

# Space-Time DPG for Compressible Navier-Stokes

Truman Ellis

Advisors: Leszek Demkowicz, Robert Moser,

Collaborators: Nate Roberts (Argonne), Jesse Chan (Rice)



# Overview of DPG

DPG is a Minimum Residual Method

Find  $u \in U$  such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator  $B : U \rightarrow V'$  defined by  $b(u, v) = \langle Bu, v \rangle_{V' \times V}$ .

This gives the operator equation

$$Bu = l \quad \in V'.$$

We wish to minimize the residual  $Bu - l \in V'$ :

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|R_V^{-1}(Bw_h - l)\|_V^2 .$$

# Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions  $\delta u \in U_h$  gives,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions  $v_{\delta u_h} := R_V^{-1}B\delta u_h$  for each trial function  $\delta u_h$ .

## Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with  $v_{\delta u_h} \in V$  that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

# Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\sigma - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\beta u - \sigma) = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \beta u - \sigma \\ u \end{pmatrix} &= f \end{aligned}$$

# Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

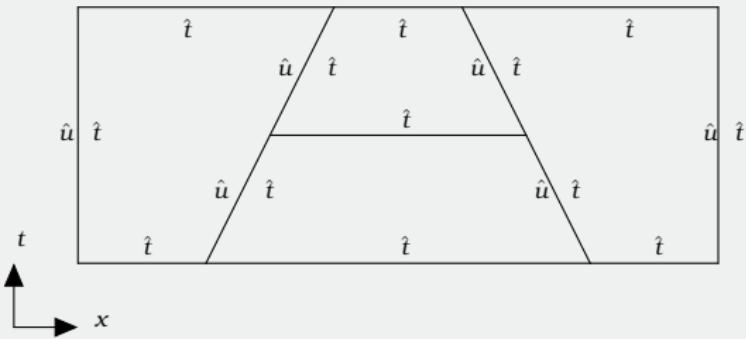
$$\begin{aligned} \left( \frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right)_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle_{\partial K} &= 0 \\ - \left( \begin{pmatrix} \beta u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right)_K + \langle \hat{t}, v \rangle_{\partial K} &= f \end{aligned}$$

where

$$\hat{u} := \text{tr}(u)$$

$$\begin{aligned} \hat{t} &:= \text{tr}(\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}_x \\ &\quad + \text{tr}(u) \cdot n_t \end{aligned}$$

## Support of Trace Variables



- Trace  $\hat{u}$  defined on spatial boundaries
- Flux  $\hat{t}$  defined on all boundaries

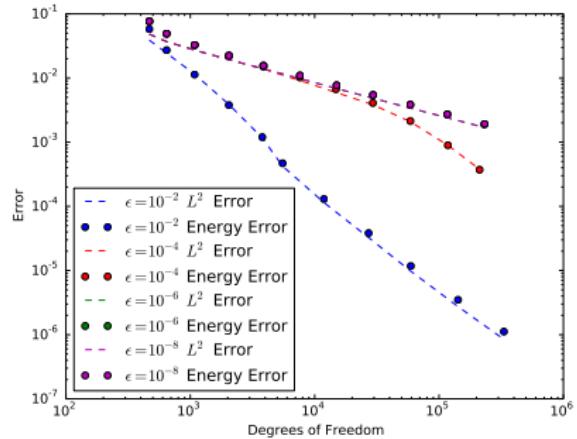
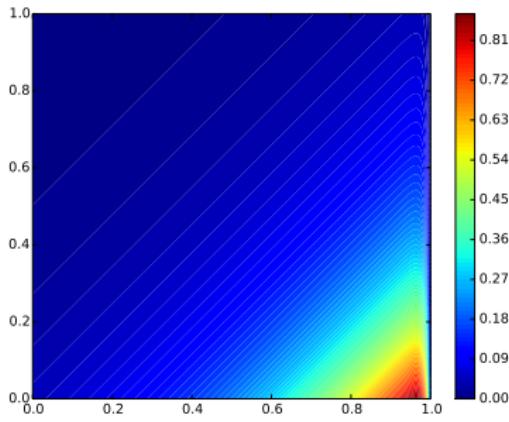
# Space-Time Convection-Diffusion

Robust Convergence to Analytical Solution

$$u = e^{-lt} (e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}), \quad \lambda_{1,2} = \frac{-1 \pm \sqrt{1 - 4l\epsilon}}{-2\epsilon}$$

Test Norm:

$$\begin{aligned} \|(\boldsymbol{v}, \boldsymbol{\tau})\|^2 &= \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\boldsymbol{\beta}} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \min \left( \frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2 \\ &\quad + \epsilon \|\nabla \boldsymbol{v}\|^2 + \|\boldsymbol{\beta} \cdot \nabla \boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2 \end{aligned}$$



# Space-Time Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$\frac{1}{\mu} \mathbb{D} - (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{2}{3} \nabla \cdot \mathbf{u} \mathbb{I} = 0$$

$$\frac{Pr}{C_p \mu} \mathbf{q} + \nabla T = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \rho R T \mathbf{u} + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e,$$

# Space-Time Navier-Stokes

Compact Notation

Conserved quantities

$$C_c := \rho$$

$$\mathbf{C}_m := \rho \mathbf{u}$$

$$C_e := \rho(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u})$$

Euler fluxes

$$\mathbf{F}_c := \rho \mathbf{u}$$

$$\mathbb{F}_m := \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I}$$

$$\mathbf{F}_e := \rho \mathbf{u} \left( C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \rho R T \mathbf{u}$$

Viscous fluxes

$$\mathbf{K}_c := \mathbf{0}$$

$$\mathbb{K}_m := \mathbb{D}$$

$$\mathbf{K}_e := -\mathbf{q} + \mathbf{u} \cdot \mathbb{D}$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \frac{1}{\mu} \mathbb{D}$$

$$\mathbf{M}_q := \frac{Pr}{C_p \mu} \mathbf{q}$$

Viscous relations

$$\mathbf{G}_{\mathbb{D}} := 2 \mathbf{u}$$

$$G_q := -T$$

Use change of variables to get conservation or entropy variables.

# Space-Time Navier-Stokes

Define Group Variables

Group terms

$$C := \{C_c, \mathbf{C}_m, C_e\}$$

$$F := \{\mathbf{F}_c, \mathbb{F}_m, \mathbf{F}_e\}$$

$$K := \{\mathbf{K}_c, \mathbb{K}_m, \mathbf{K}_e\}$$

$$M := \{\mathbb{M}_{\mathbb{D}}, \mathbf{M}_{\mathbf{q}}\}$$

$$G := \{\mathbf{G}_{\mathbb{D}}, G_{\mathbf{q}}\}$$

$$f := \{f_c, \mathbf{f}_m, f_e\}$$

Group variables

$$W := \{\rho, \mathbf{u}, T\}$$

$$\hat{W} := \{2\hat{\mathbf{u}}, -\hat{T}\}$$

$$\Sigma := \{\mathbb{D}, \mathbf{q}\}$$

$$\hat{t} := \{\hat{t}_e, \hat{\mathbf{t}}_m, , \hat{t}_e\}$$

$$\Psi := \{\mathbb{S}, \tau\}$$

$$V := \{v_c, \mathbf{v}_m, , v_e\} .$$

Navier-Stokes variational formulation is

$$(M, \Psi) + (G, \nabla \cdot \Psi) - \langle \hat{W}, \Psi \cdot \mathbf{n}_x \rangle = 0$$

$$- \left( \begin{pmatrix} F - K \\ C \end{pmatrix}, \nabla_{xt} V \right) + \langle \hat{t}, V \rangle = (f, V) .$$

# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

Mesh 1



Primitive Variables



Conservation Variables

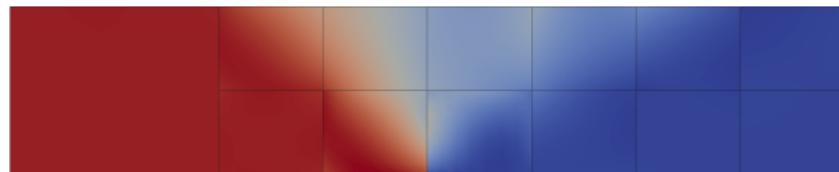


Entropy Variables

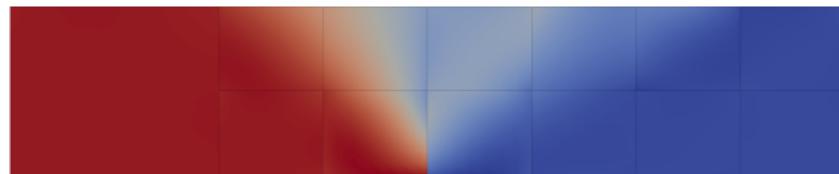
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

Mesh 2



Primitive Variables



Conservation Variables

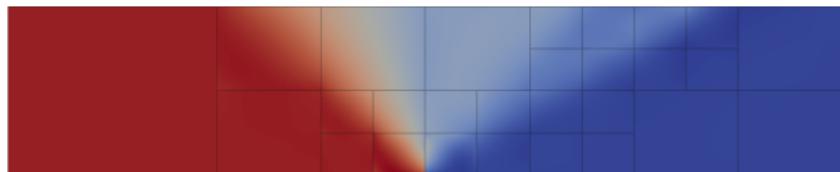


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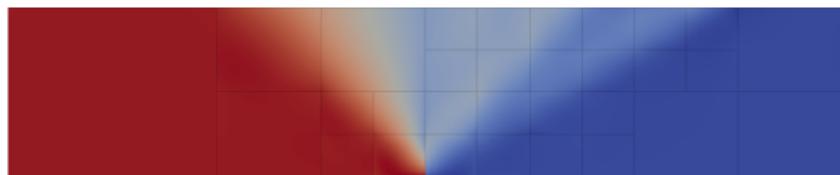
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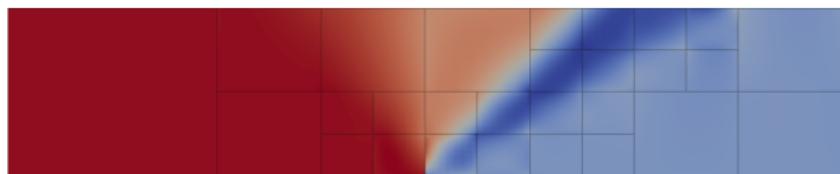
Mesh 3



Primitive Variables



Conservation Variables

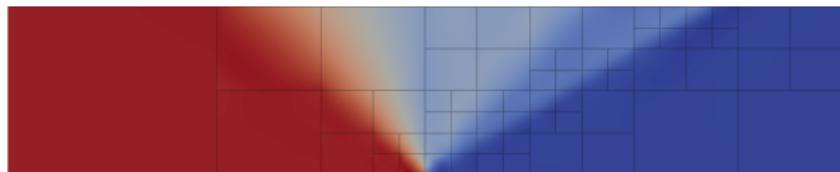


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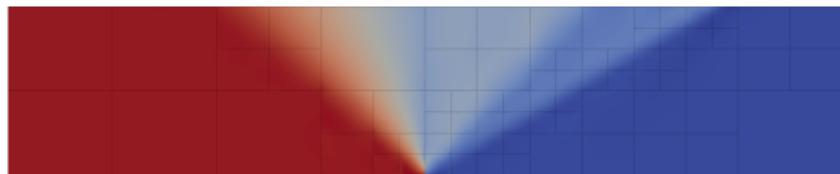
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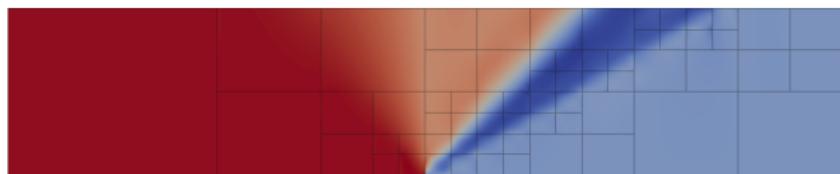
Mesh 4



Primitive Variables



Conservation Variables

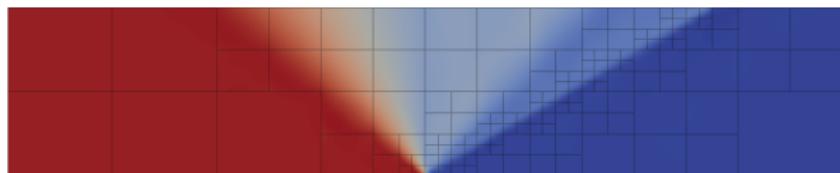


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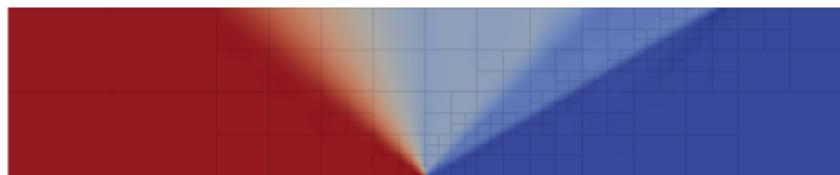
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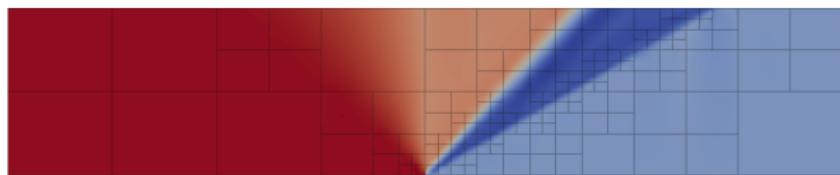
Mesh 5



