H. Chapter 11, 12

11.3

(a) $z = -5$, $|z| = 5$, $\arg z = \pi$, $\operatorname{Re} z = -5$, $\operatorname{Im} z = 0$

(b) $z = 2 - 3i$, $|z| = \sqrt{4+9} = 5$

$$\arg z = \arctan\left(-\frac{3}{2}\right)$$

$$\operatorname{Re} z = 2, \quad \operatorname{Im} z = -3$$

(c) $z = \dfrac{i}{1-i} = \dfrac{i(1+i)}{1-i^2} = \dfrac{1}{2}(-1+i) = -\dfrac{1}{2} + i\dfrac{1}{2}$

$$|z| = \frac{\sqrt{2}}{2}, \quad \arg z = \frac{3}{4}\pi$$

$$\operatorname{Re} z = -\frac{1}{2}, \quad \operatorname{Im} z = \frac{1}{2}$$

(d) $z = (1+i)^{50} = z_0^{50}$, $z_0 = (1+i)$

$$|z_0| = \sqrt{2} \implies |z| = (\sqrt{2})^{50}$$

$$\arg z = \frac{\pi}{4} \implies \arg z = 50 \cdot \frac{\pi}{4} \pmod{2\pi}$$

$$= 12\tfrac{1}{2}\pi \pmod{2\pi}$$

$$= \frac{\pi}{2}$$

$$\operatorname{Re} z = 0, \quad \operatorname{Im} z = (\sqrt{2})^{50}$$

(e) $z = \left(\dfrac{1+i}{1-i}\right)^4 = z_0^4$, $z_0 = \dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{2} = i$

$$z = i^4 = (-1)^2 = 1$$

$|z| = 1$, $\arg z = 0$, $\operatorname{Re} z = 1$, $\operatorname{Im} z = 0$

11.6 a) $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$

$$(x_1 + \lambda x_2)^2 + (y_1 + \lambda y_2)^2 \geq 0 \qquad \forall \lambda$$

$$\lambda^2(x_2^2 + y_2^2) + 2\lambda(x_1 x_2 + y_1 y_2) + x_1^2 + y_1^2 \geq 0$$

$$\therefore \quad \Delta = 4(x_1 x_2 + y_1 y_2)^2 - 4(x_1^2 + y_1^2)(x_2^2 + y_2^2) \leq 0$$

or equivalently:

$$|x_1 x_2 + y_1 y_2| \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = |z_1||z_2|$$

Now

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

is equivalent to $\quad |z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

or

$$(x_1 + x_2)^2 + (y_1 + y_2)^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$

or $\qquad x_1 x_2 + y_1 y_2 \leq |z_1||z_2|$

which was proved in the first step.

It follows from the derivation, that the equality will hold only if $(x_1, y_1) = \lambda(x_2, y_2)$, i.e. $z_1 = \lambda z_2$, for some <u>real</u> λ

b) Use induction.

12.6

a) $f(z) = (1-4z^2)^8$

$$f'(z) = 8(1-4z^2)(-8z)$$

analytic (i.e. complex differentiable) everywhere!

b) $f(z) = \dfrac{x+iy}{x^2+y^2}$

$$u(x,y) = \frac{x}{x^2+y^2} \qquad v(x,y) = \frac{y}{x^2+y^2}$$

$$u_x = \frac{1(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$v_y = \frac{1(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$u_x \neq v_y$ f is nowhere complex differentiable

c) $f(z) = \dfrac{1}{z^2+3iz-2}$

$$z^2+3iz-2 = 0 \qquad \Delta = (3i)^2 + 8 = -9+8 = -1$$

$$\sqrt{\Delta} = \pm i$$

$$z_1 = \frac{-3i+i}{2} = -i$$

$$z_2 = \frac{-3i-i}{2} = -2i$$

Thus f is complex differentiable everywhere except for $z_1 = -i$ and $z_2 = -2i$

$$f'(z) = \frac{-1}{(z^2+3iz-2)^2}(2z+3i)$$

d) $f(z) = \sin\left(\frac{1}{z}\right)$

complex differentiable everywhere except for $z=0$

$$f'(z) = \cos\left(\frac{1}{z}\right)\left(-\frac{1}{z^2}\right)$$

e) $f(z) = |z| \sin z = |z| \dfrac{e^{iz} - e^{-iz}}{2i}$

$$= -\frac{1}{2}|z|\, i\left[e^{i(x+yi)} - e^{-i(x+yi)}\right]$$

$$= -\frac{1}{2}|z|\, i\left[e^{-y}e^{ix} - e^{y}e^{-ix}\right]$$

$$= -\frac{1}{2}|z|\, i\left[e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right]$$

