Homework # 25

27.1, 3, 5abc, 6abcd, 9, 13

27.1

a) $u_{,xx} - \rho(x) u_{,tt} = h(x) u_{,t} + f(x) e^{i\omega t}$; $\rho(x) > 0$

$\rho(x) > 0$

$$\begin{vmatrix} 1 & 0 \\ 0 & -\rho \end{vmatrix} = -\rho < 0 \qquad \text{hyperbolic}$$

b) $u_{,xx} + (1-x^2) u_{,yy} = 6$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1-x^2 \end{vmatrix} = 1-x^2 \begin{cases} > 0 & \text{if } |x| < 1 & \text{elliptic} \\ = 0 & \text{if } |x| = 1 & \text{parabolic} \\ < 0 & \text{if } |x| > 1 & \text{hyperbolic} \end{cases}$$

c) $u_{,xy} + xy\, u_{,yy} - u_{,y} = 0$

$$\begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & xy \end{vmatrix} = -\frac{1}{4} < 0 \qquad \text{hyperbolic}$$

d) $u_{,xx} = \rho(x) u_{,t} + h(x) u + f(x,t)$

$$\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \qquad \text{parabolic}$$

e) $u_x = (1-u^2) u_{,yy}$

$$\begin{vmatrix} 0 & 0 \\ 0 & 1-u^2 \end{vmatrix} = 0 \qquad \text{parabolic}$$

f) $\begin{cases} e_{,x} + Ri = 0 \\ i_{,x} + C e_{,t} = 0 \end{cases}$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ dt & 0 & 0 \\ 0 & dx & dt \end{vmatrix} = -\,dt \begin{vmatrix} 1 & 0 \\ dx & dt \end{vmatrix}$$

$$= -dt\,(dt) = -dt^2$$ parabolic

27.3.

$$\begin{vmatrix} u & 1 & 2c & 0 \\ c & 0 & 2u & 2 \\ dx & dt & 0 & 0 \\ 0 & 0 & dx & dt \end{vmatrix} = -dx \begin{vmatrix} u & 1 & 0 \\ c & 0 & 2 \\ dx & dt & 0 \end{vmatrix} + dt \begin{vmatrix} u & 1 & 2c \\ c & 0 & 2u \\ dx & dt & 0 \end{vmatrix}$$

$$= -dx^2 \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + dx\,dt \begin{vmatrix} u & 0 \\ c & 2 \end{vmatrix} + dx\,dt \begin{vmatrix} 1 & 2c \\ 0 & 2u \end{vmatrix} - dt^2 \begin{vmatrix} u & 2c \\ c & 2u \end{vmatrix}$$

$$= -dx^2 \cdot 2 + dx\,dt\,(4u - dt^2 (2u^2 - 2c^2) = 0$$

$$\left(\frac{dx}{dt}\right)^2 - 2u\left(\frac{dx}{dt}\right) + (u^2 - c^2) = 0$$

$$\Delta = 4u^2 - 4(u^2 - c^2) = 4c^2 \quad , \quad \sqrt{\Delta} = 2c$$

$$\left(\frac{dx}{dt}\right)_{1,2} = \frac{2u \pm 2c}{2} = u \pm c$$

$$\begin{vmatrix} u & 1 & 2c & H_x \\ c & 0 & 2u & 0 \\ dx & dt & 0 & du \\ 0 & 0 & dx & dc \end{vmatrix} = -dx \begin{vmatrix} u & 1 & H_x \\ c & 0 & 0 \\ dx & dt & du \end{vmatrix} + dc \begin{vmatrix} u & 1 & 2c \\ c & 0 & 2u \\ dx & dt & 0 \end{vmatrix}$$

$$= -dx^2 \cancelto{0}{\begin{vmatrix} 1 & H_x \\ 0 & 0 \end{vmatrix}} + dx\,dt \begin{vmatrix} u & H_c \\ c & 0 \end{vmatrix} - dx\,du \begin{vmatrix} u & 1 \\ c & 0 \end{vmatrix}$$

$$+ dc\,dx \begin{vmatrix} 1 & 2c \\ 0 & 2u \end{vmatrix} - dc\,dt \begin{vmatrix} u & 2c \\ c & 2u \end{vmatrix}$$

$$= -dx\,dt\,H_c\,c + dx\,du \cdot c + dc\,dx\,2u - dc\,dt\left(2u^2 - 2c^2\right) = 0$$

$$-(u \pm c)\,dt^2 H_c\,c + (u \pm c)\,du\,dt + dc\,(u \pm c)\,dt\,2u - dc\,dt\,(2u^2 - 2c^2)$$

$$-H_c c\,(u \pm c)\,dt^2 + (u \pm c)\,du\,dt + 2u(u \pm c)\,dc\,dt - 2(u^2 - c^2)\,dc\,dt = 0$$

