PROPERTIES OF A σ-ALGEBRA (PROP. 3.1.1)

Given X, $S \subset \mathcal{P}(X)$ is a **σ-algebra** iff

(i) $S \neq \emptyset$

(ii) $A \in S \implies A' = X - A \in S$

(iii) $A_1, A_2, \ldots \in S \implies \bigcup_{i=1}^{\infty} A_i \in S$

The σ-algebra definition implies all of

(3.1.1 i) $A_1, \ldots, A_n \in S \implies \bigcup_{i=1}^{n} A_i \in S$

(3.1.1 ii) $\emptyset, X \in S$

(3.1.1 iii) $A_1, A_2, \ldots \in S \implies \bigcap_{i=1}^{\infty} A_i \in S$

(3.1.1 iv) $A_1, \ldots, A_n \in S \implies \bigcap_{i=1}^{n} A_i \in S$

(3.1.1 v) $A, B \in S \implies A - B \in S$

If $K \subset \mathcal{P}(X)$ then **K generates S**

$$S(K) := \bigcap \{S, \sigma - \text{algebra} : K \subset S\}$$

using that

$$S_\iota \in \mathcal{P}(X), \iota \in I \text{ are } \sigma\text{-algebras} \implies \bigcap_{\iota \in I} S_\iota \text{is a } \sigma\text{-algebra}$$

If $f : X \to Y$ and $S \subset \mathcal{P}(X)$ is a σ-algebra then $R := \left\{E \in \mathcal{P}(Y) : f^{-1}(E) \in S\right\}$ is a σ-algebra in Y.

If $f : X \to Y$ is bijective and $S \subset \mathcal{P}(X)$ is a σ-algebra then both

(a) $f(S) := \{f(A) : A \in S\}$ is a σ-algebra in Y

(b) $S(K) = S \subset \mathcal{P}(X) \implies S(f(K)) = f(S) \in \mathcal{P}(Y)$

PROPERTIES OF BOREL SETS (PROP. 3.1.4, 3.1.5 COMBINED)

$\boldsymbol{\mathcal{B}(\mathbb{R}^n)} := S(\text{open sets in } \mathbb{R}^n)$

(3.1.4) If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $B \in \mathcal{B}(\mathbb{R}^m)$ then $f^{-1}(B) \in \mathcal{B}(\mathbb{R}^n)$. Additionally, if f is a bijection with a continuous f^{-1} then $f(\mathcal{B}(\mathbb{R}^n)) = \mathcal{B}(\mathbb{R}^n)$.

(3.1.5) $E \in \mathcal{B}(\mathbb{R}^n) \wedge F \in \mathcal{B}(\mathbb{R}^m) \implies E \times F \in \mathcal{B}(\mathbb{R}^n \times \mathbb{R}^m)$

PROPERTIES OF AN (ABSTRACT) MEASURE (PROP. 3.1.6)

Function $\mu : S \to [0, \infty] \in \bar{\mathbb{R}}$ is a **measure** iff

(i) $\mu \neq \infty$

(ii) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) := \lim_{n\to\infty} \sum_{i=1}^{n} \mu(E_i)$ $\quad E_i$ pairwise disjoint

The measure definition implies all of

(3.1.6 i) $\mu(\emptyset) = 0$

(3.1.6 ii) $\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i)$ E_i pairwise disjoint

(3.1.6 iii) $E \subset F \implies \mu(E) \le \mu(F)$

(3.1.6 iv) $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu(E_i)$ $E_i \in S, i = 1, 2, \ldots$

(3.1.6 v) $\ldots \subset E_i \subset E_{i+1} \subset \ldots \implies \mu\left(\bigcup_{i=1}^{\infty}\right) = \lim_{n\to\infty} \mu(E_n)$

(3.1.6 vi) $\ldots \supset E_i \supset E_{i+1} \subset \ldots \implies \mu\left(\bigcap_{i=1}^{\infty}\right) = \lim_{n\to\infty} \mu(E_n)$

Construction of the Lebesgue measure

For a "mesh size" $k \in \mathbb{N}$, $\mathbb{R}^n$ can be partitioned by a set of half-open, half-closed "cubes", i.e.

$$\mathcal{S}^k(\mathbb{R}^n) := \left\{\left[\frac{\nu_i}{2^k}, \frac{\nu_{i+1}}{2^k}\right) \times \cdots \times \left[\frac{\nu_i}{2^k}, \frac{\nu_{i+1}}{2^k}\right) : \nu \in \mathbf{Z}^n\right\}.$$

Any open set G can be similarly partitioned $\mathcal{S}^k(G) := \left\{\sigma \in \mathcal{S}^k(\mathbb{R}^n) : \overline{\sigma} \subset G\right\}$. Define $S^k(G) := \bigcup_{\sigma\in\mathcal{S}^k(G)} \sigma$, which is monotone in k (i.e. $S^k(G) \subset S^{k+1}(G)$) and allows recovery of the original set (i.e. $G = \cup_{k=0}^{\infty} S^k(G)$). Define the **prototype for the Lebesgue measure**.

