Assignement # 26

25.3, 6b, 14, 15, 16

28.3, 6, 16

25.3

$$y' + \varepsilon y + y^2 = 0 \qquad y(0) = \cos\varepsilon$$

$$y(x) = y_0(x) + \varepsilon y_1(x) + \varepsilon^2 y_2(x) + \dots$$
$$y'(x) = y_0'(x) + \varepsilon y_1'(x) + \varepsilon^2 y_2'(x) + \dots$$

$$y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \dots + \varepsilon(y_0 + \varepsilon y_1 + \dots) + (y_0 + \varepsilon y_1 + \varepsilon y_2 + \dots)^2 = 0$$

$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \dots = 1 - \frac{\varepsilon^2}{2!} + \frac{\varepsilon^4}{4!} - \dots$$

ε^0: $\quad y_0' + y_0^2 = 0 \qquad y_0(0) = 1 \qquad (1)$

ε^1: $\quad y_1' + y_0 + 2y_0 y_1 = 0 \qquad y_1(0) = 0$

$$\Downarrow$$

$$y_1' + 2y_0 y_1 = -y_0 \qquad y_1(0) = 0 \qquad (2)$$

(1) Bernoulli's eqn (Problem 21.10, page 404)

Substitute $\quad v = y_0^{-1} \Rightarrow y_0 = v^{-1} \Rightarrow y_0' = -v^{-2}v'$

So we get $\quad -v^{-2}v' + v^{-2} = 0$

$$-v' + 1 = 0$$
$$v' = 1$$
$$v(x) = x + c$$

$$\therefore \quad y_0(x) = \frac{1}{x + c}$$

Check: $y_0' + y_0^2 = -\frac{1}{(x+c)^2} + \frac{1}{(x+c)^2} = 0$ O.K.

$y_0(0) = 1 \Rightarrow \frac{1}{c} = 1 \Rightarrow c = 1$

So: $y_0(x) = \frac{1}{x+1}$

(2) linear eqn.

$$y_1' + \frac{2}{x+1} y_1 = -\frac{1}{x+1} \qquad y_1(0) = 0$$

homog. eqn: $\frac{dy_1}{dx} = -\frac{2}{x+1} y_1$

$$\frac{dy_1}{y_1} = -\frac{2}{x+1} dx$$

$$\ln|y_1| = -2\ln(x+1) + c$$

$$|y_1|(x+1)^2 = c \quad \text{(some other } c\text{)}$$

$$y_1 = \pm \frac{c}{(x+1)^2} = \frac{c}{(x+1)^2} \quad \text{(redefined } c\text{)}$$

nonhom. eqn, variation of constant method

$$y_1' = \frac{c'(x+1)^2 - 2c(x+1)}{(x+1)^4}$$

$$y_1' + \frac{2}{x+1} y_1 = \frac{c'}{(x+1)^2} - \cancel{\frac{2c}{(x+1)^3}} + \cancel{\frac{2c}{(x+1)^3}} = -\frac{1}{x+1}$$

$$c' = -(x+1) \Rightarrow c(x) = -\frac{x^2}{2} - x - C$$

General sol'n to the nonhom. eqn:

$$y_1 = -\frac{\frac{x^2}{2}+x+c}{(x+1)^2} = -\frac{1}{2}\,\frac{x^2+2x+1+c}{(x+1)^2} \qquad (c \text{ redefined})$$

$$= -\frac{1}{2} + \frac{c}{(x+1)^2}$$

$$y_1(0) = 0 \Rightarrow 0 = -\frac{1}{2} + c \Rightarrow c = \frac{1}{2}$$

So, $$y_1(x) = \frac{1}{2}\left(\frac{1}{(x+1)^2} - 1\right)$$

and $$y(x) = \frac{1}{x+1} + \frac{\varepsilon}{2}\left[\frac{1}{(x+1)^2} - 1\right]$$

<u>Exact solution</u> (Bernoulli's eqn., problem 21.10, page 404)

$$y' + \varepsilon y + y^2 = 0 \qquad y(0) = \cos\varepsilon$$

Substitute: $v = y^{-1} \Rightarrow y = v^{-1} \Rightarrow y' = -v^{-2}v'$

$$-v^{-2}v' + \varepsilon v^{-1} + v^{-2} = 0 \quad /{-v^2}$$
$$v' - \varepsilon v - 1 = 0$$
$$v' - \varepsilon v = 1$$

$$v' - \varepsilon v = 0 \qquad r - \varepsilon = 0 \Rightarrow v(x) = c e^{\varepsilon x}$$

