Solutions!

ASE 380P2 ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANIS II

Exam 1. Monday, Mar 1, 2010

1. (a) State the Implicit Function Theorem (3 points).

   (b) Expand $z(x,y)$ about $(0,0)$, through terms of second order for the implicitly defined function,
   $$x^2 + y^2 + z^2 = 4, \quad z \geq 0$$
   (15 points).

   (c) Denoting the second-order approximation by $z_{approx}(x,y)$, sketch the original and approximating surfaces $z(x,y)$ and $z_{approx}(x,y)$, respectively (2 points).

2. (a) Consider standard spherical coordinates,
   $$\begin{cases} x = r\sin\psi\cos\theta \\ y = r\sin\psi\sin\theta \\ z = r\cos\psi \end{cases}$$
   Draw a picture representing the coordinates and the corresponding unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\psi, \boldsymbol{e}_\theta$ (3 points).

   (b) Assume that (r,ψ,θ) are functions of time t. Derive the formula for the velocity and acceleration vector in the curvilinear system of coordinates (10 points).

   (c) Use the formulas to compute the acceleration vector for a point moving on a sphere of radius R,
   $$r = R, \quad \psi = \frac{\pi}{2}t, \quad \theta = \pi t$$
   at time $t = 1$. Compute the tangential and normal acceleration at that moment (7 points).

3. (a) Define a solenoidal vector field and show that the following field is solenoidal (5 points).
   $$\boldsymbol{v} = (2y, x^2y^2, -2x^2yz)^T$$

   (b) Find a corresponding vector potential $\boldsymbol{\psi}$. Is the potential unique? Explain (15 points).

4. (a) Find a curve $y(x)$ of length L over $0 \leq x \leq 1$, with $y(0) = y(1) = 0$, so that the area under the curve is maximized. Show that, for $L \leq \pi/2$, the result is the circle

$$(x - A)^2 + (y - B)^2 = \lambda^2$$

where A, B are integration constants and λ is a Lagrange multiplier (17 points).

(b) What happens in the case when $L > \pi/2$? (3 points)

At $x = y = 0$ $\quad z = 2 > 0$

So:

$$\frac{\partial z}{\partial x} = -\frac{0}{2} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{0}{2} = 0$$

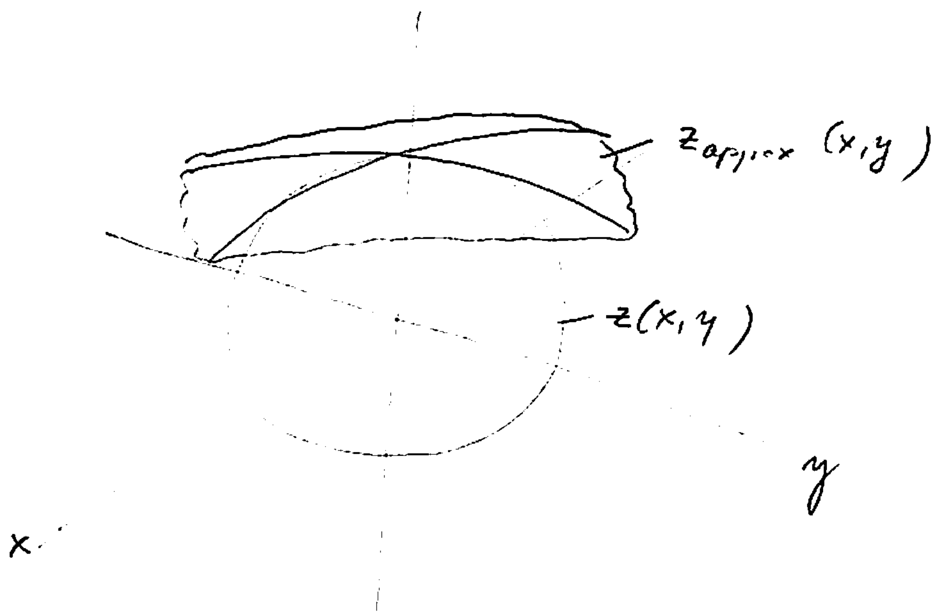
$$\frac{\partial^2 z}{\partial x^2} = -\frac{1}{2}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{1}{2}$$

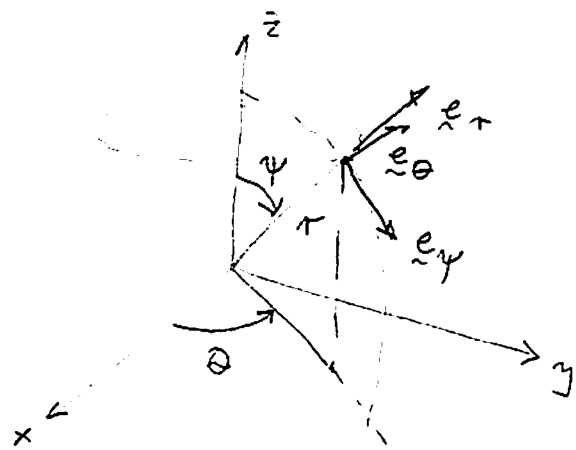
$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$z_{approx}(x,y) = 2 - \frac{1}{4}(x^2 + y^2)$$

c)



