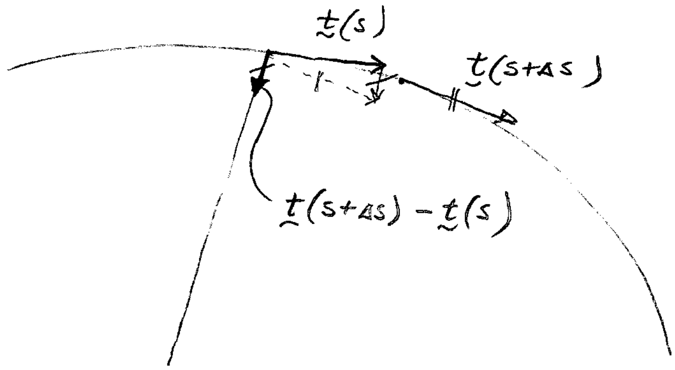
$$\underset{\sim}{t} \circ \underset{\sim}{t} = 1 \quad / \frac{d}{ds}$$

$$2\underset{\sim}{t} \circ \frac{dt}{ds} = 0$$

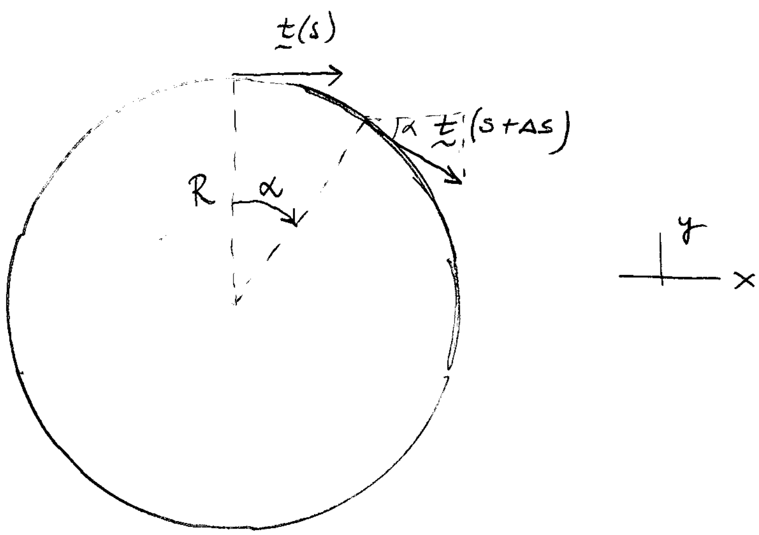
Define the curvature $\varkappa = \left| \frac{d\underset{\sim}{t}}{ds} \right|$

So: $\frac{d\underset{\sim}{t}}{ds} = \varkappa \underset{\sim}{n}$, where $|\underset{\sim}{n}| = 1$, $\underset{\sim}{n} \perp \underset{\sim}{t}$

- With the help of suitable sketches show that the multiplier $\varkappa$ can be interpreted as $1/\rho$, where ρ is the local radius of curvature of C, and that $\underset{\sim}{n}$ points towards the center of curvature



For the circle :

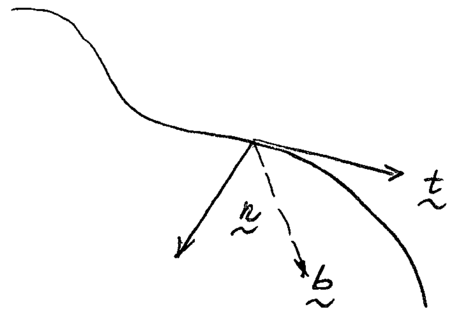


$$\underset{\sim}{t}(\alpha) = (\cos\alpha, -\sin\alpha) \qquad s = \alpha R$$

$$\frac{d\underset{\sim}{t}}{ds} = \frac{d\underset{\sim}{t}}{d\alpha}\frac{d\alpha}{ds} = (-\sin\alpha, -\cos\alpha)\,\frac{1}{R}$$

$$\frac{d\underset{\sim}{t}}{ds}(0) = (0, -1)\,\frac{1}{R}$$

Introduce $\quad \mathbf{b} \overset{\text{def}}{=} \mathbf{t} \times \mathbf{n}$



