Hw chapter 7

7.3 a)

$u = e^{x-2\sin y}$

$u_x = e^{x-2\sin y}$, $u_{xy} = e^{x-2\sin y}(-2)\cos y$

$u_y = e^{x-2\sin y}(-2)\cos y$ $u_{yx} = e^{x-2\sin y}(-2)\cos y$

$u = 4x\sin x^3y$

$u_x = 4\sin x^3y + 4x\cos x^3y \; 3x^2y$

$= 4\sin x^3y + 12x^3y\cos x^3y$

$u_y = 4x\cos x^3y \cdot x^3 = 4x^4\cos x^3y$

$u_{yx} = 16x^3\cos x^3y + 4x^4(-\sin x^3y)\; 3x^2y$

$= 16x^3\cos x^3y - 12x^6y\sin x^3y$

$u_{xy} = 4\cos x^3y\; x^3 + 12x^3\cos x^3y + 12x^3y(-\sin x^3y)\,x^3$

$= 16x^3\cos x^3y - 12x^6y\sin x^3y$

b)
$$u(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^2} & x,y \neq 0 \\ 0 & x=y=0 \end{cases}$$

$x,y \neq 0$
$$u_x(x,y) = \frac{y^3(x^2+y^2) - xy^3 2x}{(x^2+y^2)^2} = \frac{x^2y^3 + y^5 - 2x^2y^3}{(x^2+y^2)^2}$$
$$= \frac{-x^2y^3 + y^5}{(x^2+y^2)^2}$$

$$u_x(0,0) = \lim_{t\to 0}\frac{1}{t}\left(\frac{t\cdot 0}{t^2+0} - 0\right) = 0$$

So.
$$u_{xy}(0,0) = \lim_{t\to 0}\frac{1}{t}\left(u_x(0,t) - u_x(0,0)\right)$$
$$= \lim_{t\to 0}\frac{1}{t}\left(\frac{-t^5}{t^4} - 0\right) = -1$$

$(x,y) \neq (0,0) \qquad u_y(x,y) = \dfrac{3xy^2(x^2+y^2) - xy^3 2y}{(x^2+y^2)^2}$

$$= \frac{3x^3y^2 + 3xy^4 - 2xy^4}{(x^2+y^2)^2}$$

$$= \frac{3x^3y^2 + xy^4}{(x^2+y^2)^2}$$

$(x,y) = (0,0) \qquad u_y(0,0) = \lim\limits_{t \to 0} \frac{1}{t}\left(u(0,t) - u(0,0)\right)$

$$= \lim_{t \to 0} \frac{1}{t}(0 - 0) = 0$$

$$u_{yx}(0,0) = \lim_{t \to 0} \frac{1}{t}\left(u_y(t,0) - u_y(0,0)\right)$$

$$= \lim_{t \to 0} \frac{1}{t}(0 - 0) = 0$$

The point: In this case $u_{xy}(0,0) \neq u_{yx}(0,0)$!

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7.5 $u(x,t) = f(x+at) + g(x-at)$ $\qquad f = f(y)$, $g = g(y)$

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial f}{\partial y}(x+at) + \frac{\partial g}{\partial y}(x-at)$$

$$\frac{\partial^2 u}{\partial x^2}(x,t) = \frac{\partial^2 f}{\partial y^2}(x+at) + \frac{\partial^2 g}{\partial y^2}(x-at)$$

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x+at)\cdot a + \frac{\partial g}{\partial y}(x-at)(-a)$$

$$\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{\partial^2 f}{\partial y^2}(x+at)\cdot a^2 + \frac{\partial^2 g}{\partial y^2}(x-at)\, a^2$$

compare

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7.7

a)

$$f(\lambda x, \lambda y) = (\lambda x)^2 + 3(\lambda x)(\lambda y) = \lambda^2[x^2 + 3xy]$$
$$= \lambda^2 f(x,y)$$