Variation of constant: $c' e^{\varepsilon x} + \cancel{c\varepsilon e^{\varepsilon x}} - \cancel{c\varepsilon e^{\varepsilon x}} = 1$

$$c' = e^{-\varepsilon x}$$
$$c(x) = -\frac{1}{\varepsilon} e^{-\varepsilon x} + C$$

So: $$v(x) = \left(-\frac{1}{\varepsilon} e^{-\varepsilon x} + c\right) e^{\varepsilon x}$$
$$= c e^{\varepsilon x} - \frac{1}{\varepsilon}$$

$$y(x) = \frac{1}{C e^{\varepsilon x} - \frac{1}{\varepsilon}}$$

Check:

$$y' + \varepsilon y + y^2 = -\frac{1}{(Ce^{\varepsilon x} - \frac{1}{\varepsilon})^2} Ce^{\varepsilon x} \cdot \varepsilon + \frac{\varepsilon}{Ce^{\varepsilon x} - \frac{1}{\varepsilon}} + \frac{1}{(Ce^{\varepsilon x} - \frac{1}{\varepsilon})^2}$$

$$= \frac{-\varepsilon}{(Ce^{\varepsilon x} - \frac{1}{\varepsilon})^2}\left[Ce^{\varepsilon x} - \frac{1}{\varepsilon}\right] + \frac{\varepsilon}{Ce^{\varepsilon x} - \frac{1}{\varepsilon}} = 0$$

$$y(0) = \frac{1}{C - \frac{1}{\varepsilon}} = \cos\varepsilon$$

$$C - \frac{1}{\varepsilon} = \frac{1}{\cos\varepsilon} \Rightarrow C = \frac{1}{\cos\varepsilon} + \frac{1}{\varepsilon}$$

So: $$y(x) = \frac{1}{\left(\frac{1}{\cos\varepsilon} + \frac{1}{\varepsilon}\right)e^{\varepsilon x} - \frac{1}{\varepsilon}}$$

Convergence in ε $\quad (x \geq 0 \ !)$

1. For $|\varepsilon| \ll 1$ $\quad \cos\varepsilon \approx 1$ so:

$$\left(\frac{1}{\cos\varepsilon} + \frac{1}{\varepsilon}\right)e^{\varepsilon x} - \frac{1}{\varepsilon} \sim \left(1 + \frac{1}{\varepsilon}\right)e^{\varepsilon x} - \frac{1}{\varepsilon}$$

$$= e^{\varepsilon x} + \frac{1}{\varepsilon}\underbrace{(e^{\varepsilon x} - 1)}_{\vee \atop 0} > e^{\varepsilon x} + \frac{1}{\varepsilon} > 0$$

So the series $y_0 + \varepsilon y_1 + \ldots$ will converge to $y(x)$ for ε small enough.

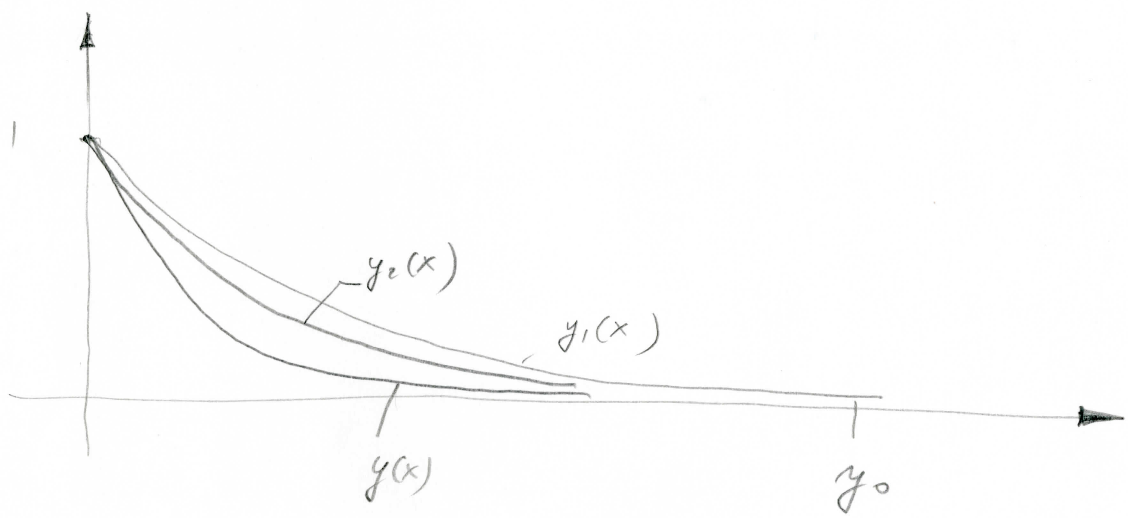
2. Is the convergence uniform in x? Most likely* yes, since for small ε

