

Space-Time Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems

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Minimum Residual and Least Squares Finite Element Methods
Workshop

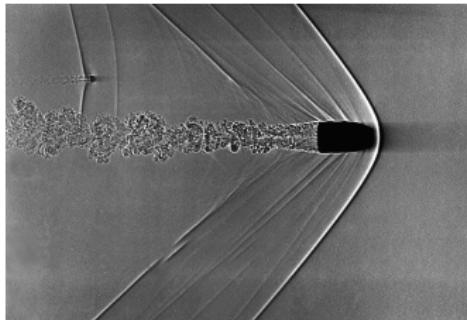


Navier-Stokes Equations

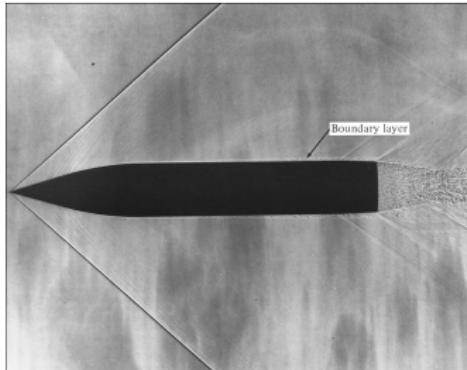
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
 - Shocks
 - Boundary layers - resolution needed for drag/load
 - Turbulence (non-localized)
- Stability of numerical schemes
 - Nonlinearity
 - Nature of PDE changes for different flow regimes
 - Coarse/adaptive grids
 - Higher order



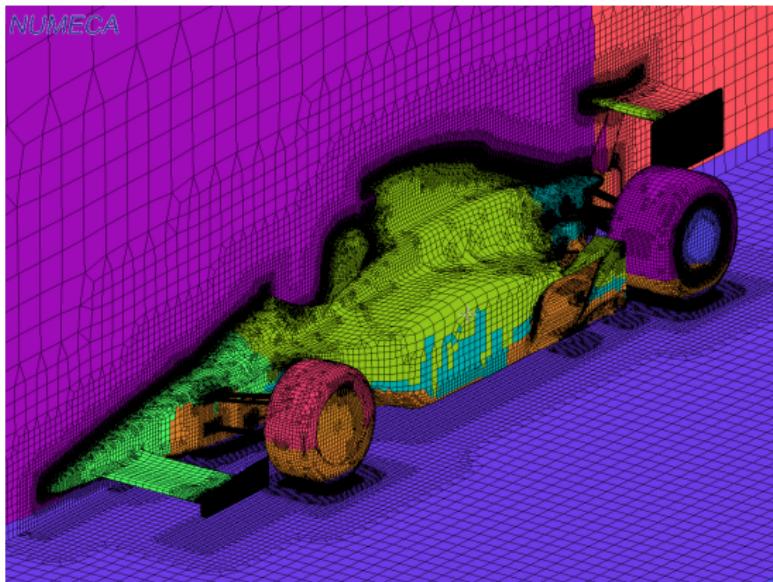
Shock



Motivation

Initial Mesh Design is Expensive and Time-Consuming

- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC

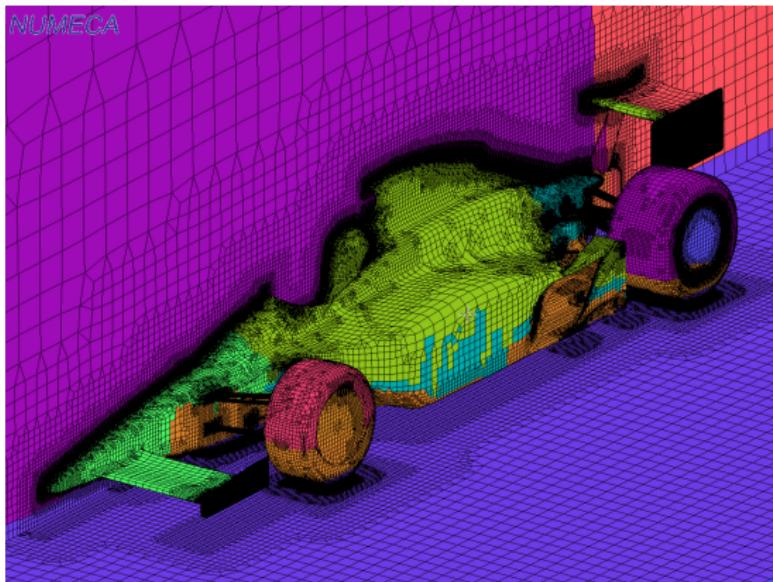


Formula 1 Mesh by Numeca

Motivation

Initial Mesh Design is Expensive and Time-Consuming

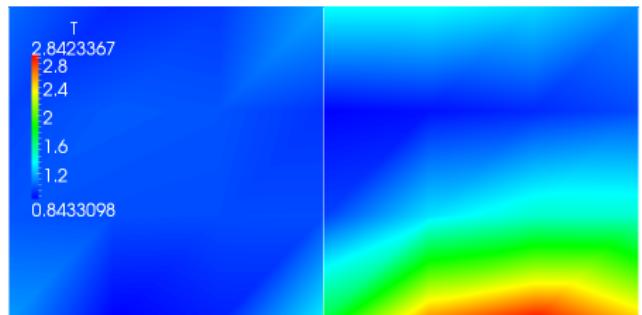
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC
- We desire an automated computational technology



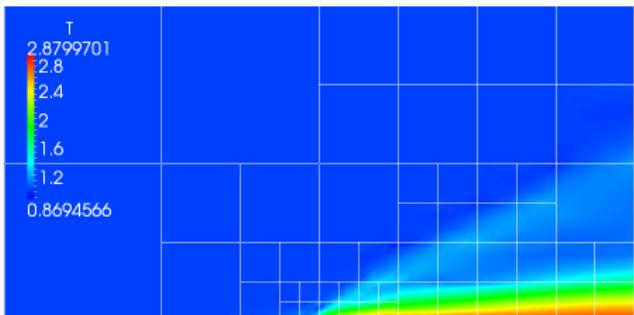
Formula 1 Mesh by Numeca

DPG on Coarse Meshes

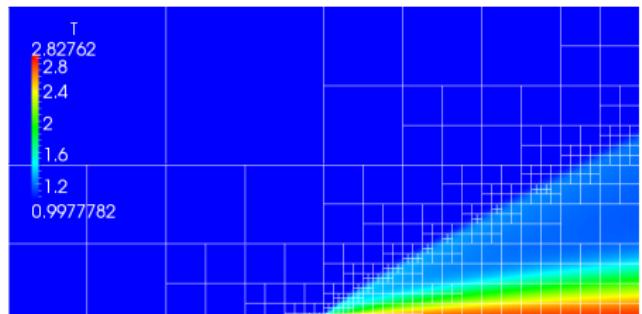
Adaptive Solve of the Carter Plate Problem¹ $Re = 1000$



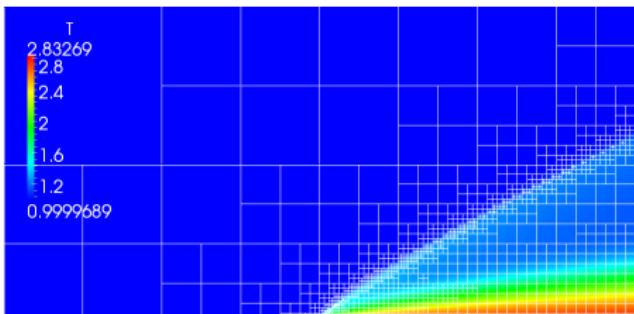
Temperature on Initial Mesh



Temperature after 4 Refinements



Temperature after 8 Refinements



Temperature after 11 Refinements

¹J.L. Chan. "A DPG Method for Convection-Diffusion Problems". PhD thesis. University of Texas at Austin, 2013.

Lessons from Other Methods

Streamline Upwind Petrov-Galerkin: Adaptively changing the test space can produce a method with better stability.

Discontinuous Galerkin: Discontinuous basis functions are a legitimate option for finite element methods.

Hybridized DG: Mesh interface unknowns can facilitate static condensation -- reducing the number of DOFs in the global solve.

Least-Squares FEM: The finite element method is most powerful in a minimum residual context (i.e. as a Ritz method).

Space-Time FEM: Highly adaptive methods should have adaptive time integration. Superior framework for problems with moving boundaries. Requires a method that is both temporally and spatially stable.

Overview of DPG

A Framework for Computational Mechanics

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \rightarrow V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.

This gives the operator equation

$$Bu = l \quad \in V'.$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|R_V^{-1}(Bw_h - l)\|_V^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h .

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

Other Features

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

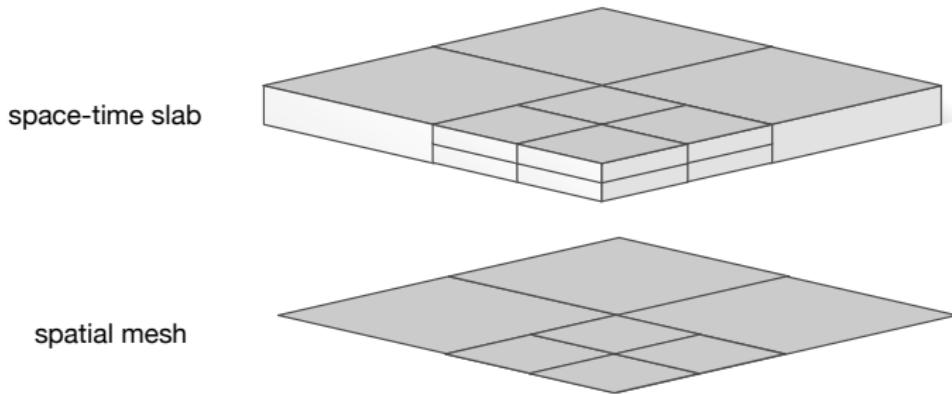
Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \|R_V^{-1}(Bu_h - l)\|_V$$

Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
 - ✓ Unified treatment of space and time
 - ✓ Local space-time adaptivity (local time stepping)
 - ✓ Parallel-in-time integration (space-time multigrid)
 - ✗ Spatially stable FEM methods may not be stable in space-time
 - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\sigma - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\beta u - \sigma) = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned}\frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \beta u - \sigma \\ u \end{pmatrix} &= f\end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\begin{aligned} \left(\frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right)_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle_{\partial K} &= 0 \\ - \left(\begin{pmatrix} \beta u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right)_K + \langle \hat{t}, v \rangle_{\partial K} &= f \end{aligned}$$

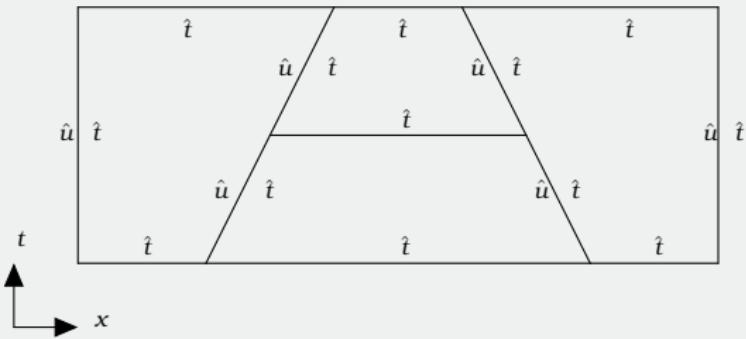
where

$$\hat{u} := \text{tr}(u)$$

$$\begin{aligned} \hat{t} &:= \text{tr}(\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}_x \\ &\quad + \text{tr}(u) \cdot n_t \end{aligned}$$

- Trace \hat{u} defined on spatial boundaries
- Flux \hat{t} defined on all boundaries

Support of Trace Variables



Space-Time Convection-Diffusion

L^2 Equivalent Norms

Bilinear form with group variables:

$$b((u, \hat{u}), v) = (u, A_h^* v)_{L^2(\Omega_h)} + \langle \hat{u}, [v] \rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^* v^* = u$

