Homework # 21

20.1

e)

$$\int_0^1 t\,\tau\, x(\tau)\, d\tau = \lambda x(t) \qquad (1)$$

$$t \underbrace{\int_0^1 \tau\, x(\tau)\, d\tau}_{\text{a constant, say } c}$$

so the solution must be

$$ct = \lambda x(t) \Rightarrow x(t) = \frac{c}{\lambda} t$$

and $$c = \int_0^1 \tau \frac{c}{\lambda} \tau\, d\tau = \frac{c}{\lambda} \int_0^1 \tau^2 d\tau = \frac{c}{\lambda} \frac{1}{3}$$

$\therefore \quad 3\lambda = 1$

so $\lambda = \frac{1}{3}$ and $x(t) = 3ct \quad , c \in R$

f)

$$\underbrace{\int_0^1 x(\tau)\, d\tau}_{\text{a constant, say } c} = \lambda x(t)$$

so $x(t) = \frac{c}{\lambda}$

$$\int_0^1 \frac{c}{\lambda} d\tau = c \qquad \frac{c}{\lambda} - c = 0 \qquad c\left(\frac{1}{\lambda} - 1\right) = 0 \Rightarrow \lambda = 1$$

so $\lambda = 1$, $x(t) = c \quad , c \in R$

✗

20.5

$$y''' + \lambda y' = 0 \qquad y(0) = y'(0) = y''(l) = 0 \qquad \lambda = \frac{P}{EI}$$

$$y(x) = e^{rx}$$

$$r^3 + \lambda r = 0 \qquad (r^2 + \lambda) r = 0$$

$$r = \pm i\sqrt{\lambda} \quad \text{or} \quad r = 0$$

general solution

$$y(x) = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x + C_3$$

$$y(0) = 0 \Rightarrow C_1 + C_3 = 0$$

$$y'(x) = -C_1\sqrt{\lambda}\sin\sqrt{\lambda}\, x + C_2\sqrt{\lambda}\cos\sqrt{\lambda}\, x$$

$$y'(0) = 0 \Rightarrow C_2\sqrt{\lambda} = 0 \Rightarrow C_2 = 0$$

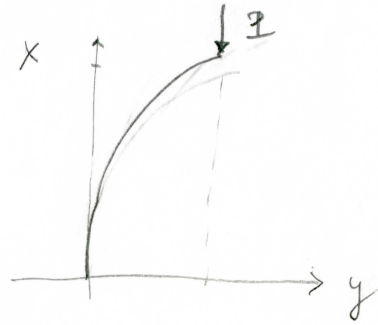
$$y''(x) = -C_1 \lambda \cos\sqrt{\lambda}\, x$$

$$y''(l) = 0 \Rightarrow \sqrt{\lambda}\, l = \frac{\pi}{2} + n\pi$$

$$\therefore \lambda = \lambda_n = \left[\left(\frac{\pi}{2} + n\pi\right)\frac{1}{l}\right]^2 \qquad n = 0, 1,$$

$$y_n(x) = C\left[\cos\sqrt{\lambda_n}\, x - 1\right] \qquad n = 0, 1, \ldots$$

First eigenmode: $y_0(x) = C\left[\cos\frac{\pi x}{2l} - 1\right]$



20.6

$$EIy'' = P[y(l) - y] - k\,y(l)(l-x) \qquad (1)$$

$$y(0) = y'(0) = 0 \qquad (2)$$

Differentiating (1) we get

$$EIy''' = -Py' + k\,y(l) \qquad (3)$$

and one more time

$$EIy^{(IV)} = -Py'' \quad \text{or} \quad y^{(IV)} + \overset{\lambda}{\overset{\shortparallel}{\frac{P}{EI}}}\, y'' = 0$$

Evaluating (1) and (3) at $x=l$ we get the two extra boundary conditions

$$y''(l) = 0 \quad \text{and} \quad EIy'''(l) = -Py'(l) + k\,y(l)$$

Characteristic equation $\quad r^4 + \lambda r^2 = 0$

Remark: Operator is self-adjoint and positive def. $\Rightarrow \lambda > 0$!

