

Discontinuous Petrov-Galerkin (DPG) Method With Optimal Test Functions

Progress Report

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World Congress on Computational Mechanics
Barcelona, July 20 - July 25, 2014



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Sandia: P.N. Bochev, K.J. Peterson, D. Ridzal and Ch. M. Siefert

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Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

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Primal DPG Method for Maxwell equations.

Assume

$$J_S^{\text{imp}} = n \times H^{\text{imp}}$$

and look for the unknown surface current on the skeleton also in the same form.

$$\left\{ \begin{array}{l} E \in H(\text{curl}, \Omega), n \times E = n \times E^{\text{imp}} \text{ on } \Gamma_1 \\ \hat{h} \in \text{tr}_{\Gamma_h} H(\text{curl}, \Omega), n \times \hat{h} = n \times (-i\omega H^{\text{imp}}) \text{ on } \Gamma_2 \\ (\frac{1}{\mu} \nabla \times E, \nabla_h \times F) + ((-\omega^2 \epsilon + i\omega \sigma) E, F) + \langle n \times \hat{h}, F \rangle_{\Gamma_h} = -i\omega (J^{\text{imp}}, F) \\ \forall F \in H(\text{curl}, \Omega_h). \end{array} \right.$$

FE discretization for curl-curl problem

Hexahedral meshes

$H(\text{curl})$ element for electric field E :

$$(\mathcal{P}^{p-1} \otimes \mathcal{P}^p \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^{p-1} \otimes \mathcal{P}^p) \times (\mathcal{P}^p \otimes \mathcal{P}^p \otimes \mathcal{P}^{p-1})$$

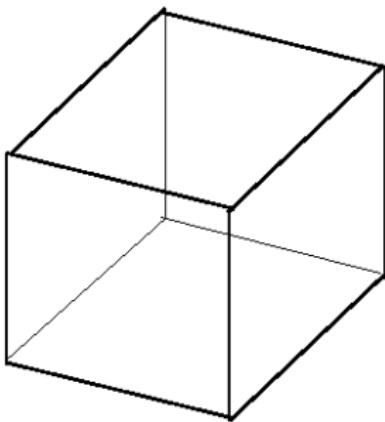
and trace of the same element for flux (surface current) \hat{h} .

Same element for the enriched space but with order $p + \Delta p$.

In reported experiments: $p = 2$, $\Delta p = 2$.

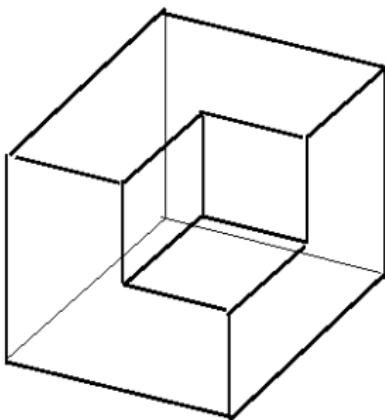
A 3D Maxwell example

Take a cube $(0, 2)^3$



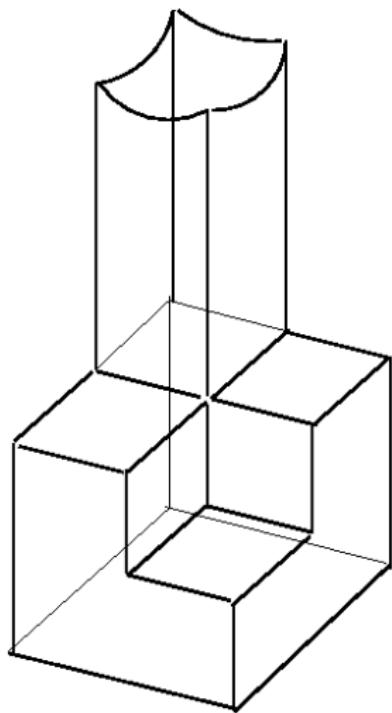
Fichera corner

Divide it into eight smaller cubes and remove one:



Fichera corner microwave

Attach a waveguide:

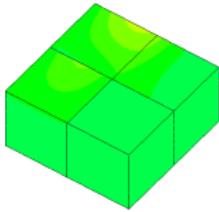
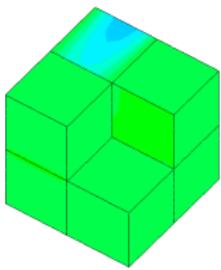
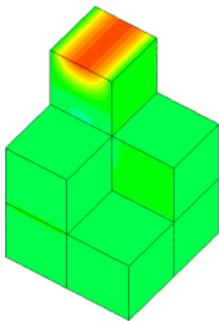
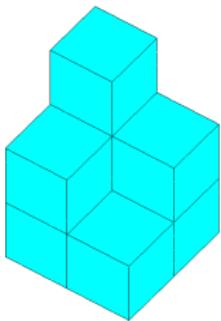


$$\epsilon = \mu = 1, \sigma = 0$$

$\omega = 5(1.6 \text{ wavelengths in the cube})$

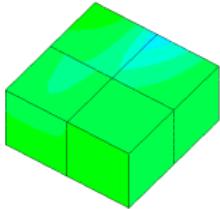
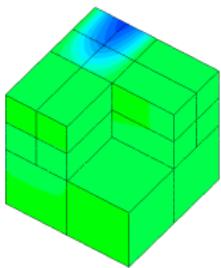
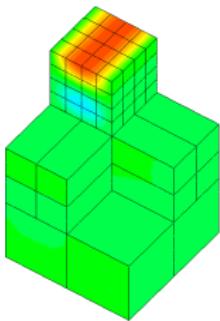
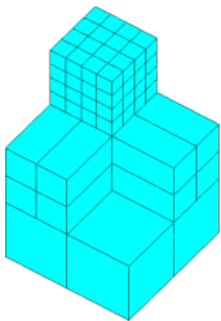
Cut the waveguide and use the lowest propagating mode for BC along the cut.

Fichera corner microwave, $p = 2$.



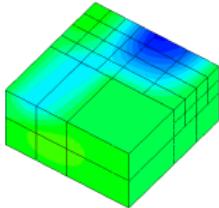
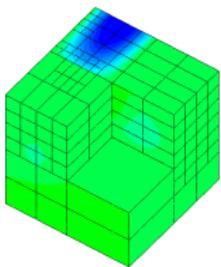
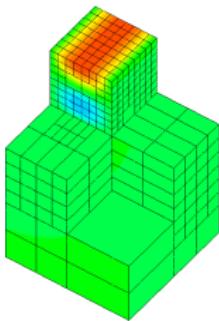
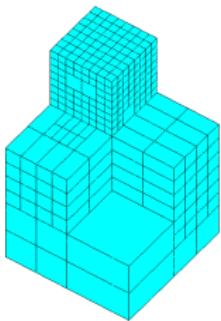
Initial mesh and real part of E_1

Fichera corner microwave, $p = 2$.



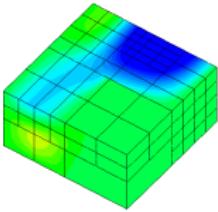
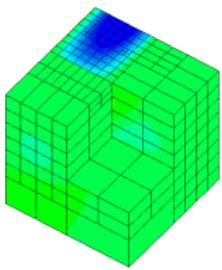
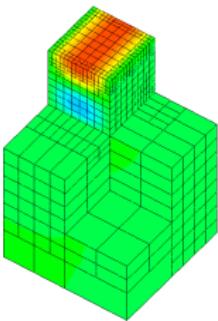
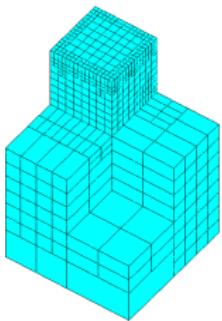
Mesh and real part of E_1 after two refinements

Fichera corner microwave, $p = 2$.



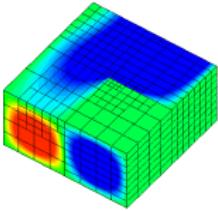
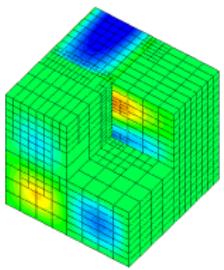
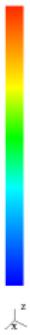
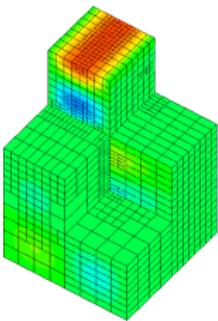
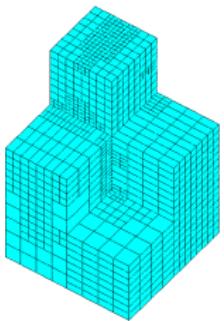
Mesh and real part of E_1 after four refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after six refinements

Fichera corner microwave, $p = 2$.



Mesh and real part of E_1 after eight refinements

From Ph.D. Dissertation of Jesse Chan: Compressible Navier-Stokes Equations: Carter's Flat Plate Problem^{1 2}



$$M_\infty = 3, Re_L = 1000, Pr = 0.72, \gamma = 1.4, \theta_\infty = 390^\circ [R]$$

¹ LD., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity.", *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

² J. Chan, L.D., R. Moser, "A DPG method for steady viscous compressible flow," *Computers and Fluids*, to appear.

Compressible Navier-Stokes

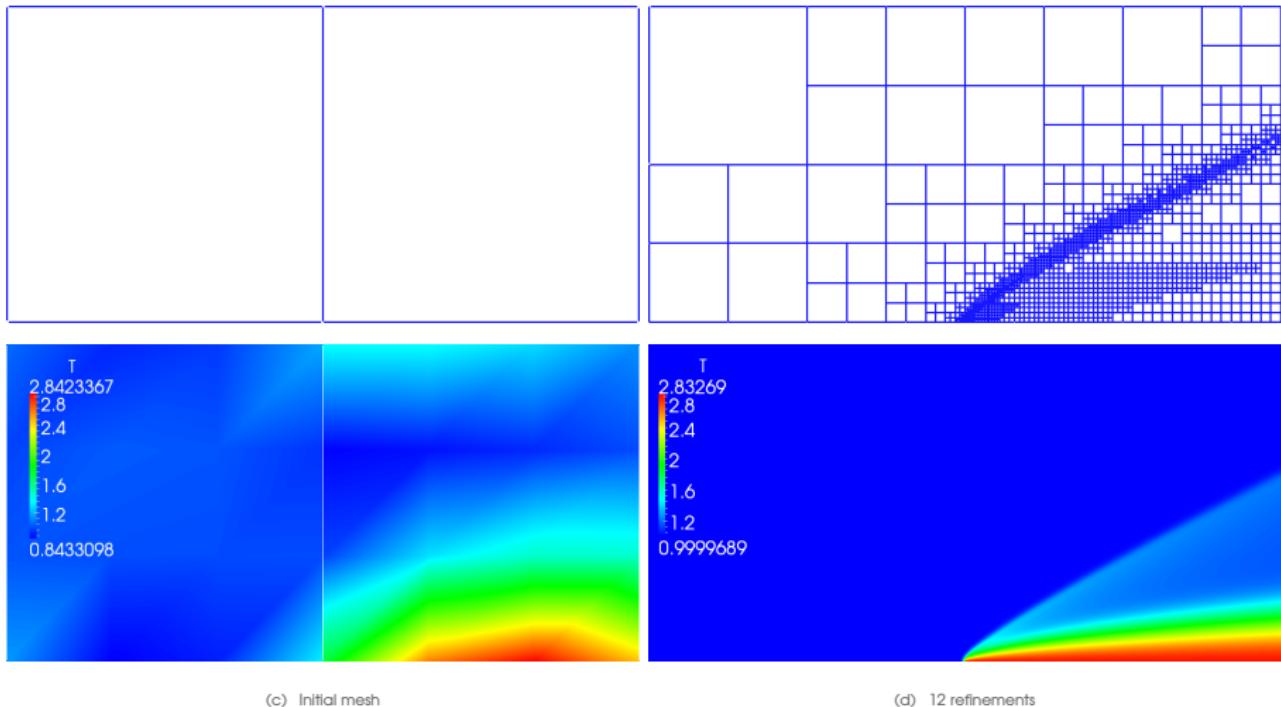


Figure : $\text{Re} = 1000$, initial and final meshes.

Compressible Navier-Stokes

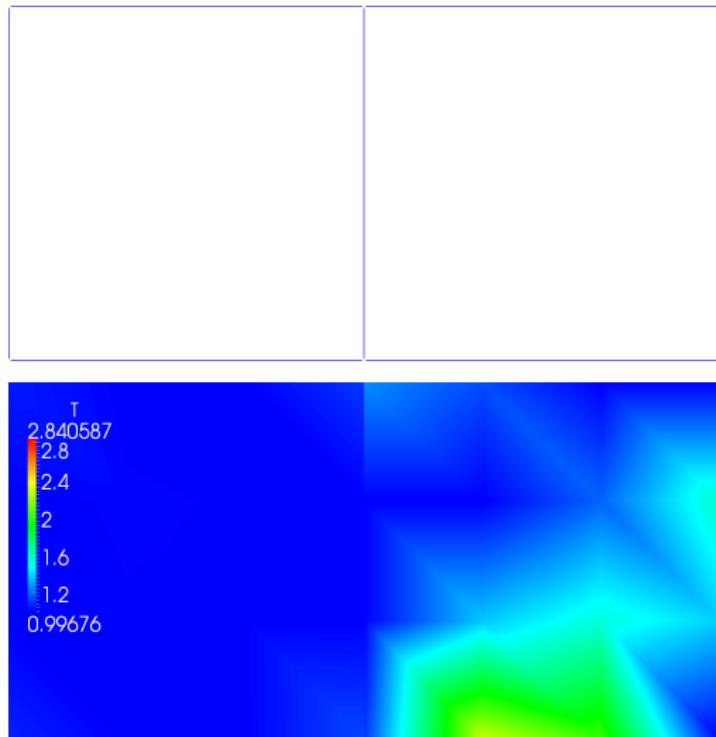


Figure : $\text{Re} = 10,000, p = 2$, initial mesh.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

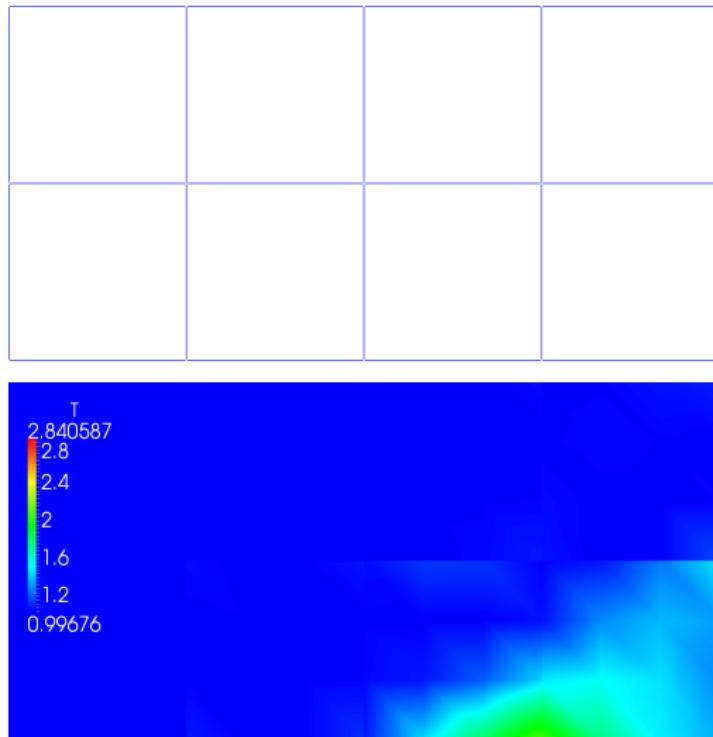


Figure : $\text{Re} = 10,000, p = 2$, refinement number 1.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

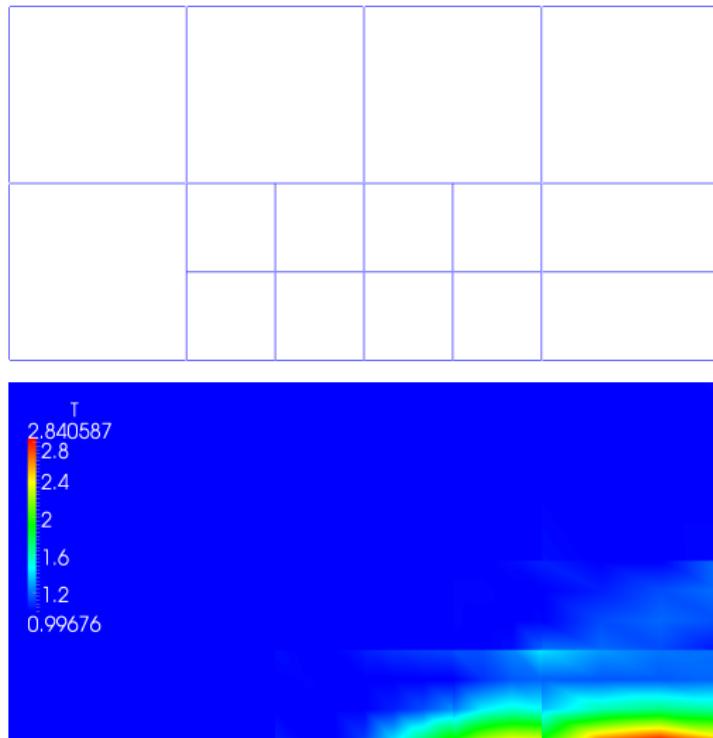


Figure : $\text{Re} = 10,000, p = 2$, refinement number 2.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

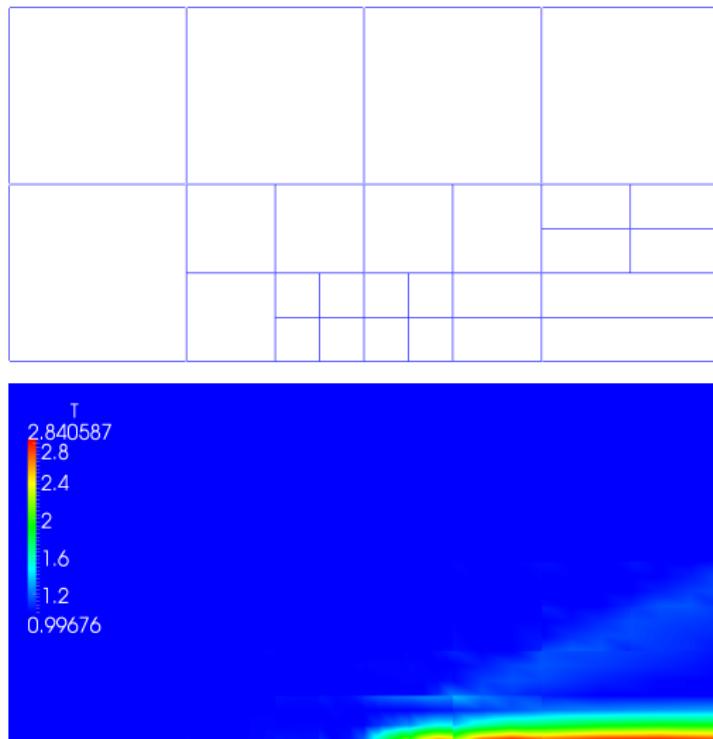


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 3.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

