Space-Time Discontinuous Petrov-Galerkin Finite Elements for Transient Computational Fluid Dynamics

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Table of Contents

1. Introduction
   - Motivation
   - Literature Review
2. DPG Overview
3. Preliminary Work
   - Local Conservation
   - Space-Time DPG
4. Proposed Work

Table of Contents

1 Introduction
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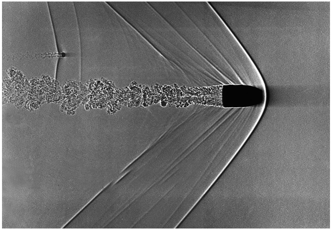
Navier-Stokes Equations

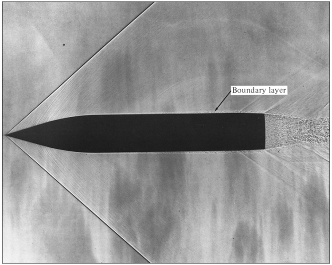
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Nonlinear convergence and uniqueness of solutions
- Stability of numerical schemes
  - Coarse/adaptive grids
  - Higher order



Shock

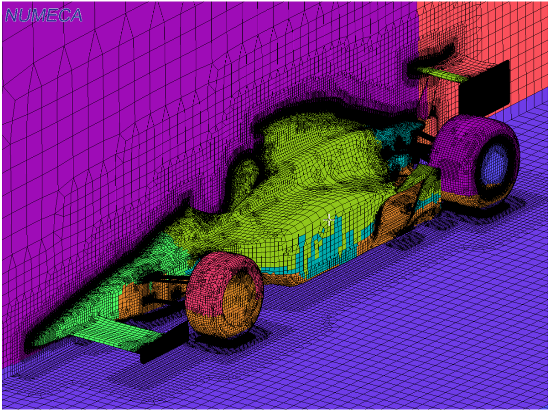


Boundary layer

Motivation

Initial Mesh Design is Expensive and Time-Consuming

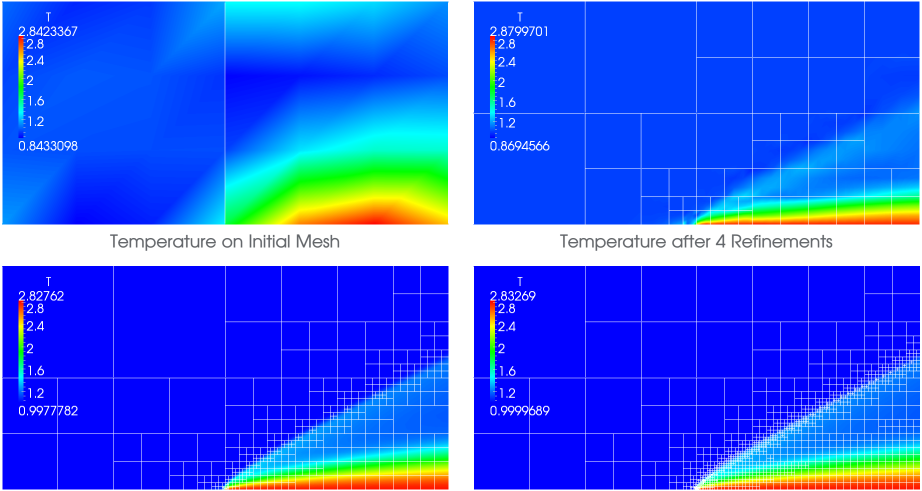
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Boundary layer meshes must respect y^+ guidelines
- Engineers often forced to work by trial and error
- Bad in the context of HPC



Formula 1 Mesh by Numeca

DPG on Coarse Meshes

Adaptive Solve of the Carter Plate Problem[1] $Re = 1000$



Temperature on Initial Mesh

Temperature after 4 Refinements

Temperature after 8 Refinements

Temperature after 11 Refinements

[1] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Table of Contents

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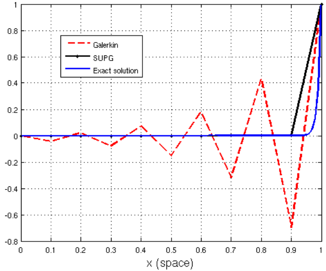
Stabilized Finite Elements for CFD

Streamline Upwind Petrov-Galerkin

- First successful finite element method for CFD[2]
- Residual based stabilization preserves consistency
- Upwind biasing of test functions
- Higher order generalizations possible
- Optimal H_0^1 approximation in 1D
- Gave rise to the field of stabilized finite elements
- Major contributors include Hughes, Franca, Johnson, Codina, Tezduyar, and many others
- Variational multiscale may be considered the spiritual successor to SUPG[3]



H_0^1 Projection

[2] A.N. Brooks and T.J.R. Hughes. ``Streamline Upwind/Petrov-Galerkin Formulations for Convection Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations''. In: *Comput. Methods Appl. Mech. Eng.* (Sept. 1990), pp. 199–259

[3] T.J.R. Hughes et al. ``The variational multiscale method -- a paradigm for computational mechanics''. In: *Comput. Methods in Appl. Mech. Eng.* 166.1 - 2 (1998). Advances in Stabilized Methods in Computational Mechanics, pp. 3 –24

Streamline Upwind Petrov-Galerkin

Two Equivalent Views on Stabilization

Convection-diffusion can be written as

$$Lu = (L_{adv} + L_{diff})u = f\,.$$

Residual Based Stabilization

$$b_{SUPG}(u, v) = l_{SUPG}(v)$$

where

$$b_{SUPG}(u, v) = b(u, v) + \sum_K \int_K \tau(L_{adv}v)(Lu - f)$$

$$l_{SUPG}(v) = l(v) + \sum_K \int_K \tau(L_{adv}v)f\,,$$

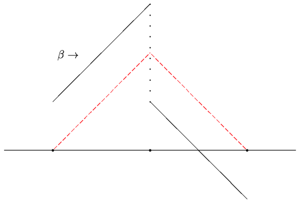
τ is the SUPG stabilization parameter.

Test Space Modification

$$b(u, \tilde{v}) = l(\tilde{v})$$

where

$$\tilde{v} = v + \tau L_{adv}v\,.$$



Stabilized Finite Elements for CFD

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Discontinuous Galerkin

- Combines elements of finite volumes and finite elements
- First proposed for neutron transport[4]
- Early contributors include Babuška, Lions, Nitsche, and Zlámal
- Develoment for CFD by Cockburn and Shu[5]
- Development for elliptic problems given by Arnold, Brezzi, Cockburn, and Marini[6]
- Nonconforming basis, locally conservative
- Naturally high order, but may require additional stabilization
- Simple to parallelizable
- Other notable contributors include Peraire, Persson, Karniadakis . . .

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[4] W.H. Reed and T.R. Hill. *Triangular mesh methods for the neutron transport equation.* Tech. rep. LA-UR-73-479. Los Alamos National Laboratory, 1973.

[5] B. Cockburn and C. Shu. ``The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V: Multidimensional Systems''. In: *J. Comp. Phys.* 141.2 (1998), pp. 199 –224.

[6] D.N. Arnold et al. ``Unified Analysis of Discontinuous Galerkin Methods for Elliptic Problems''. In: *SIAM J. Numer. Anal.* 39.5 (2001), pp. 1749–1779.

Stabilized Finite Elements for CFD

Discontinuous Galerkin

Consider 1D convection equation

$$\frac{\partial \beta(x) u}{\partial x} = f, \quad u(0) = u_0 \,.$$

Multiply by test function and integrate by parts over each element $K = [x_K, x_{x+1}]$

$$-\int_K \beta(x) u \frac{\partial v}{\partial x} + \beta u v|_{x_K}^{x_{K+1}} = \int_K f v \,.$$

Apply upwind flux

$$-\int_K \beta(x) u \frac{\partial v}{\partial x} + \beta(x_{K+1}) u(x_{K+1}^-) v(x_{K+1}^-) - \beta(x_K) u(x_K^-) v(x_K^+) = \int_K f v \,.$$

Hybridized DG (HDG) method introduces trace unknowns which facilitates static condensation, reducing interface unknowns[7].

[7] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. ``Unified Hybridization of Discontinuous Galerkin, Mixed, and Continuous Galerkin Methods for Second Order Elliptic Problems''. In: *SIAM J. Numer. Anal.* 47.2 (Feb. 2009), pp. 1319–1365.

