Euler

$$\|e_{n+1}\| \le (1+h\lambda)\|e_n\| + Ch^2$$
$$\|e_n\| \le \frac{C}{\lambda} h\left[(1+h\lambda)^n - 1\right]$$
$$\le \frac{C}{\lambda}(e^{T\lambda}-1)h$$

Trapezoid

$$\|e_{n+1}\| \le \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)\|e_n\| + \left(\frac{C}{1-\frac{1}{2}h\lambda}\right)h^3$$
$$\|e_n\| \le \frac{C}{\lambda}\left[\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1\right]h^2$$

General s-step

$$\sum_{m=0}^{s} a_m y_{n+m} = h\sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$
$$\rho(w) = \sum_{m=0}^{s} a_m w^m$$
$$\sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Order p iff

$$\rho(\xi+1) - \sigma(\xi+1)\ln(\xi+1) = c(\xi)^{p+1} + \mathcal{O}(\xi^{p+2})$$

Root Condition: all zeros reside in closed complex unit disc and all zeros of unit modulus are simple.

$$\|f(x)-p(x)\|_{L^\infty} \le \frac{(b-a)^s}{s!}\left\|f^{(s)}\right\|_{L^\infty}$$

Runge-Kutta

$$\xi_j = y_n + h\sum_{i=1}^{\nu} a_{j,i} f(t_n + c_i h, \xi_i)$$
$$y_{n+1} = y_n + h\sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j)$$

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array}$$

Collocation

$$\int_0^1 \tau^j \prod_{l=1}^{\nu}(\tau - c_l) = 0\,, \quad j = 0,1,\ldots,m-1$$

for some $m \in \{0,1,\ldots,\nu\}$. Then the collocation method is of order $\nu + m$.

$$q(t) = \prod_{j=1}^{\nu}(t-c_j)\,, \quad q_l(t) = \frac{q(t)}{t-c_l}$$
$$a_{j,i} = \int_0^{c_j}\frac{q_i(\tau)}{q_i(c_i)}\mathrm{d}\tau\,, \quad b_j = \int_0^1 \frac{q_j(\tau)}{q_j(c_j)}\mathrm{d}\tau$$

Let $c_1, c_2, \ldots, c_\nu$ be the zeros of the polynomials that are orthogonal with respect to the weight function $\omega(t) \equiv 1\,, 0 \le t \le 1$. Then the underlying collocation method is of order 2ν.

Stability of Runge-Kutta Methods

$$r(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}$$
$$(I-zA)^{-1} = \frac{\mathrm{adj}(I-zA)}{\det(I-zA)}$$

Adjugate is transpose of cofactor matrix

$|r(z)| < 1$ for all $z \in \mathbb{C}^-$ iff all the poles of r have positive real parts and $|r(it)| \le 1$ for all $t \in \mathbb{R}$.

Multistep methods

$$\eta(z,w) = \sum_{m=0}^{s}(a_m - b_m z)w^m$$

The multistep method is A-stable iff $b_s > 0$ and

$$|w_1(it)|\,, |w_2(it)|\,, \ldots, \left|w_{q(it)}(it)\right| \le 1\,, \quad t \in \mathbb{R}$$

where $w_1, w_2, \ldots, w_{q(z)}$ are the zeros of $\eta(z,\cdot)$.

Linearizing a non-linear equation

$$y' = \underbrace{f(t,y)}_{b} + \underbrace{\nabla f(t,y)}_{A}(y-\bar{y}) + \mathcal{O}(|y-\bar{y}|^2)$$

Solution of nonlinear equations

Fixed Point

$$\mathbf{w} = h\mathbf{g}(\mathbf{w}) + \boldsymbol{\beta}$$

Unique solution exists in sufficiently small neighborhood of β if $\left(\mathbf{I} - h\frac{\partial \mathbf{g}}{\partial \mathbf{w}}\right)$ is nonsingular.