$$r^2(r^2 + \lambda) = 0$$

General solution:

$$y = C_1 \cos\sqrt{\lambda}\,x + C_2 \sin\sqrt{\lambda}\,x + C_3 x + C_4$$

$$y' = -C_1\sqrt{\lambda}\,\sin\sqrt{\lambda}\,x + C_2\sqrt{\lambda}\,\cos\sqrt{\lambda}\,x + C_3$$

$$y'' = -C_1\lambda\cos\sqrt{\lambda}\,x - C_2\lambda\sin\sqrt{\lambda}\,x$$

$$y''' = C_1\lambda^{\frac{3}{2}}\sin\sqrt{\lambda}\,x - C_2\lambda^{\frac{3}{2}}\cos\sqrt{\lambda}\,x$$

$$y(0) = 0 \quad \Rightarrow \qquad c_1 + c_4 = 0$$

$$y'(0) = 0 \quad \Rightarrow \qquad c_2 \sqrt{\lambda} + c_3 = 0$$

$$y''(l) = 0 \quad \Rightarrow \qquad -c_1 \cancel{\lambda} \cos\sqrt{\lambda}\, l - c_2 \cancel{\lambda} \sin\sqrt{\lambda}\, l = 0$$

$$y'''(l) + \lambda y'(l) - \varkappa y(l) = 0 \quad \Rightarrow \qquad\qquad \varkappa = \frac{k}{EI}$$

$$\cancel{c_1 \lambda^{\frac{3}{2}} \sin\sqrt{\lambda}\, l} - \cancel{c_2 \lambda^{\frac{3}{2}} \cos\sqrt{\lambda}\, l}$$
$$- \cancel{c_1 \lambda^{\frac{3}{2}} \sin\sqrt{\lambda}\, l} + \cancel{c_2 \lambda^{\frac{3}{2}} \cos\sqrt{\lambda}\, l} + c_3 \lambda$$
$$- c_1 \varkappa \cos\sqrt{\lambda}\, l - c_2 \varkappa \sin\sqrt{\lambda}\, l - c_3 \varkappa l - c_4 \varkappa = 0$$

We end up with a 4×4 system for $c_1, \ldots, c_4$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{\lambda} & 1 & 0 \\ -\cos\sqrt{\lambda}\, l & -\sin\sqrt{\lambda}\, l & 0 & 0 \\ -\varkappa\cos\sqrt{\lambda}\, l & -\varkappa \sin\sqrt{\lambda}\, l & \lambda - \varkappa l & -\varkappa \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The system has a nontrivial solution iff $\det(\;) = 0$

Using the Laplace expansion we get

$$1\left(-\varkappa \sin\sqrt{\lambda}\, l\right) - \left(\cancel{\varkappa \cos\sqrt{\lambda}\, l \sin\sqrt{\lambda}\, l} - \cancel{\varkappa \cos\sqrt{\lambda}\, l \sin\sqrt{\lambda}\, l} \right.$$
$$\left. + (\lambda - \varkappa l)\sqrt{\lambda} \cos\sqrt{\lambda}\, l\right) = 0$$

$$(\lambda - \varkappa l)\sqrt{\lambda} \cos\sqrt{\lambda}\, l + \varkappa \sin\sqrt{\lambda}\, l = 0$$

Substituting $\lambda = \Lambda^2$, we get

$$\underline{(\Lambda^2 - \varkappa l)\Lambda \cos\Lambda l + \varkappa \sin\Lambda l = 0} \qquad (*)$$

Eliminating e.g. the fourth equation and setting $c_2 = 1$, we solve the first three equations for c_1, c_3, c_4

$$c_3 = -\Lambda$$
$$-c_1 \cos \Lambda l = \sin \Lambda l \quad \Rightarrow \quad c_1 = -\tan \Lambda l$$
$$c_4 = \tan \Lambda l$$

