CAM 389C Exercise Set I.5

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Problem 1

Let Γ be a material surface associated with the reference configuration: $\Gamma \subset \partial\Omega_t$. Let $\mathbf{g}$ be an applied force per unit area acting on Γ ($\mathbf{g} = \mathbf{g}(\mathbf{x}, t)\,, \mathbf{x} \in \Gamma$). The "traction" boundary condition on Γ at eacg $\mathbf{x} \in \Gamma$ is

$$\mathbf{Tn} = \mathbf{g}\,.$$

Show that

$$\mathbf{FSn}_0 = \mathbf{g}_0\,, \qquad \text{on } \varphi^{-1}(\Gamma)\,,$$

where $\mathbf{n}_0$ is the unit exterior normal to Γ_0 ($\Gamma = \varphi(\Gamma_0)$) and

$$\mathbf{g}_0(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X})\|\mathbf{F}^{-T}(\mathbf{Xn}_0\|\mathbf{g}(\mathbf{x})\,.$$

Solution

Please note that

$$\mathbf{T} = (\det \mathbf{F})^{-1}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T\,.$$

Therefore

$$\mathbf{Tn} = (\det \mathbf{F})^{-1}\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T\mathbf{n}$$

Recalling that

$$\mathbf{n} = \frac{\mathrm{Cof}\,\mathbf{Fn}_0}{\|\mathrm{Cof}\,\mathbf{Fn}_0\|}\,.$$

The expression becomes

$$\mathbf{Tn} = \frac{\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T\,\mathrm{Cof}\,\mathbf{F}\,\mathbf{n}_0}{\det \mathbf{F}\|\mathrm{Cof}\,\mathbf{F}\,\mathbf{n}_0\|}$$

And noting that

$$\mathrm{Cof}\,\mathbf{F} = (\det \mathbf{F})\mathbf{F}^{-T}\,,$$

we get

$$\begin{aligned}
\mathbf{T}\mathbf{n} &= \frac{\mathbf{F}\,\mathbf{S}\,\cancel{\mathbf{F}^T}\,\cancel{\mathbf{F}^{-T}}\,\cancel{\det\mathbf{F}}\,\mathbf{n}_0}{\cancel{\det\mathbf{F}}\,\|(\det\mathbf{F})\mathbf{F}^{-T}\,\mathbf{n}_0\|} \\
&= \frac{\mathbf{F}\,\mathbf{S}\,\mathbf{n}_0}{\det\mathbf{F}\,\|\mathbf{F}^{-T}\,\mathbf{n}_0\|} \\
&= \mathbf{g}(\mathbf{x})\,.
\end{aligned}$$

Moving $\det\mathbf{F}\,\|\mathbf{F}^{-T}\,\mathbf{n}_0\|$ to the right hand side,

$$\mathbf{F}\,\mathbf{S}\,\mathbf{n}_0 = \det\mathbf{F}(\mathbf{X})\,\|\mathbf{F}^{-T}(\mathbf{X})\,\mathbf{n}_0\|\,\mathbf{g}(\mathbf{x}) = \mathbf{g}_0(\mathbf{X},t)\,.$$

Problem 2

Consider an Eulerian description of the flow of a fluid in a region of $\mathbb{R}^3$. The flow is characterized by the triple $(\mathbf{v},\rho,\mathbf{T})$. The flow is said to be *potential* if the velocity is derivable as the gradient of a scalar field φ:

$$\mathbf{v} = \operatorname{grad}\varphi\,.$$

The body force acting on the fluid is said to be *conservative* if there is also a potential U such that

$$\mathbf{f} = -\rho\operatorname{grad}U\,.$$

The special case in which the stress $\mathbf{T}$ is of the form

$$\mathbf{T} = -p\mathbf{I}\,,$$

where p is a scalar field and $\mathbf{I}$ is the unit tensor, is called a *pressure* field.

Show that for potential flow, a pressure field $\mathbf{T} = -p\mathbf{I}$, and conservative body forces, the momentum equations imply that

$$\operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + U\right) + \frac{1}{\rho}\operatorname{grad}p = \mathbf{0}$$

