Space-Time Discontinuous Petrov-Galerkin Finite Elements for Transient Fluid Mechanics

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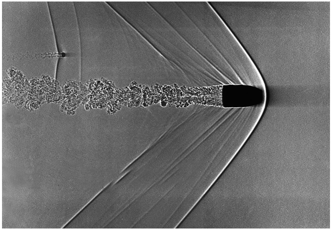
Navier-Stokes Equations

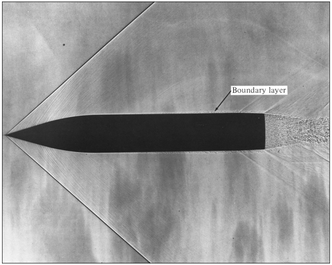
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Stability of numerical schemes
  - Nonlinearity
  - Nature of PDE changes for different flow regimes
  - Coarse/adaptive grids
  - Higher order



Shock

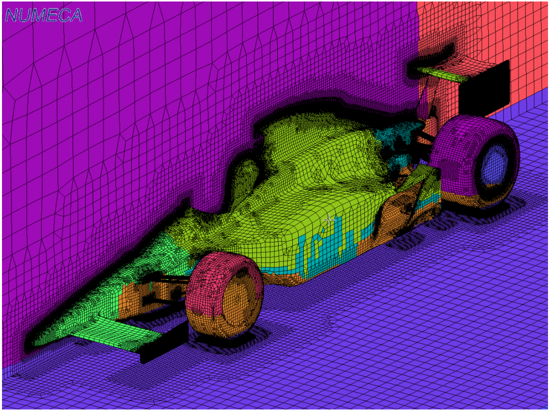


Boundary layer

Motivation

Initial Mesh Design is Expensive and Time-Consuming

- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC

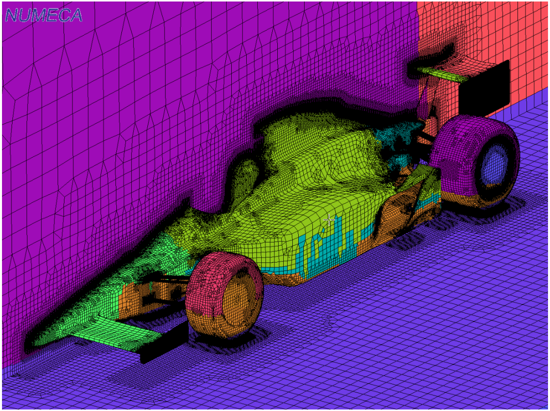


Formula 1 Mesh by Numeca

Motivation

Initial Mesh Design is Expensive and Time-Consuming

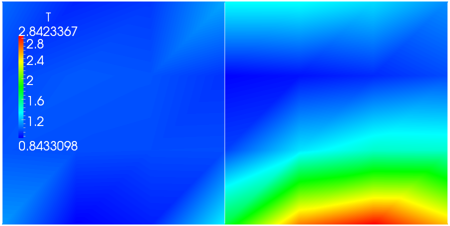
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC
- We desire an automated computational technology



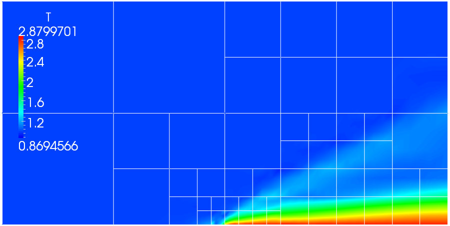
Formula 1 Mesh by Numeca

DPG on Coarse Meshes

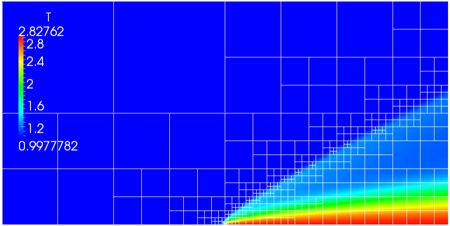
Adaptive Solve of the Carter Plate Problem[1] $Re = 1000$



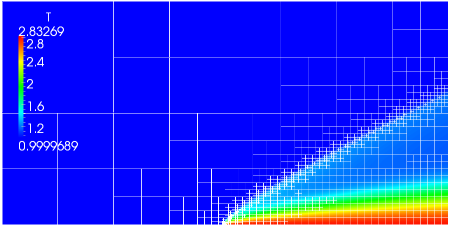
Temperature on Initial Mesh



Temperature after 4 Refinements



Temperature after 8 Refinements



Temperature after 11 Refinements

[1] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Lessons from Other Methods

Streamline Upwind Petrov-Galerkin: Adaptively changing the test space can produce a method with better stability.

Discontinuous Galerkin: Discontinuous basis functions are a legitimate option for finite element methods.

Hybridized DG: Mesh interface unknowns can facilitate static condensation -- reducing the number of DOFs in the global solve.

Least-Squares FEM: The finite element method is most powerful in a minimum residual context (i.e. as a Ritz method).

Space-Time FEM: Highly adaptive methods should have adaptive time integration. Superior framework for problems with moving boundaries. Requires a method that is both temporally and spatially stable.

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Overview of DPG

DPG is a Minimum Residual Method

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \to V'$ defined by $b(u, v) = \langle Bu, v\rangle_{V' \times V}$.
This gives the operator equation

$$Bu = l \quad \in V' .$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| Bw_h - l \right\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \underset{w_h \in U_h}{\arg\min} \frac{1}{2} \left\| R_V^{-1}(Bw_h - l) \right\|_V^2 .$$

Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$\left(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u\right)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\left\langle Bu_h - l, R_V^{-1}B\delta u_h \right\rangle = 0 \quad \forall \delta u_h \in U_h\,,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h.

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v\rangle = \langle B\delta u_h, \delta v\rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

Mixed Formulation

Identifying the error representation function:

$$\psi := R_V^{-1}(Bu_h - l)$$

allows us to develop an alternative interpretation of DPG.

DPG as a Mixed Problem

Find $\psi \in V$, $u_h \in U_h$ such that

$$\begin{aligned} (\psi, \delta v)_V - b(u_h, \delta v) &= -l(\delta v) \qquad && \forall \delta v && \in V \\ b(\delta u_h, \psi) &= 0 && \forall \delta u_h && \in U_h \end{aligned}$$

In this unconventional saddle-saint problem, the approximate solution u_h comes from a finite-dimensional trial space and plays the role of the Lagrange multiplier for the error representation function

Overview of DPG

DPG is the Most Stable Petrov-Galerkin Method

Babuška's theorem guarantees that *discrete stability and approximability imply convergence*. If bilinear form $b(u, v)$, with $M := \|b\|$ satisfies the discrete inf-sup condition with constant γ_h,

$$\sup_{v_h \in V_h} \frac{|b(u,v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U ,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U .$$

Optimal test function realize the supremum guaranteeing that $\gamma_h \geq \gamma$.

Energy Norm

If we use the energy norm, $\|u\|_E := \|Bu\|_{V'}$ in the error estimate, then $M = \gamma = 1$. Babuška's theorem implies that the minimum residual method is the most stable Petrov-Galerkin method (assuming exact optimal test functions).

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Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \left\|R_V^{-1}(Bu_h - l)\right\|_V$$

Overview of DPG

High Performance Computing

Eliminates human intervention

- Stability
- Robustness
- Adaptivity
- Automaticity
- Compute intensive
- Embarrassingly parallel local solves
- Factorization recyclable
- Low communication
- SPD stiffness matrix
- Multiphysics

Stampede Supercomputer at TACC

Mira Supercomputer at Argonne

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Locally Conservative DPG

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DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\boldsymbol{\beta} u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u = \mathbf{0}$$
$$\nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) = g$$

Multiply by test functions and integrate by parts over each element, K.

$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle u, \tau_n \rangle_{\partial K} = 0$$
$$-(\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + \langle (\boldsymbol{\beta} u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v \rangle_{\partial K} = (g, v)_K$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K}$$
$$- (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v)_K + \langle \hat{t}, v \rangle_{\partial K} = (g, v)_K$$

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K g \,,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al[2] (also Chang and Nelson[3]), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \left\| R_V^{-1}(Bu_h - l) \right\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{t}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K} \,,$$

where $\boldsymbol{\lambda} = \{\lambda_1, \cdots, \lambda_N\}$.

[2] D. Moro, N.C. Nguyen, and J. Peraire. ``A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws''. In: *Int. J. Num. Meth. Eng.* (2011).

[3] C.L. Chang and J.J. Nelson. ``Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation''. In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.

Finding the critical points of $L(u, \boldsymbol{\lambda})$, we get the following equations.

Locally Conservative Saddle Point System

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial u_h} = b(u_h, R_V^{-1} B\delta u_h) - l(R_V^{-1} B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \boldsymbol{\lambda})}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddle point problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

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Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v_u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v_u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where $\mathbf{V}$ becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable. We recently developed this modification to the *robust test norm*[4] which behaves better in the presence of singularities.

Convection-Diffusion Test Norm

$$\|(v, \boldsymbol{\tau})\|^2_{\mathbf{V},\Omega_h} = \left\| \min\left\{\frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}}\right\}\boldsymbol{\tau}\right\|^2 + \|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 \underbrace{+ \|v\|^2}_{\text{No longer necessary}}$$

[4] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014), pp. 771 –795.

Locally Conservative DPG

Optimal Test Functions

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[4] J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014), pp. 771–795.

Locally Conservative DPG

Stability and Robustness Analysis[5]

- We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{U}, p \in Q & & \\ a(\boldsymbol{u}, \boldsymbol{w}) + c(p, \boldsymbol{w}) & = l(\boldsymbol{w}) & \forall \boldsymbol{w} \in \boldsymbol{U} \\ c(q, \boldsymbol{u}) & = g(q) & \forall q \in Q \end{cases}$$

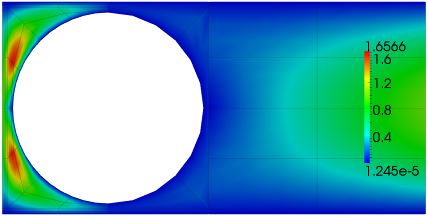
  where a, c, l, g denote the appropriate bilinear and linear forms.
- $a(\boldsymbol{u}, \boldsymbol{w}) = b(\boldsymbol{u}, R_V^{-1} B\boldsymbol{w}) = (R_V^{-1} B\boldsymbol{u}, R_V^{-1} B\boldsymbol{w})_V$
- $c(p, \boldsymbol{w}) = \sum_K \lambda_K \langle \hat{t}, 1_K \rangle_{\partial K}$
- Locally conservative DPG satisfies inf-sup and inf-sup in kernel conditions.
- Robustness is proved by switching to energy norm in Brezzi analysis.

[5] T.E. Ellis, L.F. Demkowicz, and J.L. Chan. ``Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530 –1549.

