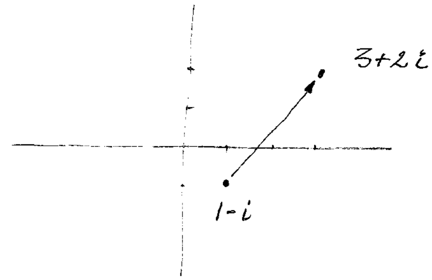
Chapter 13

13.2

a) $\displaystyle\int_{1-i}^{3+2i} \cos z \, dz$



<u>Method 1</u>

Let c be a straight line segment connecting $1-i$ with $3+2i$.

Parametrization:

$$z(t) = 1-i + t\left(3+2i-(1-i)\right)$$
$$= 1-i + t(2+3i)$$

$$\frac{dz}{dt} = 2+3i \qquad i[1-i+t(2+3i)] = i+1+t(2i-3)$$
$$= (1-3t)+i(1+2t)$$

$$\cos z(t)\frac{dz}{dt} = \cos\left(1-i+t(2+3i)\right)(2+3i)$$
$$= \frac{1}{2}\left(e^{i[1-i+t(2+3i)]} + e^{-i[1-i+t(2+3i)]}\right)(2+3i)$$

$$\int_0^1 \cos z(t)\frac{dz}{dt}\,dt = \int_0^1 \frac{1}{2}\left(e^{i[\ ]}+e^{-i[\ ]}\right)(2+3i)\,dt$$

$$= \int_0^1 \frac{1}{2}\left(e^{1-3t}e^{i(1+2t)} + e^{-(1-3t)}e^{-i(1+2t)}\right)(2+3i)\,dt$$

$$e^{1-3t}e^{i(1+2t)} = e^{1-3t}\cos(1+2t) + i\,e^{1-3t}\sin(1+2t)$$

$$\int e^{1-3t}\cos(1+2t) = \frac{1}{2}e^{1-3t}\overset{\sin(1+2t)}{\vphantom{x}} - \frac{3}{2}\int e^{1-3t}\sin(1+2t) = (*)$$

$$e^{1-3t} \qquad \frac{1}{2}\sin(1+2t)$$

$$-3e^{1-3t} \qquad \cos(1+2t)$$

$$e^{1-3t} \qquad -\tfrac{1}{2}\cos(1+2t)$$

$$-3\,e^{1-3t} \qquad \sin(1+2t)$$

$$(*) = \tfrac{1}{2} e^{1-3t}\sin(1+2t) - \tfrac{3}{2}\Big[-\tfrac{1}{2} e^{1-3t}\cos(1+2t)\Big)$$
$$- \tfrac{3}{2}\int e^{1-3t}\cos(1+2t)\,dt\,\Big]$$

So.

$$\left(1-\tfrac{9}{4}\right)\int e^{1-3t}\cos(1+2t)\,dt =$$
$$= \tfrac{1}{2} e^{1-3t}\sin(1+2t) + \tfrac{3}{4} e^{1-3t}\cos(1+2t)$$

$$\int e^{1-3t}\cos(1+2t)\,dt = -\,e^{1-3t}\left(\tfrac{2}{5}\sin(1+2t) + \tfrac{3}{5}\cos(1+2t)\right)$$

Check: $\left[-e^{1-3t}(\qquad)\right]'$

$$= 3e^{1-3t}\left(\tfrac{2}{5}\sin(\ ) + \tfrac{3}{5}\cos(\ )\right) - e^{1-3t}\left(\tfrac{4}{5}\cos(\ ) - \tfrac{6}{5}\sin(\ )\right)$$

$$= e^{1-3t}\cos(1+2t)$$

In exactly the same way one has to calculate

$$\int e^{1-3t}\sin(1+2t)$$

and then

$$\int e^{-(1-3t)}\cos(1+2t) \quad \text{and} \quad -\int e^{-(1-3t)}\sin(1+2t)$$

and sum up the integrals!

