HW Chapter 6

6.1

a) $f(x) = e^{-\alpha|x|}$, $\alpha > 0$

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} e^{-\alpha|x|} e^{-i\omega x}\,dx$$

$$= \int_{-\infty}^{0} e^{\alpha x} e^{-i\omega x}\,dx + \int_{0}^{\infty} e^{-\alpha x} e^{-i\omega x}\,dx$$

$$= \lim_{b \to -\infty} \left\{ \frac{1}{\alpha - i\omega} e^{(\alpha - i\omega)x} \Big|_b^0 \right\} + \lim_{b \to +\infty} \left\{ -\frac{1}{\alpha + i\omega} e^{-(\alpha + i\omega)x} \Big|_0^b \right\}$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

(compare the formula on the book cover)

b) $f(x) = \begin{cases} 0 & x < 0 \\ e^{-\alpha x}, & x \geq 0 \end{cases}$

$$\hat{f}(\omega) = \int_{0}^{\infty} e^{-\alpha x} e^{-i\omega x}\,dx$$

$$= \lim_{b \to \infty} \left\{ -\frac{1}{\alpha + i\omega} e^{-(\alpha + i\omega)x} \Big|_0^b \right\} = \frac{1}{\alpha + i\omega}$$

c) Careful, f is a distribution this time!

$$\mathcal{F}(\delta_{x_0})[\varphi] \stackrel{def}{=} \delta_{x_0}[\mathcal{F}(\varphi)] = \delta_{x_0}\left[\int_{-\infty}^{\infty} \varphi(\xi) e^{-ix\xi}\,d\xi \right]$$

$$= \left(\int_{-\infty}^{\infty} \varphi(\xi) e^{-ix\xi}\,d\xi \right)(x_0)$$

$$= \int_{-\infty}^{\infty} \varphi(\xi) e^{-ix_0 \xi} d\xi = \int_{-\infty}^{\infty} \varphi(\omega) e^{-ix_0 \omega} d\omega$$

$$= L_g[\varphi]$$ (regular distribution associated with function g)

where $g(\omega) = e^{-ix_0 \omega}$

So : $$\mathcal{F}(\delta_{x_0}) = e^{-ix_0 \omega}$$

6.8

a) $$\mathcal{F}\left((-ix)^n f(x)\right)(\omega) = \hat{f}^{(n)}(\omega)$$

Step 1 $$\mathcal{F}^{-1}(\hat{f}^{(n)})(x) = (-ix)^n \mathcal{F}^{-1}(\hat{f}) \qquad (*)$$

$$\text{LHS} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^{(n)} e^{ix\omega} d\omega = -(ix) \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}^{(n-1)} e^{ix\omega} d\omega$$

$\hat{f}^{(n-1)}$	$e^{ix\omega}$
$\hat{f}^{(n)}$	$(ix) e^{ix\omega}$

$$= -ix \; \mathcal{F}^{-1}(\hat{f}^{(n-1)})(x)$$

$$= (\text{by induction})(-ix)^n \mathcal{F}^{-1}(\hat{f})(x)$$

Step 2 Apply $\mathcal{F}$ to both sides of $(*)$

b) $$\mathcal{F}^{-1}\left(e^{-ia\omega}\hat{f}(\omega)\right)(x) = f(x-a)$$

$$LHS = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ia\omega}\hat{f}(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{i\underbrace{(x-a)}_{x'}\omega}\, d\omega \quad = f(x') = f(x-a)$$

c) $$\mathcal{F}^{-1}\left(\hat{f}(\omega-a)\right)(x) = e^{iax} f(x)$$

$$LHS = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega-a)\, e^{i\omega x}\, d\omega$$

$$\omega - a = \omega' \quad \Rightarrow \omega = a+\omega'$$
$$d\omega = d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega')\, e^{i(a+\omega')x}\, d\omega' = e^{iax}\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\omega')e^{i\omega' x}\,d\omega'}_{f(x)}$$

d) $$h(x) = \int_{-\infty}^{x} f(\xi)\, d\xi \quad , \quad h \to 0 \text{ as } x \to \infty$$

$$\overset{?}{\Longrightarrow} \quad \mathcal{F}(h)(\omega) = \frac{\hat{f}(\omega)}{i\omega}$$

$$h'(x) = f(x) \quad \Rightarrow \quad \mathcal{F}(f)(\omega) = \mathcal{F}(h')(\omega) = i\omega\, \mathcal{F}(h)(\omega)$$