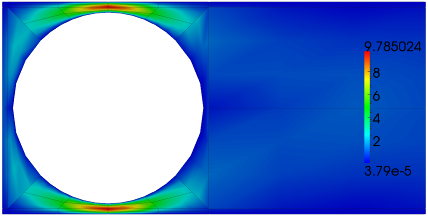
Numerical Experiments

Stokes Flow Around a Cylinder

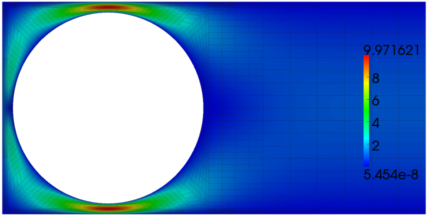
Horizontal Velocity



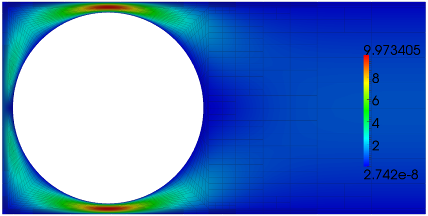
1 Refinement



1 Refinement



6 Refinements



6 Refinements

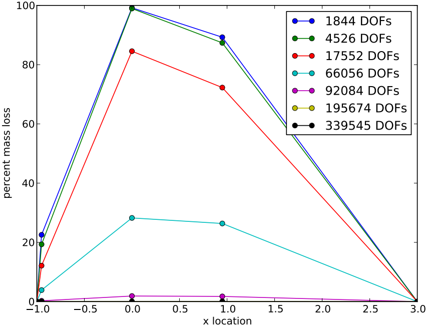
Nonconservative

Conservative

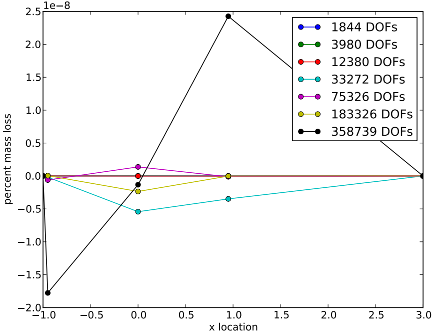
Numerical Experiments

Stokes Flow Around a Cylinder

Percent Mass Loss at $x = [-1, -0.95, 0, 0.95, 3]$



Nonconservative

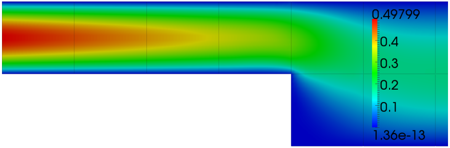


Conservative

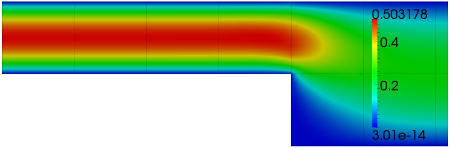
Numerical Experiments

Stokes Flow Over a Backward Facing Step

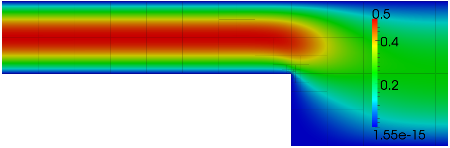
Horizontal Velocity



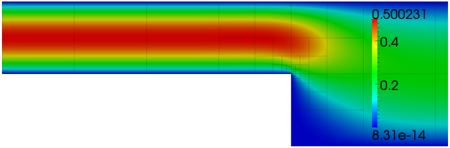
Initial Mesh



Initial Mesh



8 Refinements



8 Refinements

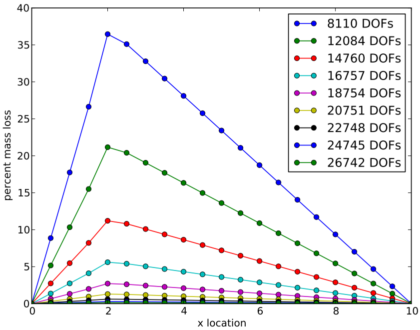
Nonconservative

Conservative

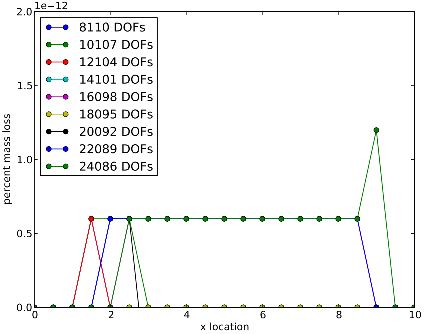
Numerical Experiments

Stokes Flow Over a Backward Facing Step

Percent Mass Loss at $x = [0, 0.5, \cdots, 9.5, 10]$



Nonconservative



Conservative

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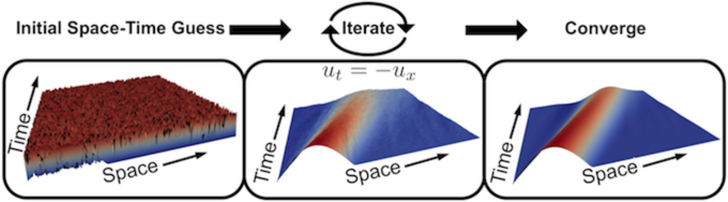
Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Courtesy of XBraid by LLNL

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \boldsymbol{\beta} \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\begin{aligned} \boldsymbol{\sigma} - \epsilon \nabla u &= 0 \\ \frac{\partial u}{\partial t} + \nabla \cdot (\boldsymbol{\beta} u - \boldsymbol{\sigma}) &= f \end{aligned}$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{\beta} u - \boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\left(\frac{1}{\epsilon}\boldsymbol{\sigma},\boldsymbol{\tau}\right)_K+(u,\nabla\cdot\boldsymbol{\tau})_K-\langle\hat{u},\boldsymbol{\tau}\cdot\boldsymbol{n}_x\rangle_{\partial K}=0$$

$$-\left(\begin{pmatrix}\boldsymbol{\beta}u-\boldsymbol{\sigma}\\ u\end{pmatrix},\nabla_{xt}v\right)_K+\langle\hat{t},v\rangle_{\partial K}=f$$

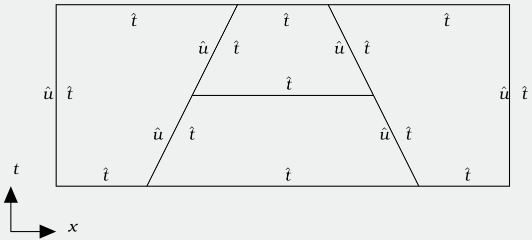
where

$$\hat{u}:=\operatorname{tr}(u)$$

$$\hat{t}:=\operatorname{tr}(\boldsymbol{\beta}u-\boldsymbol{\sigma})\cdot\boldsymbol{n}_x+\operatorname{tr}(u)\cdot n_t$$

- Trace $\hat{u}$ defined on spatial boundaries
- Flux $\hat{t}$ defined on all boundaries

Support of Trace Variables



Space-Time Convection-Diffusion

L^2 Equivalent Norms

Bilinear form with group variables:

$$b\left(\left(u,\hat{u}\right),v\right)=\left(u,A_h^*v\right)_{L^2(\Omega_h)}+\langle\widehat{u},[\![v]\!]\rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^*v^*=u$

$$\|u\|^2_{L^2(\Omega_h)}=b(u,v^*)=\frac{b(u,v^*)}{\|v^*\|_V}\|v^*\|_V$$

$$\leq\sup_{v^*\neq 0}\frac{|b(u,v^*)|}{\|v^*\|}\|v^*\|=\|u\|_E\|v^*\|_V$$

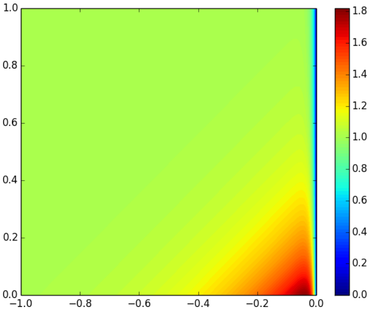
Necessary robustness condition:

$$\|v^*\|_V\lesssim\|u\|_{L^2(\Omega_h)}$$

$$\Rightarrow\|u\|_{L^2(\Omega_h)}\lesssim\|u\|_E$$

Analytical Solution

$$e^{-lt}(e^{\lambda_1(x-1)}-e^{\lambda_2(x-1)})$$
$$+\left(1-e^{\frac{1}{\epsilon}x}\right)$$



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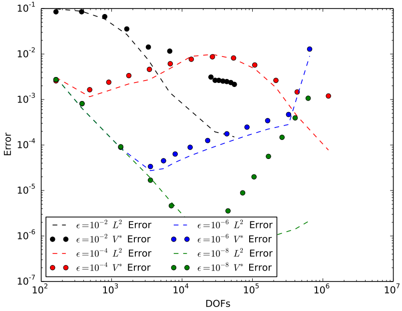
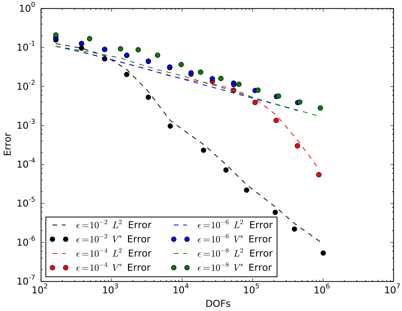
A norm should be: bounded by $\|u\|_{L^2(\Omega_h)}$, have good conditioning, not produce boundary layers in the optimal test function.



$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|^2 + \left\|\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v\right\|^2 + \|v\|^2 + \|\boldsymbol{\tau}\|^2$$

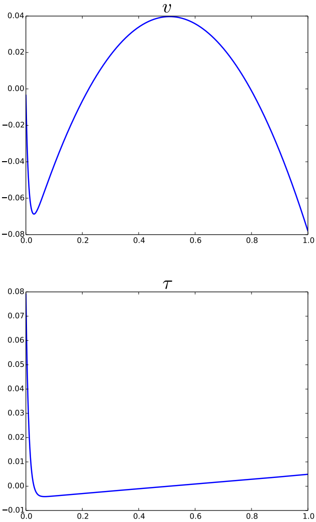


$$\|(v,\boldsymbol{\tau})\|^2 = \left\|\nabla\cdot\boldsymbol{\tau} - \tilde{\boldsymbol{\beta}}\cdot\nabla_{xt}v\right\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 + \epsilon\|\nabla v\|^2 + \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \|v\|^2$$

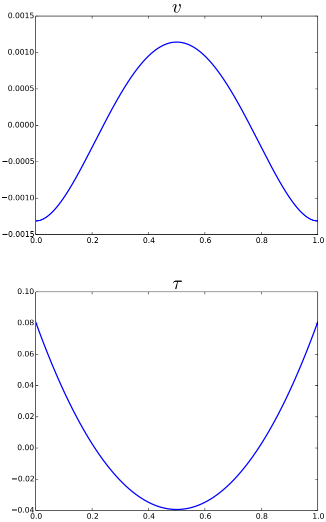
Steady Convection-Diffusion

Ideal Optimal Shape Functions

Graph Norm



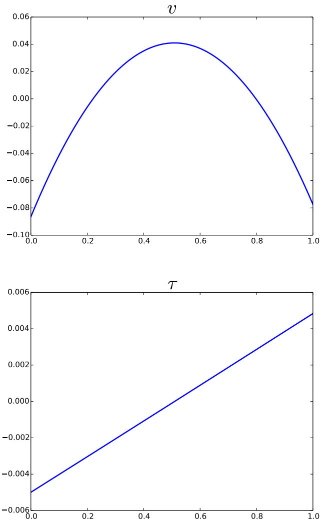
Coupled Robust Norm



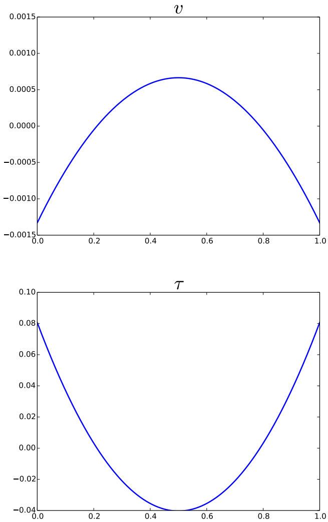
Steady Convection-Diffusion

Approximated ($p = 3$) Optimal Shape Functions

Graph Norm



Coupled Robust Norm



Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.

The robust norm was derived in[6]:

$$\|(v,\boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 + \|\nabla\cdot\boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 .$$

The case for the coupled robust norm was made in[7]:

$$\|(v,\boldsymbol{\tau})\|^2 = \|\boldsymbol{\beta}\cdot\nabla v\|^2 + \epsilon\|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2},1\right)\|v\|^2 + \|\nabla\cdot\boldsymbol{\tau} - \boldsymbol{\beta}\cdot\nabla v\|^2 + \min\left(\frac{1}{h^2},\frac{1}{\epsilon}\right)\|\boldsymbol{\tau}\|^2 .$$

[6] J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014), pp. 771 –795.

[7] J.L. Chan. ``A DPG Method for Convection-Diffusion Problems''. PhD thesis. University of Texas at Austin, 2013.

Space-Time Convection-Diffusion

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Two Robust Norms for Transient Convection-Diffusion

Let $\tilde{\boldsymbol{\beta}} := \begin{pmatrix} \boldsymbol{\beta} \\ 1 \end{pmatrix}$ and $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$.

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$\|(v, \boldsymbol{\tau})\|^2 = \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 + \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

Coupled Robust Norm

$$\|(v, \boldsymbol{\tau})\|^2 = \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 + \left\|\nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v\right\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\boldsymbol{\tau}\|^2 .$$

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Consider the problem with homogeneous boundary conditions

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\sigma} - \nabla u &= 0 \\ \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \beta_n u - \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0 . \end{aligned}$$

The adjoint operator A^* is given by

$$A^*(v, \boldsymbol{\tau}) = \left(\frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v, -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v + \nabla \cdot \boldsymbol{\tau} \right) .$$

We decompose the continuous adjoint problem $A^*(\boldsymbol{\tau}, v) = (\boldsymbol{\sigma}, u)$ into

Continuous part with forcing u

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\tau}_1 + \nabla v_1 &= 0 \\ -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= u \end{aligned}$$

$$\begin{aligned} \boldsymbol{\tau}_1 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\ v_1 &= 0 \text{ on } \Gamma_+ \\ v_1 &= 0 \text{ on } \Gamma_T \end{aligned}$$

Continuous part with forcing $\boldsymbol{\sigma}$

$$\begin{aligned} \frac{1}{\epsilon}\boldsymbol{\tau}_2 + \nabla v_2 &= \boldsymbol{\sigma} \\ -\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{\tau}_2 \cdot \boldsymbol{n}_x &= 0 \text{ on } \Gamma_- \\ v_2 &= 0 \text{ on } \Gamma_+ \\ v_2 &= 0 \text{ on } \Gamma_T \end{aligned}$$

Proofs of these lemmas can be found in[8].

Lemma (1)

If $\nabla \cdot \boldsymbol{\beta} = 0$*, we can bound*

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|u\|^2 + \epsilon \|\boldsymbol{\sigma}\|^2$$

where $v = v_1 + v_2$*.*

Lemma (2)

If $\left\|\nabla \boldsymbol{\beta} - \frac{1}{2} \nabla \cdot \boldsymbol{\beta} \boldsymbol{I}\right\|_{L^\infty} \leq C_{\boldsymbol{\beta}}$*, we can bound*

$$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim \|u\| \,.$$

[8] T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion.* Tech. rep. 15-21. ICES, Oct. 2015.