Method 2

$$\int_{1-i}^{3+2i} \cos z \, dz = \sin z \Big|_{1-i}^{3+2i} = \sin(3+2i) - \sin(1-i)$$

$$= \frac{1}{2i}\left[e^{i(3+2i)} - e^{-i(3+2i)}\right] - \frac{1}{2i}\left[e^{i(1-i)} - e^{-i(1-i)}\right]$$

$$= -\frac{i}{2}\left[e^{-2+3i} - e^{2-3i}\right] + \frac{i}{2}\left[e^{-1+i} - e^{1-i}\right]$$

$$= -\frac{i}{2}\left[e^{-2}(\cos 3 + i\sin 3) - e^{2}(\cos 3 - i \sin 3)\right]$$

$$+ \frac{i}{2}\left[e^{-1}(\cos 1 + i \sin 1) - e(\cos 1 - i \sin 1)\right]$$

$$= -i\left[-\sinh 2 \cos 3 + i \cosh 2 \sin 3\right]$$
$$+ i\left[-\sinh 1 \cos 1 + i \cosh 1 \sin 1\right]$$

$$= \left(\cosh 2 \sin 3 - \cosh 1 \sin 1\right) + i\left(\sinh 2 \cos 3 - \sinh 1 \cos 1\right)$$

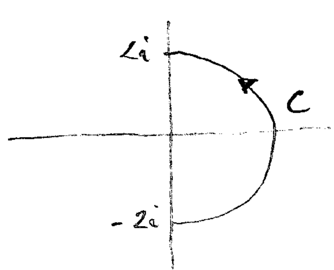
Isn't this much shorter ?

13.3

$$z = 2e^{i\theta} \qquad \theta \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$$

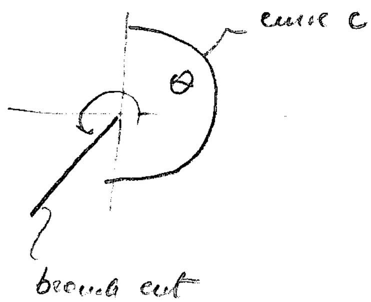
$$\frac{dz}{d\theta} = 2ie^{i\theta}$$

$$I = \int_C \frac{dz}{z} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2ie^{i\theta}}{2e^{i\theta}} \, d\theta = i\theta \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi i$$



Alternatively $\quad I = \ln z \Big|_{-2i}^{2i} = \ln(2i) - \ln(-2i)$

The choice of the branch cut for $\ln z$ is <u>not</u> arbitrary. $\ln z$ should be differentiable (and therefore continuous) in a set encircling curve C, eg.



13.6 $\qquad C_1 : |z| = 1, \qquad C_2 : |z| = 2$

a) $$\int_{C_1} (z^2 - \sin z)\, dz = \int_{C_2} (z^2 - \sin z)\, dz = 0$$

since $z^2 \cdot \sin z$ is complex differentiable in the whole $\mathbb{C}$

b) $$\int_{C_1} \frac{\sin z}{z}\, dz = \int_{C_2} \frac{\sin z}{z}\, dz = 0$$

since $\frac{\sin z}{z}$ can be extended to a complex differentiable function in the whole $\mathbb{C}$.

Indeed set $$f(z) = \begin{cases} \frac{\sin z}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

One has to check only the differentiability at 0.

$$\frac{\sin z}{z} = \frac{e^{iz} - e^{-iz}}{2iz} = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i(x+iy)}$$

$$= \frac{e^{-y+ix} - e^{y-ix}}{-2y + i2x}$$

$$= \frac{[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)](-\cancel{2}y - i\cancel{2}x)}{2\cancel{4}(x^2+y^2)}$$

$$= \frac{-1}{(x^2+y^2)}[-\sinh y\cos x + i\cosh y\sin x](y + ix)$$

$$= \underbrace{\frac{y\sinh y\cos x + x\cosh y\sin x}{x^2+y^2}}_{u(x,y)} + i\underbrace{\left(\frac{-y\cosh y\sin x + x\sinh y\cos x}{x^2+y^2}\right)}_{v(x,y)}$$

$$u_x(0,0) = \lim_{\Delta x \to 0}\left(\frac{\Delta x\sin\Delta x}{(\Delta x)^2} - 1\right)\frac{1}{\Delta x}$$

$$= \lim_{x\to 0}\frac{x\sin x - x^2}{x^3}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{\sin x + x\cos x - 2x}{3x^2}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{\cos x + \cos x - x\sin x - 2}{6x}$$

$$\overset{H}{=} \lim_{x\to 0}\frac{-2\sin x - \sin x - x\cos x}{6} = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left(v(0,\Delta y) - \cancel{v(0,0)}^{\,0} \right)$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y} \frac{0}{\Delta y^2} = 0$$