Banach Fixed Point Theorem If $G : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction: $\exists 0 \le \lambda < 1$ s.t.

$$\|G(u) - G(v)\| \le \lambda\|u - v\| \quad \forall u, v \in \mathbb{R}^d$$

Then $\exists!$ fixed point w and

$$\left\|w^i - w\right\| \le \lambda^i \left\|w^0 - w\right\|$$

If g is Lipshitz with constant Λ, we need $h \le \frac{\lambda}{\Lambda}$

Newton's Method

$$w^{i+1} = w^i - \frac{F(w^i)}{F'(w^i)}$$

$$w^{i+1} = w^i - \left(\frac{\partial F(w)}{\partial w}\right)^{-1} F(w^i)$$

<u>Continuous (bounded)</u>

$$\exists \eta^* > 0 \text{ s.t. } |a(u,v)| \le \eta^*\|u\|_H\|v\|_H\,, \forall u, v \in H$$

<u>Coercive</u>

$$\exists \eta_* > 0 \text{ s.t. } a(u,u) \ge \eta_*\|u\|_H^2\,, \forall u \in H$$

<u>Energy Norm</u> If $a(u,v)$ is bilinear, continuous, coercive, and symmetric it defines a norm

$$\|u\|_E = \sqrt{a(u,u)}$$
$$\sqrt{\eta_*}\|u\|_H \le \|u\|_E \le \sqrt{\eta^*}\|u\|_H$$

<u>Lax-Milgram</u> If H is Hilbert, $a : H \times H \to \mathbb{R}$ is bilinear, bounded, coercive and $F : H \to \mathbb{R}$ is linear, bounded, then $\exists! w \in H$ s.t.

$$a(w,v) = F(v)\,, \quad \forall v \in H$$

<u>Poincarè Inequality</u> If $\Omega \subset \mathbb{R}^d$ bounded then $\exists \quad C(\Omega)\,, C'(\Omega) > 0$ s.t.

$$\|v\|_{L_2} \le C(\Omega)\|\nabla v\|_{L_2}\,, \quad \forall v \in H_0^1(\Omega)$$
$$\|v\|_{H^1} \le C'(\Omega)\|\nabla v\|_{L_2}\,, \quad \forall v \in H_0^1(\Omega)$$

<u>BVP Solvability</u> Given $\Omega \subset \mathbb{R}^d$ bounded with Lipshitz boundary, $\Gamma_D = \Gamma$, $\tilde{g}_D \in H^1(\Omega)$ and $f \in H^{-1}(\Omega)$. Consider the problem of finding $u = \tilde{u} + \tilde{h}_d$, $\tilde{u} \in H_0^1(\Omega)$ s.t.

$$\left(\underline{\underline{a}}\nabla u\right)\nabla v + (bu)\nabla v + (cu)\,v = \langle f\,, v\rangle\,, \quad \forall v \in H_0^1(\Omega)$$

or

$$\underbrace{\left(\underline{\underline{a}}\nabla \tilde{u}\right)\nabla v + (b\tilde{u})\nabla v + (c\tilde{u})\,v}_{a(\tilde{u},v)} = \underbrace{\langle f\,, v\rangle - a(\tilde{g}_D, v)}_{F(v)}$$

1. If $\underline{\underline{a}} \in L_\infty(\Omega)^{d\times d}$, $b \in L_\infty(\Omega)^d$ and $c \in L_\infty(\Omega)$ then $a(\tilde{u},v)$ and $F(v)$ are bounded.
2. Additionally, if $\underline{\underline{a}}$ is symmetric and uniformly positive definite, $c \ge 0$ and $\|b\|_\infty < \frac{a_*}{C(\Omega)}$ then $a(\tilde{u},v)$ is coercive.
3. Additionally, if $b \equiv 0$, then $a(\tilde{u},v)$ is symmetric and there is an energy $J(\cdot)$, inner product $a(\cdot,\cdot)$ and norm $\|\cdot\|_E$ associated with the problem

By Lax-Milgram: 1. + 2. $\Rightarrow \exists!$ solution $u = \tilde{u} + \tilde{g}_D$

