Multi-Resolution Viscosity Limiter in the BLAST High-Order Finite Element Hydrodynamics Code

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Hydrodynamics

The evolution of the particles of a compressible fluid/solid in a Lagrangian reference frame is governed by the following system of differential equations:

Euler's Equations

Momentum Conservation: $$\rho \frac{\mathrm{d}v}{\mathrm{d}t} = \nabla \cdot \sigma$$

Mass Conservation: $$\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\nabla \cdot v$$

Energy Conservation: $$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = \sigma : \nabla v$$

Equation of State: $$p = EOS(e, \rho)$$

Equation of Motion: $$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$

Kinematics

x – position

v – velocity

Thermodynamics

ρ – density

e – internal energy

Stress Tensor

$$\sigma = -p\mathbf{I} + \sigma_a$$

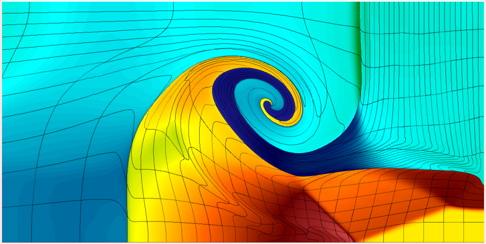
p – pressure

σ_a – artificial stress

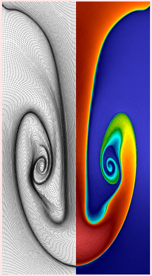
- Time derivatives are along particle trajectories
- Space derivatives are with respect to a fixed coordinate system
- Domain: $\Omega(t) = \{x(t)\}$; Total Energy: $E(t) = \int_{\Omega(t)} (\rho|v|^2/2 + \rho e)$

The BLAST High Order Finite Element Hydrodynamics Code

- Curvilinear finite elements are used to "more accurately represent deformations"
- Kinematic variables are represented with continuous fields, thermodynamic variables are represented with discontinuous fields (material interface)
- "Strong mass Conservation" is achieved by defining density as a function within a zone
- Uses the MFEM framework which allows arbitrarily high order basis functions
- Corner forces are FLOP-intensive and can be computed independently for each zone and assembled later



Triple-Point problem with Q8-Q7



Rayleigh-Taylor Instability with Q8-Q7

- https://computation.llnl.gov/casc/blast/blast.html
- Dobrev VA, Kolev TzV, Rieben RN. High order curvilinear finite element methods for Lagrangian hydrodynamics. SIAM J Sci Comput.

Artificial Viscosity in BLAST

"Our idea is to introduce (artificial) dissipative terms into the equations so as to give the shocks a thickness comparable to (but preferably somewhat larger than) the spacing of the points of the network... Then the differential equations ...may be used for the entire calculation, just as though there were no shocks at all." – Von Neumann and Richtmyer

Ideally, an artifical viscosity implementation should:

- Always reduces kinetic energy (dissipates shocks)
- Vanishes in smooth regions and in rarefaction
- Vanishes in uniform contraction and rigid rotation
- Satisfies Rankine-Hugoniot jump conditions away from the shock
- Galilean invariant

Artificial Stress in BLAST

$$\sigma_a = \mu \frac{1}{2}(\nabla v + v\nabla), \quad \text{where} \quad \mu \equiv \rho(q_2 l_s^2 |\Delta_s v| + q_1 l_s c_s)\,.$$

- This allows us to robustly handle shocks, but limits our convergence to first order in smooth flow.

Sub-Cell Shock Capturing of Persson and Peraire

- Discontinuities create oscillations in the highest polynomial modes.
- Persson and Peraire [1] do a change of basis to a hierarchical family of orthogonal polynomials, such that

$$u = \sum_{i=1}^{N(k)} u_i \psi_i \,,$$

  where u is some function of the solution (velocity, entropy, etc).
- Define a truncated expansion

$$\tilde{u} = \sum_{i=1}^{N(k-1)} u_i \psi_i \,.$$

- Compute a piece-wise constant smoothness indicator using numerical quadrature,

$$s_z = \log_{10} \frac{(u - \tilde{u}, u - \tilde{u})_z}{(u, u)_z} \,.$$

- They then turn this into an artificial viscosity that vanishes for smooth, resolved flow.
- In practice, this allows their high-order DG methods to achieve sub-zonal shock capturing.