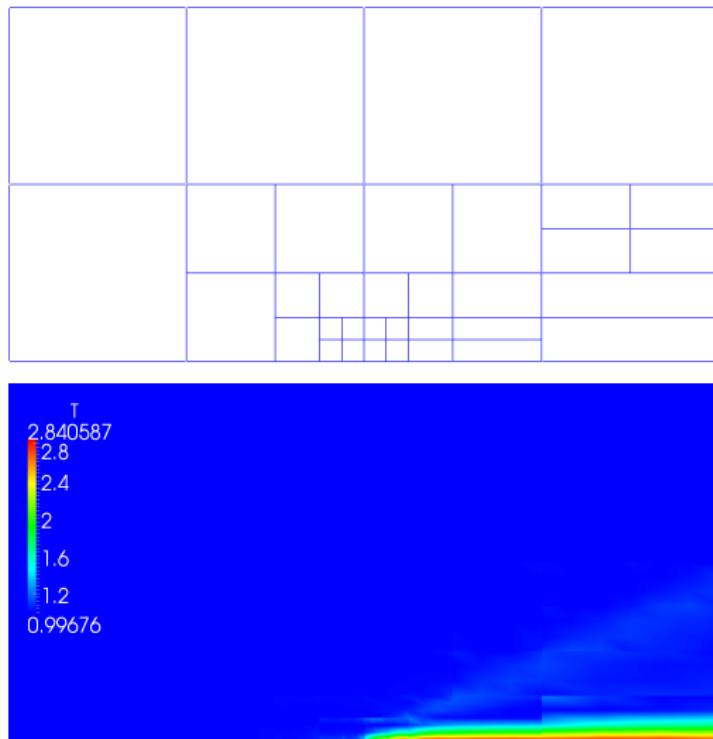


Figure : $\text{Re} = 10,000, p = 2$, refinement number 4.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

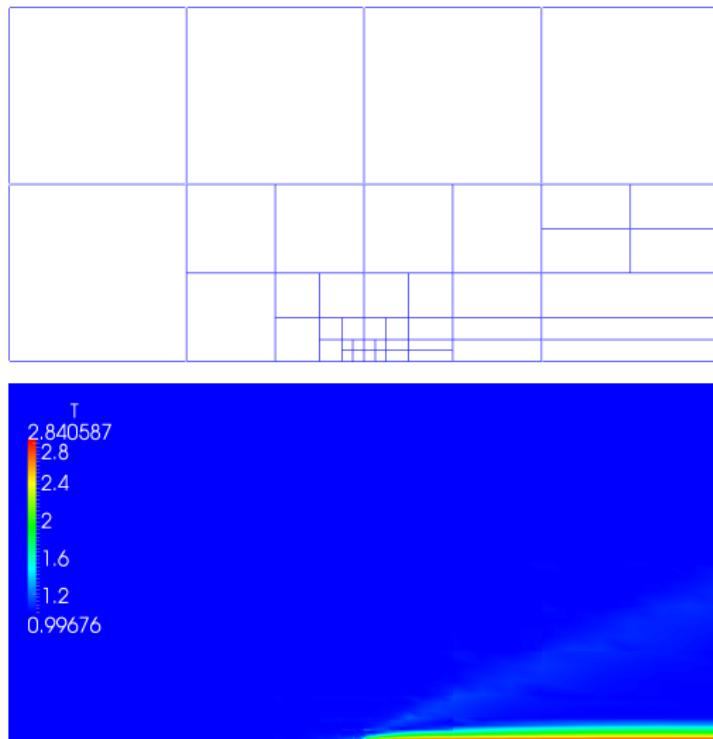


Figure : $\text{Re} = 10,000, p = 2$, refinement number 5.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

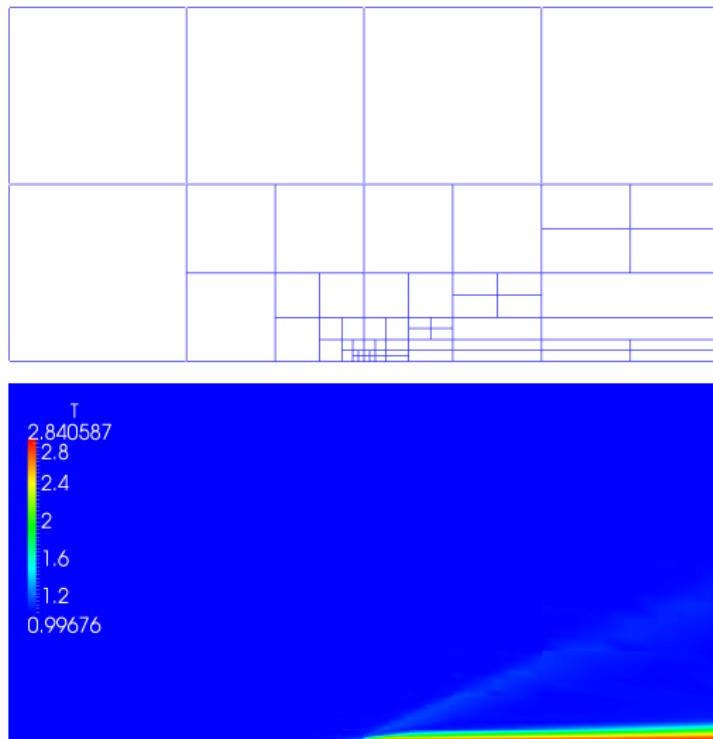


Figure : $\text{Re} = 10,000, p = 2$, refinement number 6.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

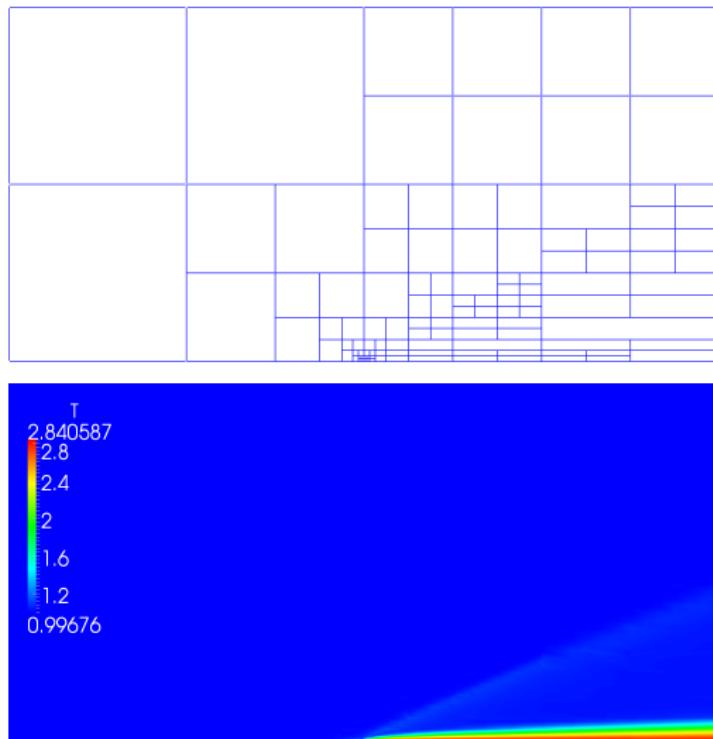


Figure : $\text{Re} = 10,000, p = 2$, refinement number 7.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

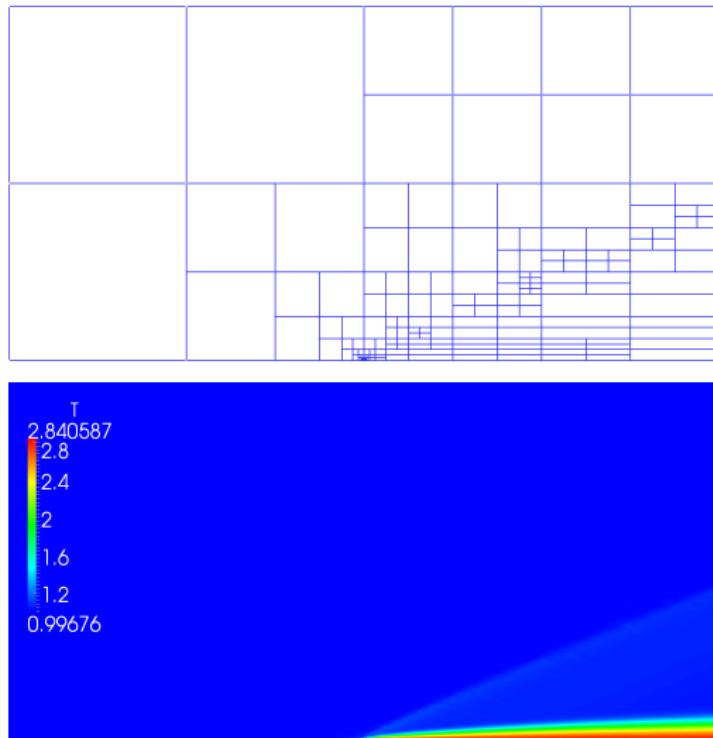


Figure : $\text{Re} = 10,000, p = 2$, refinement number 8.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

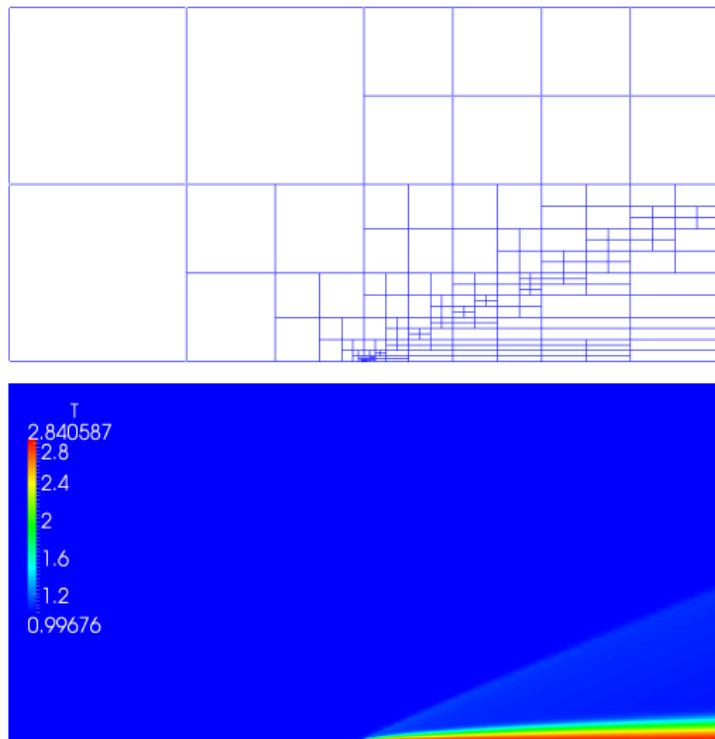


Figure : $\text{Re} = 10,000, p = 2$, refinement number 9.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

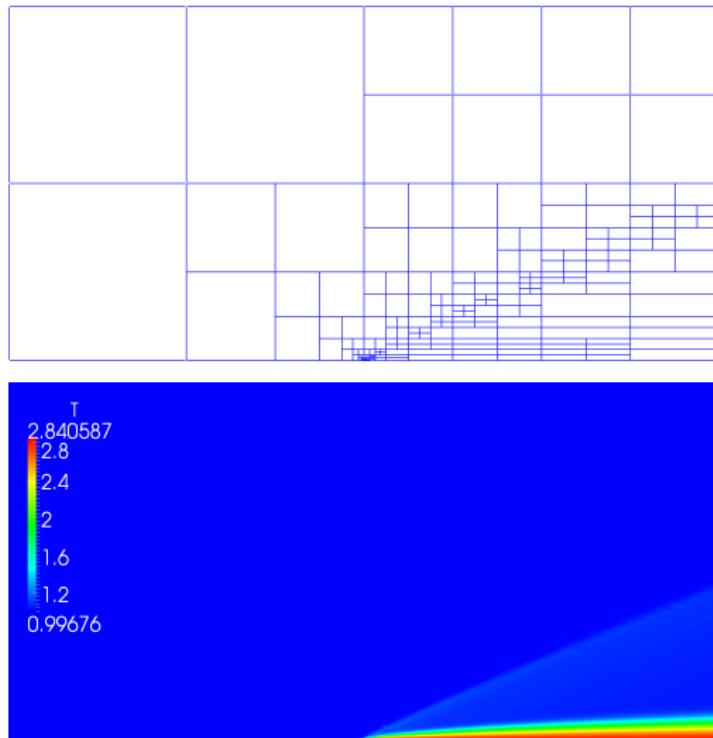


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 10.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

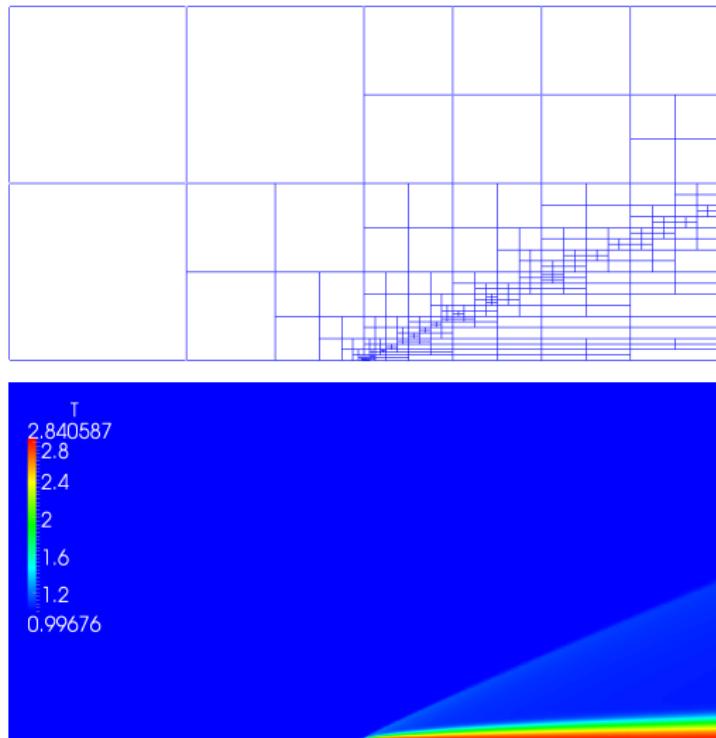


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 11.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

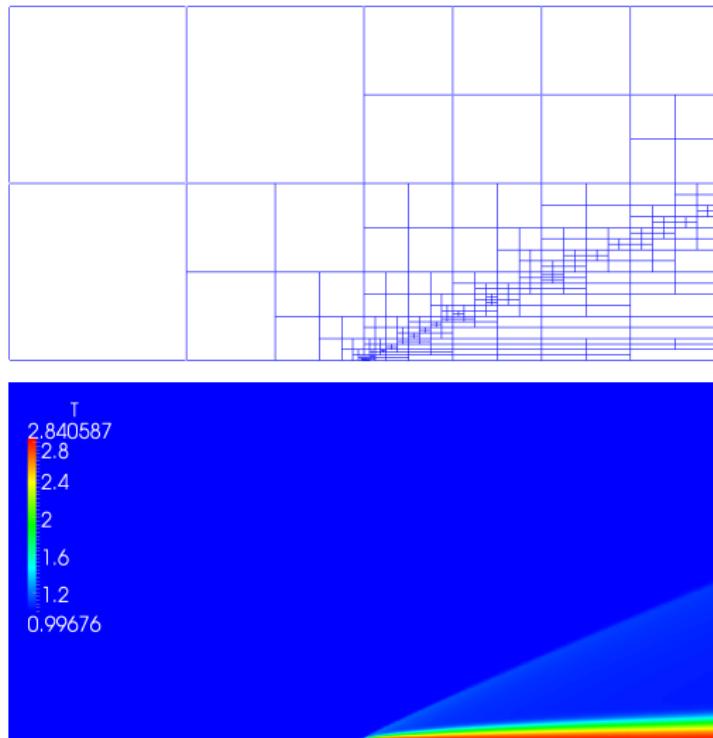


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 12.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

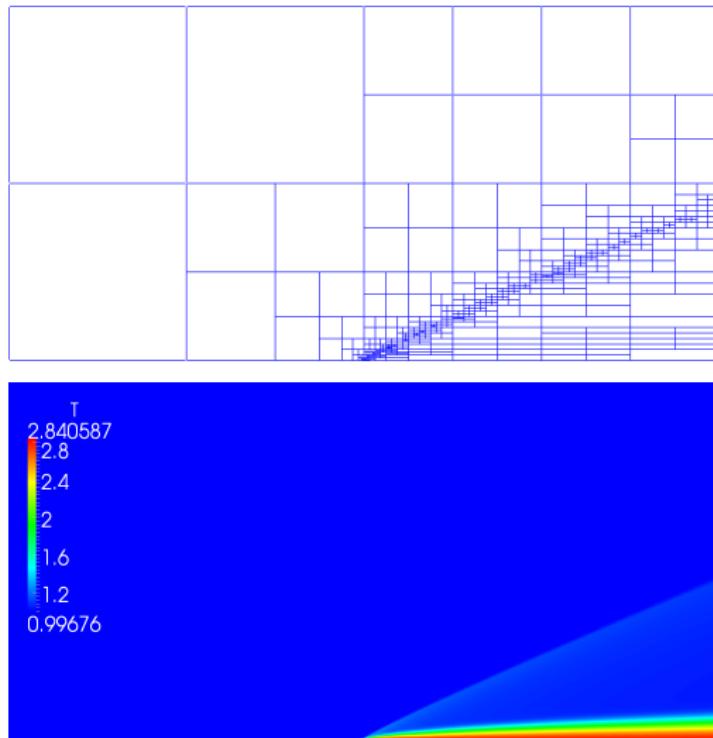


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 13.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

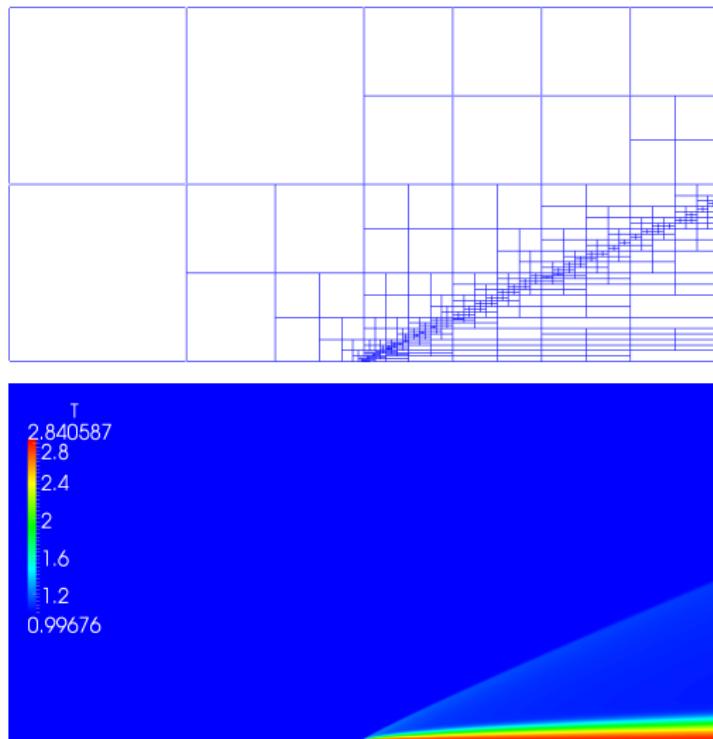


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 14.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

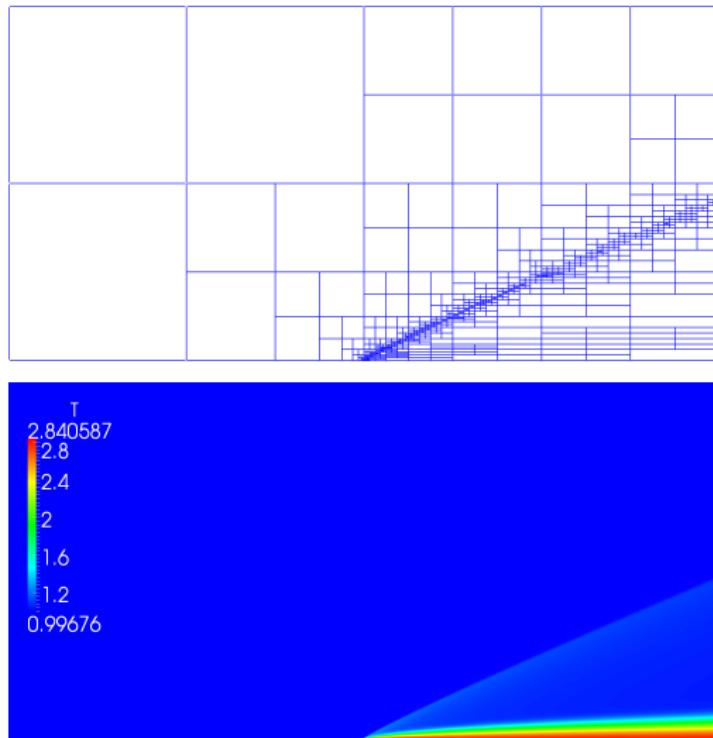


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 15.