1. + 2. + 3. $\Rightarrow \quad \tilde{u}$ is a strict minimum of J

<u>Proof of 1.</u>

$$\begin{aligned}
|a(\tilde{u},v)| &\le \left|\left(\underline{\underline{a}}\nabla\tilde{u},\nabla v\right)\right| + |(b\tilde{u},\nabla v)| + |(c\tilde{u},v)| \\
&\le \int_\Omega \left|\underline{\underline{a}}\nabla\tilde{u}\cdot v\right|\mathrm{d}\Omega + \int_\Omega |b\tilde{u}\cdot\nabla v|\,\mathrm{d}\Omega + \int_\Omega |c\tilde{u}v|\,\mathrm{d}\Omega \\
&\le \int_\Omega \left|\underline{\underline{a}}\nabla\tilde{u}\right||\nabla v|\,\mathrm{d}\Omega + \int_\Omega |b\tilde{u}||\nabla v|\,\mathrm{d}\Omega + \int_\Omega |c\tilde{u}||v|\,\mathrm{d}\Omega \\
&\le ||\underline{\underline{a}}||_\infty(|\nabla\tilde{u}|)|\nabla v| + ||b||_\infty(|\tilde{u}|)|\nabla v| + ||c||_\infty(|\tilde{u}|)|v| \\
&\le ||\underline{\underline{a}}||_\infty\|\nabla\tilde{u}\|_{L_2}\|\nabla v\|_{L_2} + ||b||_\infty\|\tilde{u}\|\|\nabla v\| + ||c||_\infty\|\tilde{u}\|\|v\| \\
&\le \underbrace{\left(||\underline{\underline{a}}||_\infty + ||b||_\infty + ||c||_\infty\right)}_{\eta^*>0}\|\tilde{u}\|_{H^1}\|v\|_{H^1}
\end{aligned}$$

$$\begin{aligned}
|F(v)| &\le |\langle f, v\rangle| + |a(\tilde{g}, v)| \\
&\le C\|v\|_{H^1} + \eta^*\|\tilde{g}_D\|_{H^1}\|v\|_{H^1}\,, \qquad f \in H^{-1} \\
&\le \hat{C}\|v\|_{H^1}
\end{aligned}$$

<u>Proof of 2.</u>

$$\begin{aligned}
a(v,v) &\ge (\underline{\underline{a}}\nabla v,\nabla v) + (bv,\nabla v) + (cv,v) \\
&\ge \underline{\underline{a}}_*\|\nabla v\|_{L_2}^2 - |(bv,\nabla v)| + (cv,v)\,, \text{ Cauchy-Schwarz} \\
&\ge \underline{\underline{a}}_*\|\nabla v\|_{L_2}^2 - \|b\|_\infty\|v\|_{L_2}\|\nabla v\|_{L_2}\,, \text{ Poincare} \\
&\ge \underline{\underline{a}}_*\|\nabla v\|_{L_2}^2 - \|b\|_\infty C(\Omega)\|\nabla v\|_{L_2} \\
&\ge \left(\frac{\underline{\underline{a}}_*}{C(\Omega} - \|b\|_\infty\right)C(\Omega)\|\nabla v\|_{L_2} \\
&\ge \left(\frac{\underline{\underline{a}}_*}{C(\Omega} - \|b\|_\infty\right)\frac{C(\Omega)}{(C'(\Omega))^2}\|v\|_{H^1} \\
&\ge \eta_*\|v\|_{H^1}^2\,, \quad \forall v \in H_0^1(\Omega)
\end{aligned}$$

<u>Stability</u> Choose $v = u_h - g \in V_h$

$$a(u_h, u_h - g) = f(u_h - g)$$
$$a(u_h - g, u_h - g) = f(u_h - g) - a(g, u_h - g)$$
$$a_*\|u_h - g\|_V^2 \le \|f\|_V\|u_h - g\|_V + a^*\|g\|_V\|u_h - g\|_V$$
$$\|u_h - g\|_V \le \frac{\|f\|_V + a^*\|g\|_V}{a_*}$$