Space-Time Finite Element Methods

Treat Time as Another Dimension to be Discretized

Space-time methods treat time as just another dimension to be discretized.

- Early contributors include Kaczkowski and Oden
- Satisfies geometric conservation laws[8]
- Tezduyar *et al.* [9] developed a Galerkin/least-squares method for moving boundaries
- Van der Vegt and Van der Ven developed a popular space-time DG method[10]
- Üngör's tent-pitcher algorithm[11] decouples space-time elements

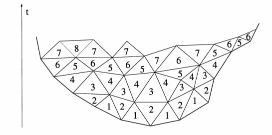

Fig. 2. Ordering of elements for an element-by-element procedure (1D×TIME). Elements with the same number can be solved independently or in parallel, elements with higher numbers depend for the solution on the output of those with lower ones.

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[8] M. Lesoinne and C. Farhat. ``Geometric conservation laws for flow problems with moving boundaries and deformable meshes, and their impact on aeroelastic computations''. In: *Comput. Methods in Appl. Mech. Eng.* 134.1 - 2 (1996), pp. 71 –90

[9] T.E. Tezduyar, M. Behr, and J. Liou. ``A new strategy for finite element computations involving moving boundaries and interfaces -- The deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests''. In: *Comput. Methods in Appl. Mech. Eng.* 94.3 (1992), pp. 339 –351

[10] J.J.W. van der Vegt and H. van der Ven. ``Space-Time Discontinuous Galerkin Finite Element Method with Dynamic Grid Motion for Inviscid Compressible Flows: I. General Formulation''. In: *J. Comp. Phys.* 182.2 (2002), pp. 546 –585

[11] A. Üngör. ``Tent-Pitcher: A meshing algorithm for space-time discontinuous Galerkin methods''. In: 9th Internat. Meshing Roundtable. 2000, pp. 111–122

Table of Contents

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Overview of DPG

DPG is a Minimum Residual Method

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bu - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1} (Bu - l) \right\|_V^2 .$$

Overview of DPG

DPG is a Minimum Residual Method

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h\,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$\left(v_{\delta u_h}, \delta v\right)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

DPG is the Most Stable Petrov-Galerkin Method

Babuška's theorem guarantees that *discrete stability and approximability imply convergence*. If bilinear form $b(u, v)$, with $M := \|b\|$ satisfies the discrete inf-sup condition with constant γ_h,

$$\sup_{v_h \in V_h} \frac{|b(u, v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U ,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U .$$

Optimal test function realize the supremum guaranteeing that $\gamma_h \geq \gamma$.

Energy Norm

If we use the energy norm, $\|u\|_E := \|Bu\|_{V'}$ in the error estimate, then $M = \gamma = 1$. Babuška's theorem implies that the minimum residual method is the most stable Petrov-Galerkin method (assuming exact optimal test functions).

Overview of DPG[12]

Other Features

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

[12] L.F. Demkowicz and J. Gopalakrishnan. ``Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)''. In: vol. 157. IMA Volumes in Mathematics and its Applications, 2014. Chap. An Overview of the DPG Method, pp. 149–180.

Overview of DPG

High Performance Computing

Eliminates human intervention

- Stability
- Robustness
- Adaptivity
- Automaticity
- Compute intensive
- Embarrassingly parallel local solves
- Factorization recyclable
- Low communication
- SPD stiffness matrix
- Multiphysics

Stampede Supercomputer at TACC

Mira Supercomputer at Argonne

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Locally Conservative DPG

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= \mathbf{0} \\ \nabla \cdot (\beta u - \boldsymbol{\sigma}) &= g \end{aligned}$$

Multiply by test functions and integrate by parts over each element, K.

$$\begin{aligned} \frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle u, \tau_n \rangle_{\partial K} &= 0 \\ -(\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle (\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v \rangle_{\partial K} &= (g, v)_K \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} &\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K} \\ &\qquad - (\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle \hat{t}, v \rangle_{\partial K} = (g, v)_K \end{aligned}$$

Locally Conservative DPG

Local Conservation for Convection-Diffusion

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[13] (also Chang and Nelson[14]), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left\| R_V^{-1}(Bu_h - l) \right\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{t}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[13] D. Moro, N.C. Nguyen, and J. Peraire. ``A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws''. In: *Int.J. Num. Meth. Eng.* (2011).

[14] C.L. Chang and J.J. Nelson. ``Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation''. In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.

Locally Conservative DPG

Locally Conservative Saddle Point System

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

Locally Conservative Saddle Point System

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B\delta u_h) - l(R_V^{-1} B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddle point problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Locally Conservative DPG

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Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable.
We recently developed this modification to the *robust test norm*[15] which behaves better in the presence of singularities.

Convection-Diffusion Test Norm

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \left\| \min\left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}} \right\} \boldsymbol{\tau} \right\|^2 + \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v\|^2$$
$$+ \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 \quad \underbrace{+ \|v\|^2}_{\text{No longer necessary}}$$

[15] J.L. Chan et al. *Robust DPG method for convection-dominated diffusion problems II: A natural inflow condition.* Tech. rep. 21. ICES, 2012.

Locally Conservative DPG

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Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

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[15] J.L. Chan et al. *Robust DPG method for convection-dominated diffusion problems II: A natural inflow condition.* Tech. rep. 21. ICES, 2012.

Locally Conservative DPG

Stability and Robustness Analysis

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- We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{U}, p \in Q \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) \quad \forall q \in Q \end{cases}$$

  where a, c, l, g denote the appropriate bilinear and linear forms.
- $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, R_V^{-1} B\boldsymbol{w}) = (R_V^{-1} B\boldsymbol{u}, R_V^{-1} B\boldsymbol{w})_V$
- $c(p, \boldsymbol{w}) = \sum_K \lambda_K \langle \hat{t}, 1_K \rangle_{\partial K}$
- Locally conservative DPG satisfies inf-sup and inf-sup in kernel conditions.
- Robustness is proved by switching to energy norm in Brezzi analysis.
- Full details can be found in *Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems*[16].

[16] T.E. Ellis, L.F. Demkowicz, and J.L. Chan. "Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems". In: *Comp. Math. Appl.* (2014). submitted.

Numerical Experiments

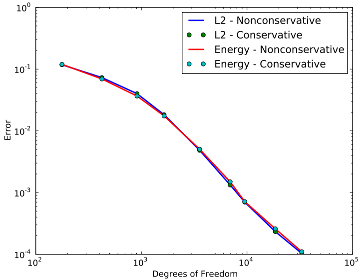
Eriksson-Johnson Problem

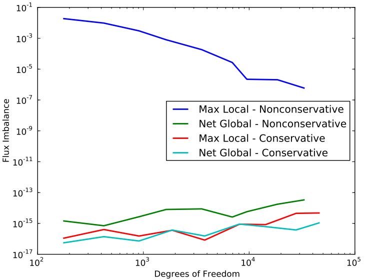
On domain $\Omega = [0, 1]^2$, with $\beta = (1, 0)^T, f = 0$ and boundary conditions