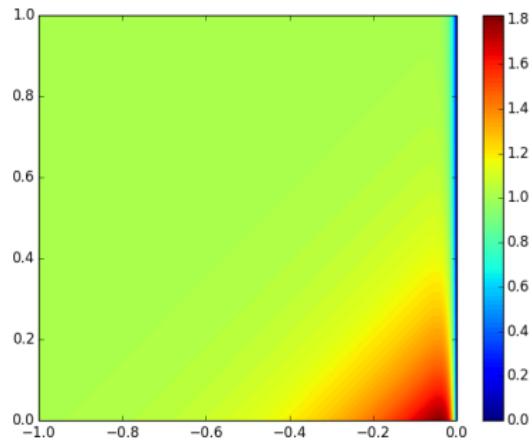
$$\begin{aligned} \|u\|_{L^2(\Omega_h)}^2 &= b(u, v^*) = \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \\ &\leq \sup_{v^* \neq 0} \frac{|b(u, v^*)|}{\|v^*\|} \|v^*\| = \|u\|_E \|v^*\|_V \end{aligned}$$

Necessary robustness condition:

$$\begin{aligned} \|v^*\|_V &\lesssim \|u\|_{L^2(\Omega_h)} \\ \Rightarrow \|u\|_{L^2(\Omega_h)} &\lesssim \|u\|_E \end{aligned}$$

Analytical Solution

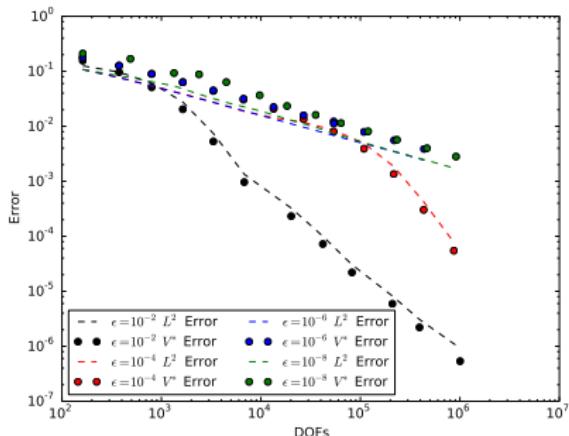
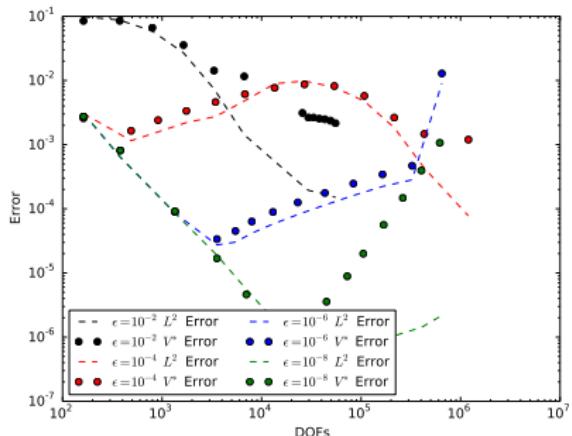
$$e^{-lt} (e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}) + \left(1 - e^{\frac{1}{\epsilon}x}\right)$$



Space-Time Convection-Diffusion

L^2 Equivalent Norms

A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.



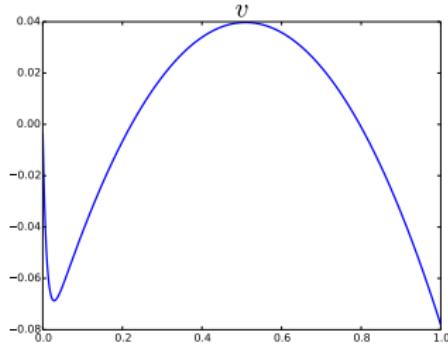
$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &\quad + \left\| \frac{1}{\epsilon} \tau + \nabla v \right\|^2 + \|v\|^2 + \|\tau\|^2 \end{aligned}$$

$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &\quad + \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\tau\|^2 \\ &\quad + \epsilon \|\nabla v\|^2 + \|\beta \cdot \nabla v\|^2 + \|v\|^2 \end{aligned}$$

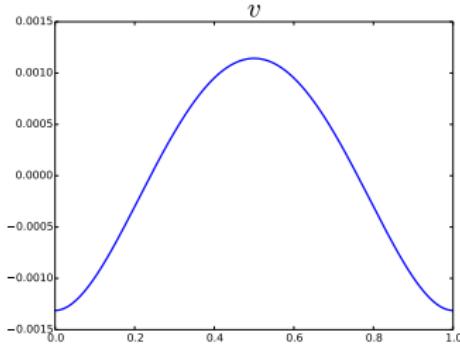
Steady Convection-Diffusion

Ideal Optimal Shape Functions

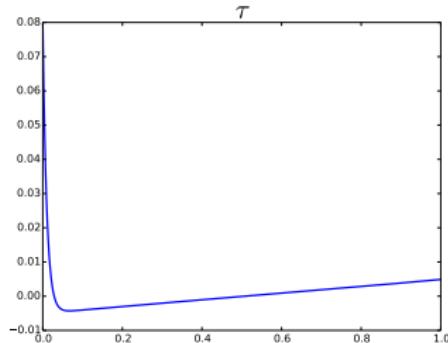
Graph Norm



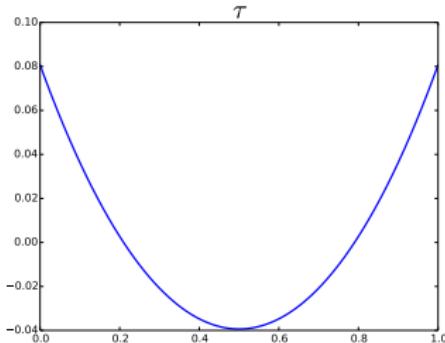
Coupled Robust Norm



τ



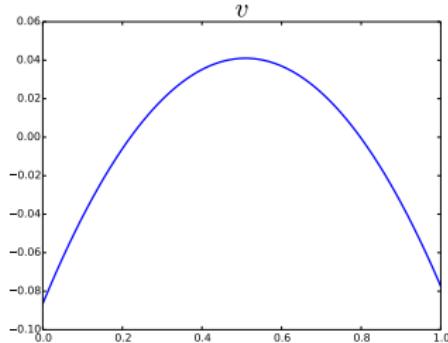
τ



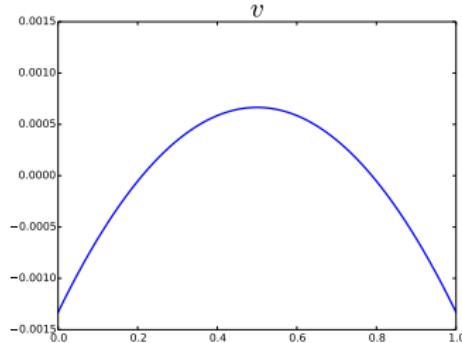
Steady Convection-Diffusion

Approximated ($p = 3$) Optimal Shape Functions

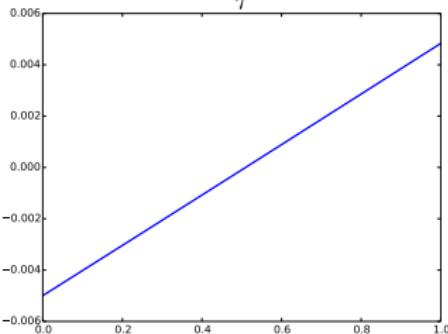
Graph Norm



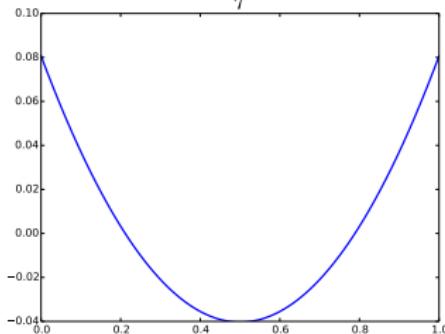
Coupled Robust Norm



τ



τ



Steady Convection-Diffusion

Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.

The robust norm was derived in²:

$$\begin{aligned} \|(v, \tau)\|^2 &= \|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 \\ &\quad + \|\nabla \cdot \tau\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\tau\|^2. \end{aligned}$$

The case for the coupled robust norm was made in³:

$$\begin{aligned} \|(v, \tau)\|^2 &= \|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 \\ &\quad + \|\nabla \cdot \tau - \beta \cdot \nabla v\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\tau\|^2. \end{aligned}$$

²J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771–795.

³J.L. Chan. "A DPG Method for Convection-Diffusion Problems". PhD thesis. University of Texas at Austin, 2013.

Space-Time Convection-Diffusion

Two Robust Norms for Transient Convection-Diffusion

Let $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$ and $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$.

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$\begin{aligned} \|(\boldsymbol{v}, \boldsymbol{\tau})\|^2 &= \left\| \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \epsilon \|\nabla \boldsymbol{v}\|^2 + \min \left(\frac{\epsilon}{h^2}, 1 \right) \|\boldsymbol{v}\|^2 \\ &\quad + \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2. \end{aligned}$$

Coupled Robust Norm

$$\begin{aligned} \|(\boldsymbol{v}, \boldsymbol{\tau})\|^2 &= \left\| \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \epsilon \|\nabla \boldsymbol{v}\|^2 + \min \left(\frac{\epsilon}{h^2}, 1 \right) \|\boldsymbol{v}\|^2 \\ &\quad + \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \min \left(\frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2. \end{aligned}$$

Robust Norms for Transient Convection-Diffusion

Adjoint Operator

Consider the problem with homogeneous boundary conditions

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \tilde{\beta} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \beta_n u - \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

The adjoint operator A^* is given by

$$A^*(v, \boldsymbol{\tau}) = \left(\frac{1}{\epsilon} \boldsymbol{\tau} + \nabla v, -\tilde{\beta} \cdot \nabla_{xt} v + \nabla \cdot \boldsymbol{\tau} \right).$$

Robust Norms for Transient Convection-Diffusion

Controlling Different Field Variables

We decompose the continuous adjoint problem $A^*(\tau, v) = (\sigma, u)$ into

Continuous part with forcing u

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\tau}_1 + \nabla v_1 &= 0 & \boldsymbol{\tau}_1 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ -\tilde{\beta} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= u & v_1 &= 0 \text{ on } \Gamma_+ \\ && v_1 &= 0 \text{ on } \Gamma_T \end{aligned}$$

Continuous part with forcing σ

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\tau}_2 + \nabla v_2 &= \sigma & \boldsymbol{\tau}_2 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ -\tilde{\beta} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0 & v_2 &= 0 \text{ on } \Gamma_+ \\ && v_2 &= 0 \text{ on } \Gamma_T \end{aligned}$$

Robust Norms for Transient Convection-Diffusion

Proved Bounds at Our Disposal

Proofs of these lemmas can be found in⁴.

Lemma (1)

If $\nabla \cdot \beta = 0$, we can bound

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|u\|^2 + \epsilon \|\sigma\|^2$$

where $v = v_1 + v_2$.

Lemma (2)

If $\|\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}\|_{L^\infty} \leq C_\beta$, we can bound

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v_1 \right\| \lesssim \|u\|.$$

⁴T.E. Ellis, J.L. Chan, and L.F. Demkowicz. Robust DPG Methods for Transient Convection-Diffusion. Tech. rep. 15-21. ICES, Oct. 2015.

Robust Norms for Transient Convection-Diffusion

Control of u

Bound on $\|(v_1, \tau_1)\|$

$$\text{Lemma (2)} \Rightarrow \|\tilde{\beta} \cdot \nabla_{xt} v_1\| \lesssim \|u\|$$

$$\text{Lemma (2)} \Rightarrow \|\nabla \cdot \tau_1\| \leq \|u\| + \|\tilde{\beta} \cdot \nabla_{xt} v_1\| \lesssim 2\|u\|$$

$$\text{Lemma (2)} \Rightarrow \|\nabla \cdot \tau_1 - \tilde{\beta} \cdot \nabla_{xt} v_1\| = \|u\|$$

$$\text{Lemma (1)} \Rightarrow \|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|u\|^2$$

$$\text{Lemma (1)} \Rightarrow \frac{1}{\epsilon} \|\tau_1\| = \epsilon \|\nabla v_1\| \leq \|u\|$$

We can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \sigma)\|_E .$$

Robust Norms for Transient Convection-Diffusion

Control of σ

Bound on $\|(v_2, \tau_2)\|$

$$\text{Definition} \Rightarrow \left\| \nabla \cdot \tau_2 - \tilde{\beta} \cdot \nabla_{xt} v_2 \right\| = 0 \leq \|\sigma\|$$

$$\text{Lemma (1)} \Rightarrow \|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\sigma\|^2$$

$$\text{Lemma (1)} \Rightarrow \frac{1}{\epsilon} \|\tau_2\| = \|\sigma\| + \epsilon \|\nabla v_2\| = (1 + \epsilon) \|\sigma\|$$

We have not been able to prove bounds on $\left\| \tilde{\beta} \cdot \nabla_{xt} v_2 \right\|$ or $\|\nabla \cdot \tau_2\|$.