<u>Error Estimation</u>

$$a(u - u_h, v) = 0\,, \quad \forall v \in V_h$$

Take $v \mapsto u - u_h - (u - v)\,, \quad v \in V_h$

$$a(u - u_h, u - u_h) = a(u - u_h, u - v)\,, \quad \forall v \in V_h$$
$$a_* \left\| u - u_h \right\|_V^2 \le a^* \left\| u - u_h \right\|_V \left\| u - v \right\|_V \,, \quad \forall v \in V_h$$
$$\left\| u - u_h \right\|_V \le \frac{a^*}{a_*} \left\| u - v \right\|_V \,, \quad \forall v \in V_h$$

<u>Cea's Lemma</u>

$$\left\| u - u_h \right\|_V \le \left(\frac{a^*}{a_*} \right) \inf_{v \in V_h} \left\| u - v \right\|_V$$

<u>Galerkin Orthogonality</u>

$$a(u - u_h, v) = 0\,, \quad \forall v \in V_h$$

<u>Theorem</u>

$$\left\| u - u_h \right\|_a = \inf_{v \in V_h} \left\| u - v \right\|_a$$

<u>Proof</u>

$$\begin{aligned} \left\| u - u_h \right\|_a^2 &= a(u - u_h, u - uh) \\ &= a(u - u_h, u - v) + \cancel{a(u - u_h, v - u_h)} \\ &\le \left\| u - u_h \right\|_a \left\| u - v \right\|_a \end{aligned}$$

<u>Ciarlet FEM Definition</u> Let

1. $E \subseteq \mathbb{R}^d$ domain with piecewise smooth boundary

2. $\mathcal{P}$ is a finite dimensional vector space of functions on E (shape functions)

3. $\mathcal{N} = \{N_1, \ldots, N_k\}$ is a basis for $\mathcal{P}'$, a set of linear functionals on $\mathcal{P}$ (nodal variables or DOFs)

Then $(E, \mathcal{P}, \mathcal{N})$ is a finite element

<u>Definition</u> Let $\{\phi_1, \ldots, \phi_k\}$ be a basis for $\mathcal{P}$ dual to $\mathcal{N}$, $(N_i(\phi_j) = \delta_{ij})$

<u>Unisolvence</u> If $\dim \mathcal{P} = k$ and $\{N_1, \ldots, N_k\} \subseteq \mathcal{P}'$

$\{N_1, \ldots, N_k\}$ is a basis iff

$N_i(v) = 0\,, \quad \forall i\,,$ then $v = 0$

<u>Peano Kernel Theorem</u> If L is a continuous linear functional on $C^{k+1}(a, b)$ s.t. $L(p) = 0\,, \forall p \in \mathbb{P}$. Then

$$L(f) = \int_a^b f^{(k+1)}(\xi) K(\xi) \mathrm{d}\xi$$
$$K(\xi) = \frac{1}{k!} L\left((\cdot - \xi)_+^k \right)$$

Stability:

$$\left(\frac{u^n - u^{n-1}}{\Delta t} \,, u^n \right) + \left(a \nabla u^n \,, \nabla u^n \right) = \left(f \,, u^n \right)$$
$$\begin{aligned} \left(u^n, u^n \right) - \left(u^{n-1}, u^{n-1} \right) &\le 2\Delta t \left\| f^n \right\|_0 \left\| u^n \right\|_0 \\ &\le \frac{\Delta t}{\epsilon} \left\| f^n \right\|_0^2 + \epsilon \left\| u^n \right\|_0^2 \Delta t \end{aligned}$$

$$\begin{aligned} \left\| u^M \right\|_0^2 - \left\| u^0 \right\|_0^2 &\le \sum_{n=1}^M \frac{\Delta t}{\epsilon} \left\| f^n \right\|_0^2 + \sum_{n=1}^M \epsilon \left\| u^n \right\|_0^2 \Delta t \\ &\le \sum_{n=1}^M \frac{\Delta t}{\epsilon} \left\| f^n \right\|_0^2 + M \Delta t \epsilon \max_n \left\| u^n \right\|_0^2 \end{aligned}$$
$$\max_n \left\| u^n \right\|_0^2 \le \frac{T}{\epsilon} \max_n \left\| f^n \right\|_0^2 + \left\| u^0 \right\|_0^2$$