$$y(x) \sim \frac{1}{\left(1+\frac{1}{\varepsilon}\right)e^{\varepsilon x} - \frac{1}{\varepsilon}} = \frac{1}{e^{\varepsilon x} + \frac{1}{\varepsilon}\left(e^{\varepsilon x} - 1\right)}$$

$$= \frac{1}{e^{z} + \frac{x}{z}\left(e^{z} - 1\right)} \qquad \varepsilon x = z, \quad \frac{1}{\varepsilon} = \frac{x}{z}$$

can be expanded in z uniformly in x

Thus the perturbation is <u>regular</u>.



(*) We could probably prove that all iterates are negative and then use the Dini theorem.

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25.66

$$(1+\varepsilon x)y'' + y = x \qquad y(0)=\varepsilon, \quad y(1)=1 \qquad (0<x<1)$$

$$(1+\varepsilon x)(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \dots) + y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots = x$$
$$y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \dots = \varepsilon$$
$$y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1) + \dots = 1$$

ε^0: $\quad y_0'' + y_0 = x \qquad y_0(0) = \varepsilon, \quad y_0(1) = 1$

$$y_0'' + y_0 = 0 \qquad r^2+1=0 \qquad r = \pm i$$
$$y_0(x) = C\cos x + D\sin x$$

general integral: $\quad y_0(x) = C\cos x + D\sin x + x$

$$y_0(0) = C = 0 \quad \Rightarrow \quad C = 0$$
$$y_0(1) = D\sin 1 + 1 = 0 \quad \Rightarrow \quad D = -\frac{1}{\sin 1}$$

$$y_0(x) = -\frac{\sin x}{\sin 1} + x$$

ε^1: $\quad y_1'' + x y_0'' + y_1 = 0 \qquad y_1(0) = 1 \qquad y_1(1) = 0$

$$y_1'' + y_1 = \frac{x \sin x}{\sin 1} - x$$

⋮

Due to the fact that $x \in (0,1)$ the perturbation seems to be regular.

25.14

Composite solution: $x^c = x^i + x^o - (x^i)^o$

where $(x^i)^o$ is the two-term outer expansion of the two-term inner expansion = two term inner expansion of the two-term outer expansion

so: $x^c = e(\cancel{1} - \underline{e^{-t/\varepsilon}}) + e[\cancel{\varepsilon} - \cancel{t} - (\varepsilon + \underline{t})e^{-t/\varepsilon}]$ (25.57a)

$\quad + \underline{e^{1-t} + \varepsilon(1-t)e^{1-t}}$ (25.57b)

$\quad - \cancel{(e - et)} - \cancel{e\varepsilon}$ (25.55)

$$= [e^{1-t} - e^{1-\frac{t}{\varepsilon}} - te^{1-\frac{t}{\varepsilon}}] + \varepsilon[(1-t)e^{1-t} - e^{1-\frac{t}{\varepsilon}}]$$

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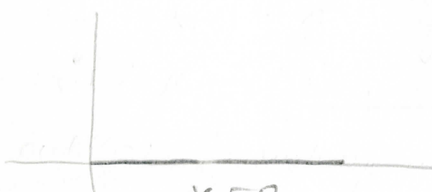
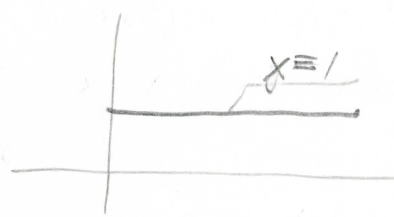
25.15

