Automating Scientific Computing with Discontinuous Petrov-Galerkin Finite Elements

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Overview of DPG

DPG is a Minimum Residual Method

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bw_h - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1} (Bw_h - l) \right\|_V^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h \,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)_K + (u,\nabla\cdot\boldsymbol{\tau})_K - \langle \hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle_{\partial K} = 0$$

$$-\left(\begin{pmatrix}\beta u-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)_K + \langle \hat{t},v\rangle_{\partial K} = f$$

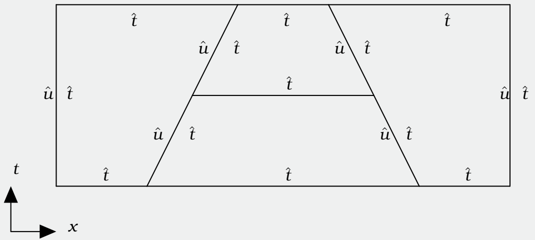
where

$$\hat{u} := \operatorname{tr}(u)$$

$$\hat{t} := \operatorname{tr}(\beta u-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x + \operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables

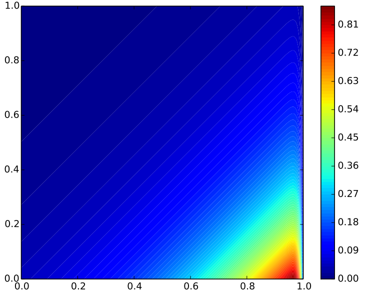


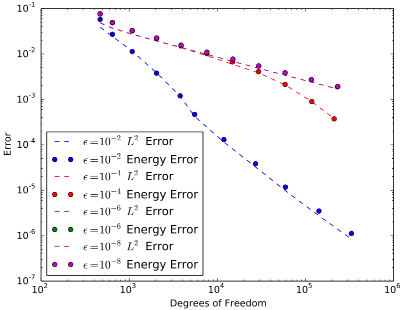
Space-Time Convection-Diffusion

Robust Convergence to Analytical Solution

$$u = e^{-lt}\left(e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}\right), \quad \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4l\epsilon}}{-2\epsilon}$$

$$\left\|(v,\boldsymbol{\tau})\right\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\left\|\boldsymbol{\tau}\right\|^2$$
$$+\,\epsilon\left\|\nabla v\right\|^2 + \left\|\boldsymbol{\beta}\cdot\nabla v\right\|^2 + \left\|v\right\|^2$$





Space-Time Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$
\begin{aligned}
\frac{1}{\mu}\mathbb{D} - \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right) + \frac{2}{3}\nabla \cdot \boldsymbol{u}\mathbb{I} &= 0 \\
\frac{Pr}{C_p \mu}\boldsymbol{q} + \nabla T &= 0 \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix} &= f_c \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \mathbb{I} - \mathbb{D} \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f}_m \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho R T \boldsymbol{u} + \boldsymbol{q} - \boldsymbol{u}\cdot\mathbb{D} \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) \end{pmatrix} &= f_e \,,
\end{aligned}
$$

Space-Time Navier-Stokes

Compact Notation

Conserved quantities

$$C_c := \rho$$

$$\boldsymbol{C}_m := \rho\boldsymbol{u}$$

$$C_e := \rho(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})$$

Euler fluxes

$$\boldsymbol{F}_c := \rho\boldsymbol{u}$$

$$\mathbb{F}_m := \rho\boldsymbol{u}\otimes\boldsymbol{u} + \rho R T\mathbb{I}$$

$$\boldsymbol{F}_e := \rho\boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho R T\boldsymbol{u}$$

Viscous fluxes

$$\boldsymbol{K}_c := \boldsymbol{0}$$

$$\mathbb{K}_m := \mathbb{D}$$

$$\boldsymbol{K}_e := -\boldsymbol{q} + \boldsymbol{u}\cdot\mathbb{D}$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \frac{1}{\mu}\mathbb{D}$$

$$\boldsymbol{M}_{\boldsymbol{q}} := \frac{Pr}{C_p\mu}\boldsymbol{q}$$

Viscous relations

$$\boldsymbol{G}_{\mathbb{D}} := 2\boldsymbol{u}$$

$$G_{\boldsymbol{q}} := -T$$

Space-Time Navier-Stokes

Define Group Variables

Group terms

$$C := \{C_c\,,\ \boldsymbol{C}_m\,,\ C_e\}$$
$$F := \{\boldsymbol{F}_c\,,\ \mathbb{F}_m\,,\ \boldsymbol{F}_e\}$$
$$K := \{\boldsymbol{K}_c\,,\ \mathbb{K}_m\,,\ \boldsymbol{K}_e\}$$
$$M := \{\mathbb{M}_{\mathbb{D}}\,,\ \boldsymbol{M}_{\boldsymbol{q}}\}$$
$$G := \{\boldsymbol{G}_{\mathbb{D}}\,,\ G_{\boldsymbol{q}}\}$$
$$f := \{f_c\,, \boldsymbol{f}_m\,, f_e\}$$

