ASE 380P2 ANALYTICAL METHODS II
EM386L MATHEMATICAL METHODS IN APPLIED MECHANIS II

Exam 3. Monday, May 3, 2010

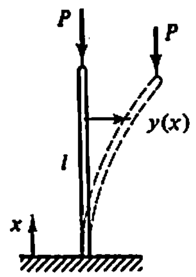
1. (a) State the Sturm–Liouville theorem (5 points).

   (b) Consider the problem of buckling a column with stiffness EI of length l, see the picture below.

   $$EIy'' = P[y(l) - y(x)], \quad y(0) = y'(0) = 0$$

   Is this a Sturm–Liouville eigenproblem ? Explain (5 points).

   (c) Determine the smallest eigenvalue P (the critical force) in terms of EI and length l (10 points).



Buckling of a column.

2. (a) Use separation of variables to determine eigenmodes of a square membrane,

   $$\begin{cases} -\Delta u &= \lambda u \qquad \text{in } D = (0,1)^2 \\ u &= 0 \qquad \text{on } \partial D \end{cases}$$

   (10 points).

   (b) Use the result to find the lowest eigenpair for the corresponding *triangular membrane* occupying triangle with vertices $(0,0), (1,0), (1,1)$. (10 points).

3. (a) Solve the following 2D boundary-value problem.

   $$-\Delta u = 1 \text{ in } 0 \le r \le 1; \quad u(1,\theta) = 0, \ \theta \in (0, 2\pi]$$

   (20 points).

4. (a) Explain why the following second order equation is hyperbolic (5 points).

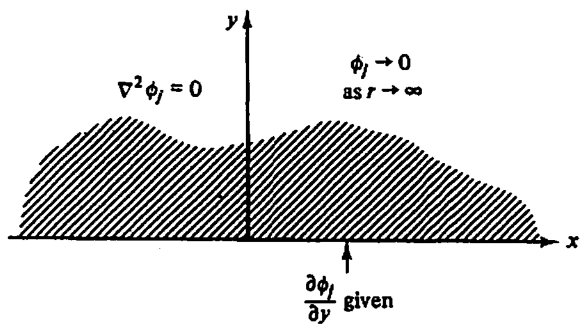
$$u_{xx} + 4u_{xy} + u_{yy} = 0$$

   (b) Replace the equation with an equivalent system of two first order equations. Is the system hyperbolic ? Explain (5 points).

   (c) Use the method of characteristics to derive the general solution to the system (10 points).

5. (a) Consider the Neumann problem for the Laplace equation in a half-plane shown below. Define the Green function for the problem (5 points).

   (b) Determine the Green function for the problem and solve it. You may use the fact that $1/2\pi \ln r$ is the 2D free–space Green function for the Laplace operator (15 points).



1a) Domain $= (a,b) \subset \mathbb{R}$

Operator $Ly = \frac{1}{w(x)} \{ -[p(x) y']' + r(x) y \}$

inner product:

$$(y, z) = \int_a^b w(x)\, y(x)\, z(x)\, dx$$

Boundary conditions:

Case I: $p(x) > 0$ in (a,b)

a) $y = 0$ (Dirichlet BC)

b) $y' = 0$ (Neumann BC)

c) $\alpha y + \beta y' = 0$, $\alpha, \beta \neq 0$ (Cauchy BC)

(one BC at $x=a$ and one at $x=b$)

Case II: periodic coefficients: $p(a) = p(b)$, $r(a) = r(b)$

$y(a) = y(b)$, $y'(a) = y'(b)$ (periodic BC)

Case III: $p = 0$ at $x=a$ or $x=b$

finite energy assumption $\Rightarrow$ y, y' finite at the point

(not a BC, really...)

$\longrightarrow\!\!\!\!\Rightarrow$

- Operator is self-adjoint, L has a countable number of eigenpairs (λ_n, y_n)
- y_n provide an L^2-orthogonal basis for the weighted L^2-space.

1b) Due to the presence of term $y(l)$ in the equation, the problem does not fit <u>directly</u> into the Sturm-Liouville framework.

1c) Differentiating the equ, we get

$$(EIy'')' + Py' = 0$$

Evaluating the original equ at $x=l$, we get an additional BC; $EIy''(l) = 0$

Consequently, substituting $y' = z$, we <u>do</u> get a Sturm-Liouville problem

$$\begin{cases} -(EIz')' = Pz \\ z(0)=0, \quad z'(l) = 0 \end{cases}$$

$$z = e^{rx} \Rightarrow EIr^2 + P = 0 \Rightarrow r = \pm i\sqrt{\tfrac{P}{EI}}$$

$$\therefore \quad z(x) = A\cos\sqrt{\tfrac{P}{EI}}\,x + B\sin\sqrt{\tfrac{P}{EI}}\,x$$

$$z(0) = 0 \Rightarrow A = 0$$

$$z' = B\sqrt{\tfrac{P}{EI}}\cos\sqrt{\tfrac{P}{EI}}\,x$$

$$z'(l) = 0 \Rightarrow \sqrt{\tfrac{P}{EI}}\,l = n\frac{\pi}{2} \qquad n = 1, 2, \ldots$$