Compressible Navier-Stokes

Extrapolation to Compressible NS Equations

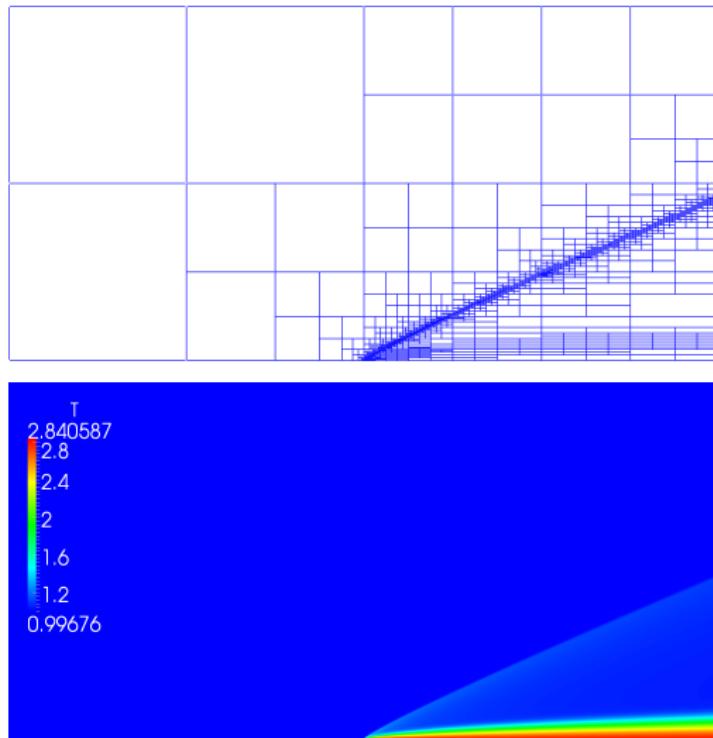


Figure : $\text{Re} = 10,000$, $p = 2$, refinement number 16.

From Ph.D. Dissertation of Jesse Chan : Compressible Navier-Stokes Equations: Holden Ramp Problem^{3 4}

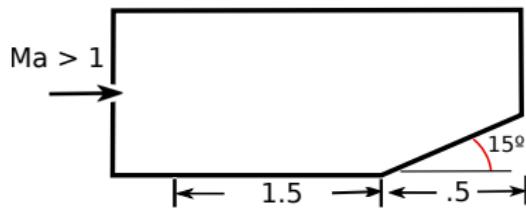


Figure : $M_\infty = 6$, $Re_{plate} = 33,936$, $Pr = 0.72$, $\gamma = 1.4$, $\theta_\infty = 390^\circ$ [R]

³LD., J.T. Oden, W. Rachowicz, "A New Finite Element Method for Solving Compressible Navier-Stokes Equations Based on an Operator Splitting Method and hp Adaptivity.", *Comput. Methods Appl. Mech. Engrg.*, **84**, 275-326, 1990.

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Compressible Navier-Stokes

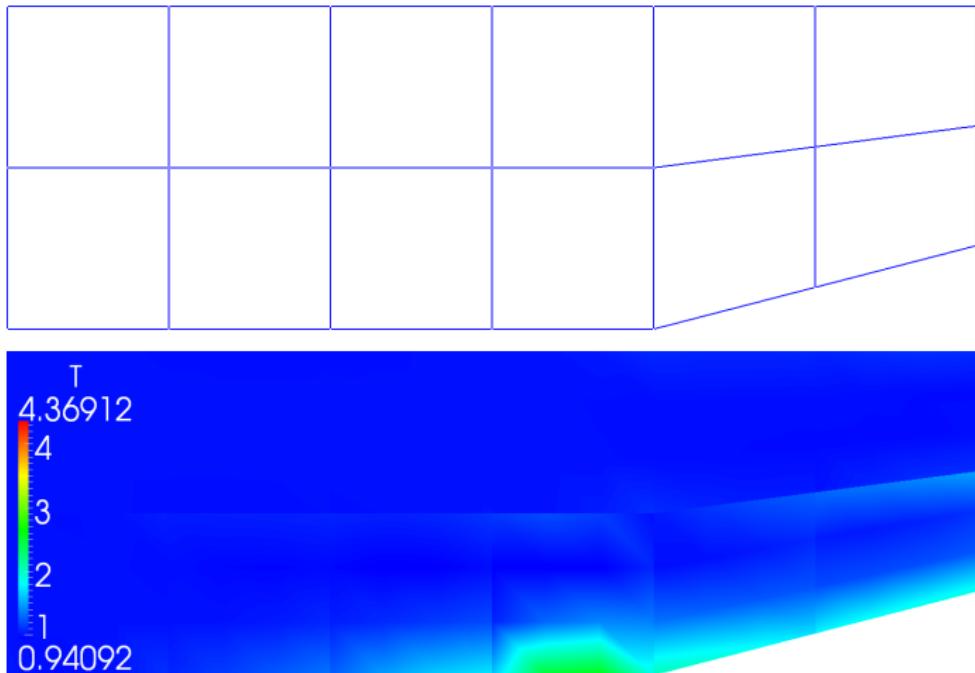


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, initial mesh.

Compressible Navier-Stokes

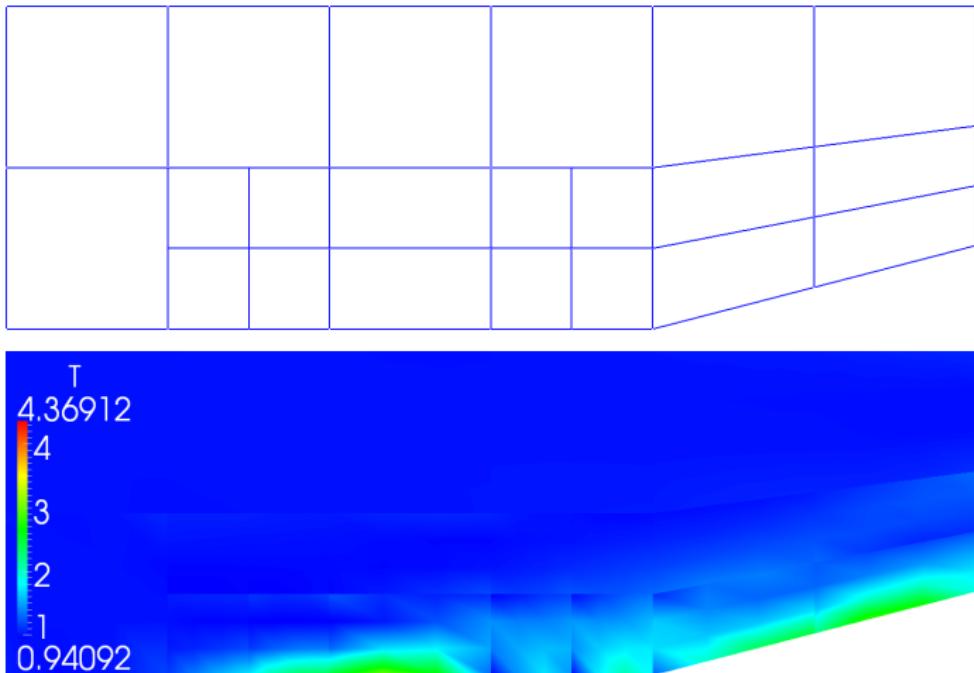


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 1.

Compressible Navier-Stokes

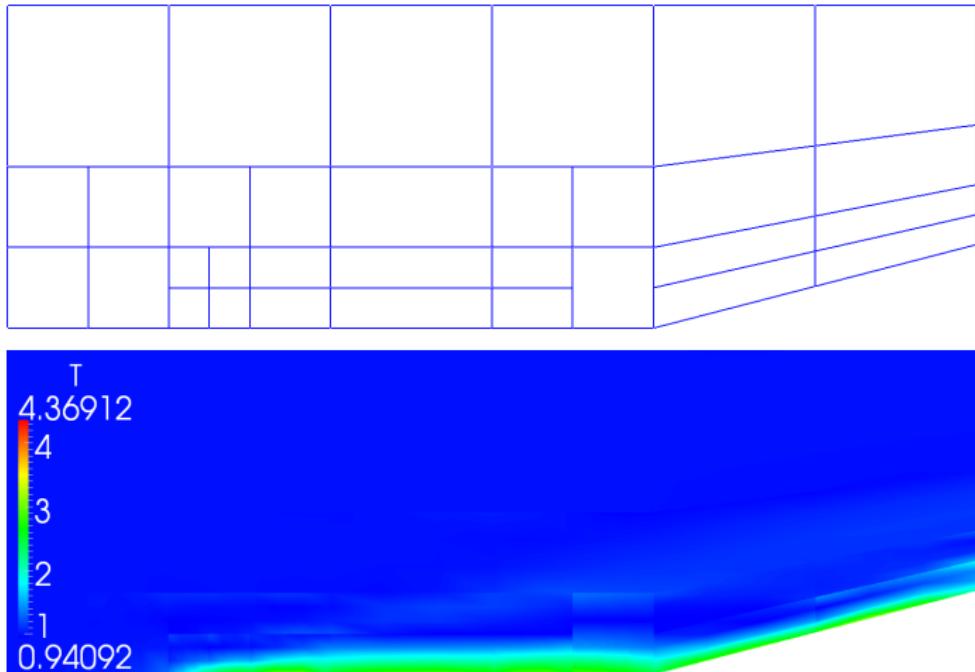


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 2.

Compressible Navier-Stokes

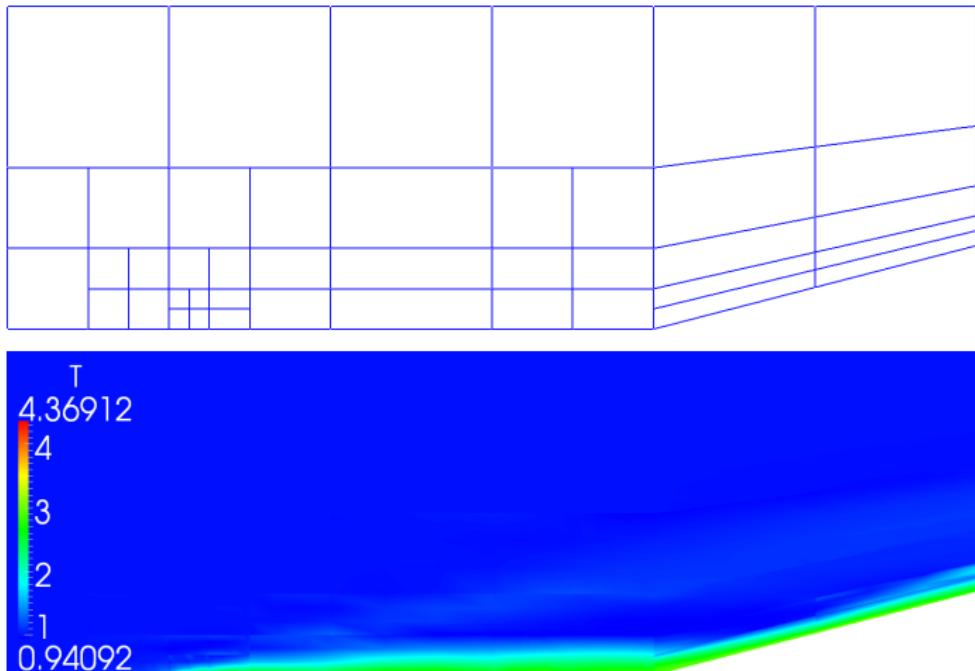


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 3.

Compressible Navier-Stokes

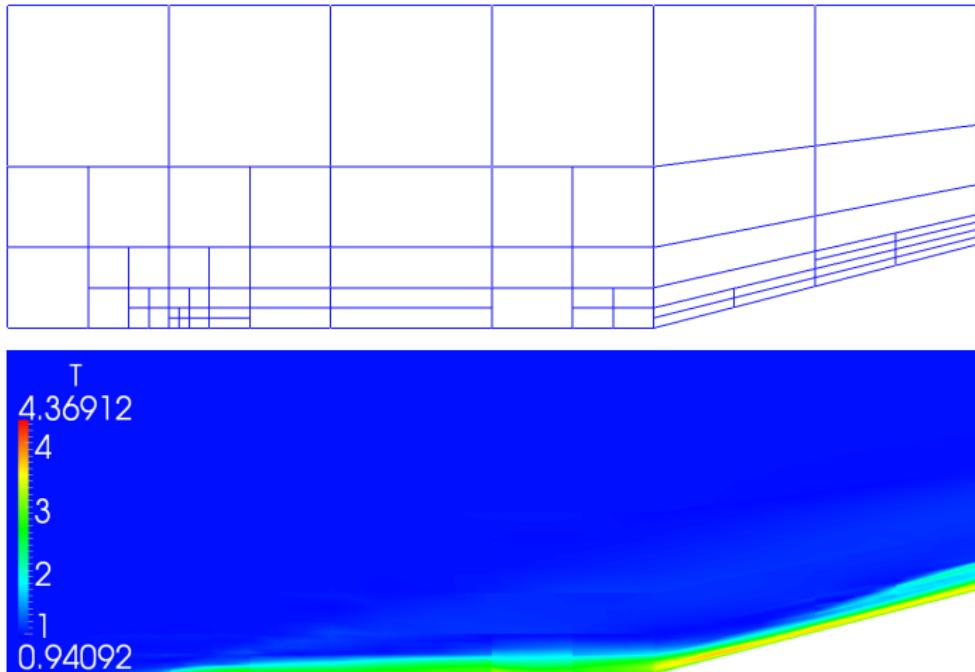


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Compressible Navier-Stokes

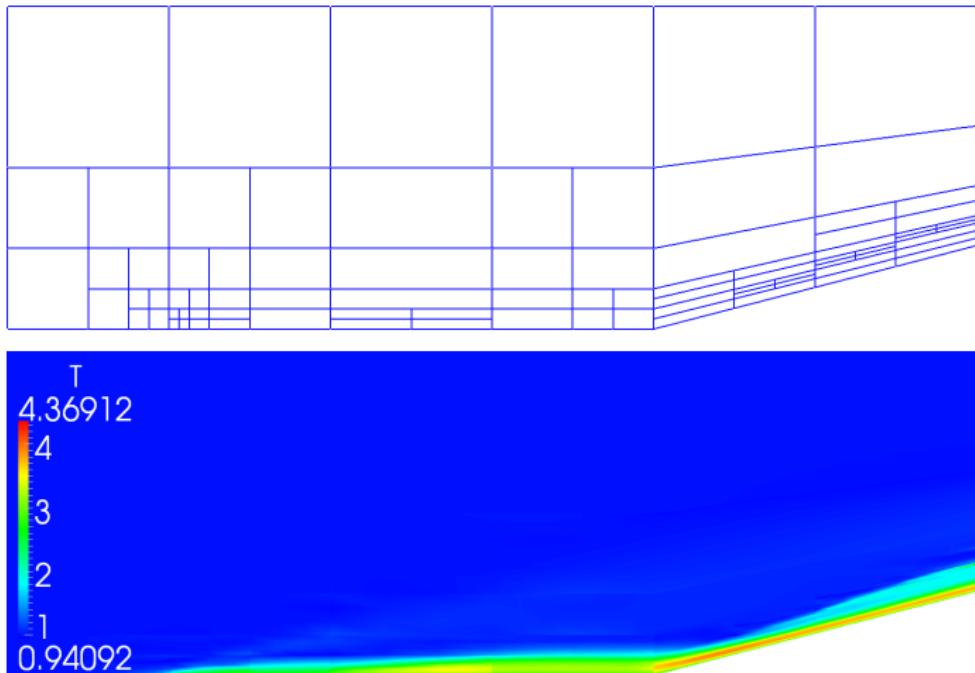


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 5.

Compressible Navier-Stokes

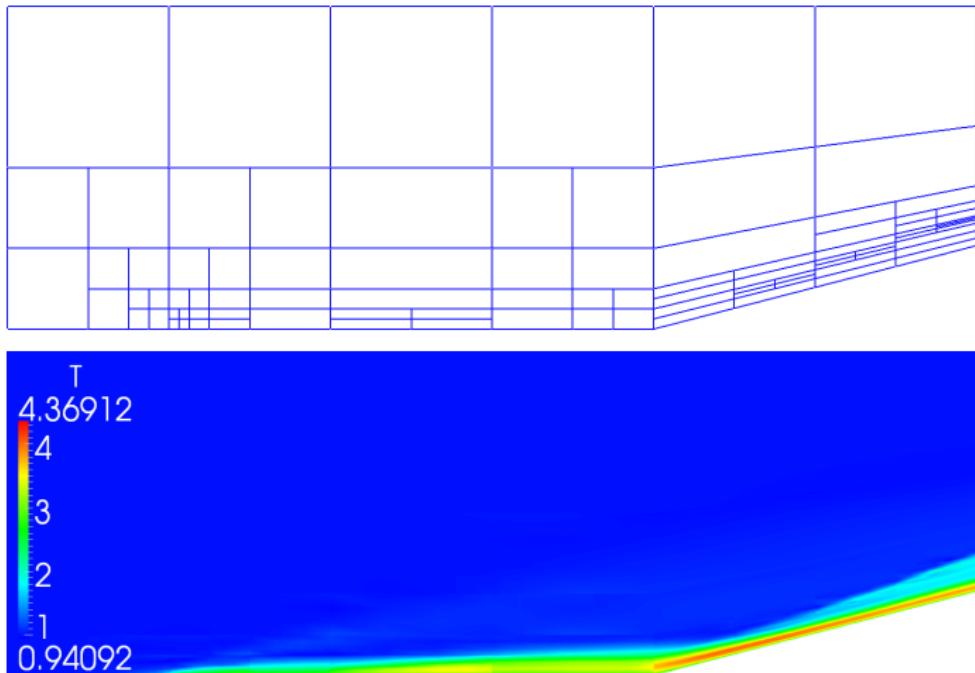


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 6.