Robust Norms for Transient Convection-Diffusion

Control of u

Bound on $\|(v_1, \boldsymbol{\tau}_1)\|$

Lemma (2) $\Rightarrow$ $$\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim \|u\|$$

Lemma (2) $\Rightarrow$ $$\|\nabla \cdot \boldsymbol{\tau}_1\| \leq \|u\| + \left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| \lesssim 2\|u\|$$

Lemma (2) $\Rightarrow$ $$\left\|\nabla \cdot \boldsymbol{\tau}_1 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_1\right\| = \|u\|$$

Lemma (1) $\Rightarrow$ $$\|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|u\|^2$$

Lemma (1) $\Rightarrow$ $$\frac{1}{\epsilon}\|\boldsymbol{\tau}_1\| = \epsilon \|\nabla v_1\| \leq \|u\|$$

We can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \boldsymbol{\sigma})\|_E .$$

Robust Norms for Transient Convection-Diffusion

Control of $\boldsymbol{\sigma}$

Bound on $\left\|(v_2, \boldsymbol{\tau}_2)\right\|$

$$\text{Definition} \Rightarrow \quad \left\|\nabla \cdot \boldsymbol{\tau}_2 - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\| = 0 \leq \|\boldsymbol{\sigma}\|$$

$$\text{Lemma (1)} \Rightarrow \quad \|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\boldsymbol{\sigma}\|^2$$

$$\text{Lemma (1)} \Rightarrow \quad \frac{1}{\epsilon} \|\boldsymbol{\tau}_2\| = \|\boldsymbol{\sigma}\| + \epsilon \|\nabla v_2\| = (1+\epsilon) \|\boldsymbol{\sigma}\|$$

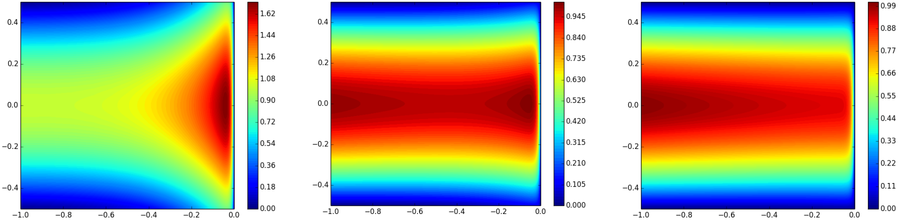
We have not been able to prove bounds on $\left\|\tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} v_2\right\|$ or $\|\nabla \cdot \boldsymbol{\tau}_2\|$.

We can **not** guarantee robust control

$$\|(0, \boldsymbol{\sigma})\|_{L^2(\Omega_h)} \not\lesssim \|(u, \boldsymbol{\sigma})\|_E \ .$$

Transient impulse decays to Eriksson-Johnson steady state solution.

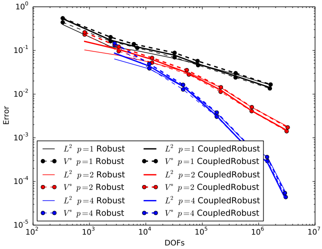
$$u = \exp(-lt)\left[\exp(\lambda_1 x) - \exp(\lambda_2 x)\right] + \cos(\pi y)\frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$



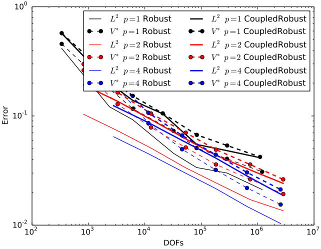
$t = 0.0$ $t = 0.5$ $t = 1.0$

Robust Norms for Transient Convection-Diffusion

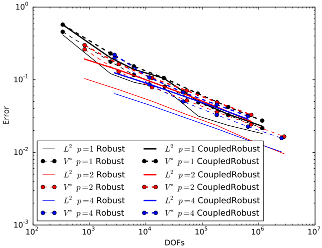
Robust Convergence to Analytical Solution



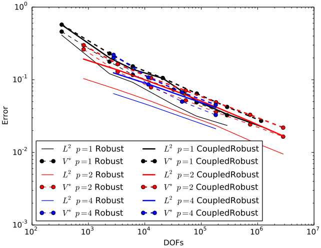
$\epsilon = 10^{-2}$



$\epsilon = 10^{-4}$



$\epsilon = 10^{-6}$



$\epsilon = 10^{-8}$

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Space-Time Incompressible Navier-Stokes

Nonlinear Form

Space-time divergence form:

$$\frac{1}{\nu}\mathbb{D} - \nabla \boldsymbol{u} = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{D} + p\mathbb{I} \\ \boldsymbol{u} \end{pmatrix} = \boldsymbol{f}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Multiply by $\mathbb{S} \in \mathbb{H}(\mathrm{div}, Q)$, $\boldsymbol{v} \in \boldsymbol{H}^1_{xt}(Q)$, $q \in H^1(Q)$, and integrate by parts:

$$\left(\frac{1}{\nu}\mathbb{D}, \mathbb{S}\right) + (\boldsymbol{u}, \nabla \cdot \mathbb{S}) - \langle \hat{\boldsymbol{u}}, \mathbb{S} \cdot \boldsymbol{n}_x \rangle = 0$$

$$-\left(\begin{pmatrix} \boldsymbol{u} \otimes \boldsymbol{u} - \mathbb{D} + p\mathbb{I} \\ \boldsymbol{u} \end{pmatrix}, \nabla_{xt} \boldsymbol{v}\right) + \langle \hat{\boldsymbol{t}}, \boldsymbol{v} \rangle = (\boldsymbol{f}, \boldsymbol{v})$$

$$-(\boldsymbol{u}, \nabla q) + \langle \hat{\boldsymbol{u}} \cdot \boldsymbol{n}, q \rangle = 0$$

Recall the adjoint and robust norm for convection-diffusion:

$$(\boldsymbol{\sigma}, \frac{1}{\epsilon}\boldsymbol{\tau} + \nabla v) + (u, \nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v - \frac{\partial v}{\partial t})$$

$$\begin{aligned} \|(v, \boldsymbol{\tau})\|_{V,K}^2 := &\left\|\boldsymbol{\beta} \cdot \nabla v + \frac{\partial v}{\partial t}\right\|_K^2 + \epsilon \|\nabla v\|_K^2 + \min\left(\frac{\epsilon}{h^2}, 1\right)\|v\|_K^2 \\ &+ \|\nabla \cdot \boldsymbol{\tau}\|_K^2 + \min\left(\frac{1}{\epsilon}, \frac{1}{h^2}\right)\|\boldsymbol{\tau}\|_K^2 \end{aligned}$$

For incompressible Navier-Stokes the adjoint comes from:

$$\begin{aligned} &\left(\Delta\mathbb{D}, \frac{1}{\nu}\mathbb{S} + \nabla \boldsymbol{v}\right) + \left(\Delta \boldsymbol{u}, \nabla \cdot \mathbb{S} - \nabla q - \left(\tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} + \tilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{v})^T + \frac{\partial \boldsymbol{v}}{\partial t}\right)\right) \\ &+ (p, -\nabla \cdot \boldsymbol{v}) \end{aligned}$$

Space-Time Incompressible Navier-Stokes

Norms for Navier-Stokes come from analogy

Convection-Diffusion		Navier-Stokes
ϵ	$\rightarrow$	ν
$\boldsymbol{\tau}$	$\rightarrow$	$\mathbb{S}$
∇v	$\rightarrow$	$\nabla \boldsymbol{v}$
$\nabla \cdot \boldsymbol{\tau}$	$\rightarrow$	$\nabla \cdot \mathbb{S} - \nabla q$
$\boldsymbol{\beta} \cdot \nabla v + \frac{\partial v}{\partial t}$	$\rightarrow$	$\tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} + \tilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{v})^T + \frac{\partial \boldsymbol{v}}{\partial t}$

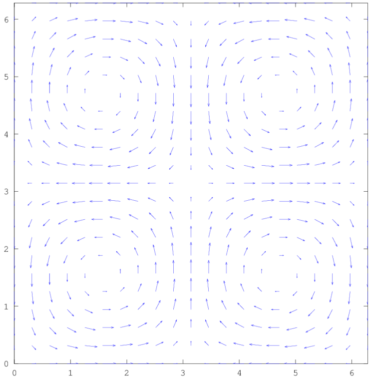
Robust norm:

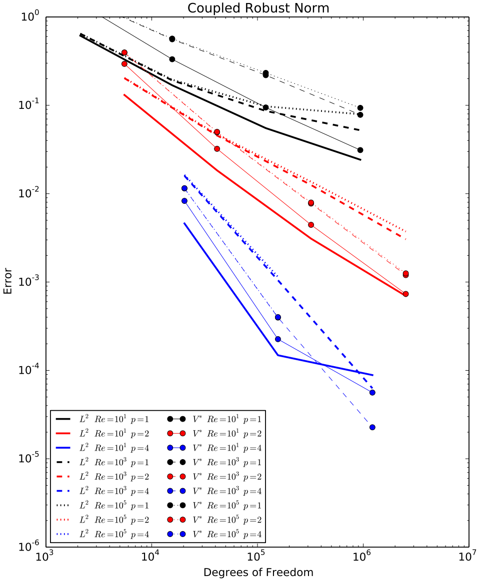
$$
\begin{aligned}
\|(\boldsymbol{v}, \mathbb{D}, q)\|_{V,K}^2 := & \left\| \tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{v} + \tilde{\boldsymbol{u}} \cdot (\nabla \boldsymbol{v})^T + \frac{\partial \boldsymbol{v}}{\partial t} \right\|_K^2 + \nu \|\nabla \boldsymbol{v}\|_K^2 \\
& + \min\left(\frac{\nu}{h^2}, 1\right) \|\boldsymbol{v}\|_K^2 + \|\nabla \cdot \mathbb{S} - \nabla q\|_K^2 \\
& + \min\left(\frac{1}{\nu}, \frac{1}{h^2}\right) \|\mathbb{S}\|_K^2 + \|\nabla \cdot \boldsymbol{v}\|_K^2 + \|q\|_K^2
\end{aligned}
$$

Space-Time Incompressible Navier-Stokes

Taylor-Green Vortex

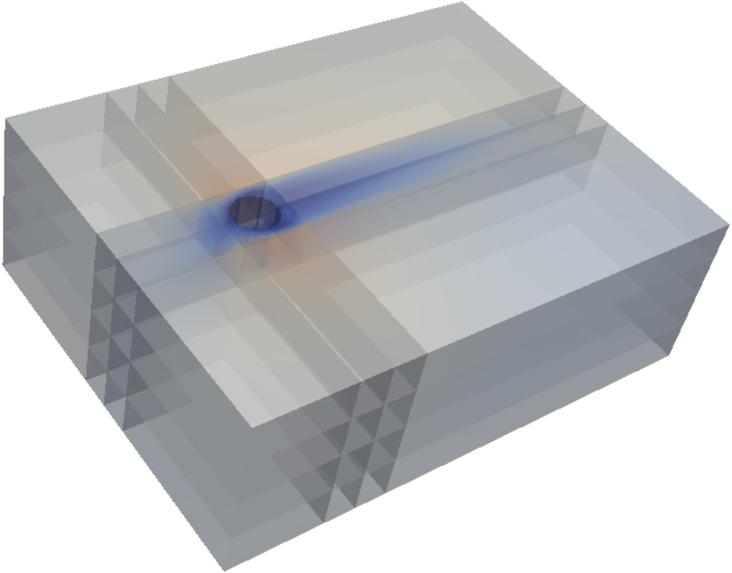
$$\boldsymbol{u} = e^{-\frac{2}{Re}t}\begin{pmatrix} \sin x \cos y \\ -\cos x \sin y \end{pmatrix}$$





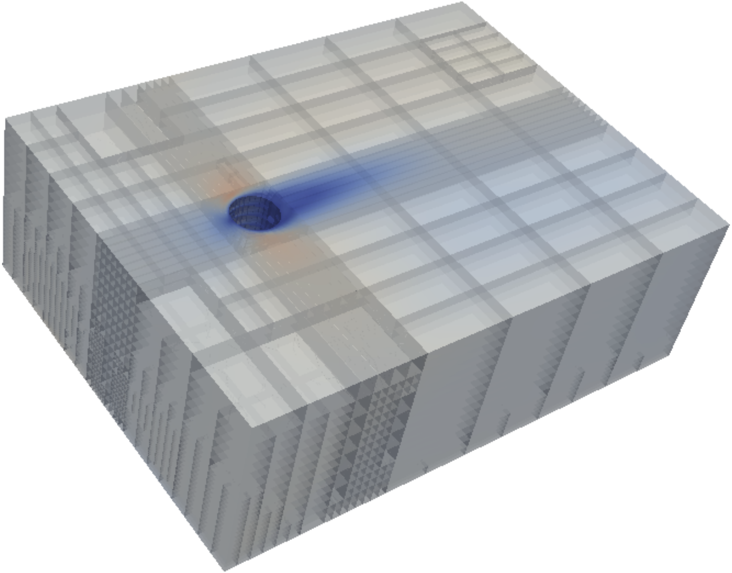
Space-Time Incompressible Navier-Stokes

Flow Over a Cylinder, Initial Mesh



Space-Time Incompressible Navier-Stokes

Flow Over a Cylinder, 4 Refinements



Space-Time Incompressible Navier-Stokes

Solve Times and Strong Scaling

Transient Flow Over a Cylinder

			1 Node	4 Nodes		32 Nodes	
Ref	Elems	DOFs	Time	Time	Scaling vs 1	Time	Scaling vs 4
0	80	31304	1772	453	3.91	451	1.01
1	605	225908	8190	3574	2.29	717	4.98
2	3013	1081598	32008	12076	2.65	2648	4.56
3	9726	3429384		28744		6319	4.54
4	11742	4144674				8510	

