Strong Form

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \quad \text{in } \Omega \\
-\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\beta} u) &= f \quad \text{in } \Omega \\
u &= u_0 \quad \text{in } \partial\Omega
\end{aligned}
$$

Weak Form

$$
\begin{aligned}
\int_K \frac{1}{\epsilon}\boldsymbol{\sigma} \cdot \boldsymbol{\tau} + u \nabla \cdot \boldsymbol{\tau} \,\mathrm{d}x - \int_{\partial K} \hat{u} \boldsymbol{\tau} \cdot n \,\mathrm{d}s &= 0 \quad \forall \boldsymbol{\tau} \\
\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u) \cdot \nabla v \,\mathrm{d}x - \int_{\partial K} (\widehat{\sigma_n - \beta_n} u) \mathrm{sgn}(n) v \,\mathrm{d}s = \int_K f v \,\mathrm{d}x &\quad \forall v
\end{aligned}
$$

Spaces

$$
\begin{array}{ccc}
u \in L^2(K) & \hat{u} \in H^1|_{\partial K} \equiv H^{\frac{1}{2}} & v \in H^1(K) \\
\boldsymbol{\sigma} \in \mathbf{L}^2(K) & (\widehat{\sigma_n - \beta_n} u) \in H(div, K)|_{\partial K} & \boldsymbol{\tau} \in H(div, K)
\end{array}
$$

Local Solve

Everything together

$$
\begin{aligned}
&\int_K \left\{ \nabla \cdot \boldsymbol{\tau} \nabla \cdot \boldsymbol{\delta\tau} + \boldsymbol{\tau} \boldsymbol{\delta\tau} + \nabla v \nabla \delta v + v \delta v \right\} w(x) \,\mathrm{d}x \\
&\quad = \int_K \frac{1}{\epsilon} \boldsymbol{\sigma} \cdot \boldsymbol{\delta\tau} + u \nabla \cdot \boldsymbol{\delta\tau} + \boldsymbol{\sigma} \cdot \nabla \delta v - u \boldsymbol{\beta} \cdot \nabla \delta v \,\mathrm{d}x \\
&\qquad - \int_{\partial K} \hat{u} \boldsymbol{\delta\tau} \cdot \mathbf{n} + (\widehat{\sigma_n - \beta_n} u) \mathrm{sgn}(n) \delta v \,\mathrm{d}s
\end{aligned}
$$

$$
K_{vv} = \int_K \left\{ \nabla \delta v \nabla \delta v + \delta v \delta v \right\} w(x) \,\mathrm{d}x
$$

$$
K_{v\tau} = 0
$$

$$
K_{\tau v} = 0
$$

$$
\begin{aligned}
K_{\tau\tau} &= \int_K \left\{ \nabla \cdot \boldsymbol{\delta\tau} \nabla \cdot \boldsymbol{\delta\tau} + \boldsymbol{\delta\tau} \boldsymbol{\delta\tau} \right\} w(x) \,\mathrm{d}x \\
&=
\end{aligned}
$$

u DOFs

$$\int_K \{\nabla\cdot\boldsymbol{\tau}\nabla\cdot\boldsymbol{\delta\tau}+\boldsymbol{\tau}\boldsymbol{\delta\tau}+\nabla v\nabla\delta v+v\delta v\}\,w(x)\,\mathrm{d}x$$
$$=\int_K u\nabla\cdot\boldsymbol{\delta\tau}-u\boldsymbol{\beta}\cdot\nabla\delta v\,\mathrm{d}x$$

$$F_v=\int_K -u\boldsymbol{\beta}\cdot\nabla\delta v\,\mathrm{d}x$$

$$F_\tau=\int_K u\nabla\cdot\boldsymbol{\delta\tau}\,\mathrm{d}x$$

$\boldsymbol{\sigma}$ DOFs

$$\int_K \{\nabla\cdot\boldsymbol{\tau}\nabla\cdot\boldsymbol{\delta\tau}+\boldsymbol{\tau}\boldsymbol{\delta\tau}+\nabla v\nabla\delta v+v\delta v\}\,w(x)\,\mathrm{d}x$$
$$=\int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\delta\tau}+\boldsymbol{\sigma}\cdot\nabla\delta v\,\mathrm{d}x$$

$$F_v=\int_K \boldsymbol{\sigma}\cdot\nabla\delta v\,\mathrm{d}x$$

$$F_\tau=\int_K \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\delta\tau}\,\mathrm{d}x$$