Convergence: Find $u^n \in V + g^n$ s.t.

$$\left(c \frac{u^n - u^{n-1}}{\Delta t} \,, v \right) + \left(a \nabla u^n \,, \nabla v \right) = \left(f^n \,, v \right)$$
$$+ \left(c \left(\frac{u^n - u^{n-1}}{\Delta t} - u_t^n \right) \,, v \right) \,, \quad \forall v \in V$$

Find $u_h^n \in V_h + g^n$ s.t.

$$\left(c \frac{u_h^n - u_h^{n-1}}{\Delta t} \,, v \right) + \left(a \nabla u_h^n \,, \nabla v \right) = \left(f^n \,, v \right) \,, \quad \forall v \in V_h$$
$$\left(c \frac{\zeta^n - \zeta^{n-1}}{\Delta t} \,, v \right) + \left(a \nabla \zeta^n \,, \nabla v \right) = \left(c \left(\frac{u^n - u^{n-1}}{\Delta t} - u_t^n \right) \,, v \right)$$

$(a \nabla (\tilde{u} - u) \,, \nabla v) = 0\,, \quad \forall v \in V_h$

$$\begin{aligned} &\left(c \tilde{\zeta}^n \,, \tilde{\zeta}^n \right) - \left(c \tilde{\zeta}^{n-1} \,, \tilde{\zeta}^{n-1} \right) + 2\Delta t \left(a \nabla \tilde{\zeta}^n \,, \nabla \tilde{\zeta}^n \right) \\ &\le 2\Delta t \left(c \left(\frac{u^n - u^{n-1}}{\Delta t} - u_t^n \right) \,, \tilde{\zeta}^n \right) + 2\Delta t \left(c \left(\tilde{u}_t^n - u_t^n \right) \,, \tilde{\zeta}^n \right) \\ &\le c^* \Delta t \left\| \frac{\tilde{u}^n - \tilde{u}^{n-1}}{\Delta t} - \tilde{u}_t^n \right\|_0^2 + c^* \Delta t \left\| \tilde{u}_t^n - u_t^n \right\|_0^2 + 2 \frac{c^*}{c_*} \Delta t \left(c \tilde{\zeta}^n \,, \tilde{\zeta}^n \right) \\ &\le \underbrace{\sum_{k=1}^n c^* \Delta t \left\{ \left\| \frac{\tilde{u}^k - \tilde{u}^{k-1}}{\Delta t} - \tilde{u}_t^k \right\|_0^2 + \left\| \tilde{u}_t^k - u_t^k \right\|_0^2 \right\}}_{a^n} \\ &+ \underbrace{\sum_{k=1}^{n-1} c^* \Delta t \left\{ \left\| \frac{\tilde{u}^k - \tilde{u}^{k-1}}{\Delta t} - \tilde{u}_t^k \right\|_0^2 + \left\| \tilde{u}_t^k - u_t^k \right\|_0^2 \right\}}_{a^{n-1}} + \underbrace{2 \frac{c^*}{c_*}}_{b^n} \Delta t \underbrace{\left(c \tilde{\zeta}^n \,, \tilde{\zeta}^n \right)}_{u^n} \end{aligned}$$

$$\begin{aligned} \left\| \frac{\tilde{u}^n - \tilde{u}^{n-1}}{\Delta t} - \tilde{u}_t^n \right\|^2 &= \frac{1}{\Delta t} \left\| \int_{t_{n-1}}^{t_n} \left(\tilde{u}_t(\tau) - \tilde{u}_t^n \right) \mathrm{d}\tau \right\|^2 \\ &= \frac{1}{\Delta t} \left\| \int_{t_{n-1}}^{t_n} - \int_\tau^{t_n} \tilde{u}_{tt}(s) \mathrm{d}s \mathrm{d}\tau \right\|^2 \\ &\le \int_{t_{n-1}}^{t_n} \left\| \tilde{u}_{tt}(\tau) \right\|^2 \mathrm{d}\tau (\Delta t)^2 \end{aligned}$$

$$\begin{aligned} u^N &\le a^N \left(1 + \sum_{n=1}^N b^n C_1 \Delta t \right) \le C_2 a^N \\ &\le C_2 \left(c^* \left\| \tilde{u}_{tt} \right\|_{L^2(L^2)}^2 \Delta t^2 + C_3 h^4 \right) \\ &\le C \cdot \mathcal{O}(\Delta t^2 + h^4) \end{aligned}$$