Consider even a simpler problem $\varepsilon x'' + x' = 0$ $x(0) = 0$, $x(1) = 1$. For $\varepsilon = 0$ the eqn degenerates to $x' = 0$ and dependently on which of the BC we use, we get one of the two solutions

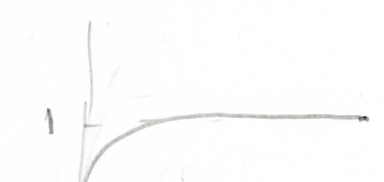


$x \equiv 0$



Adding term $\varepsilon x''$ and the second BC produces a boundary layer. A priori, we may have two situations

or



In the first case however both x'' and x' are positive, so the equation $\varepsilon x'' + x = 0$ $(\varepsilon > 0!)$ cannot be satisfied, and the second case will occur.

Discussion of the case with the zero-th order term goes along the same lines.

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25.16

$$x = \sum_{n=0}^{\infty} \varepsilon^n f_n(t) + e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n g_n(t)$$

$$\varepsilon \ddot{x} + \dot{x} + x = 0, \qquad x(0) = 0, \quad x(1) = 1$$

$$\dot{x} = \sum_{n=0}^{\infty} \varepsilon^n \dot{f}_n - \frac{1}{\varepsilon} e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n g_n + e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n \dot{g}_n$$

$$\ddot{x} = \sum_{n=0}^{\infty} \varepsilon^n \ddot{f}_n + \frac{1}{\varepsilon^2} e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n g_n - \frac{2}{\varepsilon} e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n \dot{g}_n$$

$$+ e^{-\frac{t}{\varepsilon}} \sum_{n=0}^{\infty} \varepsilon^n \ddot{g}_n$$

$$\varepsilon^2 \ddot{x} + \varepsilon \dot{x} + \varepsilon x =$$

$$= \varepsilon^2 \ddot{f}_0 + \ldots + e^{-\frac{t}{\varepsilon}} (g_0 + \varepsilon g_1 + \varepsilon^2 g_2 +) - 2\varepsilon e^{-\frac{t}{\varepsilon}} (\dot{g}_0 + \varepsilon \dot{g}_1 + \ldots)$$

$$+ \varepsilon^2 e^{-\frac{t}{\varepsilon}} (\ddot{g}_0 + \varepsilon \ddot{g}_1 + \ldots) + \varepsilon \dot{f}_0 + \varepsilon^2 \dot{f}_1 + \ldots$$

$$- e^{-\frac{t}{\varepsilon}} (g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \ldots) + \varepsilon e^{-\frac{t}{\varepsilon}} (\dot{g}_0 + \varepsilon \dot{g}_1 + \ldots)$$

$$+ \varepsilon f_0 + \varepsilon^2 f_1 + \ldots + e^{-\frac{t}{\varepsilon}} (\varepsilon g_0 + \varepsilon^2 g_1 + \ldots) = 0$$

ε^0: no terms

$\varepsilon^0 e^{-\frac{t}{\varepsilon}}$: $g_0 - g_0 = 0$ satisfied automatically !

ε^1: $\dot{f}_0 + f_0 = 0$

$\varepsilon^1 e^{-\frac{t}{\varepsilon}}$: $\cancel{g_1} - 2\dot{g}_0 - \cancel{g_1} + \dot{g}_0 + g_0 = -\dot{g}_0 + g_0 = 0$

ε^2: $\ddot{f}_0 + \dot{f}_1 + f_1 = 0$

$\varepsilon^2 e^{-\frac{t}{\varepsilon}}$: $\cancel{g_2} - 2\dot{g}_1 + \ddot{g}_0 - \cancel{g_2} + \dot{g}_1 + g_1 = -\dot{g}_1 + g_1 + \ddot{g}_0 = 0$

