Space-Time Discontinuous Petrov-Galerkin Finite Elements for Transient Computational Fluid Dynamics

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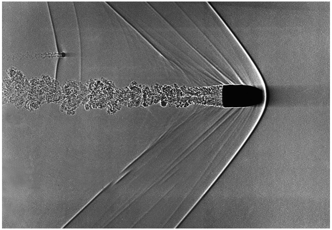
Navier-Stokes Equations

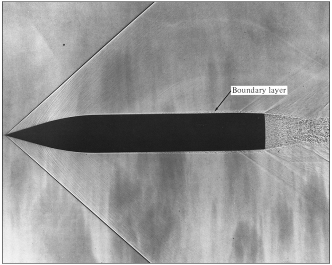
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Nonlinear convergence and uniqueness of solutions
- Stability of numerical schemes
  - Coarse/adaptive grids
  - Higher order



Shock

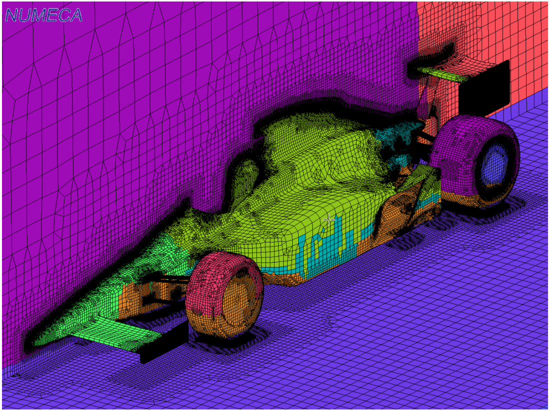


Boundary layer

Motivation

Initial Mesh Design is Expensive and Time-Consuming

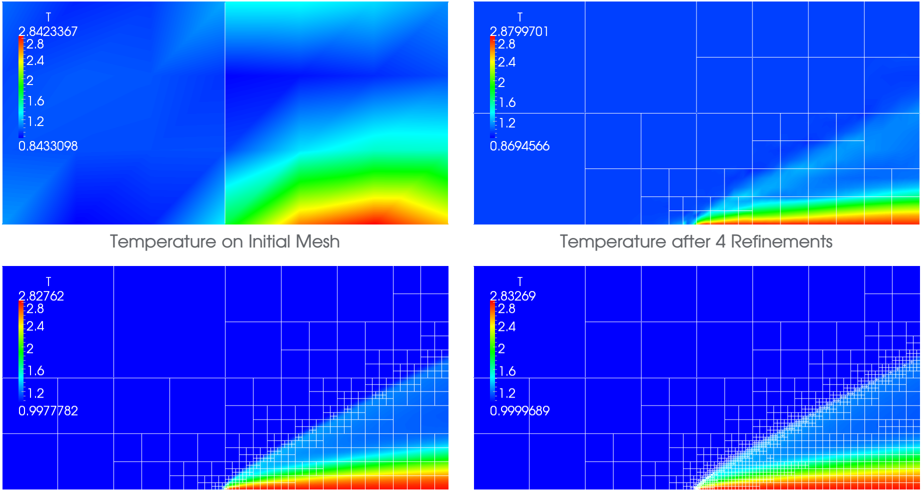
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Boundary layer meshes must respect y^+ guidelines
- Engineers often forced to work by trial and error
- Bad in the context of HPC



Formula 1 Mesh by Numeca

DPG on Coarse Meshes

Adaptive Solve of the Carter Plate Problem[1] $Re = 1000$



Temperature on Initial Mesh

Temperature after 4 Refinements

Temperature after 8 Refinements

Temperature after 11 Refinements

[1] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

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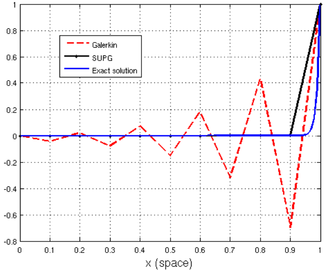
Stabilized Finite Elements for CFD

Streamline Upwind Petrov-Galerkin

- First successful finite element method for CFD[2]
- Residual based stabilization preserves consistency
- Upwind biasing of test functions
- Higher order generalizations possible
- Optimal H_0^1 approximation in 1D
- Gave rise to the field of stabilized finite elements
- Major contributors include Hughes, Franca, Johnson, Codina, Tezduyar, and many others
- Variational multiscale may be considered the spiritual successor to SUPG[3]



H_0^1 Projection

[2] A.N. Brooks and T.J.R. Hughes. ``Streamline Upwind/Petrov-Galerkin Formulations for Convection Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations''. In: *Comput. Methods Appl. Mech. Eng.* (Sept. 1990), pp. 199–259

[3] T.J.R. Hughes et al. ``The variational multiscale method -- a paradigm for computational mechanics''. In: *Comput. Methods in Appl. Mech. Eng.* 166.1 - 2 (1998). Advances in Stabilized Methods in Computational Mechanics, pp. 3 –24

Streamline Upwind Petrov-Galerkin

Two Equivalent Views on Stabilization

Convection-diffusion can be written as

$$Lu = (L_{adv} + L_{diff})u = f\,.$$

Residual Based Stabilization

$$b_{SUPG}(u,v) = l_{SUPG}(v)$$

where

$$b_{SUPG}(u,v) = b(u,v) + \sum_K \int_K \tau(L_{adv}v)(Lu - f)$$

$$l_{SUPG}(v) = l(v) + \sum_K \int_K \tau(L_{adv}v)f\,,$$

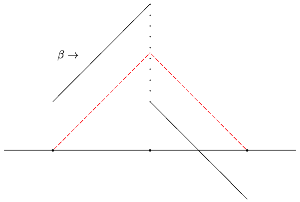
τ is the SUPG stabilization parameter.

Test Space Modification

$$b(u,\tilde{v}) = l(\tilde{v})$$

where

$$\tilde{v} = v + \tau L_{adv} v\,.$$



Stabilized Finite Elements for CFD

Discontinuous Galerkin

- Combines elements of finite volumes and finite elements
- First proposed for neutron transport[4]
- Early contributors include Babuška, Lions, Nitsche, and Zlámal
- Develoment for CFD by Cockburn and Shu[5]
- Development for elliptic problems given by Arnold, Brezzi, Cockburn, and Marini[6]
- Nonconforming basis, locally conservative
- Naturally high order, but may require additional stabilization
- Simple to parallelizable
- Other notable contributors include Peraire, Persson, Karniadakis . . .

[4] W.H. Reed and T.R. Hill. *Triangular mesh methods for the neutron transport equation*. Tech. rep. LA-UR-73-479. Los Alamos National Laboratory, 1973.

[5] B. Cockburn and C. Shu. ``The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V: Multidimensional Systems''. In: *J. Comp. Phys.* 141.2 (1998), pp. 199 –224.

[6] D.N. Arnold et al. ``Unified Analysis of Discontinuous Galerkin Methods for Elliptic Problems''. In: *SIAM J. Numer. Anal.* 39.5 (2001), pp. 1749–1779.

