Space-Time Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Minimum Residual and Least Squares Finite Element Methods Workshop

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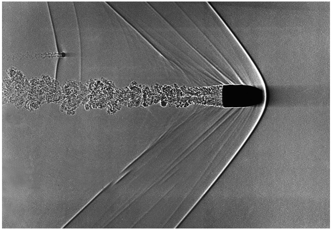
Navier-Stokes Equations

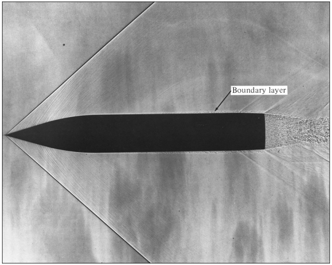
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Stability of numerical schemes
  - Nonlinearity
  - Nature of PDE changes for different flow regimes
  - Coarse/adaptive grids
  - Higher order



Shock

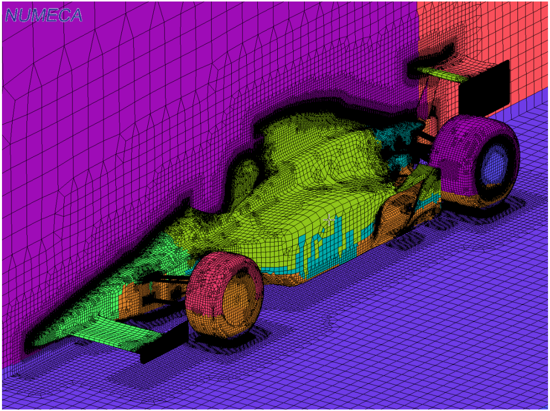


Boundary layer

Motivation

Initial Mesh Design is Expensive and Time-Consuming

- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC

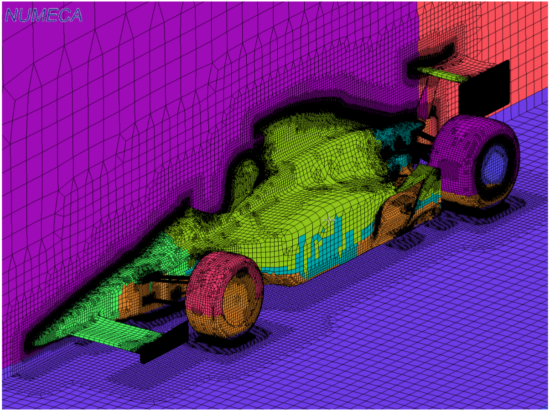


Formula 1 Mesh by Numeca

Motivation

Initial Mesh Design is Expensive and Time-Consuming

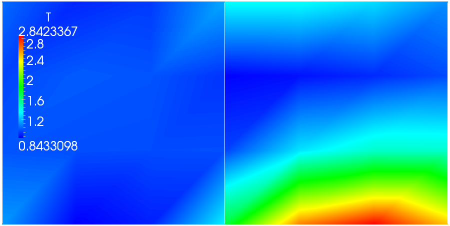
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC
- We desire an automated computational technology



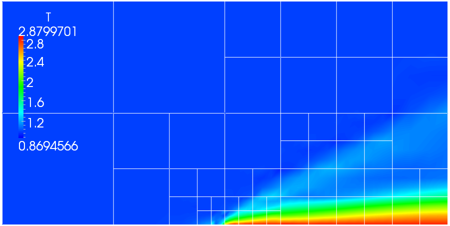
Formula 1 Mesh by Numeca

DPG on Coarse Meshes

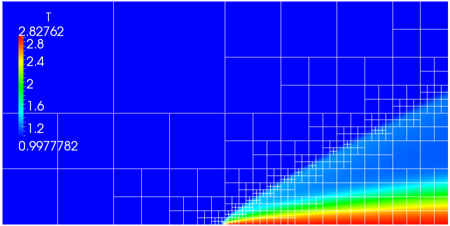
Adaptive Solve of the Carter Plate Problem[1] $Re = 1000$



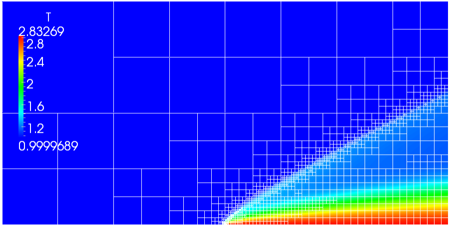
Temperature on Initial Mesh



Temperature after 4 Refinements



Temperature after 8 Refinements



Temperature after 11 Refinements

[1] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Lessons from Other Methods

Streamline Upwind Petrov-Galerkin: Adaptively changing the test space can produce a method with better stability.

Discontinuous Galerkin: Discontinuous basis functions are a legitimate option for finite element methods.

Hybridized DG: Mesh interface unknowns can facilitate static condensation -- reducing the number of DOFs in the global solve.

Least-Squares FEM: The finite element method is most powerful in a minimum residual context (i.e. as a Ritz method).

Space-Time FEM: Highly adaptive methods should have adaptive time integration. Superior framework for problems with moving boundaries. Requires a method that is both temporally and spatially stable.

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bw_h - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1} (Bw_h - l) \right\|_V^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h\right\rangle = 0 \quad \forall \delta u_h \in U_h\,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$\left(v_{\delta u_h}, \delta v\right)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

Other Features

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Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

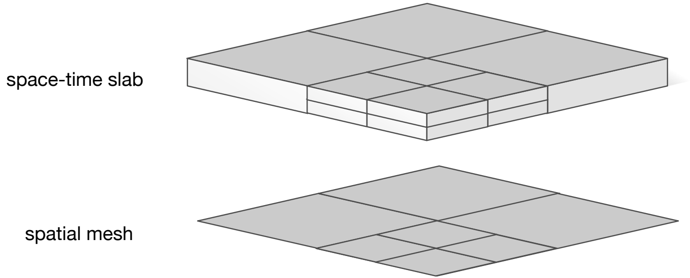
Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)_K + (u,\nabla\cdot\boldsymbol{\tau})_K - \langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle_{\partial K} = 0$$

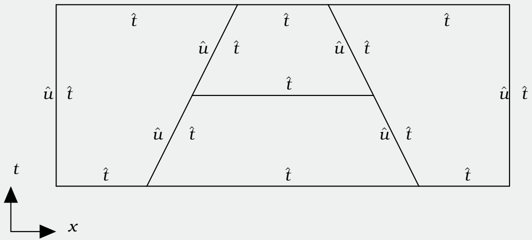
$$-\left(\begin{pmatrix}\beta u-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)_K + \langle\hat{t},v\rangle_{\partial K} = f$$

where

$$\hat{u} := \operatorname{tr}(u)$$
$$\hat{t} := \operatorname{tr}(\beta u-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x + \operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables



Space-Time Convection-Diffusion

L^2 Equivalent Norms

Bilinear form with group variables:

$$b\left(\left(u, \hat{u}\right), v\right) = \left(u, A_h^* v\right)_{L^2(\Omega_h)} + \langle \widehat{u}, [\![v]\!] \rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^* v^* = u$

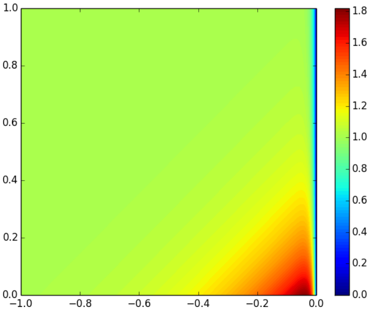
$$\|u\|^2_{L^2(\Omega_h)} = b(u, v^*) = \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V$$
$$\leq \sup_{v^* \neq 0} \frac{|b(u, v^*)|}{\|v^*\|} \|v^*\| = \|u\|_E \|v^*\|_V$$

Necessary robustness condition:

$$\|v^*\|_V \lesssim \|u\|_{L^2(\Omega_h)}$$
$$\Rightarrow \|u\|_{L^2(\Omega_h)} \lesssim \|u\|_E$$

Analytical Solution

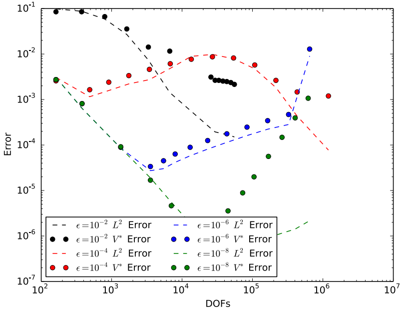
$$e^{-lt}\left(e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}\right) + \left(1 - e^{\frac{1}{\epsilon}x}\right)$$

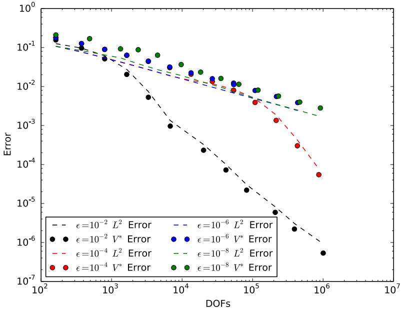


Space-Time Convection-Diffusion

L^2 Equivalent Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.





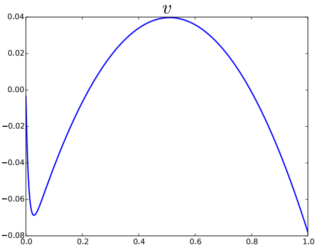
$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \left\|\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v\right\|^2 + \|v\|^2 + \|\boldsymbol{\tau}\|^2$$

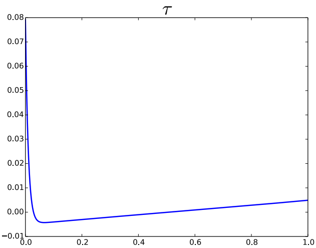
$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \|v\|^2$$

Steady Convection-Diffusion

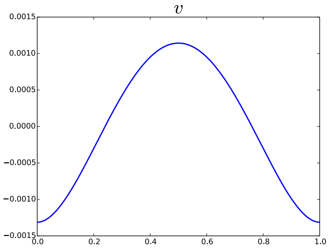
Ideal Optimal Shape Functions

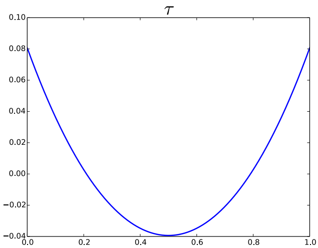
Graph Norm





Coupled Robust Norm

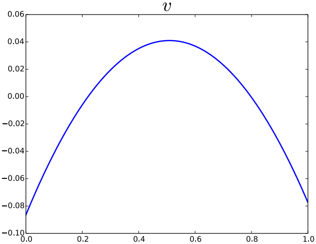


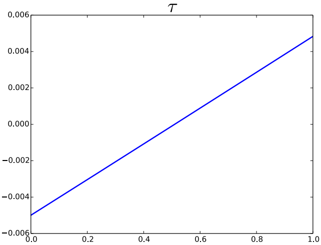


Steady Convection-Diffusion

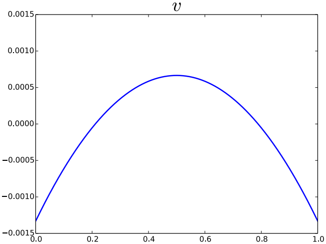
Approximated ($p = 3$) Optimal Shape Functions

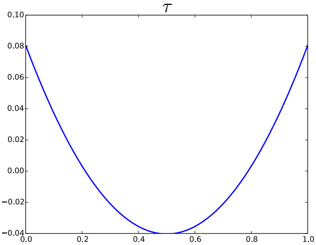
Graph Norm





Coupled Robust Norm





Steady Convection-Diffusion

Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.

The robust norm was derived in[2]:

$$\|(v,\boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\,\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 \,.$$

The case for the coupled robust norm was made in[3]:

$$\|(v,\boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\,\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 + \|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 \,.$$

[2] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.

[3] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

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Two Robust Norms for Transient Convection-Diffusion

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$.

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$
\begin{aligned}
\|(v,\boldsymbol{\tau})\|^2 = {} & \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \epsilon\,\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 \\
& + \|\nabla\cdot\boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 \,.
\end{aligned}
$$

Coupled Robust Norm

$$
\begin{aligned}
\|(v,\boldsymbol{\tau})\|^2 = {} & \left\|\tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \epsilon\,\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 \\
& + \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt} v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 \,.
\end{aligned}
$$

Consider the problem with homogeneous boundary conditions

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\
\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\
\beta_n u - \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\
u &= 0 \text{ on } \Gamma_+ \\
u &= u_0 \text{ on } \Gamma_0 .
\end{aligned}
$$

The adjoint operator A^* is given by

$$
A^*(v, \boldsymbol{\tau}) = \left(\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v, -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \nabla \cdot \boldsymbol{\tau} \right) .
$$

We decompose the continuous adjoint problem $A^*(\boldsymbol{\tau}, v) = (\boldsymbol{f}, g)$ into

Continuous part with forcing g

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 &= 0 \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= g
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_1 &= 0 \text{ on } \Gamma_+ \\
v_1 &= 0 \text{ on } \Gamma_T
\end{aligned}
$$

Continuous part with forcing $\boldsymbol{f}$

$$
\begin{aligned}
\frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 &= \boldsymbol{f} \\
-\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0
\end{aligned}
\qquad
\begin{aligned}
\boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\
v_2 &= 0 \text{ on } \Gamma_+ \\
v_2 &= 0 \text{ on } \Gamma_T
\end{aligned}
$$

Robust Norms for Transient Convection-Diffusion

Proved Bounds at Our Disposal

Proofs of these lemmas can be found in[4].