<u>BC</u> :

$$x(0) = \sum_{n=0}^{\infty} \varepsilon^n \left(f_n(0) + g_n(0) \right)$$

$\Rightarrow \quad f_n(0) + g_n(0) = 0 \qquad n = 0, 1, 2, \ldots$

$$x(1) = \sum_{n=0}^{\infty} \varepsilon^n f_n(1) + \underset{\substack{\uparrow \\ \text{small}}}{e^{-\frac{1}{\varepsilon}}} \sum_{n=0}^{\infty} \varepsilon^n g_n(1)$$

neglected

$$\approx \sum_{n=0}^{\infty} \varepsilon^n f_n(1) = 1$$

$\Rightarrow \quad f_0(1) = 1 \ , \quad f_n(1) = 0 \ , \quad n = 1, 2, \ldots$

Solving:

$f_0(t) = e^{1-t}$ $\qquad f_0(0) = e \Rightarrow g_0(0) = -e$

$g_0(t) = -e^{1+t}$

$\dot{f}_1 + f_1 = -\ddot{f}_0 = -e^{1-t} \Rightarrow f_1(t) = (c - et)e^{-t}$

$f_1(0) = 0 \Rightarrow c = 0 \Rightarrow f_1(t) = -te^{1-t}$

$\dot{g}_1 - g_1 = \ddot{g}_0 = -e^{1+t} \Rightarrow g_1(t) = (c - et)e^{t}$

$g_1(0) = 0 \Rightarrow c = 0 \Rightarrow g_1(t) = -te^{t+1}$

So: $\quad x(t) \approx f_0 + \varepsilon f_1 + e^{-\frac{t}{\varepsilon}}(g_0 + \varepsilon g_1)$

$$= e^{1-t}[1 - \varepsilon t] + e^{-\frac{t}{\varepsilon}+1+t}[-1 - \varepsilon t]$$

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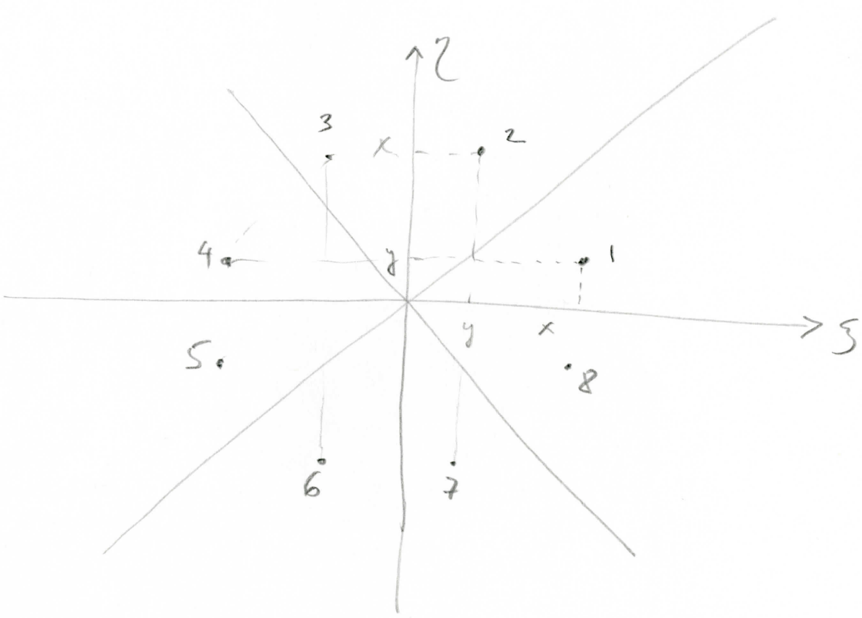
28.3

a)



We are looking for such a combination of eight free space Green functions

$$G = \sum_{i=1}^{8} c_i G_i$$

that the following conditions are satisfied:

1° $c_1 = 1$

2° $c_8 = -c_1$, $c_7 = -c_2$, $c_6 = -c_3$, $c_5 = -c_4$

(this will imply that $G \equiv 0$ along $\eta = 0$)