So the eigenmodes (to within an arbitrary multiplicative constant) are :

$$y(x) = -\tan \Lambda l \cos \Lambda x + \sin \Lambda x - \Lambda x + \tan \Lambda l$$

where Λ's are solutions of $(*)$

Using the perturbation method :

$$\Lambda = \Lambda_0 + \varkappa \Lambda_1 , \qquad 0 < \varkappa \ll 1 \qquad (\text{small})$$

$$\cos \Lambda l = \cos (\Lambda_0 + \varkappa \Lambda_1) l$$
$$= \cos \Lambda_0 l \underbrace{\cos \varkappa \Lambda_1 l}_{\simeq 1} - \sin \Lambda_0 l \underbrace{\sin \varkappa \Lambda_1 l}_{\simeq \varkappa \Lambda_1 l}$$

$$\sin \Lambda l = \sin (\Lambda_0 + \varkappa \Lambda_1) l$$
$$= \sin \Lambda_0 l \underbrace{\cos \varkappa \Lambda_1 l}_{\simeq 1} - \cos \Lambda_0 l \underbrace{\sin \varkappa \Lambda_1 l}_{\simeq \varkappa \Lambda_1 l}$$

$$(\Lambda_0^2 + 2\Lambda_0 \varkappa \Lambda_1 + \varkappa^2 \Lambda_1^2 - \varkappa l)(\Lambda_0 + \varkappa \Lambda_1)(\cos \Lambda_0 l - \sin \Lambda_0 l \, \varkappa \Lambda_1 l)$$

$$+ \varkappa (\sin \Lambda_0 l - \cos \Lambda_0 l \, \varkappa \Lambda_1 l) = 0$$

$$\left[\Lambda_0^2 + x(2\Lambda_0\Lambda_1 - l) + x^2\Lambda_1^2\right](\Lambda_0 + x\Lambda_1)(\cos\Lambda_0 l - x\sin\Lambda_0 l\Lambda_1 l)$$

$$+ x(\sin\Lambda_0 l - x\cos\Lambda_0 l\Lambda_1 l) = 0$$

$$x^0\left(\Lambda_0^2 \cdot \Lambda_0 \cos\Lambda_0 l\right)$$

$$+ x^1\Big[(2\Lambda_0\Lambda_1 - l)\Lambda_0\cos\Lambda_0 l + \Lambda_0^2\Lambda_1\cos\Lambda_0 l - \Lambda_0^2\Lambda_0\sin\Lambda_0 l\Lambda_1 l$$
$$+ \sin\Lambda_0 l\Big]$$

$$+ x^2\{ \qquad \} + \ldots = 0$$

$\Lambda_0^3 \cos\Lambda_0 l = 0 \quad \Rightarrow \quad \Lambda_0 = 0$ (would lead to the trivial solution) or $\cos\Lambda_0 l = 0 \quad \Rightarrow \quad \Lambda_0 l = \frac{\pi}{2} + n\pi, \quad n = 0, 1, \ldots$

$$\sin\Lambda_0 l(-\Lambda_0^3\Lambda_1 l + 1) = 0 \quad \Rightarrow \quad (\sin\Lambda_0 = \pm 1 \neq 0!)$$

$$\Lambda_1 l = \Lambda_0^{-3}$$

So, approximately

$$\Lambda \approx \frac{1}{l}\left(\frac{\pi}{2} + n\pi\right) + x l^2\left(\frac{\pi}{2} + n\pi\right)^{-3}, \quad n = 0, 1, \ldots$$

Critical load $(n = 0)$

$$\left(\frac{P}{EI}\right)^2 \approx \frac{\pi}{2l} + x l^2\left(\frac{\pi}{2}\right)^3$$

$$\underline{P \approx \left\{E^2 I^2\left[\frac{\pi}{2l} + x l^2\left(\frac{\pi}{2}\right)^3\right]\right\}^{\frac{1}{2}}}$$