Stabilized Finite Elements for CFD

Discontinuous Galerkin

Consider 1D convection equation

$$\frac{\partial \beta(x) u}{\partial x} = f, \quad u(0) = u_0 .$$

Multiply by test function and integrate by parts over each element $K = [x_K, x_{x+1}]$

$$-\int_K \beta(x) u \frac{\partial v}{\partial x} + \beta u v|_{x_K}^{x_{K+1}} = \int_K fv .$$

Apply upwind flux

$$-\int_K \beta(x) u \frac{\partial v}{\partial x} + \beta(x_{K+1}) u(x_{K+1}^-) v(x_{K+1}^-) - \beta(x_K) u(x_K^-) v(x_K^+) = \int_K fv .$$

Hybridized DG (HDG) method introduces trace unknowns which facilitates static condensation, reducing interface unknowns[7].

[7] B. Cockburn, J. Gopalakrishnan, and R. Lazarov. ``Unified Hybridization of Discontinuous Galerkin, Mixed, and Continuous Galerkin Methods for Second Order Elliptic Problems''. In: *SIAM J. Numer. Anal.* 47.2 (Feb. 2009), pp. 1319–1365.

Space-Time Finite Element Methods

Treat Time as Another Dimension to be Discretized

Space-time methods treat time as just another dimension to be discretized.

- Early contributors include Kaczkowski and Oden
- Satisfies geometric conservation laws[8]
- Tezduyar *et al.* [9] developed a Galerkin/least-squares method for moving boundaries
- Van der Vegt and Van der Ven developed a popular space-time DG method[10]
- Üngör's tent-pitcher algorithm[11] decouples space-time elements

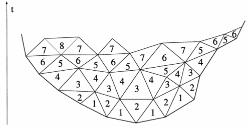


Fig. 2. Ordering of elements for an element-by-element procedure (1D×TIME). Elements with the same number can be solved independently or in parallel, elements with higher numbers depend for the solution on the output of those with lower ones.

[8] M. Lesoinne and C. Farhat. ``Geometric conservation laws for flow problems with moving boundaries and deformable meshes, and their impact on aeroelastic computations''. In: *Comput. Methods in Appl. Mech. Eng.* 134.1 - 2 (1996), pp. 71 –90

[9] T.E. Tezduyar, M. Behr, and J. Liou. ``A new strategy for finite element computations involving moving boundaries and interfaces -- The deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests''. In: *Comput. Methods in Appl. Mech. Eng.* 94.3 (1992), pp. 339 –351

[10] J.J.W. van der Vegt and H. van der Ven. ``Space-Time Discontinuous Galerkin Finite Element Method with Dynamic Grid Motion for Inviscid Compressible Flows: I. General Formulation''. In: *J. Comp. Phys.* 182.2 (2002), pp. 546 –585

[11] A. Üngör. ``Tent-Pitcher: A meshing algorithm for space-time discontinuous Galerkin methods''. In: 9th Internat. Meshing Roundtable. 2000, pp. 111–122

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Overview of DPG

DPG is a Minimum Residual Method

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bu - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1} (Bu - l) \right\|_V^2 .$$

Overview of DPG

DPG is a Minimum Residual Method

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1} B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1} B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h\,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1} B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$\left(v_{\delta u_h}, \delta v\right)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

DPG is the Most Stable Petrov-Galerkin Method

Babuška's theorem guarantees that *discrete stability and approximability imply convergence*. If bilinear form $b(u, v)$, with $M := \|b\|$ satisfies the discrete inf-sup condition with constant γ_h,

$$\sup_{v_h \in V_h} \frac{|b(u, v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U \;,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U \;.$$

Optimal test function realize the supremum guaranteeing that $\gamma_h \geq \gamma$.

Energy Norm

If we use the energy norm, $\|u\|_E := \|Bu\|_{V'}$ in the error estimate, then $M = \gamma = 1$. Babuška's theorem implies that the minimum residual method is the most stable Petrov-Galerkin method (assuming exact optimal test functions).

Overview of DPG[12]

Other Features

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

[12] **DPGOverview2**.

Overview of DPG

High Performance Computing

Eliminates human intervention

- Stability
- Robustness
- Adaptivity
- Automaticity
- Compute intensive
- Embarrassingly parallel local solves
- Factorization recyclable
- Low communication
- SPD stiffness matrix
- Multiphysics

Stampede Supercomputer at TACC

Mira Supercomputer at Argonne

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Start with the strong-form PDE.

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = \mathbf{0}$$
$$\nabla \cdot (\beta u - \boldsymbol{\sigma}) = g$$

Multiply by test functions and integrate by parts over each element, K.

$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle u, \tau_n \rangle_{\partial K} = 0$$
$$-(\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle (\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v \rangle_{\partial K} = (g, v)_K$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K}$$
$$- (\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle \hat{t}, v \rangle_{\partial K} = (g, v)_K$$

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The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[13] (also Chang and Nelson[14]), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2}\left\|R_V^{-1}(Bu_h - l)\right\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K\rangle}_{\langle \hat{t}, 1_K\rangle_{\partial K} - \langle g, 1_K\rangle_K} ,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[13] D. Moro, N.C. Nguyen, and J. Peraire. ``A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws''. In: *Int. J. Num. Meth. Eng.* (2011).

[14] C.L. Chang and J.J. Nelson. ``Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation''. In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.

Locally Conservative DPG

Locally Conservative Saddle Point System

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

Locally Conservative Saddle Point System

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B\delta u_h) - l(R_V^{-1} B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddle point problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Locally Conservative DPG

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Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. We recently developed this modification to the *robust test norm*[15] which behaves better in the presence of singularities.

Convection-Diffusion Test Norm

$$\|(v,\boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \left\|\min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right\|^2 + \|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2$$
$$+ \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 \underbrace{+ \|v\|^2}_{\text{No longer necessary}}$$

[15] J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795. ISSN: 0898-1221. DOI: http://dx.doi.org/10.1016/j.camwa.2013.06.010.

Locally Conservative DPG

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Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

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Convection-Diffusion Test Norm

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \left\| \min\left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}} \right\} \boldsymbol{\tau} \right\|^2 + \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v\|^2 + \|\boldsymbol{\beta} \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \underbrace{\left(\frac{1}{|K|} \int_K v \right)^2}_{\text{Scaling term}}$$

[15] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795. ISSN: 0898-1221. DOI: http://dx.doi.org/10.1016/j.camwa.2013.06.010.

Locally Conservative DPG

Stability and Robustness Analysis

- We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{U}, p \in Q \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) \quad \forall q \in Q \end{cases}$$

  where a, c, l, g denote the appropriate bilinear and linear forms.
- $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, R_V^{-1} B\boldsymbol{w}) = (R_V^{-1} B\boldsymbol{u}, R_V^{-1} B\boldsymbol{w})_V$
- $c(p, \boldsymbol{w}) = \sum_K \lambda_K \langle \hat{t}, 1_K \rangle_{\partial K}$
- Locally conservative DPG satisfies inf-sup and inf-sup in kernel conditions.
- Robustness is proved by switching to energy norm in Brezzi analysis.
- Full details can be found in *Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems*[16].

[16] T.E. Ellis, L.F. Demkowicz, and J.L. Chan. ``Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems''. In: *Comp. Math. Appl.* (2014).