Group variables

$$W := \{\rho\,,\ \boldsymbol{u}\,,\ T\}$$
$$\hat{W} := \left\{2\hat{\boldsymbol{u}}\,,\ -\hat{T}\right\}$$
$$\Sigma := \{\mathbb{D}, \boldsymbol{q}\}$$
$$\hat{t} := \{\hat{t}_e\,,\ \hat{\boldsymbol{t}}_m,\ ,\ \hat{t}_e\}$$
$$\Psi := \{\mathbb{S}\,,\ \boldsymbol{\tau}\}$$
$$V := \{v_c\,,\ \boldsymbol{v}_m,\ ,\ v_e\}\ .$$

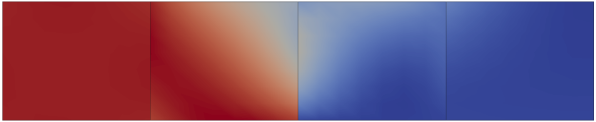
Navier-Stokes variational formulation is

$$(M, \Psi) + (G, \nabla \cdot \Psi) - \left\langle \hat{W}, \Psi \cdot \boldsymbol{n}_x \right\rangle = 0$$
$$-\left(\begin{pmatrix} F - K \\ C \end{pmatrix}, \nabla_{xt} V \right) + \langle \hat{t}, V \rangle = (f, V)\ .$$

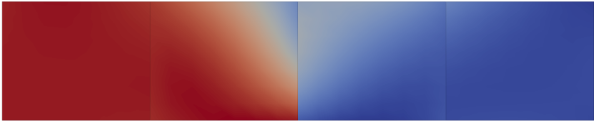
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 1



Primitive Variables



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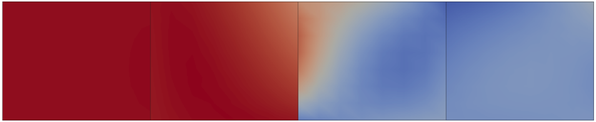
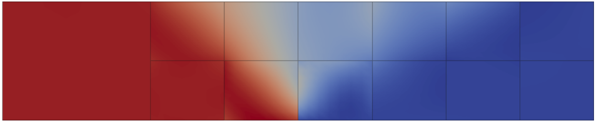


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 2



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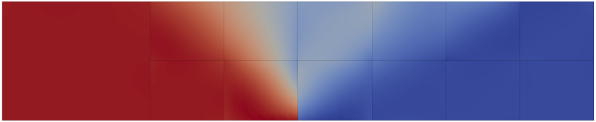

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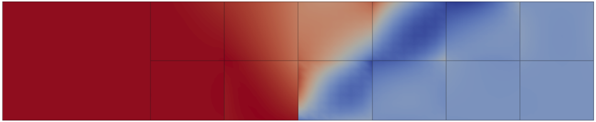
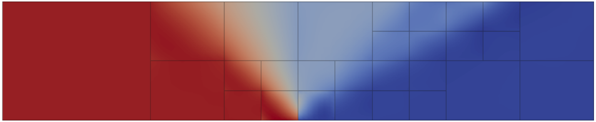


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 3



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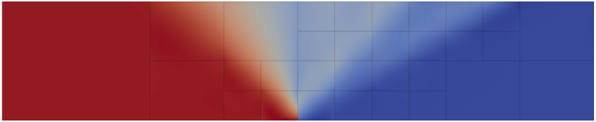

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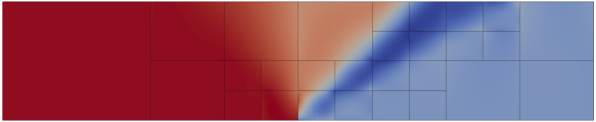
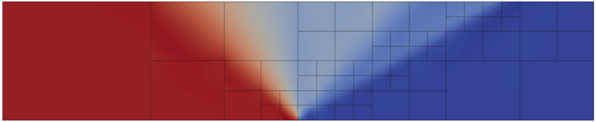


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 4



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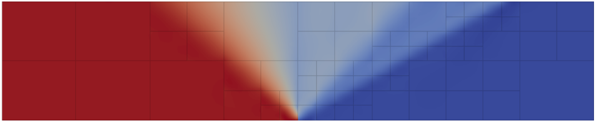