We can **not** guarantee robust control

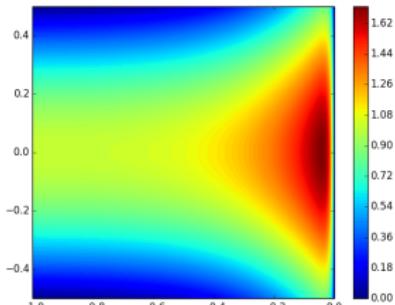
$$\|(0, \sigma)\|_{L^2(\Omega_h)} \not\lesssim \|(u, \sigma)\|_E.$$

Robust Norms for Transient Convection-Diffusion

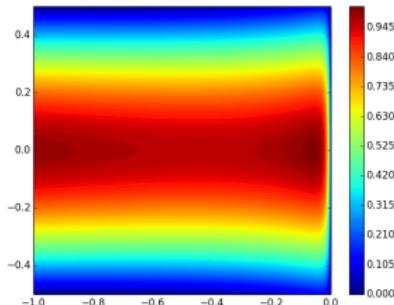
Transient Analytical Solution

Transient impulse decays to Eriksson-Johnson steady state solution.

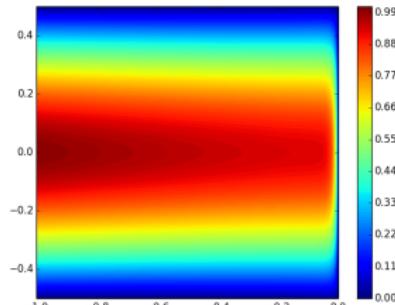
$$u = \exp(-lt) [\exp(\lambda_1 x) - \exp(\lambda_2 x)] + \cos(\pi y) \frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$



$t = 0.0$



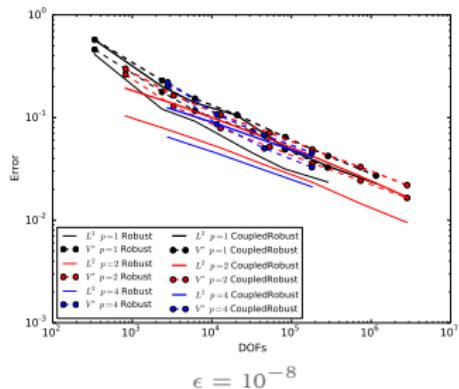
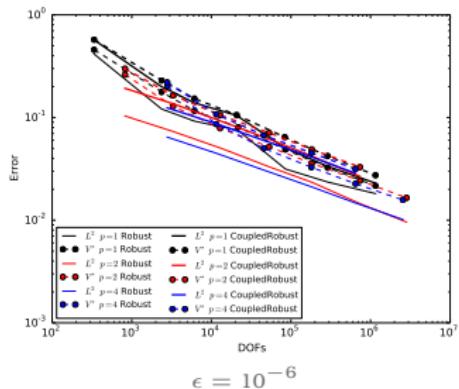
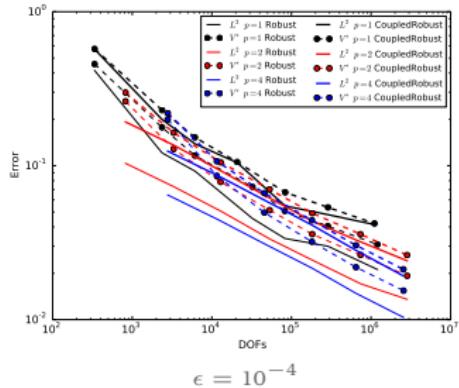
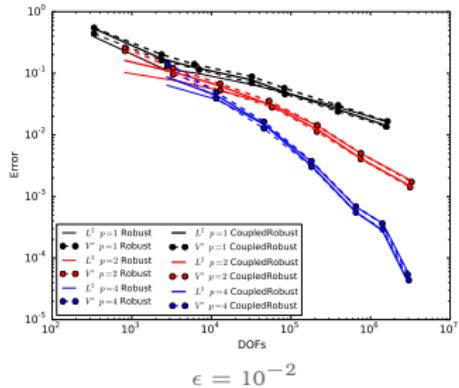
$t = 0.5$



$t = 1.0$

Robust Norms for Transient Convection-Diffusion

Robust Convergence to Analytical Solution



Space-Time Compressible Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$\frac{1}{\mu} \mathbb{D} - (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{2}{3} \nabla \cdot \mathbf{u} \mathbb{I} = 0$$

$$\frac{Pr}{C_p \mu} \mathbf{q} + \nabla T = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \rho R T \mathbf{u} + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e,$$

Space-Time Compressible Navier-Stokes

Compact Notation

Conserved quantities

$$C_c := \rho$$

$$\mathbf{C}_m := \rho \mathbf{u}$$

$$C_e := \rho(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u})$$

Euler fluxes

$$\mathbf{F}_c := \rho \mathbf{u}$$

$$\mathbb{F}_m := \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I}$$

$$\mathbf{F}_e := \rho \mathbf{u} \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \rho R T \mathbf{u}$$

Viscous fluxes

$$\mathbf{K}_c := \mathbf{0}$$

$$\mathbb{K}_m := \mathbb{D}$$

$$\mathbf{K}_e := -\mathbf{q} + \mathbf{u} \cdot \mathbb{D}$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \frac{1}{\mu} \mathbb{D}$$

$$\mathbf{M}_q := \frac{Pr}{C_p \mu} \mathbf{q}$$

Viscous relations

$$\mathbf{G}_{\mathbb{D}} := 2\mathbf{u}$$

$$G_q := -T$$

Use change of variables to get conservation or entropy variables.

Space-Time Compressible Navier-Stokes

Conservation Variables

Change of variables:

$$\rho = \rho$$

$$\mathbf{m} = \rho \mathbf{u}$$

$$E = \rho \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right)$$

Euler fluxes:

$$\mathbf{F}_c^c = \mathbf{m}$$

$$\mathbb{F}_m^c = \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} + (\gamma - 1) \left(E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \mathbf{I}$$

$$\mathbf{F}_e^c = \gamma E \frac{\mathbf{m}}{\rho} - (\gamma - 1) \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho^2} \mathbf{m}$$

Space-Time Compressible Navier-Stokes

Entropy Variables

Change of variables:

$$V_c = \frac{-E + (E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}) \left(\gamma + 1 - \ln \left[\frac{(\gamma-1)(E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m})}{\rho^\gamma} \right] \right)}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

$$\mathbf{V}_m = \frac{\mathbf{m}}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

$$V_e = \frac{-\rho}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

Euler fluxes:

$$\mathbf{F}_c^e = \left[\frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[\frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \mathbf{V}_m$$

$$\mathbb{F}_m^e = \left[\frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[\frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \left(-\frac{\mathbf{V}_m \otimes \mathbf{V}_m}{V_e} + (\gamma - 1) \mathbf{I} \right)$$

$$\mathbf{F}_e^e = \left[\frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[\frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \frac{\mathbf{V}_m}{V_e} \left(\frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m - \gamma \right)$$

Space-Time Compressible Navier-Stokes

Define Group Variables

Group terms

$$C := \{C_c, \mathbf{C}_m, C_e\}$$

$$F := \{\mathbf{F}_c, \mathbb{F}_m, \mathbf{F}_e\}$$

$$K := \{\mathbf{K}_c, \mathbb{K}_m, \mathbf{K}_e\}$$

$$M := \{\mathbb{M}_{\mathbb{D}}, \mathbf{M}_{\mathbf{q}}\}$$

$$G := \{\mathbf{G}_{\mathbb{D}}, G_{\mathbf{q}}\}$$

$$f := \{f_c, \mathbf{f}_m, f_e\}$$

Group variables

$$W := \{\rho, \mathbf{u}, T\}$$

$$\hat{W} := \{2\hat{\mathbf{u}}, -\hat{T}\}$$

$$\Sigma := \{\mathbb{D}, \mathbf{q}\}$$

$$\hat{t} := \{\hat{t}_e, \hat{\mathbf{t}}_m, , \hat{t}_e\}$$

$$\Psi := \{\mathbb{S}, \tau\}$$

$$V := \{v_c, \mathbf{v}_m, , v_e\} .$$

Navier-Stokes variational formulation is

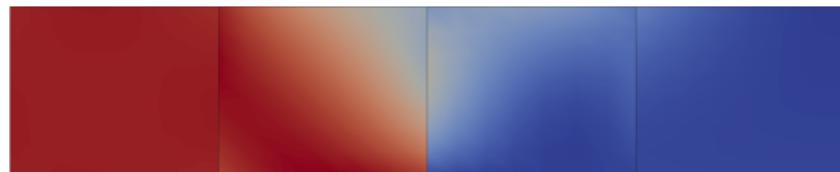
$$(M, \Psi) + (G, \nabla \cdot \Psi) - \langle \hat{W}, \Psi \cdot \mathbf{n}_x \rangle = 0$$

$$- \left(\begin{pmatrix} F - K \\ C \end{pmatrix}, \nabla_{xt} V \right) + \langle \hat{t}, V \rangle = (f, V) .$$

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 1



Primitive Variables



Conservation Variables

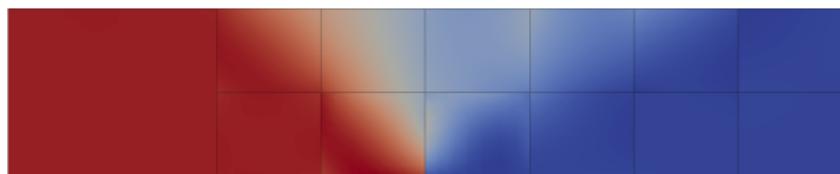


Entropy Variables

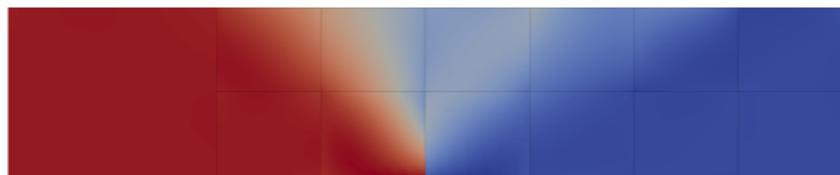
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

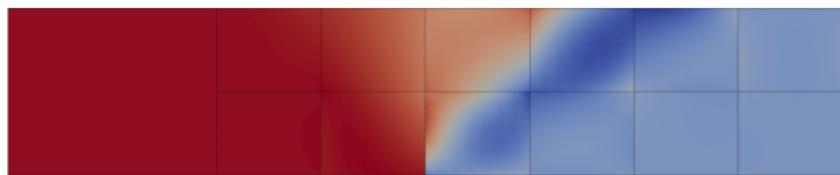
Mesh 2



Primitive Variables



Conservation Variables

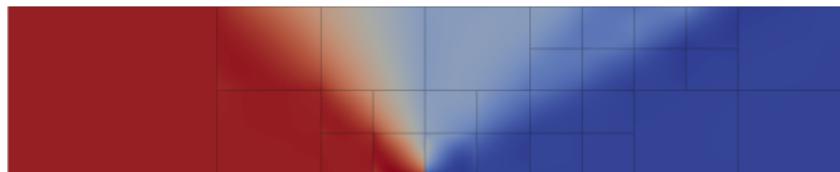


Entropy Variables

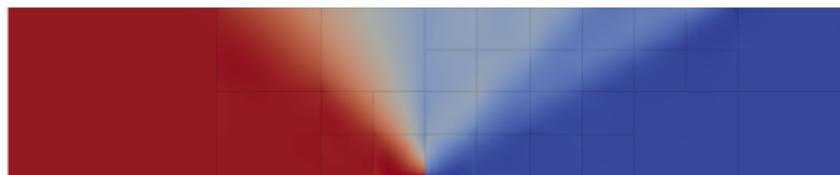
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