$$\hat{t} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

$$u(x, y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$

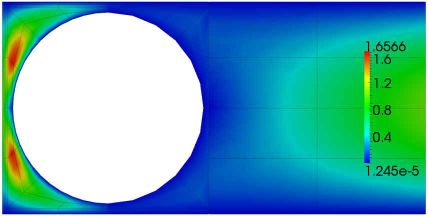




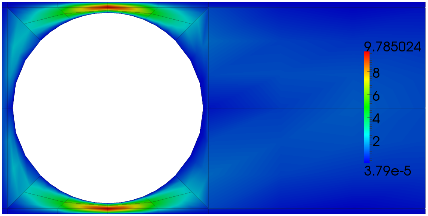
Numerical Experiments

Stokes Flow Around a Cylinder

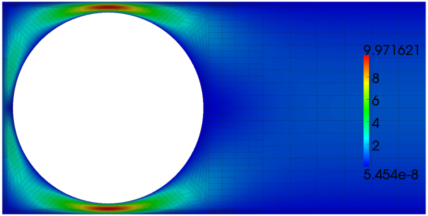
Horizontal Velocity



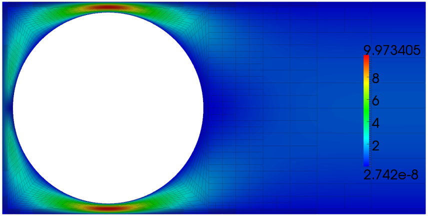
1 Refinement



1 Refinement



6 Refinements

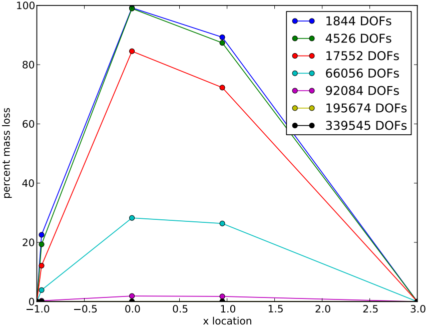


6 Refinements

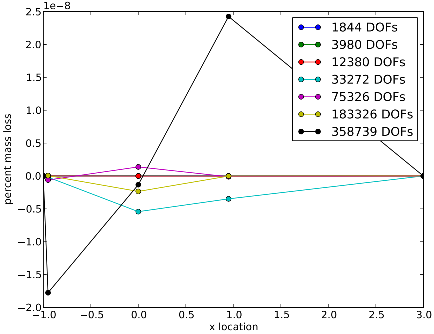
Nonconservative

Conservative

Percent Mass Loss at $x = [-1, -0.95, 0, 0.95, 3]$



Nonconservative

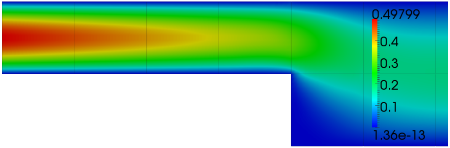


Conservative

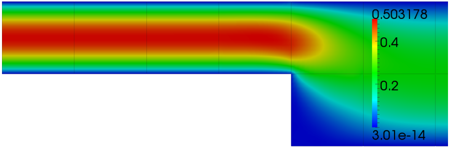
Numerical Experiments

Stokes Flow Over a Backward Facing Step

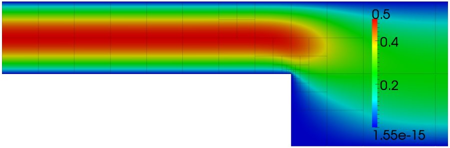
Horizontal Velocity



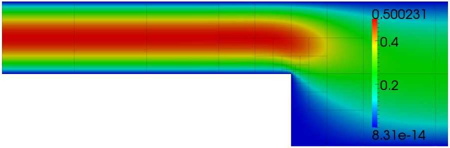
Initial Mesh



Initial Mesh



8 Refinements



8 Refinements

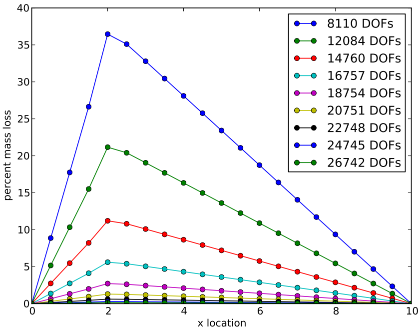
Nonconservative

Conservative

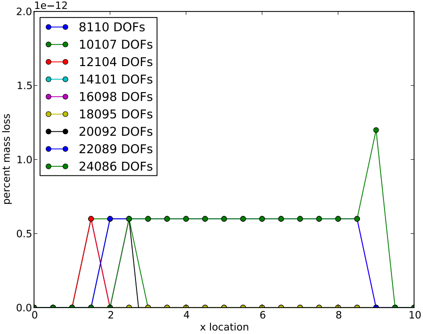
Numerical Experiments

Stokes Flow Over a Backward Facing Step

Percent Mass Loss at $x = [0, 0.5, \cdots, 9.5, 10]$



Nonconservative



Conservative

Table of Contents

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1. Introduction
   - Motivation
   - Literature Review
2. DPG Overview
3. Preliminary Work
   - Local Conservation
   - Space-Time DPG
4. Proposed Work

Space-Time DPG

Motivation

Extends the capabilities of a DPG solver

- Preserves stability and robustness of DPG method
- Unified treatment of space and time
- Local space-time adaptivity (local time stepping)
  - Small solution features require small time step
  - Global time step not limited to smallest element
- Natural framework for moving meshes

More computationally difficult

- Solve requires $d+1$ dimensions
- Mesh structure more difficult
- Need to differentiate between spatial and temporal boundaries
- Larger global solves than finite difference time stepping

Heat Equation

Simplest Nontrivial Space-Time Problem

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Heat Equation

DPG Formulation

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)+(u,\nabla\cdot\boldsymbol{\tau})-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle=0$$

$$-\left(\begin{pmatrix}-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)+\langle\hat{t},v\rangle=f$$

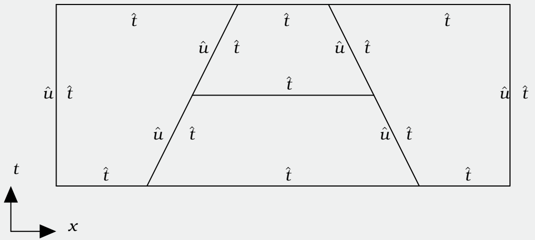
where

$$\hat{u}:=\operatorname{tr}(u)$$

$$\hat{t}:=\operatorname{tr}(-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

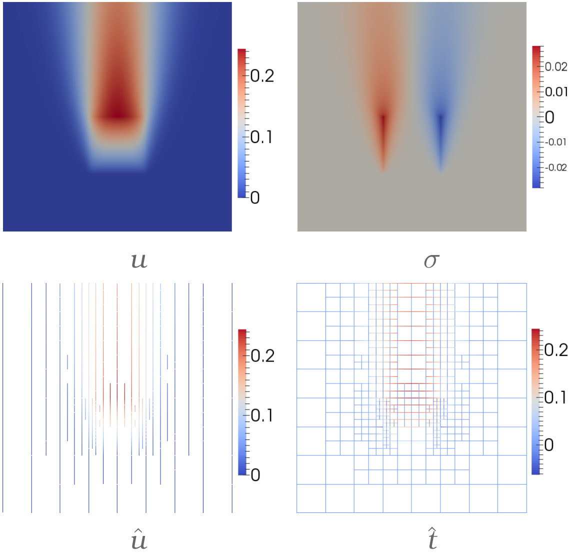
Support of Trace Variables



Heat equation

Pulsed Source Problem

- Initial condition $u = 0$.
- Apply unit source $x \in [3/8, 5/8]$, $t \in [1/4, 1/2]$.
- Should not violate causality
- Space-time adaptivity picks up areas of rapid change.



Compressible Navier-Stokes

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Strong Form

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \rho e_0\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{u}\otimes\boldsymbol{u}+p\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}e_0+\boldsymbol{u}p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\end{bmatrix}=\begin{bmatrix}f_c\\ \boldsymbol{f}_m\\ f_e\end{bmatrix},$$

where

$$\mathbb{D}=2\mu\boldsymbol{S}^*=2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-\frac{1}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I}\right],$$

$$\boldsymbol{q}=-C_p\frac{\mu}{Pr}\nabla T,$$

and (assuming an ideal gas EOS)

$$p=\rho RT.$$

Compressible Navier-Stokes

First Order Space-Time Form

Writing this in space-time in terms of ρ, $\boldsymbol{u}$, T, $\mathbb{D}$, and $\boldsymbol{q}$:

$$\begin{aligned}
\mathbb{D} - \mu\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right) + \frac{2\mu}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I} &= 0\\
\boldsymbol{q} + C_p\frac{\mu}{Pr}\nabla T &= 0\\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix} &= f_c\\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix} &= \boldsymbol{f}_m\\
\nabla_{xt}\cdot\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix} &= f_e\,.
\end{aligned}$$

Compressible Navier-Stokes

DPG Formulation

Multiplying by test functions and integrating by parts:

$$
\begin{aligned}
(\mathbb{D}, \mathbb{S}) + (2\mu \boldsymbol{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2\mu}{3}\boldsymbol{u}, \nabla \operatorname{tr} \mathbb{S}\right) - \langle 2\mu \hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \rangle + \left\langle \frac{2\mu}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \right\rangle &= 0 \\
(\boldsymbol{q}, \boldsymbol{\tau}) - \left(C_p \frac{\mu}{Pr} T, \nabla \cdot \boldsymbol{\tau}\right) + \left\langle C_p \frac{\mu}{Pr} \hat{T}, \tau_n \right\rangle &= 0 \\
-\left(\begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix}, \nabla_{xt} v_c\right) + \langle \hat{t}_c, v_c \rangle &= (f_c, v_c) \\
-\left(\begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D} \\ \rho \boldsymbol{u} \end{pmatrix}, \nabla_{xt} \boldsymbol{v}_m\right) + \langle \hat{\boldsymbol{t}}_m, \boldsymbol{v}_m \rangle &= (\boldsymbol{f}_m, \boldsymbol{v}_m) \\
-\left(\begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) \end{pmatrix}, \nabla_{xt} v_e\right) + \langle \hat{t}_e, v_e \rangle &= (f_e, v_e) \ ,
\end{aligned}
$$

where $\hat{u}$ and $\hat{T}$ are spatial traces and $\hat{t}_c$, $\hat{\boldsymbol{t}}_m$, and $\hat{t}_e$ are fluxes.