3° $c_2 = -c_1$, $c_3 = -c_8$, $c_4 = -c_7$, $c_5 = -c_6$

(this will imply that $G \equiv 0$ along $\xi = \eta$)

$$1°, 2° \Rightarrow \quad G = G_1 + c_2 G_2 + c_3 G_3 + c_4 G_4 - c_4 G_5 - c_3 G_6 - c_2 G_7 - G_8$$

$$3° \Rightarrow \quad G = G_1 - G_2 + G_3 - G_4 + G_5 - G_6 + G_7 - G_8$$

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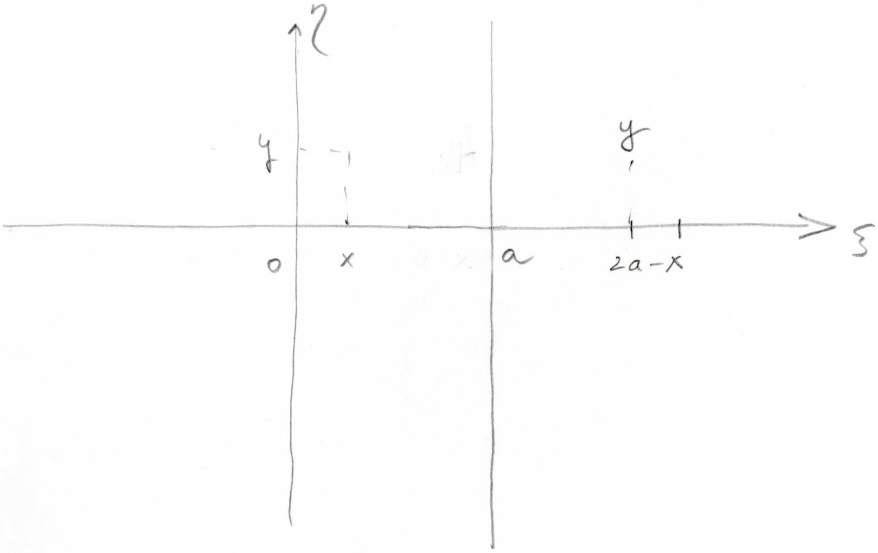
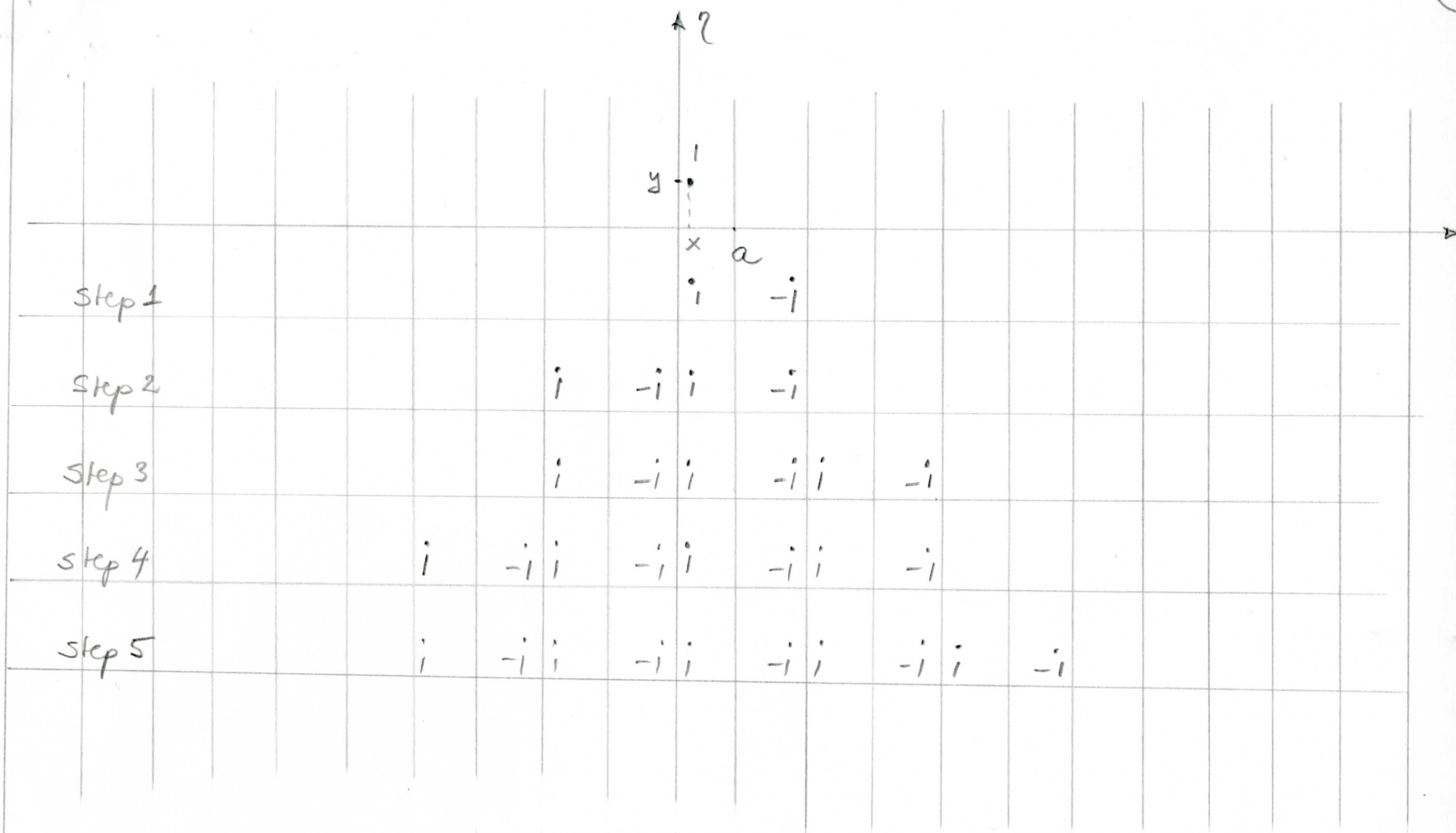
b)