$$\therefore \quad \mathcal{F}(h)(\omega) = \frac{\hat{f}(\omega)}{i\omega}$$

\#

6.12

a) $f(x) = f(-x)$

$$f(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-ix\omega}\, dx = \int_{-\infty}^{0} f(x)\, e^{-ix\omega}\, dx + \int_{0}^{\infty} f(x)\, e^{-ix\omega}\, dx$$

$$= \int_{0}^{\infty} f(x) \big(\underbrace{e^{ix\omega} + e^{-ix\omega}}_{2\cos(\omega x)}\big)\, dx$$

Analogously the remaining formulas.

6.15

a) $\bar{F}(s) = \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty t^a e^{-st}\, dt$

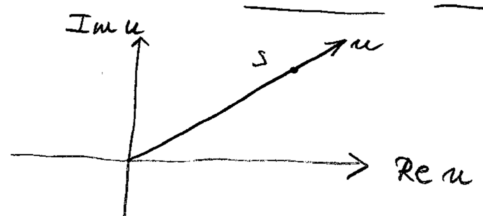
For s real, $s>0$, the calculation is elementary:

$$\int_0^\infty t^a e^{-st}\, dt = \frac{1}{s}\int_0^\infty \left(\frac{u}{s}\right)^a e^{-u}\, du = \frac{1}{s^{a+1}} \underbrace{\int_0^\infty u^a e^{-u}\, du}_{\Gamma(a+1)}$$

$st = u$ $\qquad = \dfrac{\Gamma(a+1)}{s^{a+1}}$

$s\,dt = du$

The same calculation can be carried out for s complex, except that now the integral $\int_0^\infty u^a e^{-u}\, du$ has to be interpreted as the contour integral along the path



It is a nontrivial fact that this integral is independent of Im u (for $a>0$) and therefore (case Im $u = 0$) equals $\Gamma(a+1)$.

b) $f(t) = e^{at}$

$$\bar{f}(s) = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{(a-s)t}\, dt$$

$$= \frac{1}{a-s} e^{(a-s)t} \Big|_0^\infty$$

$$= \frac{1}{a-s} \lim_{t \to \infty} e^{(a-s)t} - \frac{1}{a-s}$$

$$= -\frac{1}{a-s} \quad \text{provided } \underline{\operatorname{Re} s > a} \;!$$

c) $f(t) = \sin at$

$$\bar{f}(s) = \int_0^\infty \sin at\, e^{-st}\, dt = \frac{1}{2i}\left\{\int_0^\infty e^{iat} e^{-st} dt - \int_0^\infty e^{-iat} e^{-st} dt\right\}$$

$$e^{iat} = \cos at + i \sin at$$

$$-\quad e^{-iat} = -\cos at + i \sin at$$

$$\frac{e^{iat} - e^{-iat}}{2i} = \sin at$$

$$= \frac{1}{2i}\left\{\int_0^\infty e^{(ia-s)t} dt - \int_0^\infty e^{-(ia+s)t} dt\right\}$$

$$= \frac{1}{2i}\left\{\frac{1}{ia-s} e^{(ia-s)t}\Big|_0^\infty + \frac{1}{ia+s} e^{-(ia+s)t}\Big|_0^\infty\right.$$

$$= \frac{1}{2i}\left\{-\frac{1}{ia-s} - \frac{1}{ia+s}\right\} = -\frac{1}{2i}\left\{\frac{ia - \cancel{s} + ia + \cancel{s}}{-a^2 - s^2}\right\} = \frac{a}{a^2+s^2}$$

d) $f(t) = \cos at$

same technique as in c)

e) $f(t) = \sinh at = \dfrac{e^{at} - e^{-at}}{2}$

$$\mathcal{L}(f) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at})$$

$$= \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a}$$

$$= \frac{1}{2}\frac{\cancel{s}+a-\cancel{s}+a}{s^2-a^2} = \frac{a}{s^2-a^2}$$

f) $f(t) = \cosh at$

Same technique as in e)