Compressible Navier-Stokes

Flux and Trace Variables

Spatial traces and fluxes are defined as follows:

$$\hat{\boldsymbol{u}} = \operatorname{tr}(\boldsymbol{u})$$

$$\hat{T} = \operatorname{tr}(T)$$

$$\hat{t}_c = \operatorname{tr}(\rho\boldsymbol{u}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho)\, n_t$$

$$\hat{\boldsymbol{t}}_m = \operatorname{tr}(\rho\boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho\boldsymbol{u})\, n_t$$

$$\hat{t}_e = \operatorname{tr}\left(\rho\boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\right) \cdot \boldsymbol{n}_x + \operatorname{tr}\left(\rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right) n_t\,.$$

Linearization

Fluxes, traces, and $\boldsymbol{q}$ are linear in the above bilinear form, but we need to linearize in ρ, $\boldsymbol{u}$, T, and $\mathbb{D}$ (Jacobian and residual not shown here).

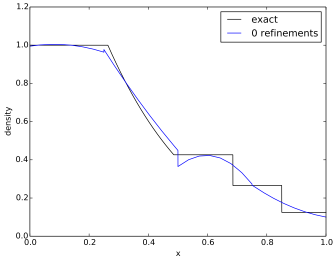
Compressible Navier-Stokes

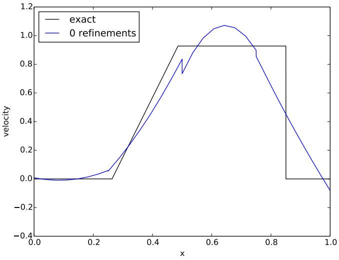
Test Norm

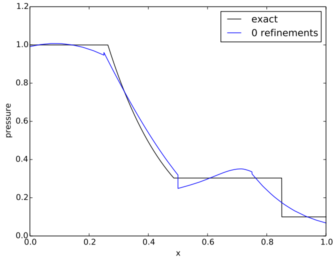
$$
\begin{aligned}
&\left\|\nabla \boldsymbol{v}_m + \nabla v_e \otimes \tilde{\boldsymbol{u}}\right\|^2 + \left\|\nabla v_e\right\|^2 \\
&+ \left\| -\tilde{\boldsymbol{u}} \cdot \nabla v_c - \frac{\partial v_c}{\partial t} - \tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}} : \nabla \boldsymbol{v}_m - R\tilde{T} \nabla \cdot \boldsymbol{v}_m - \tilde{\boldsymbol{u}} \cdot \frac{\partial \boldsymbol{v}_m}{\partial t} \right. \\
&\quad \left. - C_v \tilde{T} \tilde{\boldsymbol{u}} \cdot \nabla v_e - \frac{1}{2} \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \tilde{\boldsymbol{u}} \cdot \nabla v_e - R\tilde{T} \tilde{\boldsymbol{u}} \nabla v_e - C_v \tilde{T} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\| -\tilde{\rho} \nabla v_c - \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{v}_m - \tilde{\rho} \nabla \boldsymbol{v}_m \cdot \tilde{\boldsymbol{u}} - \tilde{\rho} \frac{\partial \boldsymbol{v}_m}{\partial t} - C_v \tilde{\rho} \tilde{T} \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}} \nabla v_e \right. \\
&\quad \left. - \frac{1}{2} \tilde{\rho} \tilde{\boldsymbol{u}} \cdot \nabla v_e \tilde{\boldsymbol{u}} - \frac{1}{2} \tilde{\rho} \nabla v_e \cdot \tilde{\boldsymbol{u}} \tilde{\boldsymbol{u}} - R\tilde{\rho} \tilde{T} \nabla v_e + \tilde{\mathbb{D}} \cdot \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\boldsymbol{u}} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\rho} \tilde{\boldsymbol{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\| -R\tilde{\rho} \nabla \cdot \boldsymbol{v}_m - C_v \tilde{\rho} \tilde{\boldsymbol{u}} \nabla v_e - R\tilde{\rho} \tilde{\boldsymbol{u}} \nabla v_e - C_v \tilde{\rho} \frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\| \frac{1}{\mu} \mathbb{S} \right\|^2 + \left\| 2 \nabla \cdot \mathbb{S} - \frac{2}{3} \nabla \operatorname{tr} \mathbb{S} \right\|^2 + \left\| \frac{Pr}{C_p \mu} \boldsymbol{\tau} \right\|^2 + \left\| \nabla \cdot \boldsymbol{\tau} \right\|^2 \\
&+ \left\| v_c \right\|^2 + \left\| \boldsymbol{v}_m \right\|^2 + \left\| v_e \right\|^2 .
\end{aligned}
$$

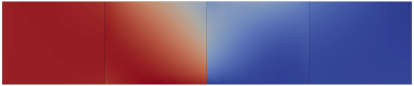
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



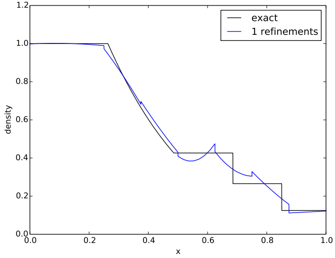


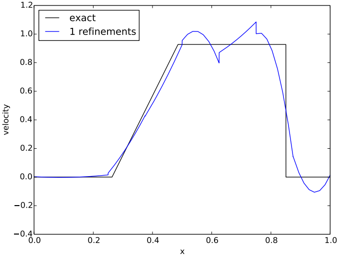


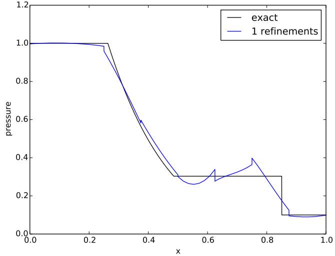


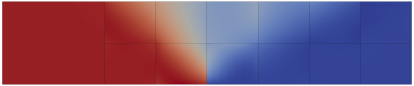
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$









Compressible Navier-Stokes

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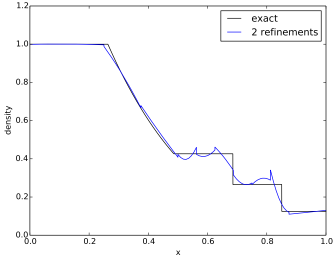

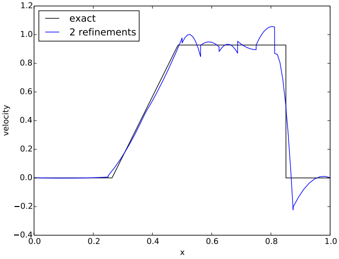

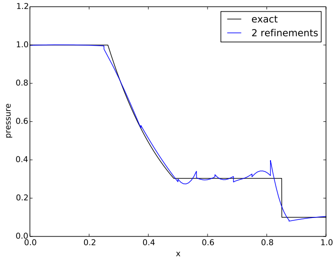

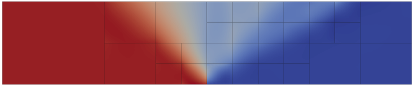

Compressible Navier-Stokes

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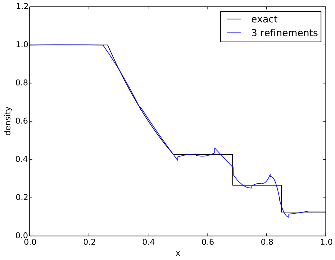