($n = -1, -2$ give the same eigenpairs)

$$\therefore \quad \frac{P}{EI} = \frac{n^2\pi^2}{4l^2}$$

$$P = P_n = \frac{\pi^2 n^2 EI}{4l^2}$$

the smallest eigenvalue (critical force)

$$P_{crit} = \frac{\pi^2 EI}{(2l)^2}$$

#

2a

$$u = X(x)\,Y(y)$$

$$-X''Y - XY'' = \lambda XY$$

$$-\frac{X''}{X} = \frac{Y''}{Y} + \lambda = \mu \quad \text{(separation constant)}$$

$LX = -X''$ with BC: $X(0) = X(1)$ self-adjoint, > 0
implies that $\mu = k^2$, $k > 0$

$-X'' - k^2 X = 0 \Rightarrow X(x) = A\cos kx + B\sin kx$

$X(0) = 0 \Rightarrow A = 0$

$X(1) = 0 \Rightarrow k = k_n = n\pi$, $n = 1, 2, \dots$

$$\frac{Y''}{Y} + \lambda = k_n^2$$

$$-Y'' = (\lambda - k_n^2)Y$$

Same reasoning for Y leads to

$$\lambda - k_n^2 = k_m^2 \qquad k_m = m\pi, \quad m = 1, 2, \dots$$

We obtain thus a two-parameter family of eigenpairs:

$$\lambda = \lambda_{nm} = k_n^2 + k_m^2$$

$$u(x,y) = \sin k_n x \,\sin k_m y$$

2b) Critical observation is that $\lambda_{nm} = \lambda_{mn}$. Consequently, for $n \neq m$, λ_{nm} are <u>double</u> eigenvalues. This implies that <u>any</u> linear combination of $\sin nx \sin my$ and $\sin mx \sin nx$ is also an eigenvector. Taking

$$u_{nm}(x,y) = \sin nx \sin my - \sin mx \sin ny$$

we obtain $u = 0$ for $y = x$. This implies that

$$\lambda_{nm} = k_n^2 + k_m^2, \quad u_{nm} \qquad n, m = 1, 2, \ldots, \; n < m$$

are also eigenpairs for the triangular membrane.

$$\min \lambda_{nm} = \lambda_{12} = 5\pi^2$$

3a)

Step 1: Find a particular solution to

$$-\Delta u = 1$$

$$u = -\frac{1}{4}(x^2+y^2) = -\frac{1}{4}r^2$$

$$u_x = -\frac{1}{2}x, \quad u_{xx} = -\frac{1}{2} \quad \Rightarrow -u_{xx} - u_{yy} = 1$$

Step 2: Use ansatz: $u = -\frac{1}{4}r^2 + v$

then
$$-\Delta v = 0$$
$$v(1,\theta) = \frac{1}{4}$$

Step 3: Separation of variables:

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0 \qquad / \cdot r^2$$

$$v = R(r)\,\Theta(\theta)$$

$$-R\Theta'' = r(rR')'\Theta$$

$$-\frac{\Theta''}{\Theta} = \frac{r(rR')'}{R} = \lambda$$

$-\Theta''$ with periodic BC self-adjoint, positive semi-definite $\Rightarrow \lambda = k^2 \quad k \geq 0$

$$\Theta = A e^{ik\theta} + B e^{-ik\theta}$$

periodic BC's $\Rightarrow k = k_n = n \quad , \quad n \in \mathbb{Z}$

$$r(rR')' - k^2 R = 0$$

(Cauchy–Euler equ., use ansatz $R = r^\alpha$)

$$R' = \alpha r^{\alpha-1}$$

$$rR' = \alpha r^\alpha$$

$$(rR')' = \alpha^2 r^{\alpha-1}$$

$$r(rR')' = \alpha^2 r^\alpha$$

$$(\alpha^2 - k^2) r^\alpha = 0 \quad \Rightarrow \quad \alpha = \pm|k|$$

$$\therefore \quad R(r) = C r^{|k|} + D r^{-|k|}$$

$R(0)$ finite $\Rightarrow$ $D = 0$

By superposition,

$$v = \sum_{n=-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$$

$$v(1,\theta) = \sum_{n=-\infty}^{\infty} A_n e^{in\theta} = \frac{1}{4}$$

L^2-orthogonality of $e^{in\theta}$ implies that $A_0 = \frac{1}{4}$

this gives $v = \frac{1}{4}$

Final solution:

$$u(x,y) = \frac{1}{4}\left[1 - (x^2+y^2)\right]$$

4a) Eqn: $A u_{xx} + 2B u_{xy} + C u_{yy} = 0$

is hyperbolic, if $\begin{vmatrix} A & B \\ B & C \end{vmatrix} < 0$

In our case $\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 < 0$

4b) Define: $\varphi = u_x$, $\psi = u_y$. We get the system:

$$\begin{cases} \varphi_x + 2\varphi_y + 2\psi_x + \psi_y = 0 \\ \varphi_y - \psi_x = 0 \end{cases}$$