Compressible Navier-Stokes

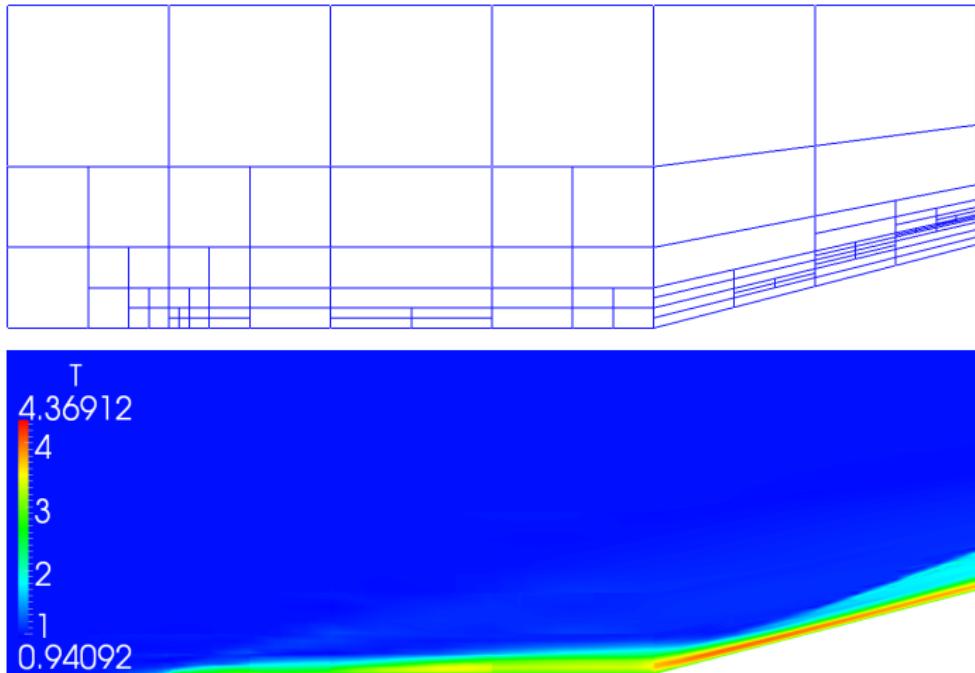


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Compressible Navier-Stokes

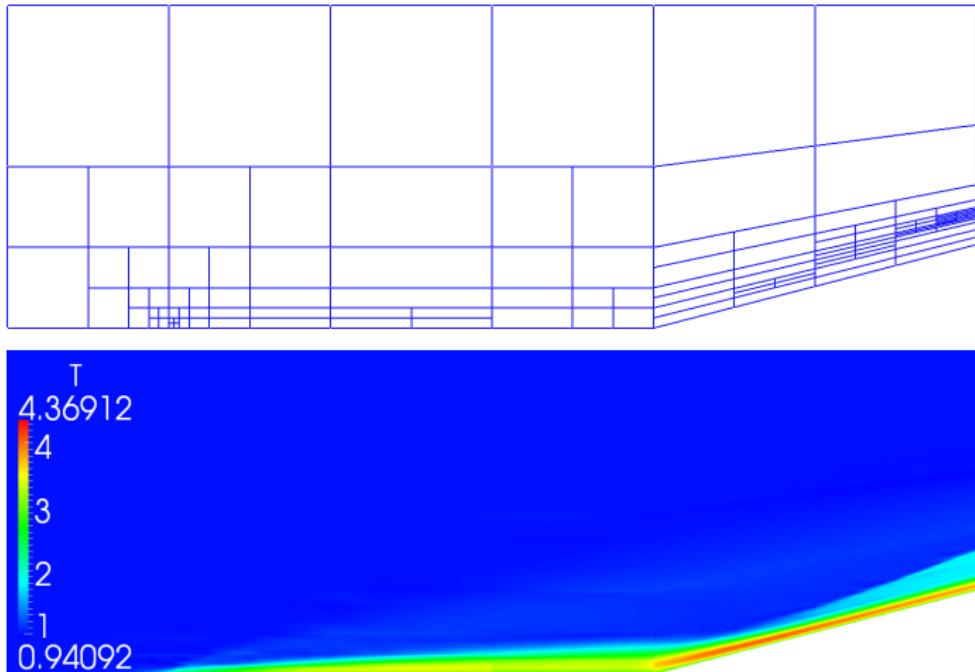


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 8.

Compressible Navier-Stokes

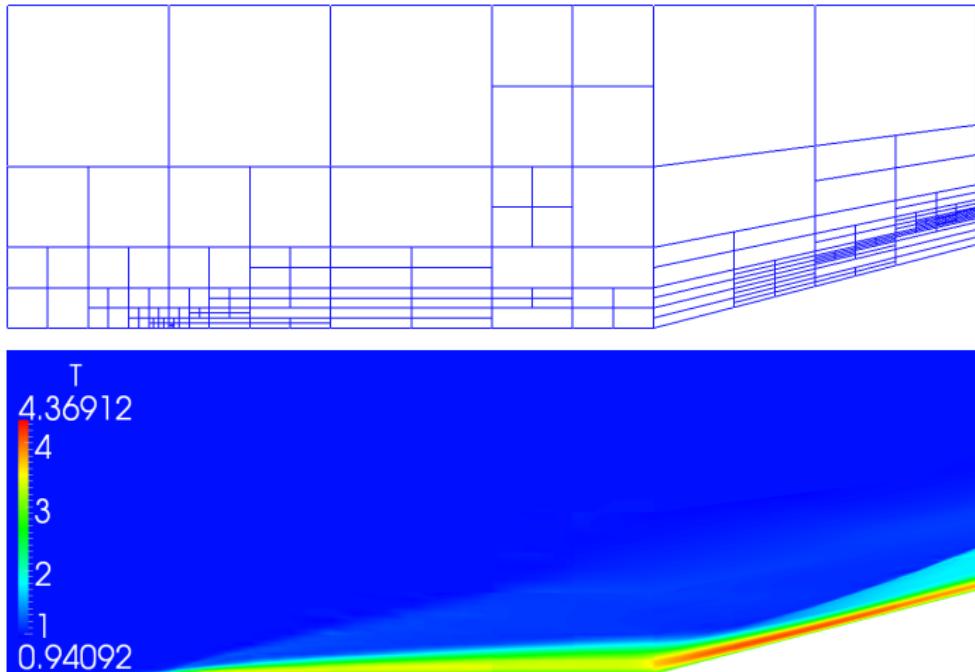


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 9.

Compressible Navier-Stokes

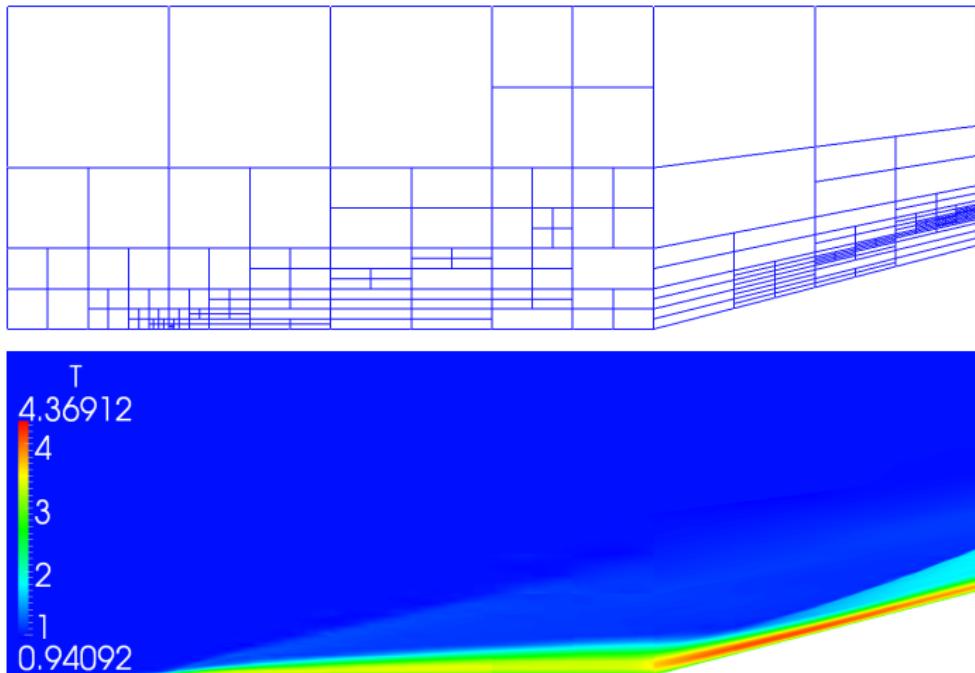


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 10.

Compressible Navier-Stokes

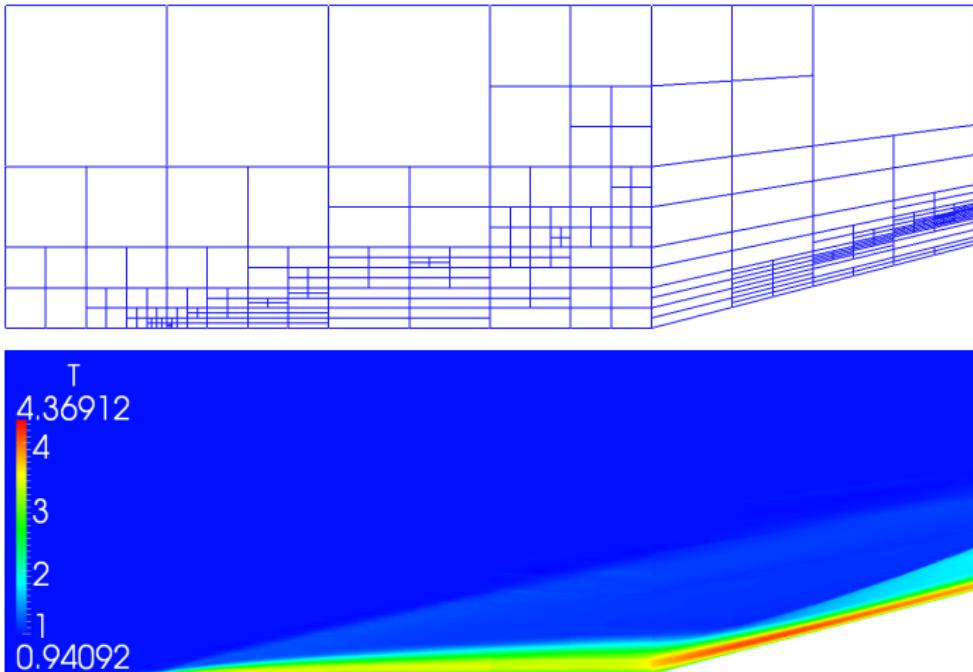


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 11.

Compressible Navier-Stokes

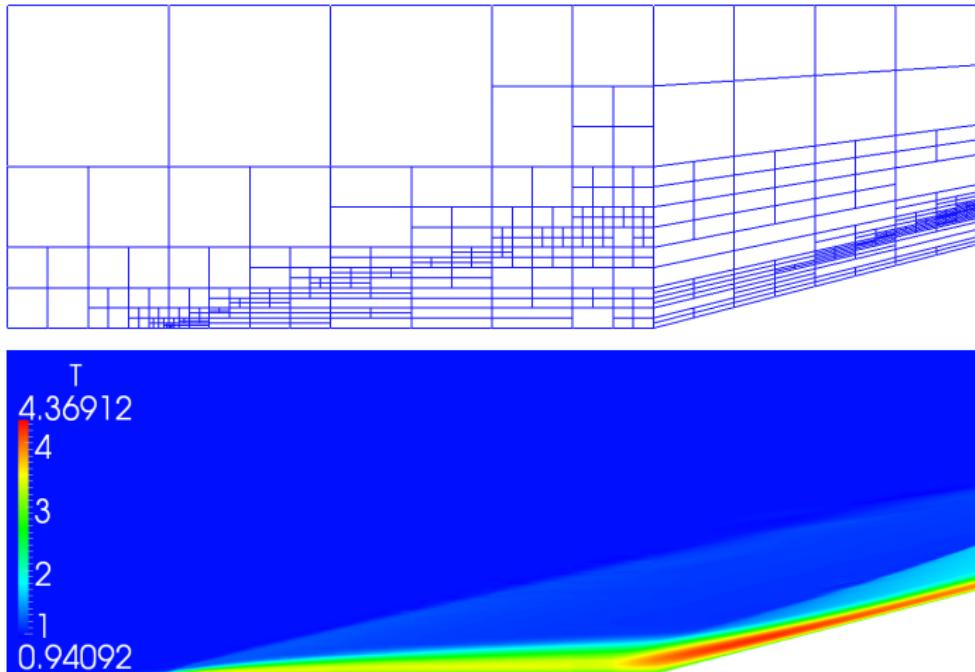


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 12.

Compressible Navier-Stokes

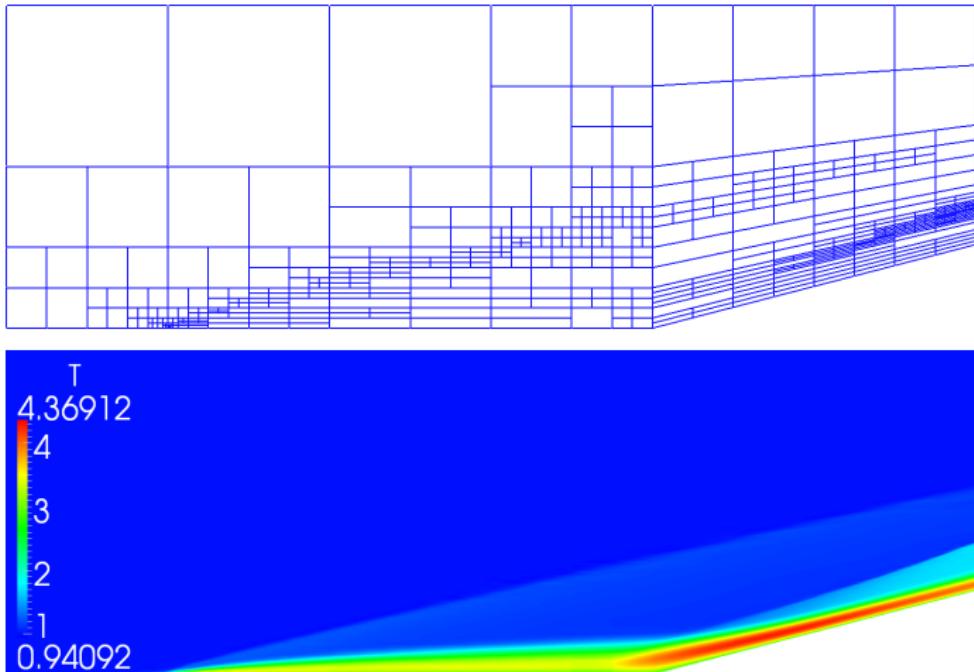


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Compressible Navier-Stokes

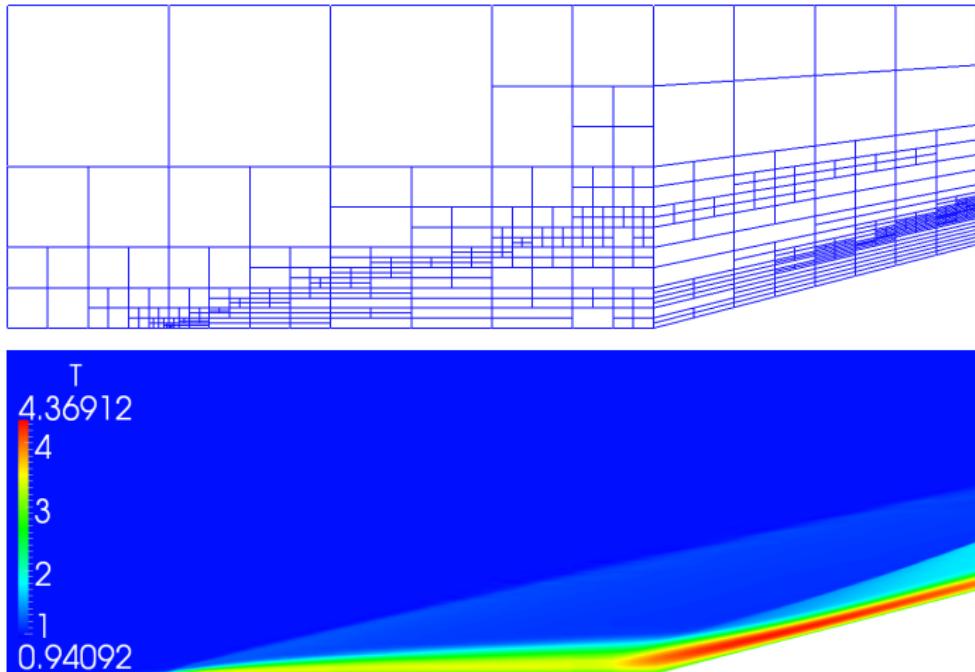


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Compressible Navier-Stokes

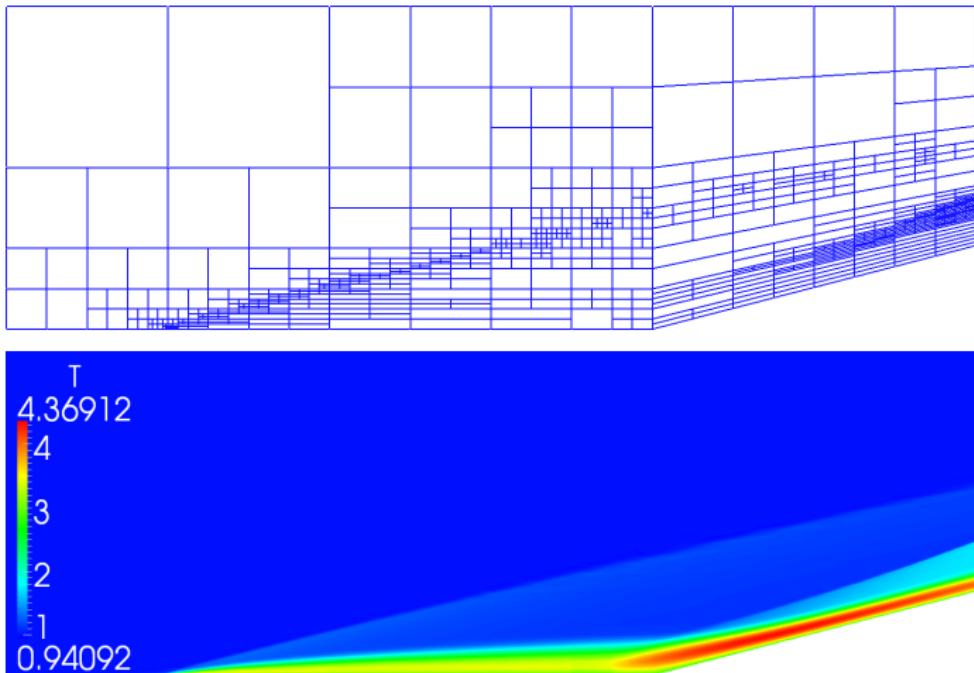


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Compressible Navier-Stokes

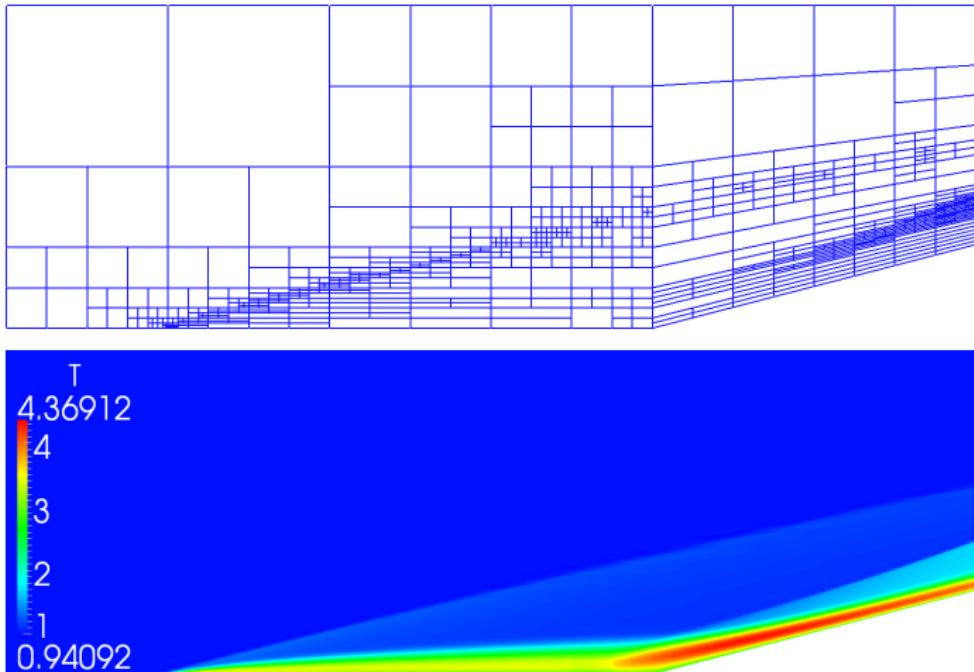


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 16.

