Chapter 8

Vectors, Surfaces, and Volumes

This chapter was intended to be about surfaces and volumes with vectors postponed until Chapter 9 on vector field theory, but vectors and vector products will be helpful in our discussion of surfaces and volumes. Thus we start the present chapter with an introduction to vectors and their manipulation.

8.1. VECTORS AND ELEMENTARY OPERATIONS

Although our notion of what constitutes a vector will be considerably broadened in Part III, when we introduce the concept of an abstract linear vector space, for the present we will regard a **vector** simply as a directed magnitude.

Graphically, it may be represented by an arrow, the length of which is adjusted to some scale to indicate the desired magnitude. Physically, it may denote a force, the position (with respect to some reference point) of some particle in space, the particle's velocity, and so on. But note that the position of the vector in space is immaterial; for instance, the vectors **A**, **B**, **C** in Fig. 8.1(a) are identical. That is not to say that the *effect* of a vector is independent of its position; it should be apparent that the motion of the body $\mathcal{B}$ induced by a force **F** [Fig. 8-1(b)] will certainly depend on its point of application (as will the stress field induced in $\mathcal{B}$, and so on).

In this book we denote vectors by boldface type, such as **A**; in writing, it is customary to underline instead. The **length** or **magnitude** of **A** is denoted $\|\mathbf{A}\|$ or simply A.

We define the following operations on and between vectors.

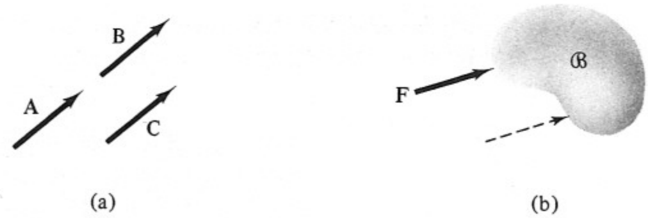

Figure 8.1. Position of a vector.

Scalar Multiplication. By $c\mathbf{A}$ (or $\mathbf{A}c$), where c is any real number, we mean the vector that is $|c|$ times as long as $\mathbf{A}$, in the same direction as $\mathbf{A}$ if c is positive and in the opposite direction if c is negative. If $c = 0$, then $c\mathbf{A} = 0\mathbf{A} \equiv \mathbf{0}$, the so-called *null vector* or *zero vector*, having zero length. In particular, if $c = -1$, then we denote $c\mathbf{A} = (-1)\mathbf{A} \equiv -\mathbf{A}$, read as "minus **A**."

In contrast with the vector quantity **A**, we call c a **scalar**.

Addition and Subtraction. Given any vectors **A** and **B** [Fig. 8.2(a)], we define $\mathbf{A} + \mathbf{B}$, called the *sum* of **A** and **B**, according to any of the (equivalent) constructions shown in Fig. 8.2(b), (c), or (d). In view of the construction of Fig. 8.2(d), we refer to this definition of addition as the **parallelogram law**. With vector addition defined, we also define the *difference* of **A** and **B** as $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B})$, as shown, for example, in Fig. 8.2(b). Note that in Fig. 8.2 **A** and **B** happen to be coplanar; both lie in the plane of the paper. Even if they were not, however, one or both could be shifted (as discussed above) so that they become coplanar and the parallelogram law could then be applied.

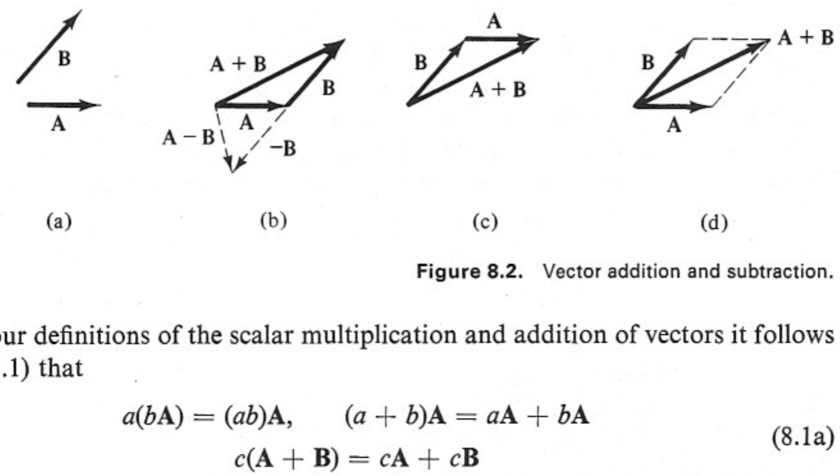

Figure 8.2. Vector addition and subtraction.

From our definitions of the scalar multiplication and addition of vectors it follows (Exercise 8.1) that

$$a(b\mathbf{A}) = (ab)\mathbf{A}, \qquad (a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$$
$$c(\mathbf{A} + \mathbf{B}) = c\mathbf{A} + c\mathbf{B} \tag{8.1a}$$

and

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$
$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}. \tag{8.1b}$$

Scalar (or Dot) Product. For any two vectors **A** and **B** we define their **scalar product** or **dot product** $\mathbf{A} \cdot \mathbf{B}$ according to

$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos\theta, \tag{8.2}$$

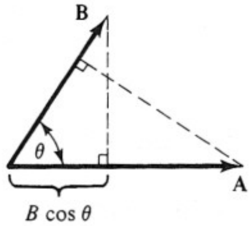

Figure 8.3. The scalar or dot product.

where A, B are the magnitudes of **A**, **B** and θ is the angle ($\leq \pi$) between **A** and **B** when they are arranged "tail to tail," as in Fig. 8.3. Observe that $B \cos\theta$ is the projection of **B** on **A**, so that $\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$ is A times the projection of **B** on **A**. Alternatively, it is B times the projection of **A** on **B**. In particular, if $\theta = \pi/2$, the projection is zero and $\mathbf{A} \cdot \mathbf{B} = 0$; we then say that **A** and **B** are **orthogonal**. ($\mathbf{A} \cdot \mathbf{B} = 0$ implies that $\mathbf{A} = \mathbf{0}$ and/or $\mathbf{B} = \mathbf{0}$ and/or $\theta = \pi/2$.) On the other hand, $\mathbf{A} \cdot \mathbf{A} = A^2$, since $\theta = 0$ in this case.

From its definition it follows that the scalar product is *commutative*

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{8.3}$$

and *linear*

$$(\alpha\mathbf{A} + \beta\mathbf{B}) \cdot \mathbf{C} = \alpha(\mathbf{A} \cdot \mathbf{C}) + \beta(\mathbf{B} \cdot \mathbf{C}), \tag{8.4}$$

where **A**, **B**, **C** are any vectors and α, β any scalars. For instance,

$$(\mathbf{A} + 2\mathbf{B}) \cdot (\mathbf{C} - \mathbf{D}) = \mathbf{A} \cdot \mathbf{C} - \mathbf{A} \cdot \mathbf{D} + 2\mathbf{B} \cdot \mathbf{C} - 2\mathbf{B} \cdot \mathbf{D} \tag{8.5}$$

as in "ordinary algebra."

Vector Product. Analogous to (8.2), we also define a **vector product** or **cross product** of any two vectors **A**, **B** according to

$$\mathbf{A} \times \mathbf{B} \equiv AB \sin\theta\,\hat{\mathbf{e}}, \tag{8.6}$$

where $\hat{\mathbf{e}}$ is a unit vector (i.e., it is of unit length, as denoted by the caret) that is perpendicular to the plane containing **A** and **B** and directed according to the "right-hand rule"; that is, curling the other four fingers of our right hand from **A** (the first vector) into **B** (the second vector), our thumb points in the $\hat{\mathbf{e}}$ direction (Fig. 8.4).

Since $B \sin\theta$ is the altitude of the parallelogram defined by **A**, **B** and A is the base, observe that $\|\mathbf{A} \times \mathbf{B}\| = AB \sin\theta$ corresponds to the area of the parallelogram.

$$\|\mathbf{A} \times \mathbf{B}\| = \text{area of the } \mathbf{A}, \mathbf{B} \text{ parallelogram.} \tag{8.7}$$

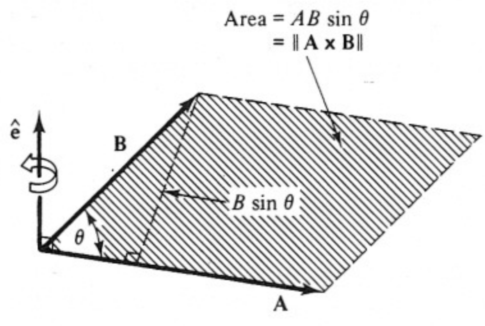

Figure 8.4. The vector or cross product.