$(+)$ $$-H_c c\,dt^2 + du\,dt + \underbrace{2u\,dc\,dt - 2(u-c)\,dc\,dt}_{2c\,dc\,dt} = 0$$

$$\underline{-H_c c\,dt + du + 2c\,dc = 0}$$

$(-)$ $$-H_c c\,dt + du + \underbrace{2u\,dc - 2(u+c)\,dc}_{-2c\,dc} = 0$$

$$\underline{-H_c c\,dt + du - 2c\,dc = 0}$$

27.5

a) $\varepsilon \varphi_{,xx} + \varphi_{,yy} = 0 \qquad \varepsilon > 0$

Characteristics: $\qquad A\varphi_{,xx} + 2B\varphi_{,xy} + C\varphi_{,yy} = F$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{\pm\sqrt{-\varepsilon}}{\varepsilon} = \pm i \frac{1}{\sqrt{\varepsilon}}$$

$$dy + i\frac{1}{\sqrt{\varepsilon}}\,dx = 0 \qquad dy - i\frac{1}{\sqrt{\varepsilon}}\,dx = 0$$

$$y + i\frac{1}{\sqrt{\varepsilon}}\,x = c \qquad y - i\frac{1}{\sqrt{\varepsilon}}\,x = c$$

so: $\psi_+(x,y) = y + i\frac{1}{\sqrt{\varepsilon}}\,x$

$\psi_-(x,y) = y - i\frac{1}{\sqrt{\varepsilon}}\,x$

change of coordinates (page 574, formula 27.41)

$$\begin{cases} \xi = y \\ \eta = \frac{1}{\sqrt{\varepsilon}}\,x \end{cases}$$

$$\varphi_{,xx} = \frac{1}{\varepsilon}\varphi_{,\eta\eta}$$

$$\varphi_{,yy} = \varphi_{,\xi\xi}$$

$$\varepsilon\varphi_{,xx} + \varphi_{,yy} = \varphi_{,\eta\eta} + \varphi_{,\xi\xi} = 0$$

b) $\varphi_{,xx} - x^2\varphi_{,yy} - x\varphi^3 = 0$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - AC}}{A} = \frac{\pm\sqrt{x^2}}{1} = \pm|x|$$

Case $x>0$

$$dy - x\,dx = 0 \qquad\qquad dy + x\,dx = 0$$
$$y - \frac{x^2}{2} = C \qquad\qquad y + \frac{x^2}{2} = C$$

So: $\psi_+(x,y) = y - \frac{x^2}{2}$

$\psi_-(x,y) = y + \frac{x^2}{2}$

$$\begin{cases} \xi = y - \frac{x^2}{2} \\ \eta = y + \frac{x^2}{2} \end{cases}$$

$$\varphi_{,x} = \varphi_{,\xi}(-x) + \varphi_{,\eta}\cdot x$$
$$\varphi_{,xx} = \varphi_{,\xi\xi}\,x^2 - 2\varphi_{,\xi\eta}\,x^2 + \varphi_{,\eta\eta}\,x^2 - \varphi_{,\xi} + \varphi_{,\eta}$$

$$\varphi_{,y} = \varphi_{,\xi} + \varphi_{,\eta}$$
$$\varphi_{,yy} = \varphi_{,\xi\xi} + 2\varphi_{,\xi\eta} + \varphi_{,\eta\eta}$$

$$\varphi_{,xx} - x^2\varphi_{,yy} - \varkappa\varphi^3 = \varphi_{,\xi\xi}\,x^2 - 2\varphi_{,\xi\eta}\,x^2 + \varphi_{,\eta\eta}\,x^2 - \varphi_{,\xi} + \varphi_{,\eta}$$
$$- \varphi_{,\xi\xi}\,x^2 - 2\varphi_{,\xi\eta}\,x^2 - \varphi_{,\eta\eta}\,x^2 - \varkappa\varphi^3$$
$$= \underline{-4x^2\varphi_{,\xi\eta} - \varphi_\xi + \varphi_\eta - \varkappa\varphi^3 = 0}$$

Case $x<0$ analogous.

c) $\varphi_{,xx} + 2\varphi_{,xt} + \varphi_{,tt} = f(x,t)$

Characteristics: $\dfrac{dt}{dx} = \dfrac{B \pm \sqrt{B^2 - AC}}{A} = \dfrac{1 \pm \sqrt{1-1}}{1} = 1$

(parabolic equ.)