#

20.10

$$y'' + \lambda y = 0$$

$$y(0) = 0$$
$$y(1) - 2y'(1) = 0$$

characteristic equation:

$$r^2 + \lambda = 0 \implies r = \pm\sqrt{\lambda}$$

general solution: $\quad y(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$

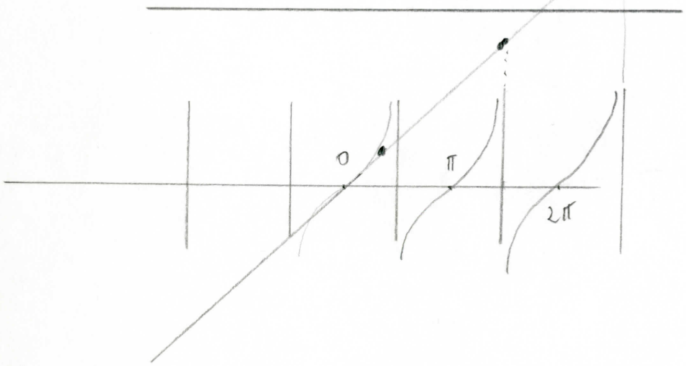
$$y(0) = 0 \implies c_1 = 0 \quad \implies y' = c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$

$$y(1) - 2y'(1) = 0 \implies$$

$$c_2 \sin\sqrt{\lambda} - 2c_2\sqrt{\lambda}\cos\sqrt{\lambda} = 0$$

So we end up with the equation

$$\underline{\sin\sqrt{\lambda} - 2\sqrt{\lambda}\cos\sqrt{\lambda} = 0} \quad \iff 2\sqrt{\lambda} = \tan\sqrt{\lambda}$$



For large n $\quad 2\sqrt{\lambda} = n\pi + \frac{\pi}{2} = \frac{1}{2}(2n+1)\pi$

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4^2} \qquad n = 0, 1, \dots$$

Generalized Fourier series expansion

$$f(x) = \sum_{n=0}^{\infty} f_n \sin\sqrt{\lambda_n}\, x$$

where $$f_n = \int_0^1 f(x) \sin\sqrt{\lambda_n}\, x$$

For $f(x) = x$, we have

$$f_n = \int_0^1 x \sin\sqrt{\lambda_n}\, x \, dx = -\frac{1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\, x \cdot x \Big|_0^1 + \frac{1}{\sqrt{\lambda_n}} \int_0^1 \cos\sqrt{\lambda_n}\, x \, dx$$

$$\begin{array}{ll} x & -\frac{1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}\, x \\ 1 & \sin\sqrt{\lambda_n}\, x \end{array}$$

$$= -\frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} + \frac{1}{\lambda_n} \sin\sqrt{\lambda_n}\, x \Big|_0^1$$

$$= -\frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} + \frac{1}{\lambda_n} \sin\sqrt{\lambda_n}$$

$$= \frac{1}{\lambda_n} \left(\sin\sqrt{\lambda_n} - \sqrt{\lambda_n} \cos\sqrt{\lambda_n} \right)$$

$$= \frac{1}{\lambda_n} \left(\cancel{\sin\sqrt{\lambda_n} - 2\sqrt{\lambda_n}} \overset{0}{\cos}\sqrt{\lambda_n} + \sqrt{\lambda_n} \cos\sqrt{\lambda_n} \right)$$

$$= \frac{1}{\sqrt{\lambda_n}} \cos\sqrt{\lambda_n}$$

So: $$x = \sum_{n=0}^{\infty} \frac{\cos\sqrt{\lambda_n}}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}\, x$$

(convergence in the L^2-sense!)