[Alternatively, we can regard $A \sin\theta$ as the altitude and B as the base; the conclusion (8.7) is the same.] $\mathbf{A} \times \mathbf{B} = \mathbf{0}$ implies that $\mathbf{A} = \mathbf{0}$ and/or $\mathbf{B} = \mathbf{0}$ and/or $\theta = 0$ or π; for example, $\mathbf{A} \times \mathbf{A} = \mathbf{0}$.

Note carefully that the vector product is *not* commutative; $\mathbf{A} \times \mathbf{B}$ is not the same as $\mathbf{B} \times \mathbf{A}$. Rather,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}, \tag{8.8}$$

since $\|\mathbf{A} \times \mathbf{B}\| = \|\mathbf{B} \times \mathbf{A}\| = AB \sin\theta$, but the $\hat{\mathbf{e}}$'s for $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ are oppositely oriented. It *is* true (Exercise 8.12), however, that the vector product is *linear*—that is,

$$(\alpha\mathbf{A} + \beta\mathbf{B}) \times \mathbf{C} = \alpha(\mathbf{A} \times \mathbf{C}) + \beta(\mathbf{B} \times \mathbf{C}) \tag{8.9}$$

for any vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and any scalars α, β.

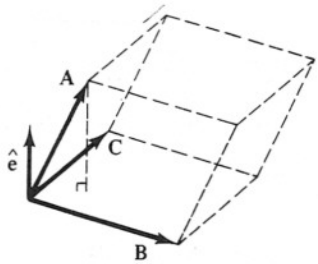

Figure 8.5. Volume interpretation of $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

Consider next the so-called **scalar triple product** $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$. Actually, the parentheses can be omitted without confusion, since $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ can't possibly be interpreted as $(\mathbf{A} \cdot \mathbf{B}) \times \mathbf{C}$ because the latter is a scalar crossed into a vector, which is meaningless. Since $\mathbf{B} \times \mathbf{C}$ is the area of the $\mathbf{B}, \mathbf{C}$ parallelogram times the unit vector $\hat{\mathbf{e}}$ (Fig. 8.5) and $\mathbf{A} \cdot \hat{\mathbf{e}}$ is the altitude of the $\mathbf{A}, \mathbf{B}, \mathbf{C}$ parallelepiped, it follows that

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \text{volume of the } \mathbf{A}, \mathbf{B}, \mathbf{C} \text{ parallelepiped.} \tag{8.10}$$

The volume is also given by $\mathbf{C} \times \mathbf{A} \cdot \mathbf{B}$ and $\mathbf{A} \times \mathbf{B} \cdot \mathbf{C}$, so that we have

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = \mathbf{C} \cdot \mathbf{A} \times \mathbf{B} = \mathbf{B} \cdot \mathbf{C} \times \mathbf{A}; \tag{8.11}$$

that is, any cyclic permutation of $\mathbf{A}, \mathbf{B}, \mathbf{C}$ in (8.10) leaves the scalar triple product unchanged.

Besides $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$ there is also the **vector triple product** $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. We should be able to express it in the form $\alpha\mathbf{B} + \beta\mathbf{C}$, since it must clearly be perpendicular to both $\mathbf{A}$ and $\mathbf{B} \times \mathbf{C}$; in other words, it must lie in the $\mathbf{B}, \mathbf{C}$ plane such that it is perpendicular to $\mathbf{A}$. To obtain this form, let $\hat{\mathbf{e}}_1$ be a unit vector aligned with $\mathbf{B}$ and let $\hat{\mathbf{e}}_2$ be a unit vector lying in the $\mathbf{B}, \mathbf{C}$ plane perpendicular to $\hat{\mathbf{e}}_1$ such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$ coincides with the $\hat{\mathbf{e}}$ shown in Fig. 8.5. Then we can express

$$\mathbf{B} = B\hat{\mathbf{e}}_1, \qquad \mathbf{C} = C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2, \qquad \mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

so that $\mathbf{B} \times \mathbf{C} = BC_2\hat{\mathbf{e}}_3$, and

$$\begin{aligned}
\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3) \times BC_2\hat{\mathbf{e}}_3 \\
&= -A_1BC_2\hat{\mathbf{e}}_2 + A_2BC_2\hat{\mathbf{e}}_1 \\
&= (A_1C_1 + A_2C_2)B\hat{\mathbf{e}}_1 - A_1B(C_1\hat{\mathbf{e}}_1 + C_2\hat{\mathbf{e}}_2) \\
&= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}
\end{aligned} \tag{8.12}$$

as claimed. That is, α turns out to be $\mathbf{A} \cdot \mathbf{C}$ and β to be $-\mathbf{A} \cdot \mathbf{B}$.

Do we need the parentheses around $\mathbf{B} \times \mathbf{C}$? Is $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$, in which case we could drop the parentheses? Well,

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = -\mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = \mathbf{C} \times (\mathbf{B} \times \mathbf{A}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A},$$

which, in general, is *not* the same as the right-hand side of (8.12), and so the parentheses are necessary.

Derivative. A vector need not be constant; it may vary over space and/or the time t. For instance, suppose that $\mathbf{A} = \mathbf{A}(t)$. We define the *derivative* of $\mathbf{A}$ with respect to t by the limit

$$\frac{d\mathbf{A}}{dt} \equiv \lim_{\Delta t \to 0} \frac{\mathbf{A}(t + \Delta t) - \mathbf{A}(t)}{\Delta t} \tag{8.13}$$

if, of course, the limit exists. We see from its definition that $d\mathbf{A}/dt$ is a vector quantity.

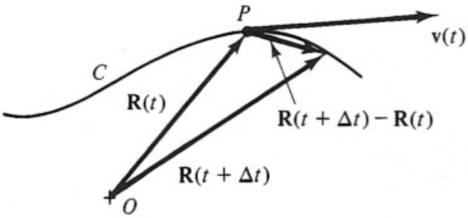

Figure 8.6. Velocity vector.

Consider, for example, the "position vector" $\mathbf{R}(t)$ from some fixed point O to a particle that is moving along some curve C. Then $d\mathbf{R}/dt$ is the limit of $[\mathbf{R}(t + \Delta t) - \mathbf{R}(t)]/\Delta t$ as $\Delta t \to 0$, and is called the *velocity* of the particle, $\mathbf{v}(t)$. Clearly, $\mathbf{v}(t)$ is tangent to C, since $\mathbf{R}(t + \Delta t) - \mathbf{R}(t)$ *defines* the tangent to C at P as $\Delta t \to 0$ (Fig. 8.6). The next derivative, $d\mathbf{v}/dt = d^2\mathbf{R}/dt^2$, is called the *acceleration*, $\mathbf{a}(t)$. In general, $\mathbf{a}$ will *not* be tangent to C; see Exercise 8.21.

What can we say about $d(\mathbf{A} \cdot \mathbf{B})/dt$ and $d(\mathbf{A} \times \mathbf{B})/dt$? Expressing $\mathbf{A}(t + \Delta t) = \mathbf{A}(t) + [\mathbf{A}(t + \Delta t) - \mathbf{A}(t)] \equiv \mathbf{A} + \Delta\mathbf{A}$, and similarly for $\mathbf{B}(t + \Delta t)$, we have [with the help of (8.3) and (8.4)]

$$\begin{aligned}\frac{d}{dt}(\mathbf{A} \cdot \mathbf{B}) &= \lim_{\Delta t \to 0} \frac{(\mathbf{A} + \Delta\mathbf{A}) \cdot (\mathbf{B} + \Delta\mathbf{B}) - \mathbf{A} \cdot \mathbf{B}}{\Delta t} \\ &= \lim_{\Delta t \to 0} \left(\frac{\Delta\mathbf{A}}{\Delta t} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{\Delta\mathbf{B}}{\Delta t} + \Delta\mathbf{A} \cdot \frac{\Delta\mathbf{B}}{\Delta t}\right) = \frac{d\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt}\end{aligned} \tag{8.14}$$

as we might have guessed. Similarly,

$$\frac{d}{dt}(\mathbf{A} \times \mathbf{B}) = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt}. \tag{8.15}$$

8.2. *COORDINATE SYSTEMS, BASE VECTORS, AND COMPONENTS*

In calculations it is often convenient to turn to a specific reference frame—a coordinate system and a corresponding set of unit base vectors. Three of the most widely used systems will now be discussed.