[1] P. Persson, J. Peraire, Sub-Cell Shock Capturing for Discontinuous Galerkin Methods, 44th AIAA Aerospace Sciences Meeting and Exhibit, Reno, Nevada, 2007. AIAA-2007-513

Multi-Resolution Viscosity Limiter in BLAST

- Persson and Peraire use discontinuous spaces, which makes the orthogonal basis expansion straight-forward.
- In BLAST, it makes more sense to do an L^2 projection using our continuous velocity space.
- For $v \in \mathcal{P}^k$, find $\tilde{\mathbf{v}} \in \mathcal{P}^{k-1}$ such that

$$\int_\Omega \tilde{\mathbf{v}}\mathbf{w} dx = \int_\Omega v w dx \quad \forall w \in \mathcal{P}^{p-1} .$$

In matrix vector form, this is

$$\mathbf{M}_{k-1}\tilde{v} = \mathbf{g}_{\mathbf{v}} .$$

- Use the difference as a shock detector which we call the smoothness.

$$s_z = \log_{10} ||v - \tilde{v}||_z^2 .$$

- Define a viscosity limiter.

$$\psi_l = \begin{cases} 0 & \text{if} \quad s_z <= s_0 - \kappa \\ \frac{1}{2}\left(1 + \sin\frac{\pi(s_z - s_0)}{2\kappa}\right) & \text{if} \quad s_0 - \kappa < s_z < s_0 + \kappa \\ 1 & \text{if} \quad s_z >= s_0 + \kappa \end{cases}$$

where s_0 and κ are tunable parameters that define the cutoff and transition region.

- Use the new limited viscosity.

$$\mu_l = \psi_l \mu$$

High-order FEM overview: Time integration

Let $Y = (\mathbf{v}; \mathbf{e}; \mathbf{x})$. Then the semi-discrete equations can be written in the form:

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \mathcal{F}(Y, t)$$

$$\mathcal{F}(Y,t) = \begin{pmatrix} \mathcal{F}_v(\mathbf{v},\mathbf{e},\mathbf{x}) \\ \mathcal{F}_e(\mathbf{v},\mathbf{e},\mathbf{x}) \\ \mathcal{F}_x(\mathbf{v},\mathbf{e},\mathbf{x}) \end{pmatrix} = \begin{pmatrix} -\mathbf{M_v}^{-1}\mathbf{F}\cdot\mathbf{1} \\ \mathbf{M_e}^{-1}\mathbf{F}^\top\cdot\mathbf{v} \\ \mathbf{v} \end{pmatrix}$$

Standard high-order time integration techniques (e.g. explicit Runge-Kutta methods) can be applied to this system of nonlinear ODEs.

The L^2 projection is done at each stage of the integration and used in the artificial viscosity when computing $\mathbf{F}$.

For example, consider the energy conserving RK2Avg method:

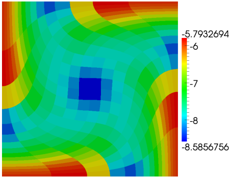
$$\begin{aligned} \tilde{\mathbf{v}} &= \mathbf{M}_{k-1}^{-1}\mathbf{g_v} \\ \mathbf{v}^{n+\frac{1}{2}} &= \mathbf{v}^n - (\Delta t/2)\,\mathbf{M_v}^{-1}\mathbf{F}^n\cdot\mathbf{1} \\ \mathbf{e}^{n+\frac{1}{2}} &= \mathbf{e}^n + (\Delta t/2)\,\mathbf{M_e}^{-1}(\mathbf{F}^n)^\top\cdot\mathbf{v}^{n+\frac{1}{2}} \\ \mathbf{x}^{n+\frac{1}{2}} &= \mathbf{x}^n + (\Delta t/2)\,\mathbf{v}^{n+\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{v}} &= \mathbf{M}_{k-1}^{-1}\mathbf{g_v} \\ \mathbf{v}^{n+1} &= \mathbf{v}^n - \Delta t\,\mathbf{M_v}^{-1}\mathbf{F}^{n+\frac{1}{2}}\cdot\mathbf{1} \\ \mathbf{e}^{n+1} &= \mathbf{e}^n + \Delta t\,\mathbf{M_e}^{-1}(\mathbf{F}^{n+\frac{1}{2}})^\top\cdot\bar{\mathbf{v}}^{n+\frac{1}{2}} \\ \mathbf{x}^{n+1} &= \mathbf{x}^n + \Delta t\,\bar{\mathbf{v}}^{n+\frac{1}{2}} \end{aligned}$$

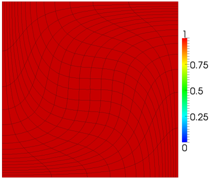
Taylor Green Vortex, $s_0 = -11$, $\kappa = 1$

- On coarse meshes, the indicator does not detect sufficient smoothness in the solution to turn off viscosity.
- With sufficient resolution, the indicator shuts down all viscosity and we recover higher order convergence rates.

Q2-Q1

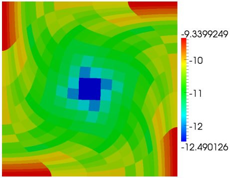


Smoothness



Limiter

Q4-Q3



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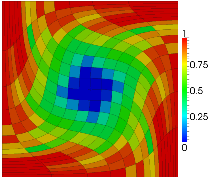

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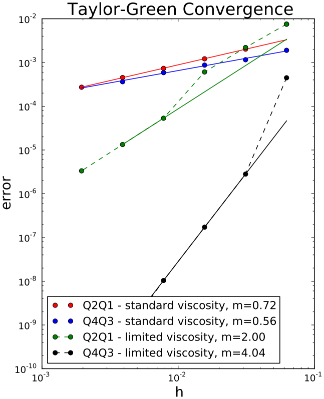

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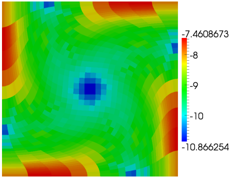

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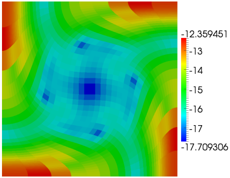

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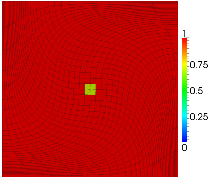

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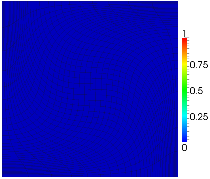

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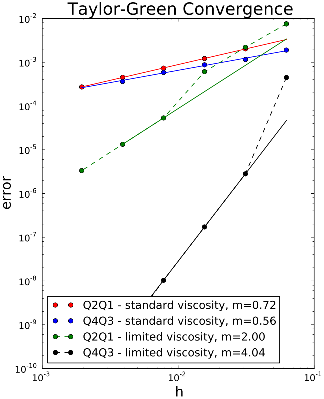

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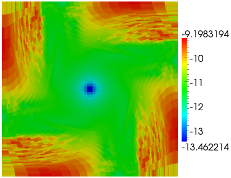

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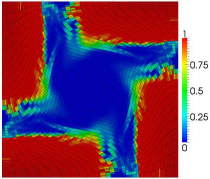

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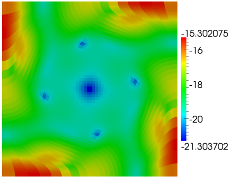