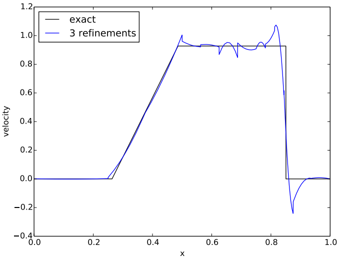

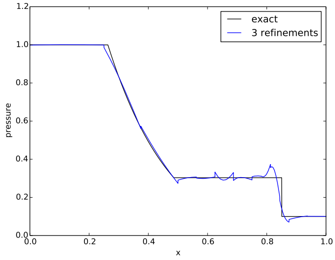

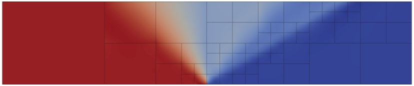

Compressible Navier-Stokes

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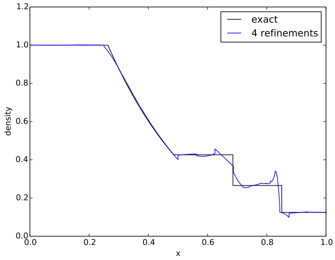

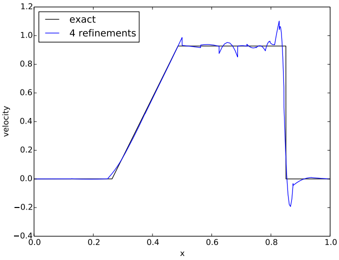

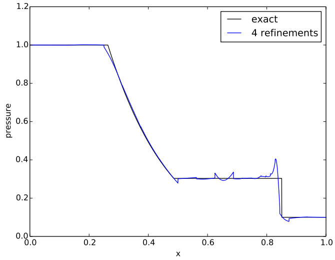

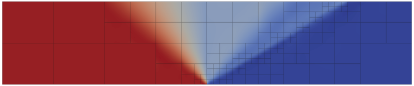

Compressible Navier-Stokes

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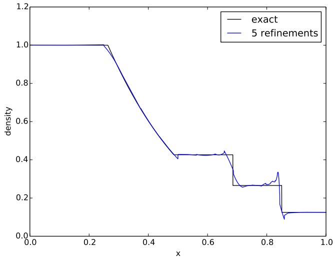

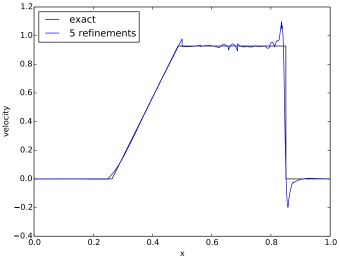

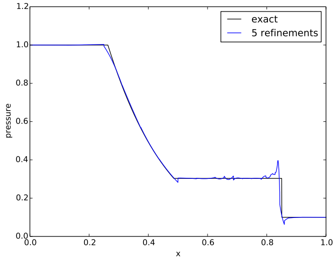

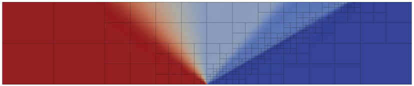

Compressible Navier-Stokes

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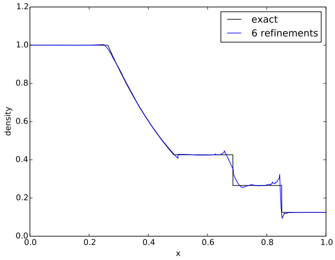
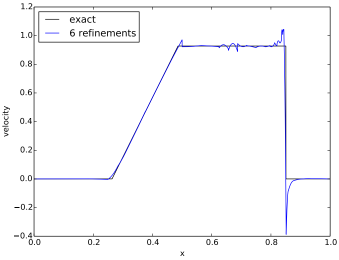
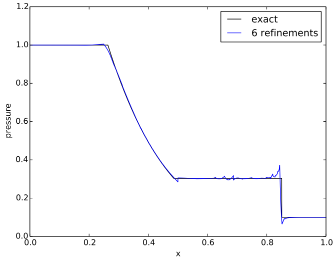
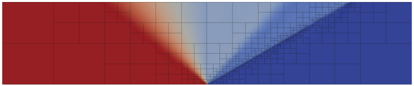


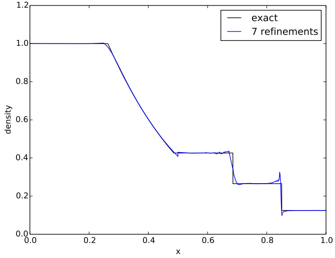


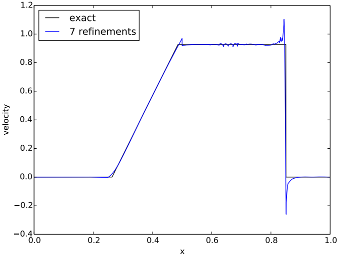


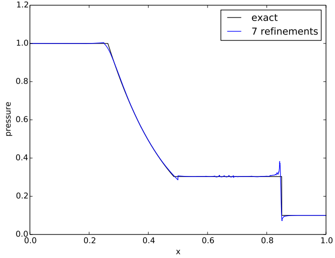


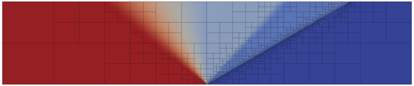
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



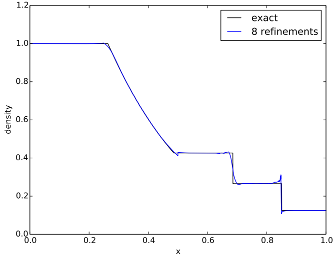


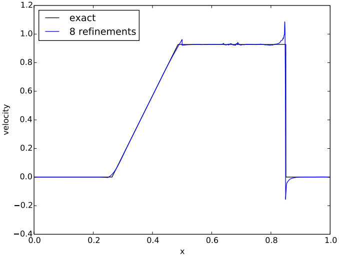


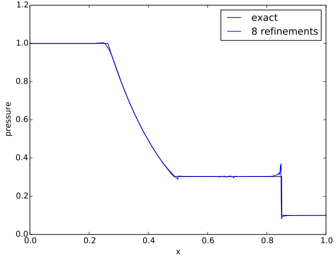


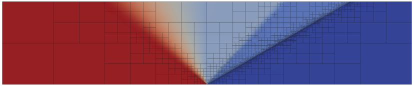
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



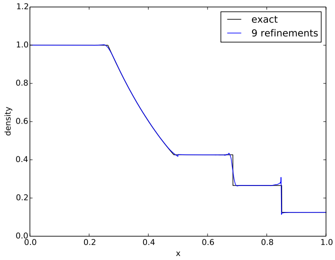


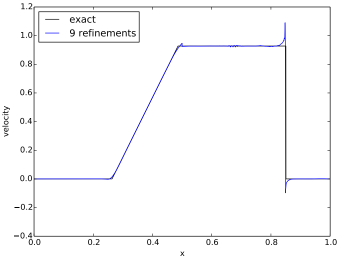


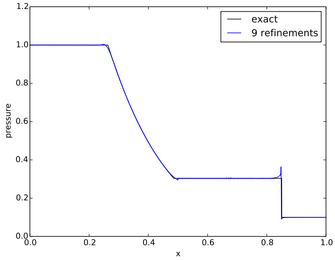


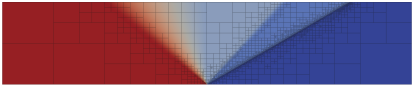
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



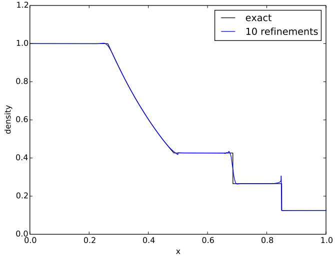


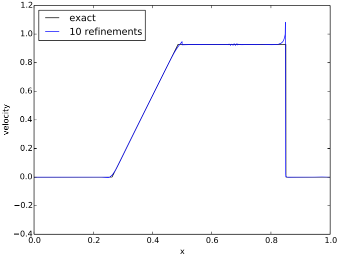


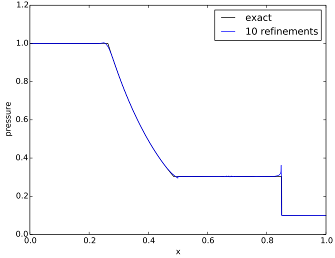


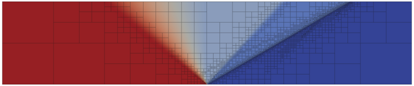
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