Cartesian. Here we refer to the rectangular x, y, z coordinates and the corresponding unit base vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ shown in Fig. 8.7. We say that the base vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are **orthonormal**, since they are both mutually orthogonal and also "normalized"; that is, they're of unit length. We have

$$\left.\begin{array}{l}\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}, \qquad \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}}, \qquad \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}},\end{array}\right\} \tag{8.16}$$

and so on.

To illustrate, the vector **A** running from the point $(-1, 4, 1)$ to $(1, 3, 4)$ in Fig. 8.7 is (by the definition of vector addition) $\mathbf{A} = \mathbf{a} + \mathbf{b} + \mathbf{c} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. More generally, we can express *any* vector **A** in the three-dimensional x, y, z space in the form

$$\mathbf{A} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \tag{8.17}$$

where the A_x, A_y, A_z are called the x, y, z **components** of **A**, respectively. "Dotting" $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ into (8.17) and recalling (8.16) yield

$$A_x = \mathbf{A} \cdot \hat{\mathbf{i}}, \qquad A_y = \mathbf{A} \cdot \hat{\mathbf{j}}, \qquad A_z = \mathbf{A} \cdot \hat{\mathbf{k}}; \tag{8.18}$$

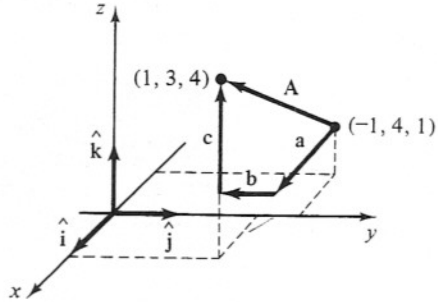

Figure 8.7. Cartesian system.

that is, A_x, A_y, A_z are the projections of **A** on the x, y, z axes, respectively.

Observe that $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are *constant* vectors, and therefore

$$\frac{d}{dt}\mathbf{A}(t) = \frac{dA_x}{dt}\hat{\mathbf{i}} + \frac{dA_y}{dt}\hat{\mathbf{j}} + \frac{dA_z}{dt}\hat{\mathbf{k}}. \tag{8.19}$$

Cylindrical. The cylindrical coordinates r, θ, z and their respective unit base vectors $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ are shown in Fig. 8.8; r, θ, z are related to x, y, z according to

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \tag{8.20}$$

Any vector in the space can be expressed in the form

$$\mathbf{A} = A_r\hat{\mathbf{e}}_r + A_\theta\hat{\mathbf{e}}_\theta + A_z\hat{\mathbf{e}}_z. \tag{8.21}$$

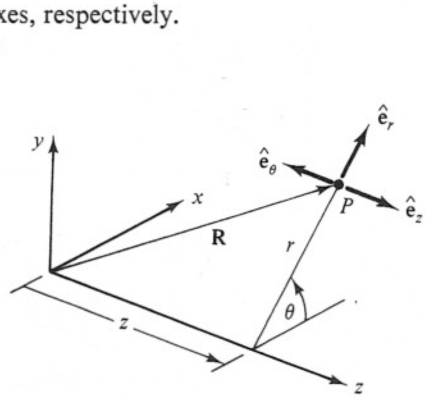

Figure 8.8. Cylindrical system.

Since the set $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ is orthonormal (i.e., $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_z = 0$ and $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_r = \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z = 1$), it follows that the r, θ, z components are given by

$$A_r = \mathbf{A} \cdot \hat{\mathbf{e}}_r, \qquad A_\theta = \mathbf{A} \cdot \hat{\mathbf{e}}_\theta, \qquad A_z = \mathbf{A} \cdot \hat{\mathbf{e}}_z. \tag{8.22}$$

Example 8.1. Obtain general expressions for the velocity and acceleration of a particle with respect to a cylindrical reference system. We start with the "position vector"

$$\mathbf{R} = z\hat{\mathbf{e}}_z + r\hat{\mathbf{e}}_r \tag{8.23}$$

from the origin out to the r, θ, z point, say P (Fig. 8.8). Note that **R** is *not* $r\hat{\mathbf{e}}_r + \theta\hat{\mathbf{e}}_\theta + z\hat{\mathbf{e}}_z$; the *units* of this expression aren't even consistent, since θ is dimensionless (it's measured in radians), whereas r and z have the correct units of length.

Even though $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are *constant* vectors, note that $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ are not. More specifically, $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ vary with θ, although not with r, whereas $\hat{\mathbf{e}}_z$ is, in fact, constant. For instance, if we imagine sliding the $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ triad to a new r location, say $r + \Delta r$, with both θ and z held *fixed*, we see that the three vectors remain unchanged because both their magnitude (unity) and direction remain unchanged, even though their *location* is different. But if we vary θ, with r and z held fixed, then $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ *rotate*. The upshot is that $\hat{\mathbf{e}}_r = \hat{\mathbf{e}}_r(\theta)$, $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\theta(\theta)$, $\hat{\mathbf{e}}_z =$ constant.

If the point P moves about according to $r = r(t)$, $\theta = \theta(t)$, $z = z(t)$, then

$$\mathbf{R}(t) = z(t)\hat{\mathbf{e}}_z + r(t)\hat{\mathbf{e}}_r[\theta(t)]. \tag{8.24}$$

Its velocity is

$$\mathbf{v}(t) = \dot{\mathbf{R}}(t) = \dot{z}\hat{\mathbf{e}}_z + \dot{r}\hat{\mathbf{e}}_r + r\dot{\hat{\mathbf{e}}}_r, \tag{8.25}$$

where the dots denote time derivatives as usual. Yet it remains to express $\dot{\hat{\mathbf{e}}}_r$ in terms of $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, $\hat{\mathbf{e}}_z$. (*All* vectors are to be expressed as linear combinations of the base vectors.) Well,

$$\frac{d\hat{\mathbf{e}}_r}{dt} = \frac{d\hat{\mathbf{e}}_r}{d\theta}\frac{d\theta}{dt}, \tag{8.26}$$

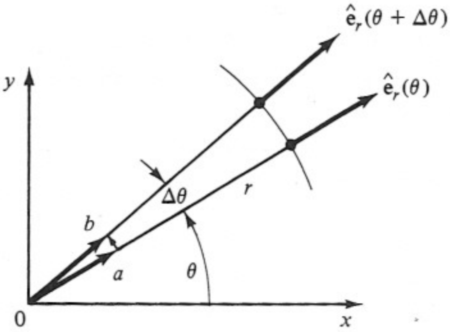

Figure 8.9. Calculation of $d\hat{\mathbf{e}}_r/d\theta$.

so that what we actually need is $d\hat{\mathbf{e}}_r/d\theta$. By definition,

$$\frac{d\hat{\mathbf{e}}_r}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\hat{\mathbf{e}}_r(\theta + \Delta\theta) - \hat{\mathbf{e}}_r(\theta)}{\Delta\theta}. \tag{8.27}$$

The vector difference in the numerator is easily evaluated if we slide the two vectors back to O (Fig. 8.9) so that they are tail to tail. Then $\hat{\mathbf{e}}_r(\theta + \Delta\theta) - \hat{\mathbf{e}}_r(\theta)$ is the little vector from a to b; its length is $1\ \Delta\theta$ and its direction is $\hat{\mathbf{e}}_\theta$. Thus

$$\frac{d\hat{\mathbf{e}}_r}{d\theta} = \lim_{\Delta\theta \to 0} \frac{\Delta\theta\hat{\mathbf{e}}_\theta}{\Delta\theta} = \hat{\mathbf{e}}_\theta, \tag{8.28}$$

and so (8.25) becomes

$$\mathbf{v}(t) = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta + \dot{z}\hat{\mathbf{e}}_z. \tag{8.29}$$

Differentiating once more, the acceleration $\mathbf{a}(t)$ is

$$\begin{aligned} \mathbf{a}(t) &= \ddot{r}\hat{\mathbf{e}}_r + \dot{r}\dot{\hat{\mathbf{e}}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta + r\dot{\theta}\dot{\hat{\mathbf{e}}}_\theta + \ddot{z}\hat{\mathbf{e}}_z \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\mathbf{e}}_\theta + \ddot{z}\hat{\mathbf{e}}_z, \end{aligned} \tag{8.30}$$

where we've used the fact that (Exercise 8.13)

$$\frac{d\hat{\mathbf{e}}_\theta}{d\theta} = -\hat{\mathbf{e}}_r \tag{8.31}$$

to express $\dot{\hat{\mathbf{e}}}_\theta = (d\hat{\mathbf{e}}_\theta/d\theta)(d\theta/dt) = -\dot{\theta}\hat{\mathbf{e}}_r$.