$$= -\frac{1}{2}|z|\left(-e^{-y}\sin x + e^{y}\sin x\right) - \frac{1}{2}|z|\, i\left(e^{-y}\cos x - e^{y}\cos x\right)$$

$$= \underbrace{-\sqrt{x^2+y^2}\,\sin x \sinh y}_{u(x,y)} + \underbrace{\sqrt{x^2+y^2}\,\cos x \sinh y}_{v(x,y)}\, i$$

$$u_x = -\frac{2x}{2\sqrt{x^2+y^2}}\sin x \sinh y - \sqrt{x^2+y^2}\cos x \sinh y$$

$$v_y = -\frac{2y}{2\sqrt{x^2+y^2}}\cos x \sinh y + \sqrt{x^2+y^2}\cos x \cosh y$$

$u_x \neq v_y \Rightarrow f$ is nowhere complex differentiable

(In fact $f(z) = |z|$ is nowhere complex differentiable)

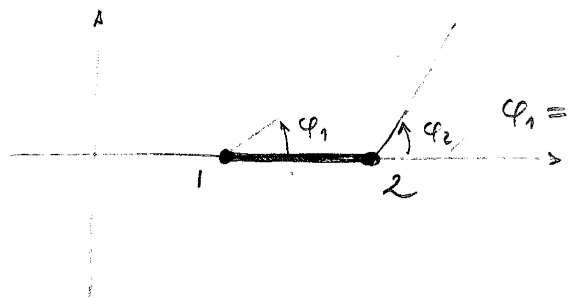
f) $f(x,y) = \underbrace{x+y}_{u(x,y)} + i\underbrace{(\sin x + \cos y)}_{v(x,y)}$

$$\left.\begin{aligned} u_x &= 1 \\ v_y &= -\sin y \end{aligned}\right\} \Rightarrow u_x \neq v_y$$

$\therefore$ f is nowhere complex differentiable

12.14

a)



$\varphi_1 = \varphi_2 = 0$

$$f(z) = \sqrt{(z-1)(z-2)} = \sqrt{r_1 e^{i\varphi_1}\, r_2 e^{i\varphi_2}}$$

$$= \sqrt{r_1 r_2}\; e^{i\,\varphi_1/2}\; e^{i\,\varphi_2/2}$$

$z = \frac{3}{2}$ $\qquad r_1 = r_2 = \frac{1}{2}$

On the upper side of the cut: $\varphi_1 = 0$, $\varphi_2 = \pi$

so: $f(z) = \frac{1}{2} e^{i0} e^{i\frac{\pi}{2}} = \frac{1}{2} \cdot 1 \cdot i = \frac{i}{2}$

On the lower side of the cut: $\varphi_1 = 2\pi$, $\varphi_2 = \pi$

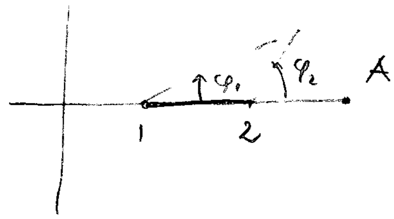
so: $f(z) = \frac{1}{2} e^{i\pi} e^{i\frac{\pi}{2}} = \frac{1}{2}(-1)i = -\frac{i}{2}$

Check: $(z-1)(z-2) = -\frac{1}{4}$

$\left(\frac{i}{2}\right)^2 = -\frac{1}{4}$, $\left(-\frac{i}{2}\right)^2 = -\frac{1}{4}$

b) No!

e.g. the values of φ_1 and φ_2 for point A



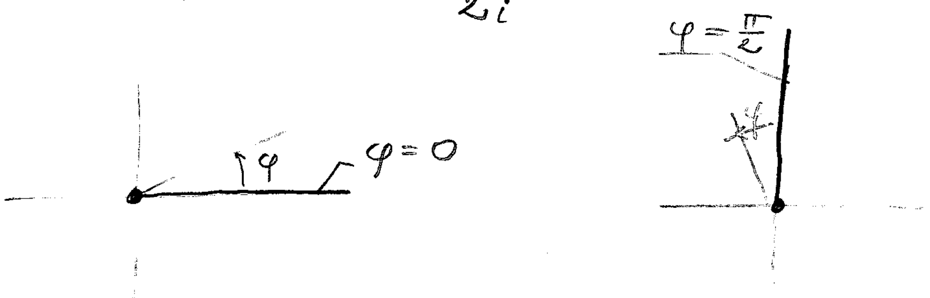
are $\varphi_1 = \varphi_2 = 2k\pi$, $k \in \mathbb{Z}$

and $f(z) = \sqrt[3]{r_1 r_2}\, e^{i\varphi_1/3}\, e^{i\varphi_2/3}$ is <u>not</u> k-independent