Lemma (1)

If $\nabla \cdot \boldsymbol{\beta} = 0$*, we can bound*

$$\left\| v \right\|^2 + \epsilon \left\| \nabla v \right\|^2 \leq \left\| g \right\|^2 + \epsilon \left\| \boldsymbol{f} \right\|^2$$

where $v = v_1 + v_2$.

Lemma (2)

If $\left\| \nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I} \right\|_{L^\infty} \leq C_{\boldsymbol{\beta}}$*, we can bound*

$$\left\| \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 \right\| \lesssim \left\| g \right\| .$$

[4] T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion.* Tech. rep. 15-21. ICES, Oct. 2015.

Robust Norms for Transient Convection-Diffusion

Control of u

Bound on $\|(v_1, \boldsymbol{\tau}_1)\|$

Lemma (2) $\Rightarrow$ $$\left\|\tilde{\beta} \cdot \nabla_{xt} v_1\right\| \lesssim \|g\|$$

Lemma (2) $\Rightarrow$ $$\|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|g\| + \left\|\tilde{\beta} \cdot \nabla_{xt} v_1\right\| \lesssim 2\|g\|$$

Lemma (2) $\Rightarrow$ $$\left\|\nabla \cdot \boldsymbol{\tau}_1 - \tilde{\beta} \cdot \nabla_{xt} v_1\right\| = \|g\|$$

Lemma (1) $\Rightarrow$ $$\|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|g\|^2$$

Lemma (1) $\Rightarrow$ $$\frac{1}{\epsilon}\|\boldsymbol{\tau}_1\| = \epsilon \|\nabla v_1\| \leq \|g\|$$

Plugging in $g = u$, we can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \boldsymbol{\sigma})\|_E .$$

Robust Norms for Transient Convection-Diffusion

Control of $\boldsymbol{\sigma}$

Bound on $\|(v_2, \boldsymbol{\tau}_2)\|$

Definition $\Rightarrow$ $\left\|\nabla \cdot \boldsymbol{\tau}_2 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\| = 0 \leq \|\boldsymbol{f}\|$

Lemma (1) $\Rightarrow$ $\|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\boldsymbol{f}\|^2$

Lemma (1) $\Rightarrow$ $\frac{1}{\epsilon}\|\boldsymbol{\tau}_2\| = \|\boldsymbol{f}\| + \epsilon\|\nabla v_2\| = (1+\epsilon)\|\boldsymbol{f}\|$

We have not been able to prove bounds on $\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\|$ or $\|\nabla \cdot \boldsymbol{\tau}_2\|$.

Plugging in $\boldsymbol{f} = \boldsymbol{\sigma}$, we can **not** guarantee robust control

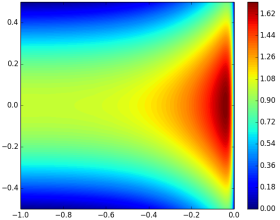
$$\|(0, \boldsymbol{\sigma})\|_{L^2(\Omega_h)} \not\lesssim \|(u, \boldsymbol{\sigma})\|_E \ .$$

Robust Norms for Transient Convection-Diffusion

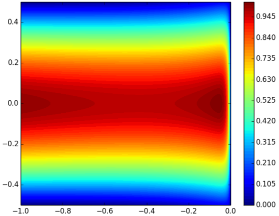
Transient Analytical Solution

Transient impulse decays to Eriksson-Johnson steady state solution.

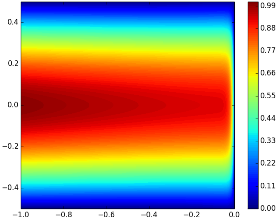
$$u = \exp(-lt)\left[\exp(\lambda_1 x) - \exp(\lambda_2 x)\right] + \cos(\pi y)\frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$



$t = 0.0$



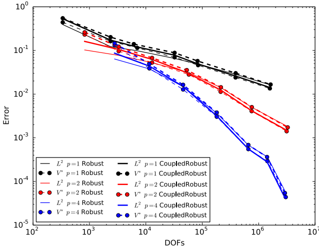
$t = 0.5$



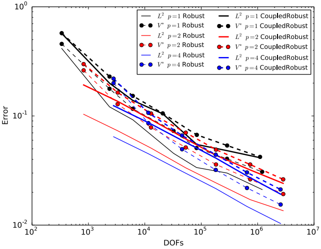
$t = 1.0$

Robust Norms for Transient Convection-Diffusion

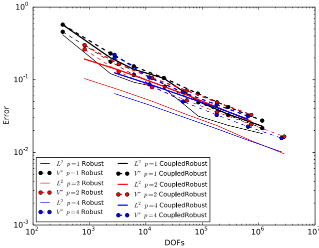
Robust Convergence to Analytical Solution



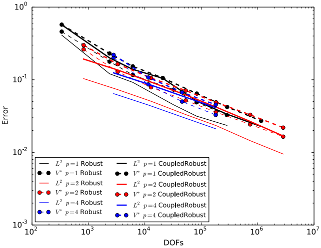
$\epsilon = 10^{-2}$



$\epsilon = 10^{-4}$



$\epsilon = 10^{-6}$



$\epsilon = 10^{-8}$

Space-Time Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$
\begin{aligned}
\frac{1}{\mu}\mathbb{D} - \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \frac{2}{3}\nabla \cdot \boldsymbol{u}\mathbb{I} &= 0 \\
\frac{Pr}{C_p\mu}\boldsymbol{q} + \nabla T &= 0 \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix} &= f_c \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \mathbb{I} - \mathbb{D} \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f}_m \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) + \rho R T \boldsymbol{u} + \boldsymbol{q} - \boldsymbol{u} \cdot \mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right) \end{pmatrix} &= f_e \,,
\end{aligned}
$$

Space-Time Navier-Stokes

Compact Notation

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Conserved quantities

$$C_c := \rho$$

$$\boldsymbol{C}_m := \rho\boldsymbol{u}$$

$$C_e := \rho(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})$$

Euler fluxes

$$\boldsymbol{F}_c := \rho\boldsymbol{u}$$

$$\mathbb{F}_m := \rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho RT\mathbb{I}$$

$$\boldsymbol{F}_e := \rho\boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho RT\boldsymbol{u}$$

Viscous fluxes

$$\boldsymbol{K}_c := \boldsymbol{0}$$

$$\mathbb{K}_m := \mathbb{D}$$

$$\boldsymbol{K}_e := -\boldsymbol{q} + \boldsymbol{u}\cdot\mathbb{D}$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \frac{1}{\mu}\mathbb{D}$$

$$\boldsymbol{M}_{\boldsymbol{q}} := \frac{Pr}{C_p\mu}\boldsymbol{q}$$

Viscous relations

$$\boldsymbol{G}_{\mathbb{D}} := 2\boldsymbol{u}$$

$$G_{\boldsymbol{q}} := -T$$

Use change of variables to get conservation or entropy variables.