$$m(G) := \lim_{k\to\infty} \frac{1}{2^{kn}} \cdot \#\mathcal{S}^k(G).$$

From the prototype Lebesgue measure definition follows all of

(i) $G \subset H \text{ open} \implies m(G) \le m(H)$

(ii) $G, H \text{ open}, G \cap H = \emptyset \implies m(G \cap H) = m(G) + m(H)$

(iii) $G \subset \mathbb{R}^n, H \subset \mathbb{R}^m \text{ open} \implies m(G \times H) = m(G) \cdot m(H)$

(iv) $m((a, b)) = b - a$

(v) $m((a_1, b_1) \times \cdots \times (a_n, b_n)) = \prod_{i=1}^{n} (b_i - a_i)$

$$G \subset \mathbb{R}^n \text{ open} \implies m(G) = \sup\left\{m(H), H \text{ open}, \overline{H} \text{ compact} \subset G\right\}$$

Subadditivity and σ-subadditivity of the measure for open sets:

(i) $m(G_1 \cup G_2) \le m(G_1) + m(G_2)$

(ii) $m\left(\bigcup_{i=1}^{n} G_i\right) \le \sum_{i=1}^{n} m(G_i)$

(iii) $m\left(\bigcup_{i=1}^{\infty} G_i\right) \le \sum_{i=1}^{\infty} m(G_i)$

$$G \text{ open} \implies \inf_F \{m(G-F) : F \text{ closed} \subset G)\} = 0$$
$$F_1, F_2 \text{ closed}, F_1 \cap F_2 = \emptyset \implies \exists G_1 \text{ open} \supset F_1, \exists G_2 \text{ open} \supset F_2, G_1 \cap G_2 = \emptyset$$

Characterization of Lebesgue measurable sets (Prop. 3.2.3, Thm 3.2.1)

Obtain the **Lebesgue measure** by extending the prototype measure:

$$m^*(E) := \inf\{m(G) : G \text{ open} \supset E\} \;\; \forall E \in \mathbb{R}^n$$

from which it follows that

(i) $$E \text{ open} \implies m^*(E) = m(E)$$

(ii) $$E \subset F \implies m^*(E) \leq m^*(F)$$

(iii) $$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$$

The following three families of sets coincide with each other

(3.2.3 a) $$\left\{E \subset \mathbb{R}^n : \inf_{G \text{ open} \supset E} m^*(G-E) = 0\right\}$$

(3.2.3 b) $$\left\{E \subset \mathbb{R}^n : \inf_{G \text{ open} \supset E \supset F \text{ closed}} m\;(G-F) = 0\right\}$$

(3.2.3 c) $$\left\{E \subset \mathbb{R}^n : \inf_{E \supset F \text{ closed}} m^*(E-F) = 0\right\}$$

and are exactly the **Lebesgue measurable sets**, $\boldsymbol{\mathcal{L}(\mathbb{R}^n)}$. The following hold:

(3.2.1 i) $$\mathcal{L} \text{ is a } \sigma\text{-algebra}, \mathcal{B}(\mathbb{R}^n) \subset \mathcal{L}(\mathbb{R}^n).$$

(3.2.1 ii) $$m := m^*\big|_{\mathcal{L}} \text{ is a measure.}$$

(Corollary) $$Z \subset \mathbb{R}^n, m^*(Z) = 0 \implies Z \in \mathcal{L}(\mathbb{R}^n)$$

Lebesgue measurable sets, $\mathcal{L}(\mathbb{R}^n)$ can also be characterized as

(a) $\{H - Z : H \text{ is } G_\delta\text{-type}, m^*(Z) = 0\}$ where $H = \bigcap_{i=1}^{\infty} G_i$ for G_i open

(b) $\{J \cup Z : J \text{ is } F_\sigma\text{-type}, m^*(Z) = 0\}$ where $J = \bigcup_{i=1}^{\infty} F_i$ for F_i closed

(c) $S\left(\mathcal{B}\left(\mathbb{R}^N\right) \cup \{Z : m^*(Z) = 0\}\right)$ generation of a σ-algebra

$$F \subset \mathbb{R}^m, Z \subset \mathbb{R}^n, m_n^*(Z) = 0 \implies m_{m+n}^*(F \times Z) = 0$$
$$E_1 \in \mathcal{L}(\mathbb{R}^n), E_2 \in \mathcal{L}(\mathbb{R}^m) \implies E_1 \times E_2 \in \mathcal{L}(\mathbb{R}^{n+m})$$
$$m_{n+m}(E_1 \times E_2) = m_n(E_1) \cdot m_m(E_2)$$

Properties of measurable (Borel) functions (Prop. 3.4.1)

$f : \mathbb{R}^n \supset E \to \overline{\mathbb{R}}$ is **measurable** iff both

(a) E is measurable

(b) $\boldsymbol{\{y < f(x)\}} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in E, y < f(x)\}$ is measurable.