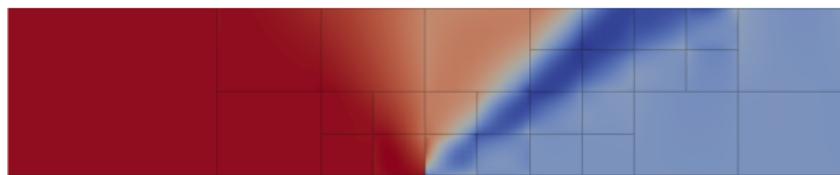
Mesh 3



Primitive Variables



Conservation Variables

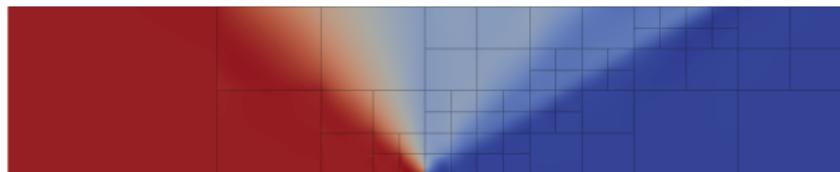


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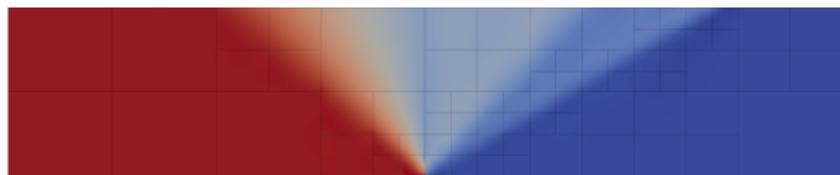
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

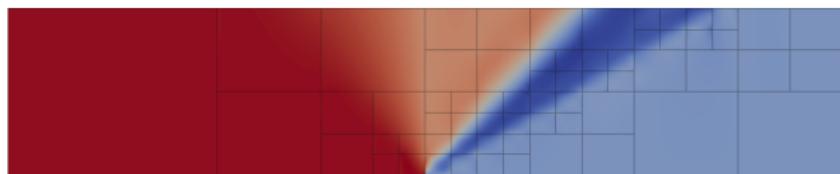
Mesh 4



Primitive Variables



Conservation Variables

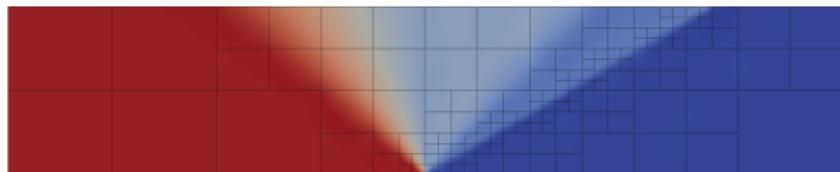


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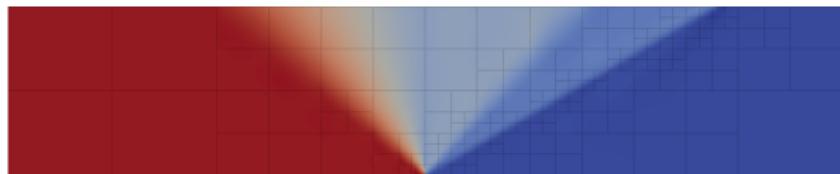
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

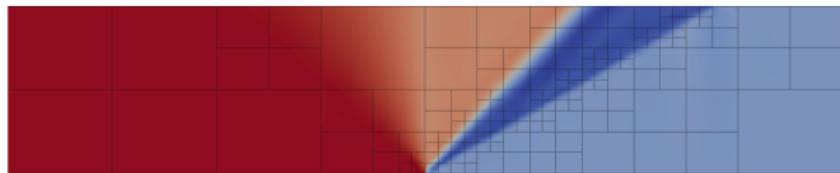
Mesh 5



Primitive Variables



Conservation Variables

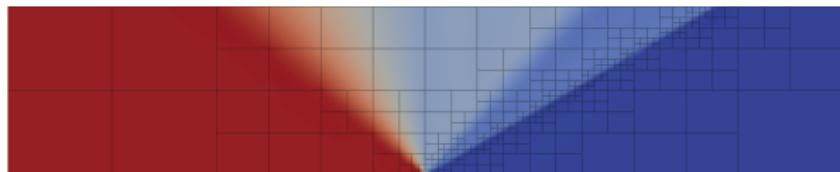


Entropy Variables

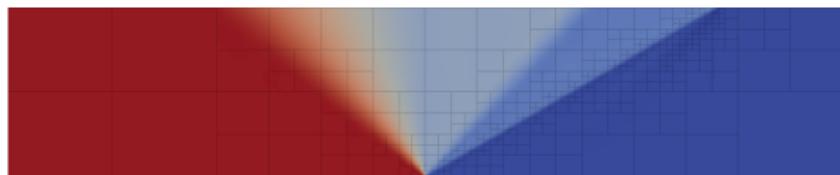
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

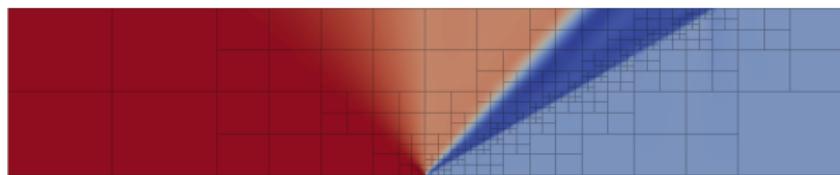
Mesh 6



Primitive Variables



Conservation Variables

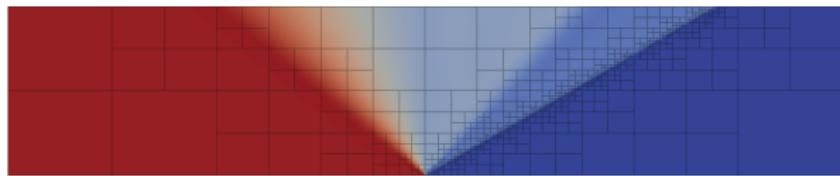


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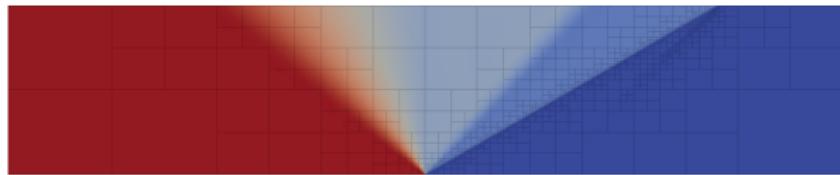
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

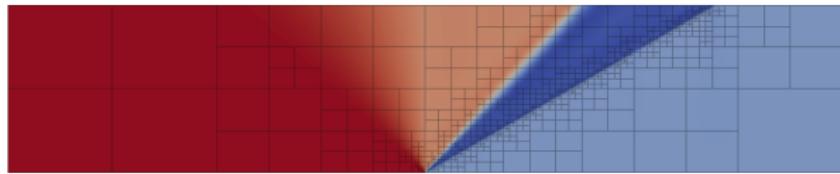
Mesh 7



Primitive Variables



Conservation Variables

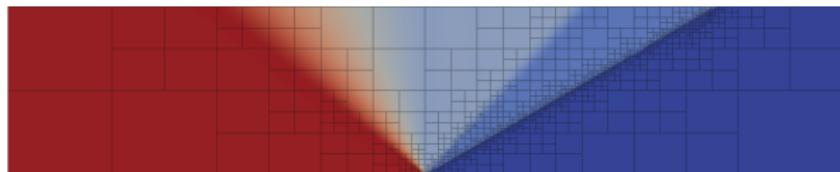


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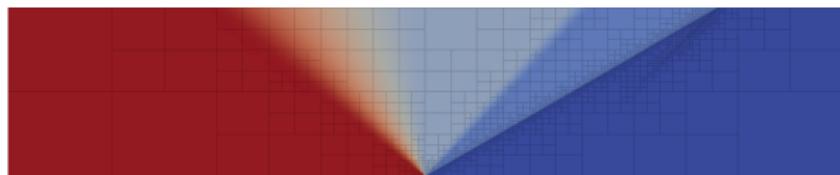
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

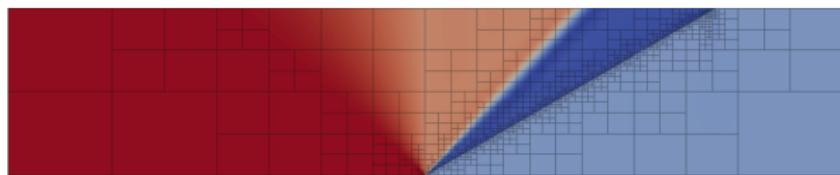
Mesh 8



Primitive Variables



Conservation Variables

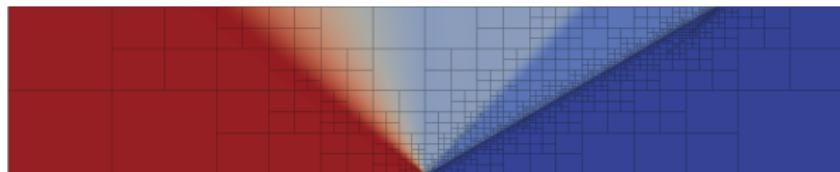


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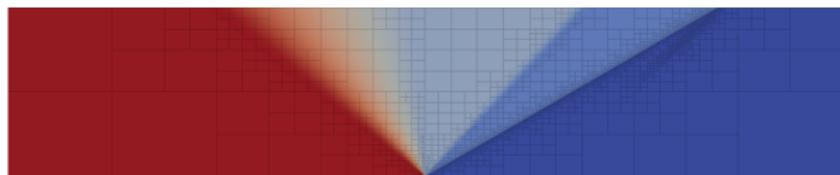
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

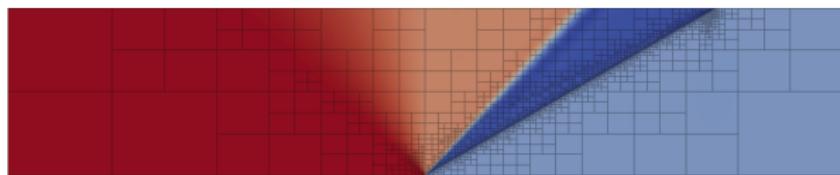
Mesh 9



Primitive Variables



Conservation Variables

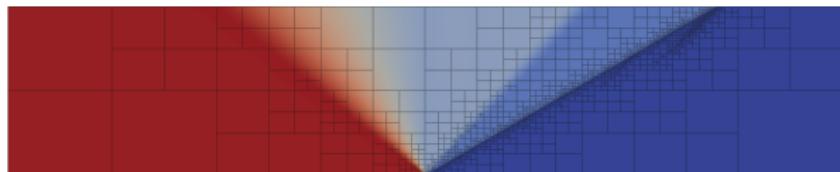


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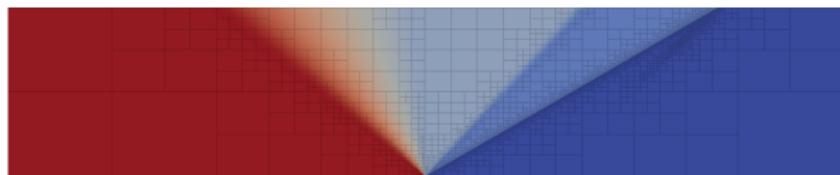
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