20.11

$$y'' + \lambda y = 0 \qquad \lambda > 0$$

$$y(0) = y'(0)$$
$$y(1) = 0$$

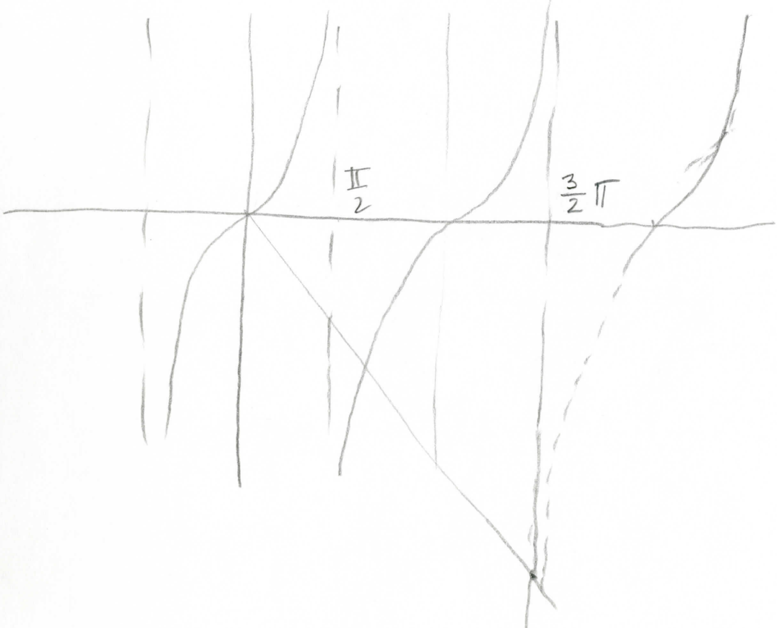
$$y = C_1 \cos\sqrt{\lambda}\, x + C_2 \sin\sqrt{\lambda}\, x$$
$$y' = -C_1\sqrt{\lambda} \sin\sqrt{\lambda}\, x + C_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x$$

$$y(0) = y'(0) \Rightarrow \qquad C_1 = C_2\sqrt{\lambda}$$

$$\therefore\ y = C\left(\sqrt{\lambda}\cos\sqrt{\lambda}\, x + \sin\sqrt{\lambda}\, x\right)$$

$$y(1) = 0 \Rightarrow$$

$$\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = 0$$
$$\Updownarrow$$
$$\tan\sqrt{\lambda} = -\sqrt{\lambda}$$



Approximately

$$\sqrt{\lambda_n} = \frac{\pi}{2} + n\pi$$
$$= \frac{1}{2}(2n+1)\pi$$

$$\lambda_n = \frac{(2n+1)^2\pi^2}{4}$$

So: $$x = \sum_{n=1}^{\infty} c_n (\Lambda_n \cos \Lambda_n x + \sin \Lambda_n x)$$ in the L^2-sense!

where Λ_n is a solution of $\tan \Lambda_n = -\Lambda_n$

$\left(\Lambda_n \approx \frac{(2n+1)\pi}{2} \right)$

and $$c_n = \int_0^1 x (\Lambda_n \cos \Lambda_n x + \sin \Lambda_n x) dx$$

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20.13

(a) $$y'' + \lambda y = 0 \qquad (1)$$

$$2y(0) - y(1) + 4y'(1) = 0 \qquad (2)$$
$$y(0) + 2y'(1) = 0 \qquad (3)$$

Case $\lambda \neq 0$

$r^2 + \lambda = 0 \qquad r = \pm i\sqrt{\lambda} \qquad \lambda \in \mathbb{C}$!

$$y(x) = C_1 e^{i\sqrt{\lambda} x} + C_2 e^{-i\sqrt{\lambda} x}$$
$$y'(x) = i\sqrt{\lambda}\, C_1 e^{i\sqrt{\lambda} x} - i\sqrt{\lambda}\, C_2 e^{-i\sqrt{\lambda} x}$$

$(2) - 2\cdot(3) = 0 \Rightarrow y(1) = 0 \qquad (4)$

So $(2)+(3)$ is equivalent to $(3) + (4)$

$(3) \Rightarrow \quad C_1 + C_2 + 2\left(i\sqrt{\lambda}\, C_1 e^{i\sqrt{\lambda}} - i\sqrt{\lambda}\, C_2 e^{-i\sqrt{\lambda}}\right) = 0$

