Chapter 10

The Calculus of Variations

One of the first problems discussed in a beginning course in calculus is the determination of the maxima and minima of a given function $F(x)$. Historically, too, such determination was an early problem in the development of the calculus. Newton, in a paper written in 1671 (published in 1736), argued that at a maximum or minimum the rate of change of $f(x)$ must be zero, and Leibnitz in 1684 made the equivalent statement that the tangent line must be horizontal there, statements that seem all too elementary to us now and that, to be appreciated, must be considered in the context of seventeenth-century mathematics.

A few years later Newton investigated a difficult problem concerning the drag on an axisymmetric body moving through a resisting medium. With x as the axial coordinate and the radius of the body denoted as $y(x)$, the problem was to find the function $y(x)$ connecting two given endpoints

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2, \tag{10.1}$$

such that the integral

$$I = \int_{x_1}^{x_2} \frac{yy'^3}{1 + y'^2}\, dx, \tag{10.2}$$

which (according to Newton's simplified physical model) is proportional to the drag, is a minimum.

Note carefully that Newton's problem involved the minimization not of a function but of a *functional* $I(y)$ over a certain domain, say the class of functions $y(x)$ that satisfy the end conditions (10.1) and are (at least) differentiable over $x_1 \leq x \leq x_2$. Although he found the solution, Newton expressed no interest in pursuing a generalization to a wider class of problems. (This situation reflects the fact that more interest attached to specific problems in those days than to the systematic development of a broad area.)

Not until 1696 was attention focused on these problems. In that year John Bernoulli put before the scholars of his time his *brachistochrone problem*—that is, finding the path from one given point to another, in the same vertical plane, along which a particle will descend (due to gravity) in the shortest time. The problem generated considerable interest and solutions were offered by Newton, Leibnitz, L'Hospital, Bernoulli himself, and his older brother James. Finally, a subsequent student of John Bernoulli, Leonhard Euler (1707–83), took James' solution and developed the *calculus of variations* as it is known today.

10.1 FUNCTIONS OF ONE OR MORE VARIABLES

Before looking at *functionals*, consider the simplest case—that is, a *function* of one variable, $F(x)$. As usual, x is a real variable and F is to be real valued.

We say that $F(x)$ has an *absolute maximum* over an interval I if there exists a point x_0 such that

$$F(x) \leq F(x_0) \tag{10.3}$$

for all x in I and a *relative* or *local maximum* at x_0 if there is some neighborhood $N(x_0)$, which is a subset of I, such that (10.3) holds in $N(x_0)$. The same is true for absolute and relative *minima* with (10.3) changed to $F(x) \geq F(x_0)$.

We are concerned here only with *relative* maxima and minima and so will dispense with the adjective "relative"; from here on "max" and "min" will mean relative maximum and relative minimum, respectively. Furthermore, since $N(x_0)$ is contained in I, it follows that we will consider only cases where x_0 is an interior point of I.

THEOREM 10.1. In order for $F(x)$ to attain a max or min at an interior point x_0 of I, it is necessary that $F'(x_0)$ either vanish or fail to exist.

That these conditions are not also sufficient is apparent from Fig. 10.1. F' fails to exist at P and Q and vanishes at O and R; yet the function has neither maximum nor minimum at the points O and Q.

As *sufficient* conditions, we state without proof[1]

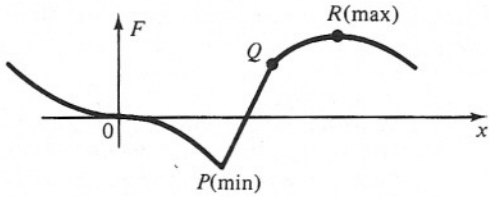

Figure 10.1. Max, min, and neither.

[1]For proof of Theorems 10.1, 10.2, and 10.4, as well as further details, see T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, Reading, Mass., 1957, pp. 91, 92, 148–157.

THEOREM 10.2. Suppose that

$$F'(x_0) = F''(x_0) = \cdots = F^{(n-1)}(x_0) = 0 \quad \text{but} \quad F^{(n)}(x_0) \neq 0,$$

where $F^{(n)}(x)$ is continuous in the interval and x_0 is an interior point. Then if n is even, F has a min at x_0 if $F^{(n)}(x_0) > 0$ and a max at x_0 if $F^{(n)}(x_0) < 0$. If n is odd, F has neither a max nor a min at x_0.

Extending the discussion to the case of n independent variables, $F(x_1, \ldots, x_n)$, we state (without proof) two additional theorems. First, some *necessary* conditions.

THEOREM 10.3. In order for $F(x_1, \ldots, x_n) \equiv$ "$F(P)$" to attain a max or min at an interior point P_0 of the region, it is necessary that $\partial F/\partial x_1, \ldots, \partial F/\partial x_n$ all vanish at P_0 or that some of them fail to exist there.

That these conditions are not also sufficient follows from the examples $F(x, y) = xy$ and $G(x, y) = x^3 + |y|$, say; $F_x(0, 0) = F_y(0, 0) = G_x(0, 0) = 0$ and $G_y(0, 0)$ fails to exist; yet both F and G have neither max nor min at $(0, 0)$.

In stating *sufficient* conditions, we limit ourselves to functions of *two* variables.

THEOREM 10.4. Suppose that

$$F_x(a, b) = F_y(a, b) = 0 \quad \text{and} \quad F_{xy}^2(a, b) - F_{xx}(a, b)F_{yy}(a, b) \equiv D,$$

where $F(x, y)$ has continuous second-order partials in the region and (a, b) is an interior point. If

(i) $F_{xx}(a, b) < 0$ and $D < 0$, then F has a max at (a, b).
(ii) $F_{xx}(a, b) > 0$ and $D < 0$, then F has a min at (a, b).
(iii) $D > 0$, then F has neither a max nor a min at (a, b).

The determinant

$$\begin{vmatrix} F_{xx}(a, b) & F_{xy}(a, b) \\ F_{yx}(a, b) & F_{yy}(a, b) \end{vmatrix} = -D$$

is sometimes called the **Hessian** of F.

Note that if $D = 0$, Theorem 10.4 provides no information and the nature of F at (a, b) will be dictated by the values of third- or higher-order partials. Sometimes the situation is transparent. For instance, $F(x, y) = x^4 + y^4$ has $F_x(0, 0) = F_y(0, 0) = 0$, but $D = 0$, so that Theorem 10.4 does not apply. Nevertheless, it is obvious from inspection that F has a min at $(0, 0)$. Similarly, it is apparent that $F = -(x^4 + y^4)$ has a max at $(0, 0)$.

Example 10.1. Consider $F(x, y) = x^2 - y^2$. Then $F_x = 2x = 0$ on the line $x = 0$ and $F_y = -2y = 0$ on the line $y = 0$. Thus $F_x = F_y = 0$ only at the origin $(0, 0)$. Since the second-order partials are continuous everywhere and $D = 0 - (2)(-2) = 4 > 0$, it follows from Theorem 10.4 that the "flat spot" at the origin is neither a max nor a min. Note that $F(x, 0) = x^2$ is a "valley" and that $F(0, y) = -y^2$ is a "hill"; therefore the F surface is shaped somewhat like a "saddle" in the neighborhood of the origin, which, in fact, is called a **saddlepoint** (Fig. 10.2).

COMMENT. Note that our $F(x, y) = x^2 - y^2$ happens to be **harmonic**; that is, it has continuous second-order partials and satisfies Laplace's equation, $F_{xx} + F_{yy} = 0$. More generally, suppose that $F(x, y)$ is *any* nonconstant harmonic function in a given region R and consider any point a, b in R at which $F_x = F_y = 0$. Then since $F_{xx} + F_{yy} = 0$, it follows that F_{xx} and F_{yy} are of opposite sign, so that $D > 0$, and hence a, b is neither a max nor a min (unless $F_{xy} = F_{xx} = F_{yy} = 0$ at a, b, in which case no conclusion can be drawn). Consider this result in the context of *heat conduction*, for instance. From (9.51) we see that $F(x, y)$ can be regarded as a steadystate temperature field in R. The fact that F cannot have a max (or min) inside R makes sense physically. Suppose that F did have a max at some point in R. Then surely heat would flow away from this point and the temperature F there would drop, thereby contradicting the assumption that F is in steady state. ∎

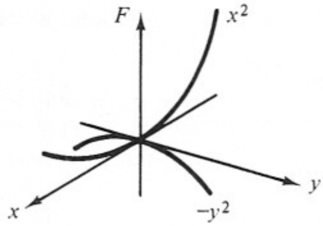

Figure 10.2. Saddlepoint.

10.2 CONSTRAINTS; LAGRANGE MULTIPLIERS

Example 10.2. Find the point(s) on the ellipse $5x^2 - 6xy + 5y^2 = 8$ that is closest to the origin; that is, find x, y such that $(x^2 + y^2)^{1/2}$ is a min, subject to the *constraint* $5x^2 - 6xy + 5y^2 = 8$. Equivalently, we want

$$F(x, y) = x^2 + y^2 = \text{min}, \tag{10.4}$$

subject to the constraint

$$g(x, y) = 5x^2 - 6xy + 5y^2 = 8. \tag{10.5}$$

Notice that if we seek x, y by setting

$$F_x = 2x = 0, \qquad F_y = 2y = 0,$$

we arrive at $x = y = 0$, which certainly minimizes F but which does not satisfy the constraint (10.5)! That is, the point $x = y = 0$ does not lie on the ellipse $5x^2 - 6xy + 5y^2 = 8$.

The key fact is that whether or not the x, y variables are independent, the fundamental necessary condition for a differentiable function $F(x, y)$ to have a max or min at a particular point x, y is that the differential $dF = 0$ there. Now

$$dF = \nabla F \cdot d\mathbf{s}. \tag{10.6}$$

If the variables x, y are *independent*, the orientation of $d\mathbf{s}$ is arbitrary, and so from (10.6) $dF = 0$ implies $\nabla F = 0$ or $F_x = F_y = 0$.[2] Then $dF = 0$ and $\nabla F = 0$ are equivalent and can be used interchangeably. But if x, y are *dependent*, the orientation of $d\mathbf{s}$ is *not* arbitrary and the condition $dF = 0$ for a max or min does *not* imply that $\nabla F = 0$.

[2]If the variables x, y are independent and $dF = F_x\,dx + F_y\,dy = 0$ at a particular point, it must be true that $F_x = F_y = 0$ there, since dx, dy are independent increments; if, for example, we choose $dx = 0$ and $dy \neq 0$, it follows from $dF = F_x\,dx + F_y\,dy = 0$ that $F_y = 0$, whereas the choice $dx \neq 0$, $dy = 0$ reveals that $F_x = 0$.