$\hat{u}$ DOFs

$$\int_K \{\nabla\cdot\boldsymbol{\tau}\nabla\cdot\boldsymbol{\delta\tau}+\boldsymbol{\tau}\boldsymbol{\delta\tau}+\nabla v\nabla\delta v+v\delta v\}\,w(x)\,\mathrm{d}x$$
$$=\int_{\partial K}\hat{u}\boldsymbol{\delta\tau}\cdot\mathbf{n}\,\mathrm{d}s$$

$$F_v=0$$

$$F_\tau=\int_{\partial K}\hat{u}\boldsymbol{\delta\tau}\cdot\mathbf{n}\,\mathrm{d}s$$

$(\widehat{\sigma_n - \beta_n} u)$ DOFs

$$\int_K \{\nabla\cdot\boldsymbol{\tau}\nabla\cdot\boldsymbol{\delta\tau} + \boldsymbol{\tau}\boldsymbol{\delta\tau} + \nabla v\nabla\delta v + v\delta v\}\, w(x)\,\mathrm{d}x$$
$$= \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)\delta v\,\mathrm{d}s$$

$$F_v = \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)\delta v\,\mathrm{d}s$$

$$F_\tau = 0$$

Element Assembly

Full Residual

$$\int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\nabla v + \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + u\nabla\cdot\boldsymbol{\tau} - fv\,\mathrm{d}x - \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)v + \hat{u}\boldsymbol{\tau}\cdot n\,\mathrm{d}s = 0$$

Residual Components

$$F_u = \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\nabla v_u + \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}_u + u\nabla\cdot\boldsymbol{\tau}_u - fv_u\,\mathrm{d}x$$
$$- \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)v_u + \hat{u}\boldsymbol{\tau}_u\cdot n\,\mathrm{d}s$$

$$F_{\boldsymbol{\sigma}} = \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\nabla v_{\boldsymbol{\sigma}} + \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}_{\boldsymbol{\sigma}} + u\nabla\cdot\boldsymbol{\tau}_{\boldsymbol{\sigma}} - fv_{\boldsymbol{\sigma}}\,\mathrm{d}x$$
$$- \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)v_{\boldsymbol{\sigma}} + \hat{u}\boldsymbol{\tau}_{\boldsymbol{\sigma}}\cdot n\,\mathrm{d}s$$

$$F_{\hat{u}} = \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\nabla v_{\hat{u}} + \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}_{\hat{u}} + u\nabla\cdot\boldsymbol{\tau}_{\hat{u}} - fv_{\hat{u}}\,\mathrm{d}x$$
$$- \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)v_{\hat{u}} + \hat{u}\boldsymbol{\tau}_{\hat{u}}\cdot n\,\mathrm{d}s$$

$$F_{(\widehat{\sigma_n - \beta_n} u)} = \int_K (\boldsymbol{\sigma} - \boldsymbol{\beta} u)\cdot\nabla v_{(\widehat{\sigma_n - \beta_n} u)} + \frac{1}{\epsilon}\boldsymbol{\sigma}\cdot\boldsymbol{\tau}_{(\widehat{\sigma_n - \beta_n} u)} + u\nabla\cdot\boldsymbol{\tau}_{(\widehat{\sigma_n - \beta_n} u)} - fv_{(\widehat{\sigma_n - \beta_n} u)}\,\mathrm{d}x$$
$$- \int_{\partial K} (\widehat{\sigma_n - \beta_n} u)\mathrm{sgn}(n)v_{(\widehat{\sigma_n - \beta_n} u)} + \hat{u}\boldsymbol{\tau}_{(\widehat{\sigma_n - \beta_n} u)}\cdot n\,\mathrm{d}s$$