Compressible Navier-Stokes

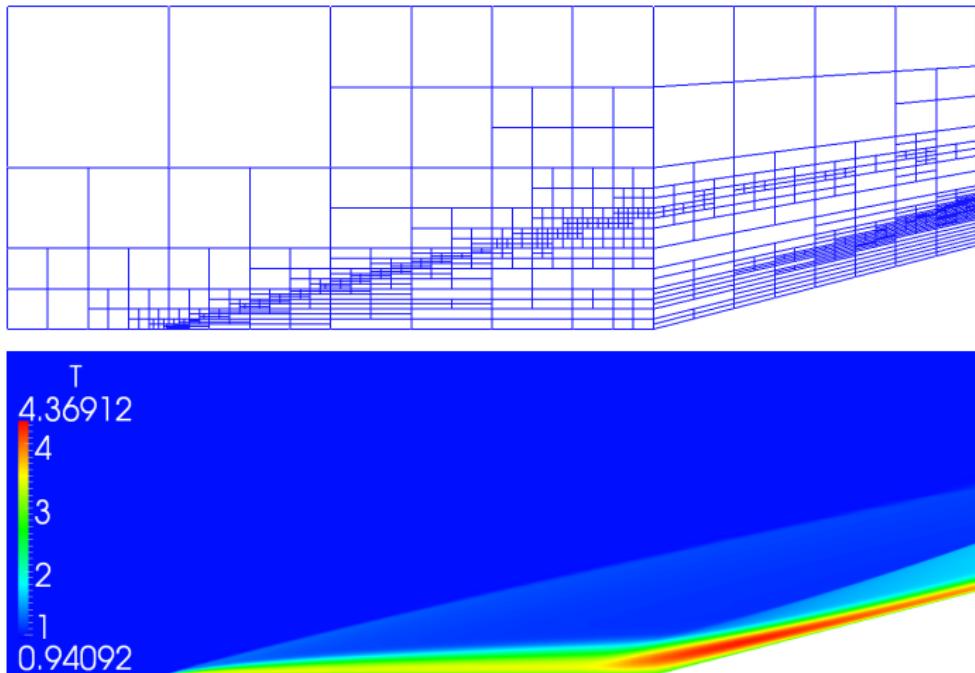


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 17.

Compressible Navier-Stokes

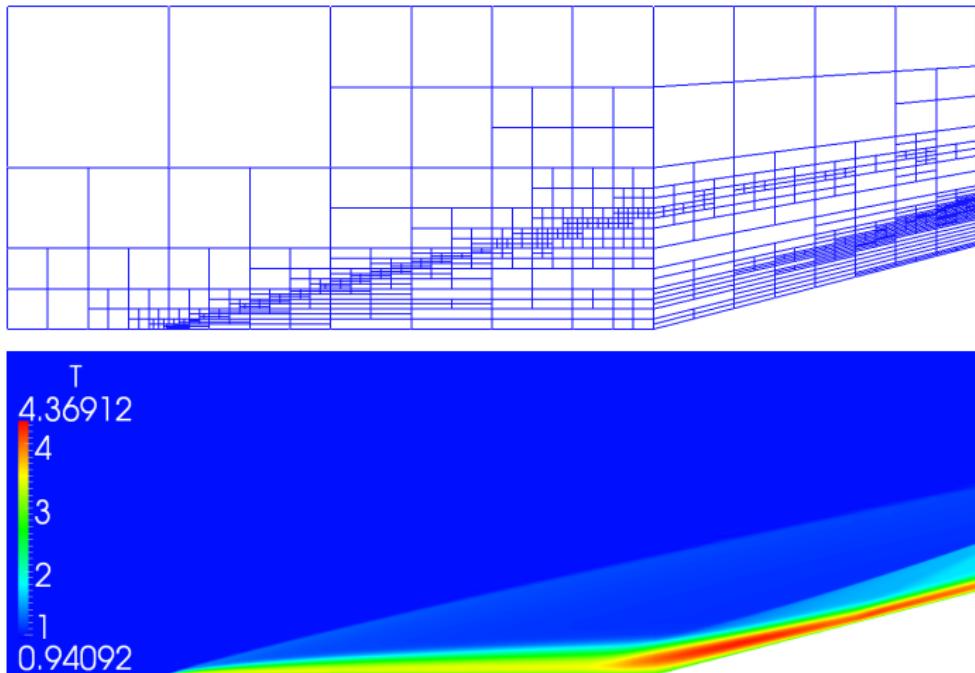


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 18.

Compressible Navier-Stokes

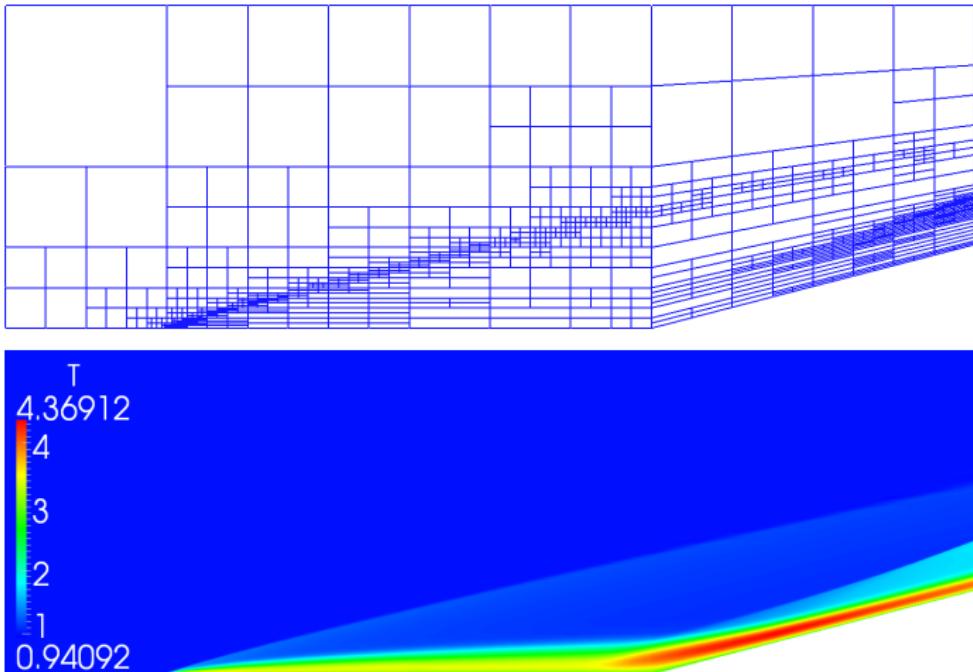


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 19.

Compressible Navier-Stokes

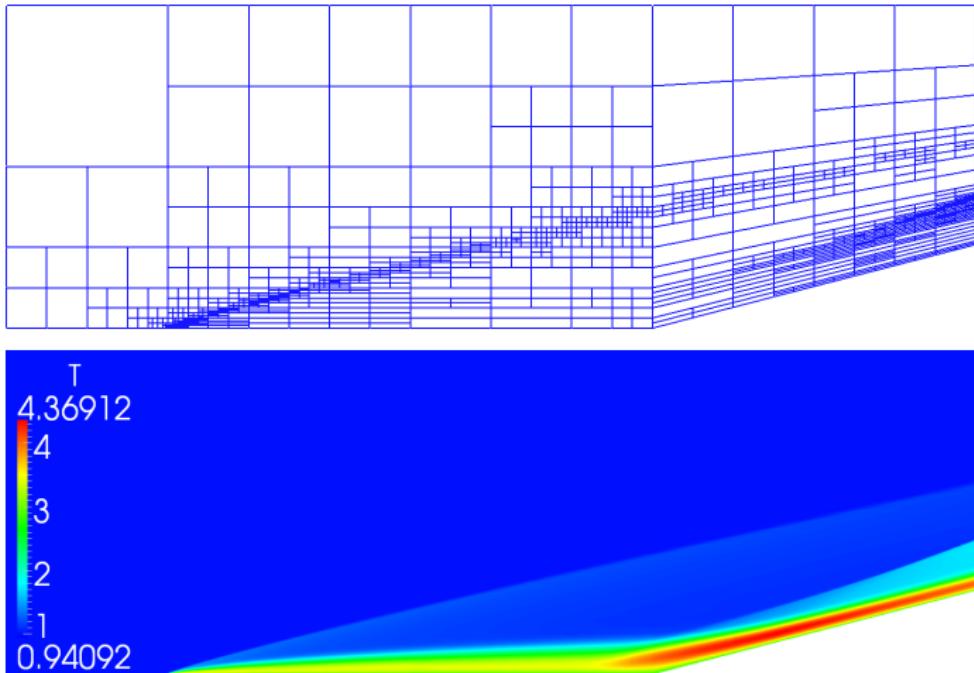


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 20.

Compressible Navier-Stokes

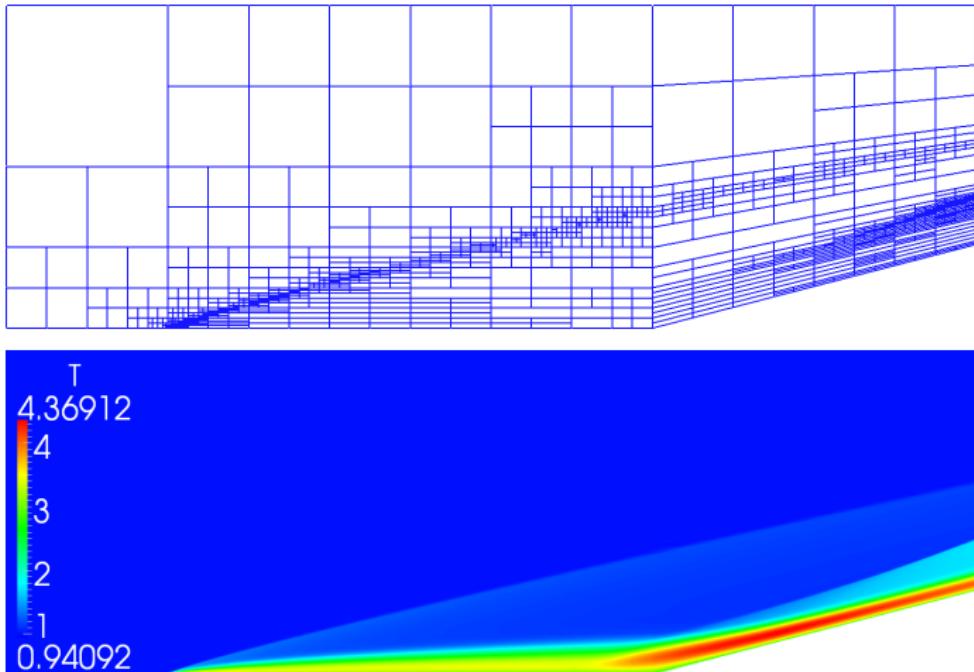


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 21.

Compressible Navier-Stokes

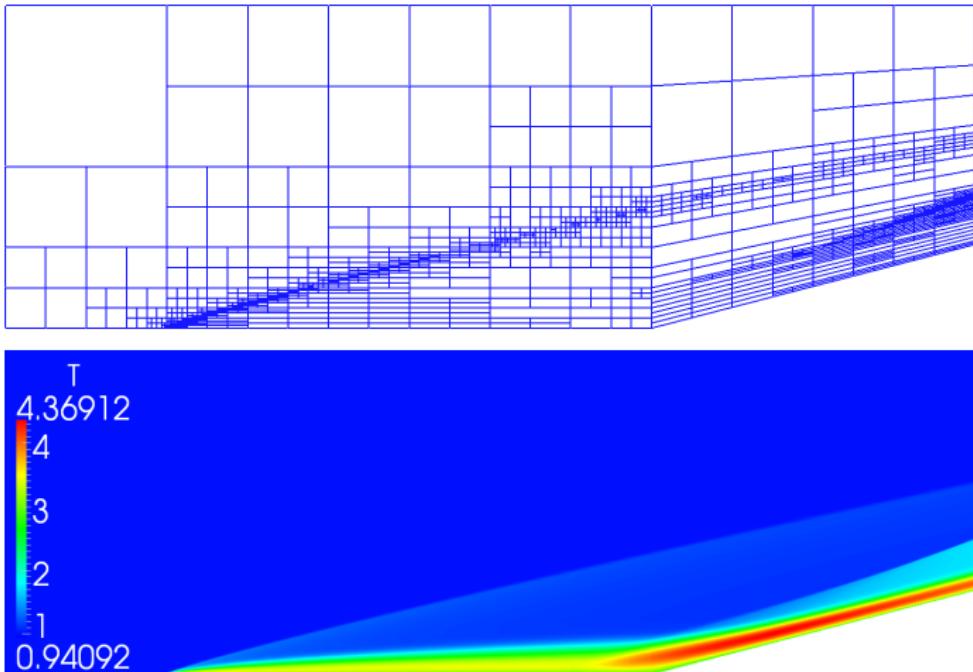


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 22.

Compressible Navier-Stokes

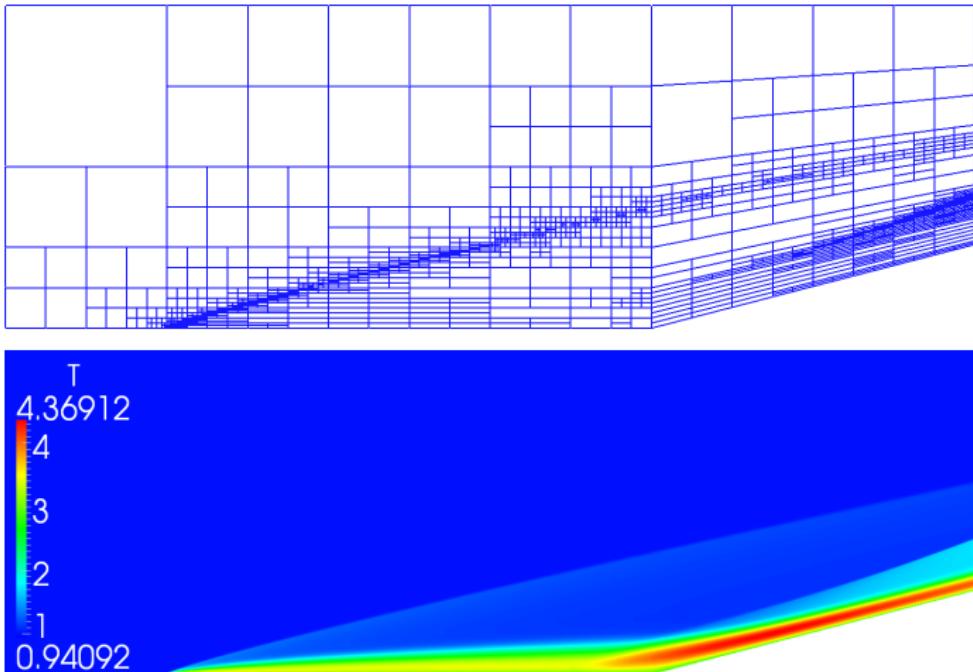


Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 23.

Compressible Navier-Stokes

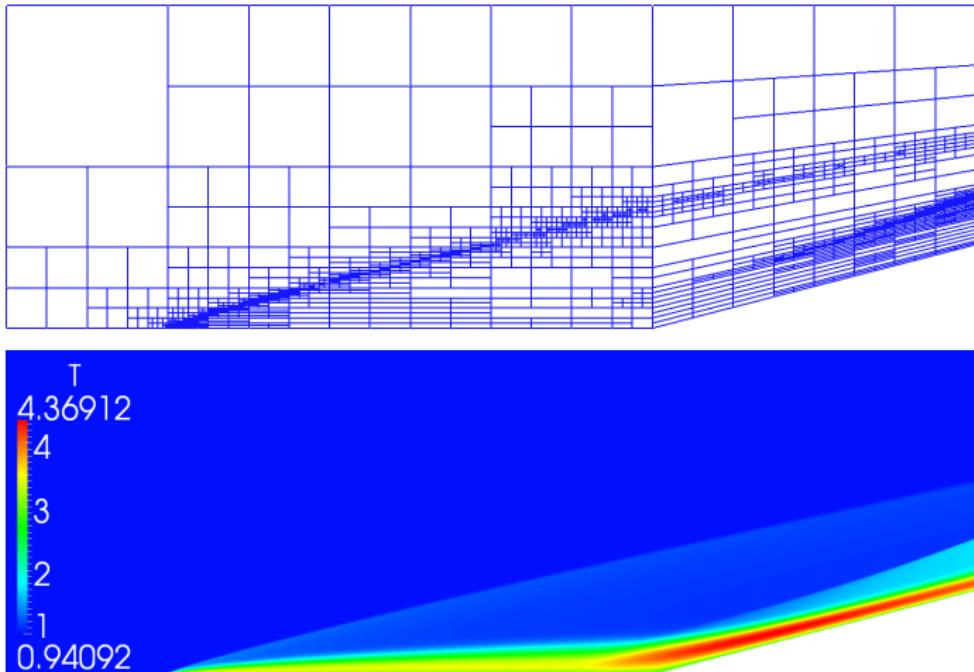


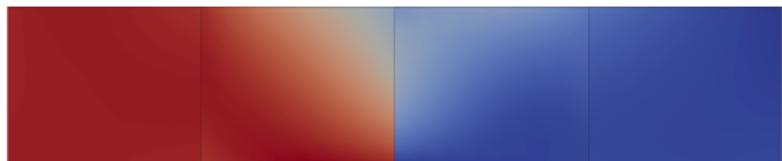
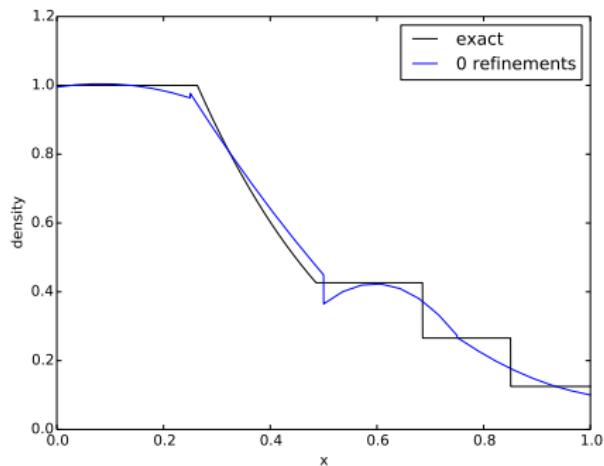
Figure : Mach 6, $\text{Re} \approx 34,000$, $p = 2$, refinement number 24.