- Show tha $\quad \dfrac{d\mathbf{b}}{ds} \parallel \mathbf{n}$

$$\frac{d\mathbf{b}}{ds} = \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds} = \overset{\mathbf{0}}{\cancel{\kappa\, \mathbf{n} \times \mathbf{n}}} + \mathbf{t} \times \frac{d\mathbf{n}}{ds}$$

$$\Rightarrow \quad \frac{d\mathbf{b}}{ds} \perp \mathbf{t}$$

At the same time $\quad |\mathbf{b}| = 1 \quad \Rightarrow \quad \dfrac{d\mathbf{b}}{ds} \perp \mathbf{b}$

So: $\quad \dfrac{d\mathbf{b}}{ds} \parallel \mathbf{t} \times \mathbf{b} \quad \Rightarrow \quad \dfrac{d\mathbf{b}}{ds} \parallel \mathbf{n}$

Define the torsion $\tau \qquad \dfrac{d\mathbf{b}}{ds} = \tau \mathbf{n}$

- Show that $\quad \dfrac{d\mathbf{n}}{ds} = -\kappa \mathbf{t} - \tau \mathbf{b}$

$$\mathbf{n} = \mathbf{b} \times \mathbf{t}$$

$$\frac{d\mathbf{n}}{ds} = \frac{d\mathbf{b}}{ds} \times \mathbf{t} + \mathbf{b} \times \frac{d\mathbf{t}}{ds} = \tau\, \mathbf{n} \times \mathbf{t} + \mathbf{b} \times \kappa \mathbf{n}$$

$$= \tau(-\mathbf{b}) + \kappa \big(\underbrace{\mathbf{b} \times \mathbf{n}}_{= -\mathbf{t}}\big)$$

- Show that

$$\varkappa = \left\| \frac{d^2\underset{\sim}{r}}{ds^2} \right\|$$

Trivial, since $\underset{\sim}{t} = \dfrac{d\underset{\sim}{r}}{ds}$

- Show that

$$\left[\underset{\sim}{t}, \frac{d\underset{\sim}{t}}{ds}, \frac{d^2\underset{\sim}{t}}{ds^2} \right] = \underset{\sim}{t} \circ \left(\frac{d\underset{\sim}{t}}{ds} \times \frac{d^2\underset{\sim}{t}}{ds^2} \right) = -\varkappa^2\tau$$

$$\frac{d^2}{ds^2}\underset{\sim}{t} = \frac{d}{ds}\frac{d\underset{\sim}{t}}{ds} = \frac{d}{ds}(\varkappa\underset{\sim}{n}) = \frac{d\varkappa}{ds}\underset{\sim}{n} + \varkappa\frac{d\underset{\sim}{n}}{ds}$$

$$= \frac{d\varkappa}{ds}\underset{\sim}{n} + \varkappa(-\tau\underset{\sim}{b} - \varkappa\underset{\sim}{t})$$

$$= -\varkappa^2\underset{\sim}{t} + \frac{d\varkappa}{ds}\underset{\sim}{n} - \varkappa\tau\underset{\sim}{b}$$

$$\frac{d\underset{\sim}{t}}{ds} = \varkappa\underset{\sim}{n}$$

$$\frac{d\underset{\sim}{t}}{ds} \times \frac{d^2\underset{\sim}{t}}{ds^2} = \varkappa\underset{\sim}{n} \times \left[-\varkappa^2\underset{\sim}{t} + \frac{d\varkappa}{ds}\underset{\sim}{n} - \varkappa\tau\underset{\sim}{b} \right]$$

$$= -\varkappa^2\underbrace{\underset{\sim}{n}\times\underset{\sim}{t}}_{= -\underset{\sim}{b}} - \varkappa^2\tau\underbrace{\underset{\sim}{n}\times\underset{\sim}{b}}_{= \underset{\sim}{t}}$$