The derivatives of the base vectors, (8.28) and (8.31), are important because they inevitably arise whenever we differentiate vectors that are expressed in terms of cylindrical coordinates.

Of the five terms constituting $\mathbf{a}(t)$, in (8.30), $-r\dot{\theta}^2\hat{\mathbf{e}}_r$ and $2\dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta$ are called the **centripetal** and **Coriolis** accelerations, respectively. ▮

Spherical. The spherical polar ρ, θ, ϕ system and its corresponding base vectors are shown in Fig. 8.10. The spherical polar and cartesian coordinates are related according to

$$x = \rho \sin\theta \cos\phi, \qquad y = \rho \sin\theta \sin\phi, \qquad z = \rho \cos\theta. \tag{8.32}$$

Observe that $\hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_\rho(\theta, \phi)$, $\hat{\mathbf{e}}_\theta = \hat{\mathbf{e}}_\theta(\theta, \phi)$, and $\hat{\mathbf{e}}_\phi = \hat{\mathbf{e}}_\phi(\phi)$. We leave it for you

(Exercise 8.14) to show that

$$\left.\begin{aligned} \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \theta} &= \hat{\mathbf{e}}_\theta, & \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \phi} &= \sin\theta\, \hat{\mathbf{e}}_\phi \\ \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} &= -\hat{\mathbf{e}}_\rho, & \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \phi} &= \cos\theta\, \hat{\mathbf{e}}_\phi \\ \frac{\partial \hat{\mathbf{e}}_\phi}{\partial \phi} &= -\sin\theta\, \hat{\mathbf{e}}_\rho - \cos\theta\, \hat{\mathbf{e}}_\theta. \end{aligned}\right\} \tag{8.33}$$

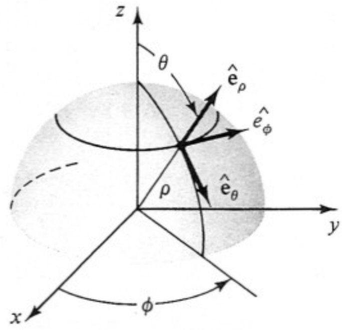

Figure 8.10. Spherical polar system.

Vector Products and Scalar Triple Products as Determinants. Let $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$ be any right-handed orthonormal set of base vectors—in other words, mutually orthogonal unit vectors such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$, $\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3 = \hat{\mathbf{e}}_1$, and $\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}_2$. For example, they might be $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ (cartesian), $\hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_z$ (cylindrical), $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ (spherical), or any other such set.

For any vectors **A**, **B** observe that we can express

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}. \tag{8.34}$$

This expression is verified easily by expanding both $(A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3) \times (B_1\hat{\mathbf{e}}_1 + B_2\hat{\mathbf{e}}_2 + B_3\hat{\mathbf{e}}_3)$ and the determinant and noting that the results are the same. Furthermore,

$$\mathbf{A} \cdot \mathbf{B} \times \mathbf{C} = (A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3) \cdot \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}. \tag{8.35}$$

8.3. ANGULAR VELOCITY OF A RIGID BODY

Consider a rigid body $\mathcal{B}$ rotating about a fixed z axis with *angular velocity* Ω (radians per unit time), as sketched in Fig. 8.11. That is, each point in $\mathcal{B}$ moves in a circular path with $\dot{\theta} = \Omega$; the center of the circle is on the axis of rotation, and the plane of the circle is perpendicular to the axis. According to the right-hand rule, we define the angular velocity vector $\mathbf{\Omega}$ as shown in the figure.

We claim that the velocity of each point in $\mathcal{B}$ can be expressed as

$$\mathbf{v} = \mathbf{\Omega} \times \mathbf{R}, \tag{8.36}$$

where **R** is the position vector from some point on the axis, say O (the origin of our cylindrical coordinate system), to the point in question. This statement is easily verified:

$$\mathbf{R} = z\hat{\mathbf{e}}_z + r\hat{\mathbf{e}}_r$$

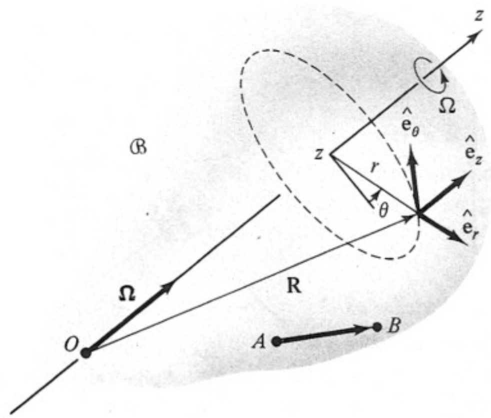

Figure 8.11. Rotation about a fixed axis.

and so

$$\mathbf{v} = \dot{\mathbf{R}} = r\dot{\hat{\mathbf{e}}}_r = r\dot{\theta}\hat{\mathbf{e}}_\theta = \Omega r\hat{\mathbf{e}}_\theta.$$

On the other hand,

$$\boldsymbol{\Omega} \times \mathbf{R} = \Omega\hat{\mathbf{e}}_z \times (z\hat{\mathbf{e}}_z + r\hat{\mathbf{e}}_r) = \Omega r\hat{\mathbf{e}}_\theta.$$

In fact, if A and B are any two points in $\mathcal{B}$ and **AB** denotes the vector from A to B (Fig. 8.11), it follows that

$$\begin{aligned}\dot{\mathbf{AB}} &= \frac{d}{dt}(\mathbf{OB} - \mathbf{OA}) = \dot{\mathbf{OB}} - \dot{\mathbf{OA}} \\ &= \boldsymbol{\Omega} \times \mathbf{OB} - \boldsymbol{\Omega} \times \mathbf{OA} = \boldsymbol{\Omega} \times (\mathbf{OB} - \mathbf{OA}) = \boldsymbol{\Omega} \times \mathbf{AB}. \qquad (8.37)\end{aligned}$$

Generalizing further, suppose that the body is undergoing an *arbitrary* motion in space, not necessarily a rotation about a fixed axis. According to **Chasle's theorem** from elementary mechanics, the motion can be decomposed into a *translation* plus a *rotation*. That is, consider any point P in $\mathcal{B}$ with velocity $\mathbf{v}_P$. Then at any given instant the motion of $\mathcal{B}$ may be regarded as a translation with velocity $\mathbf{v}_P$ (i.e., where every point in $\mathcal{B}$ has the velocity $\mathbf{v}_P$) plus a rotation with angular velocity $\boldsymbol{\Omega}$ about some axis through P. (Furthermore, $\boldsymbol{\Omega}$ is independent of the choice of P.)[1]

Thus if A and B are points in $\mathcal{B}$, then

$$\mathbf{v}_A = \mathbf{v}_P + \boldsymbol{\Omega} \times \mathbf{PA}, \qquad \mathbf{v}_B = \mathbf{v}_P + \boldsymbol{\Omega} \times \mathbf{PB},$$

so that

$$\mathbf{v}_B - \mathbf{v}_A = \boldsymbol{\Omega} \times (\mathbf{PB} - \mathbf{PA}) = \boldsymbol{\Omega} \times \mathbf{AB}$$

or

$$\dot{\mathbf{AB}} = \boldsymbol{\Omega} \times \mathbf{AB} \qquad (8.38)$$

as before [equation (8.37)]. Note carefully that **AB** is a vector of *fixed length*, since $\mathcal{B}$ is a rigid body. This situation is reflected in the fact that $\dot{\mathbf{AB}}$ is perpendicular to **AB**, per (8.38), since a component along **AB** would correspond to a stretching or contraction of the **AB** vector.