(6)

$dt - dx = 0$

$t - x = c$

$$\begin{cases} \xi = t - x \\ \eta = x \end{cases}$$

$\varphi_{,x} = -\varphi_{,\xi} + \varphi_{,\eta}$

$\varphi_{,xx} = \varphi_{,\xi\xi} - 2\varphi_{,\xi\eta} + \varphi_{,\eta\eta}$

$\varphi_{,t} = \varphi_{,\xi}$

$\varphi_{,tx} = -\varphi_{,\xi\xi} + \varphi_{,\xi\eta}$

$\varphi_{,tt} = \varphi_{,\xi\xi}$

$\varphi_{,xx} + 2\varphi_{,xt} + \varphi_{,tt} = \cancel{\varphi_{,\xi\xi}} - \cancel{2\varphi_{,\xi\eta}} + \varphi_{,\eta\eta} - \cancel{2\varphi_{,\xi\xi}} + \cancel{2\varphi_{,\xi\eta}} + \cancel{\varphi_{,\xi\xi}}$

$= \varphi_{,\eta\eta}$

$\underline{\varphi_{,\eta\eta} = f(\eta, \xi + \eta)}$

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27. 6 $\quad A\varphi_{,xx} + 2B\varphi_{,xy} + C\varphi_{,yy} = 0$

$\varphi = f(x + my)$

$\varphi_{,x} = f'' \qquad \varphi_{,xy} = f'' \cdot m$

$\varphi_{,xx} = f''$

$\varphi_{,y} = f'm \qquad \varphi_{,yy} = f''m^2$

$$A\varphi_{,xx} + 2B\varphi_{,xy} + C\varphi_{,yy} = f''(x+my)(A + 2Bm + Cm^2) = 0$$

$$A + 2Bm + Cm^2 = 0 \quad \Rightarrow \quad m = \frac{-B \pm \sqrt{B^2 - AC}}{C}$$

b) $$3\varphi_{,xx} + 8\varphi_{,xy} - 3\varphi_{,yy} = 0$$

$$m = \frac{-8 \pm \sqrt{64+9}}{-3} = \frac{8}{3} \mp \frac{\sqrt{73}}{3}$$

$$\varphi(x,y) = f\left(x + \left(\frac{8}{3} - \frac{\sqrt{73}}{3}\right)y\right) + g\left(x + \left(\frac{8}{3} + \frac{\sqrt{73}}{3}\right)y\right)$$

c) $$\varphi(x,y) = y\,h(x+my)$$

$$B^2 - AC = 0$$
$$m = -\frac{B}{C}$$

$$\varphi_{,x} = yh'$$
$$\varphi_{,xx} = yh''$$
$$\varphi_{,xy} = h' + ymh''$$

$$\varphi_{,y} = h + myh'$$
$$\varphi_{,yy} = mh' + mh' + m^2h'' = 2mh' + m^2h''$$

$$A\varphi_{,xx} + 2B\varphi_{,xy} + C\varphi_{,yy} = Ayh'' + 2Bh' + 2Bymh'' + 2Cmh' + Cm^2h''$$

$$= yh''\underbrace{(A + 2Bm + Cm^2)}_{0} + 2h'\underbrace{(B + Cm)}_{0} = 0$$

d) $$4\varphi_{,xx} - 4\varphi_{,xy} + \varphi_{,yy} = 0 \qquad m = \frac{2 \pm 0}{1} = 2$$

$$\varphi(x,y) = f(x+2y) + y\,g(x+2y)$$

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27.9 Solved in class

27.13 No! In the case of a constant solution variable coefficients are constant and, consequently, the characteristics are straight lines.

Example:

$$u_{,t} + u u_{,x} = u_{,t} + \left(\tfrac{1}{2}u^2\right)_{,x} = 0$$

$u = \text{const}$ is a solution.