$$= \varkappa^3\underset{\sim}{b} - \varkappa^2\tau\underset{\sim}{t}$$

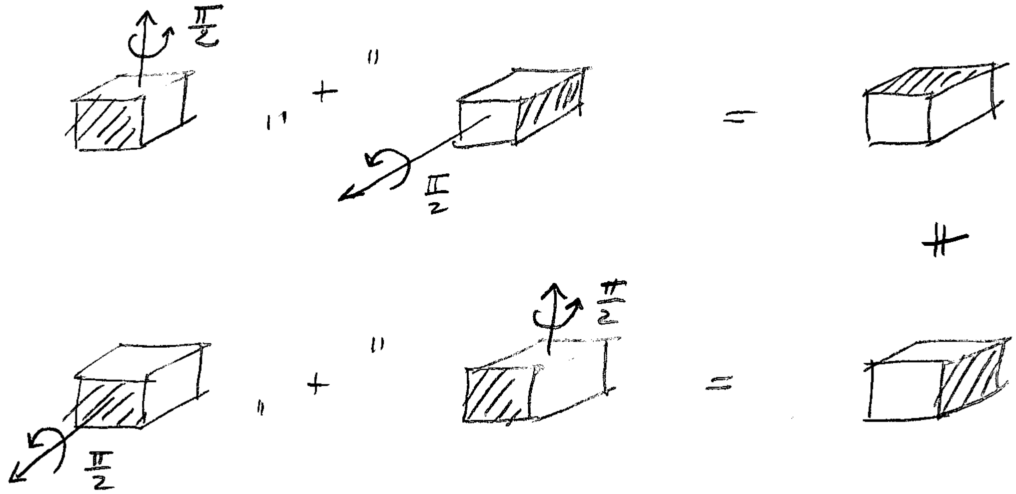
$$\underset{\sim}{t} \circ \left(\frac{d\underset{\sim}{t}}{ds} \times \frac{d^2\underset{\sim}{t}}{ds^2} \right) = \underset{\sim}{t} \circ (\varkappa^3\underset{\sim}{b} - \varkappa^2\tau\underset{\sim}{t}) = -\varkappa^2\tau$$

As $\varkappa = \left\| \dfrac{d^2\underset{\sim}{r}}{ds} \right\|$ we get

$$\tau = -\frac{\left[\underset{\sim}{t}, \frac{d\underset{\sim}{t}}{ds}, \frac{d^2\underset{\sim}{t}}{ds^2} \right]}{\left\| \frac{d^2\underset{\sim}{r}}{ds^2} \right\|^2} = -\frac{\left[\frac{d\underset{\sim}{r}}{ds}, \frac{d^2\underset{\sim}{r}}{ds^2}, \frac{d^3\underset{\sim}{r}}{ds^3} \right]}{\left\| \frac{d^2\underset{\sim}{r}}{ds^2} \right\|^2}$$

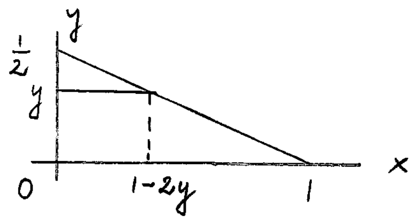
\#

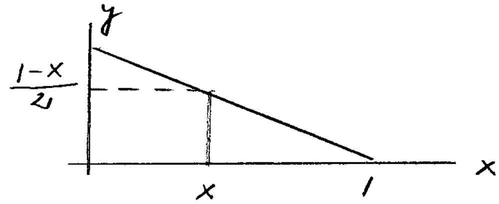
8.25



8.29a)

$$\int_0^{1/2}\int_0^{1-2y} e^{\frac{x}{x+2y}}\,dx\,dy = \int_0^1\int_0^{\frac{1-x}{2}} e^{\frac{x}{x+2y}}\,dy\,dx = (*)$$





$u = x \qquad 0<x<1 \quad \Rightarrow \quad 0<u<1$

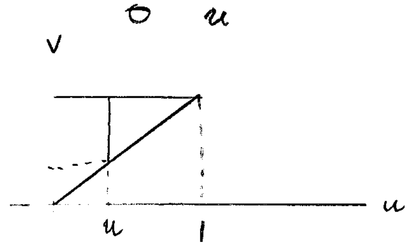
$v = x+2y$

For a fixed u, the $x=u$ will be fixed as well and y will vary from 0 to $\frac{1-x}{2} = \frac{1-u}{2}$. So v will vary from u to 1

$$u<v<1$$

Consequently, $\left(\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2 \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2}\right)$

$$(*) = \int_0^1\int_u^1 e^{\frac{u}{v}}\,dv\,du = \int_0^1\int_0^v e^{\frac{u}{v}}\,du\,dv$$