Space-Time Navier-Stokes

Define Group Variables

Group terms

$$C := \{C_c\,,\ \boldsymbol{C}_m\,,\ C_e\}$$
$$F := \{\boldsymbol{F}_c\,,\ \mathbb{F}_m\,,\ \boldsymbol{F}_e\}$$
$$K := \{\boldsymbol{K}_c\,,\ \mathbb{K}_m\,,\ \boldsymbol{K}_e\}$$
$$M := \{\mathbb{M}_{\mathbb{D}}\,,\ \boldsymbol{M}_{\boldsymbol{q}}\}$$
$$G := \{\boldsymbol{G}_{\mathbb{D}}\,,\ G_{\boldsymbol{q}}\}$$
$$f := \{f_c\,, \boldsymbol{f}_m\,, f_e\}$$

Group variables

$$W := \{\rho\,,\ \boldsymbol{u}\,,\ T\}$$
$$\hat{W} := \left\{2\hat{\boldsymbol{u}}\,,\ -\hat{T}\right\}$$
$$\Sigma := \{\mathbb{D}, \boldsymbol{q}\}$$
$$\hat{t} := \{\hat{t}_e\,,\ \hat{\boldsymbol{t}}_m,\ ,\ \hat{t}_e\}$$
$$\Psi := \{\mathbb{S}\,,\ \boldsymbol{\tau}\}$$
$$V := \{v_c\,,\ \boldsymbol{v}_m,\ ,\ v_e\}\ .$$

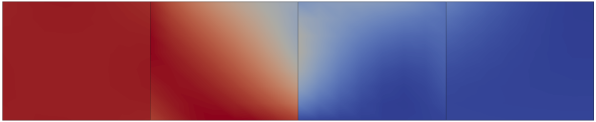
Navier-Stokes variational formulation is

$$(M, \Psi) + (G, \nabla\cdot\Psi) - \left\langle \hat{W}, \Psi\cdot\boldsymbol{n}_x \right\rangle = 0$$
$$-\left(\begin{pmatrix} F-K \\ C \end{pmatrix}, \nabla_{xt} V\right) + \langle \hat{t}, V\rangle = (f, V)\ .$$

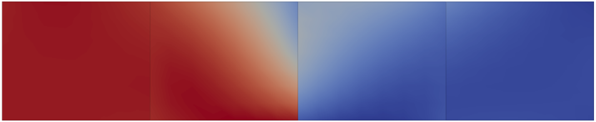
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

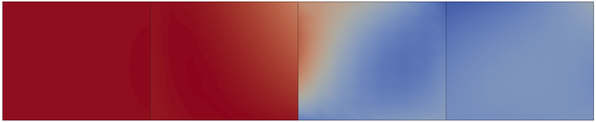
Mesh 1



Primitive Variables



Conservation Variables

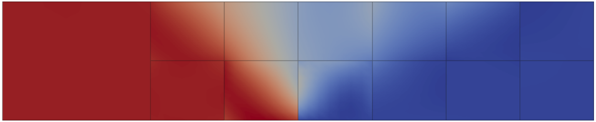


Entropy Variables

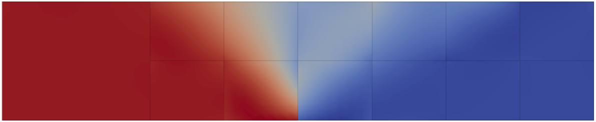
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

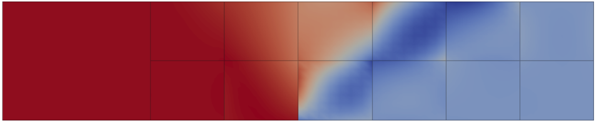
Mesh 2



Primitive Variables



Conservation Variables

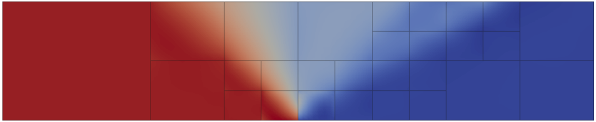


Entropy Variables

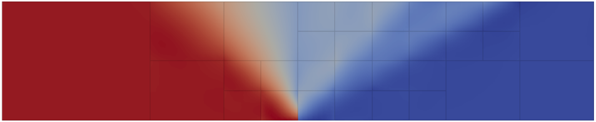
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

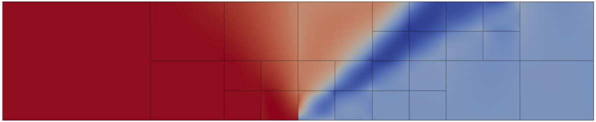
Mesh 3



Primitive Variables



Conservation Variables

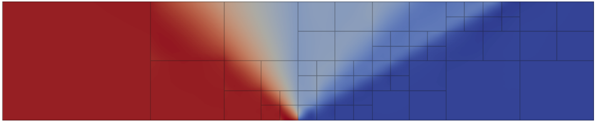


Entropy Variables

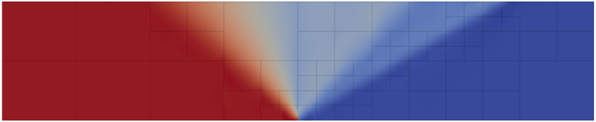
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

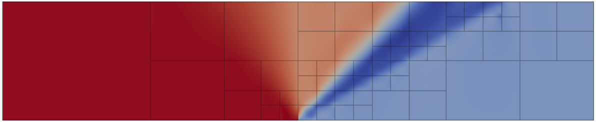
Mesh 4



Primitive Variables



Conservation Variables

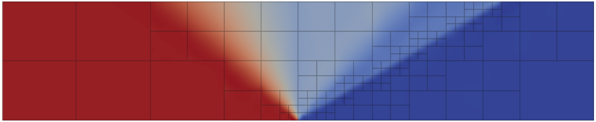


Entropy Variables

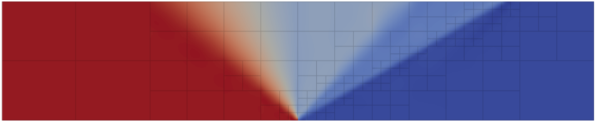
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

