CAM 389C Exercise Set II.1

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Problem 1

Ten football teams in a college conference had the following record last season: Three teams 7 and 4, four teams 8 and 3, two teams 9 and 2, and one team 11 and 0. The total number of teams in the conference:

$$N = \sum_{j=0}^{\infty} N(j) = 10\,,$$

where $N(j)$ is the number of teams with j wins. This set of teams is the sample set Ω.

a) Show that the probability that a team selected randomly has j wins is

$$\mathbb{P}(j) = \frac{N(j)}{N}\,.$$

b) Show another property,

$$\sum_{j=0}^{\infty} \mathbb{P}(j) = 1\,.$$

c) Demonstrate by a full calculation that

$$\langle j \rangle = \sum_{j=0}^{\infty} j\mathbb{P}(j)\,.$$

d) Compute the variance and standard deviation,

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2 = \sum_{j=0}^{\infty} (j - \langle j \rangle)^2 \mathbb{P}(j)\,.$$

Solution

a) There are $N = 10$ possible teams, then the probability of randomly choosing a team with j wins is the number of teams with j wins divided by the total number of teams.

b)

$$\sum_{j=0}^{\infty} \mathbb{P}(j) = \sum_{j=0}^{\infty} \frac{N(j)}{N}$$

Note that

$$N(0) = \cdots = N(6) = N(10) = N(12) = \cdots = N(\infty) = 0\,.$$

So,

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{N(j)}{N} &= \frac{N(7)}{10} + \frac{N(8)}{10} + \frac{N(9)}{10} + \frac{N(11)}{10} \\ &= \frac{3}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} = 1\end{aligned}$$

c) By definition,

$$\langle x \rangle = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x$$

but j is defined on $\mathbb{N}$, and for our problem $\rho(x) \equiv \mathbb{P}(j)$. Therefore

$$\langle j \rangle = \sum_{j=0}^{\infty} j\mathbb{P}(j) = 7(0.3) + 8(0.4) + 9(0.2) + 11(0.1) = 8.2\,.$$

d) We have two ways of calculating the variance. First, we have already calculated $\langle j \rangle$, so $\langle j^2 \rangle$ is

$$\langle j^2 \rangle = \sum_{j=0}^{\infty} j^2\mathbb{P}(j) = 7^2(0.3) + 8^2(0.4) + 9^2(0.2) + 11^2(0.1) = 68.6\,.$$

Then,

$$\sigma^2 = \langle j^2 \rangle - \langle j \rangle^2 = 68.6 - (8.2)^2 = 1.36\,.$$

Alternatively, we could calculate directly:

$$\sigma^2 = \sum_{j=0}^{\infty}(j-\langle j \rangle)^2\mathbb{P}(j) = (7-8.2)^2(0.3)+(8-8.2)^2(0.4)+(9-8.2)^2(0.2)+(11-8.2)^2(0.1) = 1.36\,.$$

The standard deviation is just the square root of the variance,

$$\sigma = \sqrt{1.36} = 1.166\,.$$

Problem 2

The Gaussian probability density function in one dimension is of the form

$$\rho(x) = C\mathrm{e}^{-\frac{\alpha}{2}(x-x_0)^2}\,,$$

where x_0 is a point on the real line and C and α are constants.
a) Determine C.
b) Determine $\langle x \rangle$, $\langle x^2 \rangle$.
c) Determine σ.
d) Sketch a graph of $\rho(x)$.

Solution

a) We need

$$\int_{-\infty}^{\infty} C\mathrm{e}^{-\frac{\alpha}{2}(x-x_0)^2}\,\mathrm{d}x = 1\,.$$

Perform a change of variable, $y = \sqrt{\frac{\alpha}{2}}(x - x_0)$, then $\mathrm{d}x = \sqrt{\frac{2}{\alpha}}$, and the integral becomes

$$C\sqrt{\frac{2}{\alpha}}\int_{-\infty}^{\infty} \mathrm{e}^{-y^2}\,\mathrm{d}y = C\sqrt{\frac{2}{\alpha}}\sqrt{\pi}\,.$$

Therefore,

$$C = \sqrt{\frac{\alpha}{2\pi}}\,.$$

b) In order to find $\langle x \rangle$, we need to integrate x against $\rho(x)$ from $-\infty$ to ∞. We use Mathematica to evaluate this more complicated integral:

$$\langle x \rangle = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} x\mathrm{e}^{-\frac{\alpha}{2}(x-x_0)^2}\,\mathrm{d}x = x_0\,.$$