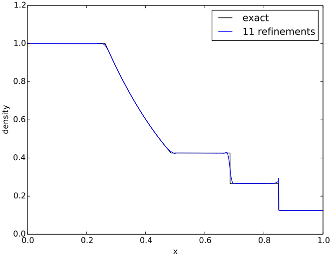


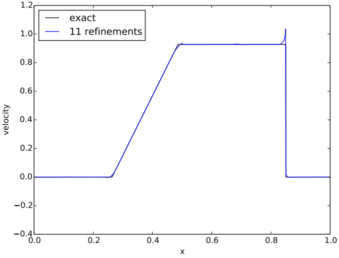


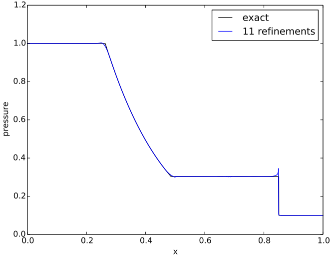


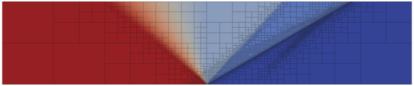
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$









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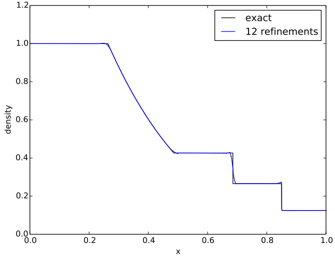

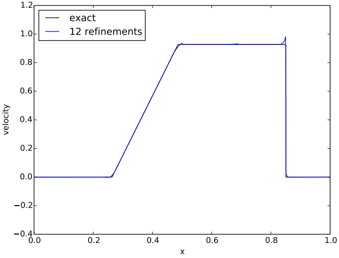

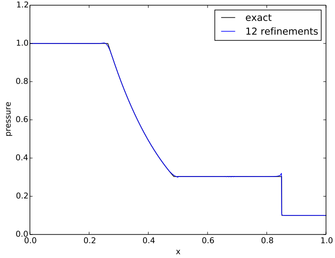

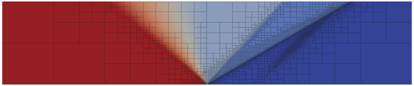

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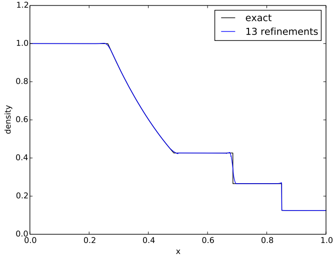

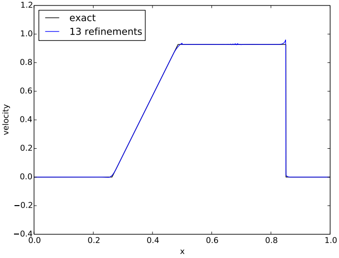

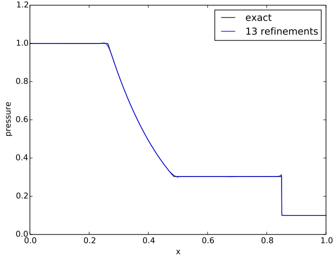

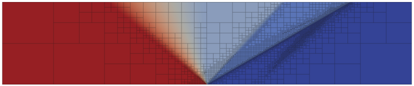

Compressible Navier-Stokes

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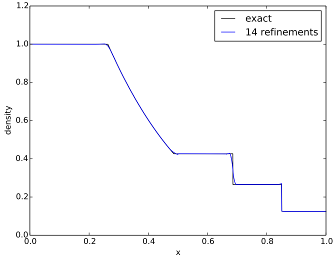
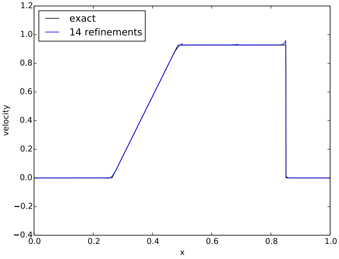
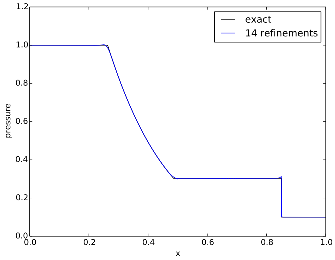
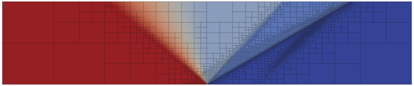




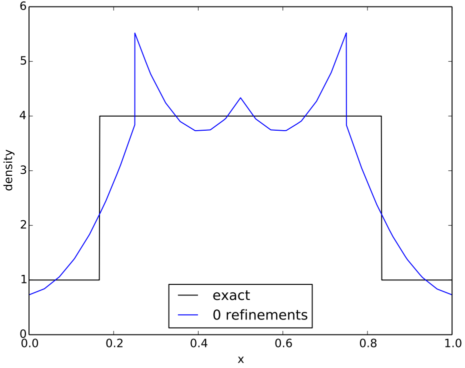


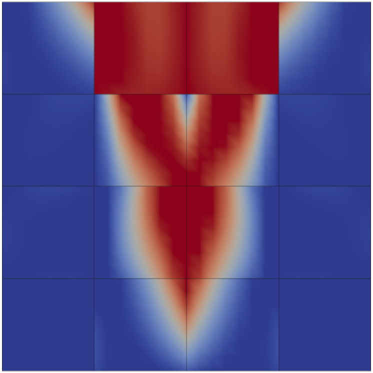


Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.





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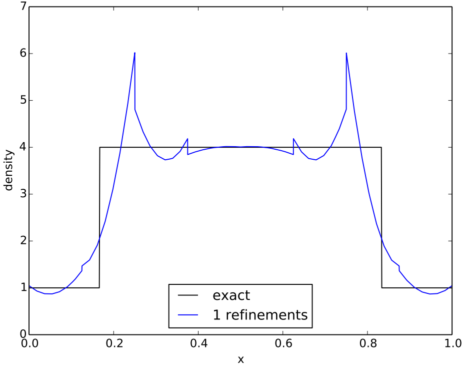

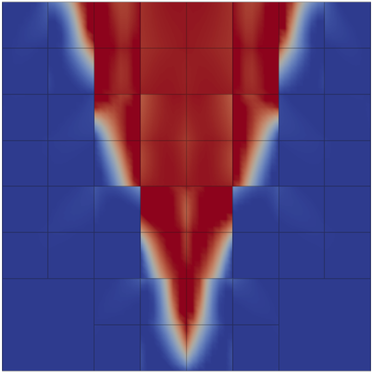

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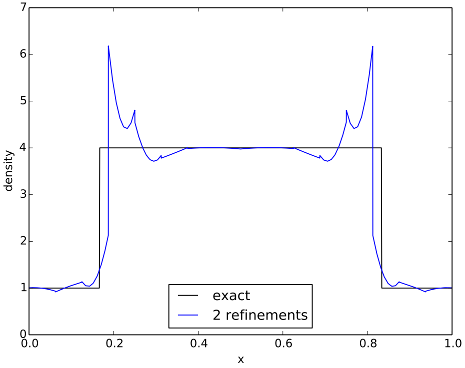

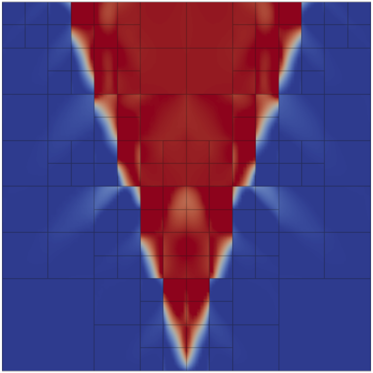

Compressible Navier-Stokes

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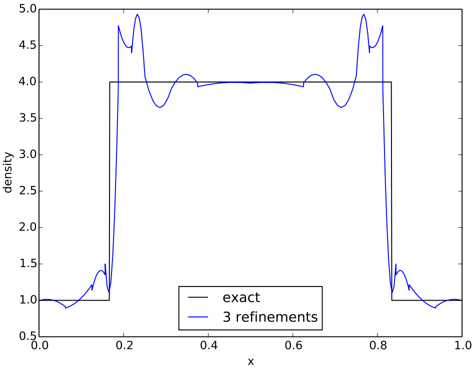