To illustrate, let us return to the present example. We have

$$dF = 2x\,dx + 2y\,dy = 0. \tag{10.7}$$

Instead of dx and dy being independent (i.e., $d\mathbf{s}$ being arbitrary), in which case it would follow that $F_x = 2x = 0$ and $F_y = 2y = 0$ (i.e., $\nabla F = 0$), dx and dy are related through the constraint (10.5), or

$$dg = (10x - 6y)\,dx + (10y - 6x)\,dy = 0. \tag{10.8}$$

Assuming, tentatively, that $10y - 6x$ is not zero (at the desired minimum), it follows from (10.8) that

$$dy = \frac{10x - 6y}{6x - 10y}\,dx,$$

and inserting this result into (10.7),

$$dF = \left[2x + 2y\left(\frac{5x - 3y}{3x - 5y}\right)\right] dx = 0.$$

Since dx all by itself *is* arbitrary, then

$$2x + 2y\left(\frac{5x - 3y}{3x - 5y}\right) = 0$$

or $$3x^2 - 5xy + 5xy - 3y^2 = 0$$

or $$y = \pm x.$$

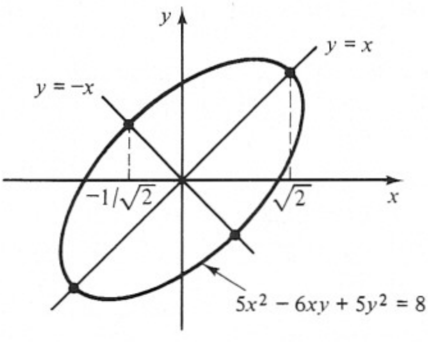

Figure 10.3.

Thus the points that we seek lie on the lines $y = \pm x$. Yet they must also lie on the ellipse (10.5), and so they are the intersections $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, 1/\sqrt{2})$, as indicated in Fig. 10.3.

It is clear from the figure that $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ correspond to maxima of F (namely, $F = 4$) and that $(1/\sqrt{2}, -1/\sqrt{2})$ and $(-1/\sqrt{2}, 1/\sqrt{2})$ correspond to minima (namely, $F = 1$) (Exercise 10.1). ▮

An elegant variation of this procedure, due to Lagrange, will be described next. Consider, for instance, the problem

$$F(v, w, x, y, z) = \max \qquad \text{(or min)}, \tag{10.9}$$

subject to the constraints

$$g(v, w, x, y, z) = a, \qquad h(v, w, x, y, z) = b. \tag{10.10}$$

First, we form the combination

$$F^* \equiv F - \lambda_1 g - \lambda_2 h, \tag{10.11}$$

where the λ's are constants. Now $dF = 0$ at a max and $dg = dh = 0$, since g and h are constant; consequently, $dF^* = dF - \lambda_1\,dg - \lambda_2\,dh = 0$ there, too; that is,

$$\begin{aligned} dF^* = (F_v - \lambda_1 g_v - \lambda_2 h_v)\,dv + (F_w - \lambda_1 g_w - \lambda_2 h_w)\,dw + \cdots \\ + (F_z - \lambda_1 g_z - \lambda_2 h_z)\,dz = 0. \end{aligned} \tag{10.12}$$

Suppose that we try to choose λ_1 and λ_2 so that $F_y - \lambda_1 g_y - \lambda_2 h_y = F_z - \lambda_1 g_z -$

$\lambda_2 h_z = 0$ at the desired max; that is,

$$g_y\lambda_1 + h_y\lambda_2 = F_y, \qquad g_z\lambda_1 + h_z\lambda_2 = F_z. \tag{10.13}$$

From Chapter 7 we note that this situation is possible [i.e., equations (10.13) have a solution] if

$$\begin{vmatrix} g_y & h_y \\ g_z & h_z \end{vmatrix} \neq 0 \tag{10.14}$$

at the desired max—in other words, if the Jacobian $\partial(g, h)/\partial(y, z) \neq 0$ there. Assuming that such is the case, (10.12) reduces to

$$\begin{aligned} dF^* = (F_v - \lambda_1 g_v - \lambda_2 h_v)\, dv + (F_w - \lambda_1 g_w - \lambda_2 h_w)\, dw \\ + (F_x - \lambda_1 g_x - \lambda_2 h_x) dx = 0. \end{aligned} \tag{10.15}$$

Having eliminated dy and dz, the remaining increments—dv, dw, and dx—*are* independent,[3] and therefore the three coefficients in parentheses in (10.15) must equal zero. This result, together with (10.13), says that

$$\begin{aligned} F_v^* &= F_v - \lambda_1 g_v - \lambda_2 h_v = 0 \\ F_w^* &= F_w - \lambda_1 g_w - \lambda_2 h_w = 0 \\ F_x^* &= F_x - \lambda_1 g_x - \lambda_2 h_x = 0 \\ F_y^* &= F_y - \lambda_1 g_y - \lambda_2 h_y = 0 \\ F_z^* &= F_z - \lambda_1 g_z - \lambda_2 h_z = 0 \end{aligned}$$

at the max. They are five equations in the seven unknowns $v, w, x, y, z, \lambda_1, \lambda_2$, and so we adjoin the two additional equations (10.10) to complete the system.

If (10.14) were *not* satisfied, it would still be all right if, for instance,

$$\begin{vmatrix} g_x & h_x \\ g_z & h_z \end{vmatrix} \neq 0$$

at the desired max; then we could eliminate the dx and dz terms instead of dy and dz.

Generalizing, suppose that

$$F(x_1, \ldots, x_m) = \max \qquad \text{(or min)}, \tag{10.16}$$

subject to the constraints

$$\begin{gathered} g_1(x_1, \ldots, x_m) = c_1 \\ \vdots \\ g_n(x_1, \ldots, x_m) = c_n, \end{gathered} \tag{10.17}$$

where $F, g_1, \ldots, g_n$ have continuous first partials and $n < m$. To solve by the method of Lagrange multipliers, we form

$$F^* \equiv F - \lambda_1 g_1 - \cdots - \lambda_n g_n \tag{10.18}$$

(where the λ's are the so-called *Lagrange multipliers*) and seek $x_1, \ldots, x_m, \lambda_1, \ldots, \lambda_n$

[3]The condition (10.14) implies that equations (10.10) can be solved for y and z in terms of v, w, x. Thus $F = F[v, w, x, y(v, w, x), z(v, w, x)]$, and in place of the five dependent variables we have the three independent variables v, w, x.

from the $m + n$ equations

$$\frac{\partial F^*}{\partial x_j} = 0 \qquad (j = 1, 2, \ldots, m) \tag{10.19}$$

$$g_j = c_j \qquad (j = 1, \ldots, n.) \tag{10.20}$$

Note carefully that (10.19) amounts to *minimizing F^* subject to no constraints.*

Example 10.3. Let us re-solve Example 10.2 by the method of Lagrange multipliers. With $F^* = F - \lambda g = x^2 + y^2 - \lambda(5x^2 - 6xy + 5y^2)$, (10.19) and (10.20) are

$$2x - 10\lambda x + 6\lambda y = 0$$

$$2y + 6\lambda x - 10\lambda y = 0$$

$$5x^2 - 6xy + 5y^2 = 8.$$

Eliminating λ from the first two gives $y = \pm x$, which, together with the last equation, yields the same result as before. ▮

Finally, a few words about an important although slightly different kind of problem. Suppose that we seek $x_1, \ldots, x_m$ so that the *linear* function

$$F(x_1, \ldots, x_m) = b_1x_1 + \cdots + b_mx_m = \max \qquad \text{(or min)}, \tag{10.21}$$

subject to the *linear* constraints

$$\begin{aligned} g_1(x_1, \ldots, x_m) &= a_{11}x_1 + \cdots + a_{1m}x_m = c_1 \\ &\;\;\vdots \\ g_n(x_1, \ldots, x_m) &= a_{n1}x_1 + \cdots + a_{nm}x_m = c_n, \end{aligned} \tag{10.22}$$

where the x_j's *must all be nonnegative*. Or the constraints may be *inequalities*, such as

$$a_{11}x_1 + \cdots + a_{1m}x_m \le c_1. \tag{10.23}$$

But this situation is equivalent because we can always introduce an additional variable x_{m+1} and rewrite (10.23) as

$$a_{11}x_1 + \cdots + a_{1m}x_m + x_{m+1} = c_1$$

with the proviso that x_{m+1} is also nonnegative; x_{m+1} is called a *slack variable*.

Such problems are of interest in the broad area known as operations research. It turns out that their solutions cannot be found by setting various derivatives equal to zero; special techniques like the *simplex method* are needed. These various techniques come under the heading of **linear programming**, which is outside the scope of this book.[4]

10.3 FUNCTIONALS AND THE CALCULUS OF VARIATIONS

Finally, we come to the extremization (i.e., maximization or minimization of *functionals* or at least of certain classes of functionals. Let us start with the following problem.

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[4]See, for example, R. M. Stark and R. L. Nichols, *Mathematical Foundations for Design*, McGraw-Hill, New York, 1972.

Find the function $y(x)$, defined over $x_1 \leq x \leq x_2$ and passing through the given endpoints

$$y(x_1) = y_1, \qquad y(x_2) = y_2, \tag{10.24a}$$

such that

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \text{extremum}. \tag{10.24b}$$

We ask that f have as many continuous partial derivatives with respect to its arguments x, y, y' as needed and seek $y(x)$ from within a certain class of admissible functions, say twice continuously differentiable. That is, y'' is to exist and be continuous over $x_1 \leq x \leq x_2$.[5]

To illustrate, suppose that we wish to find the arc of minimum length that passes through given endpoints x_1, y_1 and x_2, y_2. With $ds = [(dx)^2 + (dy)^2]^{1/2}$, the problem is as follows.

$$I = \int ds = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx = \min \tag{10.25a}$$

with

$$y(x_1) = y_1 \quad \text{and} \quad y(x_2) = y_2. \tag{10.25b}$$

As a general rule, note how new situations are often handled by recasting them in terms of old and familiar ones. For instance, in Chapter 7 we extended the idea of Taylor series to functions $f(x, y)$ of *two* variables by expressing x and y in terms of a parameter t so that $f(x, y) = f[x(t), y(t)] \equiv F(t)$ was reduced to a function of *one* variable; alternatively, we also considered expanding in *one* variable at a time.

In the same spirit we reduce problem (10.24) to the extremization of a *function* as follows. Denoting the *solution* (as yet unknown) as $y(x)$, let us consider a one-parameter family of "comparison functions" $Y(x)$, defined by

$$Y(x) = y(x) + \epsilon\eta(x), \tag{10.26}$$

where $\eta(x)$ is an arbitrary, twice continuously differentiable[6] function satisfying the end conditions

$$\eta(x_1) = \eta(x_2) = 0. \tag{10.27}$$

For example, if $y(x)$ and $\eta(x)$ are as shown in Fig. 10.4, then assigning various (positive and negative) values to the parameter ϵ produces the $Y(x)$ family indicated; of course, an infinite number of members actually exist, only a few of which have been sketched here.

Note that no matter how $\eta(x)$ is defined, the true solution $y(x)$ will certainly be a member of the Y family—namely, for $\epsilon = 0$. Note also that $Y(x_1) = y(x_1) + \epsilon\eta(x_1) = y_1 + \epsilon \cdot 0 = y_1$ and $Y(x_2) = \text{etc.} = y_2$; thus all members of the family satisfy the end conditions (10.24a) and are qualified candidates.

Consider next the quantity

$$I(\epsilon) \equiv \int_{x_1}^{x_2} f(x, Y, Y')\, dx,$$

[5]Extension to a broader class of admissible y's, including y's with "corners," is discussed in J. C. Clegg, *Calculus of Variations*, Oliver & Boyd, Edinburgh, 1968.