[1] For a proof of Chasle's theorem, see, for example, the appendix in I. Shames, *Engineering Mechanics; Dynamics*, Prentice-Hall, Englewood Cliffs, N.J., 1960.

Example 8.2. Equation (8.38) is quite helpful in differentiating base vectors. Consider, for instance, the spherical polar system shown in Fig. 8.10. Let us regard the $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ triad as a rigid body; in other words, think of the three vectors as metal rods that are welded together. Their angular velocity, for any given $\rho(t)$, $\theta(t)$, and $\phi(t)$, is

$$\begin{aligned}\mathbf{\Omega} &= \dot{\theta}\hat{\mathbf{e}}_\phi + \dot{\phi}\hat{\mathbf{e}}_z = \dot{\theta}\hat{\mathbf{e}}_\phi + \dot{\phi}[(\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_\rho)\hat{\mathbf{e}}_\rho + (\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_\theta)\hat{\mathbf{e}}_\theta + (\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_\phi)\hat{\mathbf{e}}_\phi] \\ &= \dot{\theta}\hat{\mathbf{e}}_\phi + \dot{\phi}\cos\theta\,\hat{\mathbf{e}}_\rho + \dot{\phi}\cos\left(\theta + \frac{\pi}{2}\right)\hat{\mathbf{e}}_\theta \\ &= \dot{\phi}\cos\theta\,\hat{\mathbf{e}}_\rho - \dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\theta + \dot{\theta}\hat{\mathbf{e}}_\phi.\end{aligned}$$

(Seeking to express all vectors in terms of $\hat{\mathbf{e}}_\rho, \hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$, we have expanded $\hat{\mathbf{e}}_z$ in the form $\hat{\mathbf{e}}_z = a\hat{\mathbf{e}}_\rho + b\hat{\mathbf{e}}_\theta + c\hat{\mathbf{e}}_\phi$. Dotting both sides with $\hat{\mathbf{e}}_\rho$ yields $\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_\rho = a$, and similarly for b and c, as expressed above.)

Then since $\hat{\mathbf{e}}_\rho$ is of fixed length, (8.38) applies:

$$\begin{aligned}\dot{\hat{\mathbf{e}}}_\rho &= \mathbf{\Omega} \times \hat{\mathbf{e}}_\rho = (\dot{\phi}\cos\theta\,\hat{\mathbf{e}}_\rho - \dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\theta + \dot{\theta}\hat{\mathbf{e}}_\phi) \times \hat{\mathbf{e}}_\rho \\ &= \dot{\phi}\sin\theta\,\hat{\mathbf{e}}_\phi + \dot{\theta}\hat{\mathbf{e}}_\theta,\end{aligned} \tag{8.39}$$

and similarly for $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$.

The space derivatives are also easily available. For example,

$$\dot{\hat{\mathbf{e}}}_\rho[\theta(t), \phi(t)] = \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \theta}\dot{\theta} + \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \phi}\dot{\phi}$$

and comparing this equation with (8.39), we see that

$$\frac{\partial \hat{\mathbf{e}}_\rho}{\partial \theta} = \hat{\mathbf{e}}_\theta \quad \text{and} \quad \frac{\partial \hat{\mathbf{e}}_\rho}{\partial \phi} = \sin\theta\,\hat{\mathbf{e}}_\phi,$$

as claimed earlier in (8.33). ▮

8.4. CURVILINEAR COORDINATE REPRESENTATION OF SURFACES[2]

Just as a space curve can be represented by parametric equations $x = x(u)$, $y = y(u)$, $z = z(u)$, we expect that a *surface* can be represented by a *two*-parameter family

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v). \tag{8.40}$$

In general, (8.40) yields a space curve for each fixed v; and as v varies continuously, we anticipate that this family of curves will generate a surface.

Example 8.3. Suppose that

$$x = a\sin v\cos u, \qquad y = a\sin v\sin u, \qquad z = a\cos v. \tag{8.41}$$

It follows that

$$x^2 + y^2 + z^2 = a^2, \tag{8.42}$$

[2]One especially important application of the differential geometry of curved surfaces is in the theory of shells and curved plates; see, for example, Chapter 12 in C. Wang, *Applied Elasticity*, McGraw-Hill, New York, 1953. For a more complete treatment of the mathematics of curves and surfaces, we recommend D. J. Struik, *Differential Geometry*, 2nd ed., Addison-Wesley, Reading, Mass., 1961.

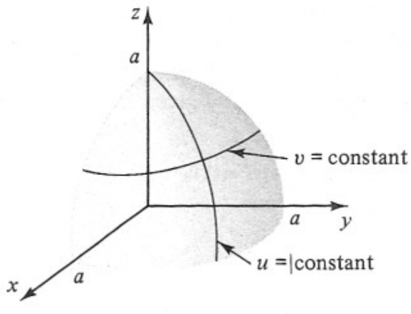

Figure 8.12. The spherical surface defined by (8.41).

and so the parametric equations (8.41) define the surface of a sphere of radius a, part of which is shown in Fig. 8.12. If $v =$ constant, then $z = a \cos v$ is constant; therefore the $v =$ constant curves are lines of "latitude" (if we regard $0, 0, a$ and $0, 0, -a$ as the "north and south poles"). On the other hand, if $u =$ constant, then $y = (\tan u)x$, and thus the $u =$ constant curves are lines of "longitude" or "meridians."[3] [If this discussion looks familiar, recall the spherical polar system (8.32) and note that u and v actually correspond to the angles ϕ and θ, respectively.]

To develop the full spherical surface, we confine u, v to the domain $0 \leq u < 2\pi$, say, and $0 \leq v \leq \pi$.

It is not hard to imagine that the u, v coordinates are more natural for calculations involving the surface of the sphere than the original cartesian variables. The "calculations" might, for instance, consist of the dynamics of a bead on the surface, a calculation of the surface area, and so on. Finally, note that the u, v coordinates in this example happen to be *orthogonal*; that is, the $u =$ constant and $v =$ constant curves intersect at right angles. The special significance of orthogonality will be discussed later in Chapter 9. ∎

Area of a Curved Surface. Part of the problem of computing the area of a curved surface is first deciding what is meant by the area of a curved surface. We know what the area of a *flat* surface means. For example, the area of a rectangle is defined to be the product of its dimensions. Extending this idea, we define a differential area dA of a *curved* surface as the area of its tangent plane approximation, as sketched in Fig. 8.13. The point of tangency is immaterial; we've chosen the corner P. The tangent vectors $\mathbf{ds}_1$ and $\mathbf{ds}_2$, which define the tangent plane parallelogram, may be expressed as follows.

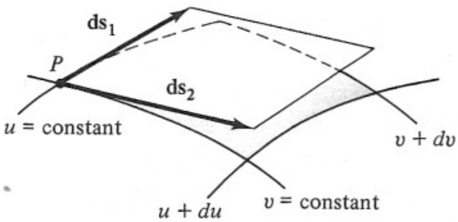

Figure 8.13. Tangent-plane definition of dA.

$$\mathbf{ds}_1 = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$

where

$$dx = x_u\,du + x_v\,dv = x_v\,dv$$
$$dy = y_u\,du + y_v\,dv = y_v\,dv$$
$$dz = z_u\,du + z_v\,dv = z_v\,dv,$$

since $\mathbf{ds}_1$ is along the $u =$ constant curve, and thus $du = 0$. So

$$\mathbf{ds}_1 = (x_v\hat{\mathbf{i}} + y_v\hat{\mathbf{j}} + z_v\hat{\mathbf{k}})\,dv \equiv \mathbf{V}\,dv, \tag{8.43}$$

and similarly

$$\mathbf{ds}_2 = (x_u\hat{\mathbf{i}} + y_u\hat{\mathbf{j}} + z_u\hat{\mathbf{k}})\,du \equiv \mathbf{U}\,du. \tag{8.44}$$

[3] A *meridian* is a curve on a surface of revolution formed by the intersection of the surface with a plane containing the axis of revolution.