from which it follows that

(3.4.1 i) $E \subset \operatorname{dom} f$ measurable , f measurable $\implies f|_E$ measurable

(3.4.1 ii)

$$f_i : E_i \to \overline{\mathbb{R}} \text{ measurable, } E_i \text{ pairwise disjoint} \implies \bigcup_{i=1}^{\infty} f_i : \bigcup_{i=1}^{E_i} \to \overline{\mathbb{R}} \text{ measurable}$$

(3.4.1 iii) f measurable, $\lambda \in \mathbb{R} \implies \lambda f$ measurable

(3.4.1 iv) $f_i : E \to \overline{\mathbb{R}}$ measurable $\implies \sup_i f_i, \ \inf_i f_i, \ \limsup_i f_i, \ \liminf_i f_i$ measurable

Similar properties hold for Borel sets.

$$\operatorname{dom} f = E \text{ open, } f \text{ continuous} \implies f \text{ Borel} \implies f \text{ measurable}$$
$$g : \mathbb{R}^n \to \mathbb{R}^n \text{ an affine isomorphism , } f \text{ measurable} \iff f \circ g \text{ measurable}$$

Property $P(x)$ is satisfied **almost everywhere** (a.e.) in $\mathbb{R}^n$ iff $m(\{x \in \mathbb{R}^n : \neg P(x)\}) = 0$.

$$f_1 = f_2 \text{ a.e. on } E \subset \mathbb{R}^n \implies (f_1 \text{ measurable} \iff f_2 \text{ measurable})$$

Properties of Lebesgue integral (Prop. 3.5.1)

For f measurable, $f \geq 0$, the **Lebesgue integral** is defined as

$$\int f\, dm := m_{n+1}(S(f)) = m_{n+1}(\{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in \operatorname{dom} f, 0 < y < f(x)\})$$

Every nonnegative measurable function is Lebesgue integrable. For f measurable with arbitrary sign, define

$$f^+(x) := \{\max(f(x), 0) : x \in \operatorname{dom} f, f(x) \geq 0\}$$
$$f^-(x) := \{\max(-f(x), 0) : x \in \operatorname{dom} f, f(x) < 0\}$$

which allows definition of the integral as

$$\int f\, dm = \int f^+\, dm - \int f^-\, dm$$

provided that $\int f\, dm \neq \infty - \infty$.

For measurable, nonnegative functions the following properties hold

(3.5.1 i) $$m(E) = 0 \implies \int_E \varphi \, dm = 0$$

(3.5.1 ii) $$\varphi : E \to \overline{\mathbb{R}}, E_i \subset E \text{ pairwise disjoint } \implies \int_{\bigcup E_i} \varphi \, dm = \sum_1^{\infty} \int_{E_i} \varphi \, dm$$

(3.5.1 iii) $$\varphi, \psi : E \to \overline{\mathbb{R}}, \varphi = \psi \text{ a.e. } \in E \implies \int_E \varphi \, dm = \int_E \psi \, dm$$

(3.5.1 iv) $$c \geq 0, E \text{ measurable} \implies \int_E c \, dm = c \, m(E)$$

(3.5.1 v) $$\varphi, \psi : E \to \overline{\mathbb{R}}, \varphi \leq \psi \text{ a.e. } \in E \implies \int_E \varphi \, dm \leq \int_E \psi \, dm$$

(3.5.1 vi) $$\lambda \geq 0 \implies \int (\lambda \varphi) \, dm = \lambda \int \varphi \, dm$$

Fatou's Lemma

Lebesgue Dominated Convergence Theorem (for non-negative functions, Thm. 3.5.2)

Hölder and Minkowski inequalities

Hölder Inequality: If $\Omega \subset \mathbb{R}^n$ measurable, $f, g : \Omega \to \overline{\mathbb{R}}$ measurable with $\int_\Omega |f|^p \, dm, \int_\Omega |g|^q \, dm < \infty$ where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$ then $\int_\Omega fg \, dm < \infty$ and

$$\left| \int_\Omega fg \, dm \right| \leq \left(\int_\Omega |f|^p \, dm \right)^{\frac{1}{p}} \left(\int_\Omega |g|^q \, dm \right)^{\frac{1}{q}}.$$

For $1 \leq p < \infty$ define $\boldsymbol{\|f\|_p := \left(\int_\Omega |f|^p \, dm\right)^{\frac{1}{p}}}$

Minkowski Inequality: If $\Omega \subset \mathbb{R}^n$ measurable and $f, g : \Omega \to \overline{\mathbb{R}}$ measurable with $\|f\|_p, \|g\|_p < \infty$ where $p > 1$ then $\|f + g\|_p < \infty$ and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Properties of open sets, properties of closed sets, properties of the operations of interior and closure (all in context of general topological spaces)

Characterization of open and closed sets in a topological subspace

Characterization of (globally) continuous functions (Prop. 4.3.2)

Properties of compact sets

The Heine-Borel Theorem

The Weierstrass Theorem

Properties of sequentially compact sets (Prop. 4.4.5)