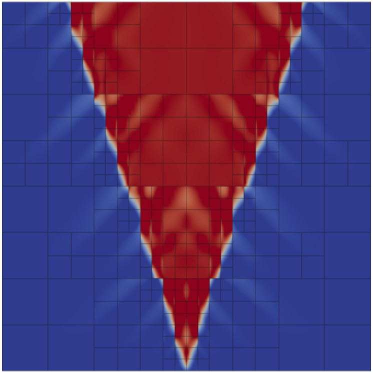

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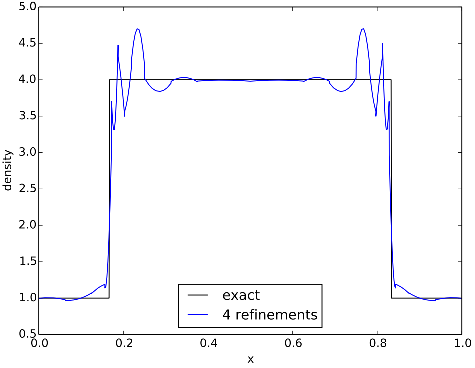

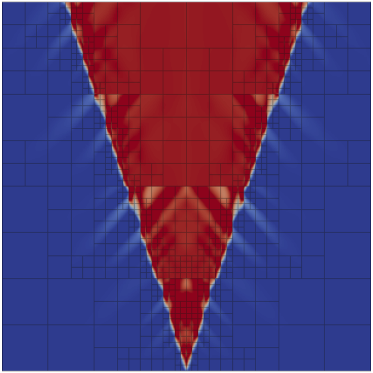

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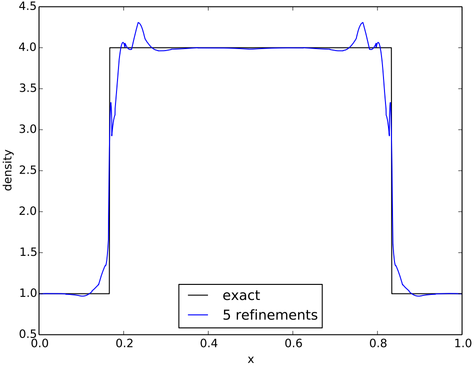

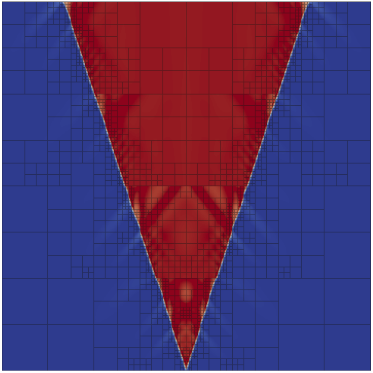

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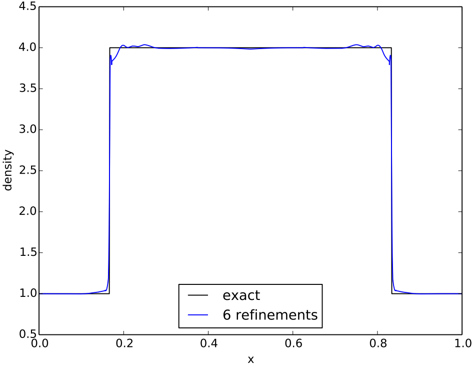

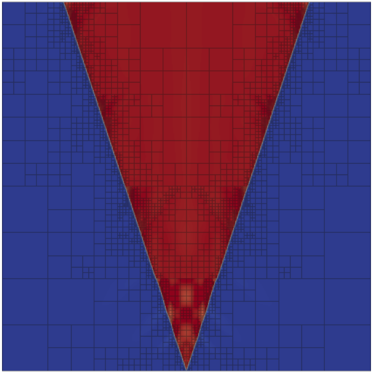

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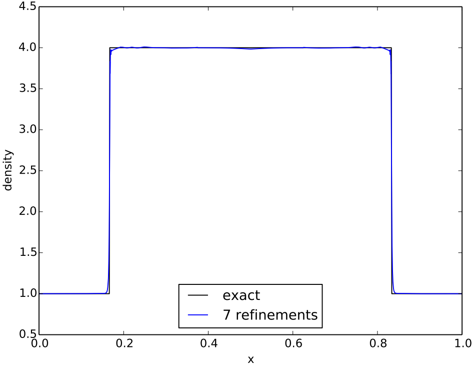

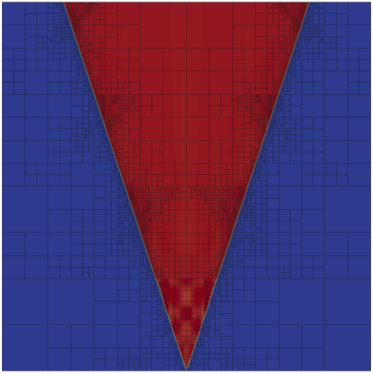

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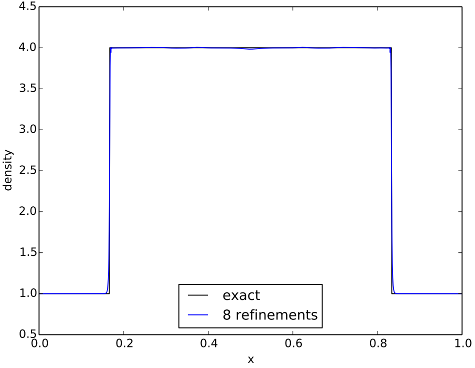

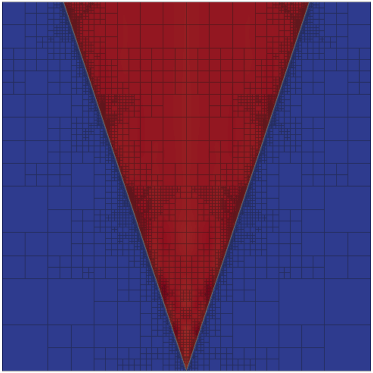

Table of Contents

THE UNIVERSITY OF TEXAS AT AUSTIN

1 Introduction
- Motivation
- Literature Review

2 DPG Overview

3 Preliminary Work
- Local Conservation
- Space-Time DPG

4 Proposed Work

Proposed Work

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Area A: Applicable Mathematics

- Robustness analysis for space-time convection-diffusion
- Explore positivity preserving techniques for compressible Navier-Stokes

Area B: Scientific Computation

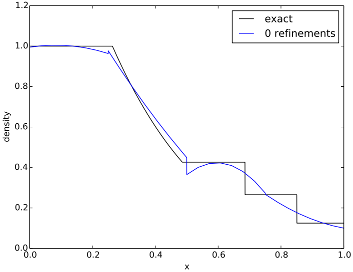
- Support development of Camellia[17]
    - Development and verification of 2D space-time simulations.
    - Implement time slabs to decrease solve size
    - Contribute to auxiliary features
- Run parallel simulations on HPC systems at TACC and ANL

Area C: Modeling and Applications

- Revisit Carter plate solve for higher Reynolds numbers
- Incompressible Taylor-Green vortex problem
- Incompressible vortex shedding problems
- Possibly investigate compressible Sedov and Noh problems

[17] N.V. Roberts. ``Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods''. In: *Comp. Math. Appl.* (2014). submitted.

Thank You!



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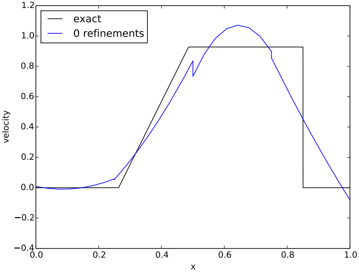

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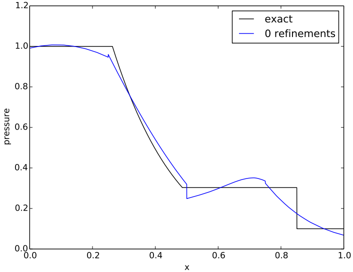

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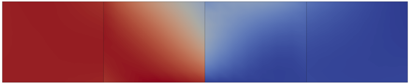

Locally Conservative DPG

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Stability Analysis

We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{U}, p \in Q \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) \quad \forall q \in Q \end{cases}$$

where a, c, l, g denote the appropriate bilinear and linear forms. Note that $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, R_V^{-1} B\boldsymbol{w}) = (R_V^{-1} B\boldsymbol{u}, R_V^{-1} B\boldsymbol{w})_V$.