Primitive Variables



Conservation Variables

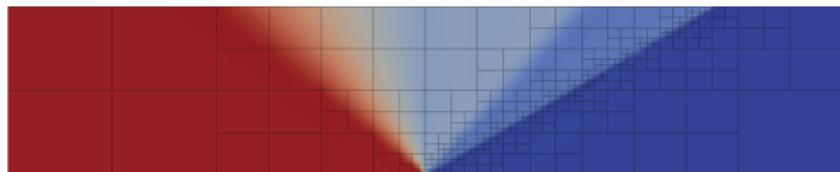


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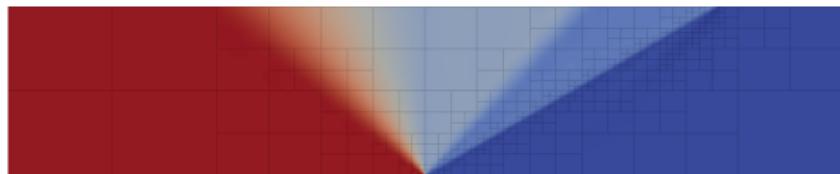
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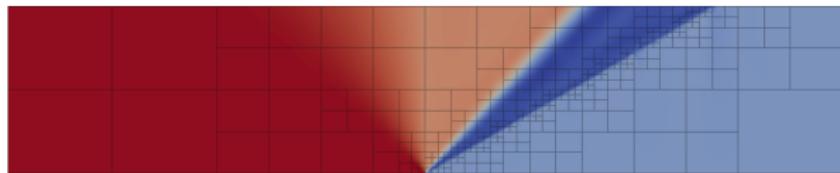
Mesh 6



Primitive Variables



Conservation Variables

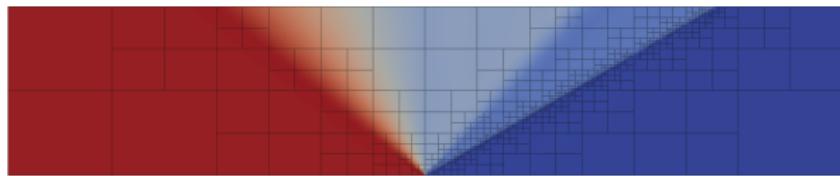


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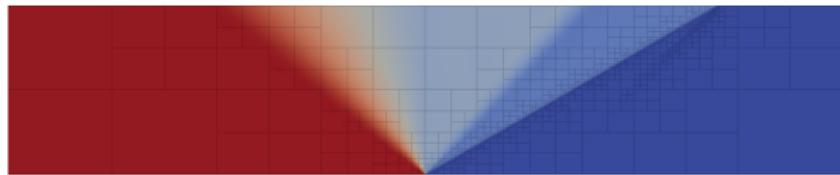
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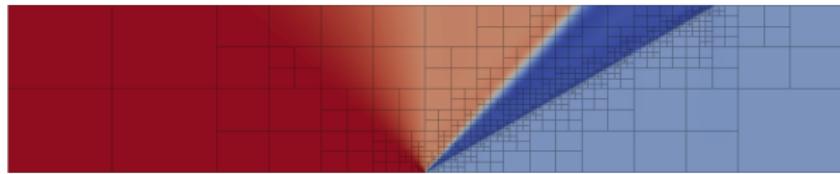
Mesh 7



Primitive Variables



Conservation Variables

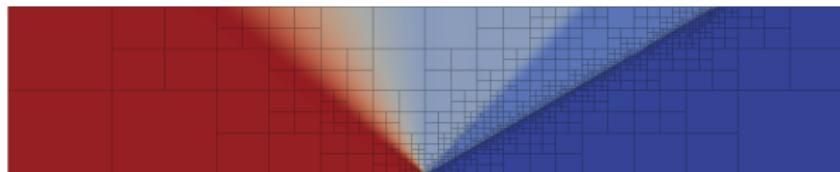


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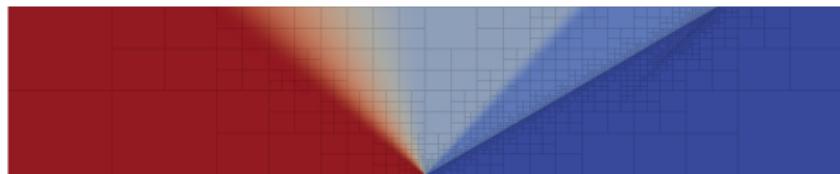
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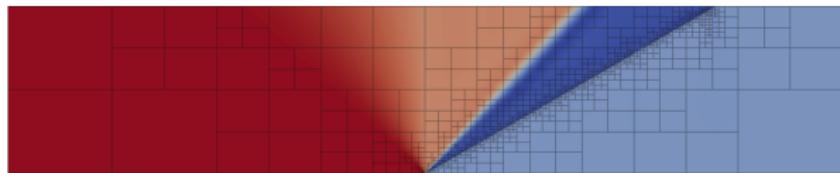
Mesh 8



Primitive Variables



Conservation Variables

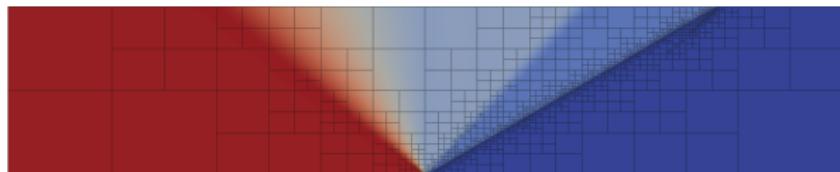


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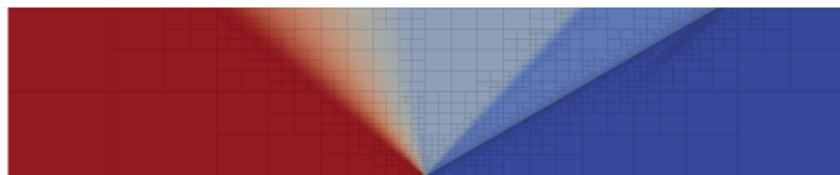
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Sod Shock Tube with  $\mu = 10^{-5}$

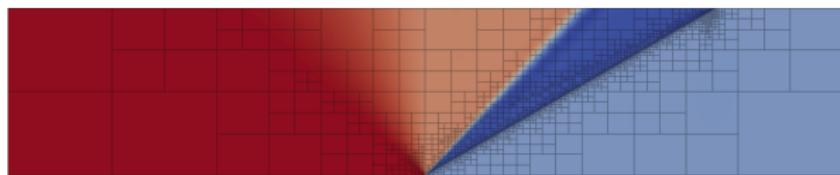
Mesh 9



Primitive Variables



Conservation Variables

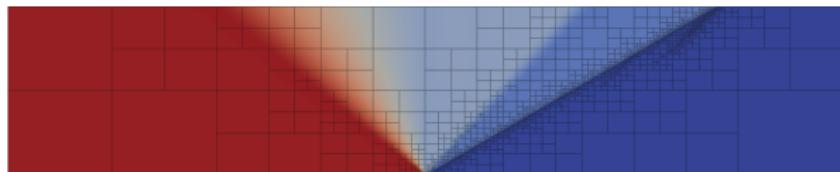


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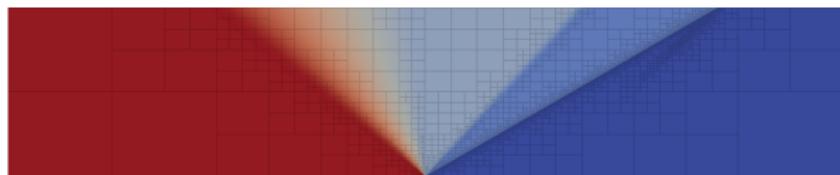
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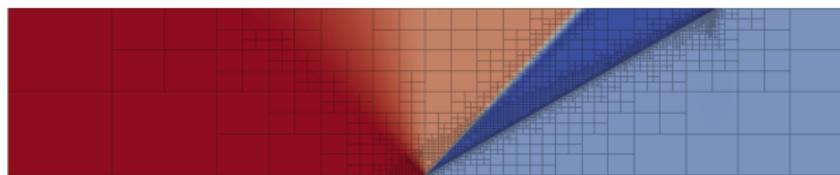
Mesh 10