2)



$$e_r = \frac{\partial r}{\partial r} = (\sin\psi\cos\theta,\ \sin\psi\sin\theta,\ \cos\psi)$$

$$e_\psi = \frac{\partial r}{\partial \psi} = (\cos\psi\cos\theta,\ \cos\psi\sin\theta,\ -\sin\psi)$$

$$\frac{\partial r}{\partial \theta} = (-\sin\psi\sin\theta,\ \sin\psi\cos\theta,\ 0)$$

$$e_\theta = (-\sin\theta,\ \cos\theta,\ 0)$$

$$\frac{\partial e_r}{\partial \psi} = (\cos\psi\cos\theta,\ \cos\psi\sin\theta,\ -\sin\psi) = e_\psi$$

$$\frac{\partial e_r}{\partial \theta} = (-\sin\psi\sin\theta,\ \sin\psi\cos\theta,\ 0) = \sin\psi\ e_\theta$$

$$\frac{\partial e_\psi}{\partial \psi} = (-\sin\psi\cos\theta,\ -\sin\psi\sin\theta,\ -\cos\psi) = -e_r$$

$$\frac{\partial e_\psi}{\partial \theta} = (-\cos\psi\sin\theta,\ \cos\psi\cos\theta,\ 0) = \cos\psi\ e_\theta$$

$$\frac{\partial e_\theta}{\partial \theta} = (-\cos\theta,\ -\sin\theta,\ 0) = -\sin\psi\ e_r - \cos\psi\ e_\psi$$

$$r = r e_r$$

$$\dot{r} = \dot{r} e_r + r\frac{\partial e_r}{\partial \psi}\dot{\psi} + r\frac{\partial e_\theta}{\partial \theta}\dot{\theta}$$

$$= \underbrace{\dot{r}}_{v_r} e_r + \underbrace{r\dot{\psi}}_{v_\psi} e_\psi + \underbrace{r\sin\psi\,\dot{\theta}}_{v_\theta} e_\theta$$

$$\ddot{\underset{\sim}{r}} = \ddot{r}\,\underset{\sim}{e}_r + \dot{r}\left(\dot{\psi}\,\underset{\sim}{e}_\psi + \sin\psi\,\dot{\theta}\,\underset{\sim}{e}_\theta\right)$$

$$+ \dot{r}\dot{\psi}\,\underset{\sim}{e}_\psi + r\ddot{\psi}\,\underset{\sim}{e}_\psi + r\dot{\psi}\left(-\dot{\psi}\,\underset{\sim}{e}_r + \cos\psi\,\dot{\theta}\,\underset{\sim}{e}_\theta\right)$$

$$+ \left(\dot{r}\sin\psi\,\dot{\theta} + r\cos\psi\,\dot{\psi}\dot{\theta} + r\sin\psi\,\ddot{\theta}\right)\underset{\sim}{e}_\theta$$

$$+ r\sin\psi\,\dot{\theta}\left(-\sin\psi\,\underset{\sim}{e}_r - \cos\psi\,\underset{\sim}{e}_\psi\right)\dot{\theta}$$

$$= \Big[\underbrace{\ddot{r} - r\dot{\psi}^2 - r\sin^2\psi\,\dot{\theta}}_{a_r}\Big]\underset{\sim}{e}_r$$

$$+ \Big[\underbrace{2\dot{r}\dot{\psi} + r\ddot{\psi} - r\sin\psi\cos\psi\,\dot{\theta}^2}_{a_\psi}\Big]\underset{\sim}{e}_\psi$$

$$+ \Big[\underbrace{2\dot{r}\sin\psi\,\dot{\theta} + 2r\cos\psi\,\dot{\psi}\,\dot{\theta} + r\sin\psi\,\ddot{\theta}}_{a_\theta}\Big]\underset{\sim}{e}_\theta$$

$$r = R, \quad \dot{r} = \ddot{r} = 0$$

$$\psi = \frac{\pi}{2}t = \frac{\pi}{2} \quad \dot{\psi} = \frac{\pi}{2}, \quad \ddot{\psi} = 0$$

$$\theta = \pi t = \pi \quad \dot{\theta} = \pi, \quad \ddot{\theta} = 0$$

$$\begin{cases} a_r = -R\dfrac{\pi^2}{4} - R\pi = -R\pi\left(1 + \dfrac{\pi}{4}\right) \\ a_\psi = 0 \\ a_\theta = 0 \end{cases}$$

$V_r = 0$, $V_\psi = R\frac{\pi}{2}$, $V_\theta = R\pi$

So $\underset{\sim}{a} \perp \underset{\sim}{v}$ at $t = 1$ $\Rightarrow$ tangential acceleration is zero and normal acceleration $\underset{\sim}{a}_n = \underset{\sim}{a}$.