Next, recalling (8.7),

$$
\begin{aligned}
dA &= \|\mathbf{ds}_2 \times \mathbf{ds}_1\| = \|(\mathbf{U}\,du) \times (\mathbf{V}\,dv)\| = \|\mathbf{U} \times \mathbf{V}\|\,du\,dv \\
&= \sqrt{(\mathbf{U} \times \mathbf{V}) \cdot (\mathbf{U} \times \mathbf{V})}\,du\,dv.
\end{aligned}
$$

With **U** and **V** as defined above, this yields

$$dA = \sqrt{EG - F^2}\,du\,dv, = \sqrt{\|r_u\|^2 \cdot \|r_\theta\|^2 - (r_u \cdot r_\theta)^2} \tag{8.45a}$$

where

$$
\begin{aligned}
E &= x_u^2 + y_u^2 + z_u^2 \\
F &= x_u x_v + y_u y_v + z_u z_v \\
G &= x_v^2 + y_v^2 + z_v^2.
\end{aligned} \tag{8.45b}
$$

Example 8.4. Returning to Example 8.3, let us use (8.45) to compute the surface area of the sphere. With $x(u, v)$, $y(u, v)$, and $z(u, v)$ given by (8.41), we find that

$$E = a^2 \sin^2 v, \qquad F = 0, \qquad G = a^2$$

so that

$$A = \int_0^\pi \int_0^{2\pi} a^2\,|\sin v|\,du\,dv. \tag{8.46}$$

Dropping the absolute value signs, since $\sin v \geq 0$ over the domain of integration, and integrating, we easily obtain $A = 4\pi a^2$, which, of course, we recognize as being correct.

The point to notice is the happy simplicity of (8.46); the integrand is quite simple and the limits are independent (i.e., the u limits are not functions of v). This situation merely reflects the fact that the curvilinear coordinates (8.41) "fit" the spherical geometry so naturally.

Finally, note the significance of $F = 0$. In general, $F = x_u x_v + y_u y_v + z_u z_v = \mathbf{U} \cdot \mathbf{V}$. Thus $F = 0$ means that $\mathbf{U} \cdot \mathbf{V} = 0$—that is, the constant u and constant v curves are *orthogonal*. ■

Special cases of (8.45).

(i) Suppose that $z = 0$—that is,

$$x = x(u, v), \qquad y = y(u, v), \qquad z = 0, \tag{8.47}$$

so that our surface is *flat* and lies in the x, y plane. Then

$$E = x_u^2 + y_u^2, \qquad F = x_u x_v + y_u y_v, \qquad G = x_v^2 + y_v^2;$$

therefore

$$EG - F^2 = x_u^2 y_v^2 - 2x_u y_v x_v y_u + x_v^2 y_u^2 = (x_u y_v - x_v y_u)^2,$$

which is none other than the *Jacobian* $J(u, v)$ squared! So we have the neat result

$$dA = |J(u, v)|\,du\,dv, \tag{8.48}$$

the absolute value signs being needed to ensure that we have the *positive* square root of $EG - F^2$, since dA must surely be positive.

For instance, if

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

then $|J| =$ etc. $= r$ (independent of whether we identify r as u and θ as v or vice versa) and $dA = r\,dr\,d\theta$. Of course, this result might have been deduced instead from the

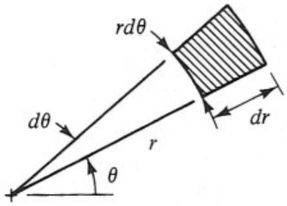

Figure 8.14. Area element for polar coordinates.

simple sketch in Fig. 8.14, but the "method of sketches" may not be practicable in more complicated cases, whereas (8.48) is always straightforward.

(ii) Suppose that we are given the equation of the surface in the form $z = f(x, y)$. That's equivalent to the system

$$x = x(u, v) = u, \qquad y = y(u, v) = v,$$
$$z = z(u, v) = f(u, v). \tag{8.49}$$

Thus

$$E = 1 + f_u^2, \qquad F = f_u f_v, \qquad G = 1 + f_v^2$$

which means that

$$dA = \sqrt{f_u^2 + f_v^2 + 1}\, du\, dv$$

or since $u = x$ and $v = y$,

$$dA = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8.50}$$

Example 8.5. Again returning to our sphere $x^2 + y^2 + z^2 = a^2$, let us compute its surface area, using (8.50). With $z = f(x, y) = +\sqrt{a^2 - x^2 - y^2}$, on the top half, say, we have (Exercise 8.26)

$$A = 2\int_{-a}^{a}\int_{-\sqrt{a^2-y^2}}^{\sqrt{a^2-y^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dx\, dy = 4\pi a^2. \quad \blacksquare$$

8.5. *CURVILINEAR COORDINATE REPRESENTATION OF VOLUMES*

You will probably not be surprised that we now consider the representation of a *volume* by a *three*-parameter family,

$$x = x(u, v, w), \qquad y = y(u, v, w), \qquad z = z(u, v, w). \tag{8.51}$$

In general, (8.51) yields a surface for each fixed w; and as w varies continuously, we expect this family of surfaces to generate a volume.

Proceeding along essentially the same lines as earlier, we have from (8.10) and Fig. 8.15 the expression

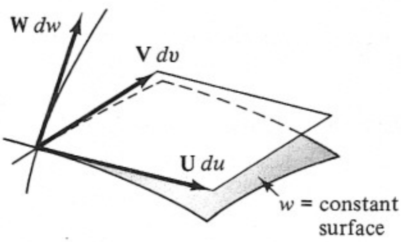

Figure 8.15. Forming the parallelepiped for dV.

$$dV = |\mathbf{U}\, du \cdot \mathbf{V}\, dv \times \mathbf{W}\, dw| = |\mathbf{U} \cdot \mathbf{V} \times \mathbf{W}|\, du\, dv\, dw$$

for the volume element, where $\mathbf{U}$ and $\mathbf{V}$ are the same as above [(8.43) and (8.44)] and $\mathbf{W}$ is $x_w\hat{\mathbf{i}} + y_w\hat{\mathbf{j}} + z_w\hat{\mathbf{k}}$. Recalling (8.35), we can express

$$dV = \left\| \begin{matrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{matrix} \right\| du\, dv\, dw,$$

where the inner set of vertical lines denotes determinant and the outer set indicates absolute value.

But the determinant is none other than the Jacobian $J(u, v, w)$, and so

$$dV = |J(u, v, w)|\, du\, dv\, dw. \tag{8.52}$$

[Note the similarity to (8.48).]

Example 8.6. To compute the *volume* of our sphere of radius a, we introduce the more natural curvilinear coordinates

$$x = w \sin v \cos u, \qquad y = w \sin v \sin u, \qquad z = w \cos v.$$

Then $|J(u, v, w)| = \text{etc.} = |w^2 \sin v|$, and therefore

$$V = \int_0^a \int_0^\pi \int_0^{2\pi} w^2 \sin v\, du\, dv\, dw = \tfrac{4}{3}\pi a^3. \quad \blacksquare$$

EXERCISES

8.1. Show that equations (8.1) follow from our definitions of addition and scalar multiplication of vectors.

8.2. Using (8.1) and the definition of $\mathbf{0}$, show carefully that $\mathbf{A} + \mathbf{0} = \mathbf{A}$ (for all As) and that $\mathbf{A} + \mathbf{B} = \mathbf{C}$ implies $\mathbf{A} = \mathbf{C} - \mathbf{B}$.

8.3. Exactly how does (8.5) follow from (8.3) and (8.4)?

8.4. If $\|\mathbf{A}\| = 1$, $\|\mathbf{B}\| = 2$, and $\|\mathbf{C}\| = 5$, can $\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{0}$? How about if $\|\mathbf{A}\| = 4$ instead? Explain.