Computations on Lonestar, 1 node = 24 processors

32008 seconds = 8.8 hours

28744 seconds = 8.0 hours

8510 seconds = 2.4 hours

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Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$
\begin{aligned}
\frac{1}{\mu}\mathbb{D} - \nabla \boldsymbol{u} &= 0 \\
\frac{Pr}{C_p \mu}\boldsymbol{q} + \nabla T &= 0 \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \\ \rho \end{pmatrix} &= f_c \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \mathbb{I} - \left(\mathbb{D} + \mathbb{D}^T - \frac{2}{3}\operatorname{tr}(\mathbb{D})\mathbb{I}\right) \\ \rho \boldsymbol{u} \end{pmatrix} &= \boldsymbol{f}_m \\
\nabla_{xt} \cdot \begin{pmatrix} \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho R T \boldsymbol{u} + \boldsymbol{q} - \boldsymbol{u} \cdot \left(\mathbb{D} + \mathbb{D}^T - \frac{2}{3}\operatorname{tr}(\mathbb{D})\mathbb{I}\right) \\ \rho\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) \end{pmatrix} &= f_e
\end{aligned}
$$

Space-Time Compressible Navier-Stokes

Compact Notation

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Conserved quantities

$$C_c := \rho$$

$$\boldsymbol{C}_m := \rho \boldsymbol{u}$$

$$C_e := \rho(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})$$

Euler fluxes

$$\boldsymbol{F}_c := \rho \boldsymbol{u}$$

$$\mathbb{F}_m := \rho \boldsymbol{u} \otimes \boldsymbol{u} + \rho R T \mathbb{I}$$

$$\boldsymbol{F}_e := \rho \boldsymbol{u}\left(C_v T + \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right) + \rho R T \boldsymbol{u}$$

Viscous fluxes

$$\boldsymbol{K}_c := \boldsymbol{0}$$

$$\mathbb{K}_m := \mathbb{D} + \mathbb{D}^T - \frac{2}{3}\operatorname{tr}(\mathbb{D})\mathbb{I}$$

$$\boldsymbol{K}_e := -\boldsymbol{q} + \boldsymbol{u}\cdot\left(\mathbb{D} + \mathbb{D}^T - \frac{2}{3}\operatorname{tr}(\mathbb{D})\mathbb{I}\right)$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \mathbb{D}$$

$$\boldsymbol{M}_{\boldsymbol{q}} := \frac{Pr}{C_p}\boldsymbol{q}$$

$$\boldsymbol{G}_{\mathbb{D}} := 2\boldsymbol{u}$$

$$G_{\boldsymbol{q}} := -T$$

Use change of variables to get conservation or entropy variables.

Change of variables:

$$\rho = \rho$$
$$\boldsymbol{m} = \rho \boldsymbol{u}$$
$$E = \rho \left(C_v T + \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right)$$

Euler fluxes:

$$\boldsymbol{F}_c^c = \boldsymbol{m}$$
$$\mathbb{F}_m^c = \frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\rho} + (\gamma - 1) \left(E - \frac{\boldsymbol{m} \cdot \boldsymbol{m}}{2\rho} \right) \mathbb{I}$$
$$\boldsymbol{F}_e^c = \gamma E \frac{\boldsymbol{m}}{\rho} - (\gamma - 1) \frac{\boldsymbol{m} \cdot \boldsymbol{m}}{2\rho^2} \boldsymbol{m}$$

Space-Time Compressible Navier-Stokes

Entropy Variables (Symmetrize the Bubnov-Galerkin Stiffness Matrix)

Change of variables:

$$V_c = \frac{-E + (E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})\left(\gamma + 1 - \ln\left[\frac{(\gamma-1)(E-\frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m})}{\rho^\gamma}\right]\right)}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}$$

$$\boldsymbol{V}_m = \frac{\boldsymbol{m}}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}$$

$$V_e = \frac{-\rho}{E - \frac{1}{2\rho}\boldsymbol{m}\cdot\boldsymbol{m}}$$

Euler fluxes:

$$\boldsymbol{F}_c^e = \left[\frac{\gamma-1}{(-V_e)^\gamma}\right]^{\frac{1}{\gamma-1}} \exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{\gamma-1}\right]\boldsymbol{V}_m$$

$$\mathbb{F}_m^e = \left[\frac{\gamma-1}{(-V_e)^\gamma}\right]^{\frac{1}{\gamma-1}} \exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{\gamma-1}\right]\left(-\frac{\boldsymbol{V}_m\otimes\boldsymbol{V}_m}{V_e} + (\gamma-1)\mathbb{I}\right)$$

$$\boldsymbol{F}_e^e = \left[\frac{\gamma-1}{(-V_e)^\gamma}\right]^{\frac{1}{\gamma-1}} \exp\left[\frac{-\gamma + V_c - \frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m}{\gamma-1}\right]\frac{\boldsymbol{V}_m}{V_e}\left(\frac{1}{2V_e}\boldsymbol{V}_m\cdot\boldsymbol{V}_m - \gamma\right)$$

Group terms

$$C := \{C_c\,,\ \boldsymbol{C}_m\,,\ C_e\}$$

$$F := \{\boldsymbol{F}_c\,,\ \mathbb{F}_m\,,\ \boldsymbol{F}_e\}$$

$$K := \{\boldsymbol{K}_c\,,\ \mathbb{K}_m\,,\ \boldsymbol{K}_e\}$$

$$M := \{\mathbb{M}_{\mathbb{D}}\,,\ \boldsymbol{M}_{\boldsymbol{q}}\}$$

$$G := \{\boldsymbol{G}_{\mathbb{D}}\,,\ G_{\boldsymbol{q}}\}$$

$$f := \{f_c\,, \boldsymbol{f}_m\,, f_e\}$$

Group variables

$$W := \{\rho\,,\ \boldsymbol{u}\,,\ T\}$$

$$\hat{W} := \left\{2\hat{\boldsymbol{u}}\,,\ -\hat{T}\right\}$$

$$\Sigma := \{\mathbb{D}, \boldsymbol{q}\}$$

$$\hat{t} := \{\hat{t}_e\,,\ \hat{\boldsymbol{t}}_m,\ ,\ \hat{t}_e\}$$

$$\Psi := \{\mathbb{S}\,,\ \boldsymbol{\tau}\}$$

$$V := \{v_c\,,\ \boldsymbol{v}_m,\ ,\ v_e\}$$

Navier-Stokes variational formulation is

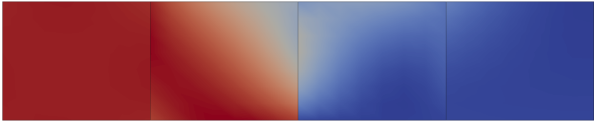
$$\left(\frac{1}{\mu}M, \Psi\right) + (G, \nabla\cdot\Psi) - \left\langle \hat{W}, \Psi\cdot\boldsymbol{n}_x \right\rangle = 0$$

$$-\left(\begin{pmatrix} F-K \\ C \end{pmatrix}, \nabla_{xt} V\right) + \langle \hat{t}, V\rangle = (f, V)$$

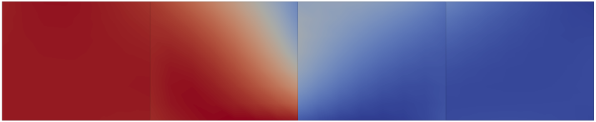
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

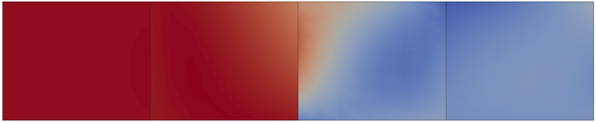
Mesh 1



Primitive Variables



Conservation Variables



Entropy Variables

Space-Time Compressible Navier-Stokes

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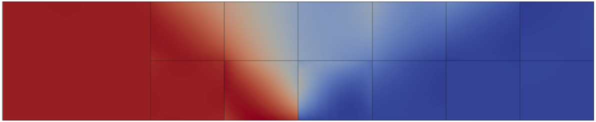
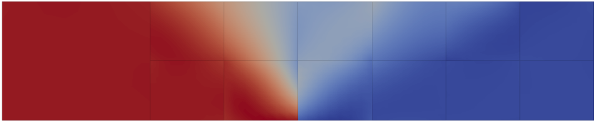
Sod Shock Tube with $\mu = 10^{-5}$

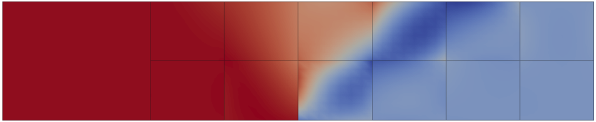
Mesh 2



Primitive Variables



Conservation Variables

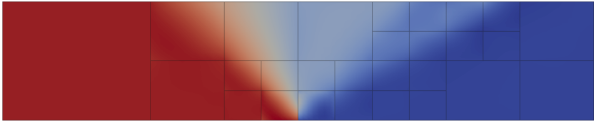


Entropy Variables

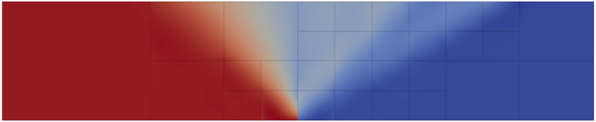
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

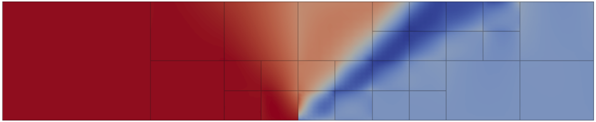
Mesh 3



Primitive Variables



Conservation Variables

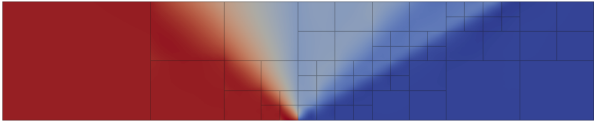


Entropy Variables

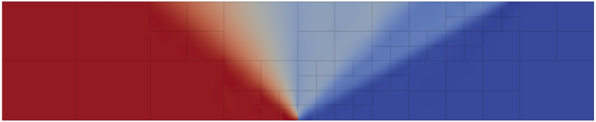
Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

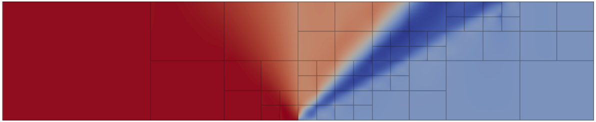
Mesh 4



Primitive Variables



Conservation Variables

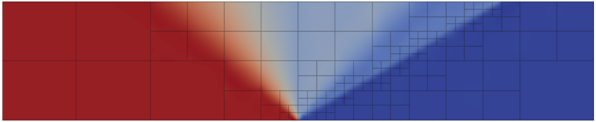


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 5



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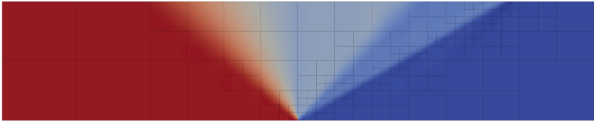

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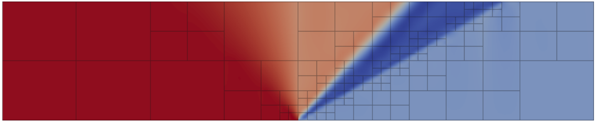
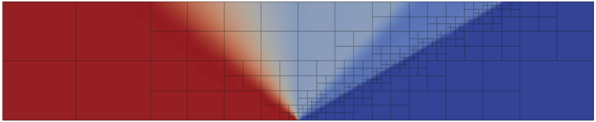


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 6



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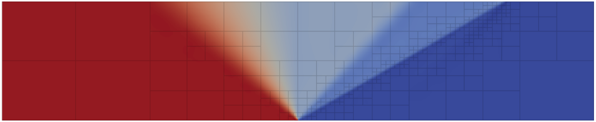

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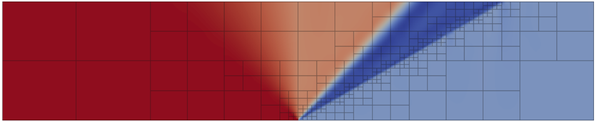
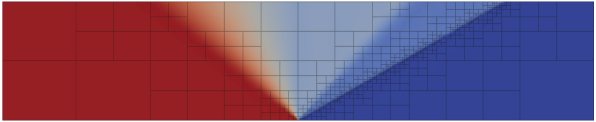


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 7



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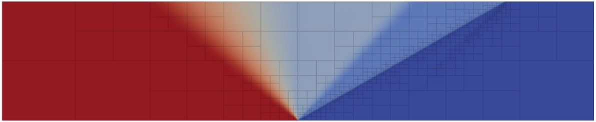