Consequently, the final Green function is

$$G(\xi,\eta,x,y) = \sum_{i=-\infty}^{i=+\infty} \left[U(\xi,\eta;\, 2ia+x, y) - U(\xi,\eta;\, 2ia-x, y) \right]$$

\#

28.6

Requesting U to be an axisymmetric solution of the Helmholtz eqn. in polar coordinates centred at (x,y) we get

$$U = U(r), \qquad U'' + \frac{1}{r}U' + k^2 U = 0$$

$$rU'' + rU' + k^2 r^2 U = 0$$

$$\therefore \quad U(r) = A\, J_0(kr) + B\, Y_0(kr)$$

Repeating the argument from class we have

$$\lim_{\varepsilon \to 0} 2\pi \int_{\varepsilon}^{\infty} U\left(\varphi'' + \frac{1}{r}\varphi' + k^2\varphi\right) r\,dr = 1$$

for any test function $\varphi = \varphi(r)$ such $\varphi \equiv 1$ in a neighborhood of 0.

Integrating by parts, we get

$$\int_{\varepsilon}^{\infty} U\left(\varphi'' + \frac{1}{r}\varphi' + k^2\varphi\right) r\,dr$$

$$= \int_{\varepsilon}^{\infty} \left\{-(U + U'r)\varphi' - U'\varphi + k^2 U\varphi r\right\} dr$$

$$+ \Big[(U + U'r)\varphi' + U\varphi\Big]\Big|_{\varepsilon}^{\infty}$$

$$= \int_{\varepsilon}^{\infty} \left\{(U' + U''r \cancel{+ U'} \cancel{- U'})\varphi + k^2 U\varphi r\right\} dr$$

$$- (U + U'r)\varphi\Big|_{\varepsilon}^{\infty} + \left[(U + U'r)\varphi' + U\varphi\right]\Big|_{\varepsilon}^{\infty}$$

$$= \int_{\varepsilon}^{\infty} \overset{0}{\cancel{\left(U'' + \frac{U'}{r} + k^2 U\right)}} \varphi r\,dr + U'(\varepsilon)\,\varepsilon\, \overset{1}{\cancel{\varphi(\varepsilon)}}$$

(Notice that $\varphi'(\varepsilon) = 0$ for sufficiently small ε, $\varphi(\infty) = \varphi'(\infty) = 0$)

So it must be: $\lim_{\varepsilon \to 0} U'(\varepsilon)\,\varepsilon = \frac{1}{2\pi}$

or $\lim_{\varepsilon \to 0} \left[A J_0'(k\varepsilon) + B Y_0'(k\varepsilon)\right] k\varepsilon = \frac{1}{2\pi}$