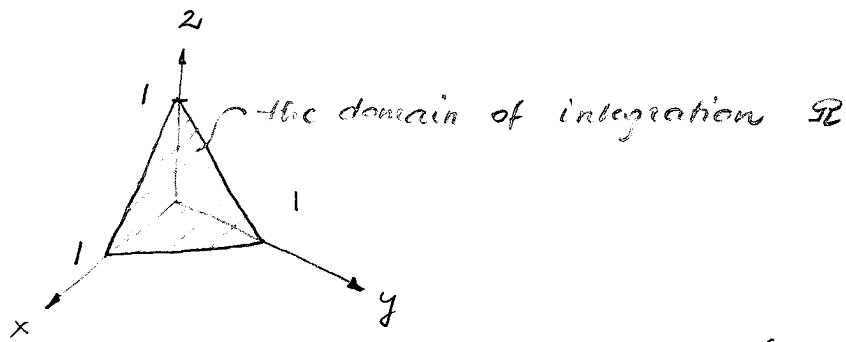


$$= \int_0^1 v e^{\frac{u}{v}}\Big|_0^v\,dv = \int_0^1 v(e-1) = \frac{e-1}{2}$$

#

8.29b

$$\int_0^1 \int_0^{1-z} \int_0^{1-y-z} e^{\frac{x}{x+y}}\, dx\, dy\, dz \quad = (*)$$

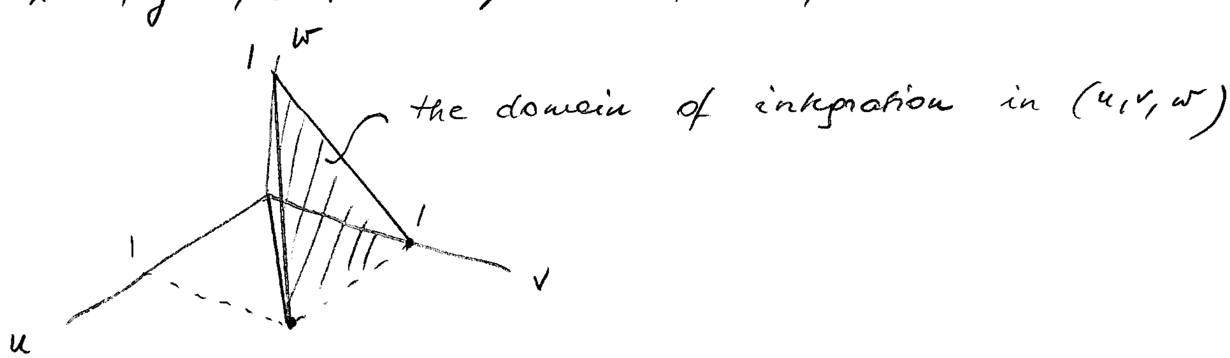


$$\Omega : \begin{cases} 0 < z < 1 \\ 0 < y < 1-z \\ 0 < x < 1-y-z \end{cases}$$

$\begin{cases} u = x \\ v = x+y \\ w = z \end{cases}$ is a linear transformation

In particular, it maps straight lines into straight lines and planes into planes, i.e. the (u, v, w) - domain must be a tetrahedron as well

$x=0, y=0, z=0 \Rightarrow u=v=w=0$
$x=1, y=0, z=0 \Rightarrow u=1, v=1, z=0$
$x=0, y=1, z=0 \Rightarrow u=0, v=1, z=0$
$x=0, y=0, z=1 \Rightarrow u=0, v=0, w=1$



$$\begin{cases} 0 < v < 1 \\ 0 < u < v \\ 0 < w < 1-v \end{cases}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad \Rightarrow \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = 1$$

$$(*) = \int_0^1 \int_0^v \int_0^{1-v} e^{\frac{u}{v}} \, dw \, du \, dv$$

$$= \int_0^1 \int_0^v e^{\frac{u}{v}} w \Big|_0^{1-v} du \, dv$$

$$= \int_0^1 \int_0^v e^{\frac{u}{v}} (1-v) \, du \, dv$$

$$= \int_0^1 v e^{\frac{u}{v}} (1-v) \Big|_0^v dv$$

$$= \int_0^1 v(1-v)[e-1] \, dv$$

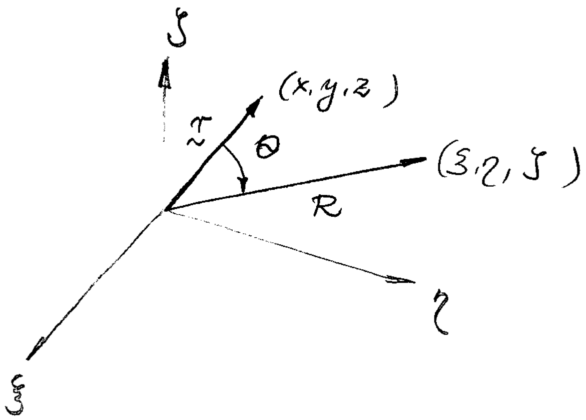
$$= (e-1) \left(\frac{v^2}{2} - \frac{v^3}{3} \right) \Big|_0^1$$