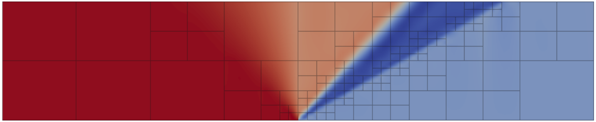
Mesh 5



Primitive Variables



Conservation Variables

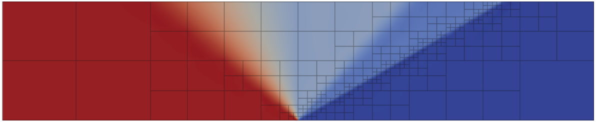


Entropy Variables

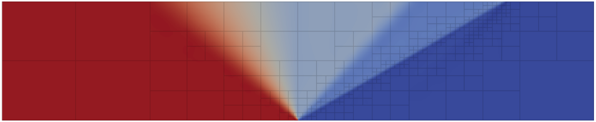
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

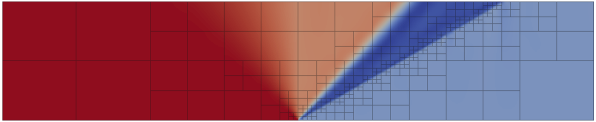
Mesh 6



Primitive Variables



Conservation Variables

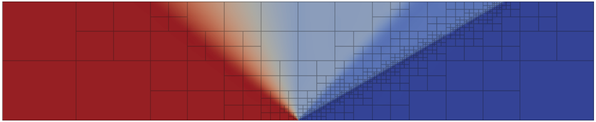


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 7



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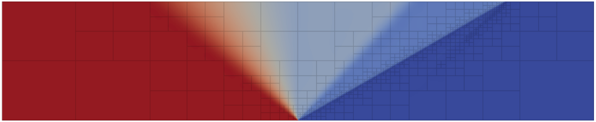

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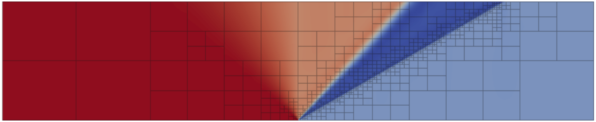
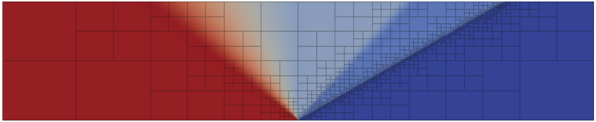


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 8



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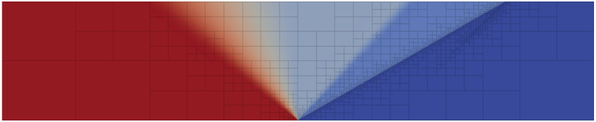

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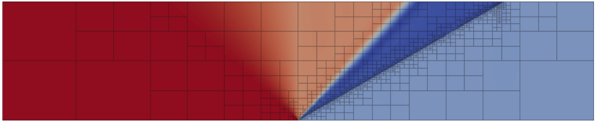
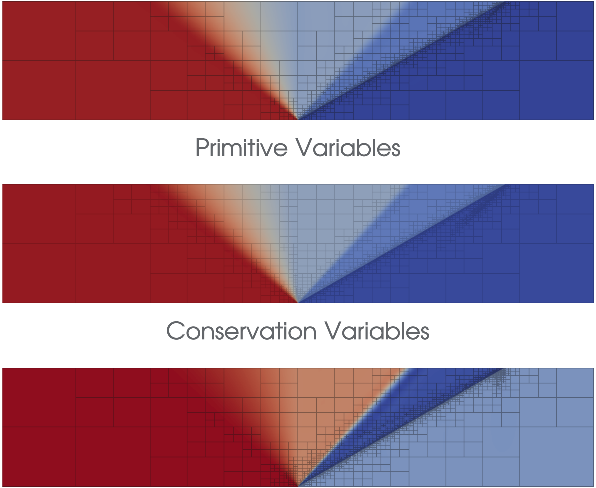


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 9



Primitive Variables

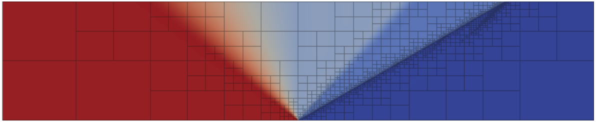
Conservation Variables

Entropy Variables

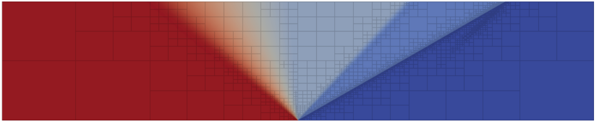
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 10



Primitive Variables



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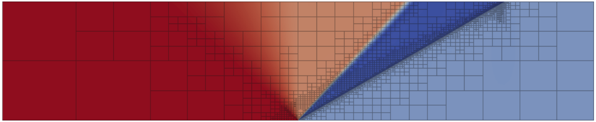
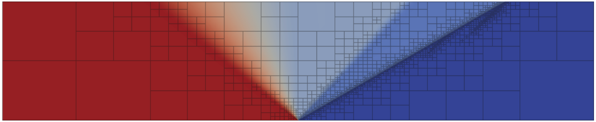


Entropy Variables

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 11



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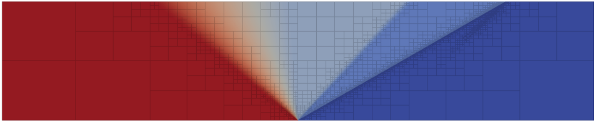

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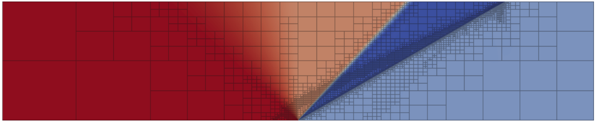
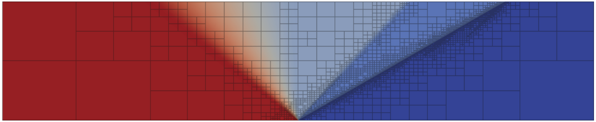


Entropy Variables

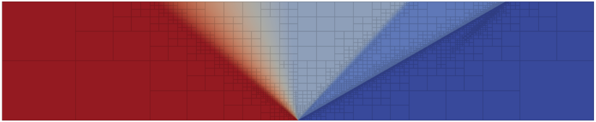
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