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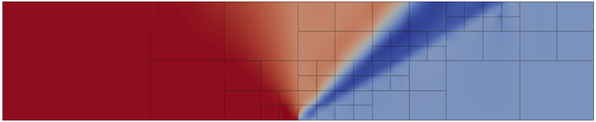
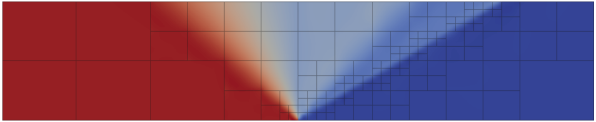


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 5



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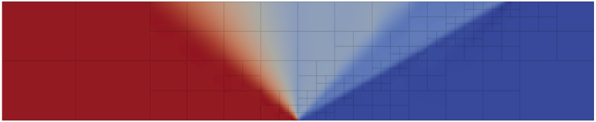

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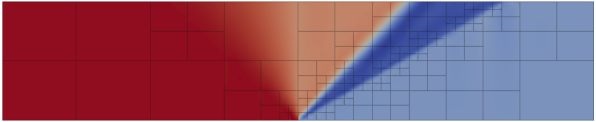
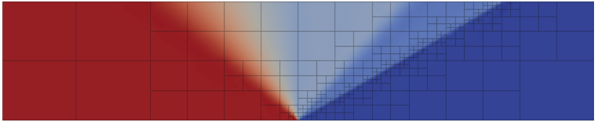


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 6



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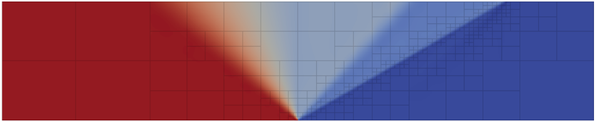

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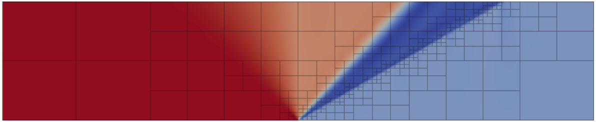
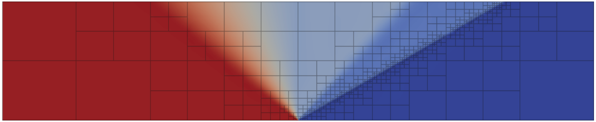


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 7



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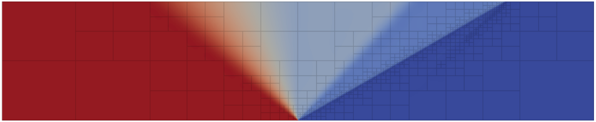

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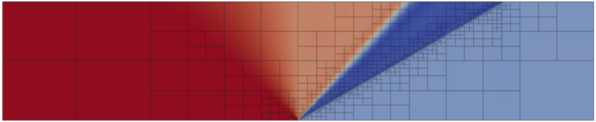
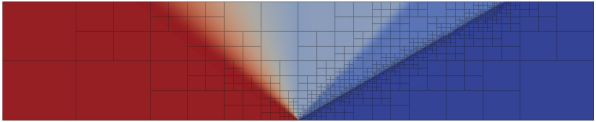


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 8



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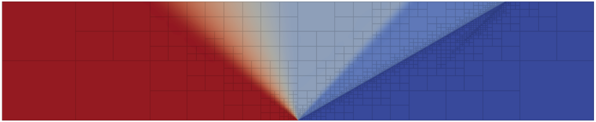

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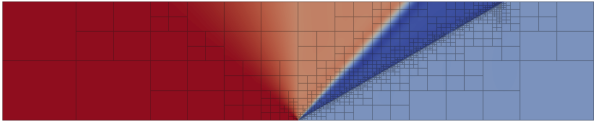
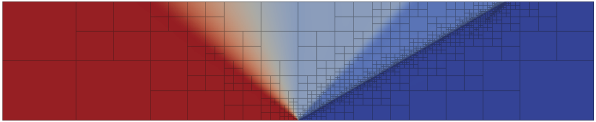


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 9



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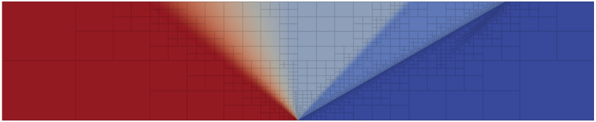