<u>yes, $k = 2$</u>

$$f(\lambda x, \lambda y) = \ln\left((\lambda x)^2 + (\lambda y)^2\right) = \ln\left(\lambda^2(x^2+y^2)\right)$$
$$= \ln \lambda^2 + \ln(x^2+y^2) \neq \lambda^k \ln(x^2+y^2) \quad \forall k$$

<u>no</u>

$$f(\lambda x, \lambda y) = \left((\lambda x)^2 - (\lambda x)(\lambda y)\right) / (2\lambda x + \lambda y)$$
$$= \lambda \frac{x^2 - xy}{2x+y} = \lambda f(x,y)$$

<u>yes, $k = 1$</u>

$$f(\lambda x, \lambda y) = (\lambda x)^2 e^{\frac{\lambda x}{2\lambda y}} = \lambda^2 x^2 e^{\frac{x}{\lambda y}} = \lambda^2 f(x,y)$$

<u>yes, $k = 2$</u>

b)

$$f(\lambda x, \lambda y, \lambda z) = \lambda^k f(x,y,z) \quad / \frac{\partial}{\partial x}$$

$$\lambda \frac{\partial f}{\partial x}(\lambda x, \lambda y, \lambda z) = \lambda^k \frac{\partial f}{\partial x}(x,y,z) \quad /:\lambda$$

$$\frac{\partial f}{\partial x}(\lambda x, \lambda y, \lambda z) = \lambda^{k-1} \frac{\partial f}{\partial x}(x,y,z)$$

c)

$$f(\lambda x_1, \lambda x_2, \dots \lambda x_n) = \lambda^k f(x_1, x_2, \dots, x_n) \quad / \frac{\partial}{\partial \lambda} \qquad k \geq 1$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\lambda x_1, \dots, \lambda x_n) \cdot x_i = k \lambda^{k-1} f(x_1, x_2, \dots, x_n)$$

Set $\lambda = 1$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_1, \dots, x_n) \cdot x_i = k f(x_1, \dots, x_n)$$

d) $f(x,y) = \sqrt{x^4+y^4}\ \sin^{-1}\frac{y}{2x}$

$$f(\lambda x, \lambda y) = \sqrt{(\lambda x)^4+(\lambda y)^4}\ \sin^{-1}\frac{\lambda y}{2\lambda x}$$

$$= \lambda^2 f(x,y) \qquad k=2$$

$$\frac{\partial f}{\partial x} = \frac{1}{2\sqrt{x^4+y^4}}\,4x^3 \sin^{-1}\frac{y}{2x} + \sqrt{x^4+y^4}\,\frac{1}{\sqrt{1-\left(\frac{y}{2x}\right)^2}}\left(-\frac{y}{2x^2}\right)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{x^4+y^4}}\,4y^3 \sin^{-1}\frac{y}{2x} + \sqrt{x^4+y^4}\,\frac{1}{\sqrt{1-\left(\frac{y}{2x}\right)^2}}\,\frac{1}{2x}$$

$$\frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y = \frac{1}{\cancel{2}\sqrt{x^4+y^4}}\,\overset{2}{\cancel{4}}(x^4+y^4)\sin^{-1}\frac{y}{2x}$$

$$+ \sqrt{x^4+y^4}\,\frac{2x}{\sqrt{4x^2-y^2}}\overset{0}{\cancel{\left(\frac{y}{2x}-\frac{y}{2x}\right)}}$$

$$= 2\sqrt{x^4+y^4}\ \sin^{-1}\frac{y}{2x} \quad = 2f(x,y)$$

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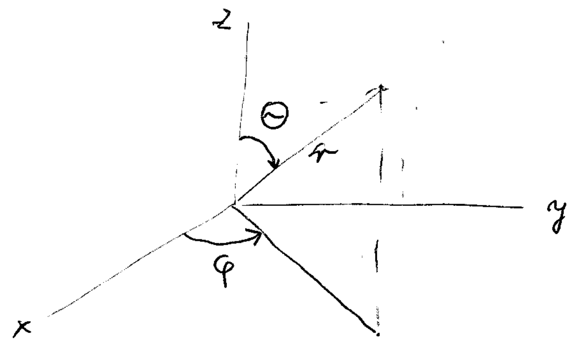
7.11

$$\begin{cases} x = r \sin\theta \cos\varphi \\ y = r \sin\theta \sin\varphi \\ z = r\cos\theta \end{cases}$$



$$f = f(x(r,\theta,\varphi), y(r,\theta,\varphi), z(r,\theta,\varphi))$$

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

$$= \underset{\sim}{\nabla} f \circ \frac{\partial \underset{\sim}{r}}{\partial r}$$