$(4) \Rightarrow C_1 e^{i\sqrt{\lambda}} + C_2 e^{-i\sqrt{\lambda}} = 0$

$$\begin{vmatrix} 1+2i\sqrt{\lambda}e^{i\sqrt{\lambda}} & 1-2i\sqrt{\lambda}e^{-i\sqrt{\lambda}} \\ e^{i\sqrt{\lambda}} & e^{-i\sqrt{\lambda}} \end{vmatrix} = 0$$

$$e^{-i\sqrt{\lambda}} + 2i\sqrt{\lambda} - e^{i\sqrt{\lambda}} + 2i\sqrt{\lambda} = 0$$

$$2i\sin\sqrt{\lambda} + 4i\sqrt{\lambda} = 0$$

$$\sin\sqrt{\lambda} + 2\sqrt{\lambda} = 0$$

$$\Lambda = \sqrt{\lambda} \qquad \sin\Lambda + 2\Lambda = 0 \qquad (*)$$

$$\Lambda = x+iy$$

$$\sin\Lambda = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y+ix} - e^{y-ix}}{2i}$$

$$= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} \quad \frac{i}{i}$$

$$= -\frac{1}{2}\left(e^{-y}(i\cos x - \sin x) - e^{y}(i\cos x + \sin x)\right)$$

$$= \left(i\cos x\,\frac{e^{y}-e^{-y}}{2} + \sin x\,\frac{e^{y}+e^{-y}}{2}\right)$$

$$= \sin x\cosh y + i\cos x\sinh y$$

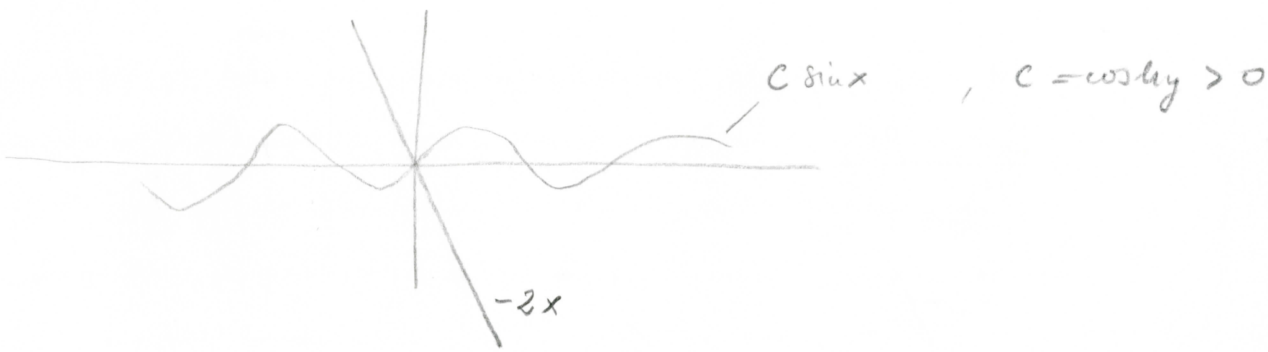
So $(*)$ is equivalent to

$$\sin x\cosh y + 2x = 0 \quad \text{and} \quad \cos x\sinh y + 2y = 0$$

But the first equation has no solution in x, for any value in y!, except for $x=0$

$$\sin x \cosh y = -2x$$

and noticing that $\cosh y > 0$ $\forall y$, we see that the two curves below does not intersect except for $x=0$



Substituting $x=0$ into the second equation we get likewise $y=0$, so $\lambda = 0$.

