Chapter 10

10.2

a) $\min \frac{1}{2}(x^2+y^2+z^2)$

$z-xy=10$

$x+y+z=1$

$$L(x,y,z,\lambda,\mu) = \frac{1}{2}(x^2+y^2+z^2) - \lambda(z-xy-10) - \mu(x+y+z-1)$$

$\frac{\partial L}{\partial x} = 0 \qquad x+\lambda y-\mu = 0 \qquad (1)$

$\frac{\partial L}{\partial y} = 0 \qquad y+\lambda x-\mu = 0 \qquad (2)$

$\frac{\partial L}{\partial z} = 0 \qquad z-\lambda-\mu = 0 \qquad (3)$

$$(1),(2) \Rightarrow \quad x = \frac{\begin{vmatrix} \mu & \lambda \\ \mu & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \lambda \\ \lambda & 1 \end{vmatrix}} = \frac{\mu(1-\lambda)}{1-\lambda^2}, \quad y = \frac{\mu(1-\lambda)}{1-\lambda^2}$$

Case 1: $\lambda^2 \neq 1 \Rightarrow x=y$

Solving $z-x^2=10$, $2x+z=1$ for x,z we find out that there is no solution to (1) (2) (3) and the constraint eqs.

Case 2: $\lambda=-1 \Rightarrow$ (1),(2) reduce to $x-y-\mu=0$ and $y-x-\mu=0$
$\Rightarrow$ no solution exists

case 3: $\lambda=1$

(1) and (2) reduce to $x+y=\mu$ (3) $\Rightarrow z=\lambda+\mu=\mu+1$

$x+y+z=1 \Rightarrow \mu+\mu+1 = 1 \Rightarrow \mu=0 \Rightarrow z=1$

$z-xy=10 \Rightarrow xy=-9$

$$\begin{cases} xy = -9 \\ x+y = 0 \end{cases} \Rightarrow x = \pm 3$$

Soln 1: $x = 3, y = -3, z = 1, \lambda = -1, \mu = 0$

Soln 2: $x = -3, y = 3, z = 1, \lambda = -1, \mu = 0$

It follows from the geometrical interpretation that a solution must exist. Both solns above are at the same distance from the origin, so we have two global minima.

b) $\min\limits_{x^2 - xy + y^2 + 2z^2 = 4} (x^2 + y^2 + z^2)$

$L(x, y, z, \lambda) = x^2 + y^2 + z^2 - \lambda(x^2 - xy + y^2 + 2z^2 - 4)$

$\frac{\partial L}{\partial x} = 0 \quad 2x - 2\lambda x + \lambda y = 0$

$\frac{\partial L}{\partial y} = 0 \quad 2y + \lambda x - 2\lambda y = 0$

$\frac{\partial L}{\partial z} = 0 \quad 2z - 4\lambda z = 0$

$$\begin{cases} (2-2\lambda)x + \lambda y = 0 & (1) \\ \lambda x + (2-2\lambda)y = 0 & (2) \end{cases}$$

$z(1 - 2\lambda) = 0 \qquad (3)$

(3) $\Rightarrow$ $z = 0$ or $\lambda = \frac{1}{2}$

Case 1 $\lambda = \frac{1}{2}$

(1), (2) $\Rightarrow$ $x = y = 0$

constraint eq $\Rightarrow$ $z = \pm\sqrt{2}$

Case 2 $z = 0$

If $\begin{vmatrix} 2-2\lambda & \lambda \\ \lambda & 2-2\lambda \end{vmatrix} \neq 0$ then $x = y = 0$ and the constraint equation cannot be satisfied

Thus

$$(2-2\lambda)^2-\lambda^2=0$$

$$4-8\lambda+3\lambda^2=0 \qquad \Delta = 64-48=16$$
$$\sqrt{\Delta}=4$$

$$\lambda_1 = \frac{8+4}{6} = 2$$
$$\lambda_2 = \frac{8-4}{6} = \frac{2}{3}$$

For $\lambda=2$ $\quad -2x+2y=0 \implies x=y$

constraint eq. $\implies x^2-x^2+x^2=4 \implies x=\pm 2$

For $\lambda=\frac{2}{3}$ $\quad \frac{2}{3}x+\frac{2}{3}y=0 \implies x=-y$

constraint eq. $\implies x^2+x^2+x^2=4$

$$x^2=\frac{4}{3}$$
$$x=\pm\frac{2}{\sqrt{3}}$$

So we have a number of stationary points here:

1. $x=y=0$, $z=\pm\sqrt{2}$ $\qquad x^2+y^2+z^2=2$
2. $x=y=\pm 2$ $\quad z=0$ $\qquad x^2+y^2+z^2=8$
3. $x=-y=\pm\frac{2}{\sqrt{3}}$ $\quad z=0$ $\qquad x^2+y^2+z^2=\frac{8}{3}$

$\therefore$ The global minimum is attained at $x=y=0$, $z=\pm\sqrt{2}$

Note that at this point we do not know whether a <u>local</u> minimum is attained at any of the remaining points.
To check that, we would have to eliminate one of the variables and use the Implicit Function Theorem to check the hessian!

#

10.2 c

$$\min_{ax+by+cz+d} \frac{1}{2}(x^2+y^2+z^2)$$

$$L(x,y,z,\lambda) = \frac{1}{2}(x^2+y^2+z^2) - \lambda(ax+by+cz+d)$$

$$x - \lambda a = 0 \qquad \Rightarrow \quad x = \lambda a$$
$$y - \lambda b = 0 \qquad\qquad y = \lambda b$$
$$z - \lambda c = 0 \qquad\qquad z = \lambda c$$

$$ax+by+cz+d = 0 \quad \Rightarrow \quad (a^2+b^2+c^2)\lambda = -d$$

$$\Rightarrow \quad x = \frac{-ad}{a^2+b^2+c^2}, \quad y = -\frac{bd}{a^2+b^2+c^2}, \quad z = -\frac{cd}{a^2+b^2+c^2}$$

It follows from the geometrical interpretation that this must be a minimum.

10.10

a) wrong!

$$\int_{x_1}^{x_2} f_{y'}\eta'\,dx = -\int_{x_1}^{x_2} (f_{y'})'\eta\,dx = 0 \qquad \forall \eta \quad \eta(x_2)=\eta(x_1)=0$$

$$\Rightarrow \quad (f_{y'})' \equiv 0 \qquad \Rightarrow \quad f_{y'} = \text{const} \quad (\text{not necessarily } 0!)$$

b) $$f_y - (f_{y'})' + (f_{y''})'' = 0 \qquad \text{E-L equation}$$

$$\left.\begin{aligned} f_{y'} - (f_{y''})' &= 0 \\ f_{y''} &= 0 \end{aligned}\right\} \text{natural B.C.}$$

Explanation:

Variational formulation:

$$\int_a^b \left(f_y \delta y + f_{y'} \delta y' + f_{y''} \delta y'' \right) dx = 0 \quad \forall \delta y \ldots$$

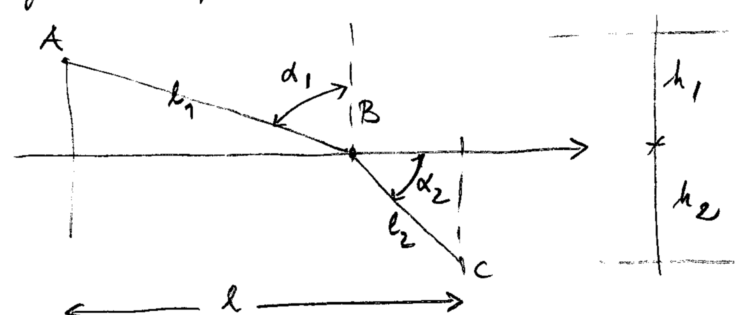
$$\Updownarrow$$

$$\int_a^b \left(f_y - (f_{y'})' + (f_{y''})'' \right) \delta y \, dx + \left[f_{y'} - (f_{y''})' \right] \delta y \Big|_a^b + f_{y''} \delta y' \Big|_a^b$$

10.13 a) speed = const $\Rightarrow$ time necessary to travel along a curve = length of the curve / speed

Thus the problem reduces to finding the shortest curve connecting two points and this must be a straight line

b)



The paths from A to B and B to C must be straight lines. (Otherwise replacing whatever it would be the curves between A and B and B and C, with straight lines, we could have shortened the time). Thus the whole problem reduces to finding position of point B.