Numerical Experiments

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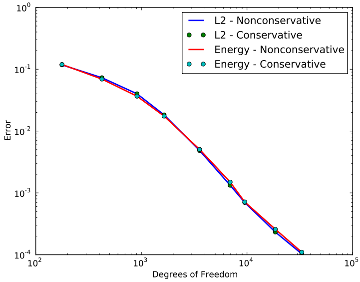
Eriksson-Johnson Problem

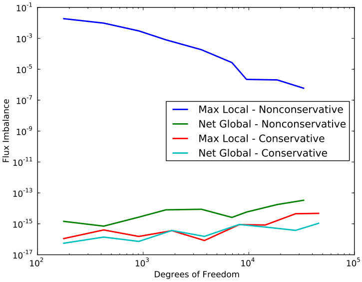
On domain $\Omega = [0,1]^2$, with $\beta = (1,0)^T, f = 0$ and boundary conditions

$$\hat{t} = u_0, \quad \beta_n \leq 0\,, \qquad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variabes gives an analytic solution

$$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x-1)) - \exp(r_1(x-1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$

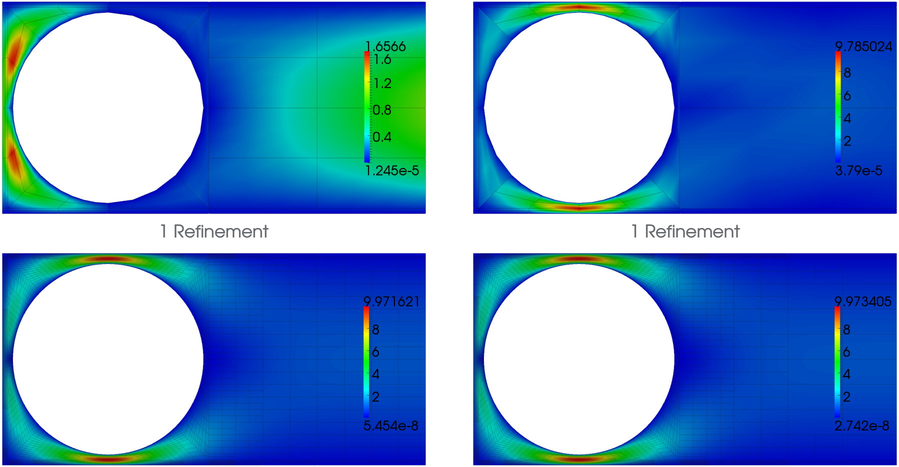




Numerical Experiments

Stokes Flow Around a Cylinder

Horizontal Velocity

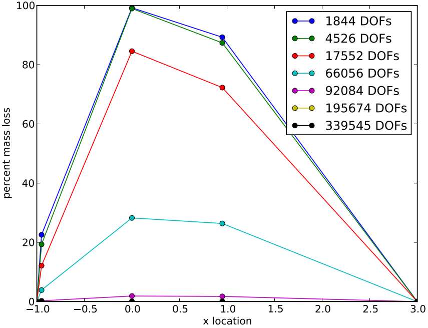


1 Refinement | 1 Refinement

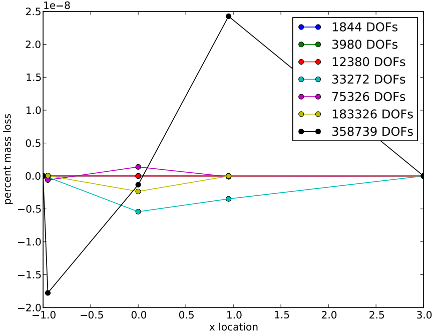
6 Refinements | 6 Refinements

Nonconservative | Conservative

Percent Mass Loss at $x = [-1, -0.95, 0, 0.95, 3]$



Nonconservative



Conservative

Numerical Experiments

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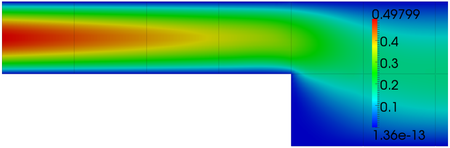
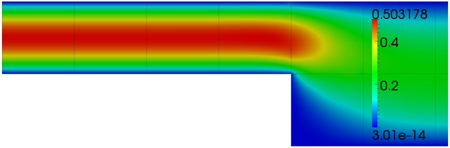
Stokes Flow Over a Backward Facing Step

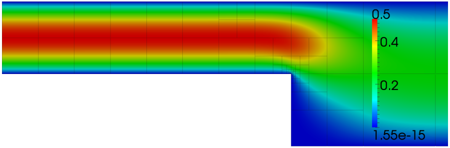
Horizontal Velocity



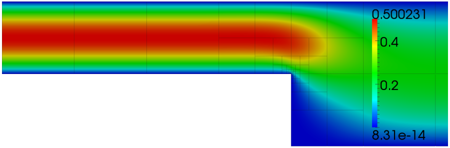
Initial Mesh



Initial Mesh



8 Refinements



8 Refinements

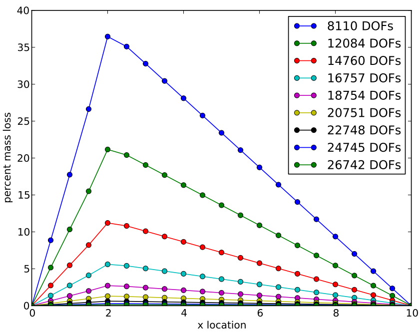
Nonconservative

Conservative

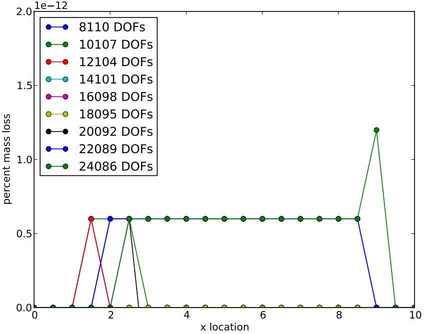
Numerical Experiments

Stokes Flow Over a Backward Facing Step

Percent Mass Loss at $x = [0, 0.5, \cdots, 9.5, 10]$



Nonconservative



Conservative

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Space-Time DPG

Motivation

Extends the capabilities of a DPG solver

- Preserves stability and robustness of DPG method
- Unified treatment of space and time
- Local space-time adaptivity (local time stepping)
  - Small solution features require small time step
  - Global time step not limited to smallest element
- Natural framework for moving meshes

More computationally difficult

- Solve requires $d+1$ dimensions
- Mesh structure more difficult
- Need to differentiate between spatial and temporal boundaries
- Larger global solves than finite difference time stepping

Heat Equation

Simplest Nontrivial Space-Time Problem

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Heat Equation

DPG Formulation

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)+(u,\nabla\cdot\boldsymbol{\tau})-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle=0$$

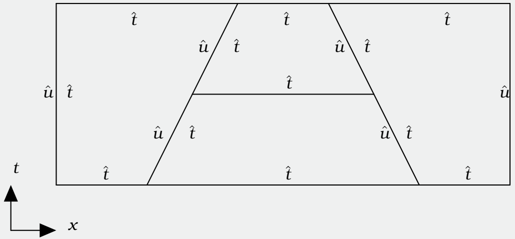
$$-\left(\begin{pmatrix}-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)+\langle\hat{t},v\rangle=f$$

where

$$\hat{u}:=\mathrm{tr}(u)$$
$$\hat{t}:=\mathrm{tr}(-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\mathrm{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