[6]$f(x)$ is said to be **continuously differentiable** (smooth) wherever $f'(x)$ exists and is continuous.

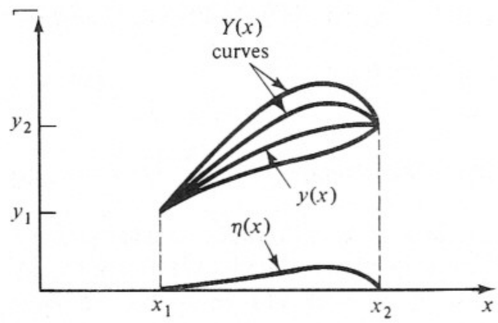

Figure 10.4. The comparison functions.

which is a *function* of the parameter ϵ (through Y and Y'). To extremize, we set $I'(\epsilon) = 0$. Unlike most extremization problems, however, we know beforehand where the minimum occurs—namely, at $\epsilon = 0$. Thus

$$I'(0) = 0 = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial Y}\frac{dY}{d\epsilon} + \frac{\partial f}{\partial Y'}\frac{dY'}{d\epsilon}\right)\bigg|_{\epsilon=0} dx.$$

But $Y = y + \epsilon\eta$ and $Y' = y' + \epsilon\eta'$, and so $dY/d\epsilon = \eta$ and $dY'/d\epsilon = \eta'$. Furthermore, all Y's and Y''s revert to y's and y''s when we set $\epsilon = 0$; consequently,

$$0 = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right) dx.$$

Integrating the second term by parts,

$$\begin{aligned} 0 &= \frac{\partial f}{\partial y'}\eta\bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx \\ &= \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\,dx, \end{aligned} \tag{10.28}$$

since $\eta(x_1) = \eta(x_2) = 0$.

Equation (10.28) must hold for *all* η's [which satisfy (10.27) and are twice continuously differentiable], and we suspect that this condition can be true only if the square bracket is zero over the interval. In fact, we have the following basic lemma.

LEMMA 10.1. If $F(x)$ is continuous and

$$\int_{x_1}^{x_2} F(x)\eta(x)\,dx = 0 \tag{10.29}$$

for **all** *continuously differentiable functions $\eta(x)$ for which $\eta(x_1) = \eta(x_2) = 0$, then it must be true that*

$$F(x) = 0$$

for all $x_1 \leq x \leq x_2$.

Outline of Proof. Suppose that $F(x)$ is *not* zero at some point ξ in the interval. Since F is continuous, there must exist an $\epsilon > 0$ such that F is of the same sign in the subinterval $(\xi - \epsilon, \xi + \epsilon)$ as at the point ξ. Using the arbitrariness of η, let η be zero outside this subinterval and positive inside (Fig. 10.5). Then the left-hand side of (10.29) will be nonzero, and we have proof by contradiction.

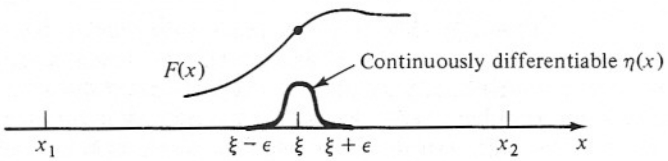

Figure 10.5. Proof of the basic lemma.

Returning to (10.28), we conclude that the desired $y(x)$ must satisfy the differential equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{10.30}$$

plus, of course, the prescribed end conditions (10.25b); (10.30) is the so-called **Euler equation** associated with the functional (10.24b). Note that the second term is *not* $\partial^2 f(x, y, y')/\partial x\, \partial y'$ but rather

$$\frac{d}{dx} f_{y'}[x, y(x), y'(x)] = f_{xy'} + f_{yy'}y' + f_{y'y'}y''.$$

In general, the Euler equation (10.30) will be a nonlinear, second-order, differential equation in the unknown $y(x)$.

Example 10.4. *Shortest Arc.* We are now able to solve the shortest arc problem, stated by (10.25). In this case, $f(x, y, y') = [1 + y'^2]^{1/2}$ happens to depend only on y'. The Euler equation (10.30) becomes

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}}\right) = 0.$$

Integrating once gives

$$\frac{y'}{\sqrt{1 + y'^2}} \equiv C.$$

Solving for y',

$$y' = \sqrt{\frac{C^2}{1 - C^2}} \equiv A,$$

say, and integrating once more, we obtain the straight line

$$y = Ax + B, \tag{10.31}$$

where A and B are determined by the end conditions (10.25b).

Comment 1. Our approach has been a little formal in several aspects. For instance, all we have actually shown is that the straight line provides a relative extremum. It's true that we can "sense" that it happens to be a *minimum*, and, in fact, an *absolute* minimum, but so far our analysis provides no grounds for such claims. Furthermore, note that in expressing $ds = [(dx)^2 + (dy)^2]^{1/2}$ as $(1 + y'^2)^{1/2}\, dx$ in (10.25a), we drop the view of x and y as independent variables and consider y as a function of x. Yet the desired curve is not necessarily expressible in the form $y = y(x)$; that is, perhaps it is not single valued. Several of these points are examined in the exercises, but generally such details lie outside our present scope.[7]

COMMENT 2. The present example is the simplest case of the more general problem of geodesics—that is, to find the arc of minimum length connecting two points on a given surface. Such an arc is called a **geodesic** for that particular surface. As an illustration, the geodesics for a sphere turn out to be great-circle arcs—that is, the intersection of the sphere with the plane containing the two given points and the center of the sphere.

If, for example, the surface is defined by $z = z(x, y)$ and we seek the geodesic between $x_1, y_1, z(x_1, y_1)$ and $x_2, y_2, z(x_2, y_2)$, we have the variational problem

$$\begin{aligned} I &= \int \sqrt{(dx)^2 + (dy)^2 + (dz)^2} \\ &= \int_{x_1}^{x_2} \sqrt{1 + y'^2 + (z_x + z_y y')^2}\, dx = \min, \end{aligned} \tag{10.32a}$$

subject to the end conditions

$$y(x_1) = y_1, \qquad y(x_2) = y_2, \tag{10.32b}$$

which is of the form (10.24).

Of course, for a *sphere*, it would be more natural to use a parametric representation of the surface, $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$. (See Exercises 10.18 and 10.19) ∎

Example 10.5. *Undetermined Endpoints.* Find the shape $y(x)$ of a hanging flexible cable that is looped around a frictionless, vertical wire at $x = a$, as sketched in Fig. 10.6.

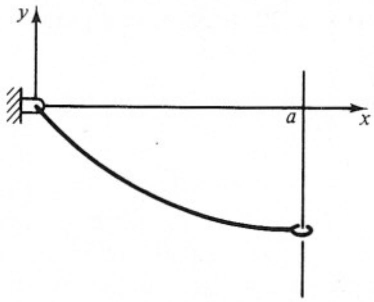

Figure 10.6. Hanging flexible cable.

The governing physical principle is that the cable will assume the shape that minimizes its potential energy V. In saying that the cable is "flexible," we mean that there is a negligible amount of elastic "strain energy" within the cable; for instance, it might be a chain or string. So there is only gravitational potential to consider. If ρ is the lineal mass density (mass per unit length), g is the acceleration of gravity, and $y = 0$ is the zero potential reference level, we have the variational problem

$$V = \int \rho g y\, ds = \int_0^a \rho g y \sqrt{1 + y'^2}\, dx = \min, \tag{10.33a}$$

where

$$y(0) = 0 \tag{10.33b}$$

but where $y(a)$ is not prescribed. Thus (10.33) is not quite of the form (10.24), and our derivation of the Euler equation needs to be reexamined.

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[7] A rigorous and detailed treatment can be found in the monograph by G. A. Bliss, *Calculus of Variations*, Open Court Publishing Company, LaSalle, Ill., 1925.

Consider, then,

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx = \text{extremum}, \tag{10.34}$$
$$y(x_1) = y_1, \qquad y(x_2) \text{ unspecified.}$$

As before, we form the family $Y = y + \epsilon\eta$, where $\eta(x_1) = 0$, but this time $\eta(x_2)$ need not equal zero; thus our comparison functions $Y(x)$ can "try out" a variety of right-end values, as sketched in Fig. 10.7. And, as earlier, we obtain

$$\begin{aligned} I'(0) = 0 &= \frac{\partial f}{\partial y'}\eta\bigg|_{x_1}^{x_2} + \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\, dx, \\ &= \frac{\partial f}{\partial y'}\bigg|_{x_2}\eta(x_2) + \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\eta(x)\, dx, \end{aligned} \tag{10.35}$$

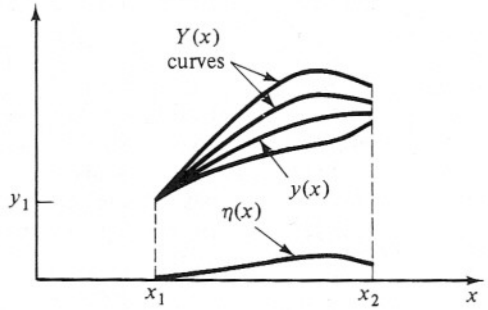

Figure 10.7. Undetermined right-end conditions.

since $\eta(x_1) = 0$. Because (10.35) is to hold for *all* (twice continuously differentiable) η's such that $\eta(x_1) = 0$, it must obviously hold for the subclass of η's that vanish at x_2 as well. Then the remaining boundary term in (10.35) drops out, and Lemma 10.1 leads to the same Euler equation as before,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0. \tag{10.36}$$

So (10.35) reduces to

$$0 = \frac{\partial f}{\partial y'}\bigg|_{x_2}\eta(x_2);$$

and since $\eta(x_2)$ *need* not equal zero, it also follows that

$$\frac{\partial f}{\partial y'}\bigg|_{x_2} = 0. \tag{10.37}$$

We call (10.37) a **natural boundary condition.**

For our hanging cable, the Euler equation and boundary conditions are therefore

$$yy'' - y'^2 - 1 = 0 \tag{10.38}$$

(after algebraic simplification) and

$$y(0) = 0, \qquad \frac{\partial f}{\partial y'} = \frac{yy'}{\sqrt{1 + y'^2}} = 0 \quad \text{at } x = a. \tag{10.39}$$

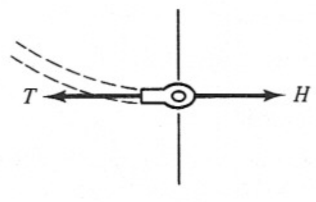

Figure 10.8. Equilibrium of the loop at $x = a$.

Apparently, from the right-hand side of (10.39), either $y(a) = 0$ or $y'(a) = 0$. Of these two, we accept the latter on the basis of a simple equilibrium argument. That is, consider an infinitesimal bit of the cable at the loop. Since the vertical wire is frictionless, the force H that it exerts on the loop must be horizontal (Fig. 10.8). This force is to be balanced by the tension T, which must therefore also be horizontal, so that $y'(a) = 0$ (Exercise 10.7).