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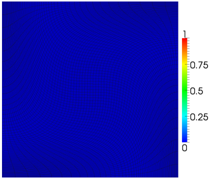

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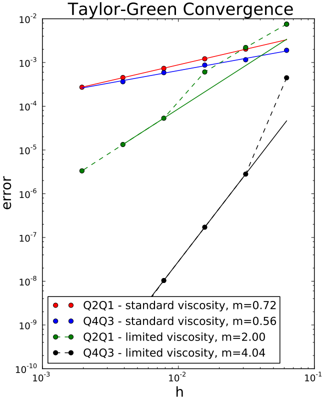

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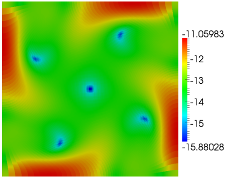

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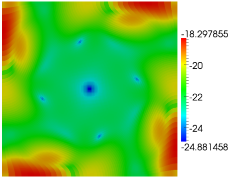

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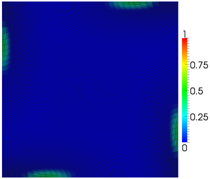

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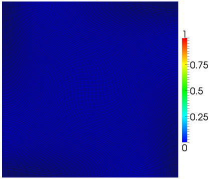
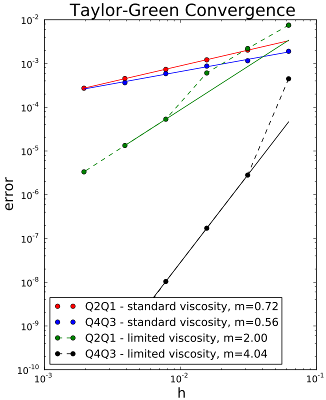
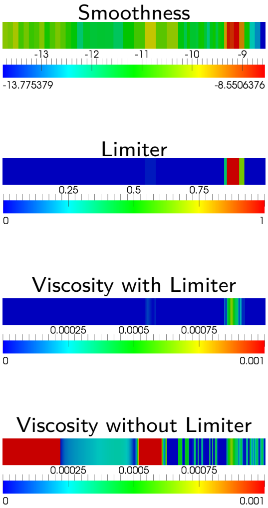