We want thus to minimize

$$\frac{l_1}{V_1} + \frac{l_2}{V_2} = \frac{h_1}{V_1 \cos\alpha_1} + \frac{h_2}{V_2 \cos\alpha_2}$$

subjected to constraint

$$h_1 \tan\alpha_1 + h_2 \tan\alpha_2 = l$$

$$L(\alpha_1, \alpha_2, \lambda) = \frac{h_1}{V_1 \cos\alpha_1} + \frac{h_2}{V_2 \cos\alpha_2} - \lambda\left(h_1 \tan\alpha_1 + h_2 \tan\alpha_2 - l\right)$$

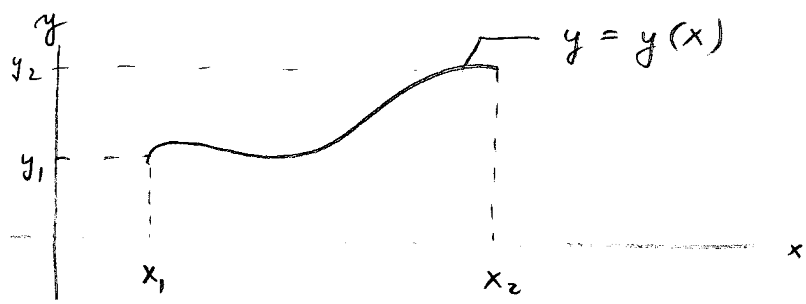
$$\frac{\partial L}{\partial \alpha_1} = 0 \Rightarrow \frac{h_1}{V_1 \cos^2\alpha_1} \sin\alpha_1 - \lambda h_1 \frac{1}{\cos^2\alpha_1} \Rightarrow \frac{\sin\alpha_1}{V_1} = \lambda$$

$$\frac{\partial L}{\partial \alpha_2} = 0 \Rightarrow \frac{h_2}{V_2 \cos^2\alpha_2} \sin\alpha_2 - \lambda h_2 \frac{1}{\cos^2\alpha_2} \Rightarrow \frac{\sin\alpha_2}{V_2} = \lambda$$

$$\therefore \quad \frac{\sin\alpha_1}{V_1} = \frac{\sin\alpha_2}{V_2}$$

#

10.14 a)



$$dl = \sqrt{1+(y')^2}\, dx$$

time to travel dl $= \dfrac{dl}{v(y)} = \dfrac{\sqrt{1+(y')^2}}{v(y)}\, dx$

Functional to minimize:

$$\int_{x_1}^{x_2} \frac{\sqrt{1+(y')^2}}{v(y)}\, dx \qquad y(x_1) = y_1, \; y(x_2) = y_2$$

E-L equation:

$$-\frac{\sqrt{1+(y')^2}}{v^2}\frac{dv}{dy} - \left(\frac{1}{v}\frac{y'}{\sqrt{1+(y')^2}}\right)' = 0$$

Transforming:

$$-\frac{\sqrt{1+y'^2}}{v^2}\frac{dv}{dy}$$

$$-\left\{-\frac{1}{v^2}\frac{dv}{dy}y'\frac{y'}{\sqrt{1+y'^2}}+\frac{1}{v}\frac{y''\sqrt{1+y'^2}-y'\frac{y'}{\sqrt{1+y'^2}}y''}{1+y'^2}\right\}=0$$

$$\Big/ \sqrt{1+y'^2}\,v^2$$

$$-(1+y'^2)\frac{dv}{dy}+\frac{dv}{dy}y'^2-vy''+v\frac{y'^2y''}{1+y'^2}=0$$

$$-\frac{dv}{dy}+vy''\left(\frac{y'^2}{1+y'^2}-1\right)=0$$

$$vy''+\frac{dv}{dy}(1+y'^2)=0$$

b) $v(y)=y \qquad y(0)=0 \qquad y(1)=0$

$$yy''+(1+y'^2)=0$$

Solution: a circle centered at $(\frac{1}{2},0)$

$$y(x)=\sqrt{x-x^2}$$

c) $v(y)=\sqrt{y}$ $\qquad y^{\frac{1}{2}}y''+\frac{1}{2}y^{-\frac{1}{2}}(1+y'^2)=0 \;\Big/ \cdot y^{\frac{1}{2}}$