$$= \frac{e-1}{6}$$

#

8.30

$$-\frac{1}{8\pi^3}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{e^{i(\xi x+\eta y+\zeta z)}}{\xi^2+\eta^2+\zeta^2}\,d\xi\,d\eta\,d\zeta \quad = \quad (*)$$



$$\begin{cases} \xi = R\sin\theta\cos\varphi & 0<\theta<\pi \\ \eta = R\sin\theta\sin\varphi & 0<\varphi<2\pi \\ \zeta = R\cos\theta & 0<R<+\infty \end{cases}$$

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(R,\theta,\varphi)} = \begin{vmatrix} \sin\theta\cos\varphi & R\cos\theta\cos\varphi & -R\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & R\cos\theta\cos\varphi & R\sin\theta\cos\varphi \\ \cos\theta & -R\sin\theta & 0 \end{vmatrix}$$

$$= -R^2\sin\theta$$

$$(*) = -\frac{1}{8\pi^3}\int_0^{\pi}\int_0^{2\pi}\int_0^{\infty}\frac{e^{irR\cos\theta}}{\cancel{R^2}}\,\cancel{R^2}\sin\theta\,dR\,d\varphi\,d\theta$$

$$= -\frac{1}{4\pi^2}\int_0^{\infty}\int_0^{\pi} e^{irR\cos\theta}\sin\theta\,d\theta\,dr$$

$$\cos\theta = \gamma$$
$$-\sin\theta \, d\theta = d\gamma$$

$$= -\frac{1}{4\pi^2}\int_0^\infty \int_{-1}^{1} e^{ir R\gamma}\, d\gamma \, dR$$

$$= -\frac{1}{4\pi^2}\int_0^\infty \frac{1}{irR}\, e^{irR\gamma}\Big|_{-1}^{1} dR$$

$$= -\frac{1}{4\pi^2}\int_0^\infty \frac{1}{irR}\left(e^{irR} - e^{-irR}\right) dR$$

$$= -\frac{1}{4\pi^2}\int_0^\infty \frac{1}{\cancel{i}rR}\, 2\cancel{i}\sin(rR)\, dR = -\frac{1}{2\pi^2 r}\int_0^\infty \frac{\sin\alpha}{\alpha}\, d\alpha$$

$$= -\frac{1}{2\pi r^2}\lim_{x\to\infty} Si(x)$$

$$rR = \alpha$$
$$dR = \frac{1}{r}\, d\alpha$$

$$= (*)$$

But (recall problem 5.7)

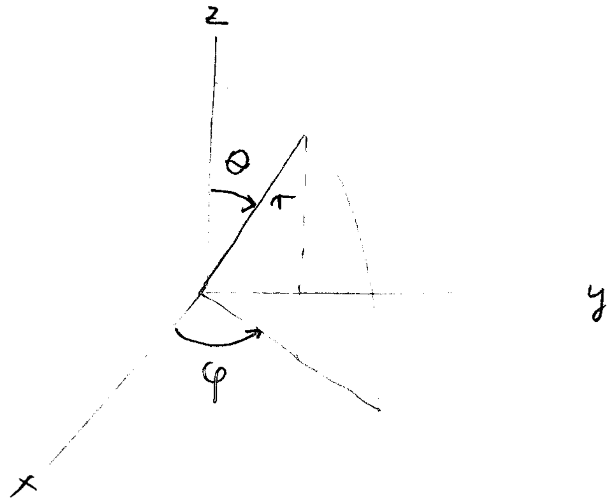
$$Si\left(\left(n+\tfrac{1}{2}\right)x\right) = \int_0^{(n+\frac{1}{2})x} \frac{\sin\alpha}{\alpha}\, d\alpha = \int_0^x \frac{\sin\left(n+\frac{1}{2}\right)t}{t}\, dt$$

$$\approx \frac{\pi}{2}\, S_n(x) \longrightarrow \frac{\pi}{2}$$

where $S_n(x)$ is the partial sum of Fourier series for the step function.

So $(*) = -\dfrac{1}{4\pi r}$ #

Derive the formulas for $\underset{\sim}{v}$ and $\underset{\sim}{a}$ in the spherical system of coordinates.