Look for α, β such that in the linear combination of the two equations

$$\alpha \varphi_x + (2\alpha+\beta)\varphi_y + (2\alpha-\beta)\psi_x + \alpha\psi_y = 0$$

vectors $(\alpha, 2\alpha+\beta)$ and $(2\alpha-\beta, \alpha)$ are LD, i.e.

there exists <u>two</u> (real) λ: $(\alpha, 2\alpha+\beta) = \lambda(2\alpha-\beta, \alpha)$

We get then a <u>single</u> equation for $\lambda\varphi + \psi$

$$(\alpha, 2\alpha+\beta)\circ(\varphi_x, \varphi_y) + (2\alpha-\beta, \alpha)\circ(\psi_x, \psi_y)$$
$$= \lambda(2\alpha-\beta, \alpha)\circ(\varphi_x, \varphi_y) + (2\alpha-\beta, \alpha)\circ(\psi_x, \psi_y)$$
$$= (2\alpha-\beta, \alpha)\circ((\lambda\varphi+\psi)_x, (\lambda\varphi+\psi)_y) = 0$$

which can be solved for $\lambda\varphi+\psi$. Once we know $\lambda\varphi+\psi$ for two values of λ, we can get both φ and ψ.

$(\alpha, 2\alpha+\beta)$ and $(2\alpha-\beta, \alpha)$ are LD iff

$$\begin{vmatrix} \alpha & 2\alpha+\beta \\ 2\alpha-\beta & \alpha \end{vmatrix} = \alpha^2 - (2\alpha+\beta)(2\alpha-\beta)$$
$$= \alpha^2 - (4\alpha^2 - \beta^2)$$
$$= \beta^2 - 3\alpha^2 = 0$$

So: $\beta = \pm\sqrt{3}\,\alpha$

For $\alpha = 1, \beta = \sqrt{3}$ $\qquad \lambda = \dfrac{2\alpha+\beta}{\alpha} = 2+\sqrt{3}$, we get

characteristic

$$\frac{dx}{2-\sqrt{3}} = dy$$

$$\frac{x}{2-\sqrt{3}} = y + C \quad \Rightarrow \quad \underbrace{\frac{x}{2-\sqrt{3}} - y}_{\text{first integral}} = C$$

So:

$$(2+\sqrt{3})\varphi + \psi = F\left(\frac{x}{2-\sqrt{3}} - y\right)$$

$\uparrow$ arbitrary function

For $\alpha = 1, \beta = -\sqrt{3}$ $\qquad \lambda = \dfrac{2\alpha+\beta}{\alpha} = 2-\sqrt{3}$, we get

$$\frac{dx}{2+\sqrt{3}} = dy \quad \Rightarrow \quad \frac{x}{2+\sqrt{3}} - y = C$$

So:

$$(2-\sqrt{3})\varphi + \psi = G\left(\frac{x}{2+\sqrt{3}} - y\right)$$

$$\varphi + (2-\sqrt{3})\psi = (2-\sqrt{3})\,F\left(\frac{x}{2-\sqrt{3}} - y\right)$$
$$\varphi + (2+\sqrt{3})\psi = (2+\sqrt{3})\,G\left(\frac{x}{2+\sqrt{3}} - y\right)$$

So:

$$\varphi = \frac{1}{2\sqrt{3}}\left\{F\left(\frac{x}{2-\sqrt{3}}-y\right) - G\left(\frac{x}{2+\sqrt{3}}-y\right)\right\}$$

$$\psi = \frac{1}{2\sqrt{3}}\left\{-(2-\sqrt{3})F\left(\frac{x}{2-\sqrt{3}}-y\right) + (2+\sqrt{3})G\left(\frac{x}{2+\sqrt{3}}-y\right)\right\}$$

$\#$

5a)

$$\int_D \underset{\sim}{\nabla} G \underset{\sim}{\nabla} \varphi = \int_D (\underbrace{-\Delta G}_{\overset{\shortparallel}{\delta}}) \varphi + \int_\Gamma \cancel{\frac{\partial G}{\partial n}}^{\,0} \varphi$$

$$\overset{\shortparallel}{\int_D \underset{\sim}{\nabla} \varphi \underset{\sim}{\nabla} G} = \int_D \cancel{(-\Delta\varphi)}^{\,0} G + \int_\Gamma \overbrace{\frac{\partial \varphi}{\partial n}}^{\text{known}} G$$

Setting $\quad -\Delta G = \delta(x-x_0, y-y_0)$

$$\frac{\partial G}{\partial n} = 0 \quad \text{on } \Gamma$$

we get

$$\varphi(x_0, y_0) = -\int_\Gamma \frac{\partial \varphi}{\partial n} G$$

5b) Use method of images to get

$$G(x,y,x_0,y_0) = \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y-y_0)^2} + \frac{1}{2\pi} \ln \sqrt{(x-x_0)^2 + (y+y_0)^2}$$

For $y=0$,

$$G(x,0,x_0,y_0) = \frac{1}{2\pi} \ln\left[(x-x_0)^2 + y_0^2\right]$$

So:

$$\varphi(x_0,y_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \ln\left[(x-x_0)^2 + y_0^2\right] \frac{\partial \varphi}{\partial n}(x) \, dx$$

#