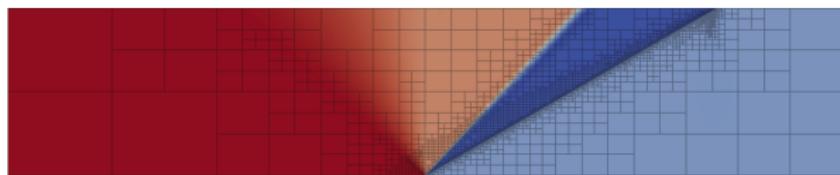
Mesh 10



Primitive Variables



Conservation Variables

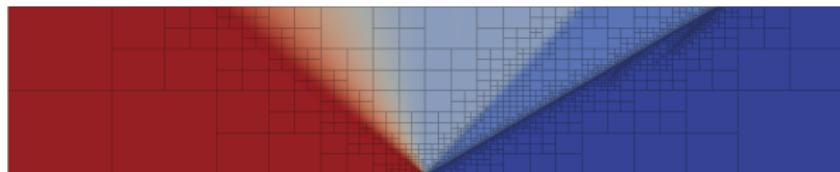


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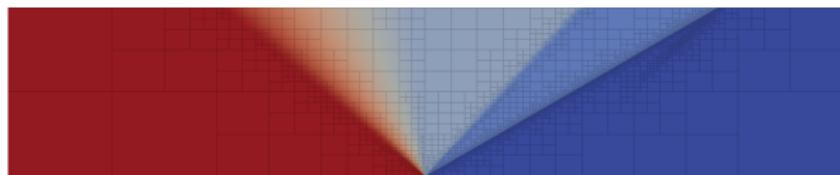
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

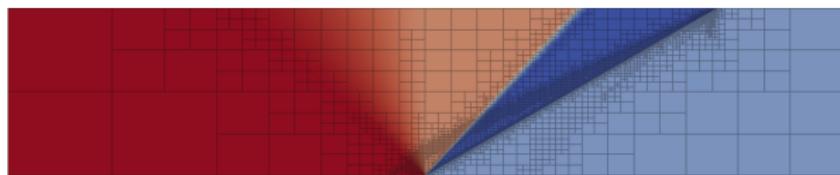
Mesh 11



Primitive Variables



Conservation Variables

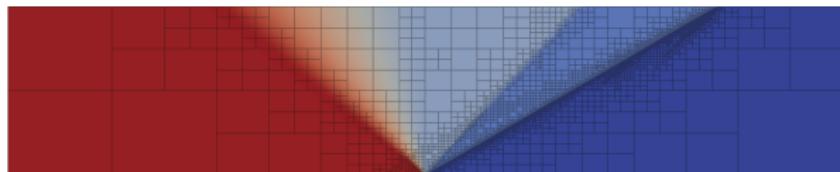


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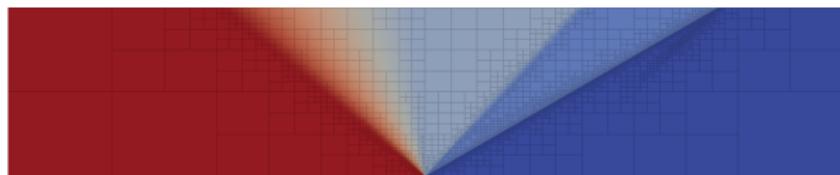
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

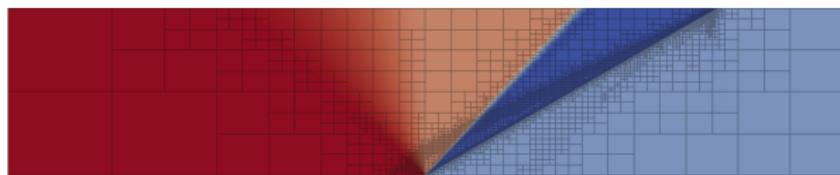
Mesh 12



Primitive Variables



Conservation Variables

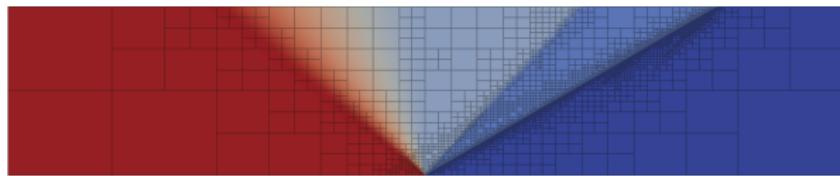


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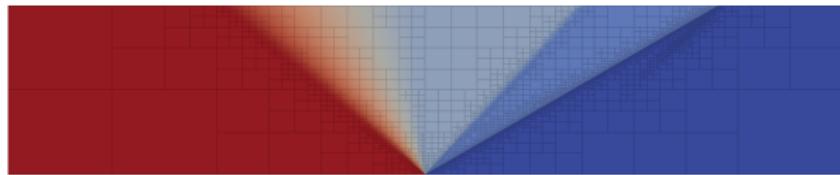
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

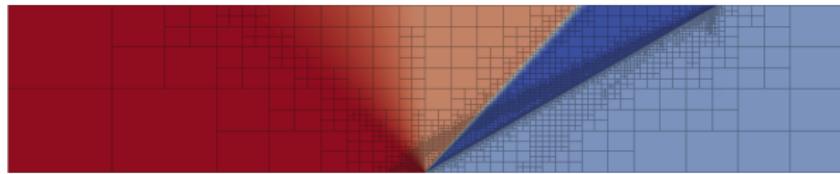
Mesh 13



Primitive Variables



Conservation Variables

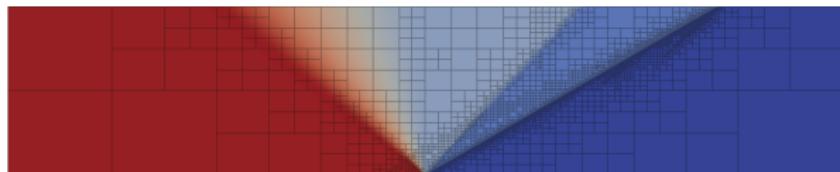


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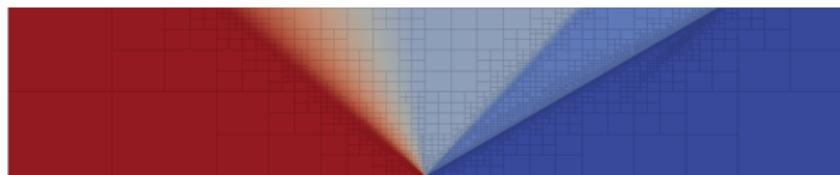
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

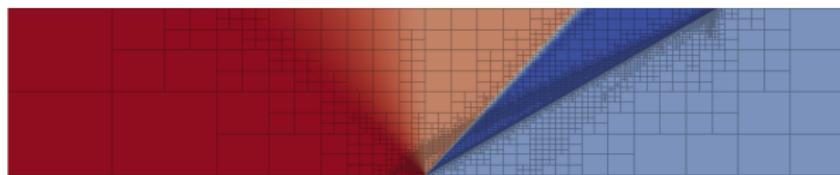
Mesh 14



Primitive Variables



Conservation Variables

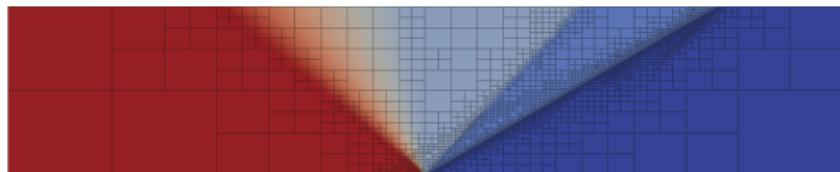


Entropy Variables

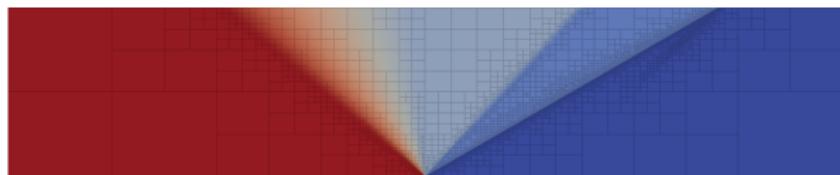
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

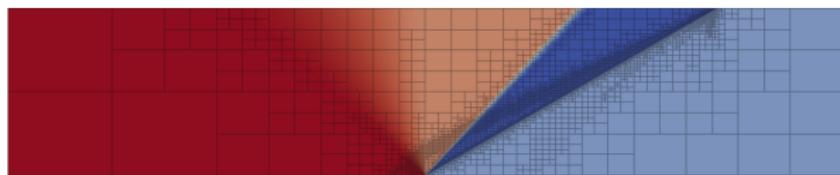
Mesh 15



Primitive Variables



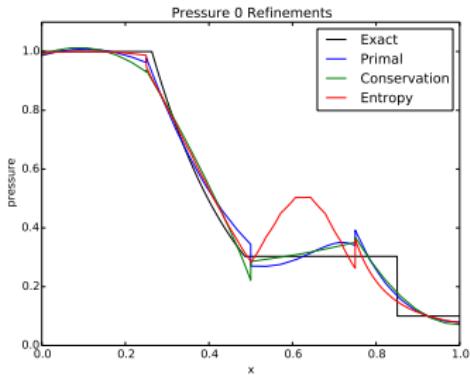
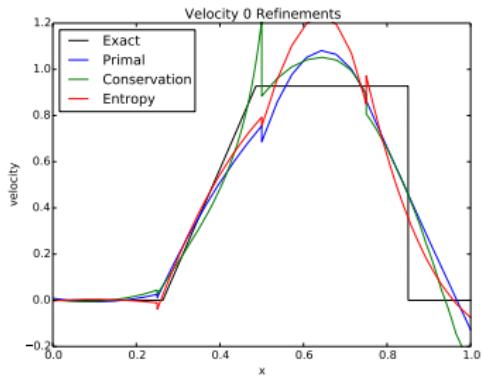
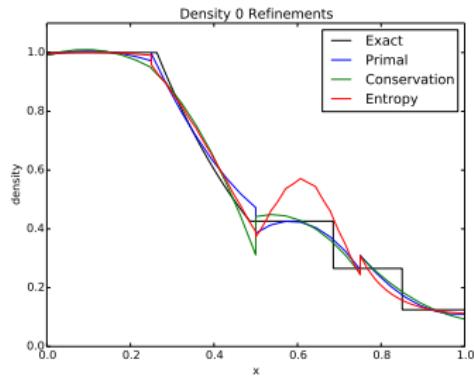
Conservation Variables



Entropy Variables

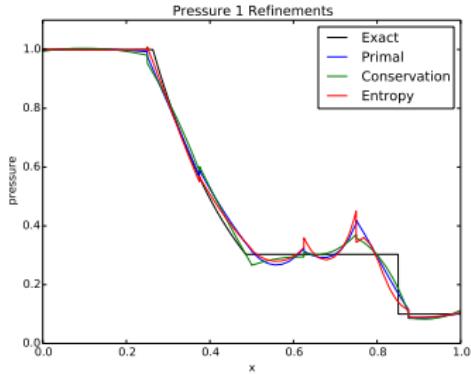
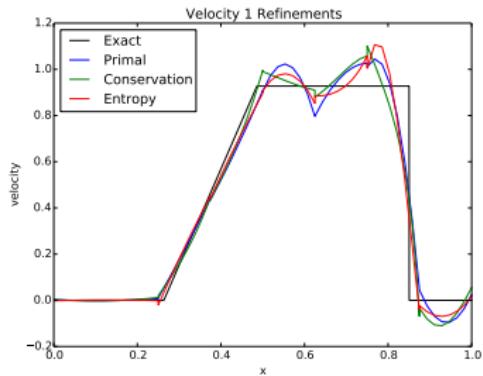
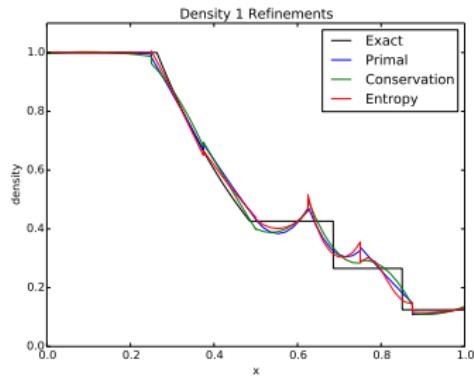
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