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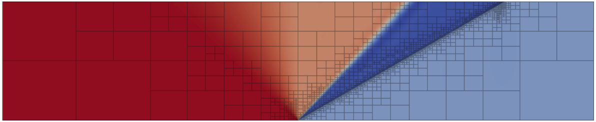
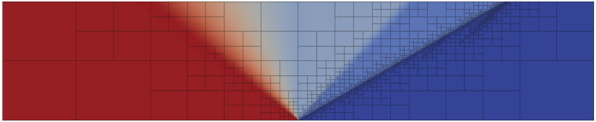


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 10



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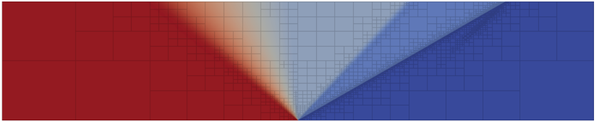

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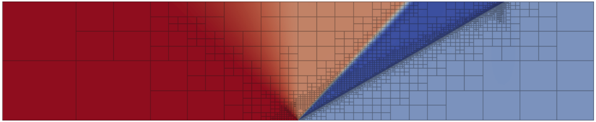
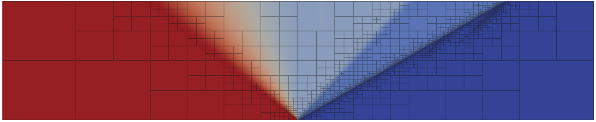


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 11



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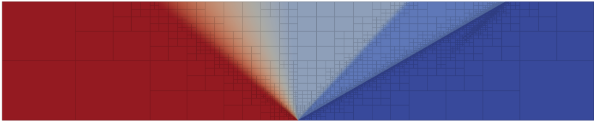

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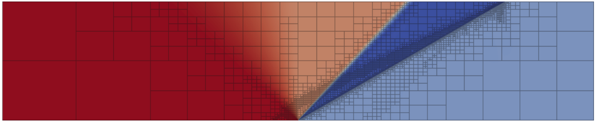
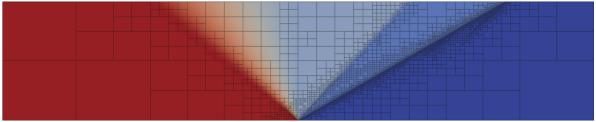


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 12



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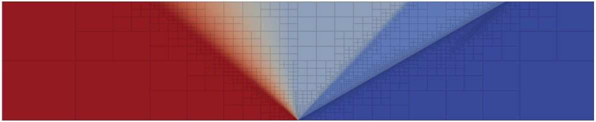

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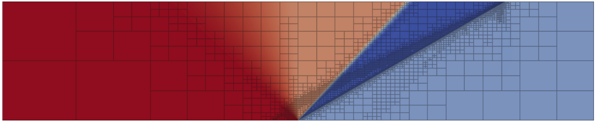
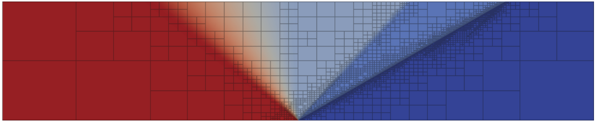


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 13



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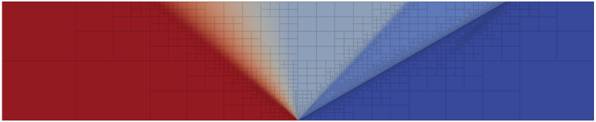

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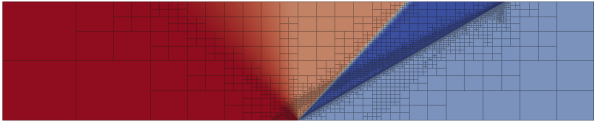

Entropy Variables

Compressible Navier-Stokes

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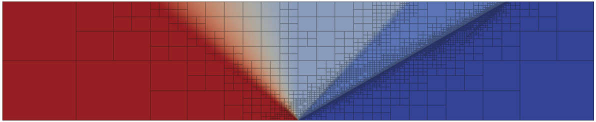
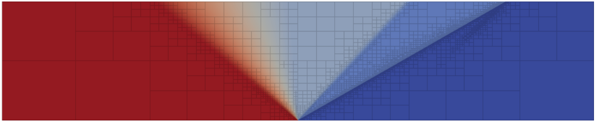
Sod Shock Tube with $\mu = 10^{-5}$

Mesh 14



Primitive Variables



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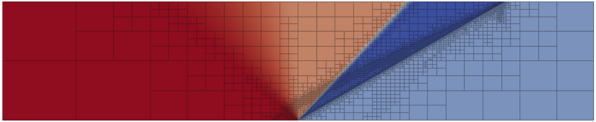
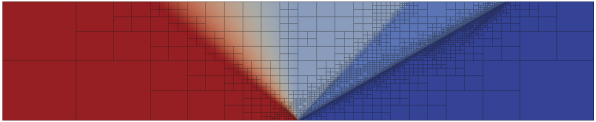


Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 15



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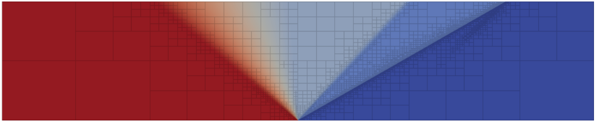

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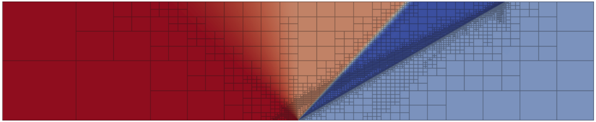
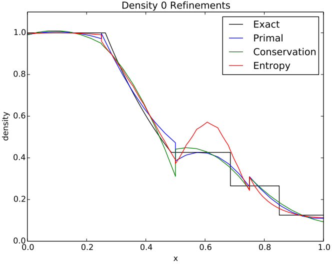
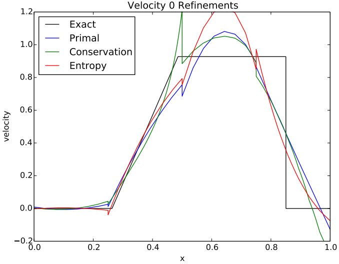


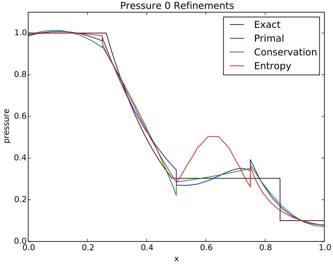
Entropy Variables

Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

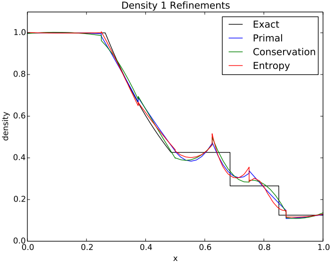


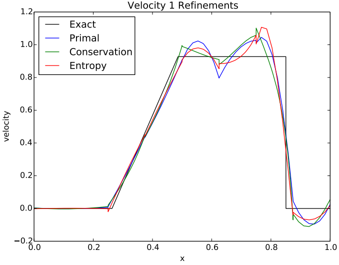


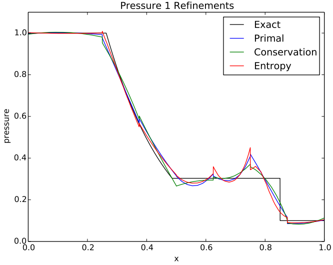


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

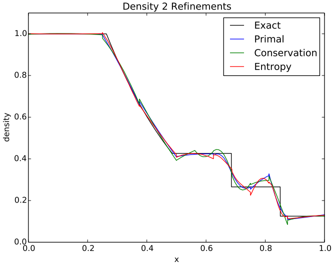


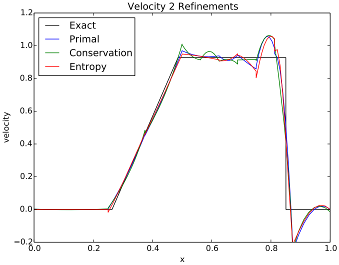


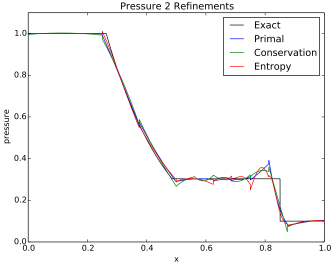


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

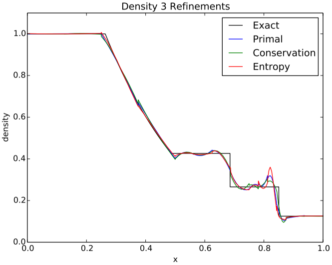


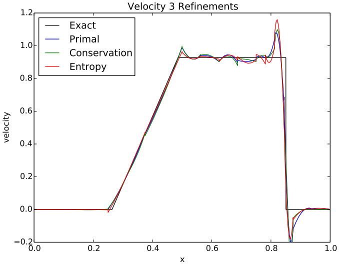


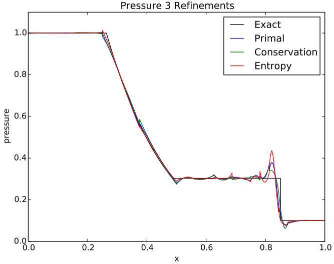


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

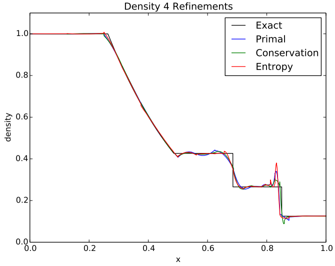


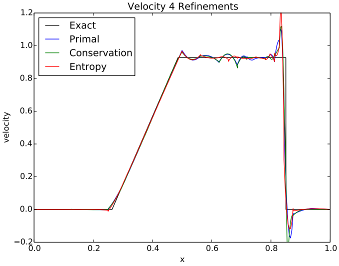


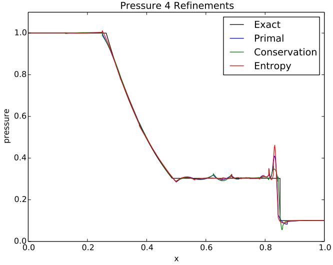