Primitive Variables



Conservation Variables

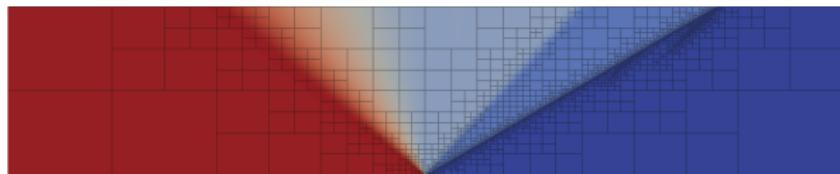


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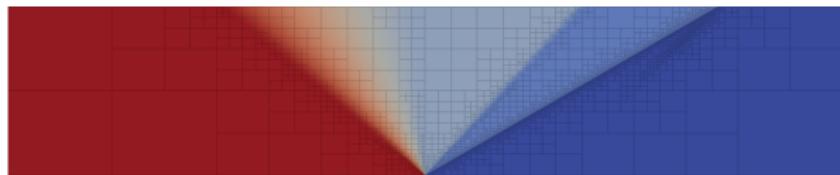
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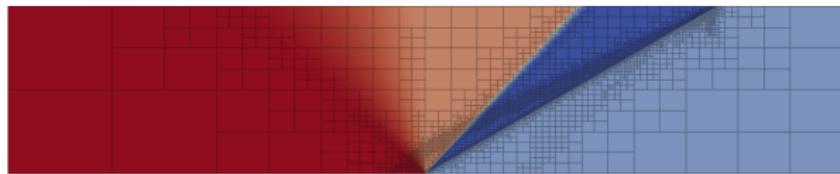
Mesh 11



Primitive Variables



Conservation Variables

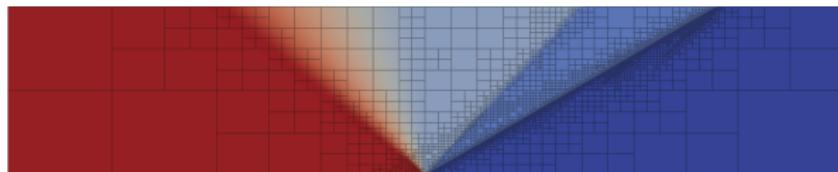


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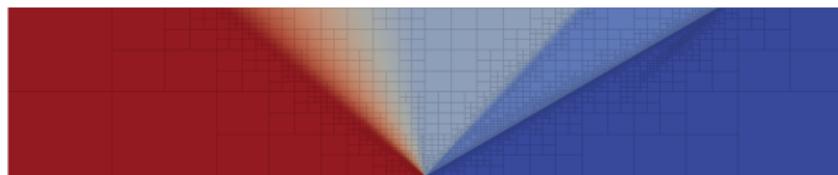
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Sod Shock Tube with  $\mu = 10^{-5}$

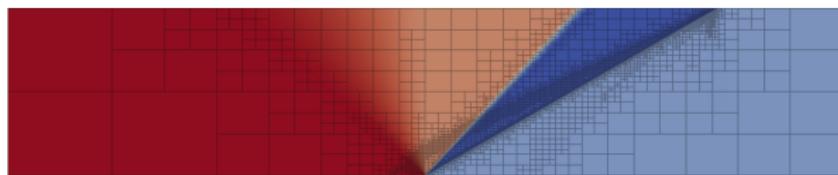
Mesh 12



Primitive Variables



Conservation Variables

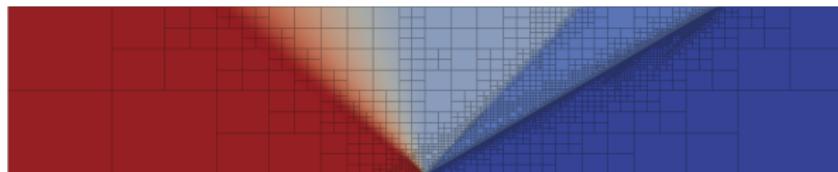


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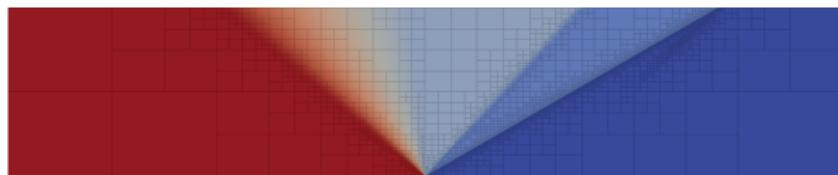
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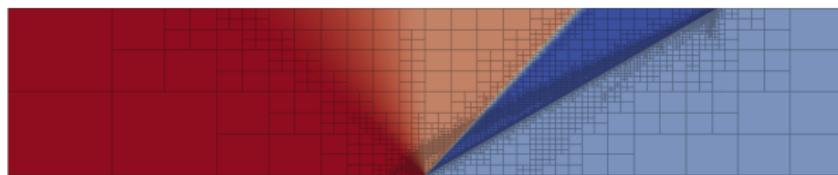
Mesh 13



Primitive Variables



Conservation Variables

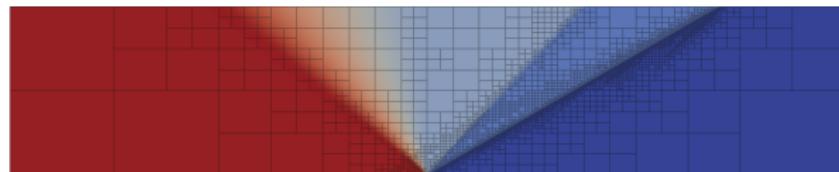


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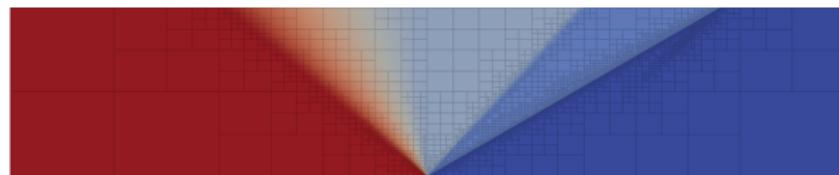
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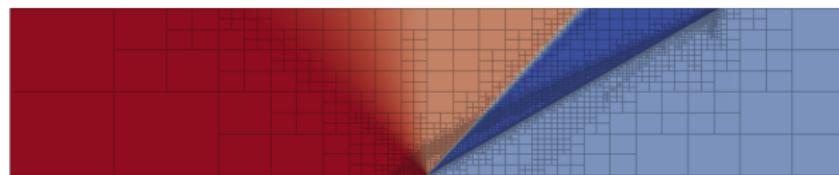
Mesh 14