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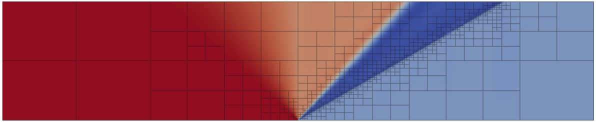
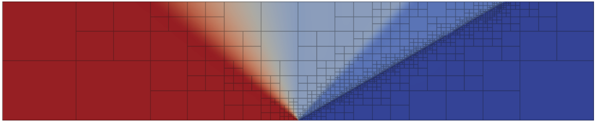


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 8



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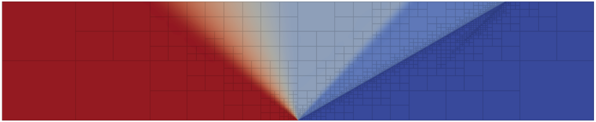

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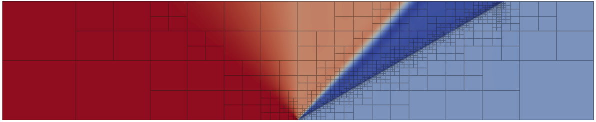
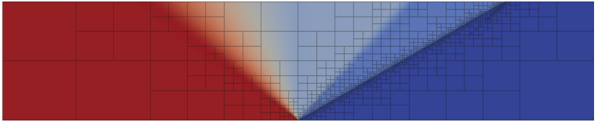


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 9



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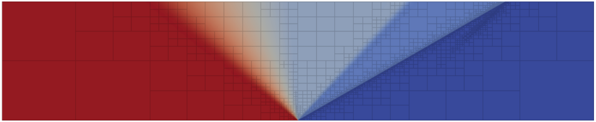

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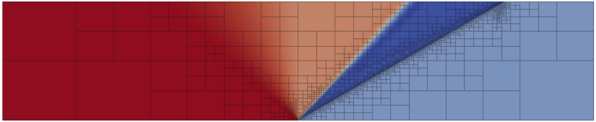
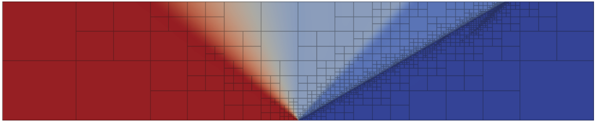


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 10



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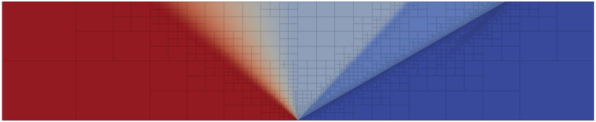

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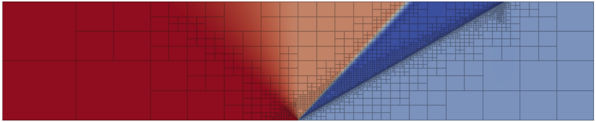
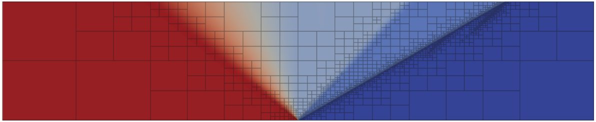


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 11



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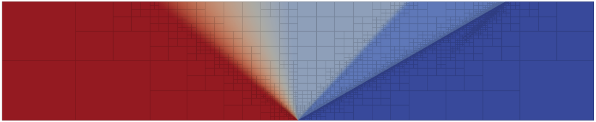

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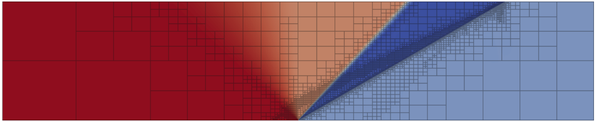
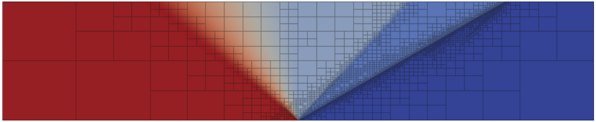


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 12



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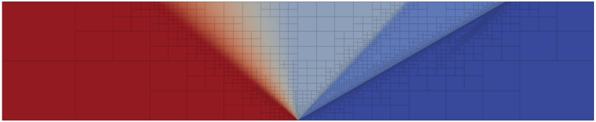

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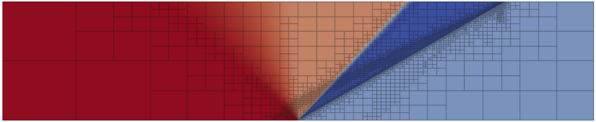
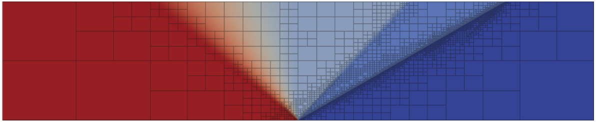


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 13



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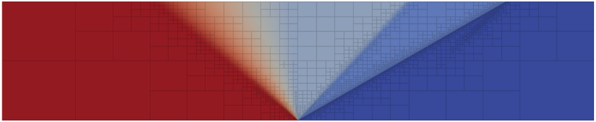

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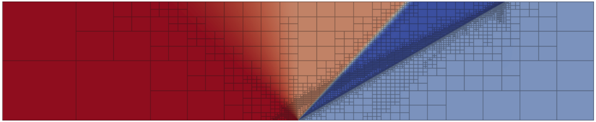
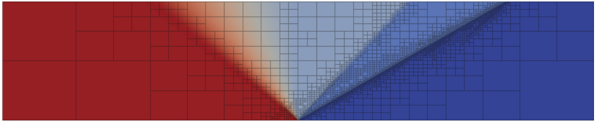


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 14



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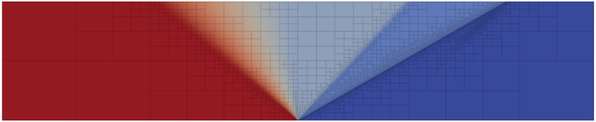

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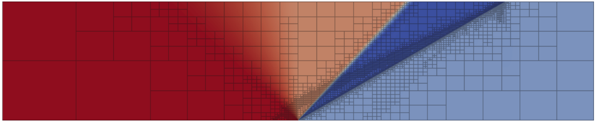
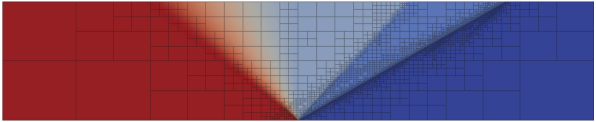


Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

Mesh 15



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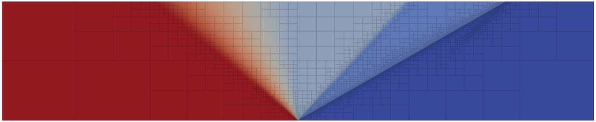

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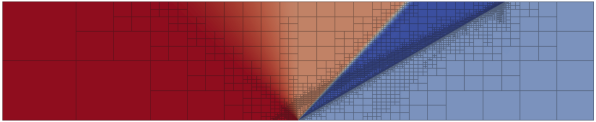
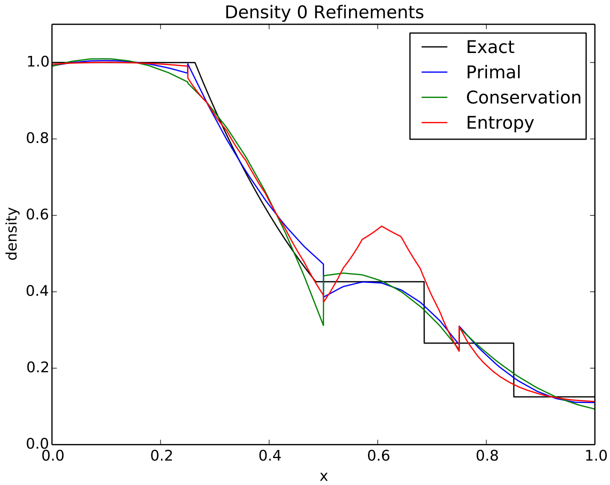
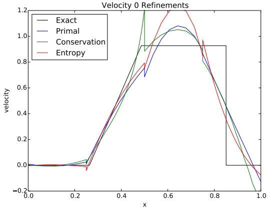


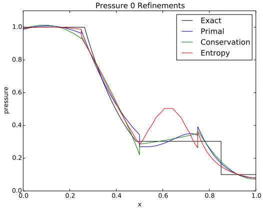
Entropy Variables

Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

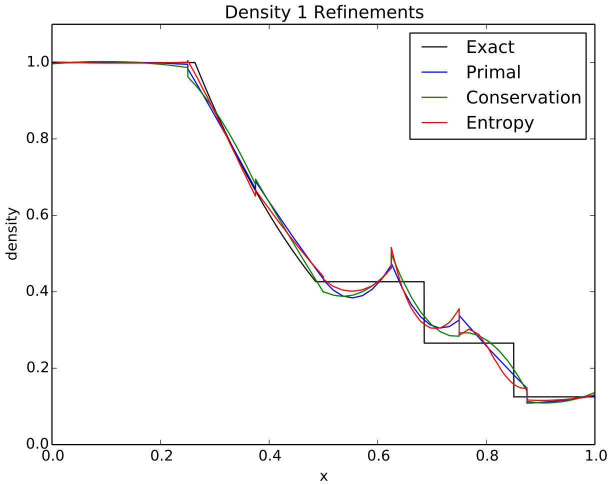


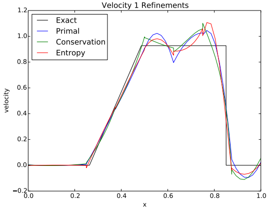


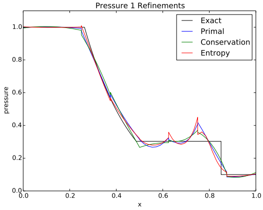


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$







Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

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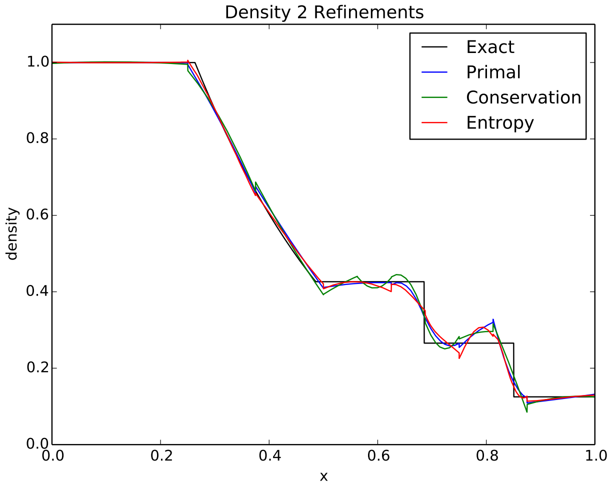
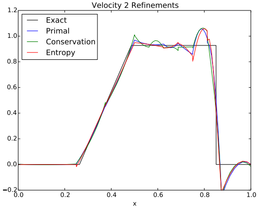
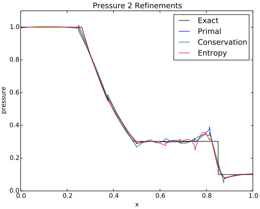
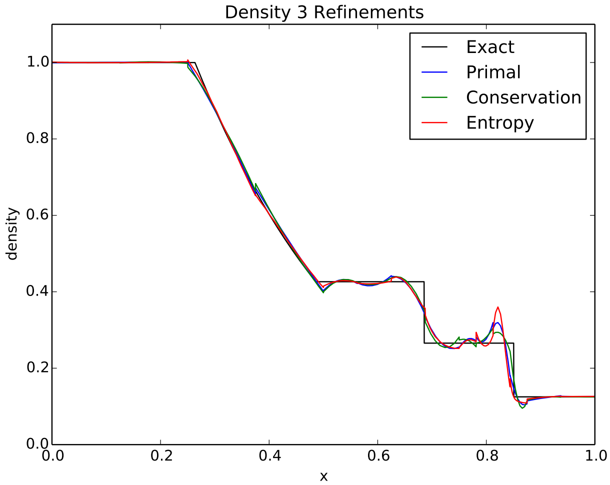


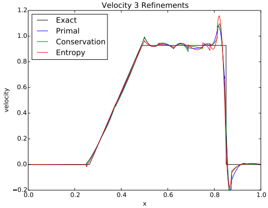


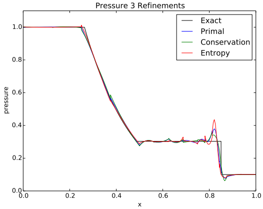


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$







Space-Time Compressible Navier-Stokes

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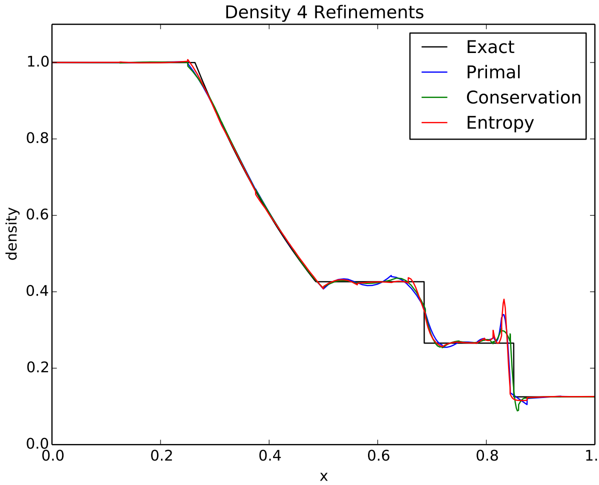
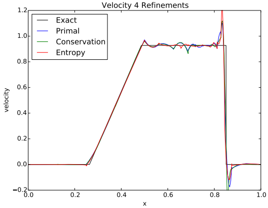
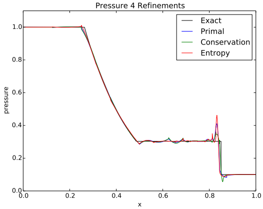
Sod Shock Tube with $\mu = 10^{-5}$