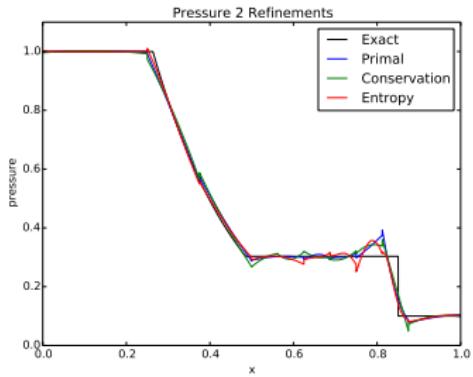
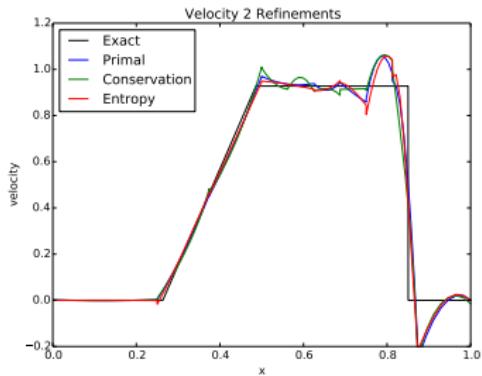
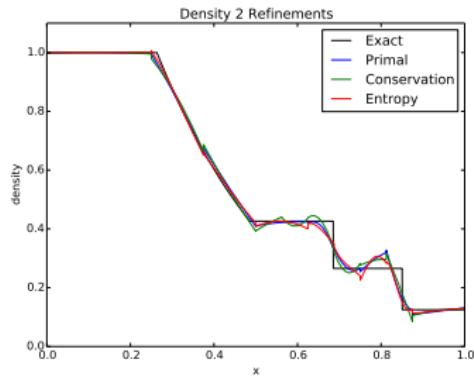
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



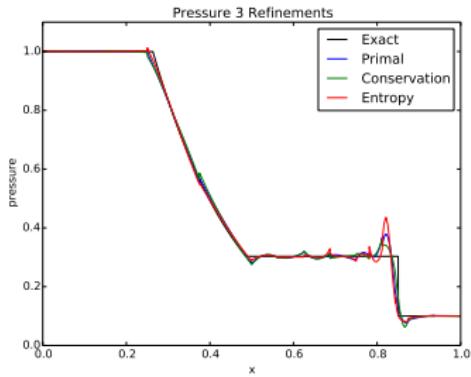
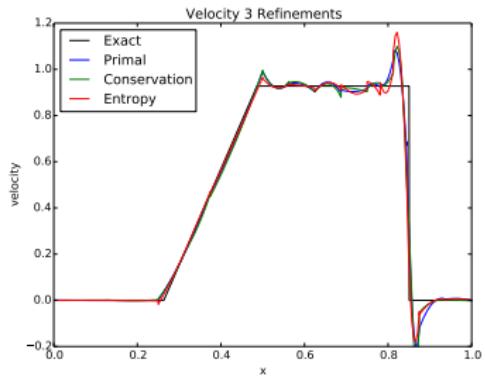
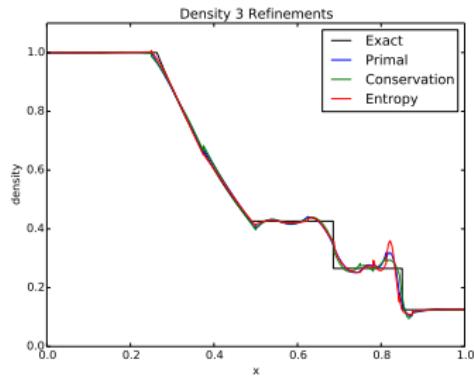
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



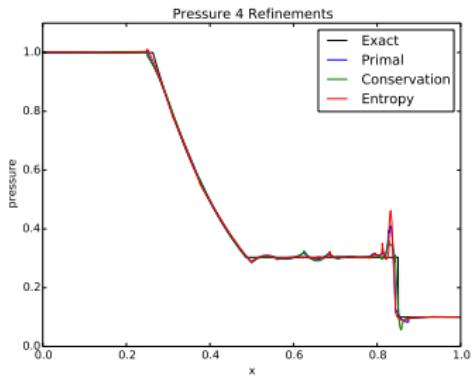
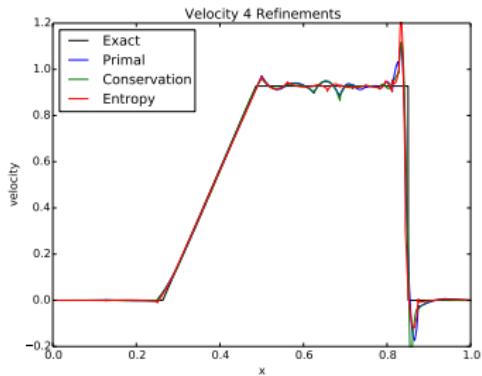
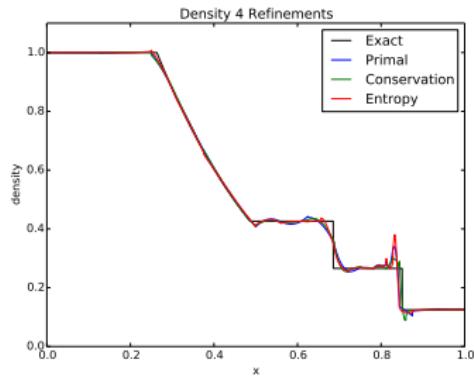
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



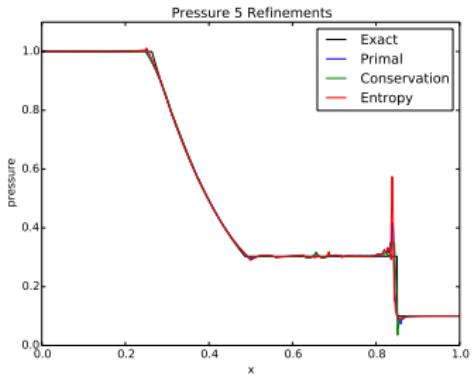
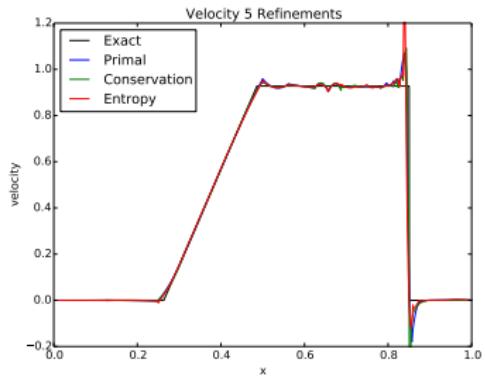
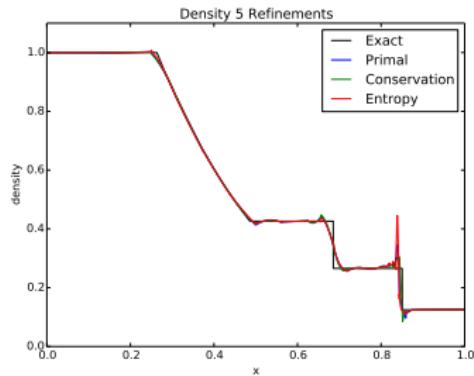
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



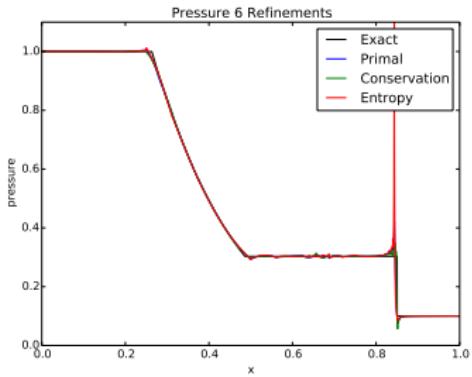
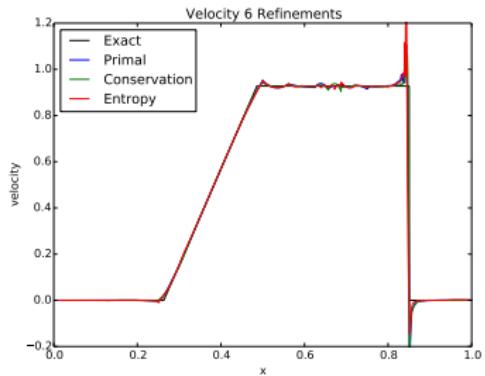
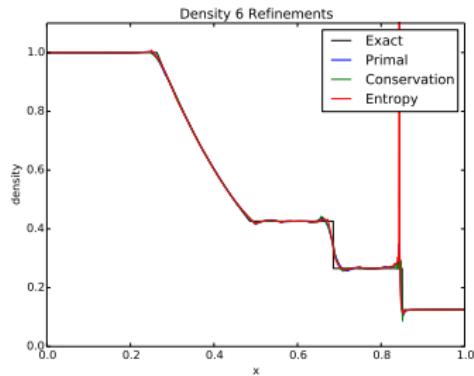
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



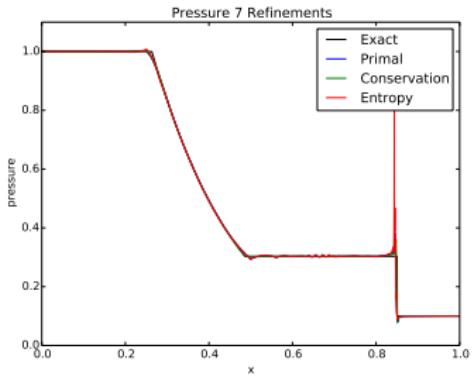
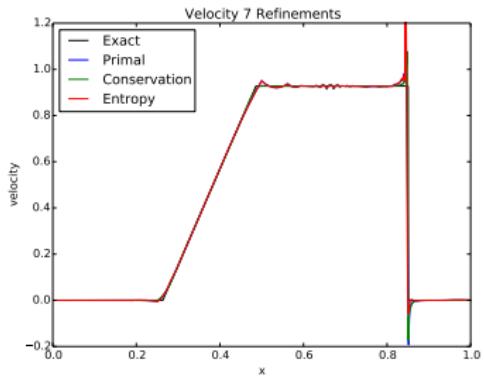
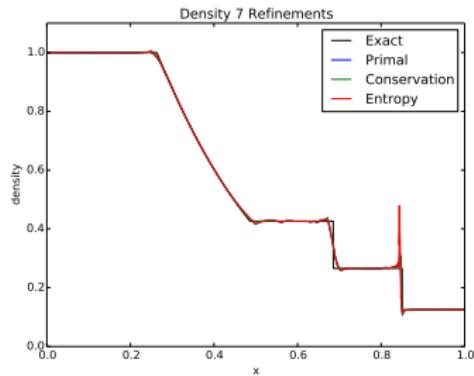
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



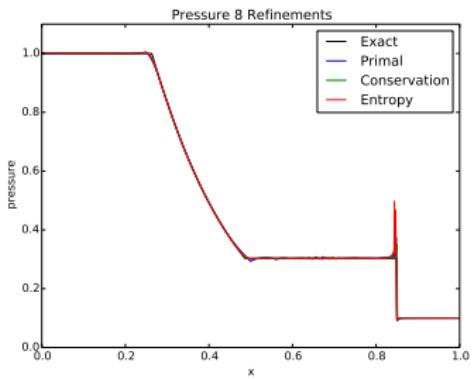
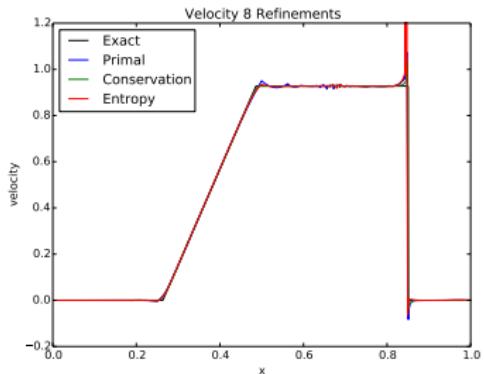
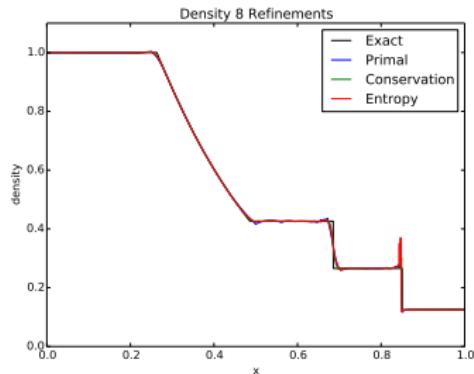
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