Unfortunately, the (nonlinear) differential equation (10.38), plus the boundary conditions $y(0) = y'(a) = 0$, has no (real) solutions; for instance, at $x = 0$ (10.38) reduces to $y'^2 = -1$! Why? The difficulty is that the problem is not properly posed until we specify how *long* the cable is. That is, the problem should have been to minimize the potential subject to the end condition $y(0) = 0$ *and the constraint*

$$\int_0^a \sqrt{1 + y'^2}\, dx = l,$$

where l is prescribed ($\geq a$). Thus we have what is called an **isoperimetric problem** (since the length or perimeter of the curve is prescribed).

Finally, a few more words about natural boundary conditions. As discussed above, observe that the bracketed terms in (10.35) necessarily vanish (thus yielding the Euler equation) whether or not $y(x)$ is prescribed at x_1 and/or x_2. It follows from (10.35), then, that

$$0 = \frac{\partial f}{\partial y'}\eta \bigg|_{x_1}^{x_2} = \frac{\partial f}{\partial y'}\bigg|_{x_2} \eta(x_2) - \frac{\partial f}{\partial y'}\bigg|_{x_1} \eta(x_1). \tag{10.40}$$

If $y(x_1)$ and/or $y(x_2)$ are unspecified, then $\eta(x_1)$ and/or $\eta(x_2)$ are arbitrary, and it follows from (10.40) that $\partial f/\partial y'|_{x_1} = 0$ and/or $\partial f/\partial y'|_{x_2} = 0$. These are referred to as *natural boundary conditions.* ∎

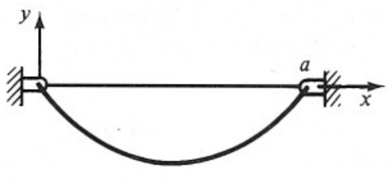

Figure 10.9.

Let us take the properly posed problem as our next example but with the end conditions $y(0) = y(a) = 0$ (Fig. 10.9), so that we need not worry about an undetermined end condition at the same time.

Example 10.6. *An Isoperimetric Problem.*

$$I = \int_0^a y\sqrt{1 + y'^2}\, dx = \min, \qquad y(0) = y(a) = 0, \tag{10.41a}$$

subject to the constraint

$$J = \int_0^a \sqrt{1 + y'^2}\, dx = l. \tag{10.41b}$$

Although the constraint may seem to complicate matters considerably, just recall the analogous situation for the extremization of a *function* F subject to a constraint $g =$ constant. We saw that the problem could be restated equivalently as $F^* = F - \lambda g =$ extremum subject to *no* constraint, where λ is the so-called Lagrange multiplier.

Considering the way that we are able to convert functionals to functions through the introduction of families of comparison functions, it is not surprising that the Lagrange method can be applied to problems like (10.41) as well.[8] So we have

$$I^* = I - \lambda J = \int_0^a (y - \lambda)\sqrt{1 + y'^2}\, dx = \min \tag{10.42a}$$

with

$$y(0) = y(a) = 0. \tag{10.42b}$$

Since $f^* = (y - \lambda)\sqrt{1 + y'^2}$ does not contain any explicit x dependence, the Euler equation has the first integral (Exercise 10.9)

$$\frac{y - \lambda}{\sqrt{1 + y'^2}} = A; \tag{10.43}$$

and integrating once more (Exercise 10.11),

$$y = \lambda - A \cosh\left(\frac{x - B}{A}\right). \tag{10.44}$$

Finally, $y(0) = y(a) = 0$ and (10.41b) yield three nonlinear algebraic equations in the unknowns A, B, and λ. Their unique solution is always possible, although perhaps numerically difficult. ∎

10.4. TWO OR MORE DEPENDENT VARIABLES; HAMILTON'S PRINCIPLE

Generalizing a little, consider the case of two or more dependent variables,

$$I = \int_{t_1}^{t_2} f(x, y, \ldots, z, \dot{x}, \dot{y}, \ldots, \dot{z}, t)\, dt = \text{extremum}, \tag{10.45}$$

where the dots denote d/dt and t is often, although not necessarily, the time.

In the discussion to follow, the variational problem will be supplied by what is known in classical mechanics as **Hamilton's principle**.

Suppose that our system is *conservative*; that is, (Chapter 9) each force that acts is derivable from a potential. Let V and T denote the *total* potential and kinetic energies, respectively. Introducing the so-called **Lagrangian**, $L \equiv T - V$, Hamilton's principle states that *the actual motion connecting two known states of the system, say at times t_1 and t_2, is the one that minimizes the integral*

$$I = \int_{t_1}^{t_2} (T - V)dt = \int_{t_1}^{t_2} L\, dt. \tag{10.46}$$

Example 10.7. Derive the equations of motion for the two-mass system shown in Fig. 10.10. Only lateral motion is considered—the allowed displacements of the two masses from their equilibrium positions are $x(t)$ and $y(t)$—and the three springs of stiffness k are assumed to be neither stretched nor compressed when $x = y = 0$

[8] See, for example, R. Weinstock, *Calculus of Variations*, McGraw-Hill, New York, 1952, pp. 48–50. A good general reference for the present chapter, it contains many important applications to engineering and physics.

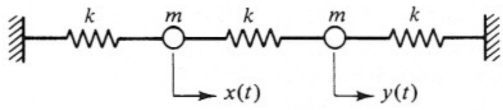

Figure 10.10. Vibration of a two-mass system.

(although the equations of motion would be the same even if there were some initial tension or compression).

Clearly, the kinetic and potential energies are

$$T = \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2 \quad \text{and}$$
$$V = \tfrac{1}{2}kx^2 + \tfrac{1}{2}k(y - x)^2 + \tfrac{1}{2}ky^2,$$

and so, according to Hamilton's principle,

$$I = \int_{t_1}^{t_2} [\tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - k(x^2 - xy + y^2)]\, dt = \text{min.} \tag{10.47}$$

In order to obtain the Euler equation(s) for the variational problem

$$I = \int_{t_1}^{t_2} f(x, y, \dot{x}, \dot{y}, t)\, dt = \text{extremum}, \tag{10.48}$$

of which our problem (10.47) is an example, we follow essentially the same lines as for the case of only one dependent variable discussed above. First, we form the comparison functions

$$X(t) = x(t) + \epsilon\xi(t), \qquad Y(t) = y(t) + \epsilon\eta(t),$$

where $\xi(t)$ and $\eta(t)$ are arbitrary, twice continuously differentiable functions satisfying the end conditions

$$\xi(t_1) = \xi(t_2) = \eta(t_1) = \eta(t_2) = 0. \tag{10.49}$$

Considering

$$I(\epsilon) = \int_{t_1}^{t_2} f(X, Y, \dot{X}, \dot{Y}, t)\, dt,$$

we set

$$\begin{aligned}
I'(0) = 0 &= \int_{t_1}^{t_2} \left(\frac{\partial f}{\partial x}\xi + \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial \dot{x}}\dot{\xi} + \frac{\partial f}{\partial \dot{y}}\dot{\eta}\right) dt \\
&= \left(\frac{\partial f}{\partial \dot{x}}\xi + \frac{\partial f}{\partial \dot{y}}\eta\right)\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)\right]\xi(t)\, dt \\
&\quad + \int_{t_1}^{t_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta(t)\, dt.
\end{aligned}$$

The boundary terms drop out because of (10.49), and the arbitrariness of ξ and η implies (with the help of Lemma 10.1) that each of the square brackets must be zero over the interval:

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \tag{10.50a}$$

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0. \tag{10.50b}$$

Thus we have the same Euler equation as before—that is, (10.30)—but governing *each* of the dependent variables.

In particular, for our case (10.47), the *equations of motion* (10.50) become

$$\left.\begin{aligned} m\ddot{x} + 2kx - ky &= 0 \\ m\ddot{y} + 2ky - kx &= 0. \end{aligned}\right\} \tag{10.51}$$

Let us ignore the *solution* of equations (10.51) for the moment, since it is not central to the present discussion.

COMMENT. Naturally, (10.51) could have been derived much more directly from Newton's second law without resorting to Hamilton's principle and the calculus of variations. Nevertheless, we emphasize that variational methods are of the greatest importance to us for at least three reasons.

First, for complicated problems in dynamics and continuum mechanics, Hamilton's principle may be somewhat simpler to apply because we need only be able to write expressions for kinetic and potential energies; for instance, we never need to know what a "Coriolis force" is! This situation is illustrated by a problem from the literature in Exercise 10.22.

Secondly, for many problems like (10.25) there *is* no (apparent) alternative *non*-variational statement, analogous to Newton's law in the present example.

Finally, the Euler equation(s) for a given problem may prove too difficult to solve, whereas the variational principle itself provides an excellent starting point for various *approximate* techniques, such as the Ritz method, which is discussed briefly in Section 10.6. ▮

10.5. *TWO OR MORE INDEPENDENT VARIABLES; VIBRATING STRINGS AND MEMBRANES*

Starting with the case of *two* independent variables x, y, we seek $w(x, y)$ in some region D with boundary C such that

$$I = \iint_D f(x, y, w, w_x, w_y)\, dx\, dy = \text{extremum}, \tag{10.52a}$$

subject to the boundary values

$$w \text{ given on } C \tag{10.52b}$$

(Fig. 10.11).

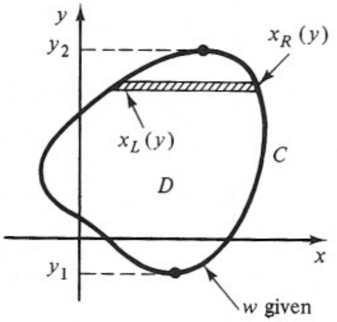

Figure 10.11. The region under consideration.

We introduce the comparison functions

$$W(x, y) = w(x, y) + \epsilon\eta(x, y),$$

where η has continuous second-order partials and satisfies

$$\eta = 0 \text{ on } C \tag{10.53}$$

but is otherwise arbitrary in D. Then

$$I(\epsilon) = \iint_D f(x, y, W, W_x, W_y)\, dx\, dy$$

is an extremum for $\epsilon = 0$; that is,

$$I'(0) = 0 = \iint_D \left(\frac{\partial f}{\partial w}\eta + \frac{\partial f}{\partial w_x}\eta_x + \frac{\partial f}{\partial w_y}\eta_y\right) dx\, dy. \tag{10.54}$$

Integrating the second term by parts,

$$\iint_D \frac{\partial f}{\partial w_x}\eta_x \, dx\, dy = \int_{y_1}^{y_2}\left\{\int_{x_L(y)}^{x_R(y)} \frac{\partial f}{\partial w_x}\eta_x \, dx\right\} dy$$

$$= \int_{y_1}^{y_2}\left\{\frac{\partial f}{\partial w_x}\eta\,\Big|_{x_L(y)}^{x_R(y)} - \int_{x_L(y)}^{x_R(y)} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right)\eta\, dx\right\} dy$$

$$= \int_C \frac{\partial f}{\partial w_x}\eta\,\frac{dy}{ds}\, ds - \iint_D \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right)\eta\, dx\, dy, \tag{10.55}$$

where s is the arc length along C measured counterclockwise.[9] Integrating the third term in the integrand of (10.54) in the same manner, (10.54) becomes

$$0 = \int_C \left(\frac{\partial f}{\partial w_x}\frac{dy}{ds} - \frac{\partial f}{\partial w_y}\frac{dx}{ds}\right)\eta\, ds + \iint_D \left[\frac{\partial f}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial w_y}\right)\right]\eta\, dx\, dy$$

and as a result of (10.53) and (a two-dimensional version of) Lemma 10.1, we arrive at the Euler equation

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial w_y}\right) = 0. \tag{10.56}$$

Example 10.8. *Vibrating String.* Starting with a perfectly flexible string of length l_0, a plot of applied tension versus length is apt to be somewhat as sketched in Fig. 10.12(a). Suppose that we tie the string as shown in Fig. 10.12(b), under tension τ, and set it in motion in the x, y plane by means of some initial disturbance. We would like the equation of motion governing the displacement $y(x, t)$.