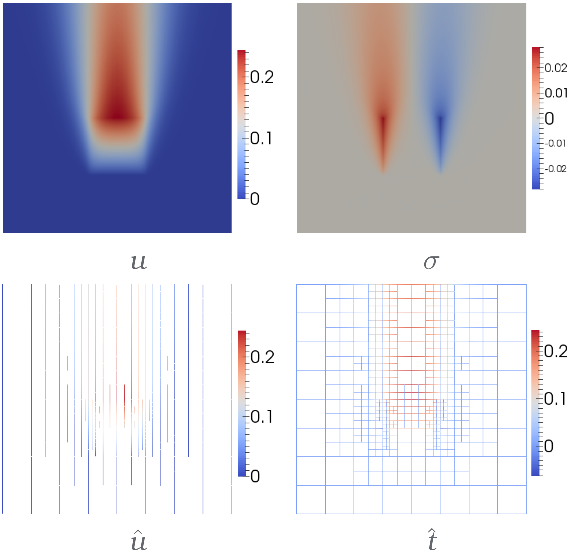
Support of Trace Variables



Heat equation

Pulsed Source Problem

- Initial condition $u = 0$.
- Apply unit source $x \in [3/8, 5/8]$, $t \in [1/4, 1/2]$.
- Should not violate causality
- Space-time adaptivity picks up areas of rapid change.



Compressible Navier-Stokes

Strong Form

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\boldsymbol{u}\\ \rho e_0\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\boldsymbol{u}\\ \rho\boldsymbol{u}\otimes\boldsymbol{u}+p\boldsymbol{I}-\mathbb{D}\\ \rho\boldsymbol{u}e_0+\boldsymbol{u}p+\boldsymbol{q}-\boldsymbol{u}\cdot\mathbb{D}\end{bmatrix}=\begin{bmatrix}f_c\\ \boldsymbol{f}_m\\ f_e\end{bmatrix},$$

where

$$\mathbb{D}=2\mu\boldsymbol{S}^*=2\mu\left[\frac{1}{2}\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T\right)-\frac{1}{3}\nabla\cdot\boldsymbol{u}\boldsymbol{I}\right],$$

$$\boldsymbol{q}=-C_p\frac{\mu}{Pr}\nabla T\,,$$

and (assuming an ideal gas EOS)

$$p=\rho RT\,.$$

Compressible Navier-Stokes

First Order Space-Time Form

Writing this in space-time in terms of ρ, $\boldsymbol{u}$, T, $\mathbb{D}$, and $\boldsymbol{q}$:

$$
\begin{aligned}
\mathbb{D} - \mu\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \frac{2\mu}{3}\nabla \cdot \boldsymbol{u}\boldsymbol{I} &= 0 \\
\boldsymbol{q} + C_p \frac{\mu}{Pr}\nabla T &= 0 \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix} &= f_c \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \boldsymbol{I} - \mathbb{D} \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f}_m \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \boldsymbol{u}\rho R T + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) \end{pmatrix} &= f_e \,.
\end{aligned}
$$

Compressible Navier-Stokes

DPG Formulation

Multiplying by test functions and integrating by parts:

$$
\begin{aligned}
(\mathbb{D}, \mathbb{S}) + (2\mu \boldsymbol{u}, \nabla\cdot\mathbb{S}) - \left(\frac{2\mu}{3}\boldsymbol{u}, \nabla \operatorname{tr}\mathbb{S}\right) - \langle 2\mu\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x\rangle + \left\langle \frac{2\mu}{3}\hat{\boldsymbol{u}}, \mathbb{S}\boldsymbol{n}_x \right\rangle &= 0\\
(\boldsymbol{q}, \boldsymbol{\tau}) - \left(C_p\frac{\mu}{Pr}T, \nabla\cdot\boldsymbol{\tau}\right) + \left\langle C_p\frac{\mu}{Pr}\hat{T}, \tau_n\right\rangle &= 0\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\\ \rho\end{pmatrix}, \nabla_{xt} v_c\right) + \langle \hat{t}_c, v_c\rangle &= (f_c, v_c)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D}\\ \rho\boldsymbol{u}\end{pmatrix}, \nabla_{xt}\boldsymbol{v}_m\right) + \langle \hat{\boldsymbol{t}}_m, \boldsymbol{v}_m\rangle &= (\boldsymbol{f}_m, \boldsymbol{v}_m)\\
-\left(\begin{pmatrix}\rho\boldsymbol{u}\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\\ \rho\left(C_vT + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\end{pmatrix}, \nabla_{xt} v_e\right) + \langle \hat{t}_e, v_e\rangle &= (f_e, v_e)\ ,
\end{aligned}
$$

where $\hat{u}$ and $\hat{T}$ are spatial traces and $\hat{t}_c$, $\hat{\boldsymbol{t}}_m$, and $\hat{t}_e$ are fluxes.

Compressible Navier-Stokes

Flux and Trace Variables

Spatial traces and fluxes are defined as follows:

$$\begin{aligned}
\hat{\boldsymbol{u}} &= \operatorname{tr}(\boldsymbol{u}) \\
\hat{T} &= \operatorname{tr}(T) \\
\hat{t}_c &= \operatorname{tr}(\rho\boldsymbol{u}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho)\, n_t \\
\hat{\boldsymbol{t}}_m &= \operatorname{tr}(\rho\boldsymbol{u} \otimes \boldsymbol{u} + \rho RT\boldsymbol{I} - \mathbb{D}) \cdot \boldsymbol{n}_x + \operatorname{tr}(\rho\boldsymbol{u})\, n_t \\
\hat{t}_e &= \operatorname{tr}\left(\rho\boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \boldsymbol{u}\rho RT + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D}\right) \cdot \boldsymbol{n}_x \\
&\quad + \operatorname{tr}\left(\rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)\right) n_t\,.
\end{aligned}$$

Linearization

Fluxes, traces, and $\boldsymbol{q}$ are linear in the above bilinear form, but we need to linearize in ρ, $\boldsymbol{u}$, T, and $\mathbb{D}$ (Jacobian and residual not shown here).

Compressible Navier-Stokes

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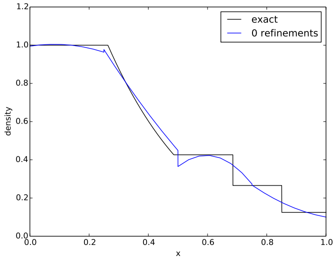
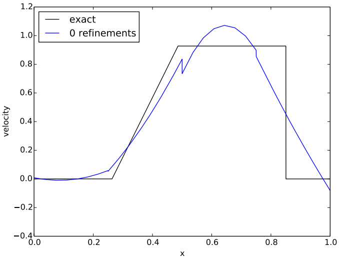
Test Norm