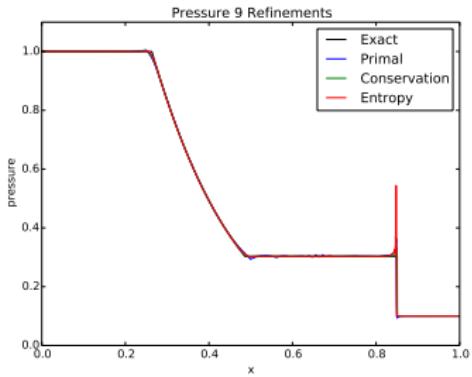
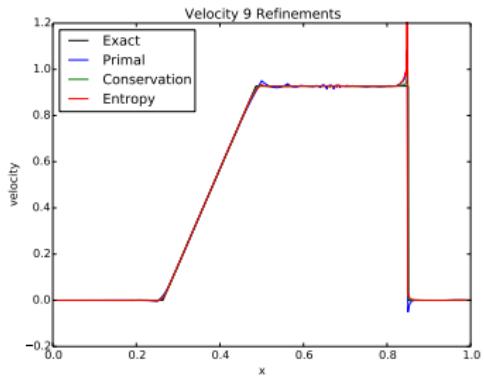
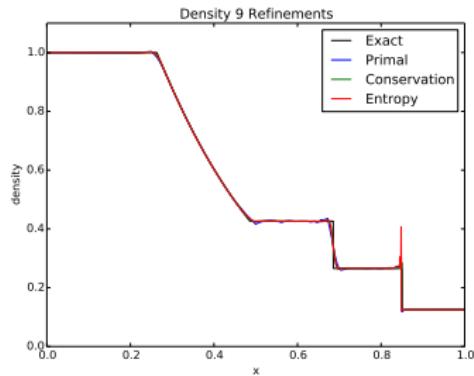
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



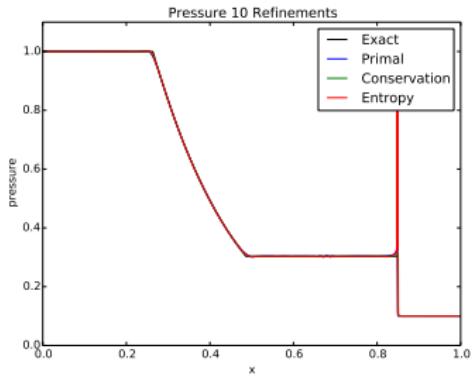
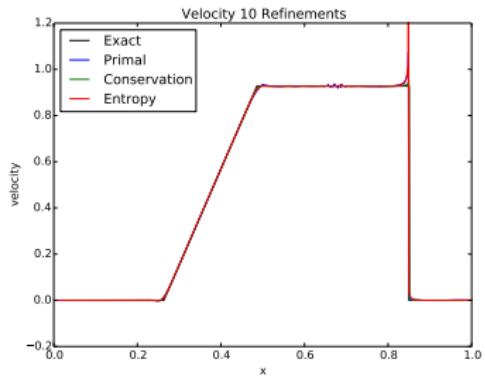
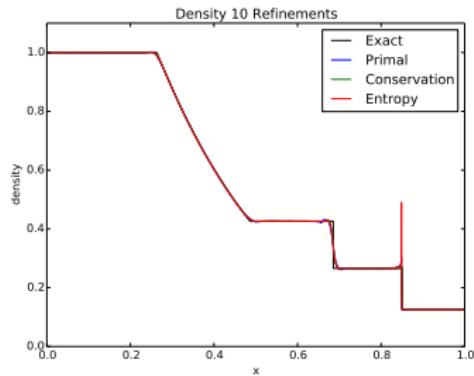
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



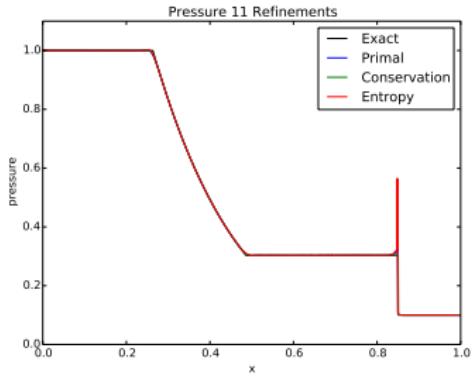
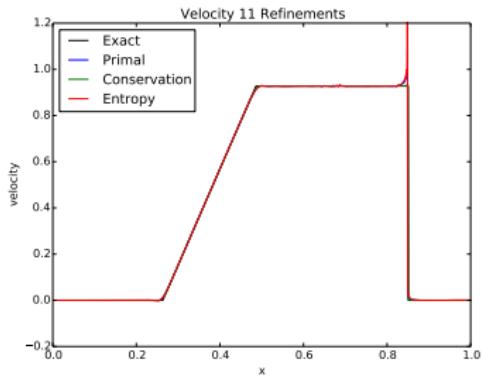
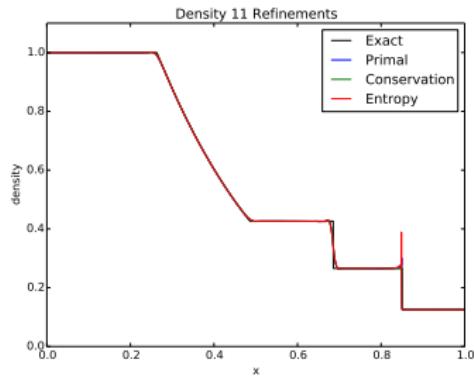
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



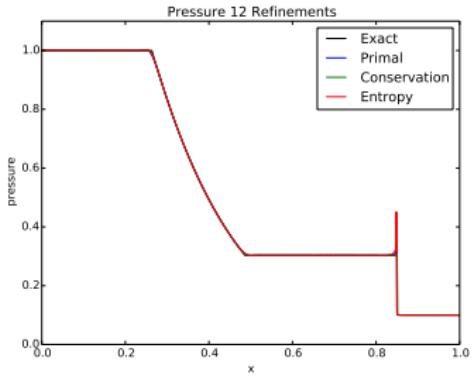
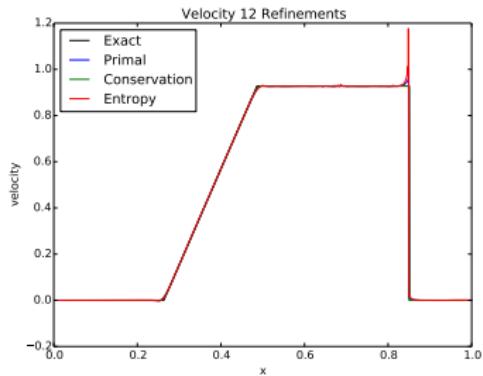
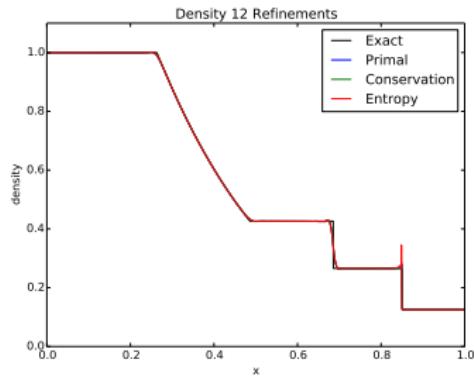
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



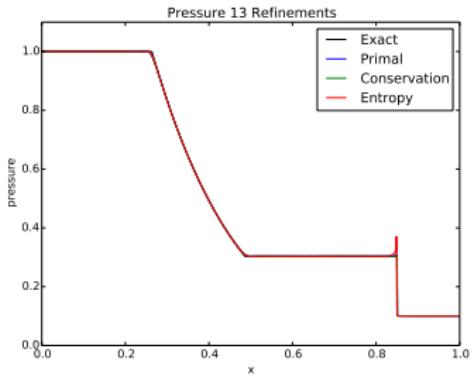
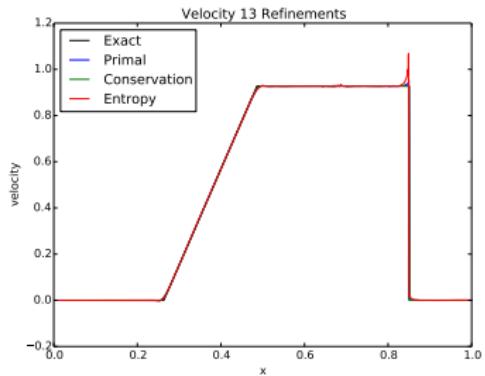
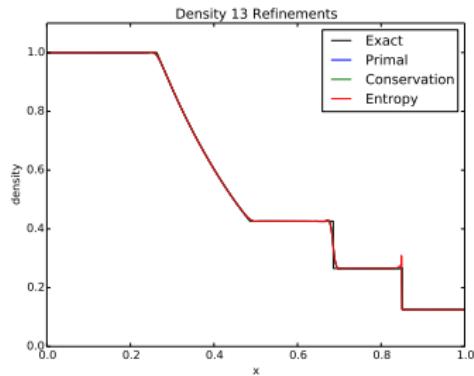
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



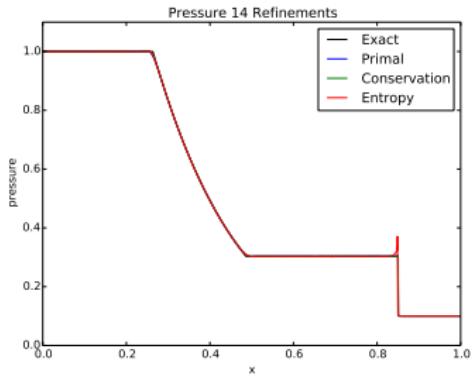
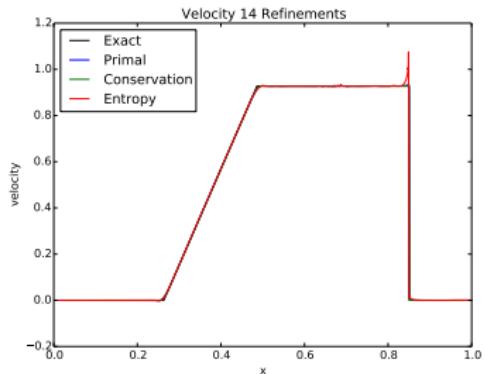
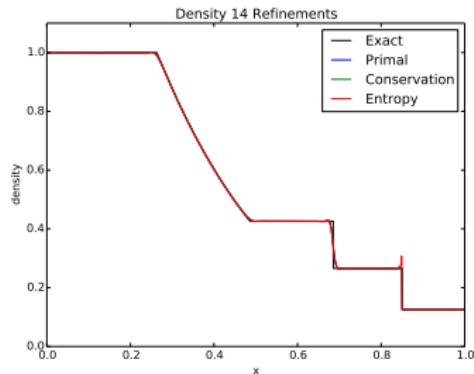
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



Space-Time Compressible Navier-Stokes

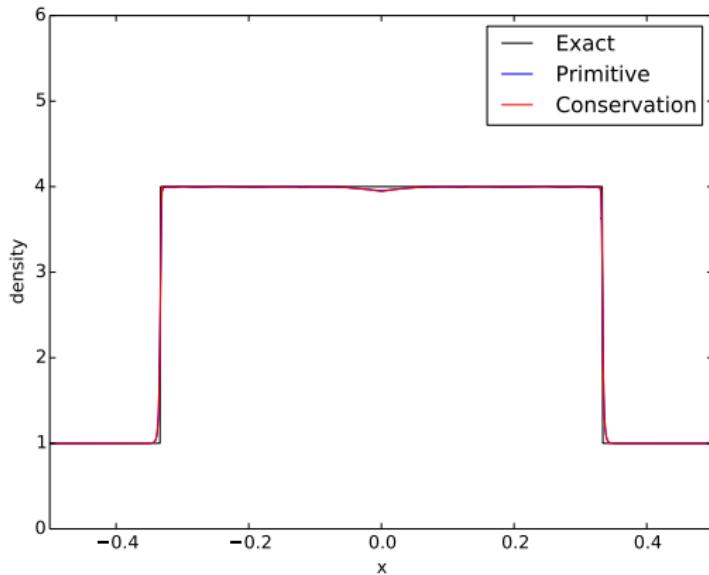
Sod Shock Tube with $\mu = 10^{-5}$



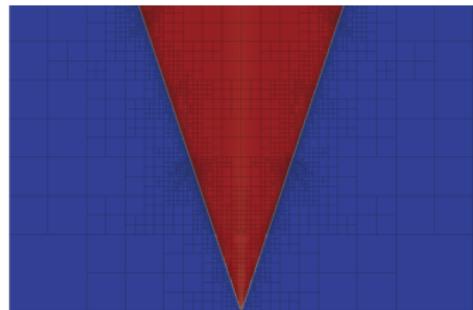
Space-Time Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

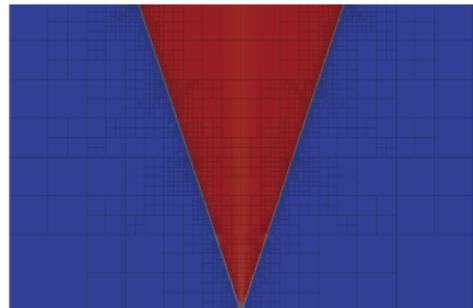
Infinitely strong shock propagation.