(try $k=0$ and $k=1$)

#

15a) $\sin\sqrt{z} = \dfrac{e^{i\sqrt{z}} - e^{-i\sqrt{z}}}{2i}$

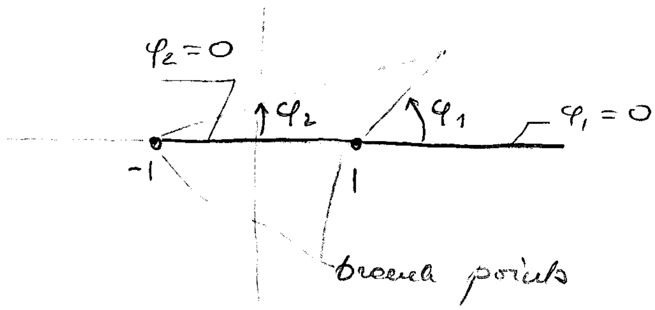


(same branch point and branch cuts as for $\sqrt{z}$)

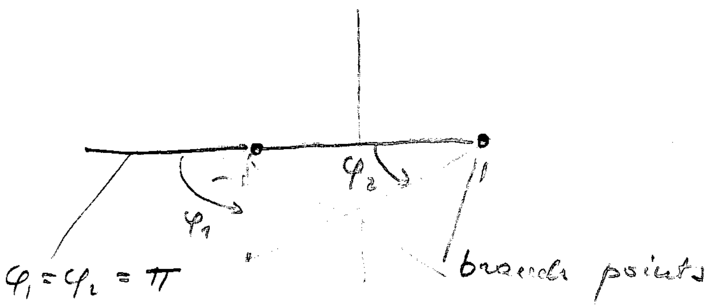
15b) $(\sqrt{z})^2$

Formally, function $\sqrt{z}$ has a branch point at $z=0$, so $(\sqrt{z})^2$ is undefined at that point. Elsewhere, $(\sqrt{z})^2 = z$ and of course $(\sqrt{z})^2$ can be extended to z at 0 as well, so $(\sqrt{z})^2$ has neither branch points nor branch cuts.

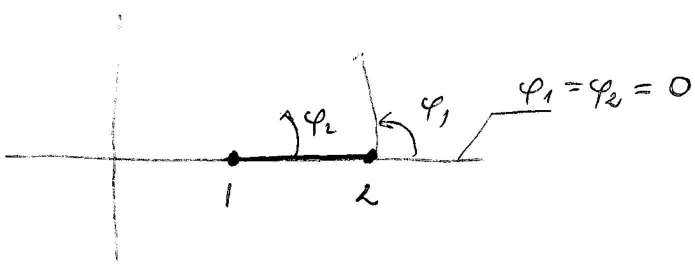
15g) $\sqrt{z+1} + \sqrt{z-1}$



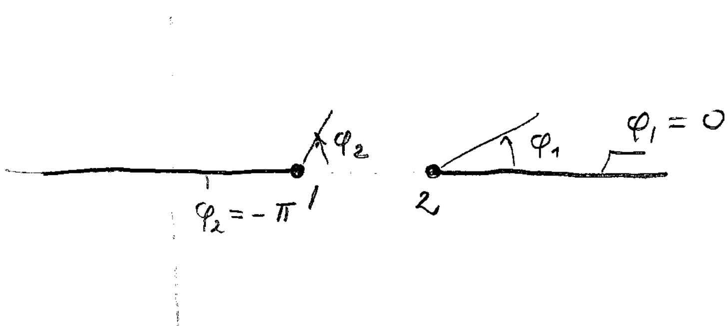
Another choice :



15h) $\sqrt{\dfrac{z-1}{z-2}}$



Another choice



(same as in Ex. 12.11)

15 i) $\cos\sqrt{z}$

(same as for $\sqrt{z}$)

16c) $(1-i)^i \overset{(def)}{=} e^{i\ln(1-i)} = e^{i\ln(e^{i\frac{3}{2}\pi})} = (*)$

$(1-i) = e^{i\frac{3}{2}\pi}$

$(*) = e^{i(i\frac{3}{2}\pi + 2k\pi)} = e^{\cancel{i2k\pi}} e^{-\frac{3}{2}\pi} = e^{-\frac{3}{2}\pi}$

16 d) $i^i = e^{i\ln i} = e^{i\ln(e^{i\frac{\pi}{2}})} = e^{i(i\frac{\pi}{2}+2k\pi)}$

$= e^{-\frac{\pi}{2}}$

sorry!