So: $u_x = v_y$ at $(0,0)$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left(u(0,\Delta y) - \cancel{u(0,0)}^{\,1} \right)$$

$$= \lim_{\Delta y \to 0} \frac{1}{\Delta y}\left(\frac{\Delta y \, \sinh \Delta y}{\Delta y^2} - 1 \right)$$

$$= \lim_{y \to 0} \frac{\sinh y - y}{y^2}$$ (changing notation Δy replaced with y)

$$\overset{H}{=} \lim_{y \to 0} \frac{\cosh y - 1}{2y}$$

$$\overset{H}{=} \lim_{y \to 0} \frac{\sinh y}{2} = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(v(\Delta x, 0) - \cancel{v(0,0)}^{\,0} \right)$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{0}{\Delta x^2} = 0$$

So: $u_y = -v_x$ at $(0,0)$

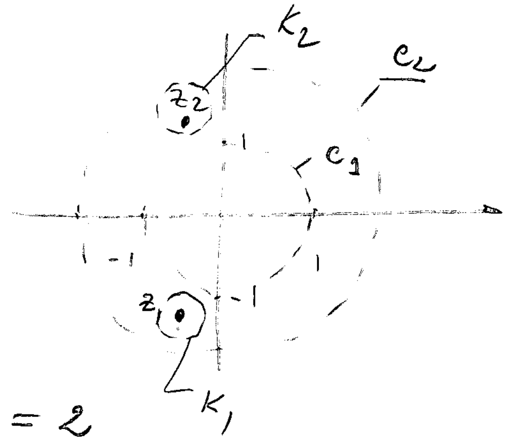
Thus $f(z)$ is complex-differentiable in the whole plane and, by Cauchy theorem, both integrals are zero

<u>Remark:</u> the fact that $z=0$ is only a non-essential (fake) singularity point for $\frac{\sin z}{z}$ follows immediately from the Taylor expansion of $\sin z$, be we cannot use it at this moment yet!

c) $$\int_{C_1} \frac{\sin z}{(z+4)^2}\, dz = \int_{C_2} \frac{\sin z}{(z+4)^2}\, dz = 0$$

Since $f(z) = \dfrac{\sin z}{(z+4)^2}$ is singular at $z=-4$ only

d) $$\int_{C_1} \frac{dz}{z^2+z+2} = 0 \quad , \text{ since :}$$



$z^2+z+2 = 0$

$\Delta = 1-8 = -7$

$z_1 = \dfrac{-1-\sqrt{-7}}{2} = -\dfrac{1}{2} - \dfrac{\sqrt{7}}{2} i$

$z_2 = -\dfrac{1}{2} + \dfrac{\sqrt{7}}{2} i$

$$\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2 = \frac{1}{4} + \frac{7}{4} = \frac{8}{4} = 2$$

and therefore both singularities are outside of C_1

The case with C_2 is more difficult, since both singular points lie within circle C_2 $\left(|z_1| = |z_2| = \sqrt{2} \right)$

According to corollary 5 to the Fundamental Cauchy theorem, given in the class, integral over C_2 equals the sum of two integrals over small circles K_1 and K_2 encircling z_1 and z_2 and contained within C_2

We have $$\frac{1}{z^2+z+2} = \frac{1}{(z-z_1)(z-z_2)} = \frac{A}{z-z_1} + \frac{B}{z-z_2}$$

$$= \frac{A(z-z_2) + B(z-z_1)}{(z-z_1)(z-z_2)} = \frac{(A+B)z - Az_2 - Bz_1}{(z-z_1)(z-z_2)}$$

So: $\quad A + B = 0 \quad \Rightarrow \quad B = -A$

$$-Az_2 - Bz_1 = A(z_1 - z_2) = 1 \quad \Rightarrow \quad A = \frac{1}{z_1 - z_2}$$

so $\quad A = \dfrac{1}{-\sqrt{7}\, i} = \dfrac{1}{\sqrt{7}}\, i$

and $$\frac{1}{z^2+z+2} = \frac{i}{\sqrt{7}} \frac{1}{z-z_1} - \frac{i}{\sqrt{7}} \frac{1}{z-z_2}$$