Sequence of 4 Time Slabs
Primitive Variables



Conservation Variables



Locally Conservative DPG

Enforce with Lagrange Multipliers

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K f,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al⁽⁴⁾ (also Chang and Nelson⁽⁵⁾), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{t}, 1_K \rangle_{\partial K} - \langle f, 1_K \rangle_K},$$

where $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_N\}$.

Locally Conservative DPG

Enforce with Lagrange Multipliers

Finding the critical points of $L(u, \lambda)$, we get the following equations⁵.

$$\frac{\partial L(u_h, \lambda)}{\partial u_h} = b(u_h, R_V^{-1}B\delta u_h) - l(R_V^{-1}B\delta u_h)$$

$$- \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \lambda)}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

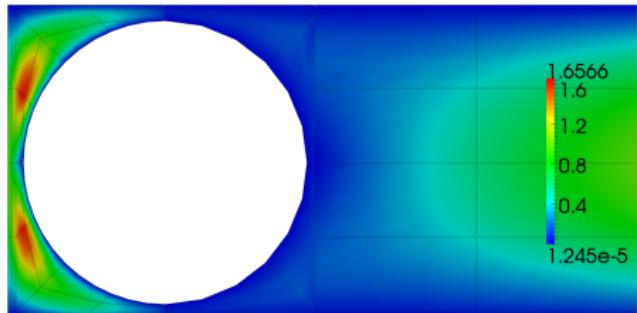
A few consequences:

- We've turned our minimization problem into a saddlepoint problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

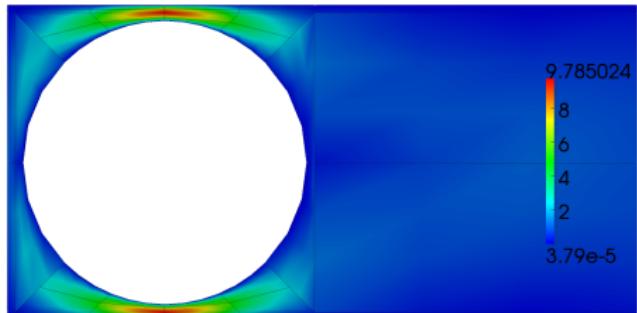
⁵ EllisDemkowiczChan2013.

Locally Conservative DPG

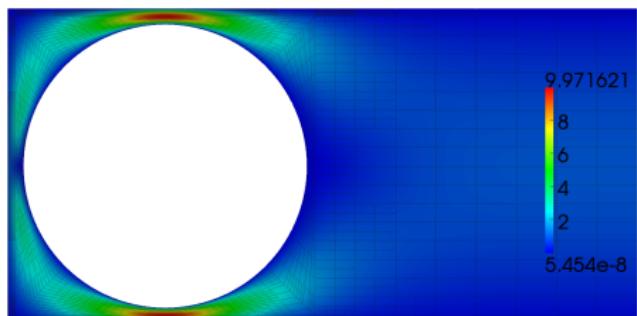
Stokes Flow Around a Cylinder



1 Refinement

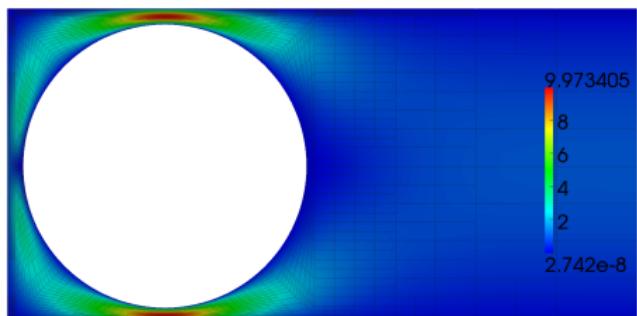


1 Refinement



6 Refinements

Nonconservative

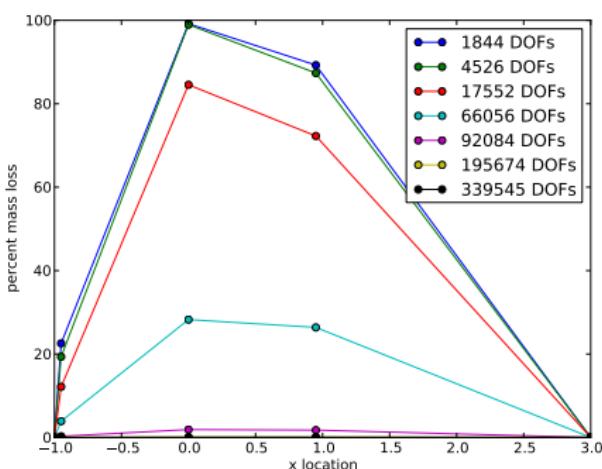


6 Refinements

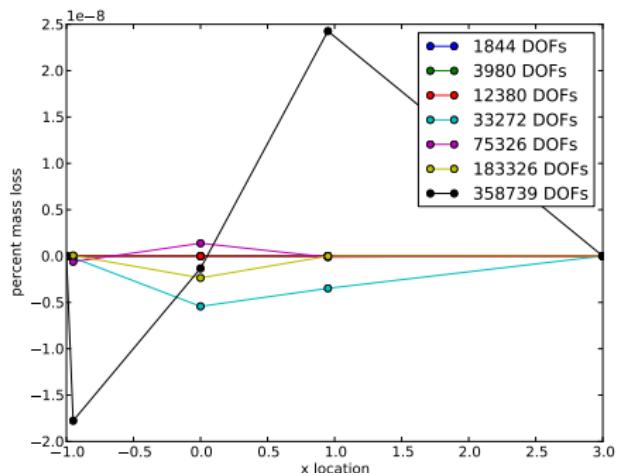
Conservative

Locally Conservative DPG

Stokes Flow Around a Cylinder



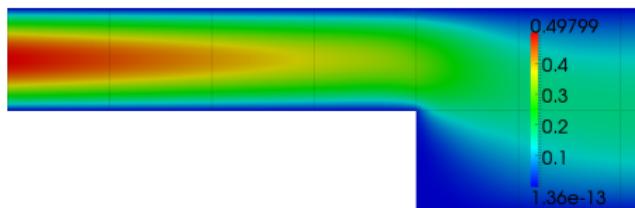
Nonconservative



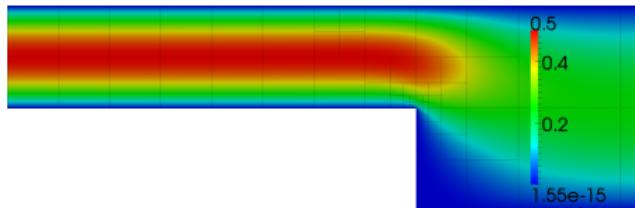
Conservative

Locally Conservative DPG

Stokes Flow Backward Facing Step

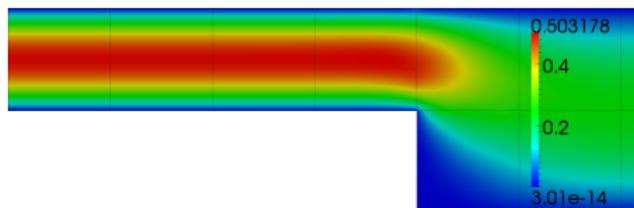


Initial Mesh

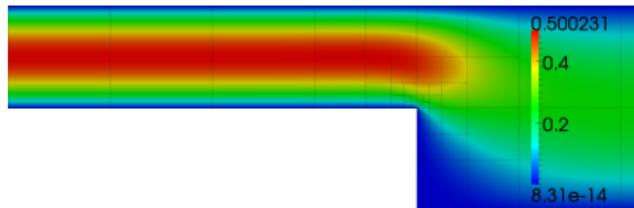


8 Refinements

Nonconservative



Initial Mesh

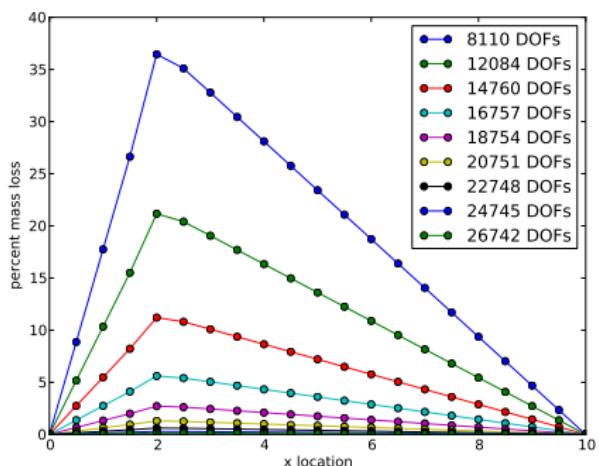


8 Refinements

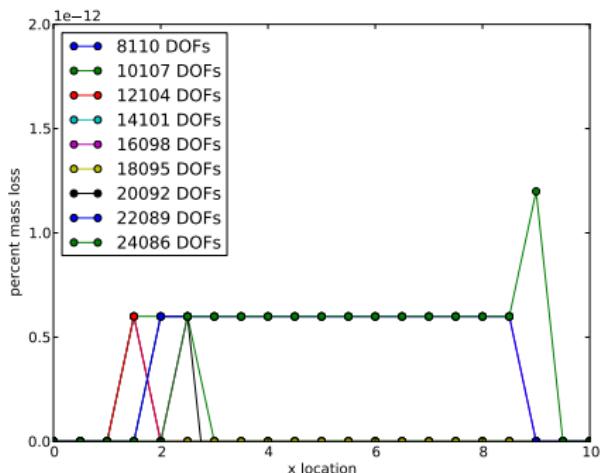
Conservative

Locally Conservative DPG

Stokes Flow Backward Facing Step



Nonconservative



Conservative

Thank You!

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- ▶ T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion*. Tech. rep. 15-21. ICES, Oct. 2015.
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- ▶ L.F. Demkowicz and J. Gopalakrishnan. "Recent Developments In Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)". In: vol. 157. *IMA Volumes in Mathematics and Its Applications*, 2014. Chap. An Overview of the DPG Method, pp. 149–180.
- ▶ L.F. Demkowicz and J. Gopalakrishnan. *Discontinuous Petrov-Galerkin (DPG) Method*. Tech. rep. 15-20. ICES, Dec. 2015.
- ▶ N.V. Roberts. "Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods". In: *Comp. Math. Appl.* 68.11 (2014). Minimum Residual and Least Squares Finite Element Methods, pp. 1581 –1604.
- ▶ N.V. Roberts, L.F. Demkowicz, and R.D. Moser. "A discontinuous Petrov-Galerkin methodology for adaptive solutions to the incompressible Navier-Stokes equations". In: *J. Comput. Phys.* 301 (2015), pp. 456 –483. issn: 0021-9991.