Homework #26

26. 3, 6, 8, 10, 11, 12

26. 3

a) Proof by contradiction: Suppose that the equations

$$\begin{cases} u_{,t} = \alpha^2 u_{,xx} & 0<x<\ell,\ t>0 \\ u(0,t) = a(t) & \\ u(\ell,t) = b(t) & t>0 \\ u(x,0) = f(x) & 0<x<\ell \end{cases}$$

are satisfied by two, different solutions u_1 and u_2. Subtracting the equations from each other and denoting the difference $u_1(x,t) - u_2(x,t) =: w(x,t)$ we get

$$\begin{cases} w_{,t} = \alpha^2 w_{,xx} \\ w(0,t) = w(\ell,t) = 0 \\ w(x,0) = 0 \end{cases}$$

Multiplying the PDE by $w(x,t)$, integrating in x and integrating the right-hand side by parts (using the BC!), we get

$$\frac{d}{dt}\frac{1}{2}\int_0^\ell w^2\,dx = \alpha^2\int_0^\ell w_{,xx}\,w\,dx = -\alpha^2\int_0^\ell w_{,x}^2\,dx$$

Integrating in time and using the IC, we arrive at

$$\frac{1}{2}\int_0^\ell w^2(x,t)\,dx = -\alpha^2\int_0^t\int_0^\ell w_{,x}^2\,dx$$

Now, the left-hand side is non-negative and the right-hand side is non-positive. Thus, it must be $w(x,t) \equiv 0$ and consequently $u_1(x,t) = u_2(x,t)$.

b) The proof is immediate. Function $w = u_1 - u_2$ satisfies the homogeneous BC and IC, so, according to the maximum and minimum principles, the max value of w = min value of $w = 0$, and consequently $w \equiv 0$.

c) As in a) we introduce $w = u_1 - u_2$ and find out that w satisfies the homogeneous equations

$$\begin{cases} \dfrac{\partial w}{\partial t} = \alpha^2 \Delta w & t>0, \ x \in D & \text{PDE} \\ a w + b \dfrac{\partial w}{\partial n} = 0 & x \in S = \partial D, \ t > 0 & \text{BC} \\ w \equiv 0 & t = 0, \ x \in D & \text{IC} \end{cases}$$

Following the same steps as in a) we get

$$\frac{d}{dt} \frac{1}{2} \int_D w^2 \, d\underset{\sim}{x} = \alpha^2 \int_D \Delta w \, w \, d\underset{\sim}{x} =$$

$$= -\alpha^2 \int_D \underset{\sim}{\nabla} w \cdot \underset{\sim}{\nabla} w \, d\underset{\sim}{x} + \alpha^2 \int_{S} \frac{\partial w}{\partial n} w \, dS$$

Splitting $S = S_1 + S_2$, $S_1 = \{x : b = 0\}$, $S_2 = \{x : b \neq 0\}$

$$= -\alpha^2 \int_D \underset{\sim}{\nabla} w \, \underset{\sim}{\nabla} w \, d\underset{\sim}{x} + \alpha^2 \int_{S_2} \left(-\frac{a}{b}\right) w^2 \, ds \leq 0$$

Integrating in time

$$\int_D w^2(x,t) \, dx \leq 0 \quad \Rightarrow \quad \underline{w(x,t) \equiv 0}$$

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26.6

$$u(x,t) = \int_{-\infty}^{\infty} H(\xi)\, \frac{e^{-(\xi-x)^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}}\, d\xi$$

$$= \int_{0}^{\infty} \frac{e^{-(\xi-x)^2/4\alpha^2 t}}{2\alpha\sqrt{\pi t}}\, d\xi \qquad = *$$

Recall (page 11) $\quad \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-\eta^2} d\eta \qquad 0 \le x < \infty$

$$\frac{\xi - x}{2\alpha\sqrt{t}} = \eta \quad \Rightarrow \quad \frac{d\xi}{2\alpha\sqrt{t}} = d\eta$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\frac{x}{2\alpha\sqrt{t}}}^{\infty} e^{-\eta^2} d\eta = \frac{1}{\sqrt{\pi}} \left(\int_{-\frac{x}{2\alpha\sqrt{t}}}^{0} + \int_{0}^{\infty} \right)$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{2\alpha\sqrt{t}}} e^{-\eta^2} d\eta + \frac{1}{\sqrt{\pi}} \int_0^{\infty} e^{-\eta^2} d\eta$$

$$= \frac{1}{2} \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}} \right) + \frac{1}{2}$$