**From Ph.D. Proposal of Truman Ellis:
Space-Time Compressible Navier-Stokes Equations:
Sod Shock Tube Problem ⁵**

⁵G.A. Sod, "A Survey of Several Finite Difference Methods for Systems of Nonlinear Hyperbolic Conservation Laws", *J. Comp. Methods Phys.*, **27**, 1-31, 1978.

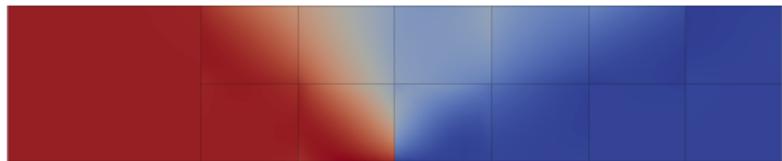
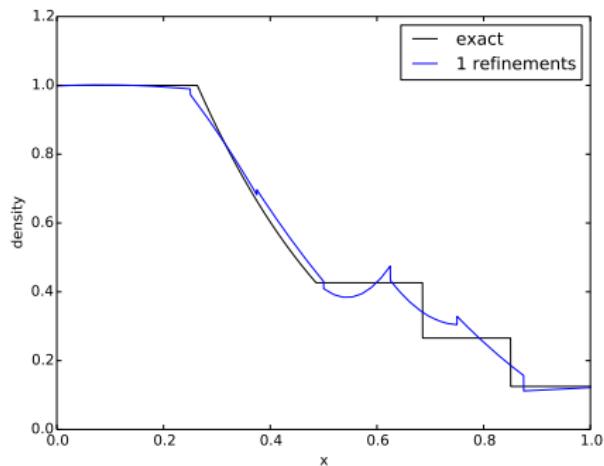
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



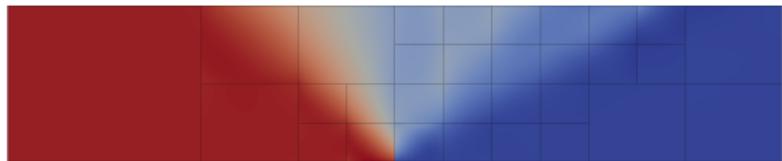
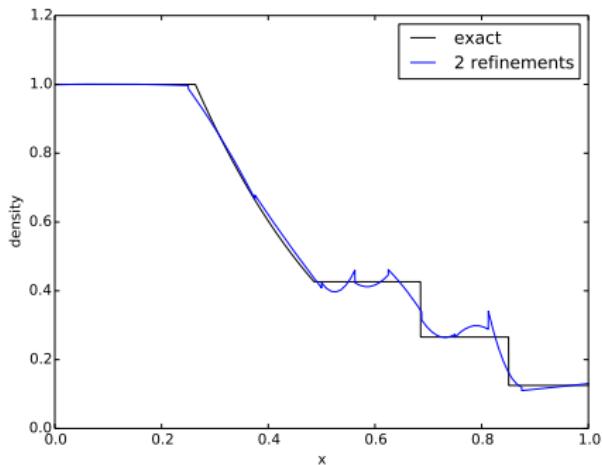
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



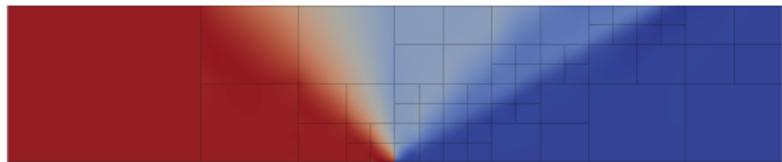
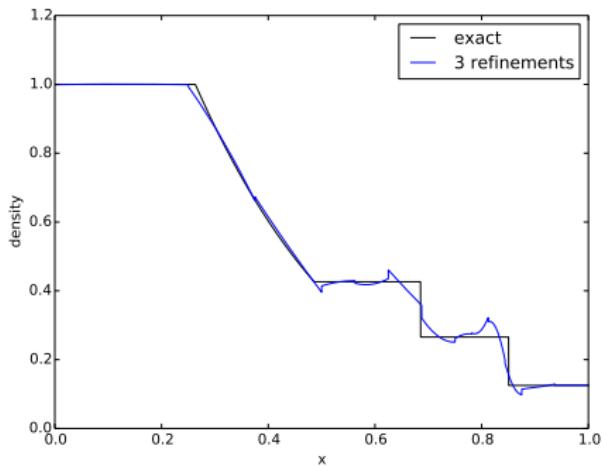
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



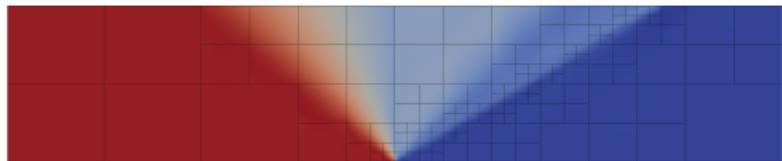
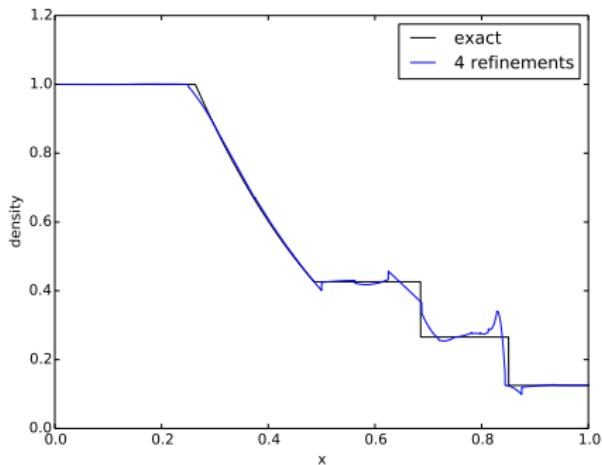
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



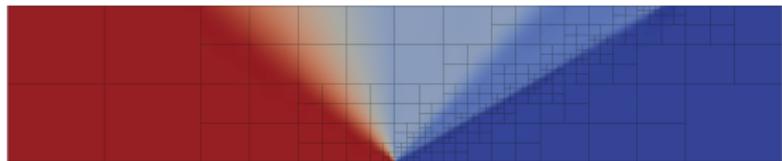
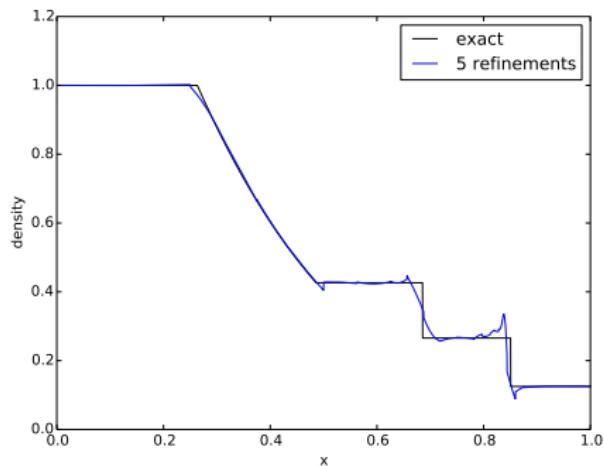
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



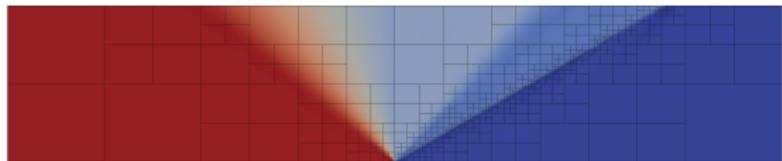
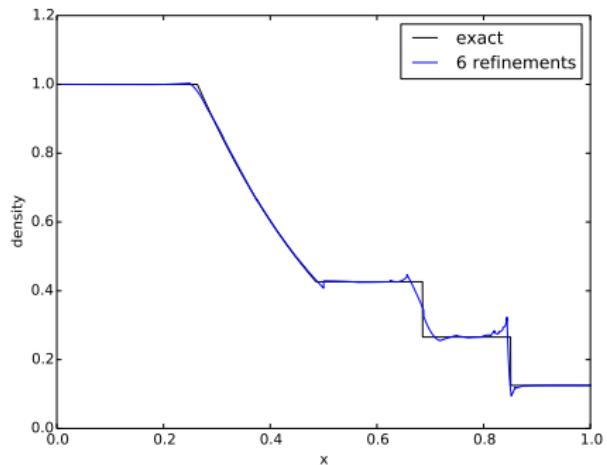
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



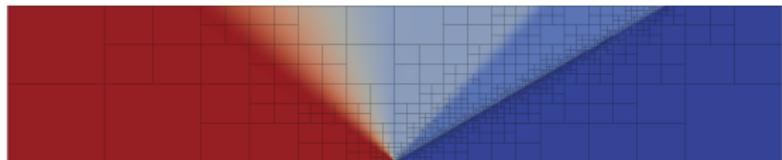
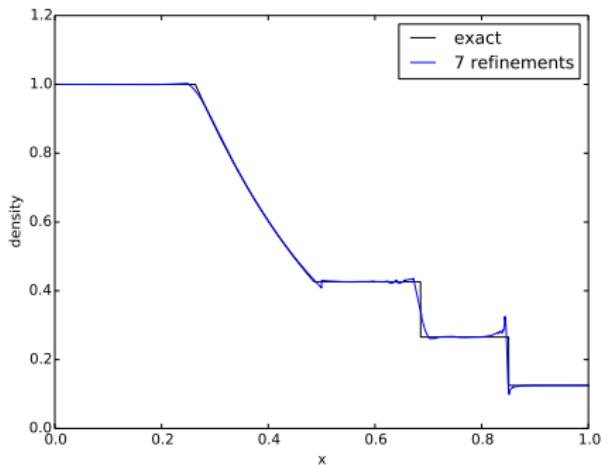
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



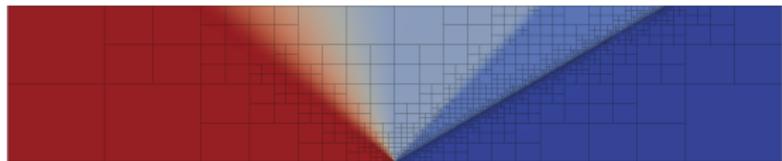
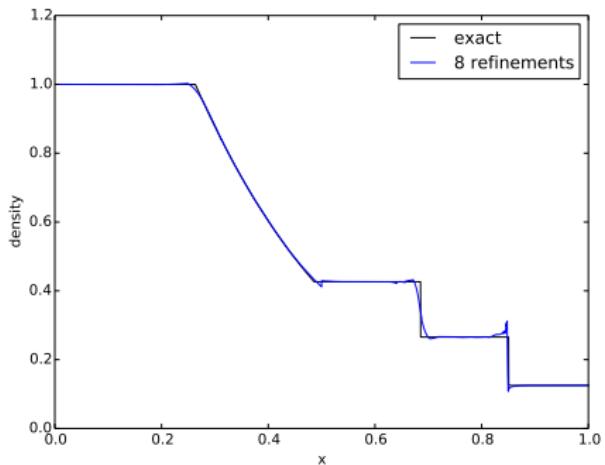
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



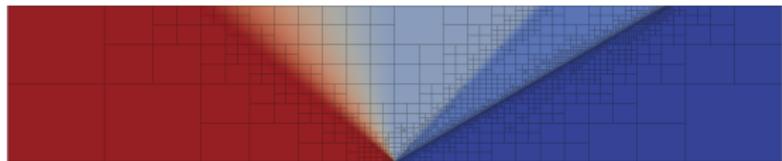
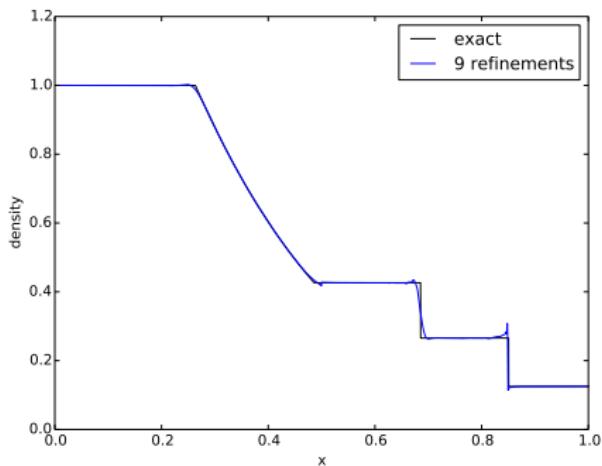
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



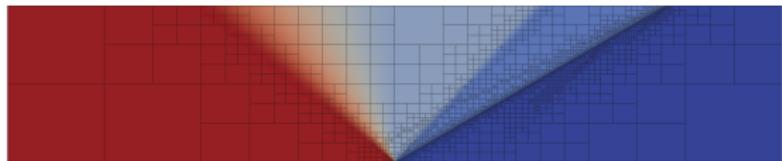
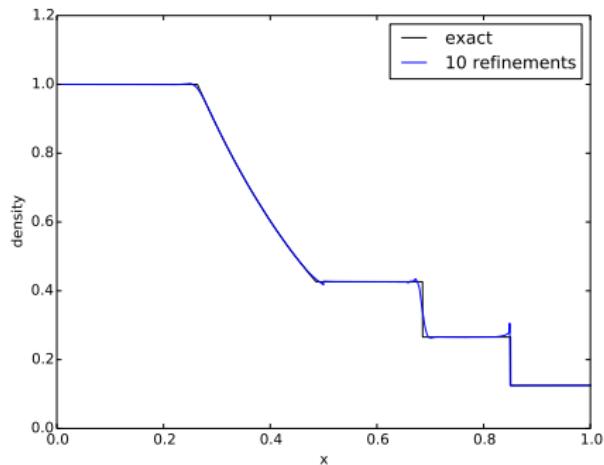
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



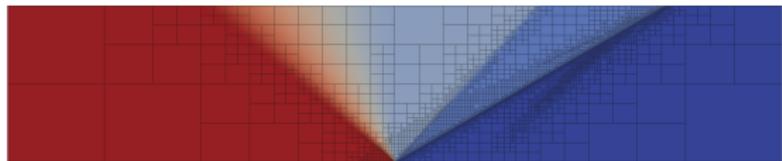
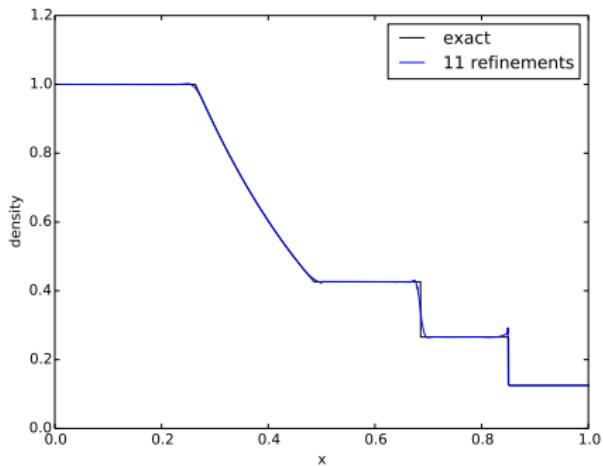
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



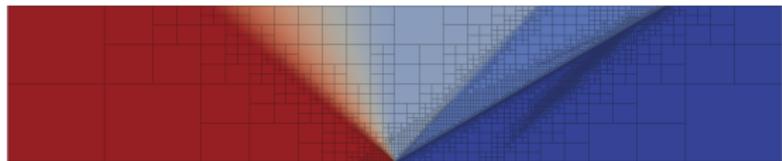
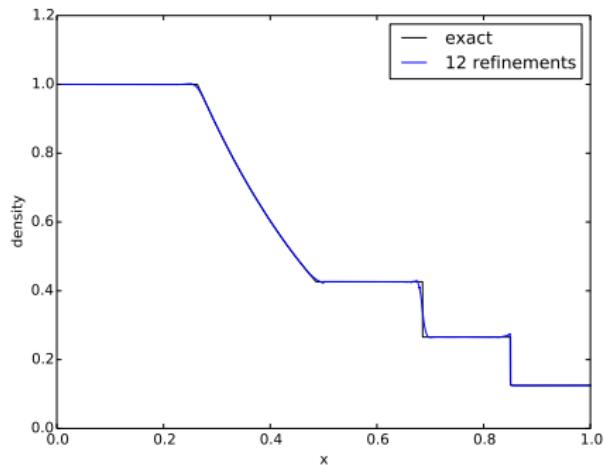
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



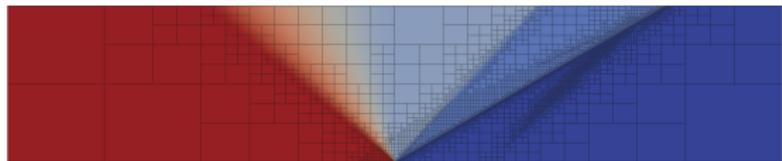
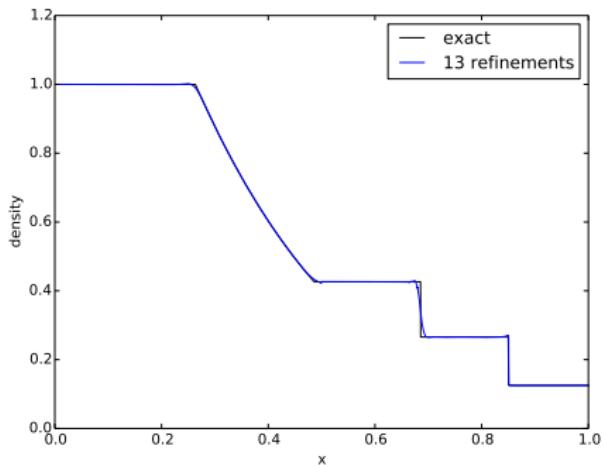
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



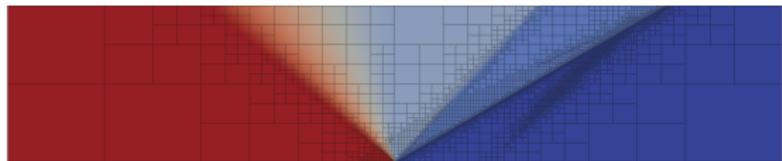
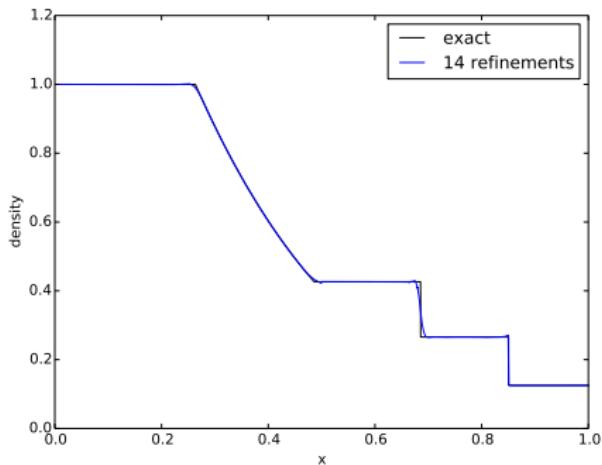
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

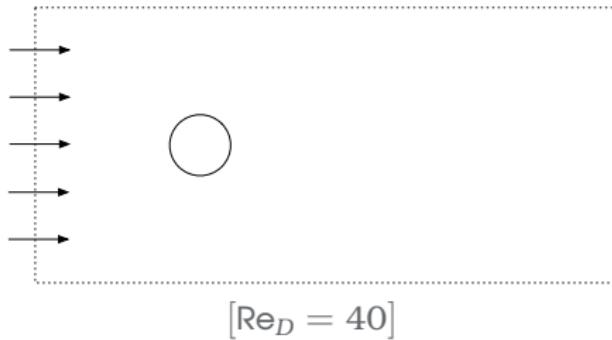


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



From Ph.D. Dissertation⁶ of Nathan V. Roberts: Incompressible Navier-Stokes: Flow Past a Cylinder⁷

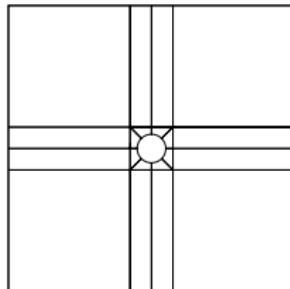
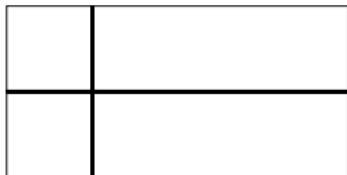


⁶Nathan V. Roberts. *A Discontinuous Petrov-Galerkin Methodology for Incompressible Flow Problems*. PhD. thesis, University of Texas at Austin, 2013.