Gronwall: If

$$u'(t) \le a(t) + b(t) u(t)\,, \quad \forall t \ge 0$$

where $b(t) \ge 0$, then

$$u(t) \le e^{\int_0^t b(\tau) \mathrm{d}\tau} \left\{ u(0) + \int_0^t a(\tau) e^{\int_0^\tau b(s) \mathrm{d}s} \mathrm{d}\tau \right\}$$

Discrete Gronwall: If

$$\frac{u^n - u^{n-1}}{\Delta t} \le \frac{a^n - a^{n-1}}{\Delta t} + b^n u^n\,, \quad \forall n \ge 1\,, \quad u^0 = a^0 = 0$$

where $a^n\,, b^n \ge 0\,, \quad \forall n$, then, for $\Delta t \ge 0$ sufficiently small,

$$u^N \le a^N + \sum_{n=1}^N a^n b^n \frac{\prod_{j=1}^{n-1} (1 - b^j \Delta t)}{\prod_{j=1}^N (1 - b^j \Delta t)} \Delta t$$

Fourier Transform

$$\hat{w}(\theta) = \sum_{j=-\infty}^{\infty} w_j e^{-ij\theta}$$
$$w_j = \frac{1}{2\pi} \int_0^{2\pi} \hat{w}(\theta) e^{ij\theta} \mathrm{d}\theta$$

Particle Velocity: $f(u)/u$

Characteristic Velocity: $f'(u)$

Characteristics: $x(t)$ where u is constant $x'(t) = f'(u(x(t), t))$

$x_\xi(t) = u_0(\xi) t + \xi$

Find CFL such that Lax-Wendroff is stable

$$\mathcal{U}_j^{n+1} = \mathcal{U}_j^n - \frac{a \Delta t}{2h} \left(\mathcal{U}_{j+1}^n - \mathcal{U}_{j-1}^n \right) + \frac{a^2 (\Delta t)^2}{2h^2} \left(\mathcal{U}_{j+1}^n - 2\mathcal{U}_j^n + \mathcal{U}_{j-1}^n \right)$$
$$\hat{\mathcal{U}}^{n+1} = \hat{\mathcal{U}}^n \left[1 - \lambda i \sin\theta + \lambda^2 (\cos\theta - 1) \right]$$
$$\left| 1 - \lambda i \sin\theta + \lambda^2 (\cos\theta - 1) \right|^2 = (1 + \lambda^2 (\cos\theta - 1))^2 + \lambda^2 \sin^2\theta \le 1$$

Rankine-Hugoniot Jump Condition $M >> st$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-M}^M u(x, t) \mathrm{d}x = (M + st) u_L + (M - st) u_R = f(u_L) - f(u_R)$$
$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R}$$

Rarefaction

$$\frac{f(u) - f(u_L)}{u - u_L} \ge s \ge \frac{f(u) - f(u_R)}{u - u_R}$$

Average flux entering $(x_{i+1/2}, x_{i+3/2})$: $\frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} f(u_{i+1/2}) \mathrm{d}x$

Conservative Method

$$\frac{\mathcal{U}_i^{n+1} - \mathcal{U}_i^n}{\Delta t} + \frac{1}{h} \left\{ F(\mathcal{U}_{i-p}^n, \mathcal{U}_{i-p+1}^n, \ldots, \mathcal{U}_{i+q}) - F(\mathcal{U}_{i-p-1}^n, \mathcal{U}_{i-p}^n, \ldots, \mathcal{U}_{i+q-1}) \right\} = 0$$