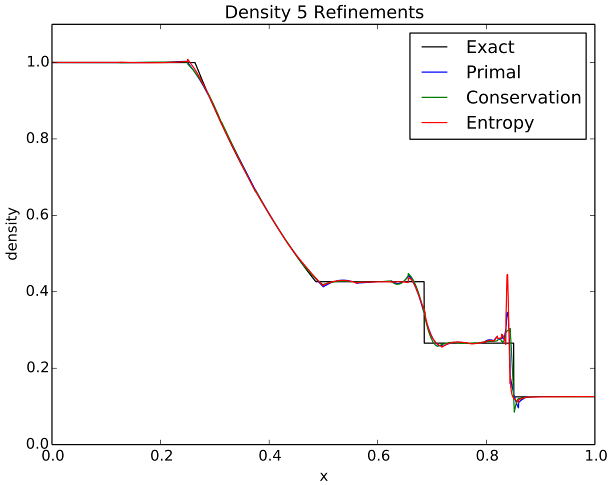


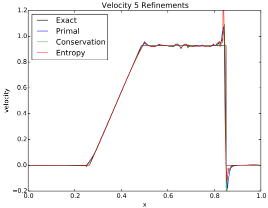


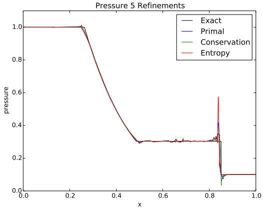


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

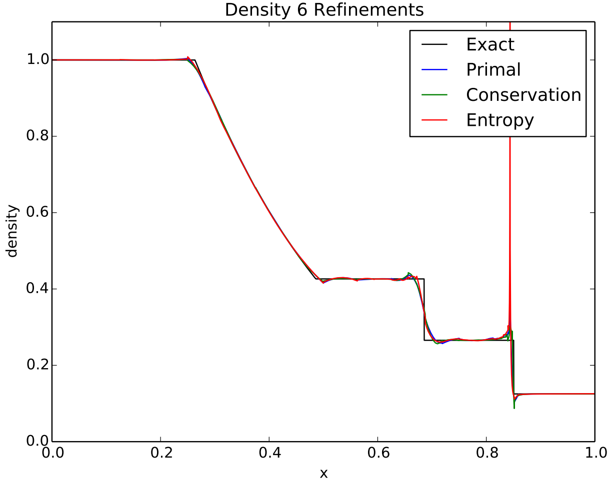


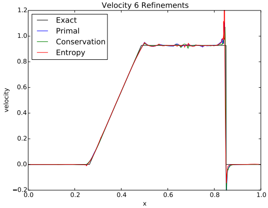


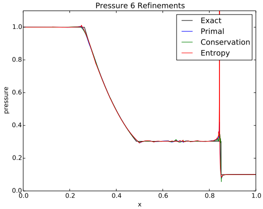


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

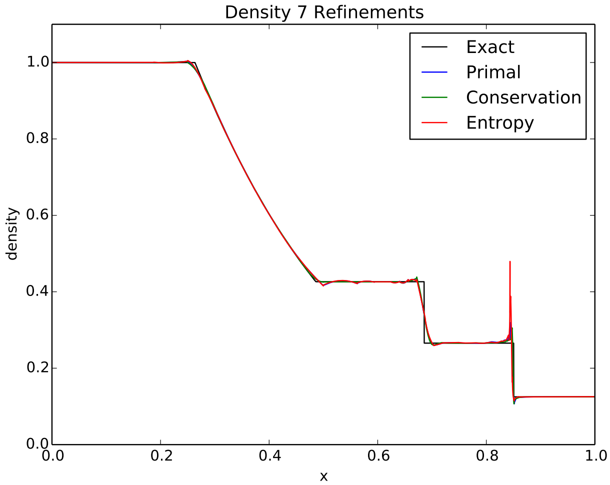


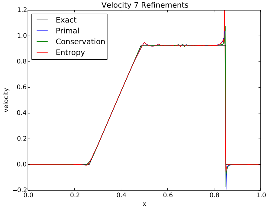


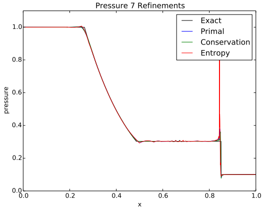


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

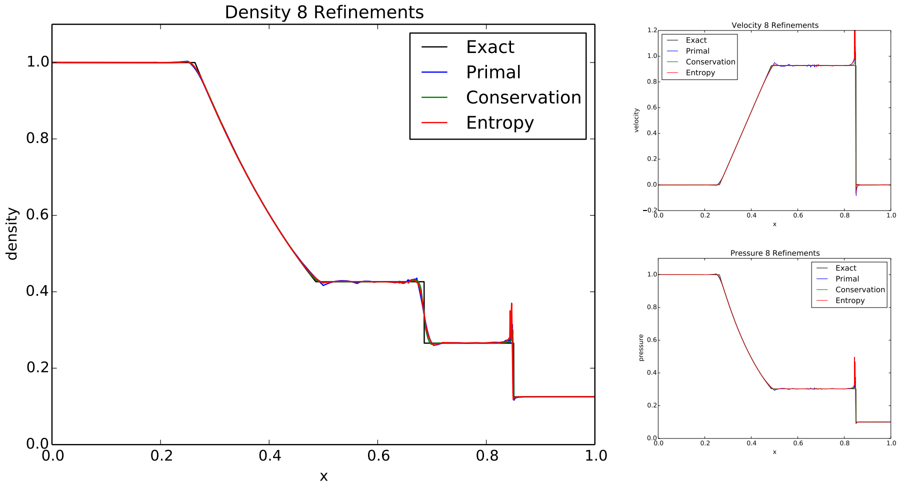






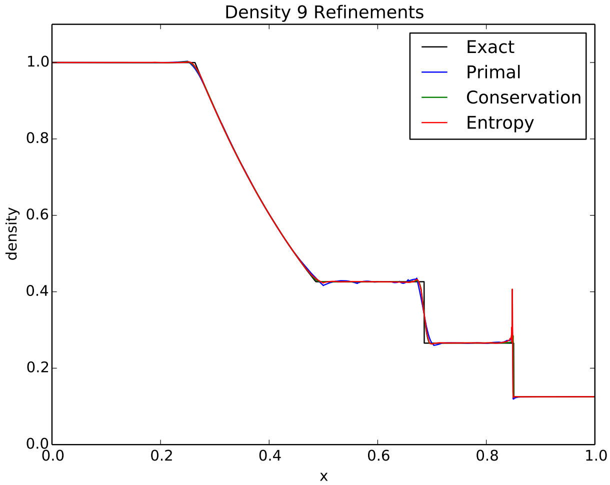
Space-Time Compressible Navier-Stokes

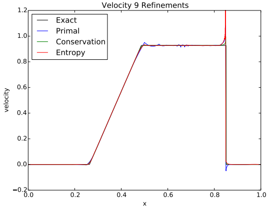
Sod Shock Tube with $\mu = 10^{-5}$

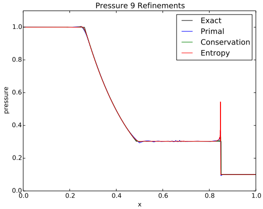


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

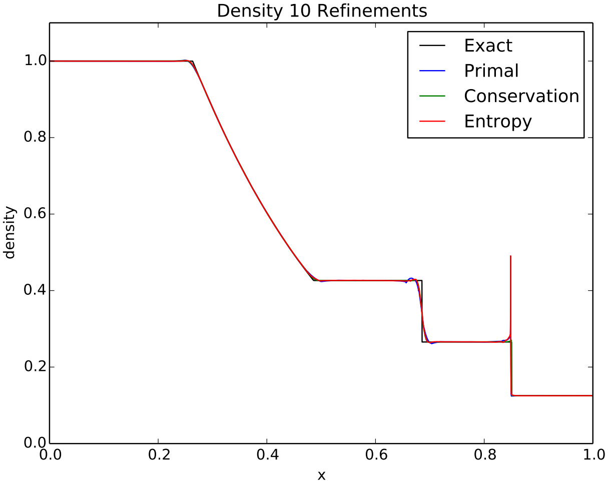


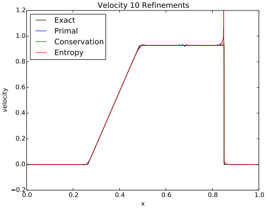


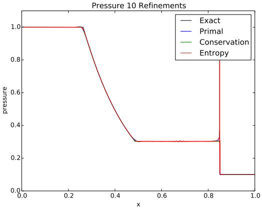


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

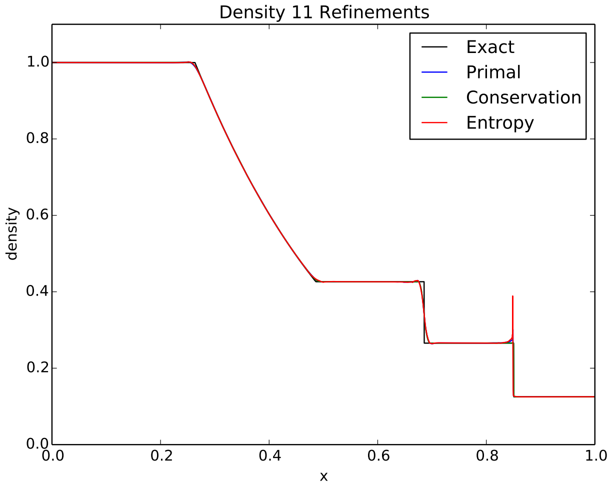


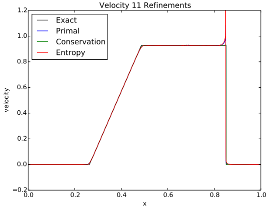


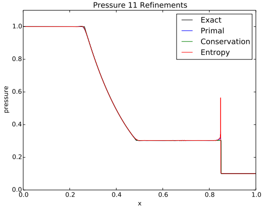


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

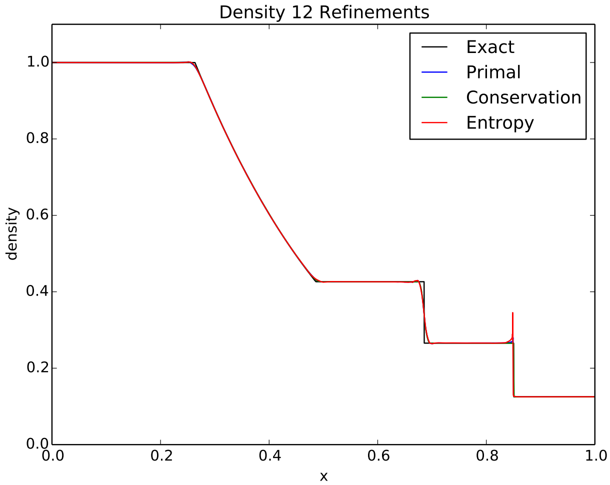


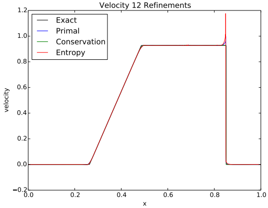


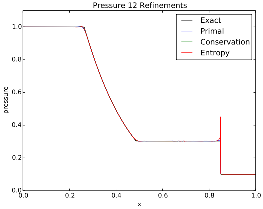


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

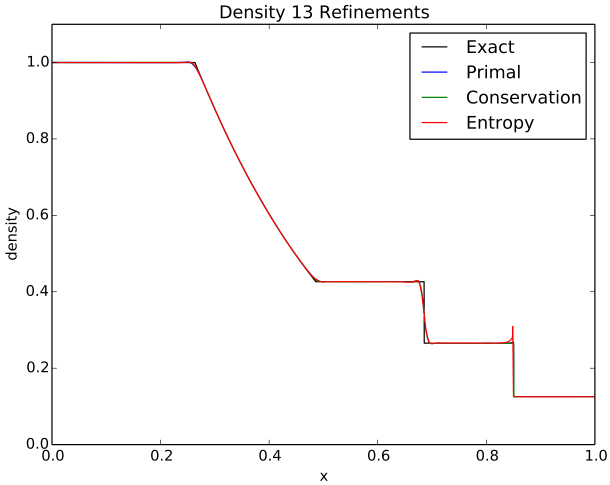


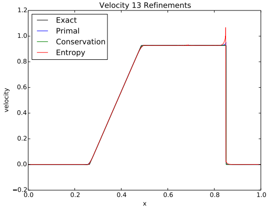


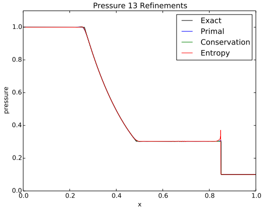


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$

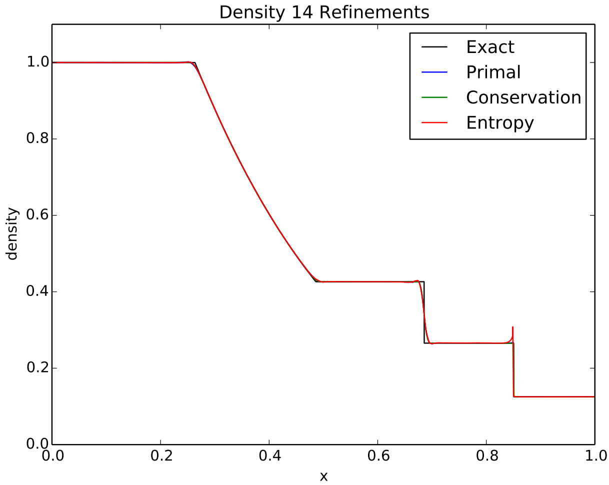


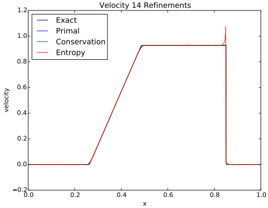


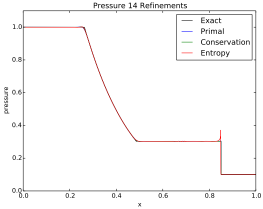


Space-Time Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$







Let W, U, and V be the set of primitive, conservation, and entropy variables.