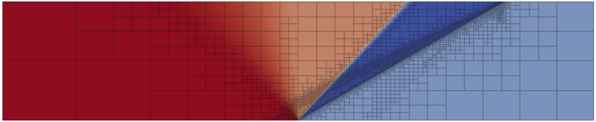
Mesh 12



Primitive Variables



Conservation Variables

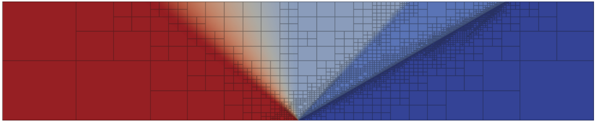


Entropy Variables

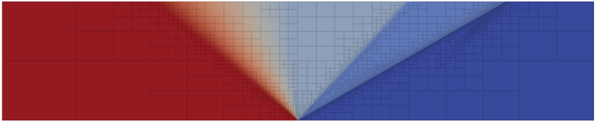
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

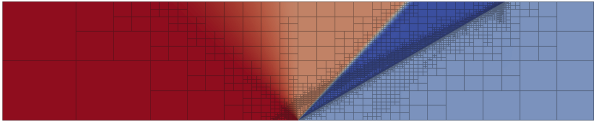
Mesh 13



Primitive Variables



Conservation Variables

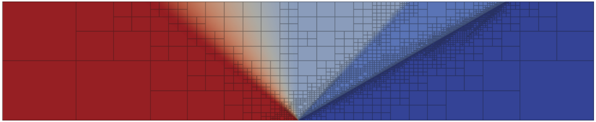


Entropy Variables

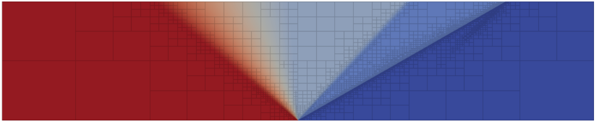
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

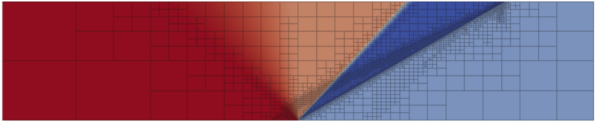
Mesh 14



Primitive Variables



Conservation Variables

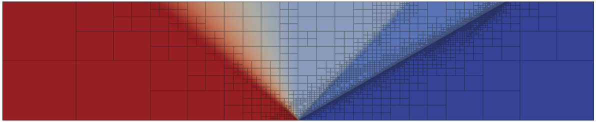


Entropy Variables

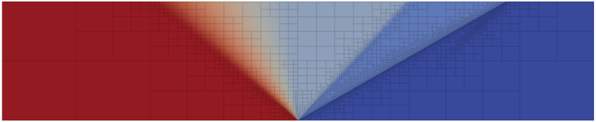
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

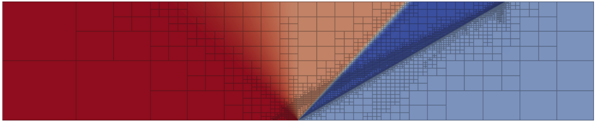
Mesh 15



Primitive Variables



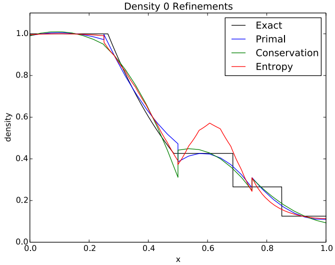
Conservation Variables

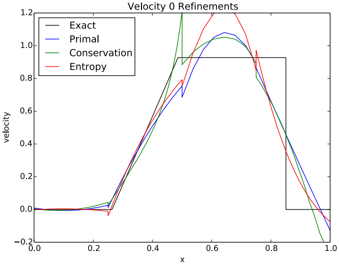


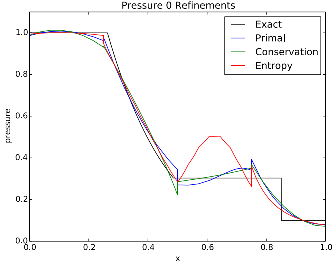
Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