Solution

The Eulerian description of the momentum equation is

$$\operatorname{div}\mathbf{T} + \mathbf{f} = \rho\frac{d\mathbf{v}}{dt}\,.$$

First note that

$$\operatorname{div}(-p\mathbf{I}) = -\operatorname{grad}p\cdot\mathbf{I} - \cancelto{0}{p\operatorname{div}\mathbf{I}} = -\operatorname{grad}p\,,$$

and that

$$
\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot \operatorname{grad}\mathbf{v} \\
&= \frac{\partial v_j}{\partial t}\mathbf{e}_j + v_i \partial_j v_i \mathbf{e}_j \\
&= \frac{\partial v_j}{\partial t}\mathbf{e}_j + \frac{1}{2}\partial_j v_i v_i \mathbf{e}_j \\
&= \frac{\partial \mathbf{v}}{\partial t} + \frac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v})\,.
\end{aligned}
$$

Substituting all of these relations into the momentum equation (with $\varphi = \operatorname{grad}\mathbf{v}$),

$$
-\operatorname{grad} p - \rho \operatorname{grad} U = \rho\left(\frac{\partial}{\partial t}\operatorname{grad}\varphi + \frac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v})\right)\,.
$$

Rearranging terms,

$$
\rho\left(\frac{\partial}{\partial t}\operatorname{grad}\varphi + \frac{1}{2}\operatorname{grad}(\mathbf{v}\cdot\mathbf{v}) + \operatorname{grad} U\right) + \operatorname{grad} p = \mathbf{0}\,.
$$

Dividing by ρ and grouping grad terms,

$$
\operatorname{grad}\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2} + U\right) + \frac{1}{\rho}\operatorname{grad} p = \mathbf{0}\,.
$$

Problem 3

Let $\mathbf{u} = \mathbf{u}(\mathbf{X}, t)$ be the displacement field. Define the quantity

$$
\psi = \frac{1}{2}\int_{\Omega_0} \rho_0 \mathbf{u}\cdot\mathbf{u}\,dX\,.
$$

Show that (if $\mathbf{f}_0 = \mathbf{0}$),

$$
\ddot{\psi} = \int_{\Omega_0} \rho_0 \dot{\mathbf{u}}\cdot\dot{\mathbf{u}}\,dX - \int_{\Omega_0} \mathbf{P} : \nabla\mathbf{u}\,dX + \int_{\Omega_0} \mathbf{u}\cdot\mathbf{P}\mathbf{n}_0\,dA_0\,.
$$

Solution

Taking the first and second time derivatives:

$$
\dot{\psi} = \frac{\cancel{1}}{\cancel{2}}\int_{\Omega_0} \rho_0(\cancel{2}\mathbf{u}\cdot\dot{\mathbf{u}})\,dX
$$

$$
\dot{\psi} = \int_{\Omega_0} \rho_0(\mathbf{u}\cdot\ddot{\mathbf{u}} + \dot{\mathbf{u}}\cdot\dot{\mathbf{u}})\,dX
$$

Substituting from the Lagrangian momentum $\rho_0 \ddot{\mathbf{u}} = \mathrm{Div}(\mathbf{P}) + \cancelto{0}{\mathbf{f}_0}$,

$$\ddot{\psi} = \int_{\Omega_0} \rho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dX + \int_{\Omega_0} \mathbf{u} \cdot \mathrm{Div}(\mathbf{P}) \, dX \,.$$

Focusing on the second term on the right hand side (noting that $\mathrm{Div}(\mathbf{Pu}) = \mathbf{u} \cdot \mathrm{Div}\, \mathbf{P} + \mathbf{P} : \nabla \mathbf{u}$),

$$\int_{\Omega_0} \mathbf{u} \cdot \mathrm{Div}(\mathbf{P}) \, dX = \int_{\Omega_0} \mathrm{Div}\, \mathbf{Pu} \, dX - \int_{\Omega_0} \mathbf{P} : \nabla \mathbf{u} \, dX \,.$$

And making use of the divergence theorem the expression becomes,

$$\int_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{P}\mathbf{n}_0 \, dA_0 - \int_{\Omega_0} \mathbf{P} : \nabla \mathbf{u} \, dX \,.$$

When we put this all together, we get

$$\ddot{\psi} = \int_{\Omega_0} \rho_0 \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} \, dX - \int_{\Omega_0} \mathbf{P} : \nabla \mathbf{u} \, dX + \int_{\partial\Omega_0} \mathbf{u} \cdot \mathbf{P}\mathbf{n}_0 \, dA_0 \,.$$