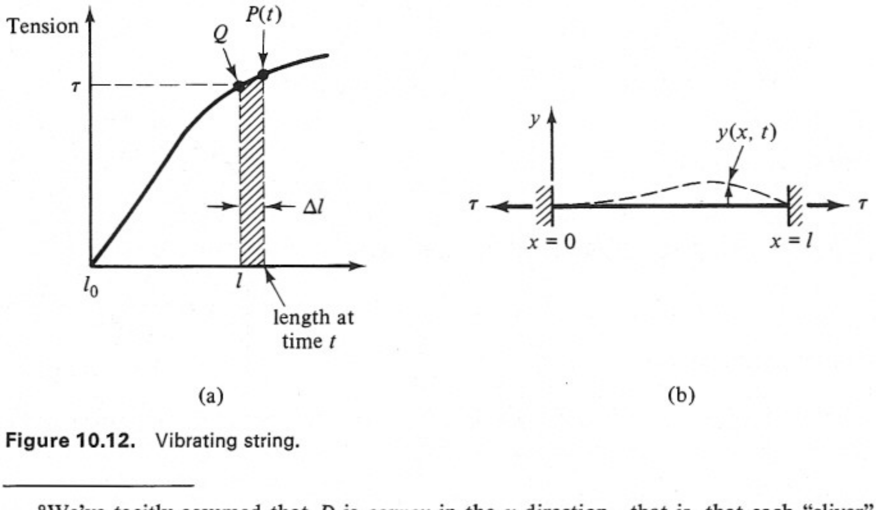

Figure 10.12. Vibrating string.

[9]We've tacitly assumed that D is *convex* in the x direction—that is, that each "sliver" running from $x_L(y)$ to $x_R(y)$ lies entirely within D. The minor modifications needed for the nonconvex case will not be pursued, since they are probably obvious and, in any case, do not affect the final result (10.55). Incidentally, rather than use integration by parts, we could have obtained (10.55) by using Green's theorem (9.60) with $M = 0$ and $N = \eta\, \partial f/\partial w_x$.

Let us suppose that the amplitude of vibration is so small that the slope $y_x(x, t)$ is small compared to unity for $0 \leq x \leq l$ and all t, that the tension is uniform in x at each instant, and that each particle of the string undergoes negligible x motion.[10]

Because of the displacement $y(x, t)$, the length of the string (and the tension in it) will vary slightly with t, as indicated by the movement of the representative point $P(t)$ in Fig. 10.12(a). The associated work is equal to the area of the shaded sliver and is stored as potential energy. Defining the equilibrium state $y = 0$ as our zero potential (reference state), we therefore have

$$V \approx \tau \Delta l = \tau \left\{ \int_0^l \sqrt{1 + y_x^2}\, dx - l \right\}$$
$$\approx \tau \left\{ \int_0^l \left(1 + \frac{1}{2} y_x^2\right) dx - l \right\} = \frac{\tau}{2} \int_0^l y_x^2\, dx. \tag{10.57}$$

Furthermore, the kinetic energy is clearly

$$T = \frac{1}{2} \int_0^l \rho(x) y_t^2\, dx, \tag{10.58}$$

where $\rho(x)$ is the lineal mass density (mass per unit length).

According to Hamilton's principle (10.46), then,

$$\int_{t_1}^{t_2} (T - V)\, dt = \frac{1}{2} \int_{t_1}^{t_2} \int_0^l (\rho y_t^2 - \tau y_x^2)\, dx\, dt = \min, \tag{10.59}$$

and so the Euler equation (10.56), with w corresponding to y and y to t, becomes the second-order partial differential equation of motion

$$\tau y_{xx} - \rho(x) y_{tt} = 0. \tag{10.60}$$

Methods of *solution* of partial differential equations are discussed in Part V.

COMMENT. Note that we have *not* assumed a linear tension-displacement curve [Fig. 10.12(a)]. In fact, its precise shape is immaterial because the process is assumed to take place in the neighborhood of the point Q, so that tension $\approx \tau$ is all that survives in the final equations. ▮

Finally, consider the following problem in *three* independent variables x, y, z. Find $w(x, y, z)$ in some region R bounded by a surface S such that

$$I = \iiint_R f(x, y, z, w, w_x, w_y, w_z)\, dx\, dy\, dz = \text{extremum}, \tag{10.61a}$$

subject to the boundary values

$$w \text{ given on } S. \tag{10.61b}$$

Referring to the one- and two-dimensional results, (10.30) and (10.56), it is not hard to imagine that the Euler equation relevant to (10.61) is (Exercise 10.23)

$$\frac{\partial f}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial w_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial w_y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial w_z}\right) = 0. \tag{10.62}$$

[10] For purposes of this example, the fact that these assumptions are "quite reasonable" will suffice. Rigorous development by means of perturbation methods (Parts IV and V) is possible but not really needed here.

Example 10.9. *Vibrating Membrane.* We consider a flexible membrane,[11] such as a drumhead, which is stretched over some region D of the x, y plane bounded by a curve C. Denote the elevation of the membrane above the x, y plane at the point x, y and the time t by $w(x, y, t)$ and suppose that w is prescribed along C for all t. The membrane is set in motion by some initial disturbance, and we seek the equation of motion governing the displacement $w(x, y, t)$.

Let us suppose that the boundary values of w on C and the initial disturbance are so small that the slope of the membrane with respect to the x, y plane in any direction is small compared to unity throughout D and for all t, that the tension τ per unit length is essentially constant throughout D and for all t, and that each particle undergoes negligible motion parallel to the x, y plane.

Let us elaborate on our assumption that the tension per unit length is essentially uniform throughout D. Consider a rectangular membrane held along its left and right edges, stretched as shown (Fig. 10.13) with a tension τ per unit length (thus with

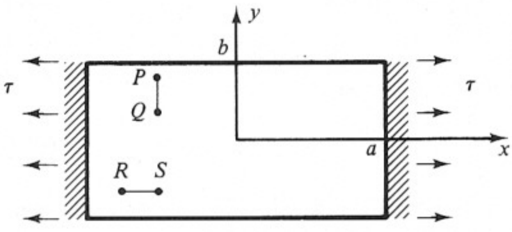

Figure 10.13. *Non*uniform membrane tension.

a total force $2b\tau$), and then also clamped along the top and bottom edges $y = \pm b$. In this case, the tension per unit length will *not* be uniform; for example, it will be τ along each vertical line (e.g., PQ) and 0 along each horizontal line (e.g., RS). Thus the present analysis would not apply to this case unless we undertook to render the tension uniform by also stretching suitably in the y direction.

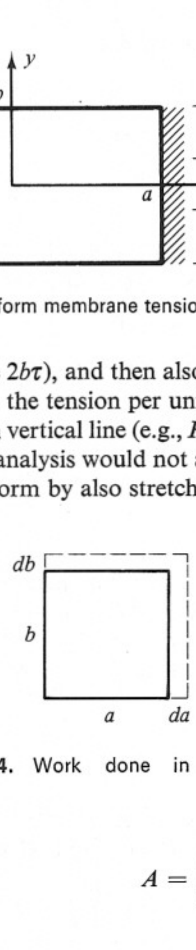

Figure 10.14. Work done in stretching.

To compute the potential V, first note that the work done by a constant and uniform tension per unit length τ in stretching the rectangular membrane element shown in Fig. 10.14 by the amounts da and db can be expressed formally as

$$\begin{aligned} d(\text{work}) &= (\tau b)\, da + (\tau a)\, db \\ &= \tau d(ab) = \tau d(\text{area}), \end{aligned} \tag{10.63}$$

and it is stored as an increase in potential, dV.

So if we recall from (8.50) that the surface area of the membrane is given at any instant by

$$A = \iint_D \sqrt{1 + w_x^2 + w_y^2}\, dx\, dy$$

[11]As with the "flexible string," the words "flexible membrane" imply that the membrane offers negligible resistance to bending.

and use the hypothetical state $w \equiv 0$ as our zero reference potential, then

$$V = \tau \left\{ \iint_D \sqrt{1 + w_x^2 + w_y^2}\, dx\, dy - \iint_D dx\, dy \right\}$$

$$\approx \tau \left\{ \iint_D \left(1 + \frac{w_x^2 + w_y^2}{2}\right) dx\, dy - \iint_D dx\, dy \right\} = \frac{\tau}{2} \iint_D (w_x^2 + w_y^2)\, dx\, dy. \tag{10.64}$$

Furthermore, the kinetic energy is

$$T = \frac{1}{2} \iint_D \rho(x, y) w_t^2\, dx\, dy, \tag{10.65}$$

where $\rho(x, y)$ is the membrane mass density (mass per unit area); consequently, Hamilton's principle becomes

$$\frac{1}{2} \int_{t_1}^{t_2} \iint_D [\rho w_t^2 - \tau(w_x^2 + w_y^2)]\, dx\, dy\, dt = \min, \tag{10.66}$$

and the equation of motion follows from (10.62) as

$$\tau(w_{xx} + w_{yy}) - \rho(x, y) w_{tt} = 0,$$

or

$$\tau\, \nabla^2 w = \rho(x, y) w_{tt}. \tag{10.67}$$

COMMENT 1. Observe that, for the *static* case, (10.67) reduces to the *Laplace* equation, and therefore we have the so-called **Dirichlet problem**

$$\nabla^2 w = 0 \text{ in } D, \; w \text{ given on } C. \tag{10.68}$$

Now, for the static case, our variational principle is simply $V = \min$ or

$$\iint_D (w_x^2 + w_y^2)\, dx\, dy = \min; \tag{10.69}$$

that is, the Laplace equation $\nabla^2 w = w_{xx} + w_{yy} = 0$ is the Euler equation for the variational principle (10.69)!