Consequently

$$\int_{K_1} \frac{dz}{z^2+z+2} = \frac{i}{\sqrt{7}} \overset{2\pi i}{\int_{K_1} \frac{dz}{z-z_1}} - \frac{i}{\sqrt{7}} \cancelto{0}{\int_{K} \frac{dz}{z-z_2}}$$

$$= -\frac{2\pi}{\sqrt{7}}$$

$$\int_{K_2} \frac{dz}{z^2+z+2} = \frac{i}{\sqrt{7}} \cancelto{0}{\int_{K_2} \frac{dz}{z-z_1}} - \frac{i}{\sqrt{7}} \cancelto{2\pi i}{\int_{K_2} \frac{dz}{z-z_2}} = \frac{2\pi}{\sqrt{7}}$$

and therefore the final integral is <u>zero</u>! $\quad$ #

e) $$\int_{C_1} \frac{\sinh z}{z}\, dz = \int_{C_2} \frac{\sinh z}{z}\, dz = 0$$

Same technique as in b)

$$\sinh z = \frac{e^z - e^{-z}}{2} = \left[\left(1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right) - \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6} + \dots\right)\right] / 2$$

$$= z + \frac{z^3}{6} + \dots$$

So $\dfrac{\sinh z}{z} = 1 + \dfrac{z^2}{6} + \dots$ is holomorfic in whole $\mathbb{C}$

f) $\int \frac{\cosh z}{z}\, dz$

$\frac{\cosh z}{z}$ has essential singularity at $z = 0$

$\left(\frac{\cosh z}{z} = \frac{1 + \frac{z^2}{2} + \dots}{z} = \frac{1}{z} + \frac{z}{2} + \dots \right)$

Using the Taylor expansion as a hint, we can represent the function as

$$\frac{\cosh z}{z} = \frac{1}{z} + \underbrace{\frac{\cosh z}{z} - \frac{1}{z}}_{\overset{\text{def}}{=}\, g(z)}$$

Using the same technique as in b), we can prove that $g(z)$ is holomorfic in whole $\mathbb{C}$. Thus

$$\int_{C_1} \frac{\cosh z}{z}\, dz = \int_{C_1} \frac{dz}{z} + \overset{0}{\cancel{\int_{C_1} g(z)\, dz}} = 2\pi i$$

Same value for C_2 !

g) $\int \frac{dz}{z^2(z^2+3)}$

Same technique as in d)

h) $\int \frac{\sin z}{z(z^2+2)}$

same technique as in d)

Remark: All problems from (13.6) can be solved much more efficiently using formula (13.20) from ~~the~~ book. Here is a few solutions:

b) $$\int_C \frac{\sin z}{z}\, dz = 2\pi i \sin(0) = 0$$

for both curves, as $\sin z$ is holomorphic in the whole $\mathbb{C}$

e) $$\int_C \frac{\sinh z}{z}\, dz = 2\pi i \sinh(0) = 0$$

for both curves, as $\sinh z$ is holomorphic in the whole $\mathbb{C}$

f) $$\int_C \frac{\cosh z}{z}\, dz = 2\pi i \cosh(0) = 2\pi i$$

for both curves, as $\cosh z$ is holomorphic in the whole $\mathbb{C}$

etc.

13.7 (was not assigned)

a) $$\int_C \frac{\sin z}{z^2}\, dz = \frac{2\pi i}{1!}\,(\sin z)'(0)$$

$$= \frac{2\pi i}{1!}\cos(0) = 2\pi i$$

b) $$\int_C \frac{\sin z}{z^3}\, dz = \frac{2\pi i}{2!}\,(\sin z)''(0) = -\pi i \sin(0) = 0$$

c) $$\int_C \frac{z^2+z}{(2z+1)^3}\, dz = \frac{1}{8}\int_C \frac{z^2+z}{\left(z+\frac{1}{2}\right)^3} = (*)$$

$2z+1 = 2\left(z+\frac{1}{2}\right)$ $\quad -\frac{1}{2} \in$ interior of unit circle!

$$(*) = \frac{1}{8}\,\frac{\cancel{2}\pi i}{\cancel{2}!}\,(z^2+z)''\left(-\frac{1}{2}\right)$$

$$= \frac{\pi i}{8}\cdot 2 = \frac{\pi i}{4}$$

d) $$\int_C \frac{e^z}{z^5}\, dz = \frac{2\pi i}{4!}\,\frac{d^4}{dz^4}(e^z)(0) = \frac{2\pi i}{24} = \frac{\pi i}{12}$$