Primitive Variables



Conservation Variables

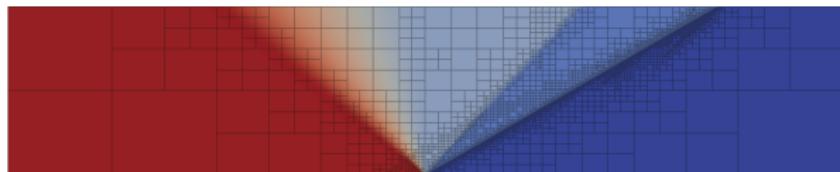


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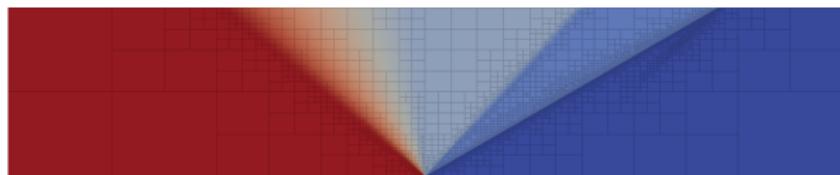
# Compressible Navier-Stokes

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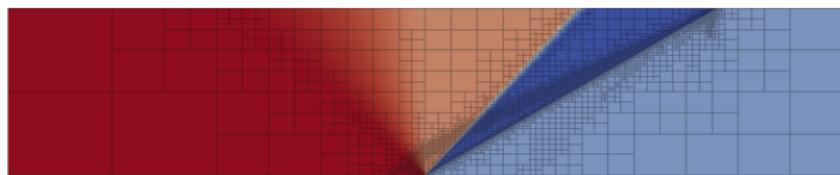
Mesh 15



Primitive Variables



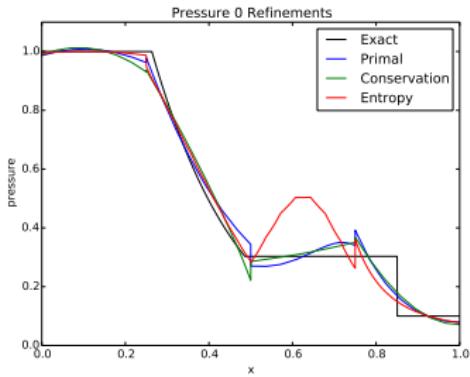
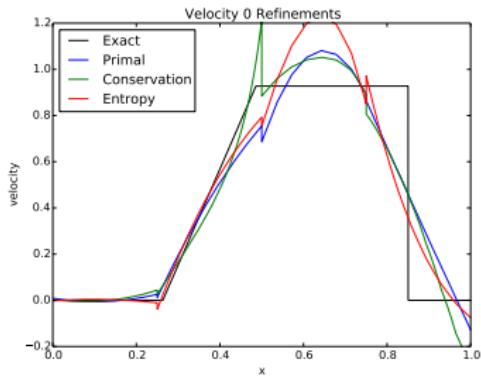
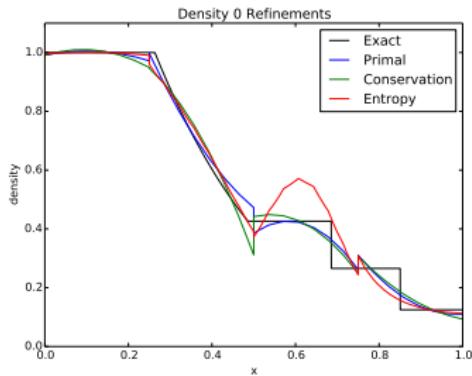
Conservation Variables



Entropy Variables

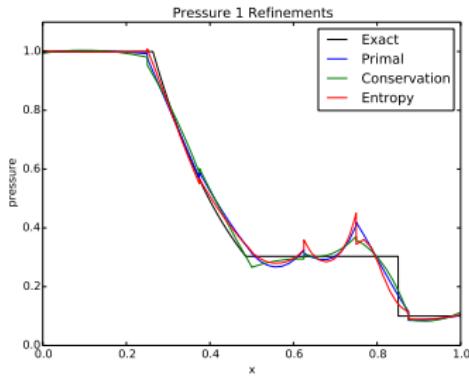
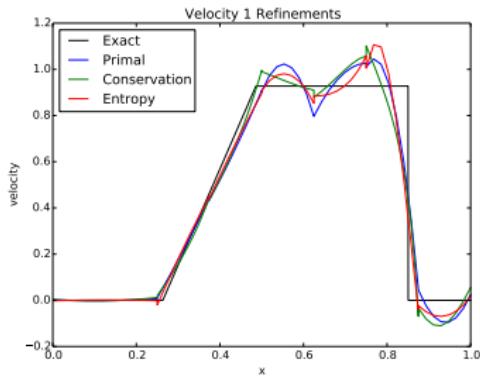
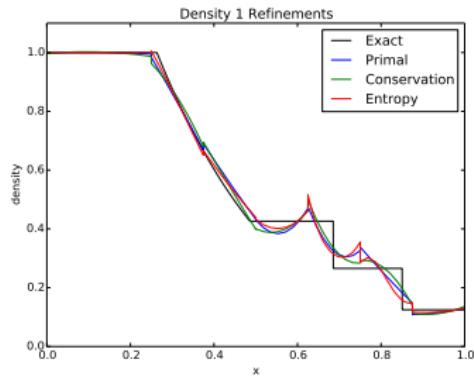
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Sod Shock Tube with  $\mu = 10^{-5}$