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3a) $\underset{\sim}{v}$ is solenoidal if $\operatorname{div} \underset{\sim}{v} = 0$

$$\begin{array}{r} \times \begin{pmatrix} \partial_x & \partial_y & \partial_z \end{pmatrix} \\ \begin{pmatrix} \psi_x & \psi_y & \psi_z \end{pmatrix} \\ \hline \end{array}$$

$$\left(\psi_{z,y} - \psi_{y,z} \, , \; \psi_{x,z} - \psi_{z,x} \, , \; \psi_{y,x} - \psi_{x,y} \right)$$

We need to find a vector potential $\underset{=}{\psi}(\psi_x, \psi_y, \psi_z)$ such that

$$\begin{cases} \psi_{z,y} - \psi_{y,z} = 2y \\ \psi_{x,z} - \psi_{z,x} = x^2y^2 \\ \psi_{y,x} - \psi_{x,y} = -2x^2yz \end{cases}$$

Try $\psi_y \equiv 0$ $\qquad \psi_{z,y} = 2y \Rightarrow \psi_z = y^2 + C(x,z)$

$\psi_{x,y} = 2x^2yz \Rightarrow \psi_x = x^2y^2z^2 + D(x,z)$

$$\psi_{x,z} - \psi_{z,x} = 2x^2y^2z + \frac{\partial D}{\partial z} - \frac{\partial C}{\partial x} = x^2y^2$$

Try $C \equiv 0 \Rightarrow \frac{\partial D}{\partial z} = x^2y^2 - 2x^2y^2z$

$\therefore \quad D = x^2y^2(z - z^2) + E(x)$

Set $E(x) \equiv 0$

So: $\underset{=}{\psi} = (x^2y^2z , 0 , y^2)$

Check:

$$\begin{array}{l} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \\ \times \left(x^2y^2z, 0, y^2 \right) \\ \hline \left(2y, x^2y^2, -2x^2yz \right) \end{array}$$

Vector potential is known only up to a gradient, since $\underset{\sim}{\nabla} \times \underset{\sim}{\nabla} \varphi = \underset{\sim}{0}$

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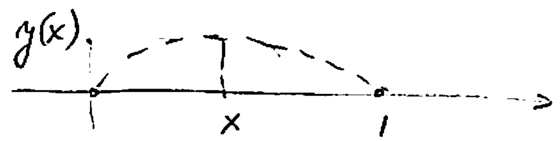
(4)



$$L(y, \lambda) = \int_0^1 y(x)\, dx - \lambda \left(\int_0^1 \sqrt{y'^2+1}\, dx - L \right)$$

<u>E-L equation:</u>

$$\lambda \left(\frac{y'}{\sqrt{y'^2+1}} \right)' + 1 = 0$$

$$\frac{y'}{\sqrt{y'^2+1}} = -\frac{x}{\lambda} + c$$

$$\frac{y'^2}{1+y'^2} = \left(\frac{x}{\lambda} - c \right)^2$$

$$y'^2 = \frac{\left(\frac{x}{\lambda} - c \right)^2}{1 - \left(\frac{x}{\lambda} - c \right)^2} = \frac{(x-\lambda c)^2}{\lambda^2 - (x-\lambda c)^2}$$

$$y' = \pm \frac{x - \lambda c}{\left[\lambda^2 - (x-\lambda c) \right]^{\frac{1}{2}}}$$

$\rightarrow$

$$y(x) = \pm \int \frac{x-\lambda c}{[\lambda^2-(x-\lambda c)^2]^{\frac{1}{2}}}\,dx + B$$

$$\lambda^2-(x-\lambda c)^2 = t$$

$$-2(x-\lambda c)\,dx = dt$$

$$= \mp \frac{1}{2}\int t^{-\frac{1}{2}}\,dt + B = \mp t^{\frac{1}{2}} + B$$

$$= \mp \left[\lambda^2-(x-\lambda c)^2\right]^{\frac{1}{2}} + B$$

so;

$$(y-B)^2 = \lambda^2 - (x-\underbrace{\lambda c}_{A})^2$$

$$(x-A)^2+(y-B)^2 = \lambda^2$$

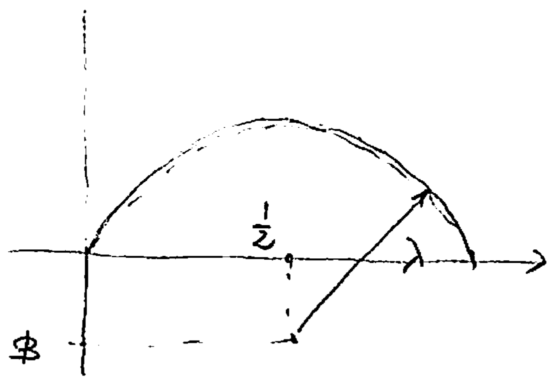
Boundary conditions:

$$y(0)=0 \quad \Rightarrow \quad A^2+B^2 = \lambda^2$$

$$y(1)=0 \quad \Rightarrow \quad (A-1)^2+B^2 = \lambda^2$$

$$A^2-(A-1)^2 = (\cancel{A}-\cancel{A}+1)(A+A-1)=0$$

$$\therefore \underline{A=\tfrac{1}{2}}$$



Coordinate B of the center and radius depend upon the length L. For $L > \frac{\pi}{2}$, the circle does not represent longer graph of a function (Problem has to be reformulated).