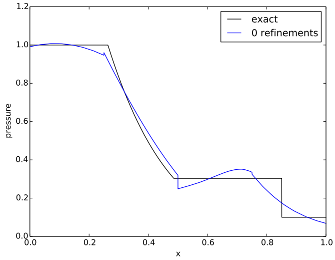
$$
\begin{aligned}
&\left\|\nabla \boldsymbol{v}_m + \nabla v_e \otimes \tilde{\boldsymbol{u}}\right\|^2 + \left\|\nabla v_e\right\|^2 \\
&+ \left\| -\tilde{\boldsymbol{u}} \cdot \nabla v_c - \frac{\partial v_c}{\partial t} - \tilde{\boldsymbol{u}} \otimes \tilde{\boldsymbol{u}} : \nabla \boldsymbol{v}_m - R\tilde{T}\nabla \cdot \boldsymbol{v}_m - \tilde{\boldsymbol{u}} \cdot \frac{\partial \boldsymbol{v}_m}{\partial t} \right. \\
&\quad \left. -C_v \tilde{T} \tilde{\boldsymbol{u}} \cdot \nabla v_e - \frac{1}{2}\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}} \cdot \nabla v_e - R\tilde{T}\tilde{\boldsymbol{u}}\nabla v_e - C_v \tilde{T}\frac{\partial v_e}{\partial t} - \frac{1}{2}\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}}\frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\| -\tilde{\rho}\nabla v_c - \tilde{\rho}\tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{v}_m - \tilde{\rho}\nabla \boldsymbol{v}_m \cdot \tilde{\boldsymbol{u}} - \tilde{\rho}\frac{\partial \boldsymbol{v}_m}{\partial t} - C_v \tilde{\rho}\tilde{T}\nabla v_e - \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}} \cdot \tilde{\boldsymbol{u}}\nabla v_e \right. \\
&\quad \left. -\frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}} \cdot \nabla v_e \tilde{\boldsymbol{u}} - \frac{1}{2}\tilde{\rho}\nabla v_e \cdot \tilde{\boldsymbol{u}}\tilde{\boldsymbol{u}} - R\tilde{\rho}\tilde{T}\nabla v_e + \tilde{\mathbb{D}} \cdot \nabla v_e - \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\frac{\partial v_e}{\partial t} - \frac{1}{2}\tilde{\rho}\tilde{\boldsymbol{u}}\frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\| -R\tilde{\rho}\nabla \cdot \boldsymbol{v}_m - C_v \tilde{\rho}\tilde{\boldsymbol{u}}\nabla v_e - R\tilde{\rho}\tilde{\boldsymbol{u}}\nabla v_e - C_v \tilde{\rho}\frac{\partial v_e}{\partial t} \right\|^2 \\
&+ \left\|\frac{1}{\mu}\mathbb{S}\right\|^2 + \left\| 2\nabla \cdot \mathbb{S} - \frac{2}{3}\nabla \operatorname{tr} \mathbb{S} \right\|^2 + \left\|\frac{Pr}{C_p \mu}\boldsymbol{\tau}\right\|^2 + \left\|\nabla \cdot \boldsymbol{\tau}\right\|^2 \\
&+ \left\|v_c\right\|^2 + \left\|\boldsymbol{v}_m\right\|^2 + \left\|v_e\right\|^2 \ .
\end{aligned}
$$

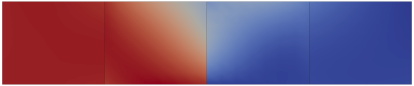
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



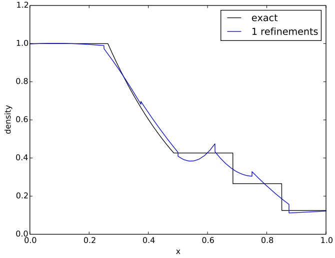


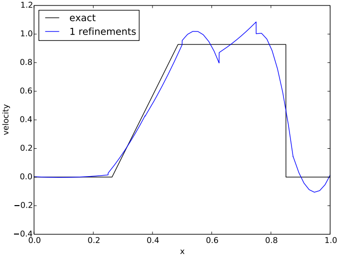


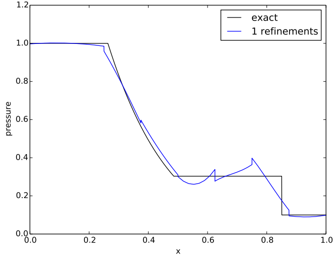


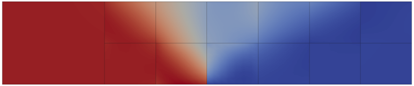
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



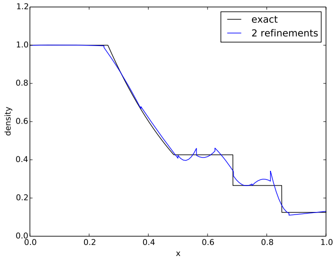


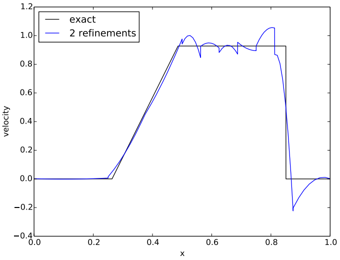


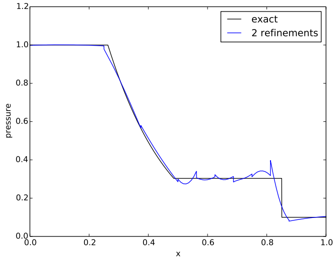


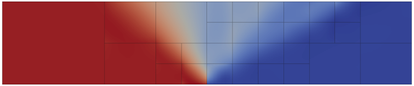
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



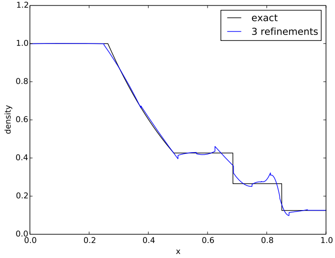


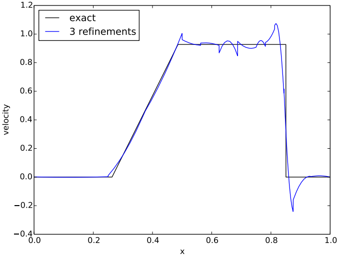


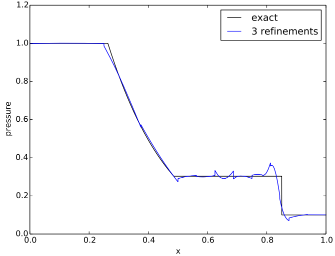


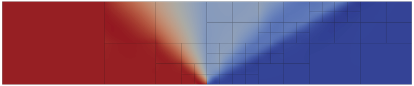
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



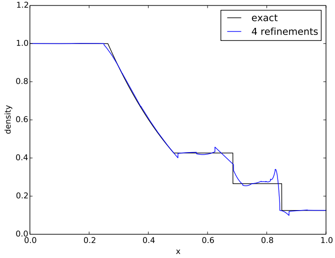


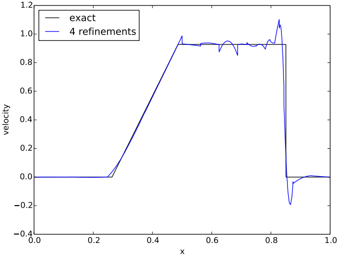


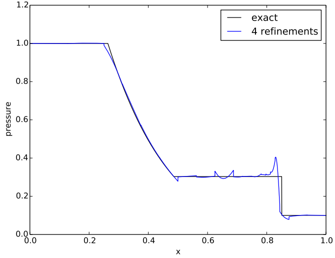


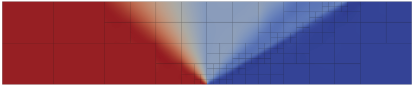
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