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

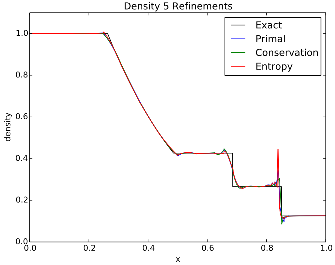


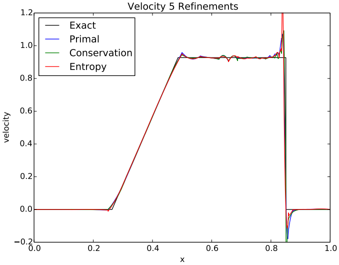


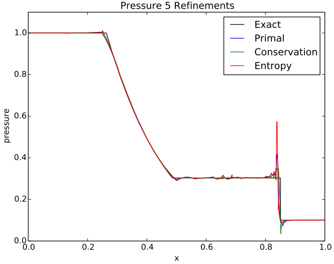


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

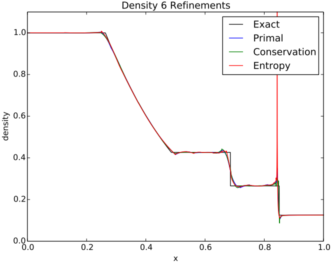


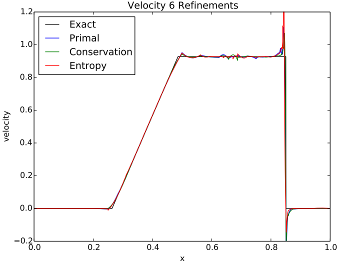


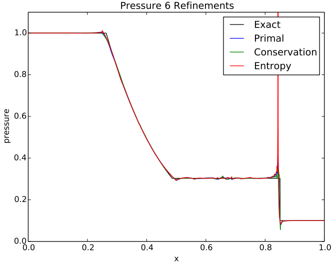


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

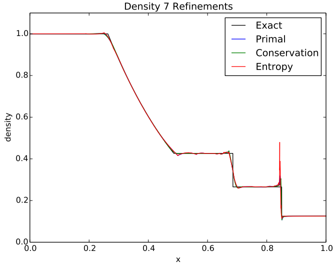


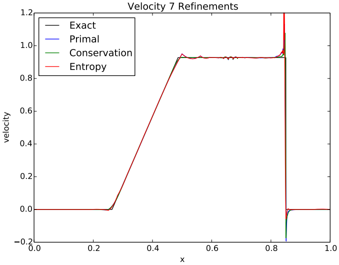


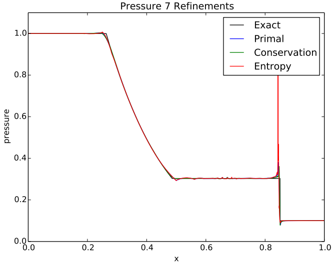


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

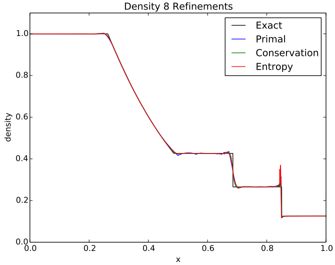


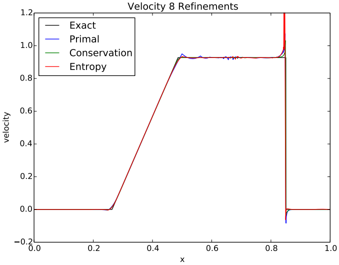


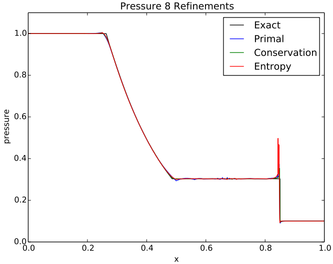


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

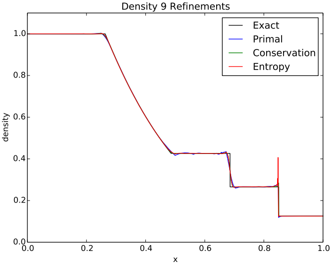


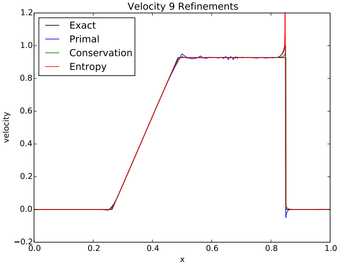


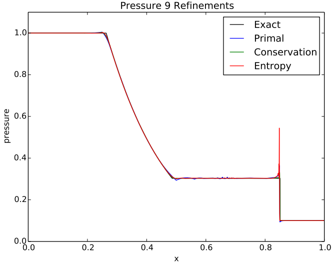


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

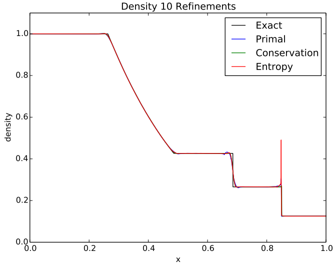


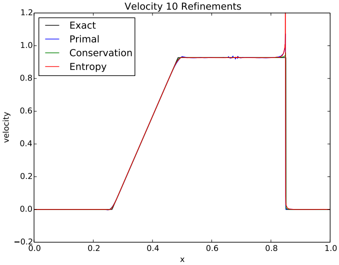


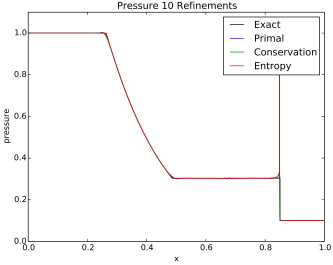


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