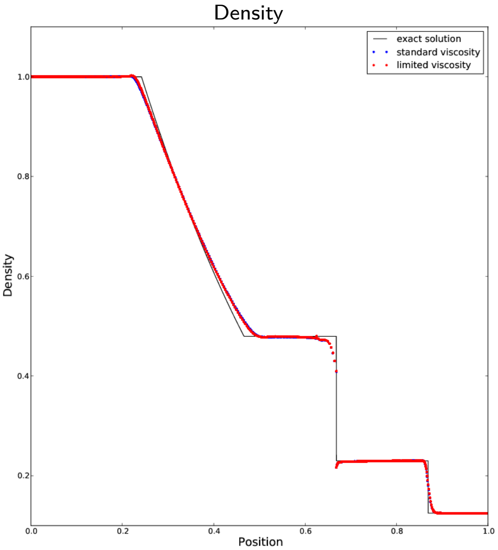
Limiter



Sod Shock Tube, $s_0 = 9.5$, $\kappa = 0.5$

Q4-Q3 elements on a 50x1 mesh





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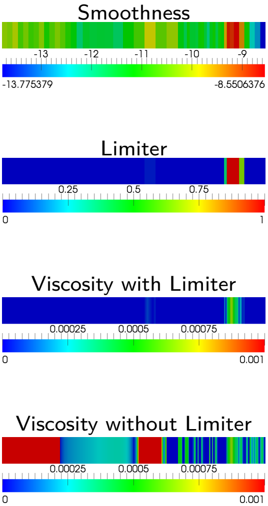
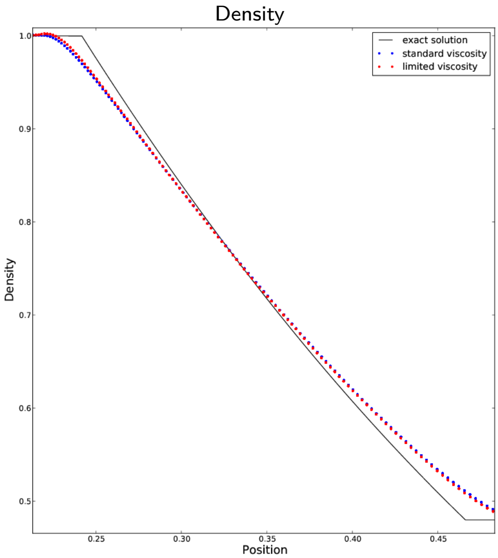
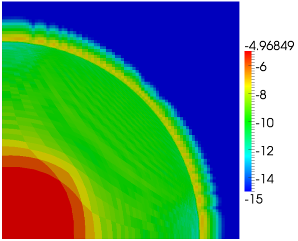


