Consistency + Stability = Convergence

$$y'' = \frac{\partial f}{\partial t} + \nabla_y f \cdot f$$

$$(a+ib)(c+id) = (ac-bd) + i(ad+bc)$$

Euler

Euler's method is convergent if $f \in C^0$ and Lipshitz in y

$$\|e_{n+1}\| \le (1+h\lambda)\|e_n\| + Ch^2$$

$$\|e_n\| \le \frac{C}{\lambda} h\left[(1+h\lambda)^n - 1\right] \le \frac{C}{\lambda}(e^{T\lambda}-1)h$$

Trapezoid

$$\|e_{n+1}\| \le \left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)\|e_n\| + \left(\frac{C}{1-\frac{1}{2}h\lambda}\right)h^3$$

$$\|e_n\| \le \frac{C}{\lambda}\left[\left(\frac{1+\frac{1}{2}h\lambda}{1-\frac{1}{2}h\lambda}\right)^n - 1\right]h^2$$

Lagrange

$$p_m(t) = \prod_{l=1, l\neq m}^{s-1} \frac{t - t_{n+l}}{t_{n+m} - t_{n+l}}$$

s-step Adam's-Bashforth

$$y_{n+s} = y_{n+s-1} + h\sum_{m=0}^{s-1} b_m f(t_{n+m}, y_{n+m})$$

$$b_m = h^{-1}\int_0^h p_m(t_{n+s-1}+\tau)\mathrm{d}\tau\,, \quad m = 0,1,\ldots,s-1$$

General s-step

$$\sum_{m=0}^{s} a_m y_{n+m} = h\sum_{m=0}^{s} b_m f(t_{n+m}, y_{n+m})$$

$$\rho(w) = \sum_{m=0}^{s} a_m w^m$$

$$\sigma(w) = \sum_{m=0}^{s} b_m w^m$$

Order p iff

$$\rho(\xi+1) - \sigma(\xi+1)\ln(\xi+1) = c(\xi)^{p+1} + \mathcal{O}(\xi^{p+2})$$

Root Condition: all zeros reside in closed complex unit disc and all zeros of unit modulus are simple.

$$\|f(x) - p(x)\|_{L^\infty} \le \frac{(b-a)^s}{s!}\left\|f^{(s)}\right\|_{L^\infty}$$

BDF

$$\sigma(w) = \beta w^s$$

$$\beta = \left(\sum_{m=1}^{s}\frac{1}{m}\right)^{-1}$$

$$\rho(w) = \beta\sum_{m=1}^{s}\frac{1}{m}w^{s-m}(w-1)^m$$

Runge-Kutta

$$\xi_j = y_n + h\sum_{i=1}^{\nu} a_{j,i} f(t_n + c_i h, \xi_i)$$

$$y_{n+1} = y_n + h\sum_{j=1}^{\nu} b_j f(t_n + c_j h, \xi_j)$$

$$\begin{array}{c|c} \mathbf{c} & A \\ \hline & \mathbf{b}^T \end{array}$$

$$LTE = \frac{y(t_{n+1}) - y(t_n)}{h} - \Phi(t_n, y(t_n), h)$$

Collocation

$$\int_0^1 \tau^j \prod_{l=1}^{\nu}(\tau - c_l) = 0\,, \quad j = 0,1,\ldots,m-1$$

for some $m \in \{0,1,\ldots,\nu\}$. Then the collocation method is of order $\nu + m$.

$$q(t) = \prod_{j=1}^{\nu}(t - c_j)\,, \quad q_l(t) = \frac{q(t)}{t - c_l}$$

$$a_{j,i} = \int_0^{c_j}\frac{q_i(\tau)}{q_i(c_i)}\mathrm{d}\tau\,, \quad b_j = \int_0^1 \frac{q_j(\tau)}{q_j(c_j)}\mathrm{d}\tau$$

Let $c_1, c_2, \ldots, c_\nu$ be the zeros of the polynomials that are orthogonal with respect to the weight function $\omega(t) \equiv 1\,, 0 \le t \le 1$. Then the underlying collocation method is of order 2ν.

A numerical method is stable if small change in the initial conditions or data, produce a correspondingly small change in the subsequent approximations.

Stability of Runge-Kutta Methods

$$r(z) = 1 + z\mathbf{b}^T(I - zA)^{-1}\mathbf{1}$$

$$(I - zA)^{-1} = \frac{\mathrm{adj}(I - zA)}{\det(I - zA)}$$

Adjugate is transpose of cofactor matrix

$|r(z)| < 1$ for all $z \in \mathbb{C}^-$ iff all the poles of r have positive real parts and $|r(it)| \le 1$ for all $t \in \mathbb{R}$.