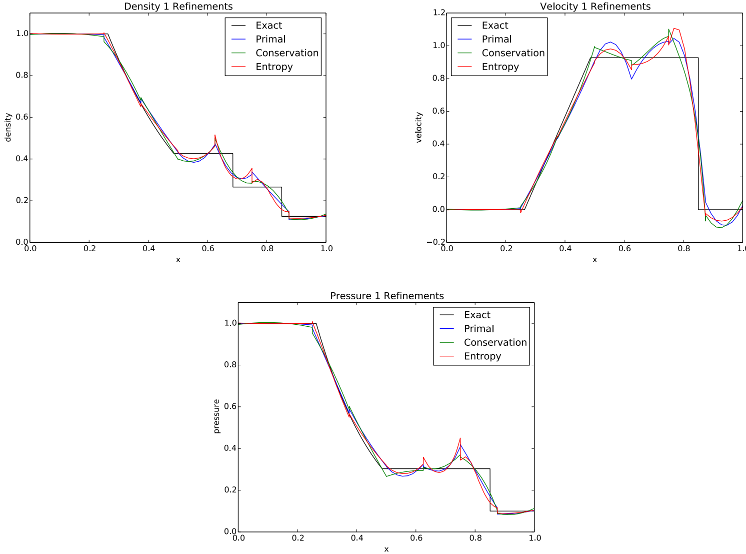






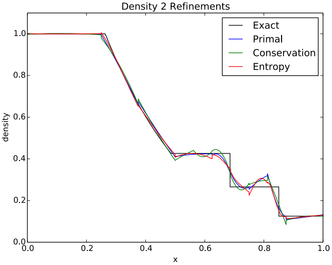
Compressible Navier-Stokes

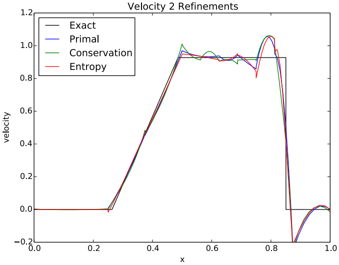
Sod Shock Tube with $\mu = 10^{-5}$

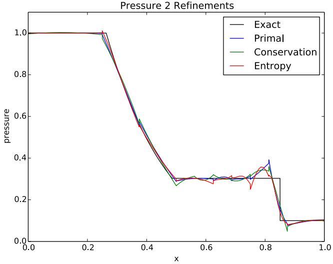


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

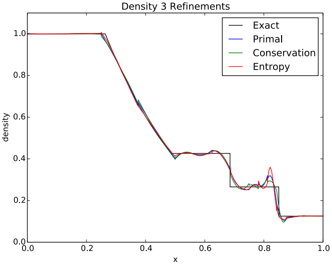


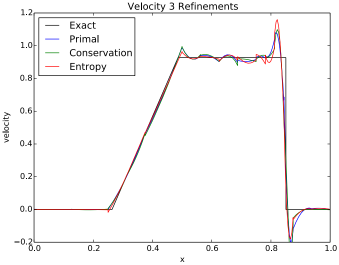


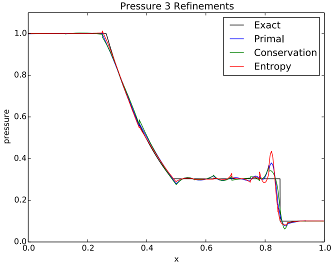


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

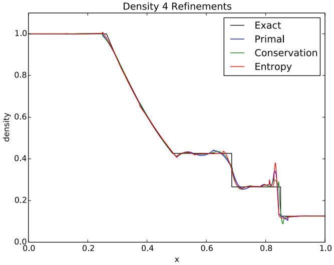


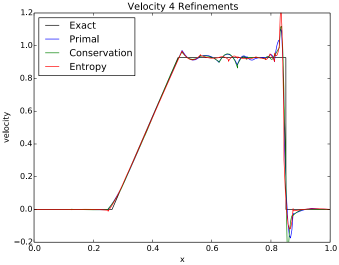


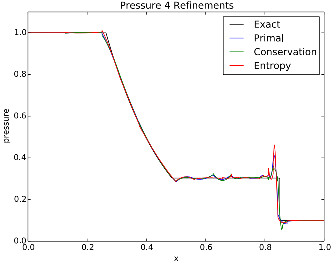


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

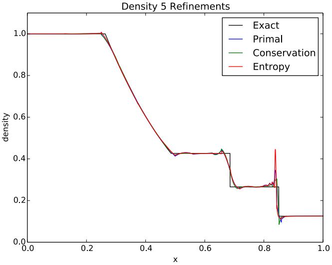


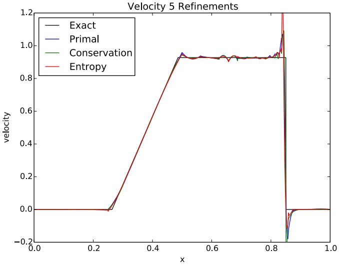


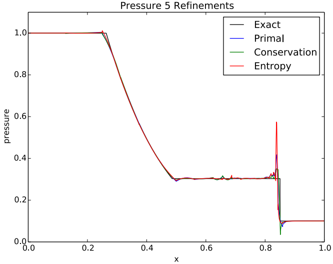


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

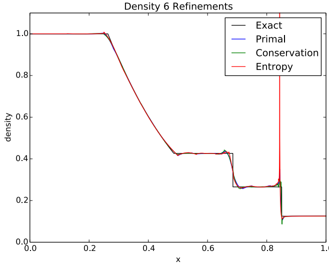


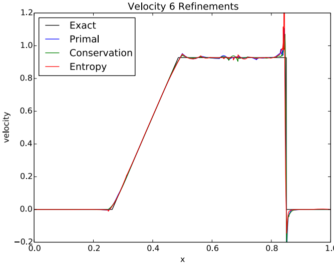


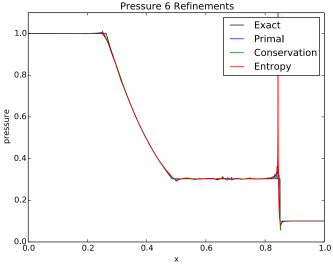


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

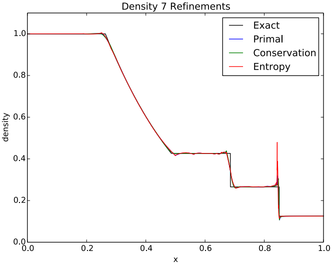


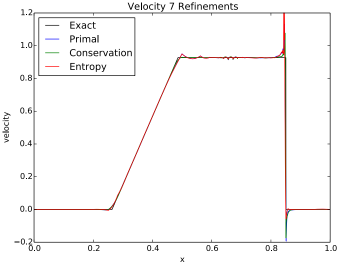


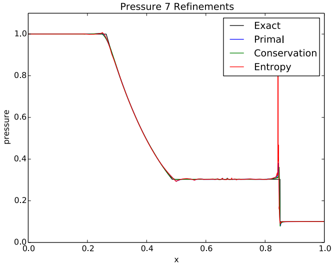


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

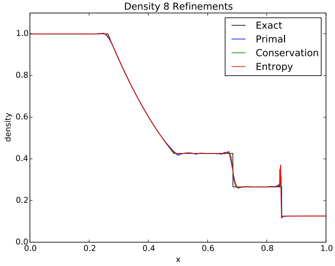


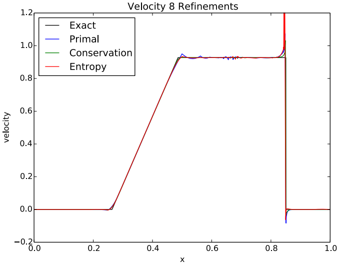


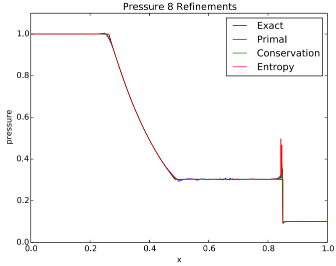