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26.8

a) $f(-x) = f(x) \quad \forall x$

Using formula (26.55)

$$u(-x,t) = \int_{-\infty}^{\infty} f(\xi)\, U(\xi + x; t)\, d\xi$$

(change of variables $\xi = -\xi$)

$$= \int_{-\infty}^{\infty} f(-\xi)\, U(-\xi + x, t)\, d\xi$$

$\left(f(-\xi) = f(\xi),\quad U(-\xi+x,t) = U(\xi - x, t) \right)$

$$= \int_{-\infty}^{\infty} f(\xi)\, U(\xi - x, t)\, d\xi = u(x,t)$$

Same technique for the antisymmetric case.

b) Use the d'Alembert solution

$$y(-x,t) = \frac{f(-x+at) + f(-x-at)}{2}$$

$$= \frac{f(-(x-at)) + f(-(x+at))}{2}$$

$$= \frac{f(x-at) + f(x-at)}{2} = y(x,t)$$

Same technique for the antisymmetric case.

26.10

$$\begin{cases} u_{,t} = \alpha^2 u_{,xx} & 0<x<l, \quad t>0 \\ u(0,t) = u(l,t), & u_x(0,t) = u_x(l,t) \\ u(x,0) = f(x) \end{cases}$$

$$u(x,t) = v(x)\, w(t)$$

$$v w' = \alpha^2 v'' w$$

$$\frac{1}{\alpha^2}\frac{w'}{w} = \frac{v''}{v} = c$$

$$v'' - cv = 0$$

This equation has periodic solutions only for

$$c = -k_n^2 \quad , \quad k_n = 2n\pi \frac{x}{l} \quad , \quad n = 0, 1, 2, \ldots$$

Then $v(x) = C \cos k_n x + D \sin k_n x$

$$w' + \alpha^2 k_n^2 w = 0 \quad \therefore \quad w(t) = Ce^{-\alpha^2 k_n^2 t}$$

Finally $u(x,t) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos k_n x + D_n \sin k_n x \right) e^{-\alpha^2 k_n^2 t}$

Imposing IC:

$$u(x,0) = C_0 + \sum_{n=1}^{\infty} \left(C_n \cos k_n x + D_n \sin k_n x \right) = f(x)$$

1. Integrating both sides $\int_0^l dx$

$$C_0 = \frac{1}{L}\int_0^l f(x)\, dx$$

2. Integrating both sides: $\int_0^l \cos k_n x \, dx$

$$\int_0^l \cos^2 k_n x = \int_0^l \frac{1+\cos 2k_n x}{2}\, dx = \frac{1}{2L}$$

$$2\cos^2\alpha - 1 = \cos 2\alpha$$

$$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$$

we get: $C_n = \dfrac{1}{2L}\displaystyle\int_0^l f(x) \cos k_n x \, dx$

3. Similarly $D_n = \dfrac{1}{2L}\displaystyle\int_0^l f(x) \sin k_n x \, dx$

Steady-state solution $u(x) \equiv C_0 = \dfrac{1}{L}\displaystyle\int_0^l f(x)\, dx$.

$\#$

26.11

$$\begin{cases} u_{,t} = \alpha^2 u_{,xx} & 0<x<\infty, \quad t>0 \\ u(x,0) = 0 & \\ u(0,t) = 100 & \end{cases}$$

Method 1

- Laplace transforming in x would require two boundary conditions at $x=0$, since

$$\left(\mathcal{L}\,\frac{\partial^2 u}{\partial x^2}\right)(s,t) = s^2\bar{u}(s,t) - s\,\underset{\underset{100}{\parallel}}{u(0,t)} - \underbrace{\frac{\partial u}{\partial x}(0,t)}_{\text{unknown !}}$$

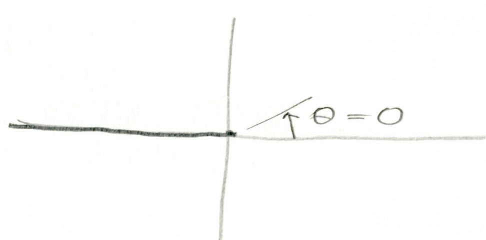
- Laplace transforming in t

$$s\,\bar{u}(x,s) - \cancelto{0}{u(x,0)} = \alpha^2\,\bar{u}_{,xx}(x,s)$$

$$\bar{u}_{,xx} + \frac{s}{\alpha^2}\,\bar{u} = 0$$

$$\bar{u}(x,s) = C e^{-\frac{\sqrt{s}}{\alpha}x} + D e^{+\frac{\sqrt{s}}{\alpha}x}$$

Branch cut for $\sqrt{s}$



8

Laplace transforming the BC

$$\bar{u}(0,s) = 100\int_0^\infty e^{-st}\,dt = \frac{100}{-s}\, e^{-st}\Big|_0^\infty = \frac{100}{s} \qquad (1)$$

($\underline{\text{Re } s > 0}$!)