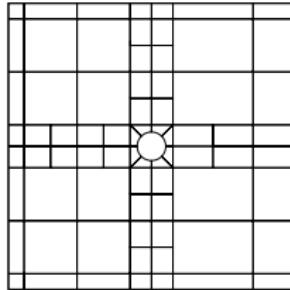
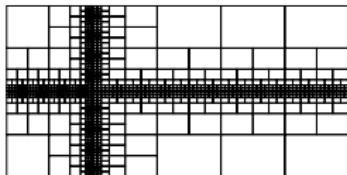
⁷L.S.G. Kovasznay. "Hot-wire investigation of the wake behind cylinders at low Reynolds numbers." *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, **198**(1053):174-190, 1949.

Incompressible Navier-Stokes

Domain: 240×120 cylinder diameters. Begin with a $p = 3$ preliminary mesh that simply captures geometry (detail):

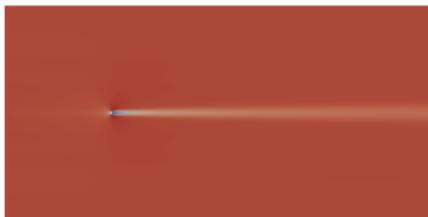
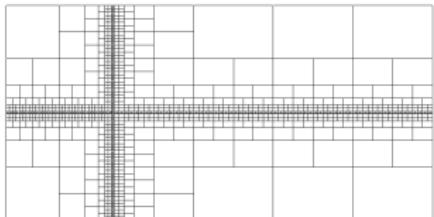


Perform initial anisotropic refinements to get aspect ratios below 2:

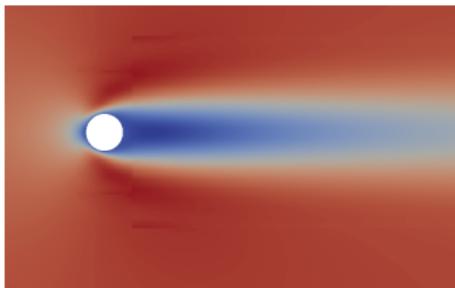
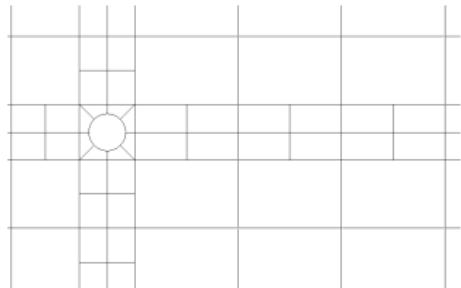


Incompressible Navier-Stokes

Initial Mesh ($p = 3$), and horizontal velocity solution:



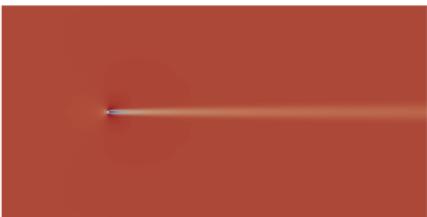
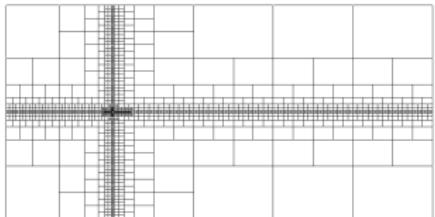
Mesh detail, and horizontal velocity solution



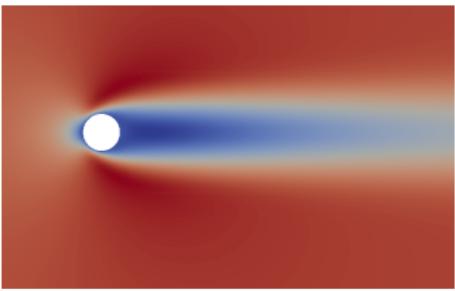
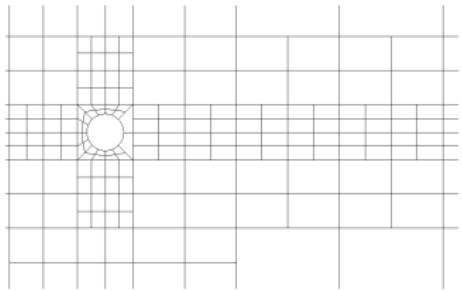
Relative energy error of solution: 9.76×10^{-2} .

Incompressible Navier-Stokes

Mesh after 1 h -refinement, and horizontal velocity solution:



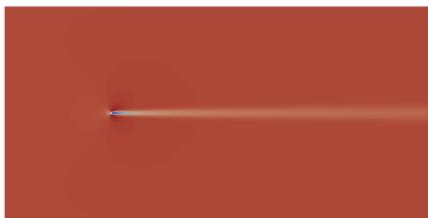
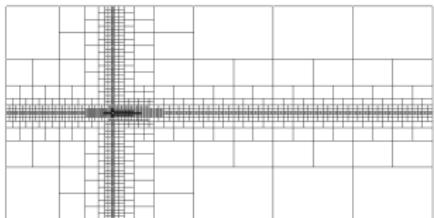
Mesh detail, and horizontal velocity solution



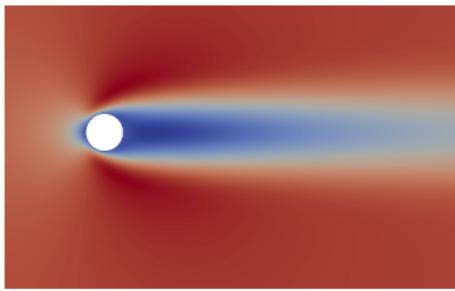
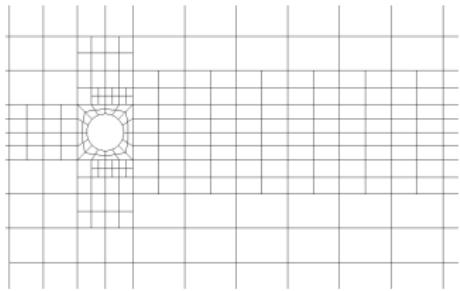
Relative energy error of solution: 1.81×10^{-2} .

Incompressible Navier-Stokes

Mesh after 2 h -refinements, and horizontal velocity solution:



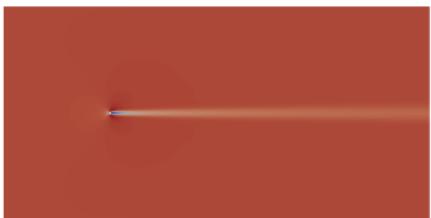
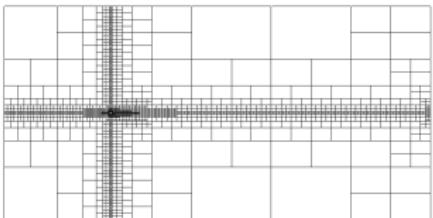
Mesh detail, and horizontal velocity solution



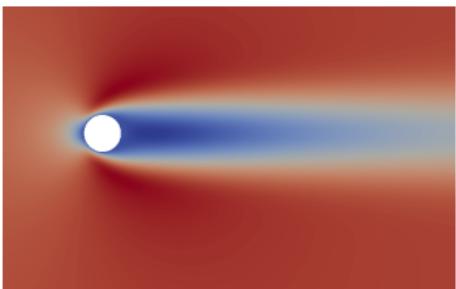
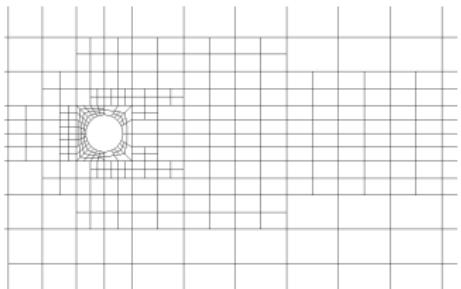
Relative energy error of solution: 3.26×10^{-3} .

Incompressible Navier-Stokes

Mesh after 3 h -refinements, and horizontal velocity solution:



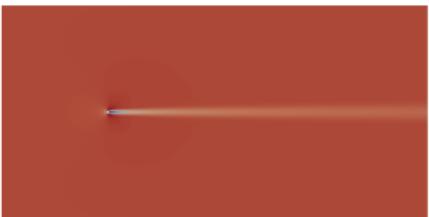
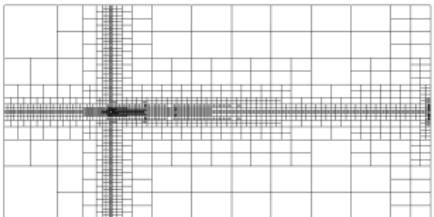
Mesh detail, and horizontal velocity solution



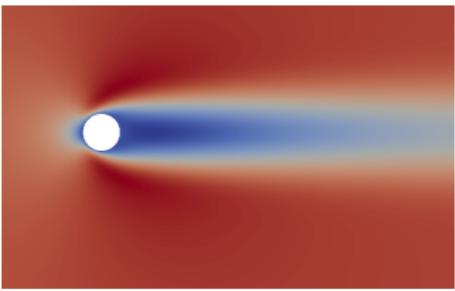
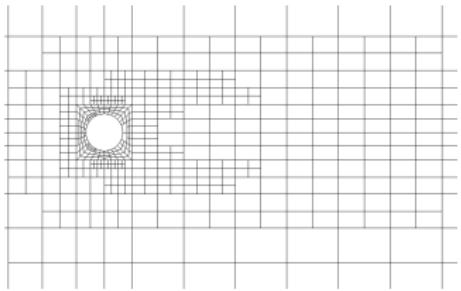
Relative energy error of solution: 1.29×10^{-3} .

Incompressible Navier-Stokes

Mesh after 4 h -refinements, and horizontal velocity solution:



Mesh detail, and horizontal velocity solution



Relative energy error of solution: 5.74×10^{-4} .

Chan, Ellis, and Roberts results were computed using Camellia.⁸

Camellia:

- is easy to use (Stokes cavity flow h -adaptive driver takes ≈ 130 lines of code)
- supports MPI, h - and p - adaptivity
- built atop Trilinos
- supports 1-3D spatial meshes (1D and 3D just added)
- space-time meshes coming soon
- open source release planned for 2015

⁸Nathan V. Roberts. "Camellia: A software framework for discontinuous Petrov-Galerkin methods." *Computers & Mathematics with Applications*, 2014 (in review).

Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

The simplest singular perturbation problem: reaction-dominated diffusion⁹

⁹L.D. and I. Harari, "Primal DPG Method for Reaction dominated Diffusion", in preparation.

The simplest singular perturbation problem:

Reaction-dominated diffusion

$$\begin{cases} u = 0 & \text{on } \Gamma \\ -\epsilon^2 \Delta u + c(x)u = f & \text{in } \Omega \end{cases}$$

where $0 < c_0 \leq c(x) \leq c_1$.

Standard variational formulation:

$$\begin{cases} u \in H^1(\Omega) \\ \epsilon^2(\nabla u, \nabla v) + (cu, v) = (f, v) \quad v \in H^1(\Omega) \end{cases}$$

Standard Galerkin method delivers the best approximation error in the energy norm:

$$\|u\|_{\epsilon^k}^2 := \epsilon^k \|\nabla u\|^2 + \|c^{1/2}u\|^2, \quad k = 2$$

Convergence in “balanced” norm

Fact: Under favorable regularity conditions, the solution is *uniformly* bounded in data f in a “balanced” norm¹⁰:

$$\|u\|_{\epsilon}^2 := \epsilon \|\nabla u\|^2 + \|c^{1/2} u\|^2$$

i.e.

$$\|u\|_{\epsilon} \lesssim \|f\|_{\text{appropriate}}$$

Question: Can we select the test norm in such a way that the DPG method will be *robust* in the balanced norm?

$$\|u - u_h\|_{\epsilon} + \|\hat{t} - \hat{t}_h\|_? \lesssim \inf_{w_h} \|u - w_h\|_{\epsilon} + \inf_{\hat{r}_h} \|\hat{t} - \hat{r}_h\|_?$$

¹⁰R. Lin and M. Stynes, “A balanced finite element method for singularly perturbed reaction-diffusion problems”, SIAM J. Numer. Anal., **50**(5): 2729–2743, 2012.

A bit of history:

Optimal test functions of Barret and Morton^{11 12}

For each $w \in U_h$, determine the corresponding v_w that solves the auxiliary variational problem:

$$\left\{ \begin{array}{l} v_w \in H_0^1(\Omega) \\ \underbrace{\epsilon^2(\nabla \delta u, \nabla v_w) + (c \delta u, v_w)}_{\text{the bilinear form we have}} = \underbrace{\epsilon(\nabla \delta u, w) + (c \delta u, w)}_{\text{the bilinear form we want}} \quad \forall \delta u \in H_0^1(\Omega) \end{array} \right.$$

With the optimal test functions, the Galerkin orthogonality for the original form changes into Galerkin orthogonality in the desired, "balanced" norm:

$$\epsilon^2(\nabla(u - u_h), \nabla v_w) + (c(u - u_h), v_w) = 0 \implies \epsilon(\nabla(u - u_h), \nabla v_u) + (c(u - u_h), w) = 0$$

Consequently, the PG solution delivers the best approximation error in the desired norm.

¹¹J.W. Barret and K. W. Morton, "Approximate Symmetrization and Petrov-Galerkin Methods for Diffusion-Convection Problems", *Comp. Meth. Appl. Mech and Engng.*, **46**, 97 (1984).

¹²L. D. and J. T. Oden, "An Adaptive Characteristic Petrov-Galerkin Finite Element Method for Convection-Dominated Linear and Nonlinear Parabolic Problems in One Space Variable", *Journal of Computational Physics*, **68(1)**: 188–273, 1986.

Theorem

Let v_u be the Barret-Morton optimal test function corresponding to u . Let $\|v_u\|_V$ be a test norm such that

$$\|v_u\|_V \lesssim \|u\|_\epsilon$$

Then

$$\|u - u_h\|_\epsilon \lesssim \|u - u_h\|_E = \inf_{w_h \in U_h} \|u - w_h\|_E \leq \text{BAE estimate}$$

Proof:

$$\begin{aligned}\|u\|_\epsilon^2 &= \epsilon(\nabla u, \nabla u) + (cu, u) = \epsilon^2(\nabla u, \nabla v_u) + (cu, v_u) \\ &= b((u, \hat{t}), v_u) \leq \frac{b((u, \hat{t}), v_u)}{\|v_u\|_V} \|v_u\|_V \\ &\leq \sup_v \frac{b((u, \hat{t}), v_u)}{\|v\|_V} \|v_u\|_V = \|(u, \hat{t})\|_E \|v_u\|_V \\ &\lesssim \|(u, \hat{t})\|_E \|u\|_\epsilon\end{aligned}$$

¹³L. D., M. Heuer, "Robust DPG Method for Convection-Dominated Diffusion Problems," *SIAM J. Num. Anal.*, **51**: 2514–2537, 2013.

Constructing optimal test norm

The point: Construction of the optimal test norm is reduced to the stability (robustness) analysis for the Barret-Morton test functions.