Jacobian Components

Now let

$$u = \sum_i u_i \phi_i\,, \qquad \boldsymbol{\sigma} = \sum_i \boldsymbol{\sigma}_i \boldsymbol{\psi}_i\,, \qquad \hat{u} = \sum_i \hat{u}_i \hat{\phi}_i\,, \qquad (\sigma_n \widehat{-\beta_n} u) = \sum_i (\sigma_n \widehat{-\beta_n} u)_i \hat{\psi}_i$$

$$K_{uu} = \int_K -\boldsymbol{\beta}\phi \cdot \nabla v_u + \phi \nabla \cdot \boldsymbol{\tau}_u \,\mathrm{d}x$$

$$K_{u\boldsymbol{\sigma}} = \int_K \boldsymbol{\psi} \cdot \nabla v_u + \frac{1}{\epsilon} \boldsymbol{\psi} \cdot \boldsymbol{\tau}_u \,\mathrm{d}x$$

$$K_{u\hat{u}} = -\int_{\partial K} \hat{\phi} \boldsymbol{\tau}_u \cdot n \,\mathrm{d}s$$

$$K_{u(\sigma_n \widehat{-\beta_n} u)} = -\int_{\partial K} \hat{\psi} \mathrm{sgn}(n) v_u \,\mathrm{d}s$$

$$K_{\boldsymbol{\sigma}u} = \int_K -\boldsymbol{\beta}\phi \cdot \nabla v_{\boldsymbol{\sigma}} + \phi \nabla \cdot \boldsymbol{\tau}_{\boldsymbol{\sigma}} \,\mathrm{d}x$$

$$K_{\boldsymbol{\sigma}\boldsymbol{\sigma}} = \int_K \boldsymbol{\psi} \cdot \nabla v_{\boldsymbol{\sigma}} + \frac{1}{\epsilon} \boldsymbol{\psi} \cdot \boldsymbol{\tau}_{\boldsymbol{\sigma}} \,\mathrm{d}x$$

$$K_{\boldsymbol{\sigma}\hat{u}} = -\int_{\partial K} \hat{\phi} \boldsymbol{\tau}_{\boldsymbol{\sigma}} \cdot n \,\mathrm{d}s$$

$$K_{\boldsymbol{\sigma}(\sigma_n \widehat{-\beta_n} u)} = -\int_{\partial K} \hat{\psi} \mathrm{sgn}(n) v_{\boldsymbol{\sigma}} \,\mathrm{d}s$$

$$K_{\hat{u}u} = \int_K -\boldsymbol{\beta}\phi \cdot \nabla v_{\hat{u}} + \phi \nabla \cdot \boldsymbol{\tau}_{\hat{u}} \,\mathrm{d}x$$

$$K_{\hat{u}\boldsymbol{\sigma}} = \int_K \boldsymbol{\psi} \cdot \nabla v_{\hat{u}} + \frac{1}{\epsilon} \boldsymbol{\psi} \cdot \boldsymbol{\tau}_{\hat{u}} \,\mathrm{d}x$$

$$K_{\hat{u}\hat{u}} = -\int_{\partial K} \hat{\phi} \boldsymbol{\tau}_{\hat{u}} \cdot n \,\mathrm{d}s$$

$$K_{\hat{u}(\widehat{\sigma_n-\beta_n u})} = -\int_{\partial K} \hat{\psi}\operatorname{sgn}(n) v_{\hat{u}}\,\mathrm{d}s$$

$$K_{(\widehat{\sigma_n-\beta_n u})u} = \int_K -\boldsymbol{\beta}\phi\cdot\nabla v_{(\widehat{\sigma_n-\beta_n u})} + \phi\nabla\cdot\boldsymbol{\tau}_{(\widehat{\sigma_n-\beta_n u})}\,\mathrm{d}x$$

$$K_{(\widehat{\sigma_n-\beta_n u})\boldsymbol{\sigma}} = \int_K \boldsymbol{\psi}\cdot\nabla v_{(\widehat{\sigma_n-\beta_n u})} + \frac{1}{\epsilon}\boldsymbol{\psi}\cdot\boldsymbol{\tau}_{(\widehat{\sigma_n-\beta_n u})}\,\mathrm{d}x$$

$$K_{(\widehat{\sigma_n-\beta_n u})\hat{u}} = -\int_{\partial K} \hat{\phi}\boldsymbol{\tau}_{(\widehat{\sigma_n-\beta_n u})}\cdot n\,\mathrm{d}s$$

$$K_{(\widehat{\sigma_n-\beta_n u})(\widehat{\sigma_n-\beta_n u})} = -\int_{\partial K} \hat{\psi}\operatorname{sgn}(n) v_{(\widehat{\sigma_n-\beta_n u})}\,\mathrm{d}s$$