Setting up the second BC at ∞

$$\lim_{x\to\infty} u(x,t) = 0$$

and Laplace transforming it, we have

$$\bar{u}(x,s) = \int_0^\infty u(x,t)\, e^{-(\text{Re}s + i\,\text{Im}\,s)t}\,dt = \int_0^\infty u(x,t)\, e^{-\text{Re}s\, t}\, e^{i\,\text{Im}\,s\,t} \qquad (2)$$

$\longrightarrow 0$ as $x \to 0$

Recall! $s = |s|\, e^{i\theta} \quad \Rightarrow \quad \sqrt{s} = \sqrt{|s|}\, e^{i\frac{\theta}{2}}$

Thus for $\theta \in (-\pi, \pi)$ (See the selected branch cut!) $\theta/2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and consequently $\text{Re}\sqrt{s} > 0$!

This implies that

$$e^{+\frac{\sqrt{s}}{\alpha}x} = e^{\frac{x}{\alpha}(\text{Re}\sqrt{s} + i\,\text{Im}\sqrt{s})}$$
$$= e^{\frac{x}{\alpha}\text{Re}\sqrt{s}}\; e^{i\,\text{Im}\sqrt{s}\,\frac{x}{\alpha}}$$

and consequently $\left|e^{\frac{\sqrt{s}}{\alpha}x}\right| \longrightarrow \infty$ as $x\to\infty$

In the same way $\left|e^{-\frac{\sqrt{s}}{\alpha}x}\right| \longrightarrow 0$ as $x\to\infty$

Therefore (2) implies that $D = 0$. Implementing (1) we get

$$C = \frac{100}{s}$$

and finally:

$$\bar{u}(x,s) = \frac{100}{s} e^{-\frac{\sqrt{s}}{\alpha} x}$$

Inverting,

$$u(x,t) = \mathcal{L}^{-1}\left(\frac{100}{s} e^{-\frac{\sqrt{s}}{\alpha} x}\right)$$

But $\mathcal{L}^{-1}\left(e^{-\frac{\sqrt{s}}{\alpha}x}\right) = \dfrac{\frac{x}{\alpha} e^{-\frac{x^2}{4\alpha^2 t}}}{2\sqrt{\pi}\, t^{3/2}}$ (see table)

$$= \frac{x}{2\alpha\sqrt{\pi}} \frac{e^{-\frac{x^2}{4\alpha^2 t}}}{t^{3/2}}$$

and consequently (page 115, property d)

$$u(x,t) = \frac{100\,x}{2\alpha\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4\alpha^2\tau}}}{\tau^{3/2}}\, d\tau$$

Substituting:

$$\frac{x^2}{4\alpha^2\tau} = u^2 \implies \frac{x}{2\alpha\tau^{\frac{1}{2}}} = u$$

$$-\frac{x^2}{4\alpha^2}\frac{1}{\tau^2}\, d\tau = 2u\, du = \frac{\cancel{2}x}{\cancel{2}\cancel{\alpha}\tau^{\frac{1}{2}}}\, du$$

$$-\frac{x}{4\alpha}\frac{1}{\tau^{\frac{3}{2}}}\, d\tau = du$$

Bounds:

$\tau \to 0 \quad \Rightarrow \quad u \to \infty$

$\tau = t \quad \Rightarrow \quad u = \dfrac{x}{2\alpha t^{\frac{1}{2}}} = \dfrac{x t^{-\frac{1}{2}}}{2\alpha}$

So:

$$u(x,t) = \frac{100 \cancel{x}}{\cancel{\alpha}} \frac{1}{\cancel{2}\sqrt{\pi}} \int_{\frac{x t^{-\frac{1}{2}}}{2\alpha}}^{\infty} \frac{\overset{2}{\cancel{4\alpha}}}{\cancel{x}} e^{-u^2} du$$

$$= 100 \frac{2}{\sqrt{\pi}} \int_{\frac{x t^{-\frac{1}{2}}}{2\alpha}}^{\infty} e^{-u^2} du$$

$$= 100 \operatorname{erfc}\left(\frac{x t^{-\frac{1}{2}}}{2\alpha}\right)$$

(see page 11 for def of erfc function!)