The entropy function

$$H = -\rho \log(p\rho^{-\gamma})$$

provides a natural residual for the Navier-Stokes equations.

$A_0 = H_{,UU} = V_{,U}$ is known as the symmetrizer and $(U, A_0 U)$ provides a natural metric for the Euler equations.

In primitive variables:

$$(u, A_0 U) = (U_{,W} W, V_{,U} U_{,W} W) = (W, U_{,W}^T V_{,U} U_{,W} W) = (W, A_0(W) W)$$

where

$$A_0(W) = U_{,W}^T V_{,U} U_{,W} = \begin{bmatrix} \frac{\gamma-1}{\rho} & 0 & 0 \\ 0 & \frac{\rho}{C_v T} & 0 \\ 0 & 0 & \frac{\rho}{T^2} \end{bmatrix}$$

Entropy Scaled Test Norms

Bilinear form with group variables:

$$b\left(\left(W,\hat{W}\right),v\right)=\left(W,A_h^* v\right)_{L^2(\Omega_h)}+\left\langle \hat{W},\llbracket v\rrbracket\right\rangle_{\Gamma_h}$$

For conforming v^* satisfying $A^* v^* = A_0 W$

$$\begin{aligned}\left\|A_0^{\frac{1}{2}}W\right\|^2 &= \frac{b(W,v^*)}{\|v^*\|_V}\|v^*\|_V\\ &\leq \sup_{v^*\neq 0}\frac{|b(W,v^*)|}{\|v^*\|}\|v^*\| = \|W\|_E\|v^*\|_V\ .\end{aligned}$$

Necessary robustness condition:

$$\begin{aligned}\|v^*\|_V &\lesssim \left\|A_0^{\frac{1}{2}}W\right\|_{L^2(\Omega_h)}\\ \Rightarrow \left\|A_0^{\frac{1}{2}}W\right\|_{L^2(\Omega_h)} &\lesssim \|W\|_E\end{aligned}$$

We load our adjoint equations with $A_0 W$:

$$\frac{1}{\mu} M^*(\Psi) + K^*(\nabla V) = 0$$

$$-\begin{pmatrix} F^* \\ C^* \end{pmatrix} (\nabla_{xt} V) + G^*(\nabla \Psi) = A_0 W$$

This leads to the entropy scaled robust norm:

$$\begin{aligned} \|(V,\Psi)\|^2_{V,K} := {} & \left\| A_0^{-\frac{1}{2}} (F^* + C^*) \right\|_K^2 + \mu \left\| A_0^{-\frac{1}{2}} K^* \right\|_K^2 \\ & + \min\left(\frac{\mu}{h^2}, 1\right) \left\| A_0^{-\frac{1}{2}} V \right\|_K^2 + \left\| A_0^{-\frac{1}{2}} G^* \right\|_K^2 \\ & + \min\left(\frac{1}{\mu}, \frac{1}{h^2}\right) \left\| A_0^{-\frac{1}{2}} M^* \right\|_K^2 \end{aligned}$$

Space-Time Compressible Navier-Stokes

Entropy Scaled Test Norms

We load our adjoint equations with $A_0 W$:

$$\frac{1}{\mu} M^*(\Psi) + K^*(\nabla V) = 0$$

$$-\begin{pmatrix} F^* \\ C^* \end{pmatrix} (\nabla_{xt} V) + G^*(\nabla \Psi) = A_0 W$$

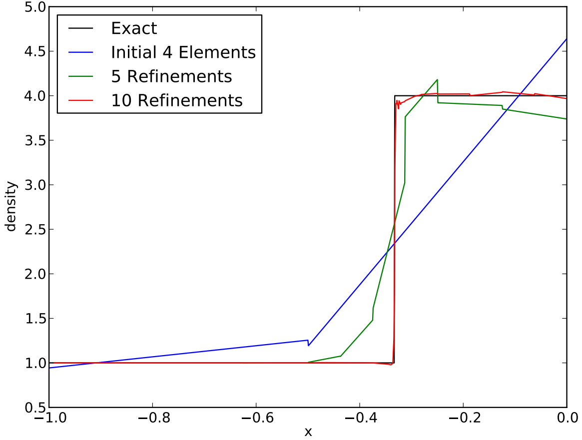
This leads to the entropy scaled robust norm:

$$\begin{aligned} \|(V,\Psi)\|^2_{V,K} := & \left\| A_0^{-\frac{1}{2}} (F^* + C^*) \right\|^2_K + \mu \left\| A_0^{-\frac{1}{2}} K^* \right\|^2_K \\ & + \min\left(\frac{\mu}{h^2}, 1\right) \left\| A_0^{-\frac{1}{2}} V \right\|^2_K + \left\| A_0^{-\frac{1}{2}} G^* \right\|^2_K \\ & + \min\left(\frac{1}{\mu}, \frac{1}{h^2}\right) \left\| A_0^{-\frac{1}{2}} M^* \right\|^2_K \end{aligned}$$

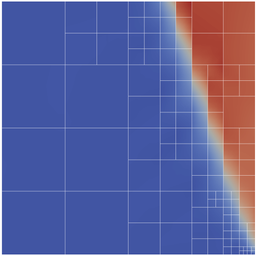
Numerical results were disappointing.

Space-Time Compressible Navier-Stokes

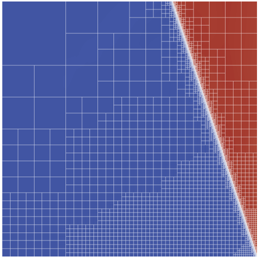
Noh Implosion with Primitive Variables, Robust Norm, $\mu = 10^{-3}$



After 5 refinements



After 10 refinements



Space-Time Compressible Navier-Stokes

Piston with $\mu = 10^{-2}$

$$\hat{t}_c = \sqrt{2}(-\rho u + \rho)$$

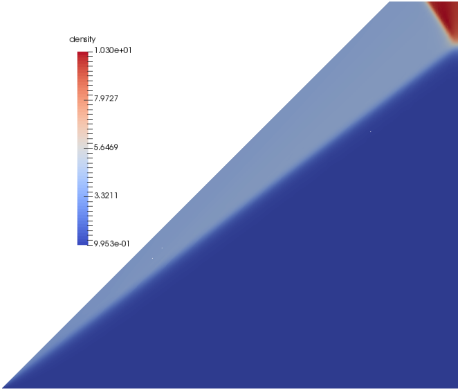
$$\hat{t}_m = \sqrt{2}(-\rho u^2 - \rho RT + \rho u)$$

$$\hat{t}_e = \sqrt{2}(-\rho u(C_v T + \frac{1}{2}u^2) - u\rho RT + \rho(C_v T + \frac{1}{2}u^2))$$

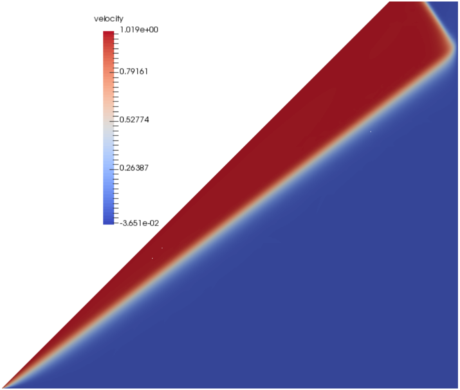
$$\hat{u} = 1$$

$$\hat{t}_c = 0$$

$$\hat{t}_m - \hat{t}_e = 0$$



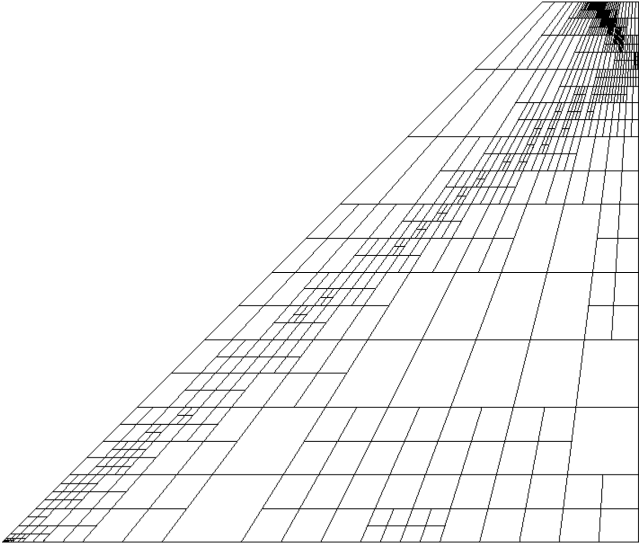
Density



Velocity

Space-Time Compressible Navier-Stokes

Piston with $\mu = 10^{-2}$



Mesh after 8 adaptive refinements

Summary of Work

- **Area A:**
  - Developed a provably robust locally conservative DPG method with appropriate test norms
  - Extended analysis of robust DPG methods for convection-diffusion to space-time
- **Area B:**
  - Contributions to parallel $hp-$adaptive Camellia[9] (space-time, parallel solution/mesh I/O and export, templating)
  - Construction of higher order adaptive solvers for convection-diffusion, Stokes, Burgers' equation, incompressible Navier-Stokes, and compressible Navier-Stokes
- **Area C:**
  - Solution of local conservation problems in convection-diffusion and Stokes flow problems
  - Solution of 2D space-time convection-diffusion and incompressible Navier-Stokes problems
  - Solution of 1D space-time compressible Navier-Stokes shock tube problems

[9] N.V. Roberts. ``Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1581–1604.

Future Directions

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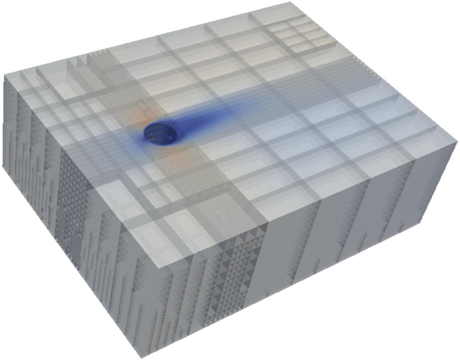
- **Improve performance:** line smoothing for multigrid
- **Shock capturing:** DPG makes no promises when it comes to Gibbs phenomenon
- **Non-Hilbert DPG:** L^1 is known to limit oscillations
- **Anisotropic refinements:** necessary for time slabs
- **More extensive 2D results:** shedding vortex problems, 2D shock problems
- **3D results:** will not be cheap



Incompressible Flow Over a Cylinder

Thank You!

Recommended References

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- D. Moro, N.C. Nguyen, and J. Peraire. ``A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws''. In: *Int. J. Num. Meth. Eng.* (2011).
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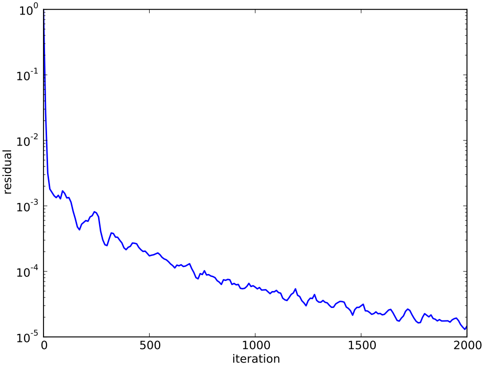
- J. Chan et al. ``A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms''. In: *Comp. Math. Appl.* 67.4 (2014), pp. 771 –795.
- T.E. Ellis, L.F. Demkowicz, and J.L. Chan. ``Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530 –1549.
- T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion*. Tech. rep. 15-21. ICES, Oct. 2015.
- N.V. Roberts. ``Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods''. In: *Comp. Math. Appl.* 68.11 (2014), pp. 1581 –1604.
- L.F. Demkowicz and J. Gopalakrishnan. ``Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)''. In: vol. 157. IMA Volumes in Mathematics and its Applications, 2014. Chap. An Overview of the DPG Method, pp. 149–180.
- L.F. Demkowicz and N. Heuer. ``Robust DPG Method for Convection-Dominated Diffusion Problems''. In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
- N. Roberts, T. Bui-Thanh, and L. Demkowicz. ``The DPG method for the Stokes problem''. In: *Comp. Math. Appl.* 67.4 (2014), pp. 966 –995.