$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

$$\frac{\partial \underset{\sim}{r}}{\partial r} = \underset{\sim}{u}_r = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\frac{\partial \underset{\sim}{r}}{\partial \theta} = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta)$$

$$\left|\frac{\partial \underset{\sim}{r}}{\partial \theta}\right| = r \quad \Rightarrow \quad \underset{\sim}{u}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\frac{\partial \underset{\sim}{r}}{\partial \varphi} = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)$$

$$\left|\frac{\partial \underset{\sim}{r}}{\partial \varphi}\right| = r\sin\theta \quad \Rightarrow \quad \underset{\sim}{u}_\varphi = (-\sin\varphi, \cos\varphi, 0)$$

So:

$$\frac{\partial \underset{\sim}{u}_r}{\partial \theta} = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) = \underset{\sim}{u}_\theta$$

$$\frac{\partial \underset{\sim}{u}_r}{\partial \varphi} = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) = \sin\theta\, \underset{\sim}{u}_\varphi$$

$$\frac{\partial \mathbf{u}_\theta}{\partial \theta} = (-\sin\theta\cos\varphi,\ -\sin\theta\sin\varphi,\ -\cos\theta) = -\mathbf{u}_r$$

$$\frac{\partial \mathbf{u}_\theta}{\partial \varphi} = (-\cos\theta\sin\varphi,\ \cos\theta\cos\varphi,\ 0) = \cos\theta\,\mathbf{u}_\varphi$$

$$\frac{d\mathbf{u}_\varphi}{d\varphi} = (-\cos\varphi,\ -\sin\varphi,\ 0) = -\cos\theta\,\mathbf{u}_\theta - \sin\theta\,\mathbf{u}_r$$

Now

$$\mathbf{r} = r\,\mathbf{u}_r$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{u}_r + r\,\dot{\mathbf{u}}_r$$
$$= \dot{r}\,\mathbf{u}_r + r\left(\frac{\partial \mathbf{u}_r}{\partial\theta}\dot{\theta} + \frac{\partial \mathbf{u}_r}{\partial\varphi}\dot{\varphi}\right)$$
$$= \dot{r}\,\mathbf{u}_r + r\dot{\theta}\,\mathbf{u}_\theta + r\dot{\varphi}\sin\theta\,\mathbf{u}_\varphi$$

So:
$$\begin{cases} v_r = u_r \\ v_\theta = r\dot{\theta} \\ v_\varphi = r\sin\theta\,\dot{\varphi} \end{cases}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{r}\,\mathbf{u}_r + \dot{r}\left(\overset{\mathbf{u}_\theta}{\cancel{\frac{\partial \mathbf{u}_r}{\partial\theta}}}\dot{\theta} + \overset{\sin\theta\,\mathbf{u}_\varphi}{\cancel{\frac{\partial \mathbf{u}_r}{\partial\varphi}}}\dot{\varphi}\right)$$
$$+ \dot{r}\dot{\theta}\,\mathbf{u}_\theta + r\ddot{\theta}\,\mathbf{u}_\theta + r\dot{\theta}\left(\overset{-\mathbf{u}_r}{\cancel{\frac{\partial \mathbf{u}_\theta}{\partial\theta}}}\dot{\theta} + \overset{\cos\theta\,\mathbf{u}_\varphi}{\cancel{\frac{\partial \mathbf{u}_\theta}{\partial\varphi}}}\dot{\varphi}\right)$$
$$+ \dot{r}\dot{\varphi}\sin\theta\,\mathbf{u}_\varphi + r\ddot{\varphi}\sin\theta\,\mathbf{u}_\varphi + r\dot{\varphi}\cos\theta\,\dot{\theta}\,\mathbf{u}_\varphi$$
$$+ r\dot{\varphi}\sin\theta\ \overset{(-\cos\theta\,\mathbf{u}_\theta - \sin\theta\,\mathbf{u}_r)}{\cancel{\frac{d\mathbf{u}_\varphi}{d\varphi}}}\dot{\varphi}$$

$$= \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\mathbf{u}_r$$
$$+ \left(2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\cos\theta\sin\theta\right)\mathbf{u}_\theta$$
$$+ \left(2\dot{r}\dot{\varphi}\sin\theta + r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta + r\dot{\varphi}\dot{\theta}\cos\theta\right)\mathbf{u}_\varphi$$