But $J_0'(k\varepsilon) \longrightarrow 0$ as $\varepsilon \to 0$ (formula 22.79)

and $Y_0'(k\varepsilon) \simeq \frac{2}{\pi}\left[J_0'(k\varepsilon)\left(\ln \frac{k\varepsilon}{2} + \gamma\right) + J_0(k\varepsilon)\frac{1}{k\varepsilon} \right]$

$$\to \frac{2}{\pi k \varepsilon} \quad \text{(formula 22.84)}$$

Consequently: $B\frac{2}{\pi} = \frac{1}{2\pi}$

and $B = \frac{1}{4}$

There is a conceptual mistake in the book at this point. Without an extra assumption about the behavior of Green function at ∞ we cannot eliminate A. In acoustics, for instance, we implement the Sommerfeld BC (see problem 26.24) and, as a result of it, the Green function reduces to

$$\frac{1}{4i} H_0(kr) \quad \text{(The Hankel function)}$$

I guess, for some BC at ∞, we can live just with the Neumann function and assume $A = 0$.

b) $-\Delta u - k^2 u = 0 \qquad \eta > 0$

$u(\xi, 0) = f(\xi)$

The Green function (comp. Example solved in the class) is:

$$\frac{1}{4}\left(Y_0\left(k\sqrt{(\xi-x)^2+(\eta-y)^2}\right) - Y_0\left(k\sqrt{(\xi-x)^2+(\eta+y)^2}\right)\right) =: G(\xi,\eta;x,y)$$

And the final solution will be

$$u(x,y) = -\int_{\Gamma_\infty} u \frac{\partial G}{\partial n}$$

$$= \int_{-\infty} f(\xi) \frac{\partial G}{\partial \eta}$$

$$\frac{\partial G}{\partial \eta} = \frac{1}{4}\Big[Y_0'\left(k\sqrt{(\xi-x)^2+(\eta-y)^2}\right) \frac{k\,\cancel{2}(\eta-y)}{\cancel{2}\sqrt{(\xi-x)^2+(\eta-y)^2}}$$
$$- Y_0'\left(k\sqrt{(\xi-x)^2+(\eta+y)^2}\right) \frac{k\,\cancel{2}(\eta+y)}{\cancel{2}\sqrt{(\xi-x)^2+(\eta+y)^2}} \Big]$$

$$\frac{\partial G}{\partial \eta}(\xi,0) = -\frac{1}{2} Y_0'\left(k\sqrt{(\xi-x)^2+y^2}\right) \frac{y}{\sqrt{(\xi-x)^2+y^2}}$$

$$= -\frac{y}{2}\int_{-\infty}^{\infty} \frac{f(\xi)\, Y_0'\left(k\sqrt{(\xi-x)^2+y^2}\right)}{\sqrt{(\xi-x)^2+y^2}}$$

#

28.16

$$\int_\Omega -\overset{0}{\cancel{\Delta u}}\, G + \int_\Gamma \frac{\partial u}{\partial n} G = \int_\Omega \nabla u \nabla G = \underbrace{\int_\Omega u\,(-\overset{\delta}{\cancel{\Delta G}})}_{u(x,y)} + \int_\Gamma u \overset{0}{\cancel{\frac{\partial G}{\partial n}}}$$

def Green function

$$\begin{cases} -\Delta G = \delta(\xi-x, \eta-y) & \text{in } \Omega \\ \dfrac{\partial G}{\partial n} = 0 & \text{on } \Gamma \end{cases}$$

$$\Rightarrow \quad u(x,y) = \int_\Gamma \frac{\partial u}{\partial n} G$$

The free space Green function (see the lecture notes) is

$$U(\xi,\eta\,;x,y) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \qquad r = \sqrt{(\xi-x)^2+(\eta-y)^2}$$

By symmetry

$$G(\xi,\eta\,;x,y) = \frac{1}{2\pi}\left[\ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}}\right) + \ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta+y)^2}}\right)\right]$$

$$= \frac{1}{2\pi}\ln\left(\frac{1}{\sqrt{(\xi-x)^2+(\eta-y)^2}\,\sqrt{(\xi-x)^2+(\eta+y)^2}}\right)$$

So:

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\frac{\partial u}{\partial n}}_{= -UT'(\xi) \text{ on } (0,a),\ 0 \text{ elsewhere}} \ln\frac{1}{(\xi-x)^2+y^2}\,d\xi$$

$$= \frac{U}{2\pi}\int_0^a T'(\xi)\ln\left[(\xi-x)^2+y^2\right]d\xi$$

$\divideontimes$