Similarly

$$\frac{\partial f}{\partial \theta} = \underset{\sim}{\nabla} f \circ \frac{\partial \underset{\sim}{r}}{\partial \theta} \quad , \quad \frac{\partial f}{\partial \varphi} = \underset{\sim}{\nabla} f \circ \frac{\partial \underset{\sim}{r}}{\partial \varphi}$$

where

$$\frac{\partial \underset{\sim}{r}}{\partial r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\frac{\partial \underset{\sim}{r}}{\partial \theta} = (r\cos\theta\cos\varphi, r\cos\theta\sin\varphi, -r\sin\theta)$$

$$\frac{\partial \underset{\sim}{r}}{\partial \varphi} = (-r\sin\theta\sin\varphi, r\sin\theta\cos\varphi, 0)$$

Note that the three vectors are orthogonal!

Representing $\underset{\sim}{\nabla} f$ in the form

$$\underset{\sim}{\nabla} f = A \frac{\partial \underset{\sim}{r}}{\partial r} + B \frac{\partial \underset{\sim}{r}}{\partial \theta} + C \frac{\partial \underset{\sim}{r}}{\partial \varphi} \qquad (*)$$

we get A, B, C by multiplying $(*)$ (in the scalar sense) by vectors $\frac{\partial \underset{\sim}{r}}{\partial r}, \frac{\partial \underset{\sim}{r}}{\partial \theta}, \frac{\partial \underset{\sim}{r}}{\partial \varphi}$ resp.

$$\frac{\partial f}{\partial r} = A \frac{\partial \underset{\sim}{r}}{\partial r} \cdot \frac{\partial \underset{\sim}{r}}{\partial r} = A \cdot 1 \quad \Rightarrow \quad A = \frac{\partial f}{\partial r}$$

$$\frac{\partial f}{\partial \theta} = B \frac{\partial \underset{\sim}{r}}{\partial \theta} \cdot \frac{\partial \underset{\sim}{r}}{\partial \theta} = B r^2 \quad \Rightarrow \quad B = \frac{1}{r^2} \frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial \varphi} = C \frac{\partial \underset{\sim}{r}}{\partial \varphi} \cdot \frac{\partial \underset{\sim}{r}}{\partial \varphi} = C r^2 \sin^2 \theta \quad \Rightarrow \quad C = \frac{1}{r^2 \sin^2 \theta} \frac{\partial f}{\partial \varphi}$$

or switching to the unit vectors, we can write

$$\underset{\sim}{\nabla} f = \frac{\partial f}{\partial r} \underset{\sim}{u}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \underset{\sim}{u}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \underset{\sim}{u}_\varphi$$

where $\underset{\sim}{u}_r = \frac{\partial \underset{\sim}{r}}{\partial r} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$

$\underset{\sim}{u}_\theta = (\cos\theta \cos\varphi, \cos\theta \sin\varphi, -\sin\theta)$