$$yy''+\frac{1}{2}(1+y'^2)=0$$

Step 1. Substitution: $y' = u(y)$

$$y'' = \frac{du}{dy} y' = \frac{du}{dy} u$$

$$2 y u \frac{du}{dy} + (1+u^2) = 0$$

$$\frac{2u\,du}{1+u^2} = -\frac{dy}{y}$$

$$\int \frac{2u\,du}{1+u^2} = -\int \frac{dy}{y}$$

$$\ln(1+u^2) = -\ln y + c$$

$$\ln[(1+u^2)y] = c$$

$$(1+u^2)y = c \quad \text{(another constant)}$$

Returning to the original variables

$$(1+y'^2)y = c$$

$$1+y'^2 = \frac{c}{y}$$

$$y'^2 = \frac{c}{y} - 1 = \frac{c-y}{y}$$

$$\frac{dy}{dx} = \sqrt{\frac{c-y}{y}}$$

$$\int_0^{y(x)} \sqrt{\frac{y}{c-y}}\,dy = \int_0^x dx$$

The left hand side integral reduces to an integral of a rational function (see page 8a)

$$\sqrt{\frac{y}{c-y}} = u$$

$$\frac{y}{c-y} = u^2$$

$$y = (c-y)u^2 = cu^2 - yu^2$$

$$y(1+u^2) = cu^2$$

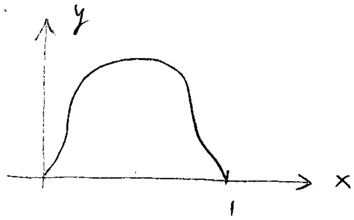
$$y = \frac{cu^2}{1+u^2}$$

$$dy = \frac{2cu(1+u^2) - cu^2 \cdot 2u}{(1+u^2)^2}\, du$$

$$= 2C \frac{u}{(1+u^2)^2}\, du$$

$$\int \sqrt{\frac{y}{c-y}}\, dy = 2C \int \frac{u^2}{(1+u^2)^2}\, du$$

10.16



Functional to minimize : $\displaystyle\int_0^1 y\,dx$

Constraint : $\displaystyle\int_0^1 \sqrt{1+y'^2}\,dx = L$

Lagrangian :

$$L(y,\lambda) = \int_0^1 y\,dx - \lambda\left(\int_0^1 \sqrt{1+y'^2}\,dx - L\right)$$

$$= \int_0^1 \left(y - \lambda\sqrt{1+y'^2}\right)dx + \lambda L$$

E-L equation :

$$\lambda\left(\frac{y'}{\sqrt{1+y'^2}}\right)' + 1 = 0 \qquad y(0) = y(1) = 0$$

$$\lambda\,\frac{y''\sqrt{1+y'^2} - y'\,\frac{2y'}{2\sqrt{1+y'^2}}\,y''}{1+y'^2} + 1 = 0$$

$$\frac{\lambda y''\sqrt{1+y'^2}\left(1 - \frac{y'^2}{1+y'^2}\right)}{1+y'^2} + 1 = 0$$

$$\frac{\lambda y''}{(1+y'^2)^{\frac{3}{2}}} + 1 = 0 \quad \Rightarrow \quad \underbrace{\frac{y''}{(1+y'^2)^{\frac{3}{2}}}}_{\text{curvature}} = \underset{c}{-\frac{1}{\lambda}}$$

the solution must be a circle

Now, more formally:

$$\lambda \left(\frac{y'}{\sqrt{1+y'^2}} \right)' + 1 = 0$$

implies that

$$\frac{y'}{\sqrt{1+y'^2}} = -\frac{1}{\lambda} x + C = -\frac{1}{\lambda}(x - A)$$