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

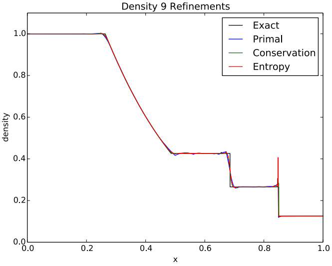


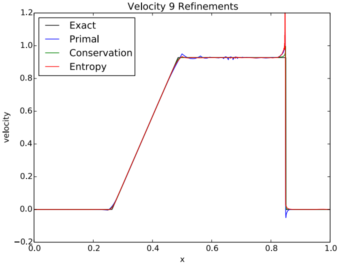


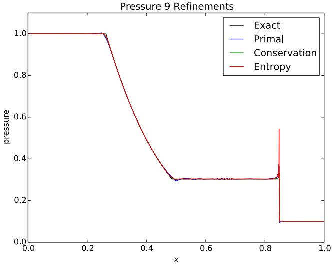


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

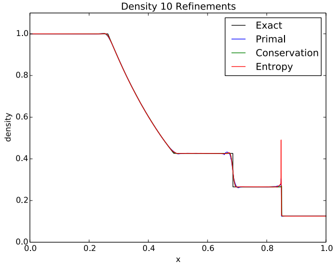


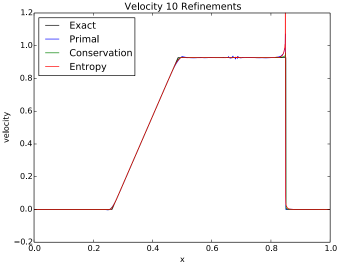


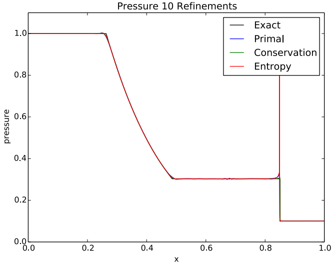


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

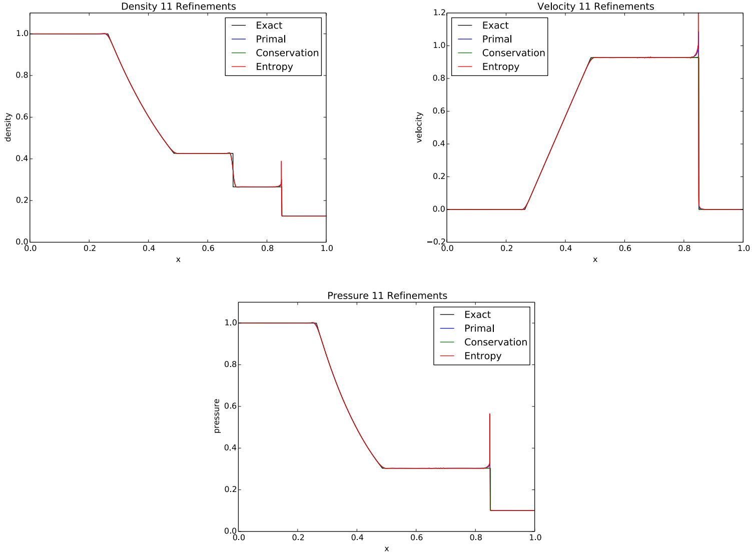






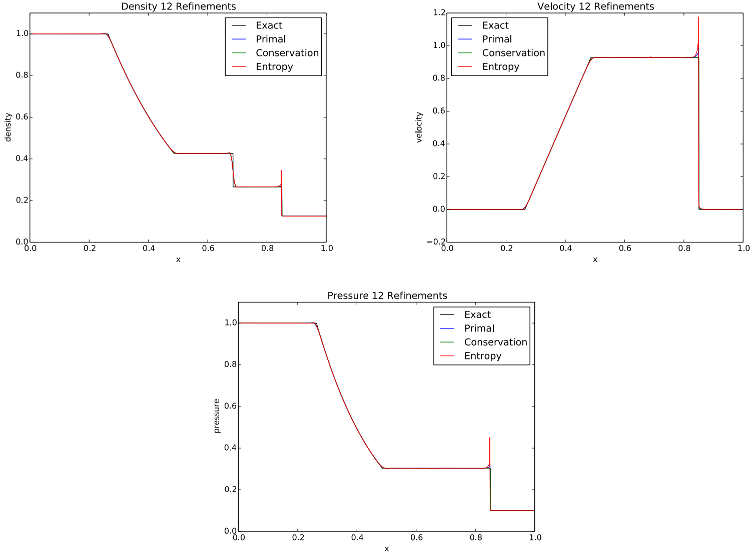
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



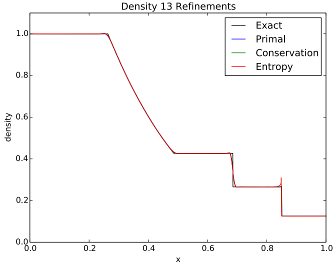
Compressible Navier-Stokes

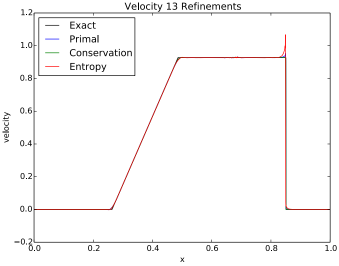
Sod Shock Tube with $\mu = 10^{-5}$

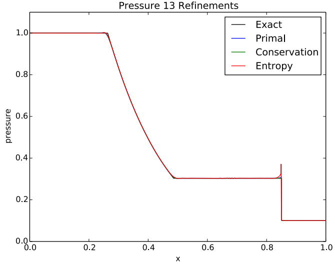


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

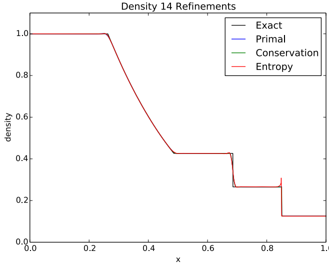


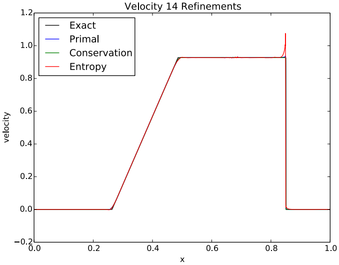


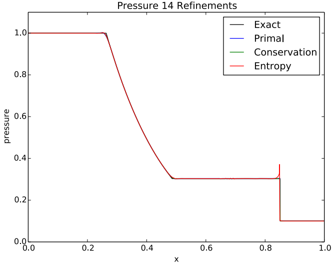


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



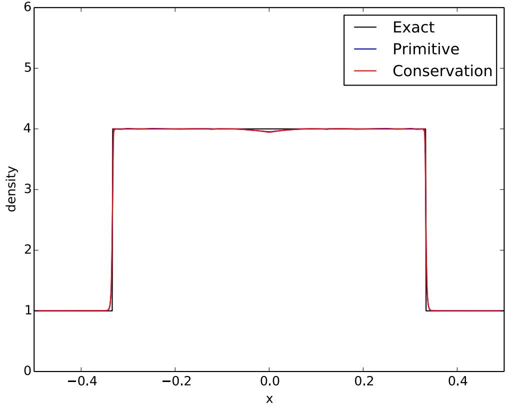




Compressible Navier-Stokes

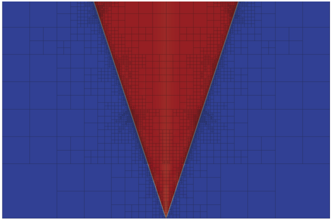
Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.

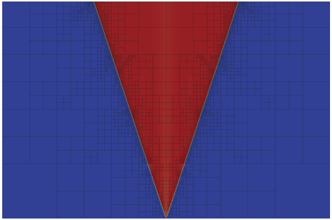


Sequence of 4 Time Slabs

Primitive Variables



Conservation Variables



Thank You!

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