$\underset{\sim}{u}_\varphi = (-\sin\varphi, \cos\varphi, 0)$

Laplacian. (compare page 159)

$$\Delta f = \nabla^2 f = \underset{\sim}{\nabla} \circ (\underset{\sim}{\nabla} f)$$

$$= \left(\frac{\partial}{\partial r}\,\underset{\sim}{u}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\,\underset{\sim}{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\,\underset{\sim}{u}_\varphi\right)$$
$$\circ\left(\frac{\partial f}{\partial r}\,\underset{\sim}{u}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\underset{\sim}{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\,\underset{\sim}{u}_\varphi\right)$$

$$\frac{\partial}{\partial r}(\underset{\sim}{\nabla} f) = \frac{\partial^2 f}{\partial r^2}\,\underset{\sim}{u}_r + \frac{\partial f}{\partial r}\cancel{\frac{\partial u_r}{\partial r}}^{\,0}$$
$$- \frac{1}{r^2}\frac{\partial f}{\partial \theta}\,\underset{\sim}{u}_\theta + \frac{1}{r}\frac{\partial^2 f}{\partial r\partial \theta}\,\underset{\sim}{u}_\theta + \frac{1}{r}\frac{\partial f}{\partial \theta}\cancel{\frac{\partial u_\theta}{\partial r}}^{\,0}$$
$$- \frac{1}{r^2\sin^2\theta}\frac{\partial f}{\partial \varphi}\,\underset{\sim}{u}_\varphi + \frac{1}{r\sin\theta}\frac{\partial^2 f}{\partial r\partial \varphi}\,\underset{\sim}{u}_\varphi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\cancel{\frac{\partial u_\varphi}{\partial r}}^{\,0}$$

$$\frac{\partial}{\partial r}(\underset{\sim}{\nabla} f)\circ\underset{\sim}{u}_r = \frac{\partial^2 f}{\partial r^2}$$

$$\frac{\partial}{\partial \theta}(\underset{\sim}{\nabla} f) = \frac{\partial^2 f}{\partial \theta\partial r}\,\underset{\sim}{u}_r + \frac{\partial f}{\partial r}\cancel{\frac{\partial \underset{\sim}{u}_r}{\partial \theta}}^{\,\underset{\sim}{u}_\theta}$$
$$+ \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}\,\underset{\sim}{u}_\theta + \frac{1}{r}\frac{\partial f}{\partial \theta}\cancel{\frac{\partial u_\theta}{\partial \theta}}^{\,-\underset{\sim}{u}_r}$$
$$- \frac{1}{r\sin^2\theta}\cos\theta\,\frac{\partial f}{\partial \varphi}\,\underset{\sim}{u}_\varphi + \frac{1}{r\sin\theta}\frac{\partial^2 f}{\partial \theta\partial \varphi}\,\underset{\sim}{u}_\varphi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\cancel{\frac{\partial u_\varphi}{\partial \theta}}^{\,0}$$

$$\frac{\partial \underset{\sim}{u}_r}{\partial \theta} = (\cos\theta\cos\varphi,\ \cos\theta\sin\varphi,\ -\sin\theta) = \underset{\sim}{u}_\theta$$

$$\frac{\partial \underset{\sim}{u}_\varphi}{\partial \theta} = 0 \qquad \frac{\partial \underset{\sim}{u}_\theta}{\partial \theta} = -\underset{\sim}{u}_r$$

$$\frac{\partial}{\partial \theta}(\underset{\sim}{\nabla} f)\circ\underset{\sim}{u}_\theta = \frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2}$$

$$\frac{\partial}{\partial \varphi}(\underset{\sim}{\nabla} f) = \frac{\partial^2 f}{\partial \varphi\partial r}\,\underset{\sim}{u}_r + \frac{\partial f}{\partial r}\frac{\partial u_r}{\partial \varphi}$$
$$+ \frac{1}{r}\frac{\partial^2 f}{\partial \varphi\partial \theta}\,\underset{\sim}{u}_\theta + \frac{1}{r}\frac{\partial f}{\partial \theta}\frac{\partial u_\theta}{\partial \varphi}$$
$$+ \frac{1}{r\sin\theta}\frac{\partial^2 f}{\partial \varphi^2}\,\underset{\sim}{u}_\varphi + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\frac{\partial u_\varphi}{\partial \varphi}$$

$$\frac{\partial \underset{\sim}{u}_r}{\partial \varphi} = (-\sin\theta \sin\varphi,\ \sin\theta\cos\varphi,\ 0) = \sin\theta\, \underset{\sim}{u}_\varphi$$