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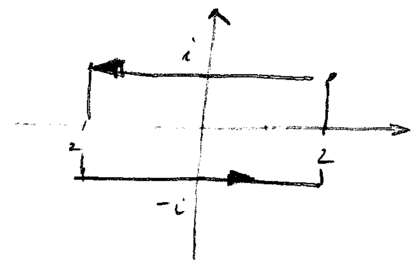
13.8

a) $\displaystyle\int_C \left(\frac{z+4}{z-4}\right)\frac{e^z}{\sin z}\,dz$



$$\left(\frac{z+4}{z-4}\right)\frac{e^z}{\sin z} = \frac{f(z)}{z}$$

where $\displaystyle f(z) = \frac{z+4}{z-4}\,\frac{e^z}{\sin z}\,z$

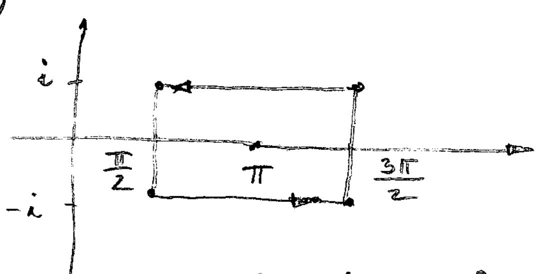
is analytic in a domain containing the curve

According to formula (13.20)

$$\int_C \frac{f(z)}{z} = 2\pi i\, f(0) = 2\pi i\,(-1)\cdot 1 = -2\pi i$$

(More precisely $\frac{z}{\sin z}$ is extended to analytic function by setting its value to 1 at 0, compare problem 13.6 b)

b)



Wrong!

Same approach. $\displaystyle\left(\frac{z+4}{z-4}\right)\frac{e^z}{\sin z} = \frac{f(z)}{z-\pi}$

where $\displaystyle f(z) = \frac{z+4}{z-4}\,e^z\,\frac{z-\pi}{\sin z} = -\frac{z+4}{z-4}\,e^z\,\frac{z-\pi}{\sin(z-\pi)}$

and function $\displaystyle\frac{z-\pi}{\sin(z-\pi)}$ can be extended by 1 to an analytic function in a domain containing the curve. Thus, by formula (13.20) again $\displaystyle\int_C \frac{f(z)}{z-\pi} = f(\pi) = -\frac{\pi+4}{\pi-4}\,e^{\pi}$

#

13.10

a) From (13.20)

$$|f^{(n)}(z)| \leq \frac{n!}{2\pi} \int_C \frac{|f(\zeta)|}{\rho^{n+1}}\, dc$$

$$\leq \frac{n!}{\cancel{2\pi}} \frac{M}{\rho^{n+1}} \cancel{2\pi} \rho = \frac{n!M}{\rho^n}$$

b) From (a) we have:

$$|f'(z)| \leq \frac{M}{\rho^n} \qquad \text{for every } \rho$$

Passing with $\rho \to \infty$ we get $f'(z) = 0$, for every z, so

$$f(z) = \text{const}$$

c) Proof by contradiction:

Suppose that $P(z)$ is nonzero everywhere. Then, since $P(z)$ is analytic, $1/P(z)$ must be analytic in the entire complex plane. At the same time

$$\frac{z^n}{P(z)} = \frac{z^n}{a_0 + a_1 z + \ldots + a_n z^n} =$$

$$= \frac{1}{\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + a_n} \longrightarrow \frac{1}{a_n} \quad \text{as } z \to \infty$$

So

$$\left| \frac{z^n}{P(z)} \right| \longrightarrow \frac{1}{|a_n|}$$

and, consequently, for any $\varepsilon > 0$

$$\left| \frac{1}{P(z)} \right| \leq \frac{1+\varepsilon}{|a_n|} \; \frac{1}{|z|^n}$$

for $|z|$ sufficiently large. This proves that $\frac{1}{P(z)}$ is bounded. By Liouville's theorem, therefore,

$$\frac{1}{P(z)} = \text{const}$$

and, consequently,

$$P(z) = \text{const}$$

, a contradiction

\#

13.11

Obviously not. Look for instance at problem 13.6 d)