COMMENT 2. Again, for the static case, suppose that the membrane were "pumped up" by a uniform net pressure p on its underside. In order to incorporate this feature into our formulation, we ask: Is there a *potential* associated with p? Well, consider a little element dA of the membrane (Fig. 10.15), subjected to a pressure force $p\, dA$. In place of the pressure we can induce exactly the same effect on the element by attaching a weightless string to it and hanging a weight of magnitude $p\, dA$ over some frictionless pulleys (Fig. 10.15); the w displacement of the membrane element and the height of the weight are related simply as shown. The gravitational potential is $-(p\, dA)w$. Integrating over the membrane and adding the result to (10.64), we have

$$V = \iint_D \left[\frac{\tau}{2}(w_x^2 + w_y^2) - pw\right] dx\, dy = \min \tag{10.70}$$

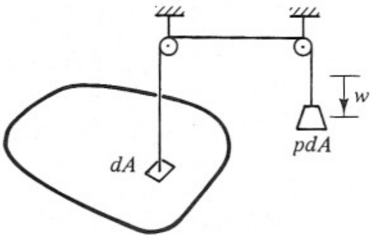

Figure 10.15. Associating a potential with p.

or from (10.62) the Poisson equation[12]

$$\nabla^2 w = -\frac{p}{\tau}. \quad \blacksquare \tag{10.71}$$

10.6. THE RITZ METHOD

Consider, for definiteness, the variational problem (10.52). The partial differential Euler equation (10.56), plus boundary conditions (10.52b), may, depending on f and the shape of D, be intractable to the point where we would gladly settle for a good *approximate* solution, or a good approximation may be all that we actually need in the first place. The *Ritz method* is a simple and powerful technique for obtaining such approximate solutions.

We seek

$$w \approx c_1\phi_1(x, y) + \cdots + c_n\phi_n(x, y), \tag{10.72}$$

where the $\phi_j(x, y)$'s are "suitably chosen functions" and the c_j's are constants to be determined. Generally the ϕ_j's are chosen so that (10.72) satisfies the given boundary conditions for all possible c_j's; for instance, if $w = 0$ on C, then $\phi_1(x, y) = \cdots = \phi_n(x, y) = 0$ on C. Then, we have $I = I(c_1, \ldots, c_n)$, and we extremize I with respect to the c_j's by setting $\partial I/\partial c_1 = \cdots = \partial I/\partial c_n = 0$. This process provides n algebraic equations in the n c_j's.

Example 10.10. *Pumped-Up Membrane.* Let D in (10.70) be the rectangle $-a \leq x \leq a$, $-b \leq y \leq b$ and suppose that $w = 0$ on its boundary. Let us try

$$w \approx (x^2 - a^2)(y^2 - b^2)(c_1 + c_2x^2 + c_3y^2 + c_4x^4 + c_5x^2y^2 + c_6y^4 + \cdots). \tag{10.73}$$

These ϕ's, which are of the form $(x^2 - a^2)(y^2 - b^2)x^my^n$, are a good choice for three reasons. They are zero on the boundary, as desired, and they are *simple*, so that the integration of V will be straightforward. Finally, note that $(x^2 - a^2)(y^2 - b^2)$ provides what we expect to be the right basic shape (dotted lines in Fig. 10.16), and the power series factor $c_1 + c_2x^2 + c_2y^2 + \cdots$ allows for the necessary "adjustments." Odd powers like $x, y, xy, x^2y, \ldots$ have been omitted because it is obvious that the solution should be symmetric in both x and y.

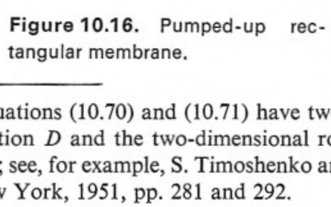

Figure 10.16. Pumped-up rectangular membrane.

Depending on the accuracy desired and how ambitious we feel, we cut (10.73) off at some point. Suppose that we simply keep the first term, $w \approx (x^2 - a^2)\ (y^2 - b^2)c_1$. Putting it into (10.70) and integrating, we obtain

$$V(c_1) = \frac{16a^3b^3[4\tau(a^2 + b^2)c_1^2 - 5pc_1]}{45}.$$

[12]Equations (10.70) and (10.71) have two well-known analogs in mechanics; torsion of a bar of cross section D and the two-dimensional rotational flow of an inviscid incompressible fluid in the region D; see, for example, S. Timoshenko and J. N. Goodier, *Theory of Elasticity*, 2nd ed., McGraw-Hill, New York, 1951, pp. 281 and 292.

Minimizing, we set

$$\frac{dV}{dc_1} = \frac{16a^3b^3[8\tau(a^2 + b^2)c_1 - 5p]}{45} = 0,$$

so that $c_1 = 5p/8\tau(a^2 + b^2)$ and

$$w \approx \frac{5p(x^2 - a^2)(y^2 - b^2)}{8\tau(a^2 + b^2).}. \tag{10.74}$$

As a partial measure of the accuracy of this result, we note that if the volume under the membrane, $\iint w\,dx\,dy$ over D, is computed by using (10.74) and compared with the exact value [using equation (26.109a) of Part V], it is found that the error is 1.2% for $b/a = 1$ and 11.9% for $b/a = 10$. Why does the accuracy drop off as b/a becomes large (Exercise 10.26)?

Seeking better accuracy, let us try

$$w \approx (x^2 - 1)(y^2 - 1)(c_1 + c_2x^2 + c_3y^2), \tag{10.75}$$

where we've set $a = b = 1$ for definiteness. Setting $\partial V/\partial c_1 = \partial V/\partial c_2 = \partial V/\partial c_3 = 0$ leads to the simultaneous equations (Exercise 10.27)

$$\begin{aligned} \frac{16}{45}c_1 + \frac{32}{525}c_2 + \frac{32}{525}c_3 &= \frac{p}{9\tau} \\ \frac{32}{105}c_1 + \frac{304}{945}c_2 + \frac{16}{329}c_3 &= \frac{p}{9\tau} \\ \frac{32}{105}c_1 + \frac{16}{329}c_2 + \frac{304}{945}c_3 &= \frac{p}{9\tau}, \end{aligned} \tag{10.76}$$

and so $c_1 = 0.2920p/\tau$ (compared with $0.3125p/\tau$ before) and $c_2 = c_3 = 0.0597p/\tau$. This time it turns out that $\iint w\,dx\,dy$ is in error by only 0.2%. It seems plausible that *any* desired accuracy can be achieved, in this example at least, by including enough terms in (10.73). This fundamental question of convergence was examined by Ritz and is discussed in the book by Mikhlin.[13] ∎

Observe that the Ritz method proceeds from a variational principle. Often, however, we have an ordinary or partial differential equation to solve that was not obtained as the Euler equation of some variational problem. In this case, it *may* be possible to find a corresponding variational principle to which the Ritz method can then be applied; this topic is considered briefly in Exercise 10.28 and at length in the book by Finlayson.[14] Alternatively, a **method of weighted residuals**, such as the *Galerkin method*, can be applied directly to the governing differential equation(s). The methods of weighted residuals and the Ritz method are among the most important *direct methods*—that is, methods of approximate solution that reduce the problem to a finite system of *algebraic* equations. The subject is discussed further in Parts IV and V. For a detailed account, including many applications to problems in fluid mechanics and heat and mass transfer, we recommend Finlayson.[15]

[13] S. G. Mikhlin, *Variational Methods in Mathematical Physics*, Macmillan, New York, 1964.

[14] B. A. Finlayson, *The Method of Weighted Residuals and Variational Principles*, Academic, New York, 1972. See especially Chapters 8 and 9.

[15] *Ibid.*

Let us illustrate with an example.

Example 10.11. *Liquid Level Control System.* Suppose that a certain liquid is pumped into a tank of uniform cross-sectional area, say unity, at a steady rate of Q m³/min and that it flows out the bottom at a rate that is proportional to the square root of the depth x (Fig. 10.17). (For instance, if the efflux at the bottom is at atmospheric pressure, then the exit velocity is known to be $\sqrt{2gx}$ where g is the acceleration of gravity, or perhaps some fraction of this value due to various losses.)

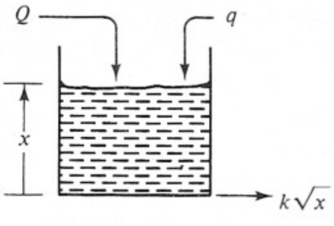

Figure 10.17. Liquid level control system.

Equating inflow to outflow, it is clear that the steady-state level x_s is given by $Q = k\sqrt{x_s}$. Although it is desired to operate at the equilibrium value x_s, x may, for various reasons, sometimes fall below this value, and we can attempt to bring it back in an optimal way by an additional input at a suitable rate $q(t)$.

The optimal control problem, then, consists of deciding how to manage the *control variable* $q(t)$. Naturally, doing so will depend on exactly what we mean by "optimal." Since we are told that it is important to operate close to equilibrium, it is reasonable to require that the square deviation $\int_0^T [x(t) - x_s]^2\, dt$ over the control time T be small. Additionally, we would not like the control expenditure, which we might measure by $\int_0^T q^2(t)\, dt$, to be uneconomically large. So given an initial state $x(0) \neq x_s$, we seek the control "policy" $q(t)$ such that

$$\int_0^T [(x - x_s)^2 + \alpha^2 q^2]\, dt = \min, \tag{10.77}$$

where the weighting factor α^2 (squared to emphasize that it is positive) is a measure of the relative importance that we attach to the square error and control expenditure.

Of course, we cannot expect to proceed much further without a statement of the dynamics of the system, and this statement is given by the mass balance

$$\dot{x} = Q + q - k\sqrt{x}\,; \tag{10.78}$$

that is d(volume)$/dt$ equals the rate in minus the rate out.

Before continuing, let us simplify subsequent steps (Exercise 10.30) by "linearizing" (10.78)—that is, the nonlinear $\sqrt{x}$ term—about the equilibrium state x_s. In other words, we expand

$$\sqrt{x} = \sqrt{x_s} + \frac{1}{2\sqrt{x_s}}(x - x_s) + \cdots$$

and break the series off after the linear term on the grounds that x deviates little from x_s ($x - x_s$ is reasonably small), so that the linear or "tangent line" approximation

$$\sqrt{x} \approx \sqrt{x_s} + \frac{1}{2\sqrt{x_s}}(x - x_s)$$

should suffice (Fig. 10.18). Putting this approximation into (10.78), recalling that $Q = k\sqrt{x_s}$, and defining the error $x - x_s \equiv y$, we have the simpler (linear) equation

$$\dot{y} = -Ay + q(t), \tag{10.79}$$

where $A \equiv k/2\sqrt{x_s}$.

So by inserting $q = \dot{y} + Ay$ into (10.77), we find

$$\int_0^T [y^2 + \alpha^2(\dot{y} + Ay)^2]\, dt = \text{min}, \tag{10.80a}$$

$$y\,(0) = x(0) - x_s \equiv y_0, \qquad y(T) \text{ unspecified}; \tag{10.80b}$$

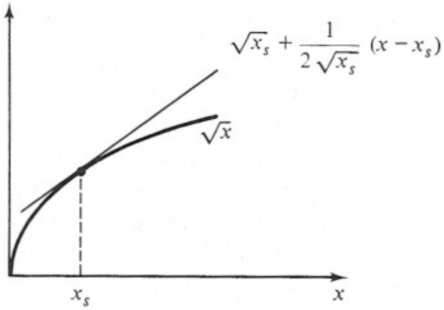

Figure 10.18. Tangent line linearization.

hence the Euler equation and boundary conditions are

$$\ddot{y} - \left(\frac{1 + \alpha^2 A^2}{\alpha^2}\right) y = 0; \tag{10.81a}$$

$$y(0) = y_0, \qquad \dot{y}(T) + Ay(T) = 0 \quad \text{(natural boundary condition).} \tag{10.81b}$$

It is not hard to solve (10.81) for $y(t)$ and thus to find the desired policy $q(t)$ from $q(t) = \dot{y} + Ay$.