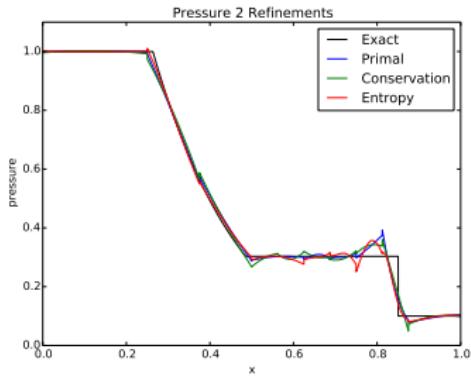
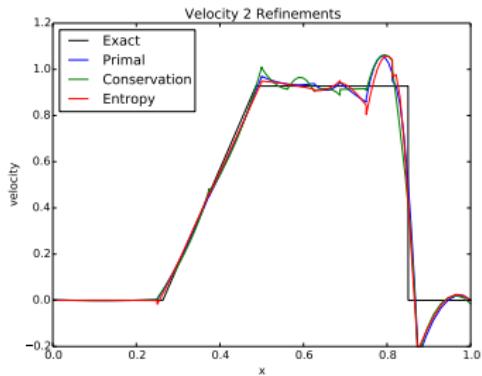
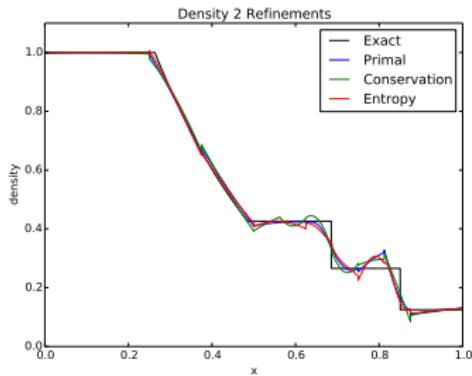
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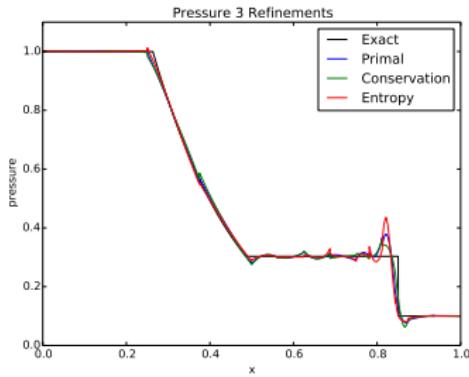
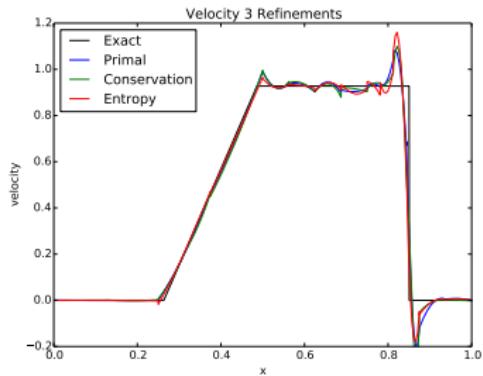
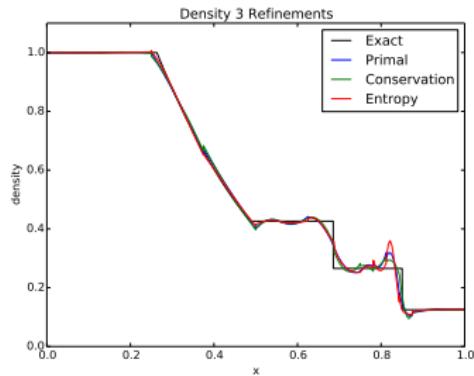
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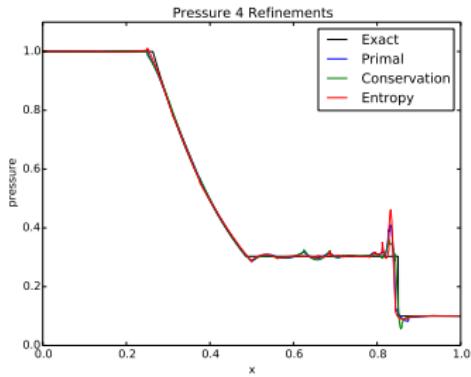
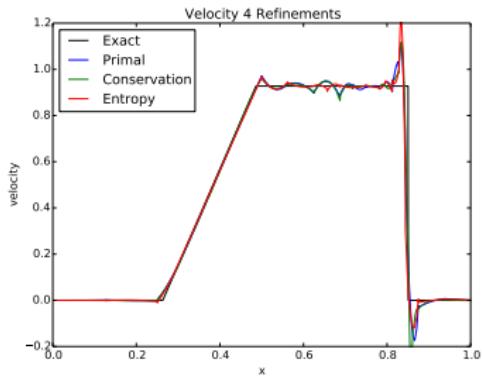
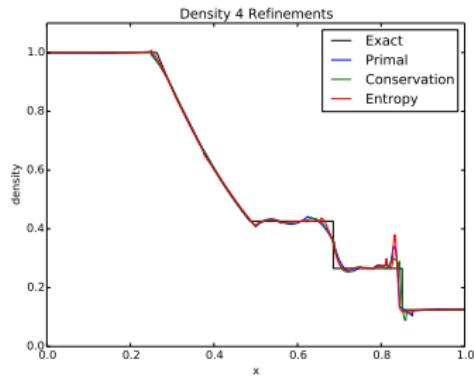
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



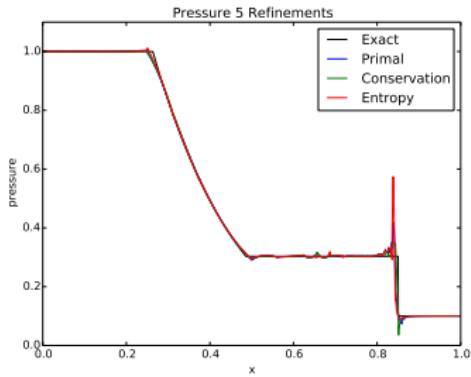
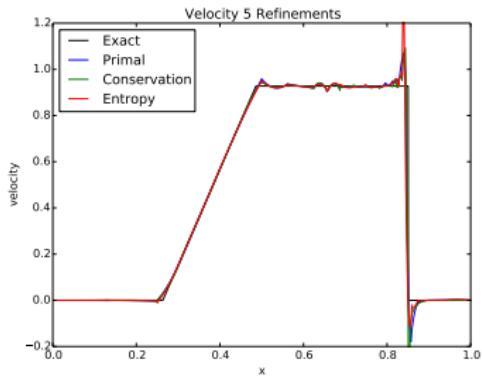
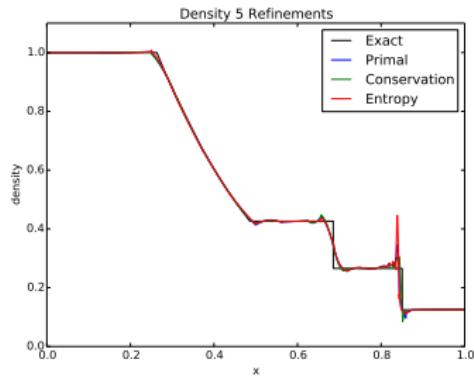
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



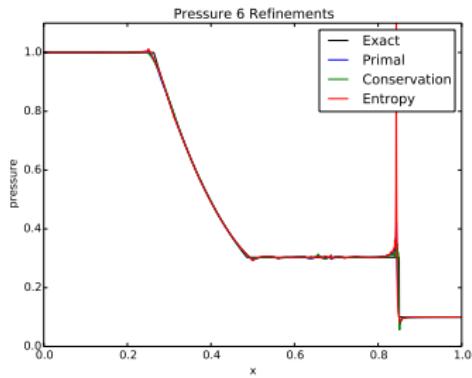
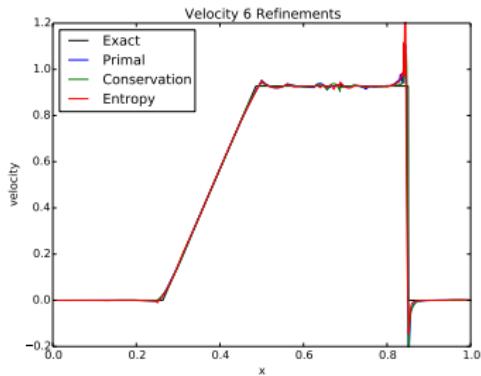
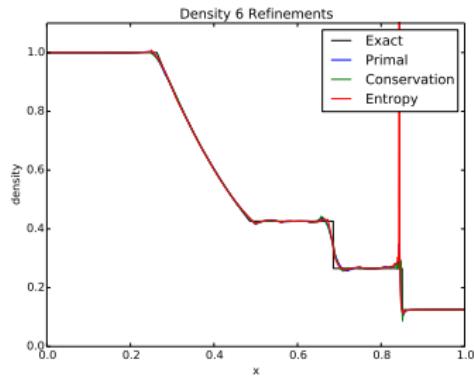
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