(just a choice for C)

$$\frac{y'^2}{1+y'^2} = \frac{1}{\lambda^2}(x-A)^2$$

$$y'^2 = (1+y'^2)\frac{1}{\lambda^2}(x-A)^2$$

$$y'^2\left[1 - \frac{(x-A)^2}{\lambda^2}\right] = \frac{1}{\lambda^2}(x-A)^2$$

$$y'^2 = \frac{(x-A)^2}{\lambda^2 - (x-A)^2}$$

$$y' = \frac{x-A}{\sqrt{\lambda^2-(x-A)^2}} \qquad \text{Ass: } x - A \geq 0$$

$$\int \frac{x-A}{\sqrt{\lambda^2-(x-A)^2}}\,dx = -\frac{1}{2}\int u^{-\frac{1}{2}}\,du = -\cancel{\frac{1}{2}}\,\frac{u^{\frac{1}{2}}}{\cancel{\frac{1}{2}}} + B$$

$$\lambda^2 - (x-A)^2 = u \qquad\qquad = -\sqrt{\lambda^2-(x-A)^2} + B$$

$$-2(x-A)\,dx = du$$

$$\therefore \quad y - B = -\sqrt{\lambda^2-(x-A)^2}$$

$$(x-A)^2 + (y-B)^2 = \lambda^2$$

Same for the case $x - A \leq 0$

When $L > \frac{\pi}{2}$, the circle is not a solution (we cannot satisfy the constraint equation!)

10.18

a) $$L(x,y,z,\lambda) = \int_{t_1}^{t_2} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\,dt - \int_{t_1}^{t_2} \lambda(x^2+y^2+z^2-R^2)\,dt$$

$$(x,y,z)(t_1) = (x_1,y_1,z_1)$$
$$(x,y,z)(t_2) = (x_2,y_2,z_2)$$

E-L equations:

$$\begin{cases} -\left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)^{\cdot} - 2x = 0 & (1) \\ -\left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)^{\cdot} - 2y = 0 & (2) \\ -\left(\dfrac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)^{\cdot} - 2z = 0 & (3) \\ x^2+y^2+z^2 = R^2 & (4) \end{cases}$$

(1)(2) $\Rightarrow$ $$\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)^{\cdot} y = x\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\right)^{\cdot} \quad / + \frac{\dot{x}\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}$$

$$\left(\frac{\dot{x}y}{\sqrt{}}\right)^{\cdot} = \left(\frac{x\dot{y}}{\sqrt{}}\right)^{\cdot}$$

$$\left(\frac{\dot{x}y - x\dot{y}}{\sqrt{}}\right)^{\cdot} = 0$$

$\therefore$ $$\dot{x}y - x\dot{y} = A\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

Similarly $$\dot{z}y - z\dot{y} \quad B\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

So: $$\frac{1}{A}(\dot{x}y - x\dot{y}) = \frac{1}{B}(\dot{z}y - z\dot{y}) \quad / : y^2$$

$\therefore$ $$\frac{1}{A}\left(\frac{x}{y}\right)^{\cdot} = \frac{1}{B}\left(\frac{z}{y}\right)^{\cdot}$$

$$\frac{1}{A}\left(\frac{x}{y}\right) - \frac{1}{B}\left(\frac{z}{y}\right) = \frac{1}{C} \quad (=\text{const})$$

$$\frac{1}{A}x - \frac{1}{B}z = \frac{1}{C}y$$

∴ The curve must lie on a plane passing through origin
⇒ it must be the great circle!

b)

$$\int_{\theta_1}^{\theta_2} R\underbrace{\sqrt{1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2}}_{F(\theta,\, \cancel{\varphi},\, \frac{d\varphi}{d\theta})}\, d\theta \longrightarrow \min$$

E-L equ:

$$-\frac{d}{d\theta}\left\{\frac{\sin^2\theta \,\frac{d\varphi}{d\theta}}{\sqrt{1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2}}\right\} = 0$$

$$\sin^2\theta\,\frac{d\varphi}{d\theta} = c\sqrt{1+\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2} \qquad /(\;)^2$$

$$\sin^4\theta\left(\frac{d\varphi}{d\theta}\right)^2 = c^2 + c^2\sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2$$

$$\left(\frac{d\varphi}{d\theta}\right)^2\left[c^2\sin^2\theta - \sin^4\theta\right] = -c^2$$

$$\left(\frac{d\varphi}{d\theta}\right)^2\sin^2\theta\left[c^2 - \sin^2\theta\right] = -c^2$$

$$\left(\frac{d\varphi}{d\theta}\right)^2 = -\frac{c^2}{\sin^2\theta\,(c^2-\sin^2\theta)}$$

$$\frac{1}{A}\left(\frac{x}{y}\right) - \frac{1}{B}\left(\frac{z}{y}\right) = \frac{1}{C} \quad (= \text{const})$$

$$\frac{1}{A}x - \frac{1}{B}z = \frac{1}{C}y$$

$\therefore$ The curve must lie on a plane passing through the origin
$\Rightarrow$ it must be the great circle!