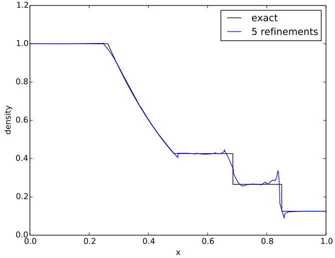


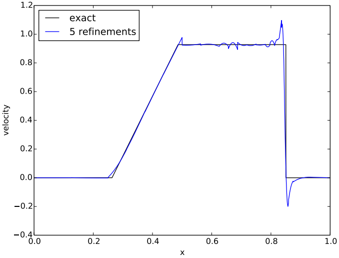


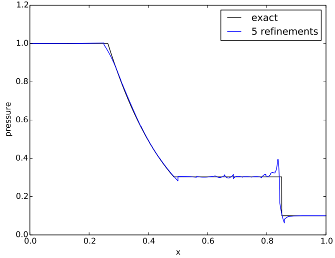


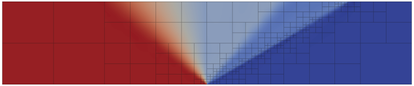
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



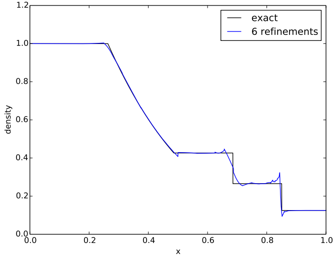


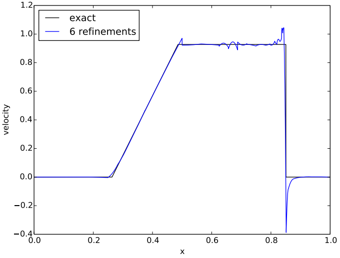


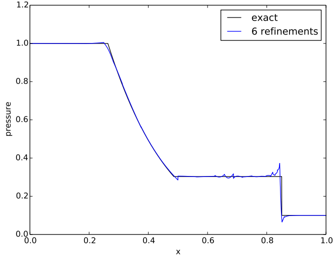


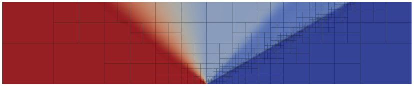
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



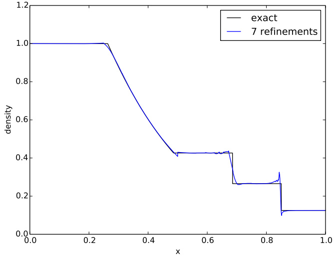


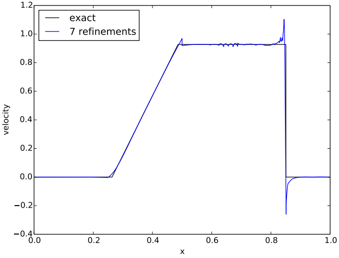


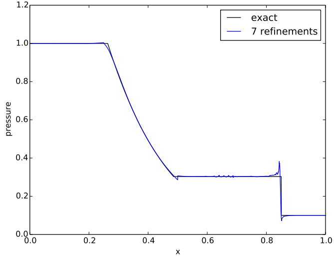


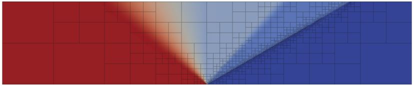
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



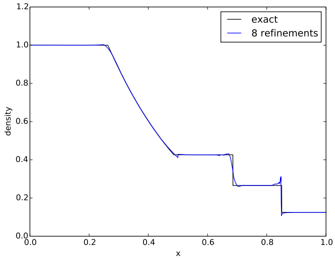


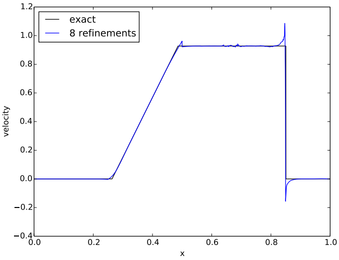


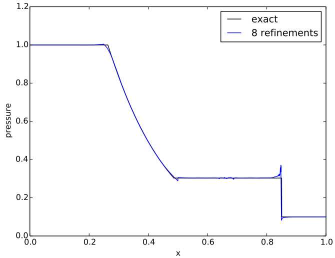


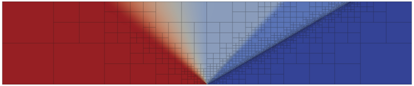
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



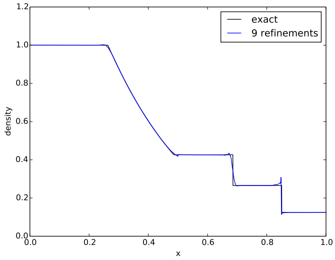


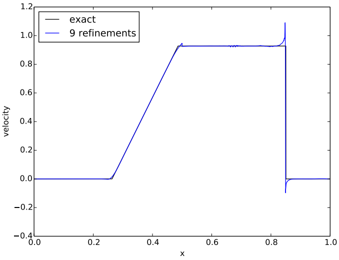


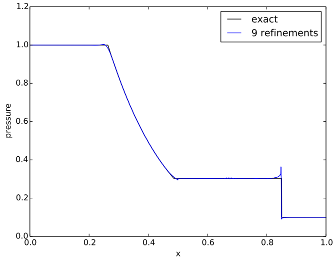


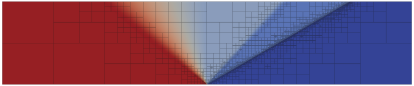
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



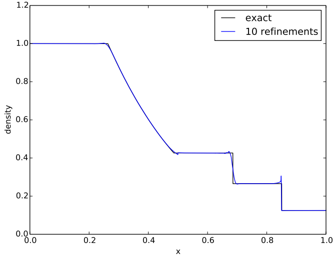


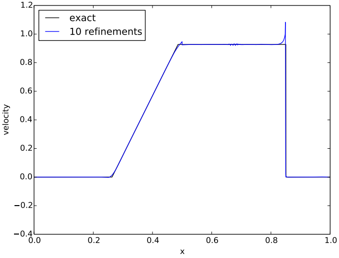


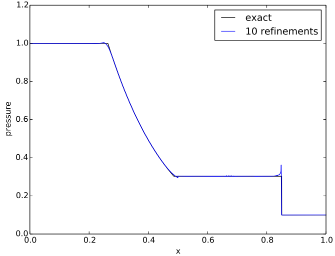


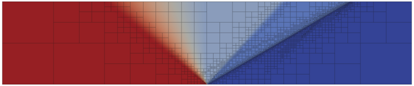
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



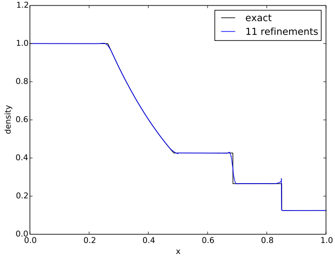


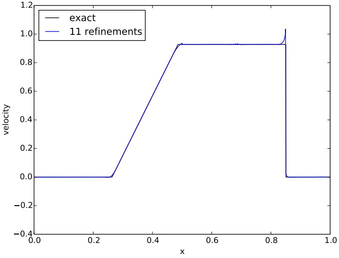


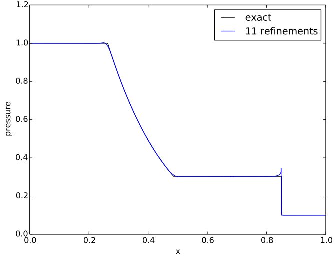


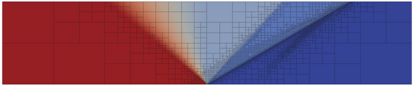
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