Case $\lambda = 0$

We get $y'' = 0 \Rightarrow y = ax + b$

$y' = a$

(2) $\Rightarrow$ $2b - (a+b) + 4a = 0$

(3) $\Rightarrow$ $b + 2a = 0$

$$\left.\begin{array}{l} 3a + b = 0 \\ 2a + b = 0 \end{array}\right\} \Rightarrow a = b = 0$$

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20.13 6)

$$y'' + \lambda y = 0 \qquad (1)$$
$$y(0) - y(1) = 0 \qquad (2)$$
$$y'(0) + y'(1) = 0 \qquad (3)$$

Case $\lambda = 0$ $\qquad y = ax + b$

$(2) \Rightarrow \quad b - (a+b) = 0 \quad \Rightarrow a = 0$

$(3) \Rightarrow \quad a + a = 0 \quad \Rightarrow a = 0$

So $y = \text{const}$ is an eigenvector

Case $\lambda \neq 0$, $\lambda \in \mathbb{C}$! $\qquad r^2 + \lambda = 0 \Rightarrow r = \pm i\sqrt{\lambda}$

$$y(x) = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}$$

$(2) \Rightarrow \quad c_1 + c_2 - \left(c_1 e^{i\sqrt{\lambda}} + c_2 e^{-i\sqrt{\lambda}}\right) = 0$

$(3) \Rightarrow \quad c_1 i\sqrt{\lambda} - c_2 i\sqrt{\lambda} + \left(c_1 i\sqrt{\lambda} e^{i\sqrt{\lambda}} - c_2 i\sqrt{\lambda} e^{-i\sqrt{\lambda}}\right) = 0$

$$\begin{pmatrix} 1 - e^{i\sqrt{\lambda}} & 1 - e^{-i\sqrt{\lambda}} \\ i\sqrt{\lambda}(1 + e^{i\sqrt{\lambda}}) & -i\sqrt{\lambda}(1 + e^{-i\sqrt{\lambda}}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-i\sqrt{\lambda}\left(1 - e^{i\sqrt{\lambda}}\right)\left(1 + e^{-i\sqrt{\lambda}}\right) - i\sqrt{\lambda}\left(1 + e^{i\sqrt{\lambda}}\right)\left(1 - e^{-i\sqrt{\lambda}}\right) = 0$$

$$-i\sqrt{\lambda}\left[1 - e^{i\sqrt{\lambda}} + e^{-i\sqrt{\lambda}} - 1 + 1 + e^{i\sqrt{\lambda}} - e^{-i\sqrt{\lambda}} - 1\right] = 0$$

is automatically satisfied for every $\lambda \in \mathbb{C}$!

$\ast$

20.42

$$y^{(0)} = x(\pi - x)$$

First iteration

$$-y'' = x(\pi - x) = \pi x - x^2$$
$$y'' = -\pi x + x^2$$
$$y' = -\pi \frac{x^2}{2} + \frac{x^3}{3} + C$$
$$y = -\pi \frac{x^3}{6} + \frac{x^4}{12} + Cx + D$$

$$y(0) = 0 \Rightarrow D = 0$$
$$y(\pi) = 0 \Rightarrow -\frac{\pi^4}{6} + \frac{\pi^4}{12} + C\pi = 0$$
$$C\pi = \frac{\pi^4}{12}$$
$$C = \frac{\pi^3}{12}$$

So: $$y^{(1)} = \frac{1}{12}x^4 - \frac{\pi}{6}x^3 + \frac{\pi^3}{12}x$$

The eigenvalue may be estimated from the Rayleigh formula

$$\overset{-y''}{\overset{\|}{(Ly}}, y) = \lambda(y, y)$$
$$\|$$
$$-\int y'' y$$
$$\|$$
$$\int y'^2 dx$$

so: $$\lambda^{(1)} = \frac{\int_0^1 \left(3x^3 - \frac{\pi}{2}x^2 + \frac{\pi^3}{12}\right)^2 dx}{\int_0^1 \left(\frac{1}{12}x^4 - \frac{\pi}{6}x^3 + \frac{\pi^3}{12}x\right)^2 dx}$$

We proceed identically with the second iteration.

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