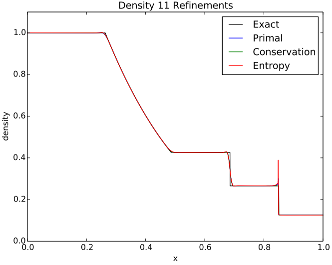


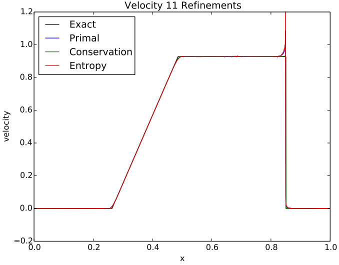


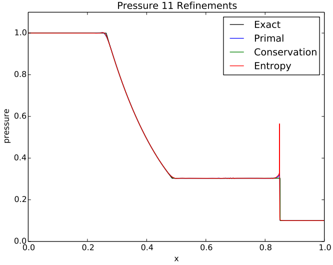


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

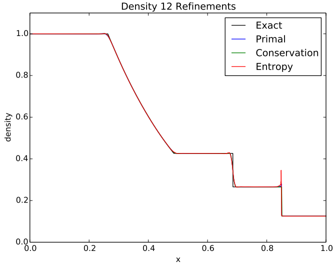


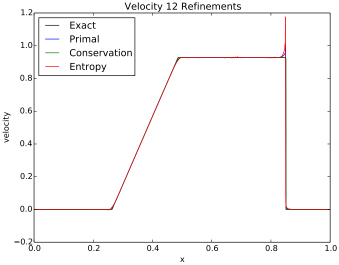


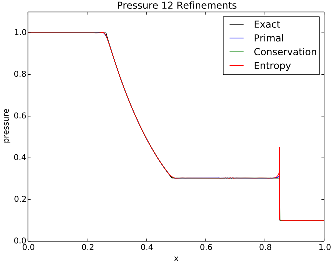


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

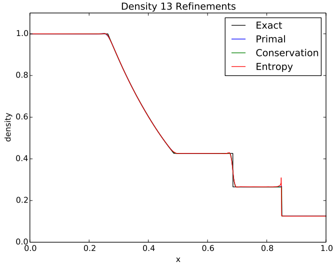


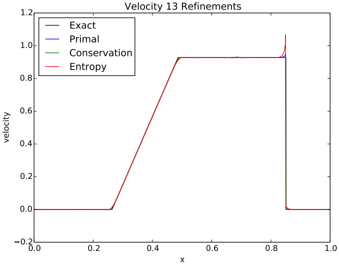


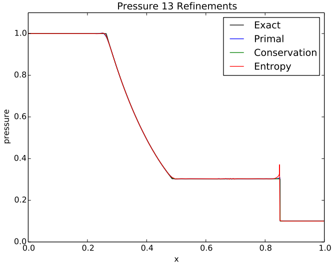


Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

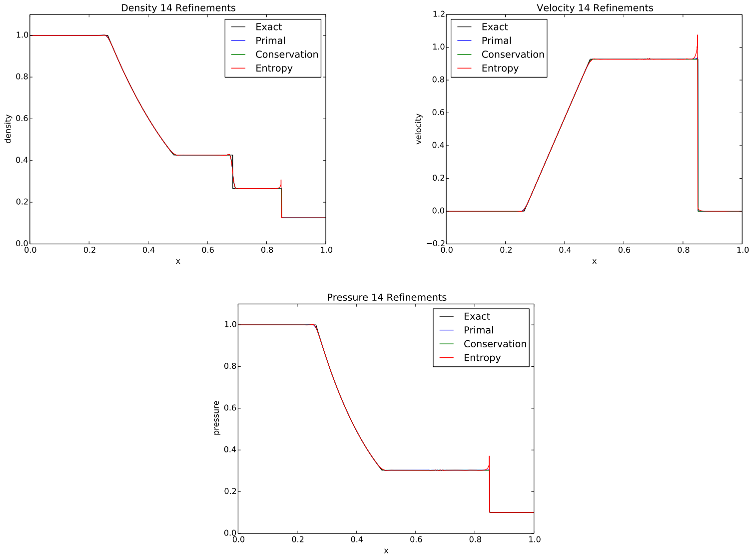






Compressible Navier-Stokes

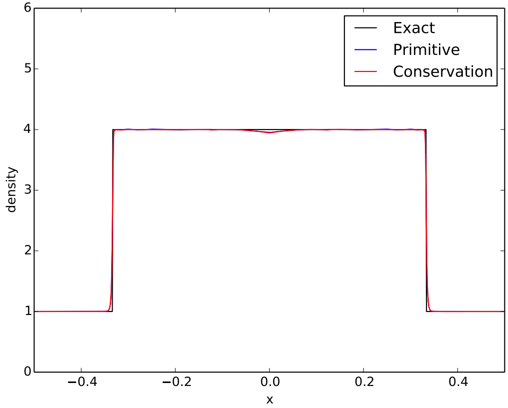
Sod Shock Tube with $\mu = 10^{-5}$

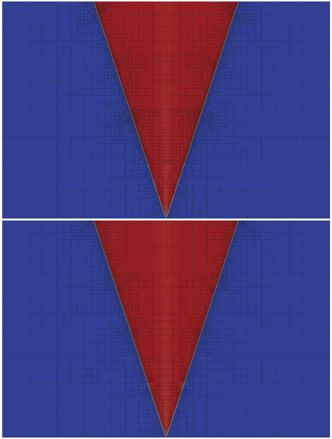


Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

Infinitely strong shock propagation.





Sequence of 4 time slabs

Related Research

Past and Present Topics in DPG Research

- Multiphysics
  - Heat conduction (Poisson and Heat equation)
  - Wave problems (Helmholtz and Maxwell)
  - Linear elasticity and plate problems
  - Convection-Diffusion, Stokes, incompressible Navier-Stokes, compressible Navier-Stokes, Euler
- Natively nonlinear DPG
- DPG for non-Hilbert L^p spaces
- Local conservation
- Iterative solvers
- Entropy scaling for physically meaningful test norms
- General polyhedral elements

Thank You!

Recommended References

- L.F. Demkowicz and J. Gopalakrishnan. ``Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)''. In: vol. 157. IMA Volumes in Mathematics and its Applications, 2014. Chap. An Overview of the DPG Method, pp. 149–180.
- T.E. Ellis, L.F. Demkowicz, and J.L. Chan. ``Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530 –1549.
- N.V. Roberts. ``Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods''. In: *Comp. Math. Appl.* 68.11 (2014). Minimum Residual and Least Squares Finite Element Methods, pp. 1581 –1604.
- L.F. Demkowicz and N. Heuer. ``Robust DPG Method for Convection-Dominated Diffusion Problems''. In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
- J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.
- N. Roberts, T. Bui-Thanh, and L. Demkowicz. ``The DPG method for the Stokes problem''. In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 966 –995.