Lemma

Let

$$\|v\|_V^2 := \epsilon^3 \|\nabla v\|^3 + \|c^{1/2} v\|^2$$

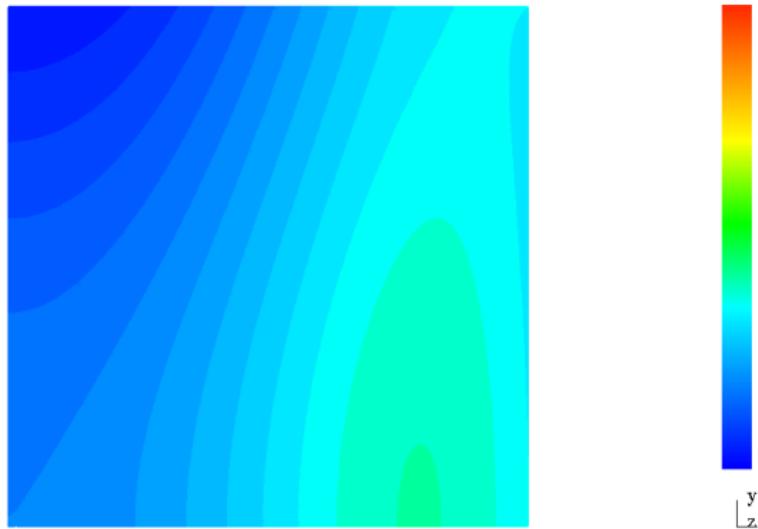
Then

$$\|v_u\| \lesssim \|u\|_\epsilon$$

In order to avoid boundary layers in the optimal test functions (that we cannot resolve using simple enriched space) we scale the reaction term with a mesh-dependent factor:

$$\|v\|_{V,mod}^2 := \epsilon^3 \|\nabla v\|^3 + \min\left\{1, \frac{\epsilon^3}{h^2}\right\} \|c^{1/2} v\|^2$$

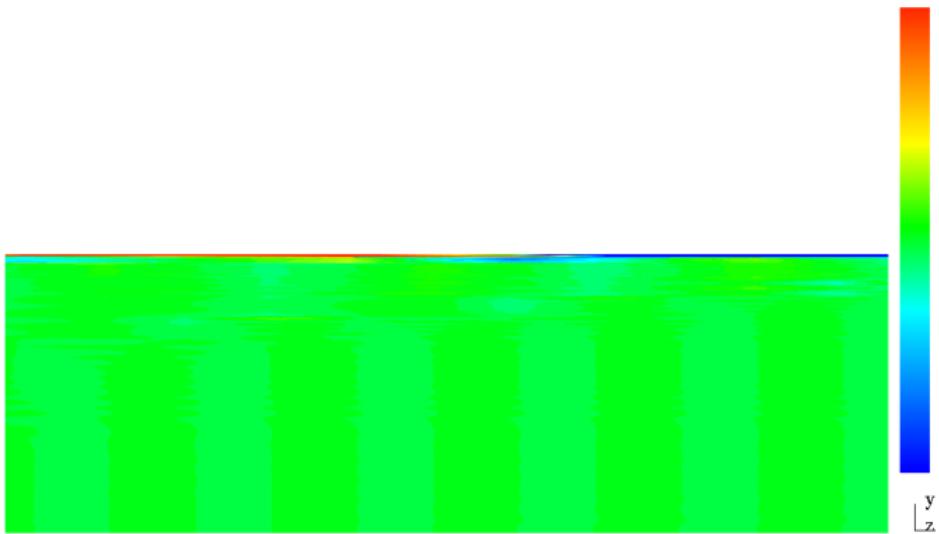
Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$



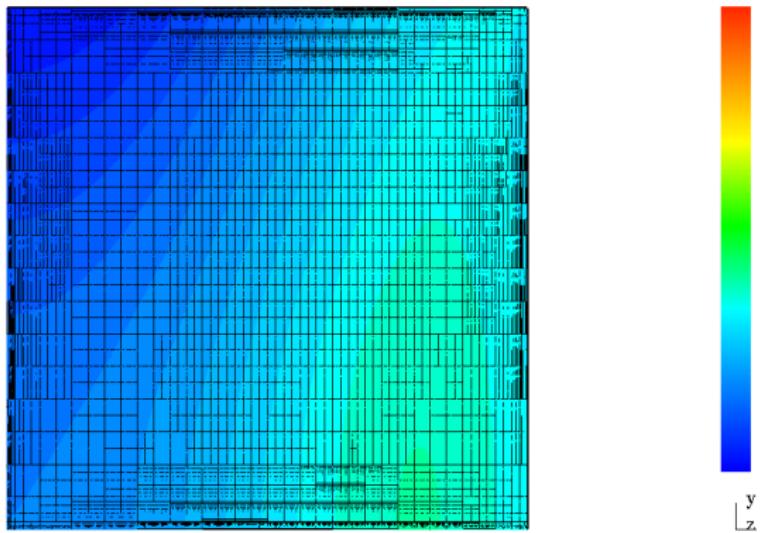
The function exhibits strong boundary layers invisible in this scale.

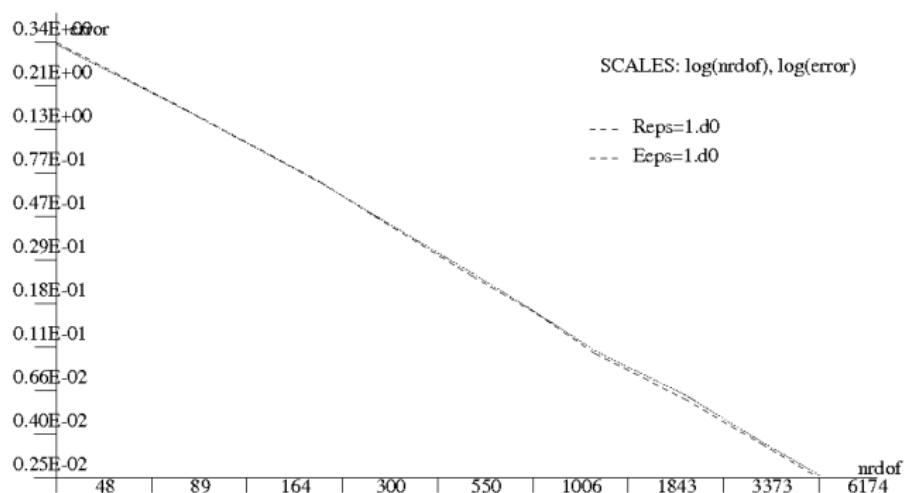
Range: (-0.6,0.6)

Manufactured solution of Lin and Stynes, $\epsilon = 10^{-1}$

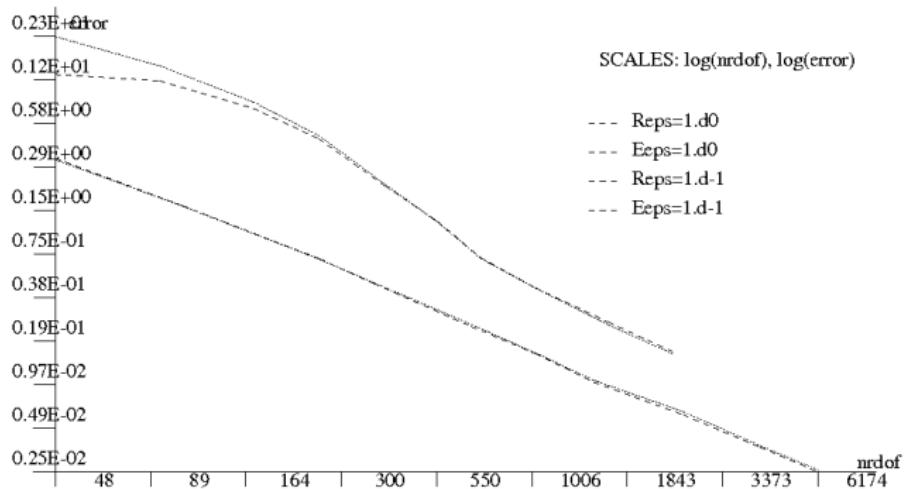


Zoom on the north boundary layer.

Optimal mesh for $\epsilon = 10^{-1}$ Optimal h -adaptive mesh and numerical solution for $\epsilon = 10^{-1}$

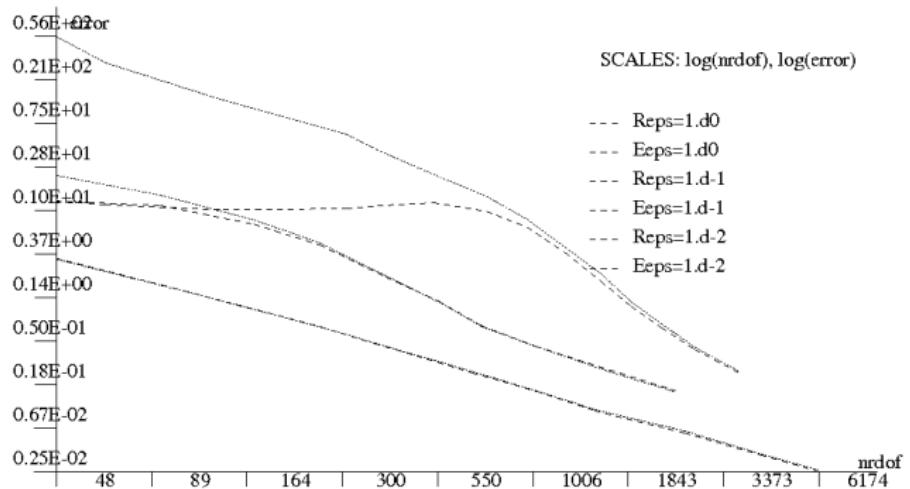
Lin/Stynes example, $\epsilon = 1$ 

Residual and “balanced” error of u for h -adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}$ 

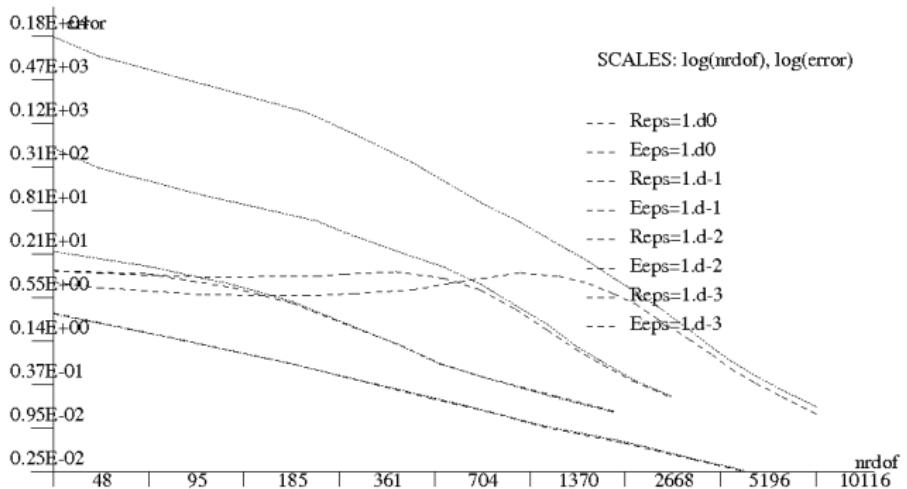
Residual and “balanced” error of u for h -adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}$.



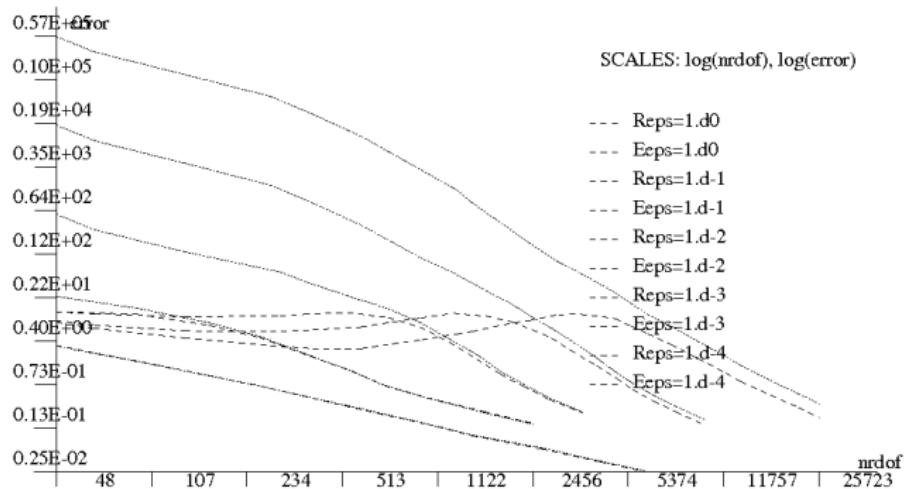
Residual and "balanced" error of u for h -adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}$.



Residual and “balanced” error of u for h -adaptive solution, $p = 2$

Lin/Stynes example, $\epsilon = 10^0, 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$

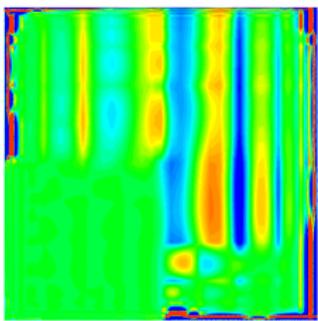
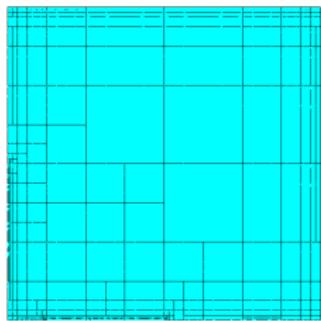


Residual and “balanced” error of u for h -adaptive solution, $p = 2$

Other tricks we can play: zooming on the solution

Question: Can we select the test norm in such a way that the DPG method would deliver high accuracy in a preselected subdomain, e.g. $(0, \frac{1}{2})^2 \subset (0, 1)^2$?

Answer: Yes!



Optimal mesh and the corresponding pointwise error (range $(-0.001 - 0.001)$).

Three DPG Punchlines

- 1 DPG Method is a Ritz method. It supports adaptivity with no preasymptotic behavior.
- 2 You can control the norm in which you want to converge.
- 3 DPG is easy to code.

Other Applications

- Elasticity, shells (volumetric, shear, membrane locking)

Other Applications

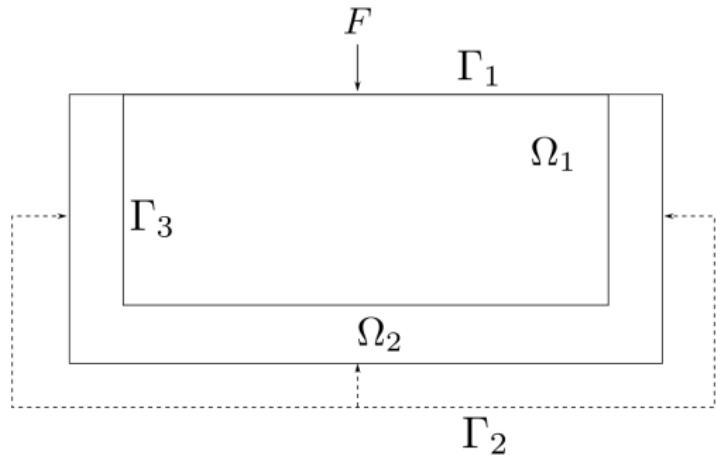
- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials

Other Applications

- Elasticity, shells (volumetric, shear, membrane locking)
- Metamaterials
- Wave propagation problems (sonars, cloaking, full wave form inversion in geomechanics)

Seismic Inversion

Tomography Setting



Seismic Inversion

Inverse Algorithm

- Solve state at current guess m using DPG at each frequency for each source

$$\begin{cases} -\Delta u_{\omega,n} - \omega^2 m u_{\omega,n} = 0 & \text{in } \Omega \\ \nabla u_{\omega,n} \cdot \hat{n} = f_n & \text{on } \Gamma_1 \\ u_{\omega,n} = 0 & \text{on } \Gamma_2 \end{cases}$$

- Solve adjoint at current guess m and solution u using DPG at each frequency for each source

$$\begin{cases} -\Delta p_{\omega,n} - \omega^2 m p_{\omega,n} = \overline{u_{\omega,n}^{\text{obs}} - u_{\omega,n}} & \text{in } \Omega \\ \nabla p_{\omega,n} \cdot \hat{n} = 0 & \text{on } \Gamma_1 \\ p_{\omega,n} = 0 & \text{on } \Gamma_2 \end{cases}$$

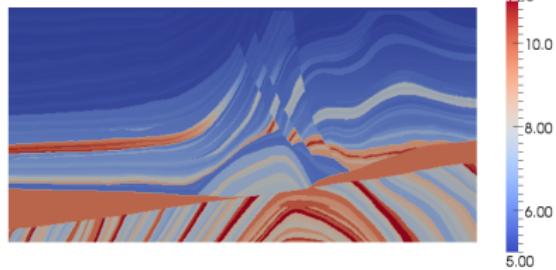
- Calculate derivative in direction \tilde{m} for use in numerical inversion algorithms

$$(g, \tilde{m})_{L^2(\Omega)} = \gamma (\nabla m, \nabla \tilde{m})_{L^2(\Omega)} - \sum_{\omega} \sum_n \omega^2 \Re(\tilde{m} u_{\omega,n}, p_{\omega,n})_{L^2(\Omega)}$$

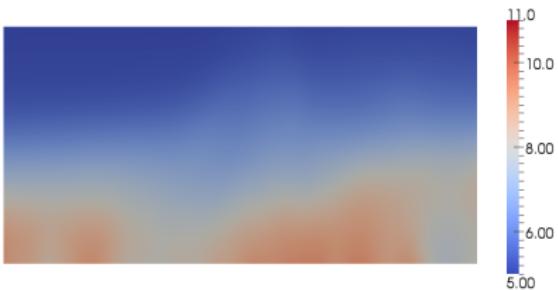
- DPG can invert over higher frequencies resulting in better resolution

Seismic Inversion

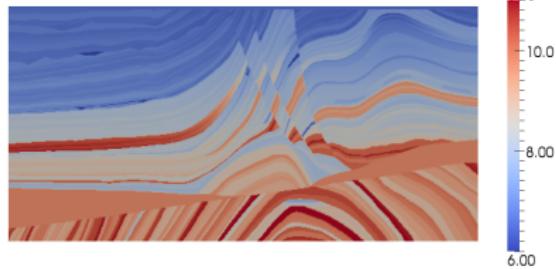
Elastic Marmousi Problem



(a) Exact μ



(b) Initial μ



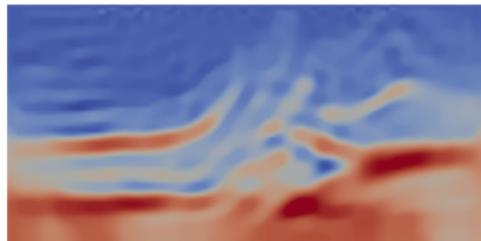
(c) Exact λ



(d) Initial λ

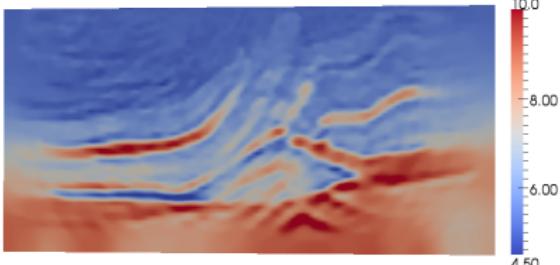
Seismic Inversion

Elastic Marmousi Problem

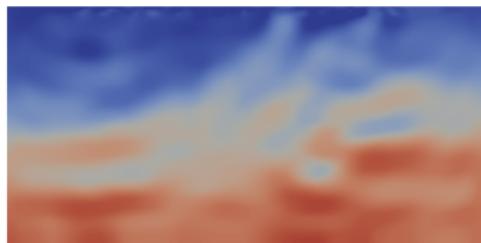


10.0
8.00
6.00
4.50

(e) CG μ , low frequency

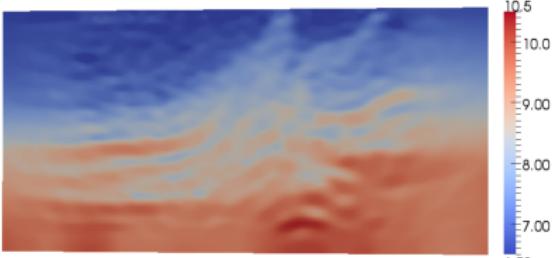


(f) DPG μ , high frequency



10.5
10.0
9.00
8.00
7.00
6.50

(g) CG λ , low frequency



10.5
10.0
9.00
8.00
7.00
6.50

(h) DPG λ , high frequency