b) Given two points on the sphere, say A and B we want to determine the <u>shortest</u> arc <u>on</u> the sphere connecting them. We shall set up the spherical coordinates in such a way that points A and B will have the <u>same</u> coordinate φ, say, equal 0. This leads to the minimization problem:

$$R \int_{\theta_1}^{\theta_2} \sqrt{1 + \sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2}\, d\theta \longrightarrow \min$$

subjected to the conditions

$$\varphi(\theta_1) = 0, \quad \varphi(\theta_2) = 0$$

the E-L eqn is:

$$-\frac{d}{d\theta}\left\{ \frac{\sin^2\theta \dfrac{d\varphi}{d\theta}}{\sqrt{1+\sin^2\theta\left(\dfrac{d\varphi}{d\theta}\right)^2}} \right\} = 0$$

Thus

$$\frac{\sin^2\theta \dfrac{d\varphi}{d\theta}}{\sqrt{1+\sin^2\theta\left(\dfrac{d\varphi}{d\theta}\right)^2}} = \text{const} = c$$

This in particular implies that $\dfrac{d\varphi}{d\theta}$ has a <u>constant</u> sign, namely $\text{sign}\left(\frac{d\varphi}{d\theta}\right) = \text{sign } c$.

But

$$0 = \varphi(\theta_2) - \varphi(\theta_1) = \int_{\theta_1}^{\theta_2} \left(\frac{d\varphi}{d\theta}\right) d\theta$$

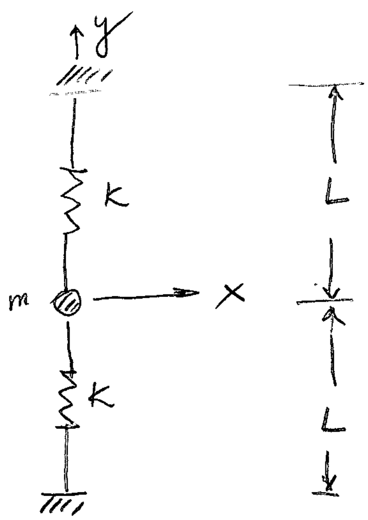
So, it must be $\frac{d\varphi}{d\theta} = 0$, i.e. $c = 0$!

And finally, according to the BC, $\varphi \equiv 0$.

But this corresponds to the great circle again !.

\#

10.20



Hamiltonian $H = T - V$

kinetic energy, potential energy

For a position (x, y), the elongation of both springs are

- for the upper spring $\Delta L = \sqrt{x^2 + (y-L)^2} - L$
- for the lower spring $\Delta l = \sqrt{x^2 + (y+L)^2} - L$

so

$$H(x,y,\dot{x},\dot{y}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{k}{2}\left\{\left[\sqrt{x^2+(y-L)^2}-L\right]^2+\left[\sqrt{x^2+(y+L)^2}-L\right]^2\right\}$$

the minimization problem:

$$\int_{t_1}^{t_2} H(x,y,\dot{x},\dot{y})\,dt \longrightarrow \min$$

E-L equs:

$$\begin{cases} m\ddot{x} - k\left\{\left[\sqrt{x^2+(y-L)^2}-L\right]\dfrac{x}{\sqrt{x^2+(y-L)^2}} + \left[\sqrt{x^2+(y+L)^2}-L\right]\dfrac{x}{\sqrt{x^2+(y+L)^2}}\right\} = 0 \\ m\ddot{y} - k\left\{\left[\sqrt{x^2+(y-L)^2}-L\right]\dfrac{(y-L)}{\sqrt{x^2+(y-L)^2}} + \left[\sqrt{x^2+(y+L)^2}-L\right]\dfrac{(y+L)}{\sqrt{x^2+(y+L)^2}}\right\} = 0 \end{cases}$$

<u>10.29.</u>

Yes, up to some details on regularity assumptions.
The test function $\eta(x)$ may not be C^2.