$$\frac{\partial \underset{\sim}{u}_\theta}{\partial \varphi} = (-\cos\theta \sin\varphi,\ \cos\theta \sin\varphi,\ 0) = \cos\theta\, \underset{\sim}{u}_\varphi$$

$$\frac{\partial \underset{\sim}{u}_\varphi}{\partial \varphi} = (-\cos\varphi,\ -\sin\varphi,\ 0) = -\sin\theta\, \underset{\sim}{u}_r - \cos\theta\, \underset{\sim}{u}_\theta$$

$$\frac{\partial}{\partial\varphi}(\underset{\sim}{\nabla} f)\circ \underset{\sim}{u}_\varphi = \frac{\partial f}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial f}{\partial\theta}\cos\theta + \frac{1}{r\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Summing up:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial f}{\partial\theta}\cos\theta\right.$$

$$\left. + \frac{1}{r\sin\theta}\frac{\partial^2 f}{\partial\varphi^2}\right)$$

$$= \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} + \frac{1}{r^2}\operatorname{ctg}\theta\frac{\partial f}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Checking with the formula in the book:

$$\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial u}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial u}{\partial\varphi}\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[2r\cancel{\sin\theta}\frac{\partial u}{\partial r} + r^2\cancel{\sin\theta}\frac{\partial^2 u}{\partial r^2} + \cancel{\cos\theta}\frac{\partial u}{\partial\theta} + \cancel{\sin\theta}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{\sin\theta}\frac{\partial^2 u}{\partial\varphi^2}\right]$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\operatorname{ctg}\theta}{r^2}\frac{\partial u}{\partial\theta} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}$$

<u>Ok.</u>

7.14

$$f(x,y,z) = 0$$

- $y = y(x,z) \Rightarrow f(x, y(x,z), z) \ / \frac{\partial}{\partial x}$

$$f_x + f_y \frac{\partial y}{\partial x} = 0$$

$$\therefore \quad \frac{\partial y}{\partial x} = -\frac{f_x}{f_y}$$

- Similarly $\frac{\partial x}{\partial y} = -\frac{f_y}{f_x}$, so $\frac{\partial x}{\partial y}\frac{\partial y}{\partial x} = 1$

- $\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}\frac{\partial z}{\partial x} = \left(-\frac{f_y}{\cancel{f_x}}\right)\left(-\frac{f_z}{\cancel{f_y}}\right)\left(-\frac{\cancel{f_x}}{\cancel{f_z}}\right) = -1$

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7.16

a) $f(x,y) = \sin(x+y) \qquad f(0,0) = 0$

$\frac{\partial f}{\partial x}(x,y) = \cos(x+y) \qquad \frac{\partial f}{\partial x}(0,0) = 1$

$\frac{\partial f}{\partial y}(x,y) = \cos(x+y) \qquad \frac{\partial f}{\partial y}(0,0) = 1$

$\frac{\partial^2 f}{\partial x^2}(x,y) = -\sin(x+y) \qquad \frac{\partial^2 f}{\partial x^2}(0,0) = 0$

$\frac{\partial^2 f}{\partial y^2}(x,y) = -\sin(x+y) \qquad \frac{\partial^2 f}{\partial y^2}(0,0) = 0$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -\sin(x+y) \qquad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$

All third order derivatives $= -\cos(x+y)$, at $(0,0)$, $= -1$

So: $\sin(x+y) = x+y + \frac{1}{6}\left[x^3 + 3x^2y + 3xy^2 + y^3\right] + \dots$

$$= (x+y) + \frac{1}{6}(x+y)^3 + \dots$$

The same result can be obtained by expanding $\sin t$ (around 0) and then substituting $t = x+y$