COMMENT 1. What is the "physical significance" of the natural boundary condition in (10.81b)? Recalling (10.79), we see that $\dot{y}(T) + Ay(T) = 0$ is the same as saying that $q(T) = 0$, which simply says that the control time is T!

COMMENT 2. Actually, a **feedback control system** would be operationally more convenient—that is, where the deviation $y = x - x_s$ is monitored and the control q is some optimally prescribed function of the state y; thus $q = q(y)$ rather than $q(t)$.

To illustrate, consider the simple case of *proportional control* $q(y) = My$, where the constant M is called the *gain*.[16] Then (10.79) becomes $\dot{y} + (A - M)y = 0$, and so

$$y = y_0 e^{-(A-M)t},$$

where we choose M so that

$$\begin{aligned}\int_0^T (y^2 + \alpha^2 q^2)\, dt &= (1 + \alpha^2 M^2) \int_0^T y^2\, dt \\ &= (1 + \alpha^2 M^2) y_0^2 \int_0^T e^{-2(A-M)t}\, dt = \text{etc.}\end{aligned}$$

is a minimum. Setting d/dM of this equation equal to zero, we obtain (for the case where T is considered to be quite large) the optimal gain

$$M = \frac{A\alpha - \sqrt{1 + A^2\alpha^2}}{\alpha}. \quad ∎$$

[16]The present example involves a *first*-order system; its state is prescribed by the single quantity y. Suppose, however, that you are to drive a car on a straight one-lane road so as to follow a "lead car" by a constant distance, say 20 m. If y is the deviation of the distance between cars from the desired value, then (roughly speaking) we have a second-order system with feedback control. That is, your eyes monitor the two state variables $y, \dot{y}$ (the error in spacing and its rate of change), your brain evaluates some $q(y, \dot{y})$, and your foot activates the brake or gas pedal accordingly. Perhaps $q(y, \dot{y}) \approx M(y - 20) + N\dot{y}$, where M and N are determined from experience.

The foundations of the mathematical theory of optimal control date back to the late 1950s and include major contributions from L. S. Pontryagin, L. W. Neustadt, J. P. LaSalle, and a group of mathematicians led by R. Bellman.

The subject is sophisticated, and even the central results are outside our present scope. If interested, a good place to start might be the 36-page introduction in the book by Boltyanskii,[17] which includes the method of **dynamic programming, Pontryagin's maximum principle**, and **discontinuous control**. In addition, we recommend the very readable account by Bellman[18] and the book by Denn,[19] which includes numerous practical applications of the theory to various chemical engineering processes.

EXERCISES

10.1. Verify, by suitable examination of second derivative(s), that $(\sqrt{2}, \sqrt{2})$ and $(-\sqrt{2}, -\sqrt{2})$ correspond to maxima of F and that $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$ correspond to minima of F in Example 10.2 as claimed.

10.2. Where possible, verify that the extremum is indeed a minimum.
(a) Find the point on the curve of intersection of $z - xy = 10$ and $x + y + z = 1$ that is closest to the origin.
(b) Find the point(s) on the ellipsoid $x^2 - xy + y^2 + 2z^2 = 4$ that is closest to the origin.
(c) Find the point on the plane $ax + by + cz = d$ that is closest to the origin.

10.3. If $x_1 x_2 \cdots x_n = 1$, where each x_i is positive, show that $x_1 + x_2 + \cdots + x_n \geq n$.

10.4. (a) Show that of all parallelograms with a prescribed area A, the one with the least perimeter is a square.
(b) Furthermore, show that if their perimeter is prescribed instead, then the one with the largest area is a square.

10.5. (*Steepest descent*) (a) In discussing (necessary) conditions for the existence of extrema of functions of several variables, we have thus far said nothing about *computational* aspects; for instance, consider

$$f(x, y, \ldots) = \min \tag{10.82}$$

or

$$f_x(x, y, \ldots) = f_y(x, y, \ldots) = \ldots = 0. \tag{10.83}$$

Equation (10.83) involves the solution of simultaneous algebraic equations, which may be nonlinear and intractable. Generally we resort to one of several standard iterative schemes for their solution. The methods of *successive approximation* and *Newton–Raphson* are discussed in Part IV; here we will consider only the so-called **method of steepest descent** for the (10.82) version of the problem. Although refinements are possible, it goes basically like this. Consider, for definiteness, the two-dimensional case $f(x, y) = \min$. Choose an initial guess-estimate x^0, y^0 to the desired solution. To move toward the minimum, it is reasonable to strike out in the direction of steepest descent of $f(x, y)$—that

[17] V. G. Boltyanskii, *Mathematical Methods of Optimal Control*, Holt, Rinehart and Winston, New York, 1971.

[18] R. Bellman, *Introduction to the Mathematical Theory of Control Processes*, Vol. 1, *Linear Equations and Quadratic Criteria*, Academic, New York, 1967.

[19] M. Denn, *Optimization by Variational Methods*, McGraw-Hill, New York, 1969.

is, in the direction of ∇f at the point x^0, y^0. The question is how *far* should we move; in other words, what is the scale factor α so that

$$f(x^0 + \alpha f_x^0, y^0 + \alpha f_y^0) \approx f^0 + \alpha[(f_x^0)^2 + (f_y^0)^2] \\ + \frac{\alpha^2}{2}[f_{xx}^0(f_x^0)^2 + 2f_{xy}^0 f_x^0 f_y^0 + f_{yy}^0(f_y^0)^2]$$

is a minimum, where $(\quad)^0$ means $(\quad)$ evaluated at x^0, y^0? Setting $d/d\alpha$ of the right-hand side equal to zero, solve for α and deduce the iterative scheme

$$x^{n+1} = x^n + \alpha f_x^n, \qquad y^{n+1} = y^n + \alpha f_y^n, \tag{10.84}$$

where
$$\alpha = -\frac{(f_x^n)^2 + (f_y^n)^2}{f_{xx}^n(f_x^n)^2 + 2f_{xy}^n f_x^n f_y^n + f_{yy}^n(f_y^n)^2}.$$

(b) Use (10.84) to locate the equilibrium point x, y that is near the origin, in the gravitational field induced by point unit masses at (1, 0), (0, 5), and (−1, 0). Is it stable or unstable?

(c) Use (10.84) to solve the simultaneous equations $x = \sin(x + y)$, $y = \cos(x - y)$ by defining $f = [x - \sin(x + y)]^2 + [y - \cos(x - y)]^2$ and starting with $x^0 = y^0 = 0.5$, say.

10.6. (a) Verify that the straight-line solution (10.31) is, in fact, a *minimum* by showing that $I''(0) > 0$.

(b) Lifting the assumption that y be a single-valued function of x, re-solve the shortest arc Example 10.4 by describing the arc parametrically by $x = x(t)$, $y = y(t)$.

10.7. In our discussion of the equilibrium of the loop shown in Fig. 10.8 we omitted the weight of the element. Is this omission an approximation or is it strictly correct? Explain.

10.8. Derive the general solution $y = A \cosh[(x - B)/A]$ of the Euler equation (10.38). *Hint:* Set $y' = p$ and hence $y'' = p\, dp/dy$.

10.9. (*First integral of Euler equation*) Show that if f in (10.24b) does not depend explicitly on x—that is, $f = f(y, y')$—then the Euler equation (10.30) admits the first integral

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

10.10. (a) Asked in an examination to find the Euler equation for the problem

$$\int_{x_1}^{x_2} f(x, y')\, dx = \min; \qquad y(x_1), \quad y(x_2) \text{ given},$$

one student showed that $0 = \int_{x_1}^{x_2} f_{y'}\eta'\, dx$ and concluded that the Euler equation is $f_{y'} = 0$. Evaluate these claims.

(b) Determine the Euler equation and natural boundary conditions for the problem

$$\int_{x_1}^{x_2} f(x, y, y', y'')\, dx = \min.$$

10.11. Integrate (10.43) to obtain (10.44).

10.12. (*John Bernoulli's brachistochrone problem*) We seek the curve $y(x)$ from (0, 0) to (a, b) along which a bead of mass m will descend under the action of gravity (no friction) in the shortest time (Fig. 10.19). Noting, from conservation of energy, that

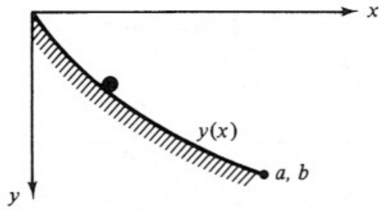

Figure 10.19. Brachistochrone.

the bead's velocity is $v = \sqrt{2gy}$, show that the variational problem may be stated as

$$\int_0^a \sqrt{\frac{1+y'^2}{2gy}}\,dx = \min; \qquad y(0) = 0, \quad y(a) = b.$$

Show (recall Exercise 10.9) that a first integral of the Euler equation is

$$y' = \sqrt{\frac{1 - Cy}{Cy}}.$$

[The next integration is more tedious; it would reveal that $y(x)$ is a *cycloid.*]

10.13. (*Fermat's principle*) Consider two different, optically homogeneous media above and below the plane P (Fig. 10.20) and suppose that there is a point light source at

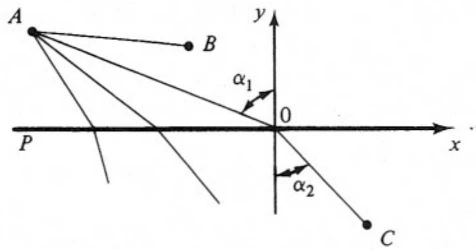

Figure 10.20. Snell's law.

A. Fermat's principle of optics states that *the path taken by a ray of light will be such that the travel time t is a minimum.* If the speed of light at each point of the upper medium, in any direction, is v_1 (a constant) and v_2 in the lower medium, show that

(a) the path taken from A to any point B in the upper medium is a straight line.

(b) the path from A to any point C in the lower medium is the broken straight line with the (Snell's) law of refraction

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{v_1}{v_2}.$$

10.14. (*More about Fermat's principle*) First read Exercise 10.13. Now instead of the simple two-layer medium, suppose that $v = v(y)$ varies continuously.

(a) Show that the paths $y(x)$ are determined by

$$v(y)y'' + v'(y)(1 + y'^2) = 0.$$

(b) Find the path from (0, 0) to (1, 0) if $v(y) = y$. (First use your intuition to sketch the solution and then see how good your intuition is.)

(c) Repeat part (b) for $v(y) = \sqrt{y}$.

*10.15. Suppose that the length of the cable in Fig. 10.6 is l and that a weight W is hung from the loop. Derive the Euler equation and the natural boundary condition at $x = a$.

10.16. (*An isoperimetric exercise*) Find a curve $y(x)$ of length L over $0 \leq x \leq 1$, with $y(0) = y(1) = 0$, so that the area under it is maximized. Show that the result is the circle $(x - A)^2 + (y - B)^2 = \lambda^2$, where A and B are integration constants and λ is a Lagrange multiplier. Comment on the state of affairs when $L > \pi/2$.