18 b) $\sin z = 0$

$$\frac{e^{iz}-e^{-iz}}{2i} = 0$$

$$e^{iz} = e^{-iz}$$

$$\frac{e^{iz}}{e^{-iz}} = e^{2iz} = 1$$

$$e^{2iz} = e^{2i(x+yi)} = e^{-2y}e^{2ix}$$

$$= e^{-2y}(\cos 2x + i\sin 2x) = 1$$

$\Rightarrow y=0 \qquad 2x = 2k\pi \Rightarrow x = k\pi$

$\underline{z = k\pi, \quad k\in\mathbb{Z}}$

18 c) $\cos z^3 = 0$

$\cos w = 0 \implies w = \frac{\pi}{2} + k\pi \quad , k \in \mathbb{Z}$

$z^3 = \frac{\pi}{2} + k\pi$

$z = re^{i\varphi} \quad , z^3 = r^3 e^{i3\varphi} = \frac{\pi}{2} + k\pi$

$\therefore \quad r^3 = \frac{\pi}{2} + k\pi \quad , \quad 3\varphi = 2l\pi \quad k, l \in \mathbb{Z}$

So: $\quad r = \sqrt[3]{\frac{\pi}{2} + k\pi} \quad , \quad k \in \mathbb{Z}$

$\varphi = \frac{2\pi}{3} l \quad , \quad l \in \mathbb{Z}$

18 d) $\sinh z = \dfrac{e^z - e^{-z}}{2}$

$e^z = e^{-z}$

$e^{2z} = 1$

$e^{2x} e^{i2y} = 1$

$x = 0, \quad y = 2k\pi$

So: $\quad z = i\,2k\pi \quad , \quad k \in \mathbb{Z}$

18 f) $\ln z = 0$

$z = e^0 = 1$

19 a) $e^z = -1$

$e^x e^{iy} = -1$

$x = 0 \quad y = -\pi + 2k\pi \quad , \quad k \in \mathbb{Z}$

$\therefore \quad z = i(2k-1)\pi$

19 d) $z^3 = -1$

$z = re^{i\varphi} \quad z^3 = r^3 e^{i3\varphi} = -1$

$r = 1 \quad 3\varphi = \pi + k2\pi \quad , \quad k \in \mathbb{Z}$

$\varphi = \frac{\pi}{3} + k\frac{2\pi}{3} \quad , \quad k \in \mathbb{Z}$

$\therefore \quad z = e^{i\frac{\pi}{3} + k\frac{2\pi}{3}} \quad , \quad k \in \mathbb{Z}$

19 e) $z^3 = -i$

$r = 1 \quad 3\varphi = \frac{3\pi}{2} + 2k\pi \quad , \quad k \in \mathbb{Z}$

$z = e^{i\left(\frac{\pi}{2} + k\frac{2\pi}{3}\right)} \quad , \quad k \in \mathbb{Z}$

---

20) $z^{\frac{3}{2}}$

$z = re^{i(\varphi + 2k\pi)} \qquad 0 \leqslant \varphi < 2\pi$

$z^{\frac{3}{2}} = r^{\frac{3}{2}} e^{i\frac{3}{2}(\varphi + 2k\pi)}$

$$e^{i\left(\frac{3}{2}\varphi + 3k\pi\right)} = \begin{cases} e^{i\frac{3}{2}\varphi} & \text{for } k \text{ even} \\ e^{i\left(\frac{3}{2}\varphi + \pi\right)} & \text{for } k \text{ odd} \end{cases}$$

Therefore $z^{\frac{3}{2}}$ is only <u>double</u> <u>valued</u>

Exactly the same situation holds for $z^{\frac{1}{2}}$, as

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{1}{2}(\varphi + 2k\pi)} \qquad k \in \mathbb{Z} \qquad 0 \leq \varphi < 2\pi$$

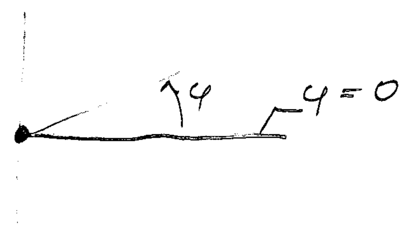
and

$$e^{i\frac{1}{2}(\varphi + 2k\pi)} = \begin{cases} e^{i\frac{\varphi}{2}} & \text{for } k \text{ even} \\ e^{i(\frac{\varphi}{2} + \pi)} & \text{for } k \text{ odd} \end{cases}$$

In order to make sense for the relation

$$\frac{d}{dz}\left(z^{\frac{3}{2}}\right) = \frac{3}{2} z^{\frac{1}{2}}$$

we have to use the same branch cut for both $z^{\frac{1}{2}}$ and $z^{\frac{3}{2}}$, e.g



$\#$