b) Similarity transformation:

Set $\eta = x t^{\alpha}$ and look for $u(x,t) = u(\eta)$

Then: $u_{,t} = \frac{du}{d\eta}\frac{d\eta}{dt} = \alpha x t^{\alpha-1}\frac{du}{d\eta} = \alpha\frac{\eta}{t}\frac{du}{d\eta}$

$$u_{,x} = \frac{du}{d\eta}\frac{d\eta}{dx} = t^{\alpha}\frac{du}{d\eta}$$

$$u_{,xx} = t^{\alpha}\frac{d^2u}{d\eta^2}\frac{d\eta}{dx} = t^{2\alpha}\frac{d^2u}{d\eta^2}$$

So: $u_{,t} = a^2 u_{,xx}$ implies

$$\alpha\frac{\eta}{t}\frac{du}{d\eta} = a^2 t^{2\alpha}\frac{d^2u}{d\eta^2}$$

or $$\alpha\eta\frac{du}{d\eta} = a^2 t^{2\alpha+1}\frac{d^2u}{d\eta^2}$$

Selecting $2\alpha+1=0$, i.e. $\alpha = -\frac{1}{2}$ we have

$$-\frac{1}{2}\eta\frac{du}{d\eta} = a^2\frac{d^2u}{d\eta^2}$$

Denoting $\frac{du}{d\eta} = v$, we have

$$-\frac{1}{2}\eta v = a^2\frac{dv}{d\eta}$$

or $$\frac{dv}{v} = -\frac{1}{2a^2}\eta\, d\eta$$

so $$\ln v = -\frac{\eta^2}{4a^2} + C$$

$$v(\eta) = C e^{-\frac{\eta^2}{4a^2}}$$

And $u(\eta) = C \int_0^{\eta} e^{-\frac{\xi^2}{4a^2}} d\xi + D = \underbrace{C\, 2a}_{\text{new } C} \int_0^{\frac{\eta}{2a}} e^{-\beta^2} d\beta + D$

$\frac{\xi}{2a} = \beta$

$\frac{1}{2a} d\xi = d\beta$ $\qquad = \underset{\uparrow \atop \text{new } C}{C} \operatorname{erf}\left(\frac{\eta}{2a}\right) + D$

So $u(x,t) = C \operatorname{erf}\left(\frac{x t^{-\frac{1}{2}}}{2a}\right) + D$

Implementing boundary condition :

$x \to 0 \Rightarrow \frac{x t^{-\frac{1}{2}}}{2a} \to 0 \Rightarrow \operatorname{erf}\left(\frac{x t^{-\frac{1}{2}}}{2a}\right) \to 0$

$\Rightarrow D = 100$

Implementing initial condition :

$t \to 0 \Rightarrow \frac{x t^{-\frac{1}{2}}}{2a} \to \infty \Rightarrow \operatorname{erf}\left(\frac{x t^{-\frac{1}{2}}}{2a}\right) \to 1$

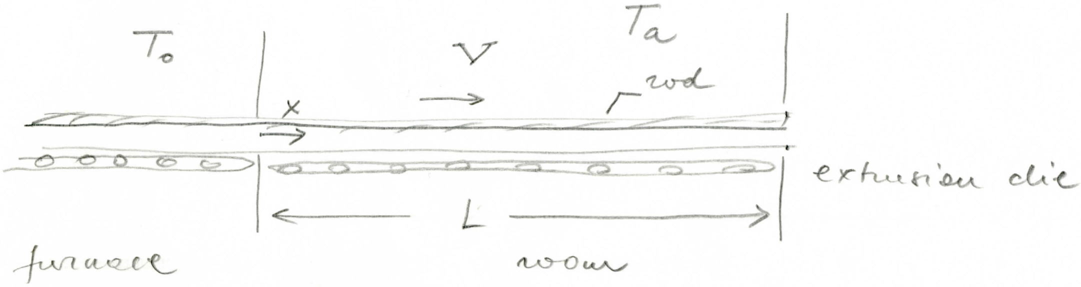
$\Rightarrow C = -100$

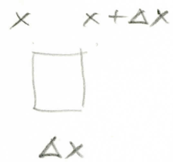
Finally : $u(x,t) = 100 \left[1 - \operatorname{erf}\left(\frac{x t^{-\frac{1}{2}}}{2a}\right)\right]$

$= 100 \operatorname{erfc}\left(\frac{x t^{-\frac{1}{2}}}{2a}\right)$

(See page 11 for definitions of erf and erfc functions).