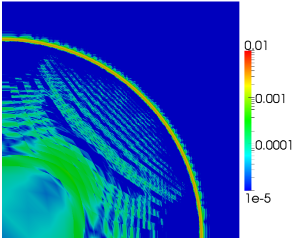


Sedov Blast Wave, $s_0 = -9.5$, $\kappa = 0.5$

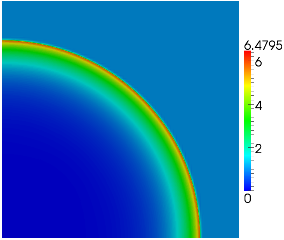
Q4-Q3 elements on an 80x80 mesh



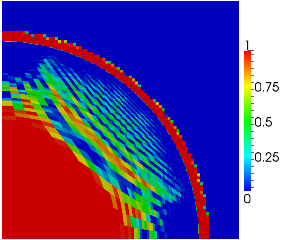
Smoothness



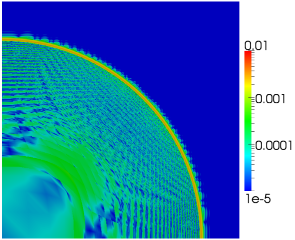
Viscosity with Limiter



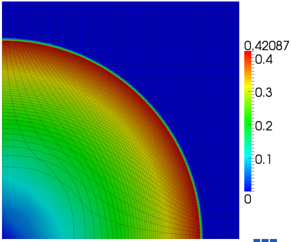
Density with Limiter



Limiter



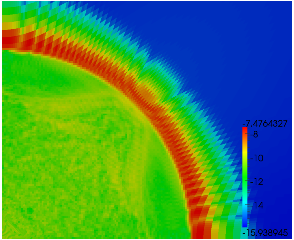
Viscosity without Limiter



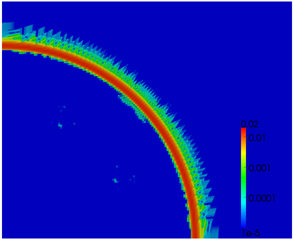
Velocity with Limiter

Noh Implosion, $s_0 = -9.5$, $\kappa = 0.5$

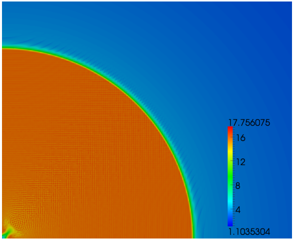
Q2-Q1 elements on an 128x128 mesh



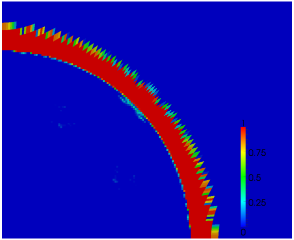
Smoothness



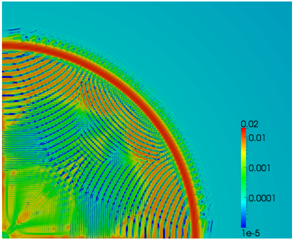
Viscosity with Limiter



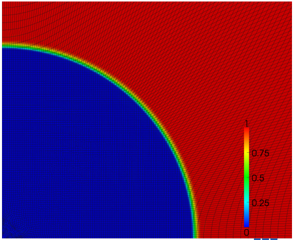
Density with Limiter



Limiter

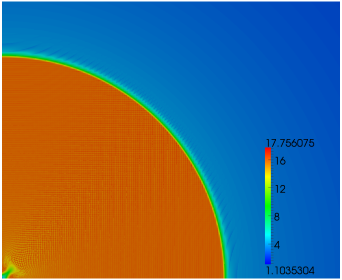


Viscosity without Limiter

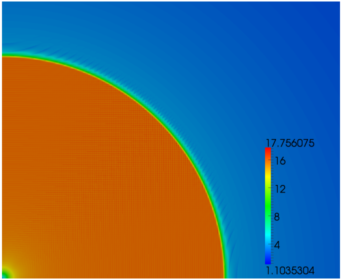


Velocity with Limiter

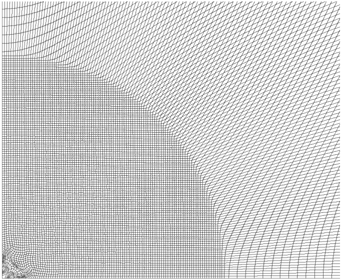
Noh Implosion, $s_0 = -9.5$, $\kappa = 0.5$



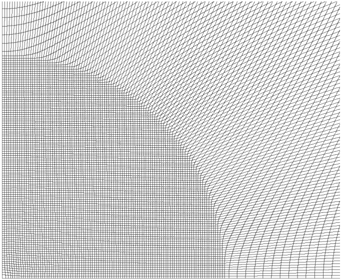
Density with Limiter



Density without Limiter



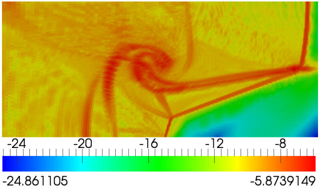
Mesh with Limiter



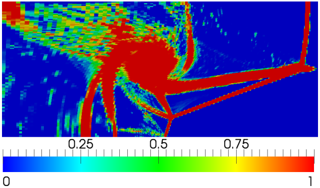
Mesh without Limiter

Triple Point Shockwave, $s_0 = -9.5$, $\kappa = 0.5$

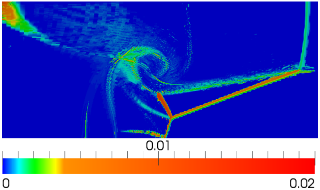
Smoothness



Limiter



Viscosity with Limiter



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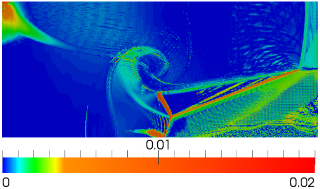

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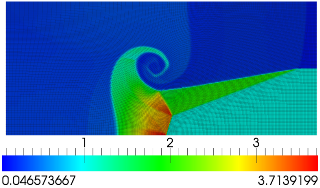
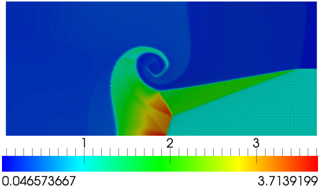


Density without Limiter



Conclusions

Future work

- Try using physical mass matrices in L^2 projection.
  - Don't need to recompute at each time step.
- Try expanding velocity in terms of orthonormal Legendre polynomials as per Persson and Peraire.
- Consider making s_0 and *kappa* functions of h and/or p.