Multistep methods

$$\eta(z, w) = \sum_{m=0}^{s}(a_m - b_m z)w^m$$

The multistep method is A-stable iff $b_s > 0$ and

$$|w_1(it)|\,, |w_2(it)|\,, \ldots, \left|w_{q(it)}(it)\right| \le 1\,, \quad t \in \mathbb{R}$$

where $w_1, w_2, \ldots, w_{q(z)}$ are the zeros of $\eta(z, \cdot)$.

Linearizing a non-linear equation

$$y' = \underbrace{f(t,y)}_{b} + \underbrace{\nabla f(t,y)}_{A}(y - \bar{y}) + \mathcal{O}(|y - \bar{y}|^2)$$

Solution of nonlinear equations

Fixed Point

$$\mathbf{w} = h\mathbf{g}(\mathbf{w}) + \boldsymbol{\beta}$$

Unique solution exists in sufficiently small neighborhood of β if $\left(\mathbf{I} - h\frac{\partial \mathbf{g}}{\partial \mathbf{w}}\right)$ is nonsingular.

Banach Fixed Point Theorem If $G : \mathbb{R}^d \to \mathbb{R}^d$ is a contraction: $\exists 0 \le \lambda < 1$ s.t.

$$\|G(u) - G(v)\| \le \lambda\|u - v\| \quad \forall u, v \in \mathbb{R}^d$$

The $\exists!$ fixed point w and

$$\left\|w^i - w\right\| \le \lambda^i\left\|w^0 - w\right\|$$

If g is Lipshitz with constant Λ, we need $h \le \frac{\lambda}{\Lambda}$

Newton's Method

$$w^{i+1} = w^i - \frac{F(w^i)}{F'(w^i)}$$

$$w^{i+1} = w^i - \left(\frac{\partial F(w)}{\partial w}\right)^{-1} F(w^i)$$

Psuedo-Code

$t_n \leftarrow t_0$
$\mathbf{y}_n \leftarrow \mathbf{y}_0$
for $n = 1$ to Nsteps **do**
  $\mathbf{w}^i \leftarrow \mathbf{y}_n + h\mathbf{g}(t_n, \mathbf{y}_n)$
  for $i = 1$ to maxIter **do**
    $\mathbf{w}^{i+1} = \mathbf{w}^i - \left(\mathbf{I} - h\frac{\partial \mathbf{g}}{\partial \mathbf{y}}(t_{n+1}, \mathbf{w}^i)\right)^{-1}\left(\mathbf{w}^i - h\mathbf{g}(t_{n+1}, \mathbf{w}^i) - \mathbf{y}_n\right)$
    if $\frac{\|\mathbf{w}^{i+1} - \mathbf{w}^i\|}{\|\mathbf{y}_n\|} \le$ tol **then**
      break
    end if
    $\mathbf{w}^i \leftarrow \mathbf{w}^{i+1}$
  end for
  $\mathbf{y}_{n+1} \leftarrow \mathbf{w}^i$
end for

Cohn-Schur criterion Both zeros of the quadratic $\alpha w^2 + \beta w + \gamma$, where $\alpha, \beta, \gamma \in \mathbb{C}$, $\alpha \neq 0$, reside in the closed complex unit disc iff

$$|\alpha| \ge |\gamma|\,, \quad \left||\alpha|^2 - |\gamma|^2\right| \ge \left|\alpha\bar{\beta} - \beta\bar{\gamma}\right| \text{ and } \alpha = \gamma \neq 0 \implies |\beta| \le 2|\alpha|$$

The two stage BDF

$$\mathbf{y}_{n+2} - \frac{4}{3}\mathbf{y}_{n+1} + \frac{1}{3}\mathbf{y}_n = \frac{2}{3}h\mathbf{f}(t_{n+2}, \mathbf{y}_{n+2})$$

is A-stable.

$$\eta(z, w) = \left(1 - \frac{2}{3}z\right)w^2 - \frac{4}{3}w + \frac{1}{3}$$

The first condition is satisfied by $b_2 = \frac{2}{3} > 0$.

$$\eta(it, w) = \left(1 - \frac{2}{3}it\right)w^2 - \frac{4}{3}w + \frac{1}{3}$$

$$|\alpha|^2 - |\gamma|^2 = 1 + \frac{4}{9}t^2 - \frac{1}{9} = \frac{4}{9}(2 + t^2) > 0$$

$$\left(|\alpha|^2 - |\gamma|^2\right)^2 - \left|\alpha\bar{\beta} - \beta\bar{\gamma}\right|^2 = \frac{16}{81}t^4 \ge 0$$