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7.16 b)

$$\sin(x+y+z^2) \qquad (x_0, y_0, z_0) = (0, 0, 1)$$

$$a = x - x_0,\ b = y - y_0,\ c = z - z_0$$

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$

$$\varphi(t) = \sin\left(x_0 + at + y_0 + bt + (z_0 + ct)^2\right)$$

$$\varphi'(t) = \cos(\quad)[a + b + 2(z_0 + ct)c]$$

$$\varphi''(t) = -\sin(\quad)[a + b + 2(z_0 + ct)c]^2 + \cos(\quad)\cdot 2c^2$$

$$\varphi(0) = \sin(x_0 + y_0 + z_0^2) = \sin 1$$
$$\varphi'(0) = \cos 1\,[a + b + 2c]$$
$$\varphi''(0) = -\sin 1\,[a + b + 2c]^2 + \cos 1\ 2c^2$$

$$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{\varphi''(0)}{2} + \dots$$

$$\sin(x + y + z^2) = \sin 1 + \cos 1\,[x + y + 2(z-1)]$$

$$-\frac{1}{2}\sin 1\,[x + y + 2(z-1)]^2 + \frac{\cos 1}{2}\,2(z-1)^2 + \dots$$

7.20

$$(y-1)e^y = \sqrt{x} - 1 \quad /()' \qquad (1)$$

$$y'e^y + (y-1)e^y y' = \frac{1}{2\sqrt{x}}$$

$$y'e^y(\cancel{1}+y\cancel{-1}) = \frac{1}{2\sqrt{x}} \qquad (2)$$

$$y' = \frac{1}{2\sqrt{x}\, y\, e^y}$$

Differentiating (2)

$$y''e^y y + y'e^y y' y + y'e^y y' = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$y'' = \frac{-1}{e^y y}\left(\frac{1}{4}x^{-\frac{3}{2}} + (y')^2 e^y (y+1)\right) \qquad (3)$$

Select, say, $x = 1$

From (1) $\quad y = 1$

From (2) $\quad y' = \frac{1}{2 \cdot 1 e} = \frac{1}{2e}$

From (3)

$$y'' = \frac{-1}{e}\left(\frac{1}{4} + \frac{1}{4e^{\cancel{2}}}\cancel{e} \cdot 2\right)$$

$$= -\frac{1}{e}\left(\frac{1}{4} + \frac{1}{2e}\right)$$

$$= -\frac{1}{e}\left(\frac{e+2}{4e}\right) = -\frac{e+2}{4+e^2}$$

So:

$$y = 1 + \frac{1}{2e}(x-1) - \frac{e+2}{2(4+e^2)}(x-1)^2 + O\left((x-1)^3\right)$$

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7.23

$$\begin{cases} Au_x + Bu_y + Cv_x + Dv_y = 0 \\ Eu_x + Fu_y + Gv_x + Hv_y = 0 \end{cases}$$

$$A = A(u,v), \dots, H = H(u,v)$$

$$\begin{cases} u(x(u,v), y(u,v)) = u & (1) \\ v(x(u,v), y(u,v)) = v & (2) \end{cases}$$

Differentiating (1) wrt u and v we get

$$\begin{cases} u_x x_u + u_y y_u = 1 \\ u_x x_v + u_y y_v = 0 \end{cases}$$

$$\begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$u_x = \frac{y_v}{x_u y_v - y_u x_v} = y_v \cdot J \quad \text{, where } J^{-1} = (x_u y_v - y_u x_v)$$

$$u_y = - x_v J$$

Similarly, differentiating (2) wrt u and v we get

$$\begin{cases} v_x x_u + v_y y_u = 0 \\ v_x x_v + v_y y_v = 1 \end{cases} \quad \text{or} \quad \begin{pmatrix} x_u & y_u \\ x_v & y_v \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So: $\quad v_x = - y_u J \quad , \quad v_y = x_u J$

Now

$$u_x v_y - u_y v_x = J^2(y_v x_u - x_v y_u) = J \quad \text{, i.e.}$$

$$J = u_x v_y - u_y v_x$$

Substituting formulas for $u_x, \ldots, v_y$ into the original equations and subdividing by J we get the result required.

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