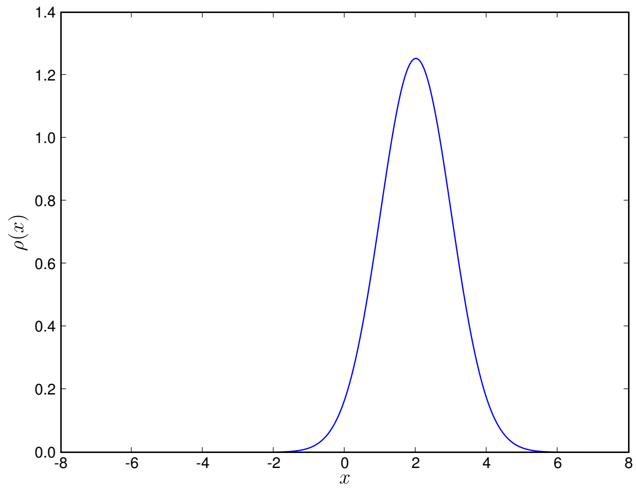
In the same manner we find $\langle x^2 \rangle$,

$$\langle x^2 \rangle = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty} x^2\mathrm{e}^{-\frac{\alpha}{2}(x-x_0)^2}\,\mathrm{d}x = \frac{1}{\alpha} + x_0^2\,.$$

c) We know that

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{\alpha} + x_0^2 - x_0^2} = \frac{1}{\sqrt{\alpha}}\,.$$

d) Choosing $\alpha = 1$ and $x_0 = 2$, we can sketch the probability density function.



Problem 3

Let

$$\Psi(x) = \begin{cases} A(1+x) & -1 \le x \le 0\,, \\ A(1-x) & 0 \le x \le 1\,, \\ 0 & x \notin [-1,1]\,. \end{cases}$$

a) Determine A.
b) Determine $\langle x \rangle$, $\langle x^2 \rangle$.
c) Determine σ_x.
d) Determine $\langle p \rangle$.

Solution

a) We know that

$$\rho(x) = \Psi(x)^* \Psi(x) = \begin{cases} A^2(1+2x+x^2) & -1 \le x \le 0\,, \\ A^2(1-2x+x^2) & 0 \le x \le 1\,, \\ 0 & x \notin [-1,1]\,. \end{cases}$$

We must scale A so that

$$\begin{aligned} \int_{-\infty}^{\infty} \rho(x)\,\mathrm{d}x &= \int_{-1}^{0} A^2(1+2x+x^2)\,\mathrm{d}x + \int_{0}^{1} A^2(1-2x+x^2)\,\mathrm{d}x \\ &= \frac{2}{3}A^2 = 1\,. \end{aligned}$$

Therefore,

$$A = \pm\sqrt{\frac{3}{2}}\,.$$

b) Now we can find

$$\begin{aligned}\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x &= \int_{-1}^{0} A^2(x+2x^2+x^3)\,\mathrm{d}x + \int_0^1 A^2(x-2x^2+x^3)\,\mathrm{d}x \\ &= -\frac{1}{8}+\frac{1}{8} = 0\,,\end{aligned}$$

and

$$\begin{aligned}\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho(x)\,\mathrm{d}x &= \int_{-1}^{0} A^2(x^2+2x^3+x^4)\,\mathrm{d}x + \int_0^1 A^2(x^2-2x^3+x^4)\,\mathrm{d}x \\ &= \frac{1}{20}+\frac{1}{20} = \frac{1}{10}\,.\end{aligned}$$

c) This gives us

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{1}{10}} = 0.316\,.$$

d) The equation for $\langle p\rangle$ is

$$\langle p\rangle = m\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \int_{-\infty}^{\infty} \Psi^* p\Psi\,\mathrm{d}x\,,$$

where $p\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi$, and

$$\frac{\partial}{\partial x}\Psi = \begin{cases} A & -1\le x\le 0\,, \\ -A & 0\le x\le 1\,, \\ 0 & x\notin[-1,1]\,. \end{cases}\,.$$

Therefore,

$$\langle p\rangle = \int_{-\infty}^{\infty}\Psi^* p\Psi\,\mathrm{d}x = \frac{\hbar}{i}\left[\int_{-1}^{0} A^2(1+x)\,\mathrm{d}x + \int_0^1 -A^2(1-x)\,\mathrm{d}x\right] = 0\,.$$

Problem 4

The state of a quantum system is given by

$$\Psi(x,t) = \alpha\mathrm{e}^{-\Lambda}\,,$$

where

$$\Lambda = \beta\hbar^{-1}(mx^2 + i\gamma t)\,,$$

and in which $\alpha\,,\beta$, and γ are constants.
a) Find the potential energy $V(x)$ of this system.
b) Calculate the expected values of x, x^2, p, and p^2.
c) Calculate σ_x and σ_p. Are these consistent with the Heisenberg principle?
d) What is α?