8.5. If $\mathbf{A} = \hat{\mathbf{i}} - \hat{\mathbf{k}}$, $\mathbf{B} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, and $\mathbf{C} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$, evaluate $2\mathbf{A} - 3\mathbf{B}$, $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{B} \cdot \mathbf{A}$, $\mathbf{A} \times \mathbf{B}$, $\mathbf{B} \times (3\mathbf{A})$, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$, the (acute) angle between $\mathbf{A}$ and $\mathbf{B}$, the projection of $\mathbf{B}$ on $\mathbf{A}$, and the projection of $\mathbf{A}$ on $\mathbf{B}$. Find a scalar α such that $\mathbf{A} + \alpha\mathbf{B}$ is orthogonal to $\mathbf{A}$. Find scalars α, β such that $\mathbf{A} + \alpha\mathbf{B} + \beta\mathbf{C}$ is orthogonal to both $\mathbf{A}$ and $\mathbf{B}$.

8.6. What (acute) angle does $\mathbf{A} = \hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ make with the normal to the plane containing the vectors $\mathbf{B} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{C} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$?

8.7. Given distinct vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$, which are tail to tail, give a necessary and sufficient condition(s) for their heads all to lie in a common plane.

8.8. To show that the diagonals of a parallelogram bisect each other, proceed as follows. Note from Fig. 8.16 that $\mathbf{A} + \mathbf{B} = \mathbf{C}$, $\mathbf{A} - \alpha\mathbf{D} = \beta\mathbf{C}$, $\mathbf{A} = \mathbf{B} + \mathbf{D}$. Eliminating $\mathbf{A}$ and $\mathbf{B}$, say, we obtain $(2\beta - 1)\mathbf{C} = (1 - 2\alpha)\mathbf{D}$. Since $\mathbf{C}$ and $\mathbf{D}$ are not aligned, it must be true that $2\beta - 1 = 1 - 2\alpha = 0$—that is, $\alpha = \beta = \frac{1}{2}$. Use this same general procedure to show that a line from one vertex of a parallelogram to the midpoint of a nonadjacent side trisects a diagonal.

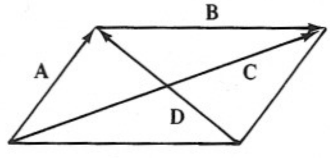

Figure 8.16. Parallelogram.

8.9. In mechanics we call $\mathbf{OA} \times \mathbf{F} = \mathbf{M}$ the *moment* of the force $\mathbf{F}$ (which acts through the point A) about the point O, where $\mathbf{OA}$ is the vector from O to A.

(a) If $\mathbf{F}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ acts through (0, 3, 2) and $\mathbf{F}_2 = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ acts through (1, −1, 4), find (if possible) the force $\mathbf{F}_3$ needed to act through (2, 2, 1) such that the system has no net moment about the point (1, 1, 0). Show that the net moment induced by $\mathbf{F}_1$ and $\mathbf{F}_2$ about (1, 1, 0) is not orthogonal to the position vector from (1, 1, 0) to (2, 2, 1). How is this fact pertinent?

(b) If $\mathbf{F}_1$ and $\mathbf{F}_2$ act through distinct points A_1 and A_2, derive a condition on $\mathbf{F}_1$ and $\mathbf{F}_2$ that will ensure that their resultant moment about O is independent of the location of O. Such a system of forces is said to constitute a pure *couple*. Can you give a specific example of *three* (nonzero) forces through distinct points that constitute a couple?

8.10. (a) If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ form a triangle such that $\mathbf{A} = \mathbf{B} + \mathbf{C}$, and α, β, γ are the angles opposite the sides $\mathbf{A}, \mathbf{B}, \mathbf{C}$, respectively, derive the *law of cosines*, $C^2 = A^2 + B^2 - 2AB \cos \alpha$, by starting with the identity $\mathbf{C} \cdot \mathbf{C} = (\mathbf{A} - \mathbf{B}) \cdot (\mathbf{A} - \mathbf{B})$.

(b) Derive the *law of sines*, $(\sin \alpha)/A = (\sin \beta)/B$, by starting with $\mathbf{C} \times \mathbf{C} = \mathbf{C} \times (\mathbf{A} - \mathbf{B})$.

8.11. Derive **Lagrange's identity,**

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} \mathbf{A} \cdot \mathbf{A} & \mathbf{A} \cdot \mathbf{B} \\ \mathbf{A} \cdot \mathbf{B} & \mathbf{B} \cdot \mathbf{B} \end{vmatrix}$$

by noting that $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} \times (\mathbf{A} \times \mathbf{B})$ (why?) and using (8.11) and (8.12).

8.12. (a) Show that the statement of linearity, (8.9), is equivalent to the two separate statements $(\alpha\mathbf{A}) \times \mathbf{B} = \alpha(\mathbf{A} \times \mathbf{B})$ and $(\mathbf{A} + \mathbf{B}) \times \mathbf{C} = \mathbf{A} \times \mathbf{C} + \mathbf{B} \times \mathbf{C}$.

(b) Prove that each of these two statements is correct.

8.13. Derive (8.31) by taking the limit of the difference quotient as we did in deriving (8.28).

8.14. Derive (8.33) by taking the limit of the difference quotient as we did in deriving (8.28).

8.15. Derive (8.28) and (8.31) by the method employed in Example 8.2.

8.16. Finish Example 8.2; that is, compute $\partial\hat{\mathbf{e}}_\theta/\partial\theta$, $\partial\hat{\mathbf{e}}_\theta/\partial\phi$, and $\partial\hat{\mathbf{e}}_\phi/\partial\phi$ and compare your results with (8.33).

8.17. Compute $d\mathbf{A}/dt$ if $\mathbf{A} = \sin\theta\, \hat{\mathbf{e}}_r - r\theta^2\hat{\mathbf{e}}_\theta + r\hat{\mathbf{e}}_z$, with $r = t^2$ and $\theta = 3t$.

8.18. A particle moves in the x, y plane according to $x = t^2$, $y = 2$. Compute its velocity and acceleration in terms of $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$.

8.19. Consider a particle P moving under the influence of a *central force field*—that is, where the force $\mathbf{F}$ on P is always aligned with the position vector $\mathbf{R}$ from some fixed point, say O, as shown in Fig. 8.17. Starting with Newton's law $\mathbf{F} = m\ddot{\mathbf{R}}$ and noting that the moment $\mathbf{R} \times \mathbf{F}$ about O is zero, show that the so-called *moment of momentum*

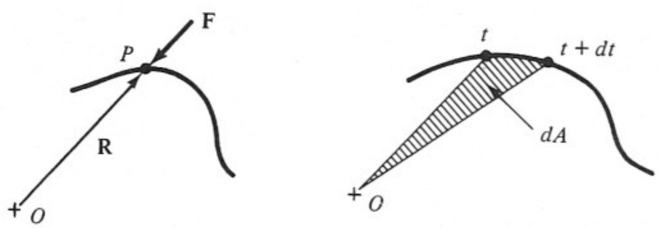

Figure 8.17. Central force motion.

$\mathbf{R} \times m\dot{\mathbf{R}}$ is a constant vector. Using this fact, show that the motion of P must lie in a *plane* and that the *areal velocity* dA/dt (i.e., the rate at which area is swept out by the $\mathbf{R}$ vector) must be constant.

8.20. (**Frenet's formulas**) Consider a space curve C defined by $\mathbf{R} = \mathbf{R}(s)$, where s is the arc length along C, measured from some reference point. The length of the tangent vector $d\mathbf{R} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ is $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$, or ds, so that $d\mathbf{R}/ds \equiv \hat{\mathbf{t}}$ is a *unit tangent* vector. Since $\hat{\mathbf{t}}$ is of unit length, show that $d\hat{\mathbf{t}}/ds$ must be normal to it. Thus we write

$$\frac{d\hat{\mathbf{t}}}{ds} \equiv \kappa\hat{\mathbf{n}} \qquad (\kappa \geq 0), \tag{8.53}$$

where the unit vector $\hat{\mathbf{n}}$ is called the **principal normal**; the plane of $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ is called the **osculating plane**. With the help of suitable sketches show that the multiplier κ is the **curvature** of C—that is, $1/\rho$, where ρ is the local **radius of curvature** of C, and that $\hat{\mathbf{n}}$ points toward the center of curvature. (If C is straight, then ρ is infinite and $\kappa = 0$.) We introduce a third unit vector, the **binormal** $\mathbf{b} \equiv \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. (Sketch all this.) Taking d/ds of $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$, show that $d\hat{\mathbf{b}}/ds$ is parallel to $\hat{\mathbf{n}}$, so that

$$\frac{d\hat{\mathbf{b}}}{ds} = \tau\hat{\mathbf{n}}. \tag{8.54}$$

The factor τ is called the **torsion**, since it is a measure of the rate at which C twists out of its osculating plane. Next, show that

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa\hat{\mathbf{t}} - \tau\hat{\mathbf{b}}. \tag{8.55}$$

Equations (8.53) to (8.55) are called the *Frenet–Serret formulas*. Finally, show that

$$\kappa = \|\hat{\mathbf{t}}'\| = \|\mathbf{R}''\| \tag{8.56}$$

and that $\hat{\mathbf{t}} \cdot \hat{\mathbf{t}}' \times \hat{\mathbf{t}}'' = -\kappa^2\tau$, and thus

$$\tau = -\frac{\mathbf{R}' \cdot \mathbf{R}'' \times \mathbf{R}'''}{\mathbf{R}'' \cdot \mathbf{R}''}, \tag{8.57}$$

where the primes denote d/ds.