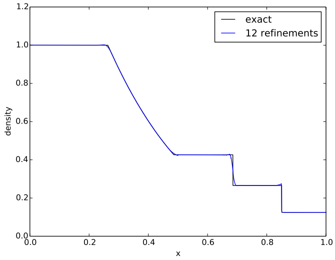


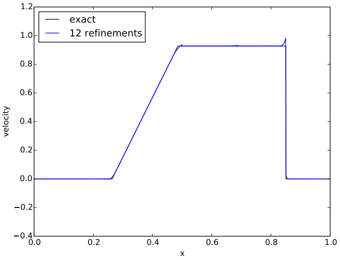


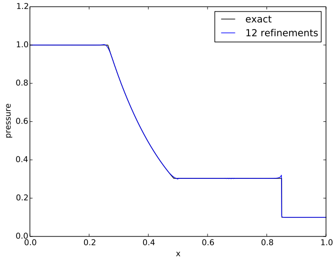


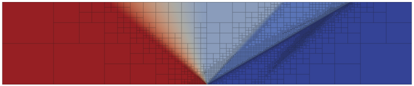
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$









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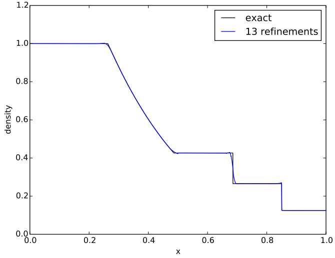

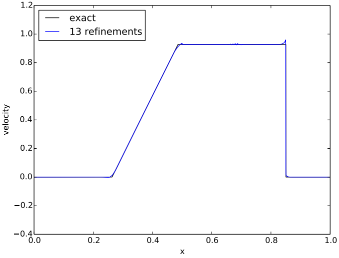

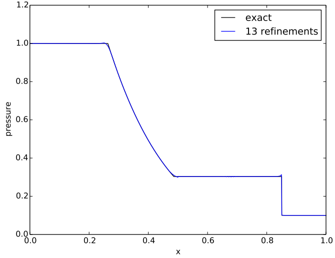

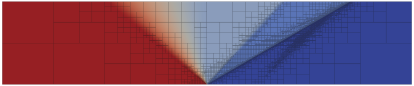

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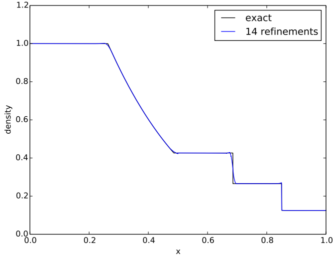
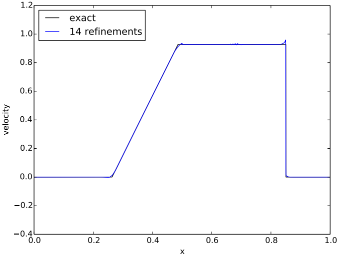
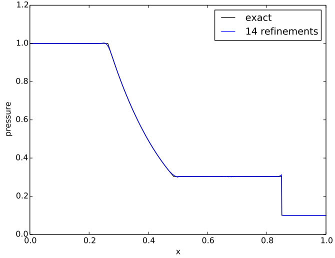
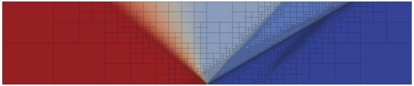




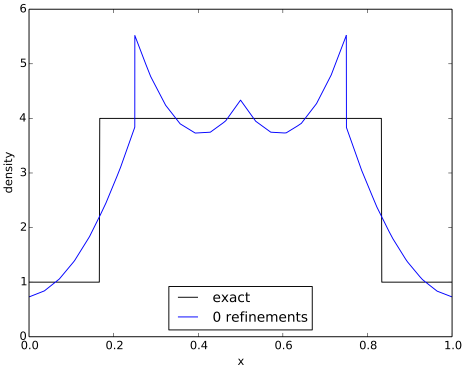


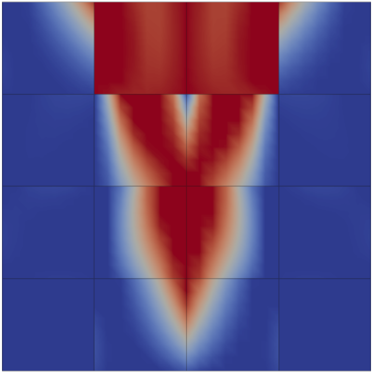


Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.





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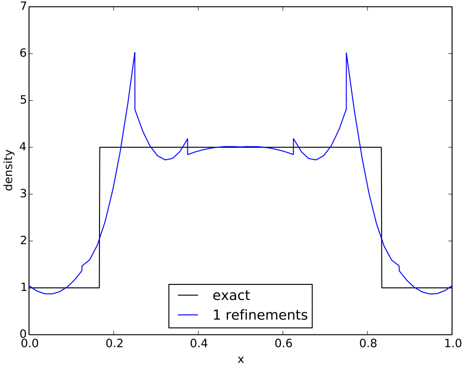

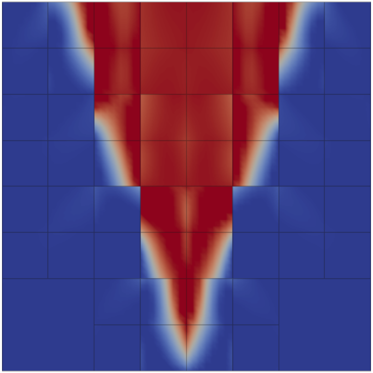

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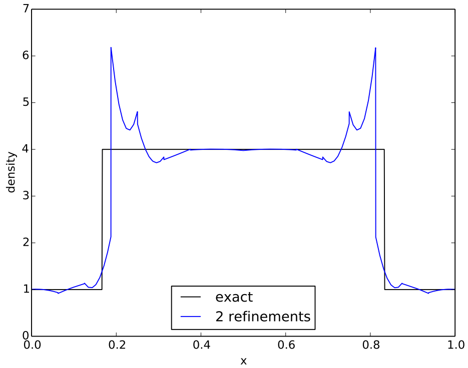

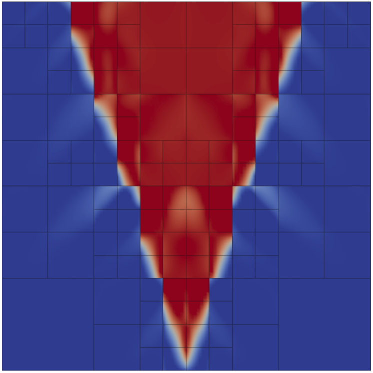

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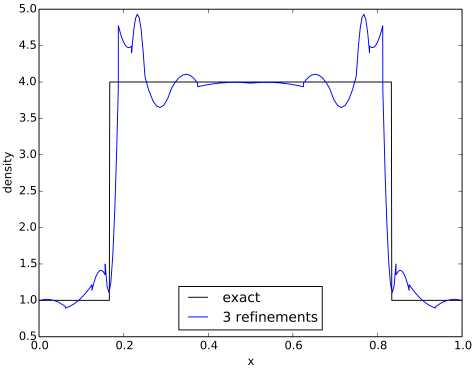