Let ψ denote the $\boldsymbol{H}(\text{div}, \Omega)$ extension of flux $\hat{t}$ that realizes the minimum in the definition of the quotient (minimum energy extension) norm.

The norm for the Lagrange multipliers λ_K is implied by the quotient norm used for $H^{-1/2}(\Gamma_h)$ and continuity bound for form $c(p, \boldsymbol{w})$:

$$||\boldsymbol{\lambda}|| := \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2}$$

Locally Conservative DPG

Inf Sup Condition

The inf-sup condition relating spaces $\boldsymbol{U}$ and Q is

$$\sup_{\boldsymbol{w}\in\boldsymbol{U}} \frac{|c(p,\boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \beta ||p||_Q$$

Let

$$R : L^2(\Omega) \ni q \to \psi \in \boldsymbol{H}(\text{div},\Omega) \cap \boldsymbol{H}^1(\Omega) = \boldsymbol{H}^1(\Omega)$$

be the continuous right inverse of the divergence operator constructed by Costabel and McIntosh[18]. Let ψ_h denote the classical, lowest order Raviart-Thomas (RT) interpolant of function

$$\psi = R(\sum_K \lambda_K 1_K)$$

Note that $\text{div}\psi_h = \text{div}\psi = \lambda_K$ in element K.

[18] M. Costabel and A. McIntosh. ``On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains''. In: *Math. Z.* 265.2 (2010), pp. 297–320. DOI: 10.1007/s00209-009-0517-8. URL: http://dx.doi.org/10.1007/s00209-009-0517-8.

Locally Conservative DPG

Inf Sup Condition

Classical h-interpolation error estimate for the lowest error Raviart-Thomas elements and continuity of operator R imply the stability estimate:

$$\begin{aligned} ||\psi_h|| &\leq ||\psi_h - \psi|| + ||\psi|| \\ &\leq Ch||\psi||_{H^1} + ||\psi|| \\ &\leq C||\mathrm{div}\psi|| = C(\textstyle\sum_K \mu(K)\lambda_K^2)^{1/2} \end{aligned}$$

Let $\hat{t}$ be the trace of ψ_h, then

$$\begin{aligned} \sup_{\hat{t}\in H^{-1/2}(\Gamma_h)} \frac{|\sum_K \lambda_K \langle \hat{t}, 1_K\rangle_{\partial K}|}{||\hat{t}||_{H^{-1/2}(\Gamma_h)}} &\geq \frac{|\sum_K \lambda_K \int_K \mathrm{div}\psi_h \, 1_K|}{||\psi_h||_{H(\mathrm{div},\Omega)}} \\ &\geq \frac{1}{C}(\sum_K \mu(K)\lambda_K^2)^{1/2} \end{aligned}$$

Locally Conservative DPG

Inf Sup in Kernel Condition

We characterize the ``kernel'' space:

$$\boldsymbol{U}_0 := \{\boldsymbol{w} \in \boldsymbol{U} \,:\, c(q, \boldsymbol{w}) = 0 \quad \forall q \in Q\}$$
$$= \{(u, \boldsymbol{\sigma}, \hat{u}, \hat{t}) \,:\, \langle \hat{t}, 1_K \rangle = 0 \quad \forall K\}$$

With $\boldsymbol{u} \in \boldsymbol{U}_0$, we have then:

$$\sup_{\boldsymbol{w} \in \boldsymbol{U}_0} \frac{|a(\boldsymbol{u}, \boldsymbol{w})|}{||\boldsymbol{w}||_{\boldsymbol{U}}} \geq \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||\boldsymbol{u}||} = \frac{|b(\boldsymbol{u}, T\boldsymbol{u})|}{||T\boldsymbol{u}||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||}$$
$$= \sup_{(v, \boldsymbol{\tau})} \frac{|b((u, \boldsymbol{\sigma}, \hat{u}, \hat{t}), (v, \boldsymbol{\tau}))|}{||(v, \boldsymbol{\tau})||} \frac{||T\boldsymbol{u}||}{||\boldsymbol{u}||} \geq \gamma^2 ||(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})||$$

where γ is the stability constant for the standard DPG formulation.
The FE error is bounded by the best approximation error. Note that the exact Lagrange multipliers are zero, so the best approximation error involves only the solution $(u, \boldsymbol{\sigma}, \hat{u}, \hat{t})$.

Locally Conservative DPG

Robustness Analysis

- We prove robustness of the restricted DPG method by switching to the energy norm in Brezzi's stability analysis.
- The inf-sup in kernel condition is simple. Upon replacing the original norm of solution $\boldsymbol{u}$ with the energy norm, γ and the continuity constant become one.
- In order to investigate the robustness of inf-sup constant β, we need to understand what the energy norm of flux variable $\hat{t}$ is.
- For an element, K, we solve for the optimal test functions, $v_K \in H^1(K)$, and $\boldsymbol{\tau}_K \in \boldsymbol{H}(\text{div}, K)$ corresponding to flux $\hat{t}$:

$$((v_K, \boldsymbol{\tau}_K), (\delta v, \delta \boldsymbol{\tau}))_V = \langle \hat{t}, \delta v \rangle \quad \forall \delta v \in H^1(K),\, \delta \boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, K)$$

- The energy norm for $\hat{t}$ is then

$$||\hat{t}||_E^2 = \sum_K ||(v_K, \boldsymbol{\tau}_K)||_V^2$$

Locally Conservative DPG

Robustness Analysis

- Need to establish conditions under which the inf-sup constant is independent of viscosity.

$$\sup_{\hat{t}} \frac{|\sum_K \lambda_K \langle \hat{t}, 1_K \rangle|}{||\hat{t}||_E} \geq \beta \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2}$$

- Select $\hat{t}$ as the trace of the Raviart-Thomas interpolant ψ_h of $\psi = R(\sum_K \lambda_K 1_K)$.
- Proceed as in the previous analysis, but evaluation of the norm of $\hat{t}$ requires a local solve.

$$\begin{aligned} ((v, \boldsymbol{\tau}),(\delta v, \delta \boldsymbol{\tau}))_V = \langle \hat{t}, \delta v \rangle_{\partial K} = \int_K \text{div} \psi_h \delta v = \int_K \text{div} \psi \delta v \\ = \int_K \lambda_K \delta v = \lambda_K (1_K, \delta v)_K \quad \forall \delta v \in H^1(K) \, \forall \delta \boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, K) \end{aligned}$$

Locally Conservative DPG

Robustness Analysis

- We need an upper bound for the energy norm of $(v_h, \boldsymbol{\tau}_h)$:

$$\left(\sum_K ||(v, \boldsymbol{\tau})||_V^2\right)^{1/2}$$

- Substituting $(v, \boldsymbol{\tau})$ for $(\delta v, \delta\boldsymbol{\tau})$ in the previous slide:

$$||(v, \boldsymbol{\tau})||_V^2 = \lambda_K(1_K, v_k)$$

- With a robust stability estimate $(1_K, v_K) \leq C\mu(K)^{1/2}||(v, \boldsymbol{\tau})||_K$,

$$||(v, \boldsymbol{\tau})||_V \leq C\mu(K)^{1/2}|\lambda_K|$$

$$\sum_K ||(v, \boldsymbol{\tau})||_V^2 \leq C^2 \sum_K \mu(K)\lambda_K^2$$

which leads to the robust estimate of inf-sup constant β.

Locally Conservative DPG

Robustness Analysis

- Finally, we need a continuity estimate on

$$\sum_K \lambda_K \langle \hat{t}, 1_K \rangle$$

- Testing with $(1_K, \mathbf{0})$ in the local problem, we obtain

$$((v, \boldsymbol{\tau}), (1_K, \mathbf{0}))_V = \langle \hat{t}, 1_K \rangle_{\partial K}$$

- With a robust estimate $|((v, \boldsymbol{\tau}), (1_K, \mathbf{0}))_V| \leq C\mu(K)^{1/2} ||(v, \boldsymbol{\tau})||_V$,

$$\begin{aligned}
|\sum_K \lambda_K \langle \hat{t}, 1_K \rangle| &\leq C \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2} \left(\sum_K ||(v, \boldsymbol{\tau})||_V^2 \right)^{1/2} \\
&= C \left(\sum_K \mu(K) \lambda_K^2 \right)^{1/2} ||\hat{t}||_E
\end{aligned}$$