g) $tH(t-1)$

$$\bar{f}(s) = \int_0^\infty tH(t-1)e^{-st}\,dt = \int_1^\infty te^{-st}\,dt$$

$t-1 = u$
$dt = du$

$$= \int_0^\infty (u+1)e^{-s(u+1)}\,du = e^{-s}\int_0^\infty (u+1)e^{-su}\,du$$

$$= e^{-s}\Big\{\underbrace{\int_0^\infty ue^{-su}\,du}_{= \frac{\Gamma(2)}{s^2}} + \underbrace{\int_0^\infty e^{-su}\,du}_{= \frac{\Gamma(1)}{s}}\Big\} = e^{-s}\Big\{\frac{1}{s^2} + \frac{1}{s}\Big\}$$

h) $f(t) = \delta(t-a)$

Be careful, this problem involves the generalization of the Laplace transform to distributions

$$\mathcal{L}(f)[\varphi] \overset{\text{def}}{=} f[\mathcal{L}\varphi]$$

$$f = \delta_a$$

$$\mathcal{L}(f)[\varphi] = \delta_a[\mathcal{L}\varphi] = \left(\int_0^\infty \varphi(t)e^{st}\,dt\right)(a)$$

$$= \int_0^\infty \varphi(t)e^{at}\,dt = \underbrace{L_{e^{at}}}_{}[\varphi]$$

regular distribution corresponding to function e^{at}

$$\mathcal{L}(\delta_a) = e^{at}, \quad a \in \mathbb{C}$$

6.16

a) $$\mathcal{L}[(-t)^n f(t)] = \bar{f}^{(n)}(s)$$

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$$

$\frac{d\bar{f}}{ds}(s)$ = (formally, a double limit theorem is involved!)

$$= \int_0^\infty f(t)\,\frac{d}{ds}(e^{-st})\,dt$$

$$= \int_0^\infty f(t)(-t)e^{-st}\,dt = \mathcal{L}[(-t)f(t)]$$

General case by induction.

b) $$\mathcal{L}^{-1}[e^{-as}\bar{f}(s)] = H(t-a)f(t-a)$$

Equivalently: $e^{-as}\bar{f}(s) = \mathcal{L}[H(t-a)f(t-a)]$

$$\mathcal{L}[H(t-a)f(t-a)] = \int_0^\infty H(t-a)f(t-a)e^{-st}\,dt$$

$$= \int_a^\infty f(t-a)e^{-st}\,dt \underset{t-a=u}{=} \int_0^\infty f(u)e^{-s(a+u)}\,du$$

$$= e^{-as}\int_0^\infty f(u)e^{-su}\,du = e^{-as}\bar{f}(s)$$

c) $\mathcal{L}^{-1}[\bar{f}(s+a)] = e^{-at} f(t)$

Equivalently: $\bar{f}(s+a) = \mathcal{L}[e^{-at} f(t)]$

$$\bar{f}(s+a) = \int_0^\infty f(t)\, e^{-(s+a)t}\, dt = \int_0^\infty \underbrace{f(t)e^{-at}}\; e^{-st}\, dt$$

$$= \mathcal{L}[f(t)e^{-at}]$$

d) $\mathcal{L}^{-1}\left[\frac{\bar{f}(s)}{s}\right] = \int_0^t f(\tau)\, d\tau$

Equivalently:

$$\frac{\bar{f}(s)}{s} = \mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right]$$

$$\mathcal{L}\left[\int_0^t f(\tau)\, d\tau\right] = \int_0^\infty \underbrace{\int_0^t f(\tau)\, d\tau}_{\varphi(t)}\; e^{-st}\, dt$$

$\varphi(t) \qquad -\frac{1}{s} e^{-st}$

$\varphi'(t) = f(t) \qquad e^{-st}$

$$= \cancel{-\frac{1}{s} \int_0^t f(\tau)\, d\tau\, e^{-st} \Big|_0^\infty}^{\,0}$$

$$+ \frac{1}{s} \int_0^\infty f(t)\, e^{-st}\, dt = \frac{1}{s} \bar{f}(s)$$

e) $$L\left[\frac{f(t)}{t}\right] = \int_s^{\infty} \bar{f}(s)\, ds$$

Equivalently (fundamental theorem of the complex integral calculus, page 253) differentiating both sides.

$$-\bar{f}(s) = \frac{d}{ds} L\left[\frac{f(t)}{t}\right]$$

use then a) result.

f) $f(t)$ periodic with period a

$$\bar{f}(s) = \frac{1}{1-e^{-as}} \int_0^a f(t) e^{-st}\, dt$$

$$\bar{f}(s) = \int_0^{\infty} f(t)\, e^{-st}\, dt = \sum_{n=0}^{\infty} \int_{na}^{(n+1)a} f(t)\, e^{-st}\, dt$$