26.12





$$T = T(x,t) \qquad x = x(t)$$

$$\frac{D}{Dt}\left(\rho \frac{\pi d^2}{4}\Delta x\, c\, T(x(t),t)\right) + \pi d\,\Delta x\, h(T-T_a)$$

$$= \frac{\pi d^2}{4} k\left(\frac{\partial T}{\partial x}(x+\Delta x,t) - \frac{\partial T}{\partial x}(x,t)\right) \qquad \text{(Fourier's law)}$$

$$= \frac{\pi d^2}{4} k\left(\frac{\partial^2 T}{\partial x^2}(x,t) + O(\Delta x)\right)\Delta x$$

Subdividing by Δx and passing to the limit with $\Delta x \to 0$,

$$\frac{D}{Dt}\left(\rho \frac{\cancel{\pi} d^{\cancel{2}}}{4} c\, T(x(t),t)\right) + \cancel{\pi}\cancel{d}\, h(T-T_a) = \frac{\cancel{\pi} d^{\cancel{2}}}{4} k \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\rho d c}{4}\left(\frac{\partial T}{\partial x}\underbrace{\frac{dx}{dt}}_{= V} + \frac{\partial T}{\partial t}\right) + h(T-T_a) = \frac{dk}{4}\frac{\partial^2 T}{\partial x^2}$$

So finally:

$$\frac{\partial^2 T}{\partial x^2} = \underbrace{\frac{\rho c}{k}}_{= \frac{1}{\alpha^2}}\left(\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x}\right) + \underbrace{\frac{4h}{kd}}_{= \sigma}(T-T_a)$$

For a steady-state solution:

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha 2}\left(v \frac{\partial T}{\partial x}\right) + \sigma (T - T_a)$$

Homogeneous eqn:

$$\frac{\partial^2 T}{\partial x^2} - \frac{v}{\alpha^2}\frac{\partial T}{\partial x} - \sigma T = 0$$

$$r^2 - \frac{v}{\alpha^2} r - \sigma = 0$$

$$\Delta = \frac{v^2}{\alpha^4} + 4\sigma$$

$$r_{1,2} = \frac{1}{2}\left(\frac{\sqrt{v}}{\alpha} \mp \sqrt{\frac{v^2}{\alpha^4} + 4\sigma}\right)$$

General solution

particular solution to the nonhom. eqn.

$$T(x) = A e^{r_1 x} + B e^{r_2 x} + T_a$$

$$T(0) = A + B + T_a = T_0 \Rightarrow A + B = T_0 - T_a$$

$$T'(x) = A r_1 e^{r_1 x} + B r_2 e^{r_2 x}$$

$$T'(0) = A r_1 + B r_2 = 0 \Rightarrow B = -A \frac{r_1}{r_2}$$

so: $$A\left(1 - \frac{r_1}{r_2}\right) = T_0 - T_a \Rightarrow A = \frac{r_2 (T_0 - T_a)}{r_2 - r_1}$$

and finally: $$\underline{T(x) = \frac{r_2 (T_0 - T_a)}{r_2 - r_1} e^{r_1 x} - \frac{r_1 (T_0 - T_a)}{r_2 - r_1} e^{r_2 x} + T_a}$$

the second formulation corresponds to the case of an infinite beam with half of it ($x<\infty$) sitting in the furnace and the other half ($x>\infty$) in the room temperature. The boundary conditions at $-\infty$ and $+\infty$ enforce the furnace and room temperature and the two compatibility conditions correspond to the assumption that both temperature and heat flux (temperature gradient) behave continuously at $x=0$. Solving the two equations we get

$$T_1(x) = A_1 e^{r_1^\circ x} + B_1 e^{r_2^\circ x} + T_0 \qquad x \leq 0$$

and

$$T_2(x) = A_2 e^{r_1 x} + B_2 e^{r_2 x} + T_a \qquad x \geq 0$$

with r_1 and r_2 as before and r_1°, r_0° given by the same formulas but σ_0 replacing σ.

$T_1(-\infty) = T_0$ implies $A_1 = 0$ since $r_1^\circ x > 0$

$T_2(\infty) = T_a$ implies $B_2 = 0$ since $r_1 x < 0$

The two compatibility equations have to be solved for B_1 and A_2

$$\begin{cases} B_1 + T_0 = A_2 + T_a \\ B_1 r_2^\circ = A_2 r_1 \end{cases} \Rightarrow A_2, B_1 = \ldots$$

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