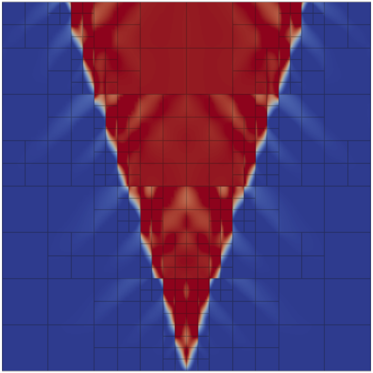

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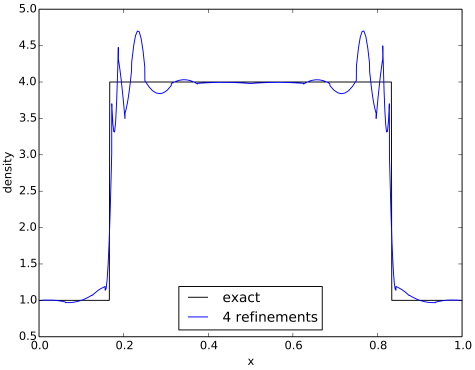

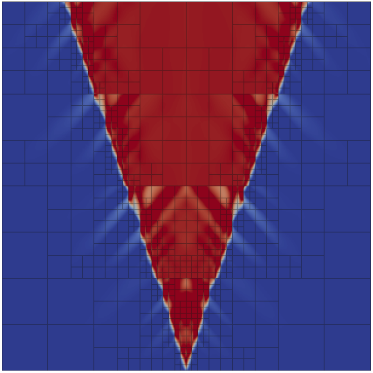

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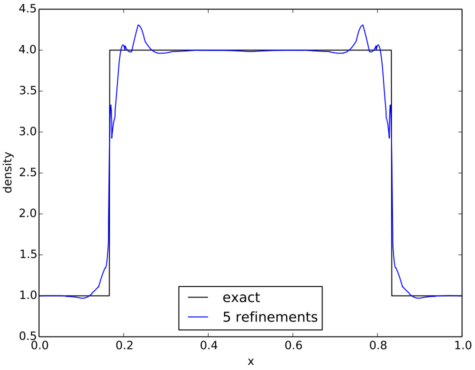

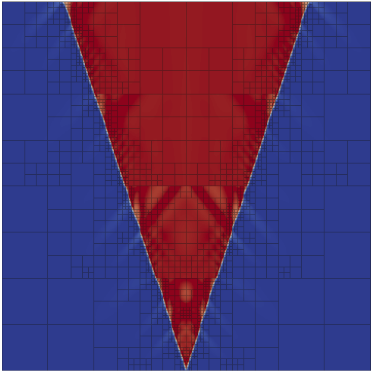

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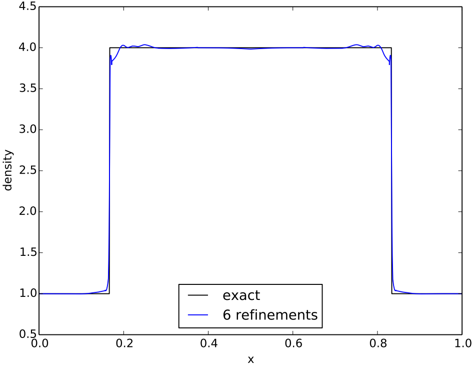

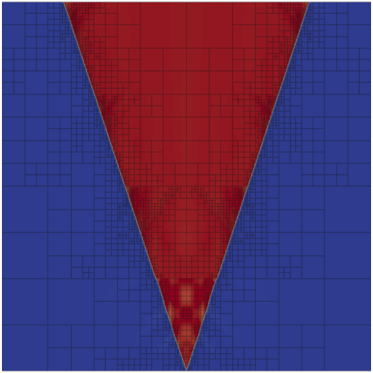

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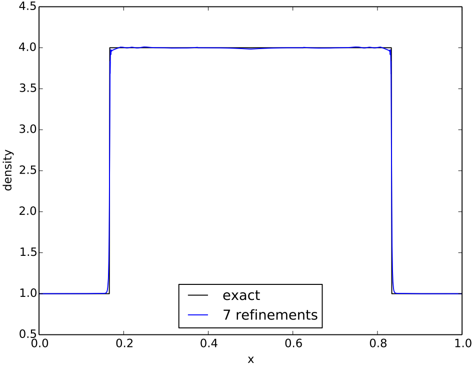

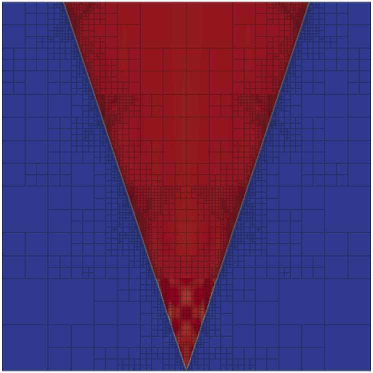

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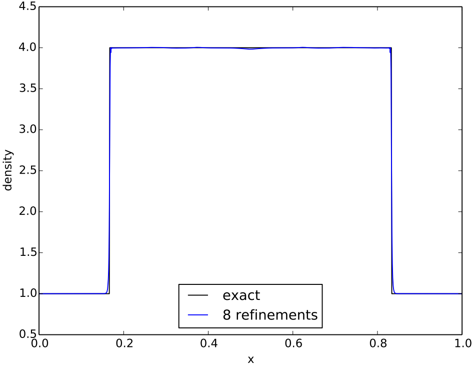

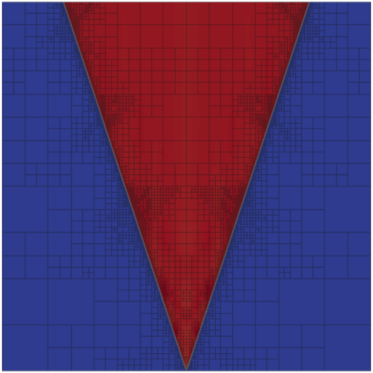

Table of Contents

Proposed Work

Area A: Applicable Mathematics

- Robustness analysis for space-time convection-diffusion
- Explore positivity preserving techniques for compressible Navier-Stokes

Area B: Scientific Computation

- Support development of Camellia[17]
  - Development and verification of 2D space-time simulations.
  - Implement time slabs to decrease solve size
  - Contribute to auxiliary features
- Run parallel simulations on HPC systems at TACC and ANL

Area C: Modeling and Applications

- Revisit Carter plate solve for higher Reynolds numbers
- Incompressible Taylor-Green vortex problem
- Incompressible vortex shedding problems
- Possibly investigate compressible Sedov and Noh problems

[17] N.V. Roberts. ``Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods''. In: *Comp. Math. Appl.* (2014). submitted.

Thank You!

THE UNIVERSITY OF TEXAS AT AUSTIN