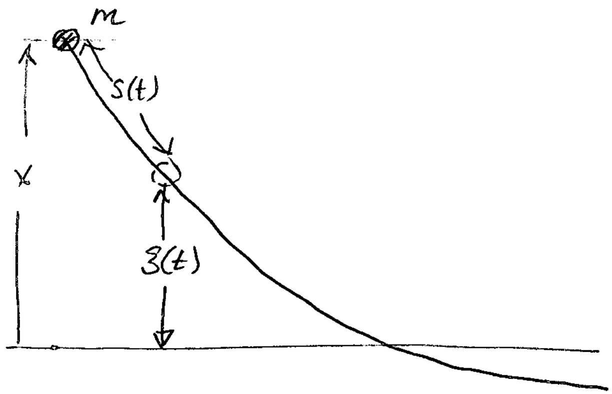
$t - na = u$ $$= \underbrace{\sum_{n=0}^{\infty} \left(e^{-as}\right)^n}_{\frac{1}{1-e^{-as}}} \int_0^a f(t)\, e^{-su}\, du$$

$e^{-st} = e^{-s(u+na)}$

(sum of a geometric series)

g) proved in the book

6.19



$$v(t) = \dot{s}(t) = \sqrt{2g(x-\xi)}$$

$$\frac{ds}{dt} = v \quad \Rightarrow \quad \frac{dt}{ds} = \frac{1}{v}$$

$$\therefore \quad \frac{dt}{d\xi} = \frac{dt}{ds}\,\frac{ds}{d\xi} = \frac{1}{v}\, s'(\xi)$$

$$t = \int_0^x \frac{dt}{d\xi}\, d\xi = \int_0^x \frac{s'(\xi)}{\sqrt{2g(x-\xi)}}\, d\xi = \frac{1}{\sqrt{2g}} \int_0^x \frac{s'(\xi)}{\sqrt{x-\xi}}\, d\xi$$

The Abel integral equation :

$$\int_0^x \frac{y(\xi)}{(x-\xi)^\alpha}\, d\xi = f(x) \quad / \mathcal{L}$$

$$\bar{y}(s) \cdot \underbrace{\mathcal{L}\left(x^{1-\alpha-1}\right)}_{\Gamma(1-\alpha)\cdot\frac{1}{s^{1-\alpha}}} = \bar{f}(s)$$

$$\therefore \quad \bar{y}(s) = \frac{s^{1-\alpha}\bar{f}(s)}{\Gamma(1-\alpha)}$$

6.18 a)

$$\bar{f}(s) = \frac{e^{-s}}{s(s+3)^{3/2}}$$

$$\mathcal{L}^{-1}(\bar{f})(t) = \mathcal{L}^{-1}\left(e^{-s}\,\bar{g}(s)\right)(t)$$

$$= H(t-1)\,g(t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+3)^{3/2}}\right)(t) = \frac{t^{\frac{1}{2}}e^{-3t}}{\Gamma(\frac{3}{2})} \qquad \text{table}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s(s+3)^{3/2}}\right)(t) = \int_0^t \frac{\tau^{\frac{1}{2}}\,e^{-3\tau}}{\Gamma(3/2)}\,d\tau$$

So finally

$$f(t) = H(t-1)\int_0^t \frac{\tau^{\frac{1}{2}}e^{-3\tau}}{\Gamma(3/2)}\,d\tau$$

#

6.18 b)

$$\bar{f}(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$$

$$\mathcal{L}^{-1}(\bar{f})(t) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+4}\right)(t)$$

$$= e^{-t}\,\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right)(t) \qquad \text{property 6.16 c)}$$

$$= e^{-t}\,\frac{\sin 2t}{2} \qquad \text{table}$$

6.18 c)

$$\bar{f}(s) = \frac{1}{s(s^2+4)}$$

$$\mathcal{L}^{-1}(\bar{f})(t) = \mathcal{L}^{-1}\left(\frac{1}{s(s^2+4)}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{\bar{g}(s)}{s}\right) \qquad \text{where } g(t) = \frac{\sin 2t}{2}$$

$$= \int_0^t g(\tau)\,d\tau = \int_0^t \frac{\sin 2\tau}{2}\,d\tau \qquad \text{property 6.16 e}$$