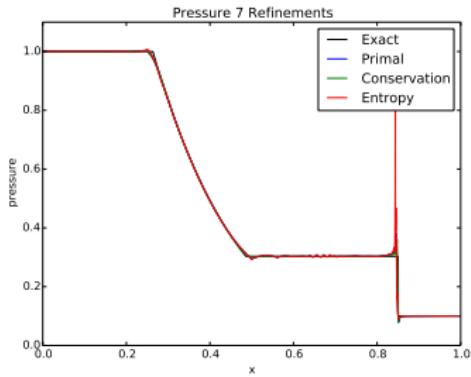
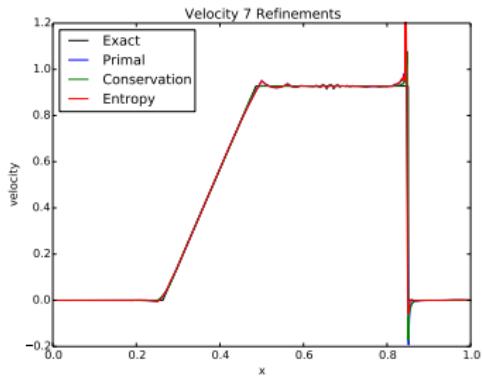
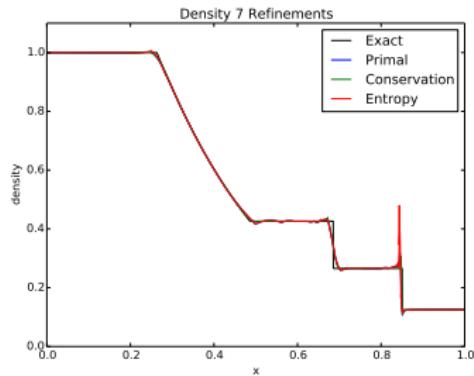
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Sod Shock Tube with  $\mu = 10^{-5}$



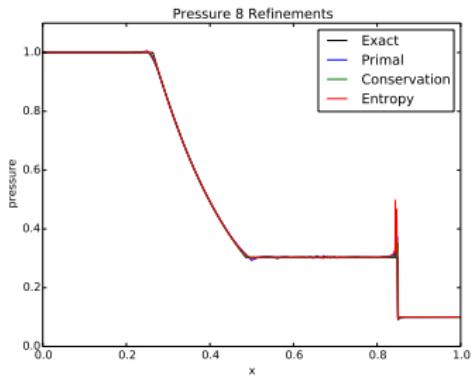
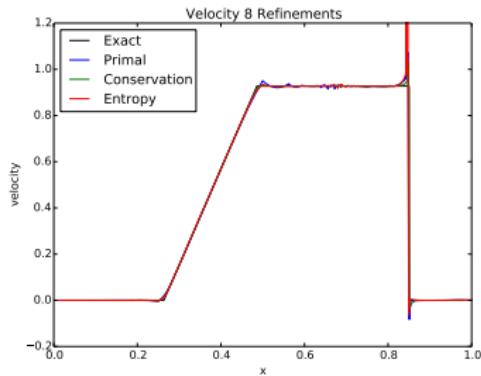
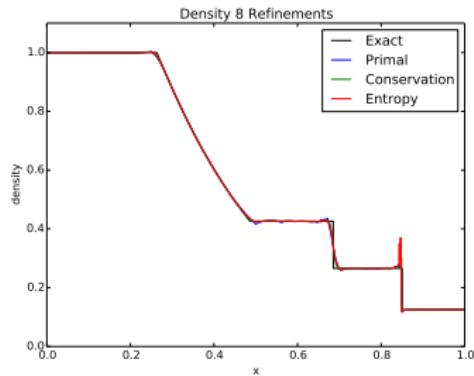
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



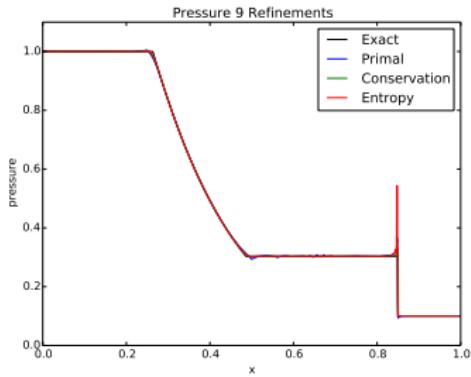
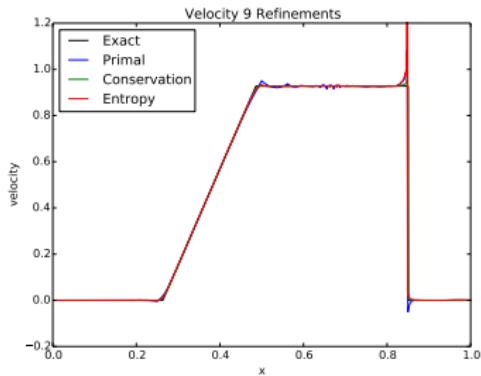
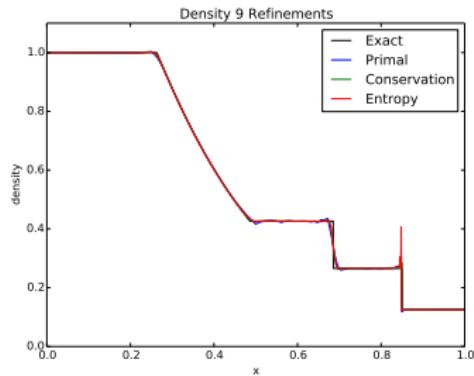
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



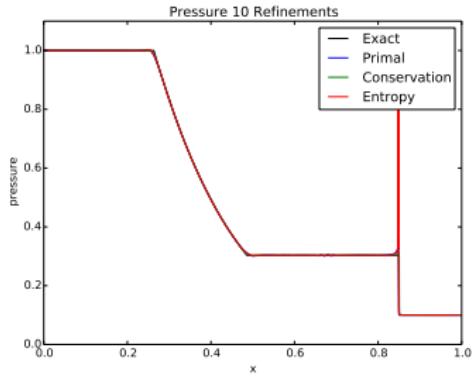
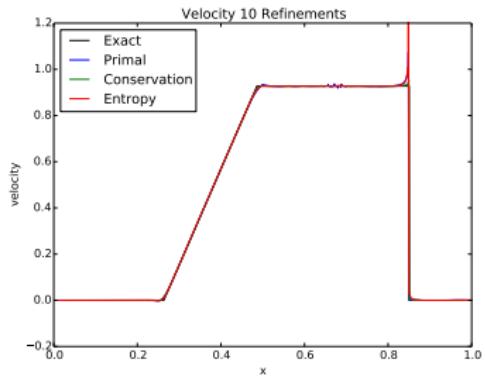
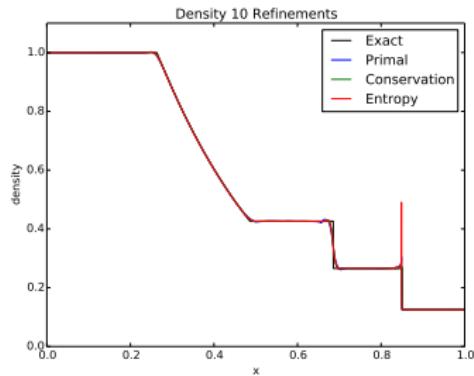
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Sod Shock Tube with  $\mu = 10^{-5}$