Problem 4

A cylindrical rubber plug 1 cm in diameter and 1 cm long is glued to a rigid foundation. Then it is pulled by external forces so that the flat cylindrical upper face $\Gamma_0 = \{(X_1, X_2, X_3) : \, X_3 = 1, \, (X_1^2 + X_2^2)^{1/2} \leq 1/2\}$ is squeezed to a flat circular diameter 1/4 cm with normal $\mathbf{n} = \mathbf{e}_2$, as shown at position $\mathbf{x} = \mathbf{x}^*$. Suppose that the stress vector at $\mathbf{x} = \mathbf{x}^*$ is uniform and normal to Γ:

$$\sigma(\mathbf{n}, \mathbf{x}^*) = \sigma(\mathbf{e}_2, \mathbf{x}^*) = 1000 \, \mathbf{e}_2 \, \mathrm{kg/cm}^2 \quad \forall \mathbf{x}^* \in \Gamma \,.$$

Suppose that the corresponding Piola-Kirchhoff stress $\mathbf{p}_0 = \mathbf{P}\mathbf{n}_0$ is uniform on Γ_0.

- Determine the Piola-Kirchoff stress vector $\mathbf{p}_0 = \mathbf{P}\mathbf{n}_0$ on Γ_0.
- Determine one possible tensor $\mathrm{Cof}\, \mathbf{F}(X_1, X2, 1)$ for this situation.

Solution

From our limited amount of information, we know that $\mathbf{p}_0$ must be in the $\mathbf{e}_3$ direction with a value such that the total force on the face of the reference configuration matches the total force on the current configuration. Therefore, let F_s denote the total scalar value of force on the face. Then on the current configuration,

$$F_s = 1000 \, \frac{\mathrm{kg}}{\mathrm{cm}^2} \left(\frac{\pi}{4} (.25 \, \mathrm{cm})^2 \right) = 49.087 \, \mathrm{kg} \,.$$

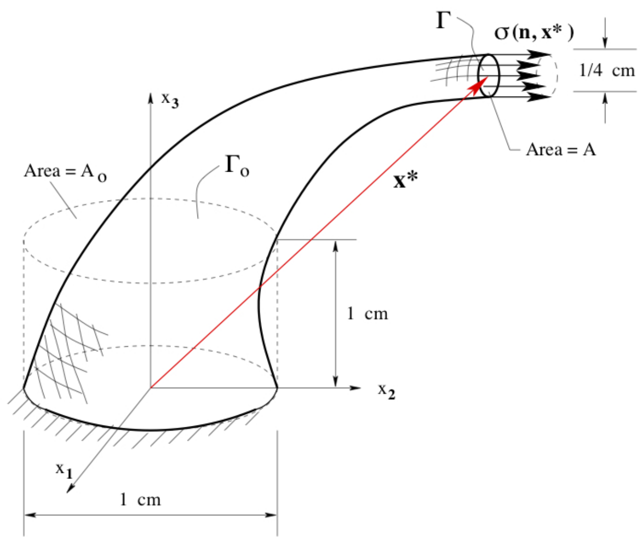

Figure 1: Illustrative sketch of the rubber plug.

We know that $\mathbf{p}_0$ is uniform on Γ_0, therefore

$$\mathbf{p}_0 = \frac{49.087\,\mathrm{kg}}{\frac{\pi}{4}(1\,\mathrm{cm})^2}\mathbf{e}_3 = 62.5\mathbf{e}_3\,\frac{\mathrm{kg}}{\mathrm{cm}^2}$$

We have very limited knowledge of the stress tensor, but there are a few things that we can discern from the given information and schematic. We can gather that $T_{22}(\mathbf{x}^*) = 1000$. Also, since there is no motion in the x_1 direction, the first column must be zeros. Also, we know that the stress at $\mathbf{x}^*$ is normal to the face, therefore $T_{12} = T_{32} = 0$. There may be vertical shear stresses at the face, so we will denote the third column entries with anonymous $\cdot$'s. Therefore

$$\mathbf{T}(\mathbf{x}^*) = \begin{bmatrix} 0 & 0 & \cdot \\ 0 & 1000 & \cdot \\ 0 & 0 & \cdot \end{bmatrix}.$$

Furthermore, we know that $\mathbf{P}(\mathbf{X}) = \mathbf{T}(\mathbf{x})\,\mathrm{Cof}\,\mathbf{F}(\mathbf{X})$, and $P_{23} = 62.5$. Therefore $\mathrm{Cof}\,\mathbf{F}$ should scale $\mathbf{T}$ by $1/16$ and flip the second and third columns. Again, the first column doesn't really matter because there is no motion in that direction. Therefore, one possible tensor for this situation is

$$\mathrm{Cof}\,\mathbf{F}(X_1, X_2, 1) = \begin{bmatrix} 1 & \cdot & 0 \\ 0 & \cdot & 1/16 \\ 0 & \cdot & 0 \end{bmatrix}.$$