8.21. (First read the previous exercise.) If we move along the curve C defined by $\mathbf{R} = \mathbf{R}(s)$ according to $s = s(t)$, where t is the time, show that our velocity and acceleration are

$$\mathbf{v} = \dot{s}\hat{\mathbf{t}} \quad \text{and} \quad \mathbf{a} = \ddot{s}\hat{\mathbf{t}} + \frac{\dot{s}^2}{\rho}\hat{\mathbf{n}}.$$

8.22. A tiny elephant moves along the path shown in Fig. 8.18 with speed $\dot{s} = 6t$. Using the results of the previous exercise, give an expression for its acceleration $\mathbf{a}$ over each

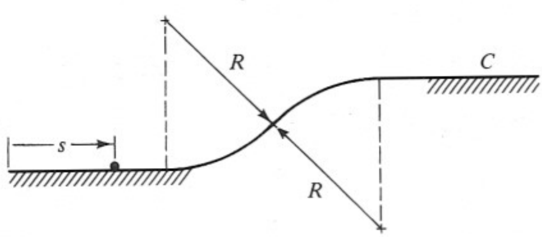

Figure 8.18.

segment of C—that is, the two linear parts and the two circular arcs. Where does the maximum acceleration occur?

8.23. For the helix defined parametrically by

$$x = a\cos\zeta, \qquad y = a\sin\zeta, \qquad z = \omega\zeta \quad (a, \omega \text{ constants})$$

(a) Compute the arc length of the helix from $\zeta = 0$ out to an arbitrary ζ.

(b) Give expressions for $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ (see Exercise 8.20) in terms of $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. *Hint:* $\hat{\mathbf{t}} = (d\mathbf{R}/d\zeta)(d\zeta/ds)$, where $d\zeta/ds$ is found simply as $1/\| d\mathbf{R}/d\zeta \|$, since $\hat{\mathbf{t}}$ is to be of unit length, or from your solution to part (a).

(c) Evaluate κ, ρ and τ. Check your solution for the special limiting cases where $\omega \longrightarrow 0$ and $\omega \longrightarrow \infty$. Do your answers look right? For example, would you expect κ, ρ, τ to vary along the helix or to be constant? Would you expect τ to be positive or negative? Why?

8.24. For the curve of intersection of the circular cylinder $x^2 + y^2 = 1$ with the circular cylinder $y^2 + z^2 = 1$, find the radius of curvature at the points (0, 1, 0) and (1, 0, 1). (See Exercise 8.20.)

***8.25.** In Section 8.3 we defined the angular velocity vector $\mathbf{\Omega}$. Can we also define an angular *rotation* vector, say $\boldsymbol{\theta}$? That is, $\boldsymbol{\theta}$ would be the amount of angular rotation θ about some axis, times a unit vector along that axis in the direction assigned by the "right-hand rule." *Hint:* Show that vector addition would not be commutative.

8.26. Fill in the omitted steps in Example 8.5 without using an integral table.

8.27. For polar coordinates, we can bypass the expression $dA = |J|\, dr\, d\theta$ and obtain $dA = r\, dr\, d\theta$ simply from Fig. 8.14. Try this for the curvilinear u, v coordinates defined by $x = u + v$, $y = u - v$ and compare with the result given by $|J|\, du\, dv$.

8.28. Derive the volume element $dV = \rho^2 \sin\theta\, d\rho\, d\theta\, d\phi$ for spherical polars first from a simple three-dimensional sketch and then from the expression $dV = |J|\, d\rho\, d\theta\, d\phi$.

8.29. Evaluate the integrals shown.

(a) $I = \int_0^{1/2} \int_0^{1-2y} e^{x/(x+2y)}\, dx\, dy.$

Hint: Set $u = x$, $v = x + 2y$. In computing the needed Jacobian $J(u, v) = x_u y_v - x_v y_u$, note that $J(u, v) = [J(x, y)]^{-1} = (u_x v_y - u_y v_x)^{-1}$, per Exercise 7.18, is more convenient, since we know $u(x, y)$ and $v(x, y)$, not $x(u, v)$ and $y(u, v)$. [In this example there's little difference because we can easily solve for $x(u, v)$ and $y(u, v)$, but this inversion is not always simple.] The same comment applies to the *volume* element, where we need the Jacobian $J(u, v, w)$. Again, it's true (Exercise 7.18) that $J(u, v, w) = [J(x, y, z)]^{-1}$.

(b) $I = \int_0^1 \int_0^{1-z} \int_0^{1-y-z} e^{x/(x+y)}\, dx\, dy\, dz.$

(c) $I = \int_0^\infty \int_y^\infty e^{-(x^2+y^2)}\, dx\, dy.$

(d) $I = \int_0^1 \int_0^{1-y} \sin\left(\dfrac{x-y}{x+y}\right) dx\, dy.$

***8.30.** We meet the integral

$$I = -\frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i(\xi x + \eta y + \zeta z)}}{\xi^2 + \eta^2 + \zeta^2}\, d\xi\, d\eta\, d\zeta$$

when we solve the so-called Poisson equation by the Fourier transform. Show that $I = -1/4\pi r$, where $r = \sqrt{x^2 + y^2 + z^2}$. *Hint:* Note that $\exp i(\xi x + \eta y + \zeta z) =$

$\exp i\mathbf{R} \cdot \mathbf{r} = \exp irR\cos\theta$, where $\mathbf{R}$ and $\mathbf{r}$ are shown in Fig. 8.19, and change over to spherical polars R, θ, ϕ with $\mathbf{r}$ as the polar axis.

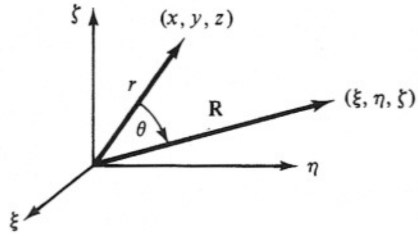

Figure 8.19. The coordinates.

8.31. Compute the moment of inertia about the y axis, $I_y = \iint x^2\, dA$, of the area enclosed by the circle $(x-a)^2 + y^2 = a^2$ by changing to polar coordinates.

8.32. The gravitational force of attraction induced per unit mass at the origin by the right circular cone of height h and (uniform) mass density σ shown in Fig. 8.20 is given by

$$F = \iiint_V \frac{\sigma z\, dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}},$$

where V is the volume of the cone.

(a) Evaluate F by changing to cylindrical coordinates.

(b) Repeat, using spherical polars.

(c) Evaluate F if V is instead the *hemisphere* $x^2 + y^2 + z^2 < a^2$ for $0 \le z \le a$.

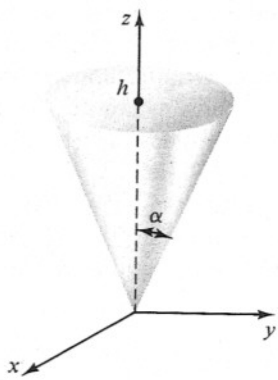

Figure 8.20. Right circular cone.

8.33. Show that the paraboloid $x^2 + y^2 = a^2 z$ can be represented by the two-parameter family $x = av\cos u$, $y = av\sin u$, $z = v^2$. Sketch the paraboloid and the constant u and constant v curves on it.

8.34. Make up two examination-type problems on this chapter.