Scaling Issues

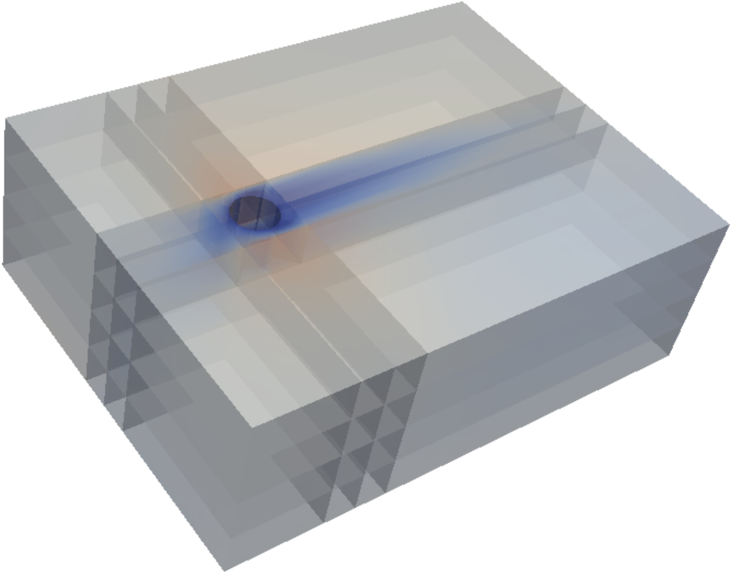
Multigrid and Convection-Diffusion

- Convection-diffusion, $\epsilon = 10^{-2}, 64 \times 64$ mesh
- Conjugate gradient
- Geometric multigrid preconditioner
- Multiplicative V-cycle
- Overlapping additive Schwarz smoother
- Hierarchy of $p-$coarsening followed by $h-$coarsening



Scaling Issues

Incompressible Flow Over a Cylinder, Initial Mesh



Scaling Issues

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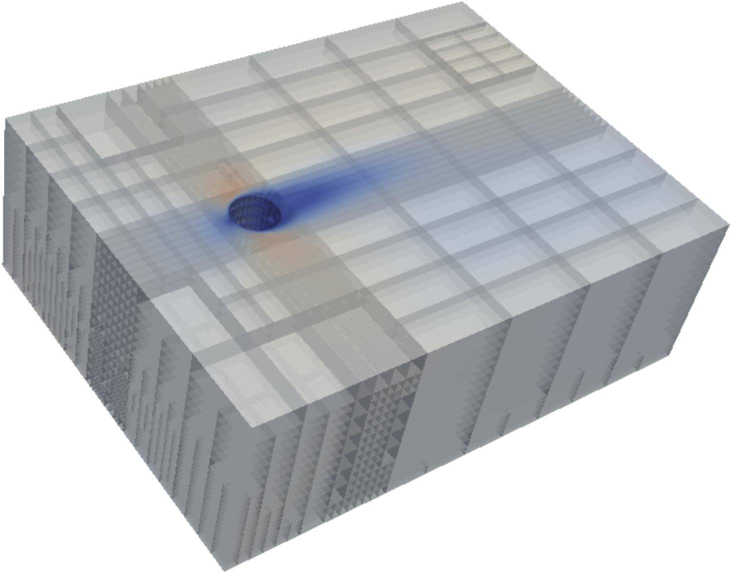
Incompressible Flow Over a Cylinder, 4 Refinements



Scaling Issues

Solve Times and Strong Scaling

Transient Flow Over a Cylinder

			1 Node	4 Nodes		32 Nodes	
Ref	Elems	DOFs	Time	Time	Scaling vs 1	Time	Scaling vs 4
0	80	31304	1772	453	3.91	451	1.01
1	605	225908	8190	3574	2.29	717	4.98
2	3013	1081598	32008	12076	2.65	2648	4.56
3	9726	3429384		28744		6319	4.54
4	11742	4144674				8510	

Computations on Lonestar, 1 node = 24 processors

32008 seconds = 8.8 hours

28744 seconds = 8.0 hours

8510 seconds = 2.4 hours

Scaling Issues

Solve Times and Strong Scaling

Taylor-Green Vortex

			1 Node	4 Nodes	
Ref	Elems	DOFs	Time	Time	Scaling vs 1
0	60	21302	331	140	2.35
1	312	108410	945	290	3.25
2	2020	691834	4880	1363	3.58
3	9244	3043024		6171	

Computations on Lonestar, 1 node = 24 processors

4880 seconds = 1.4 hours

6171 seconds = 1.7 hours

Assumptions:

- The maximum required spatial resolution is much finer than the required temporal resolution.
- Regions requiring high spatial resolution are concentrated in relatively compact parts of the domain.
- Only isotropic refinements are permitted.
- The number of time slabs is a power of 2.

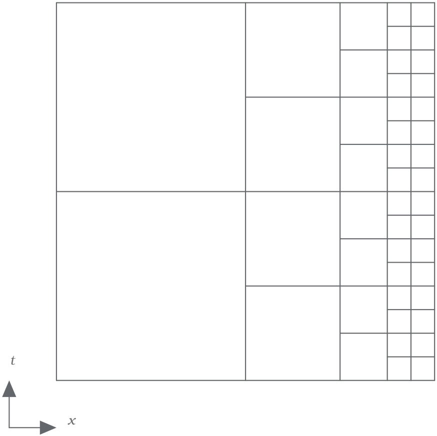
Test problem: convection diffusion with exact solution

$$u = 1 - e^{\frac{x}{\epsilon}}$$

on space-time domain $[-1, 0] \times [0, 1]$.

Scaling Issues

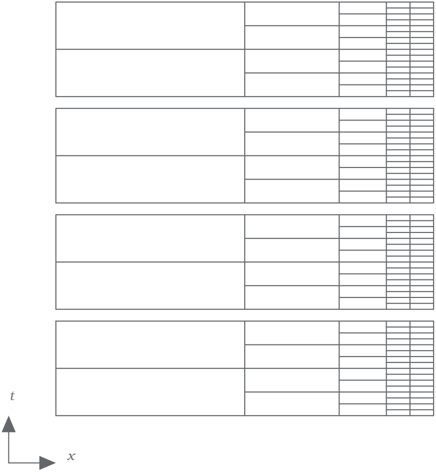
Space-Time Slabs



Single Slab Strategy

Scaling Issues

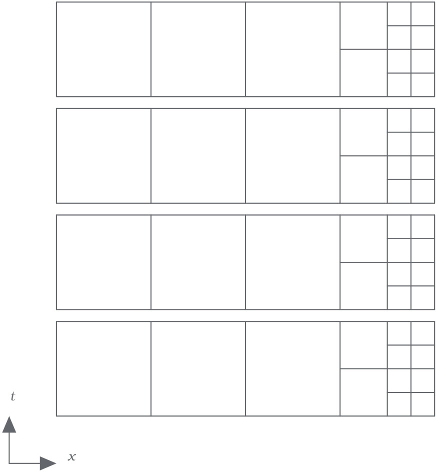
Space-Time Slabs



Naive Time Slab Strategy

Scaling Issues

Space-Time Slabs

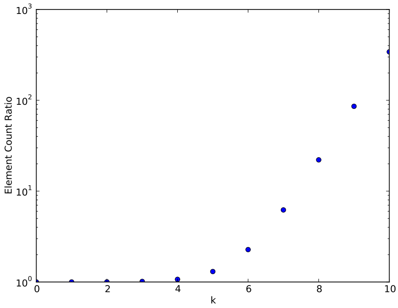


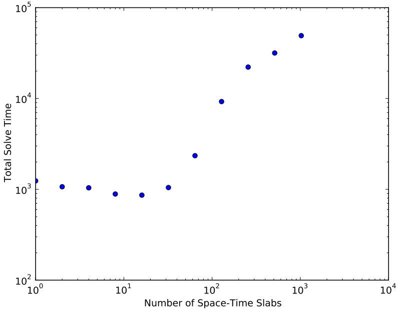
Smarter Time Slab Strategy

Scaling Issues

Space-Time Slabs

Number of time slabs = 2^k.





Ratio of total element counts

Total solve time using smart time slabs

- Without anisotropic refinements, time slabs don't significantly speed up computations.
- Time slabs could be useful for memory constrained problems.

Camellia: DPG for the Masses

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Design Goal

Make DPG research and experimentation as simple as possible, while maintaining computational efficiency and scalability.

Built on Trilinos (Teuchos, Intrepid, Shards, Epetra, etc).

Mature support for:

- Rapid specification of DPG variational forms, inner products, etc.
- Distributed computation of stiffness matrix
- 1D - 3D geometries
- Curvilinear elements
- h- and p-refinements (anisotropic in h)

Experimental support for:

- Space-time computations
- Iterative solvers (tested up to 32,768 cores)

Convection-Diffusion in Three Slides

Building the Bilinear Form

```
VarFactory vf;
//fields:
VarPtr u = vf.fieldVar("u", L2);
VarPtr sigma = vf.fieldVar("sigma", VECTOR_L2);

// traces:
VarPtr u_hat = vf.traceVar("u_hat");
VarPtr t_n = vf.fluxVar("t_n");

// test:
VarPtr v = vf.testVar("v", HGRAD);
VarPtr tau = vf.testVar("tau", HDIV);

double eps = .01;
FunctionPtr beta_x = Function::constant(1);
FunctionPtr beta_y = Function::constant(2);
FunctionPtr beta = Function::vectorize(beta_x, beta_y);

BFPtr bf = Teuchos::rcp( new BF(vf) );

bf->addTerm((1/eps) * sigma, tau);
bf->addTerm(u, tau->div());
bf->addTerm(-u_hat, tau->dot_normal());

bf->addTerm(sigma - beta * u, v->grad());
bf->addTerm(t_n, v);

RHSPtr rhs = RHS::rhs();
```

Find $u \in L^2(\Omega_h)$, $\boldsymbol{\sigma} \in \boldsymbol{L}^2(\Omega_h)$, $\hat{u} \in H^{\frac{1}{2}}(\Gamma_h)$, $\hat{t}_n \in H^{-\frac{1}{2}}(\Gamma_h)$ such that

$$\frac{1}{\epsilon}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle - (\boldsymbol{\beta} u - \boldsymbol{\sigma}, \nabla v) + \langle \hat{t}_n, v \rangle = (f, v)$$

for all $v \in H^1(K)$, $\boldsymbol{\tau} \in \boldsymbol{H}(\text{div}, K)$,

where $\epsilon = 10^{-2}$, $\boldsymbol{\beta} = (1, 2)^T$ and

$f = 0$.

Convection-Diffusion in Three Slides

Boundary Conditions and Mesh

```
int k = 2;
int delta_k = 2;
MeshPtr mesh = MeshFactory::quadMesh(bf, k+1, delta_k);

BCPtr bc = BC::bc();

SpatialFilterPtr y_equals_one = SpatialFilter::matchingY(1.0);
SpatialFilterPtr y_equals_zero = SpatialFilter::matchingY(0);
SpatialFilterPtr x_equals_one = SpatialFilter::matchingX(1.0);
SpatialFilterPtr x_equals_zero = SpatialFilter::matchingX(0.0);

FunctionPtr zero = Function::zero();
FunctionPtr x = Function::xn(1);
FunctionPtr y = Function::yn(1);
bc->addDirichlet(t_n, y_equals_zero, -2 * (1-x));
bc->addDirichlet(t_n, x_equals_zero, -1 * (1-y));
bc->addDirichlet(u_hat, y_equals_one, zero);
bc->addDirichlet(u_hat, x_equals_one, zero);
```

Create a square mesh $[0,1] \times [0,1]$ with boundary conditions

- $\hat{t}_n = 2x - 2$ on $y = 0$
- $\hat{t}_n = x - 1$ on $x = 0$
- $\hat{u} = 0$ on $y = 1$
- $\hat{u} = 0$ on $x = 1$

Note

- Can subclass SpatialFilter to match any geometry
- Adding new mesh readers is straightforward

Convection-Diffusion in Three Slides

Test Norm, Solving, and Adaptivity

```
IPPtr ip = bf->graphNorm();

SolutionPtr soln = Solution::solution(mesh, bc, rhs, ip);

double threshold = 0.20;
RefinementStrategy refStrategy(soln, threshold);

int numRefs = 10;

ostringstream refName;
refName << "ConvectionDiffusion";
HDF5Exporter exporter(mesh,refName.str());

for (int refIndex=0; refIndex < numRefs; refIndex++) {
  soln->solve();

  double energyError = soln->energyErrorTotal();
  cout << "After " << refIndex << " refinements, energy error is " << energyError << endl;

  exporter.exportSolution(soln, vf, refIndex);

  if (refIndex != numRefs)
    refStrategy.refine();
}
```

Convection-Diffusion in Three Slides

Computed Solution