Solution

a) Schrodinger's equation with potential energy is

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) + \frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi = 0\,.$$

Isolating $V(x)$,

$$V(x)\Psi = \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi - \frac{\hbar}{i}\frac{\partial}{\partial t}\Psi\,.$$

We can take the partial derivatives of Ψ,

$$\frac{\partial^2}{\partial x^2}\Psi = -2\frac{\beta m}{\hbar}\Psi + 4\frac{\beta^2 m^2 x^2}{\hbar^2}\Psi\,,$$

and

$$\frac{\partial}{\partial t}\Psi = -\frac{i\beta\gamma}{\hbar}\Psi\,.$$

Therefore

$$\begin{aligned} V(x) &= \Psi^{-1}\left(\frac{\hbar^2}{2m}\left(-2\frac{\beta m}{\hbar}\Psi + 4\frac{\beta^2 m^2 x^2}{\hbar^2}\Psi\right) - \frac{\hbar}{i}\left(-\frac{i\beta\gamma}{\hbar}\Psi\right)\right) \\ &= \beta(-h + 2\beta m x^2) - \beta\gamma \\ &= \beta(-h + 2\beta m x^2 + \gamma)\,. \end{aligned}$$

b) Assuming α, β, γ are real, the complex conjugate of Ψ is

$$\Psi^* = \alpha \mathrm{e}^{-\frac{\beta}{\hbar}(mx^2 - i\gamma t)}\,,$$

and the probability density function is

$$\begin{aligned} \rho(x) = \Psi^*\Psi &= \alpha^2 \mathrm{e}^{\frac{\beta}{\hbar}(mx^2 - i\gamma t) - \frac{\beta}{\hbar}(mx^2 + i\gamma t)} \\ &= \alpha^2 \mathrm{e}^{-2\frac{\beta}{\hbar}mx^2}\,. \end{aligned}$$

We need to normalize this so that

$$\int_{-\infty}^{\infty} \alpha^2 \mathrm{e}^{-2\frac{\beta}{\hbar}mx^2} = \alpha^2\sqrt{\frac{\hbar\pi}{2\beta m}} = 1\,.$$

This implies that

$$\alpha = \pm\left(\frac{2\beta m}{\hbar\pi}\right)^{\frac{1}{4}}\,.$$

Now we can find (via Mathematica),

$$\langle x\rangle = \int_{-\infty}^{\infty} x\rho(x)\,\mathrm{d}x = \alpha^2\int_{-\infty}^{\infty} x\mathrm{e}^{-2\frac{\beta}{\hbar}mx^2}\,\mathrm{d}x = 0\,,$$

and

$$\langle x^2\rangle = \int_{-\infty}^{\infty} x^2\rho(x)\,\mathrm{d}x = \alpha^2\int_{-\infty}^{\infty} x\mathrm{e}^{-2\frac{\beta}{\hbar}mx^2}\,\mathrm{d}x = \frac{\cancel{\alpha^2}\hbar}{4\beta m}\cancel{\sqrt{\frac{\hbar\pi}{2\beta m}}} = \frac{\hbar}{4\beta m}\,.$$

Also the expected value of momentum is

$$\langle p\rangle = \int_{-\infty}^{\infty}\Psi^* p\Psi\,\mathrm{d}x\,,$$

where

$$p\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi = \frac{\hbar}{i}\left(-\frac{2\beta m x}{\hbar}\right)\Psi = 2i\beta m x\Psi\,.$$

Therefore

$$\langle p\rangle = 2i\beta m\int_{-\infty}^{\infty} x\Psi^*\Psi\,\mathrm{d}x = 0\,.$$

We can also calculate

$$\langle p^2\rangle = \int_{-\infty}^{\infty}\Psi^* p^2\Psi\,\mathrm{d}x\,,$$

where

$$p^2\Psi = -\hbar^2\frac{\partial^2}{\partial x^2}\Psi = -\hbar^2\left(-\frac{2\beta m}{\hbar} + \frac{4\beta m^2x^2}{\hbar^2}\right)\Psi = (2\beta\hbar m - 4\beta m^2x^2)\Psi\,.$$

Therefore

$$\begin{aligned}
\langle p^2\rangle &= \int_{-\infty}^{\infty}(2\beta\hbar m - 4\beta^2m^2x^2)\Psi^*\Psi\,\mathrm{d}x \\
&= 2\beta\hbar m\underbrace{\int_{-\infty}^{\infty}\rho(x)\,\mathrm{d}x}_{1} - 4\beta^2m^2\underbrace{\int_{-\infty}^{\infty}x^2\rho(x)\,\mathrm{d}x}_{\langle x^2\rangle} \\
&= 2\beta\hbar m - 4\beta^2m^2\left(\frac{\hbar}{4\beta m}\right) \\
&= 2\beta\hbar m - \beta m\hbar \\
&= \beta\hbar m\,.
\end{aligned}$$

c) Now we can calculate the standard deviations,

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}\sqrt{\frac{\hbar}{\beta m}}\,,$$

and

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\beta\hbar m}\,.$$

These are consistent with the Heisenberg principle because

$$\sigma_x\sigma_p = \frac{1}{2}\sqrt{\frac{\hbar}{\beta m}}\sqrt{\beta\hbar m} = \frac{1}{2}\hbar \geq \frac{1}{2}\hbar\,.$$

d) From earlier, we calculated that in order to normalize the probability density function correctly,

$$\alpha = \pm \left(\frac{2\beta m}{\hbar \pi} \right)^{\frac{1}{4}} .$$