$$= -\frac{\cos 2\tau}{4}\Big|_0^t = -\frac{\cos 2t}{4} + \frac{1}{4}$$

$$= \frac{1-\cos 2t}{4}$$

6.18 d)

$$\bar{f}(s) = \frac{e^{-\pi s} + e^{-2\pi s}}{s^2+1}$$

$$\mathcal{L}^{-1}(\bar{f})(t) = \mathcal{L}^{-1}\left(\frac{e^{-\pi s}}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{e^{-2\pi s}}{s^2+1}\right) \qquad \text{additivity}$$

$$= H(t-\pi)\sin t + H(t-2\pi)\sin t \qquad \text{tables}$$

$$= \left[H(t-\pi) + H(t-2\pi)\right]\sin t$$

6.28 a)

$$\ddot{x} + 2\dot{x} = H(t-1) \quad /\mathcal{L} \qquad x(0) = 0 \,, \quad \dot{x}(0) = 1$$

$$s^2 \bar{x}(s) - s\,x(0) - \dot{x}(0) + 2\left(s\,\bar{x}(s) - x(0)\right) = \frac{e^{-s}}{s}$$

$$\left(s^2 + 2s\right)\bar{x}(s) - (s+2)\overset{0}{\cancel{x(0)}} - \overset{1}{\dot{x}(0)} = \frac{e^{-s}}{s}$$

$$\left(s^2 + 2s\right)\bar{x}(s) = \frac{e^{-s}}{s} + 1$$

$$\bar{x}(s) = \frac{e^{-s}}{s^2(s+2)} + \frac{1}{s(s+2)}$$

$$\frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right)$$

$$\frac{e^{-s}}{s^2(s+2)} = -\frac{e^{-s}}{4s} + \frac{1}{2}\frac{e^{-s}}{s^2} + \frac{1}{4}\frac{e^{-s}}{s+2}$$

$$x(t) = \frac{1}{4} H(t-1) + \frac{1}{2} H(t-1)\frac{t}{\cancel{\Gamma(2)}}1 + \frac{1}{4} H(t-1)\, e^{-2t}$$

$$= \frac{1}{4} H(t-1)\left\{1 + 2t + e^{-2t}\right\}$$

6.28 b

$$\ddot{x} + x = |\sin t| \qquad x(0) = \dot{x}(0) = 0$$

Step. 1

$$|\sin t| = [H(t) - H(t-\pi)]\sin t$$
$$- [H(t-\pi) - H(t-2\pi)]\sin t$$
$$+ [H(t-2\pi) - H(t-3\pi)]\sin t$$
$$- \dots$$

$$= \sin t - 2H(t-\pi)\underbrace{\sin t}_{-\sin(t-\pi)} + 2H(t-2\pi)\underbrace{\sin t}_{\sin(t-2\pi)} - \dots$$

$$= \sin t + 2H(t-\pi)\sin(t-\pi) + 2H(t-2\pi)\sin(t-2\pi)$$
$$= \sin t + 2\sum_{n=1}^{\infty} H(t-n\pi)\sin(t-n\pi)$$

Step 2 Solution to

$$\ddot{x} + x = H(t-n\pi)\sin(t-n\pi) \Big/ \mathcal{L} \qquad x(0) = \dot{x}(0) = 0$$

$$(s^2+1)\bar{x}(s) = e^{-n\pi s}\frac{1}{s^2+1}$$

$$\bar{x}(s) = e^{-n\pi s}\frac{1}{(s^2+1)^2} = e^{-n\pi s}\frac{1}{s^2+1}\;\frac{1}{s^2+1}$$

$$x(t) = H(t-n\pi)\int_0^{t-n\pi} \underbrace{\sin\tau\;\sin(t-n\pi-\tau)}_{-\frac{1}{2}(\cos(t-n\pi)-\cos(2\tau-(t-n\pi)))}\,d\tau$$

$$= \frac{1}{2}H(t-n\pi)\left(\sin(t-n\pi) - (t-n\pi)\cos(t-n\pi)\right)$$

<u>Step 3</u>

Using the superposition:

$$x(t) = \frac{1}{2}(\sin t - t\cos t)$$

$$+ \sum_{n=1}^{\infty} H(t-n\pi)\left[\sin(t-n\pi) - (t-n\pi)\cos(t-n\pi)\right]$$