10.17. (*Nonintegral constraints*) Consider the problem

$$I = \int_{t_1}^{t_2} f(t, x, y, \dot{x}, \dot{y})\, dt = \text{extremum},$$

with x and y prescribed at t_1 and t_2, subject to the nonintegral constraint $g(t, x, y) = c$; see (10.41b). Introducing $X = x + \epsilon\xi$ and $Y = y + \epsilon\eta$, as in Section 10.4, show that

$$0 = \int_{t_1}^{t_2} \left\{ \left[f_x - \frac{d}{dt}(f_{\dot{x}}) - \lambda g_x \right] \xi + \left[f_y - \frac{d}{dt}(f_{\dot{y}}) - \lambda g_y \right] \eta \right\} dt.$$

Thus show that

$$f_x - \frac{d}{dt}(f_{\dot{x}}) - \lambda(t) g_x = 0, \qquad f_y - \frac{d}{dt}(f_{\dot{y}}) - \lambda(t) g_y = 0.$$

Note carefully that here the Lagrange multiplier λ is a function of t unlike the case of an integral constraint, where it was a constant.

10.18. (*Geodesics for a sphere*) Show that the geodesics for a sphere of radius R are great circles. Do it two ways.

(a) With the help of Exercise 10.17, start with the problem statement

$$\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \min$$

with the constraint

$$g(t, x, y, z) = x^2 + y^2 + z^2 = R^2.$$

Hint: Show that $y\, d(dx/ds) = x\, d(dy/ds)$, so that $d(y\, dx/ds) = d(x\, dy/ds)$, and hence $y\, dx - x\, dy = A\, ds$. Similarly, obtain $z\, dx - x\, dz = B\, ds$. Eliminate ds and divide by x^2.

(b) In terms of spherical polars ρ, θ, ϕ, we have $(ds)^2 = (d\rho)^2 + \rho^2(d\theta)^2 + \rho^2 \sin^2 \theta (d\phi)^2$, and so on $\rho = R$ we have

$$\int ds = \int_{\theta_1}^{\theta_2} R\sqrt{1 + (\sin^2 \theta)(d\phi/d\theta)^2}\, d\theta = \min,$$

subject to *no* constraint. *Hint:* This solution can be quite short.

10.19. (*Geodesics for a cylinder*) Find the geodesics for the circular cylinder $x^2 + y^2 = R^2$. Do it two ways.

(a) With the help of Exercise 10.17, start with

$$\int_{t_1}^{t_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, dt = \min$$

with the constraint

$$g(t, x, y, z) = x^2 + y^2 = R^2.$$

(b) With $x = R\cos\theta$, $y = R\sin\theta$, $z = z$, start with

$$\int ds = \int \sqrt{R^2(d\theta)^2 + (dz)^2} = \int_{\theta_1}^{\theta_2} \sqrt{R^2 + z'^2}\, d\theta = \text{min},$$

subject to *no* constraints, and recall Exercise 10.9.

10.20. Derive the equations of motion (not necessarily small) on $x(t)$, $y(t)$ for the system shown in Fig. 10.21 by means of Hamilton's principle. The springs are of length L and are neither stretched nor compressed when $x = y = 0$. Suppose that the x, y plane is horizontal, so that gravity is not relevant.

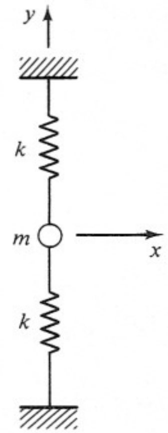

Figure 10.21. Two-degree-of-freedom oscillation.

10.21. Derive the equations of motion on $r(t)$, $\theta(t)$ for the system shown in Fig. 10.22 by means of Hamilton's principle. The unstretched length of the spring is L.

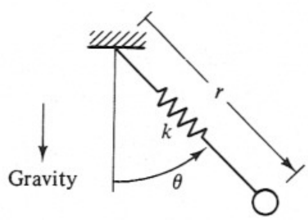

Figure 10.22. Spring pendulum.

10.22. (*Vibrating pipeline*) A nice problem from the literature[20] concerns the transverse vibrations of an above-ground oil pipeline that is supported at intervals of L feet. Let v be the (constant) velocity of the oil within the pipe, ρ the mass per unit length of the pipe, μ the mass per unit length of the fluid, E Young's modulus for the pipe, and I the moment of inertia of the pipe cross section about its neutral axis. Assume

[20] G. W. Housner, Bending Vibrations of a Pipe Line Containing Flowing Liquid, *Journal of Applied Mechanics*, Vol. 19, June 1952, p. 205.

small displacements of the pipe, that the pipe can be considered as a pinned-end beam (Fig. 10.23), and that the gravitational potential can be neglected compared with the strain energy of the pipe, so that

$$V = \int_0^L \tfrac{1}{2} EIy_{xx}^2 \, dx,$$

where $y(x, t)$ is the deflection of the pipe's centerline. Write an expression for T and thus deduce the equation of motion

$$EIy_{xxxx} + (\rho + \mu)y_{tt} + \mu v^2 y_{xx} + 2\mu v y_{xt} = 0. \tag{10.85}$$

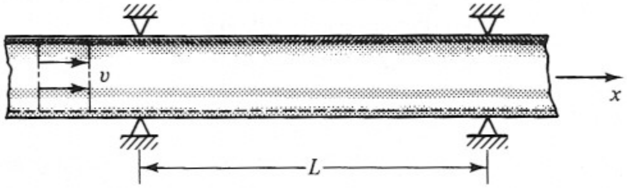

Figure 10.23. Vibrating pipeline.

Note that if $\mu \longrightarrow 0$, (10.85) reduces to the classical vibrating-beam equation (which you may have already encountered in mechanics); the third and fourth terms can be interpreted as centripetal and Coriolis terms, respectively. Note also that the only difficulty in this problem is in writing down the kinetic energy T, the expression for V being "standard" (in the *Euler beam theory*). By comparison, you will probably find that a nonvariational approach, based directly on Newton's law, is more difficult.

10.23. Provide a derivation of (10.62).

10.24. Reexpress (10.69) in terms of polar coordinates r, θ and write out the corresponding Euler equation.

10.25. (*The Ritz method*) Apply the Ritz method to find approximate solutions to the problems below.

(a) $\nabla^2 w = 3x^2 y$ in the triangle with corners at (0, 0), (1, 0), and (0, 1), where $w = 0$ on the boundary.

(b) The same as in part (a) but for the rectangular region $0 < x < 1, 0 < y < 1$.

10.26. Why does the accuracy of (10.74) drop off as b/a becomes large (or small)?

10.27. Go through the calculations that lead to equations (10.76). Incidentally, if b/a had been 10, say, instead of unity, might the three-term approximation

$$w \approx (x^2 - a^2)(y^2 - b^2)(c_1 + c_2 y^2 + c_3 y^4)$$

be more promising than

$$w \approx (x^2 - a^2)(y^2 - b^2)(c_1 + c_2 x^2 + c_3 y^2)?$$

Explain.

10.28. (*Working backward*) (a) In discussing the Ritz method, we pointed out that it would be nice to be able to take a given ordinary or partial differential equation and "work backward" to a corresponding variational principle. For instance, given the equation $y'' + y = F(x)$ over $x_1 \leq x \leq x_2$, a look at the derivation of (10.30) suggests that we proceed as follows.

$$0 = \int_{x_1}^{x_2} (y'' + y - F)\eta\, dx = y'\eta \Big|_{x_1}^{x_2} + \int_{x_1}^{x_2} (-y'\eta' + y\eta - F\eta)\, dx$$

$$= \int_{x_1}^{x_2} [(y - F)\eta + (-y')\eta']\, dx = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right] dx$$

so that $\partial f/\partial y = y - F$ and $\partial f/\partial y' = -y'$. Integrating, $f = (y^2/2) - Fy - (y'^2/2)$, and thus the variational principle is

$$\int_{x_1}^{x_2} \left(\frac{y^2}{2} - Fy - \frac{y'^2}{2}\right) dx = \text{extremum}$$

with $y'\eta = 0$ at x_1 and x_2—that is, $y' = 0$ (natural boundary condition) or y prescribed at each endpoint.

In the same manner, derive

$$\int_{x_1}^{x_2} (py'^2 - qy^2 + 2Fy)\, dx = \text{extremum}, \tag{10.86}$$

plus $py'\eta = 0$ at x_1 and x_2, for the more general differential equation

$$\frac{d}{dx}\left(p\frac{dy}{dx}\right) + qy = F(x), \tag{10.87}$$

or

$$py'' + p'y' + qy = F. \tag{10.88}$$

But if our given equation is

$$ay'' + by' + cy = d, \tag{10.89}$$

where $b \neq a'$, then it is not of the ("self-adjoint") form (10.88). Nevertheless, show that we can multiply (10.89) through by s so that $(sb) = (sa)'$—that is, $s = \exp \int [(b - a')/a]\, dx$—and hence throw it into the form (10.88).

(b) Thus show that the problem

$$xy'' + 2y' + y = 1; \qquad y(0) = 0, \quad y(1) = 1$$

corresponds to

$$\int_0^1 (x^2y'^2 - xy^2 + 2xy)\, dx = \text{extremum}; \qquad y(0) = 0, \quad y(1) = 1.$$

Using the Ritz method, seek an approximate solution of the form

$$y \approx x(x - 1)(c_1 + c_2x + c_3x^2 + \cdots) + x.$$

[This is slightly different from (10.72) due to the x term at the end, which is inserted because $y(1) = 1$, not 0.) Cut the $c_1 + c_2x + \cdots$ factor off after c_2x, say.

(c) Again "working backward," derive the variational principle corresponding to the two-dimensional Poisson equation $\nabla^2 w = \phi(x, y)$.

10.29. Instead of appealing to Lemma 10.1 to obtain the Euler equation from (10.28), suppose that we argue as follows. Set

$$\eta(x) = (x - x_1)^2(x - x_2)^2\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right].$$

Then (10.28) becomes

$$0 = \int_{x_1}^{x_2} (x - x_1)^2(x - x_2)^2 \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]^2 dx,$$

and since the integrand is nonnegative (and continuous), it follows that the square bracket must be identically zero over $x_1 < x < x_2$. Is this assumption correct? Discuss.

10.30. Is it really necessary that we linearize (10.78)? What eventual difficulties would arise if we did not?

10.31. Rework Example 10.11, using proportional control $q = My$, for the case where the tank is as shown in Fig. 10.24; in other words, the cross-sectional area varies as $a + bx$.

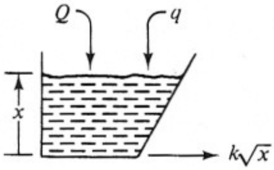

Figure 10.24. Nonconstant tank cross section.

10.32. (*Terminal process control*) Suppose that the optimal control criterion (10.80) is changed to

$$\int_0^T [y^2 + \alpha^2(\dot{y} + Ay)^2]\, dt + \beta^2 y^2(T) = \min$$

with $y(0) = y_0$ and $y(T)$ unspecified. What (in words) might be the motivation for including the $\beta^2 y^2(T)$ term? Show that the resulting Euler equation and boundary conditions are

$$\ddot{y} - \left(\frac{1 + \alpha^2 A^2}{\alpha^2}\right) y = 0;$$

$$y(0) = y_0, \qquad \alpha^2 \dot{y}(T) + (\alpha^2 A + \beta^2) y(T) = 0.$$

10.33. Make up two examination-type problems on this chapter.

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