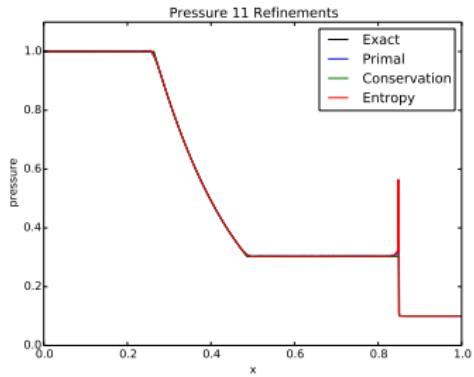
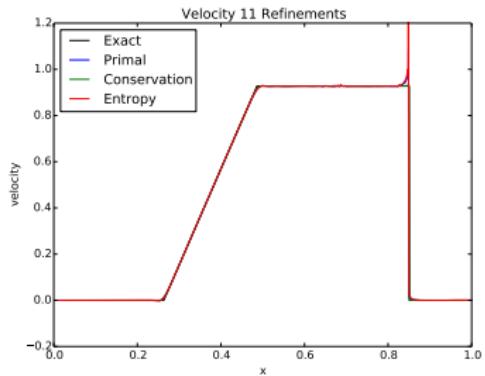
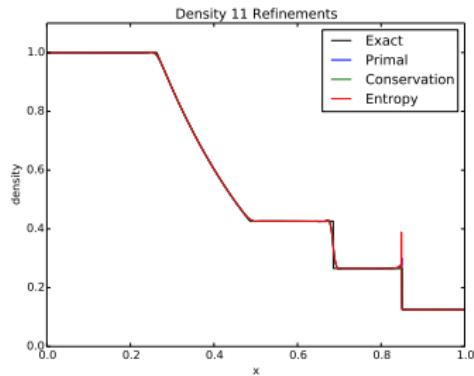
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Sod Shock Tube with  $\mu = 10^{-5}$



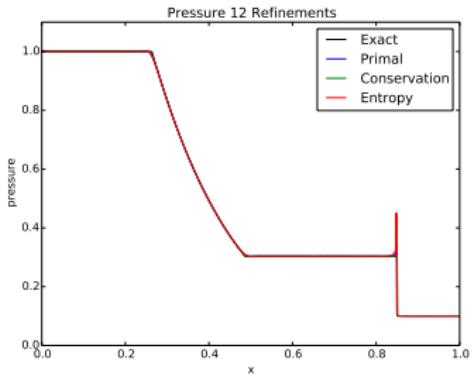
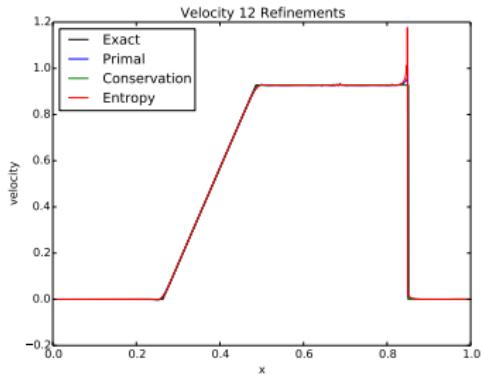
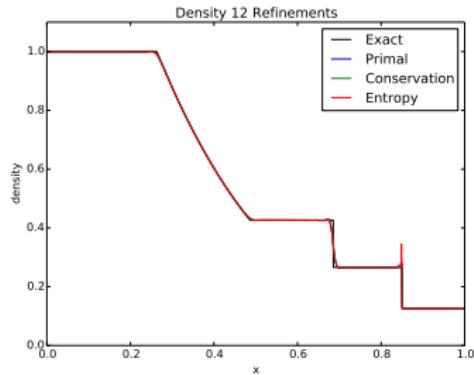
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Sod Shock Tube with  $\mu = 10^{-5}$



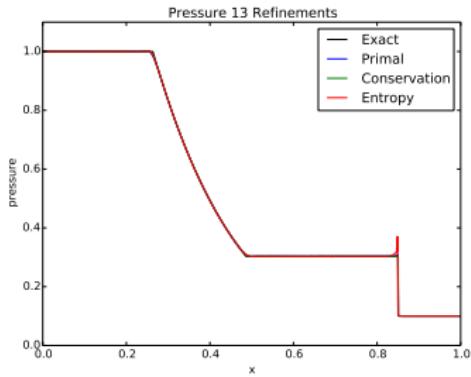
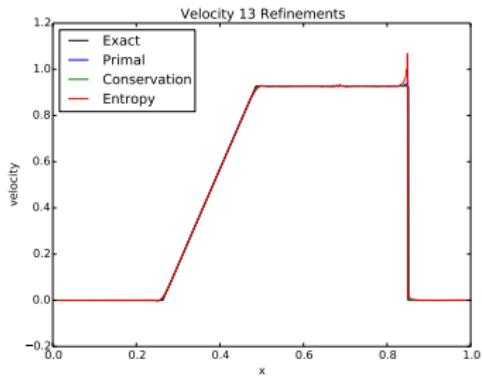
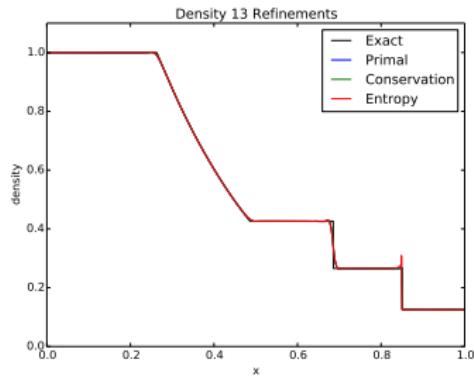
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



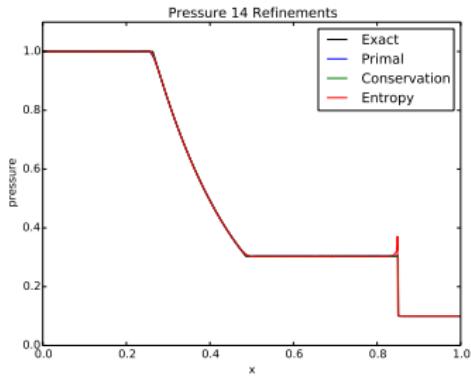
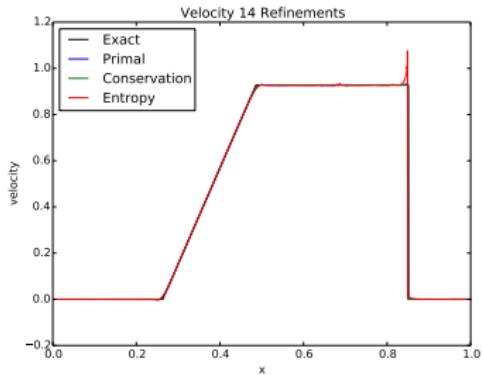
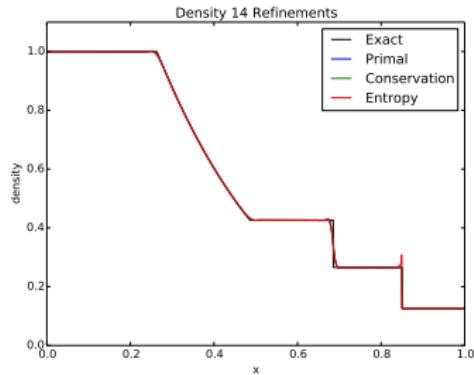
# Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



# Compressible Navier-Stokes

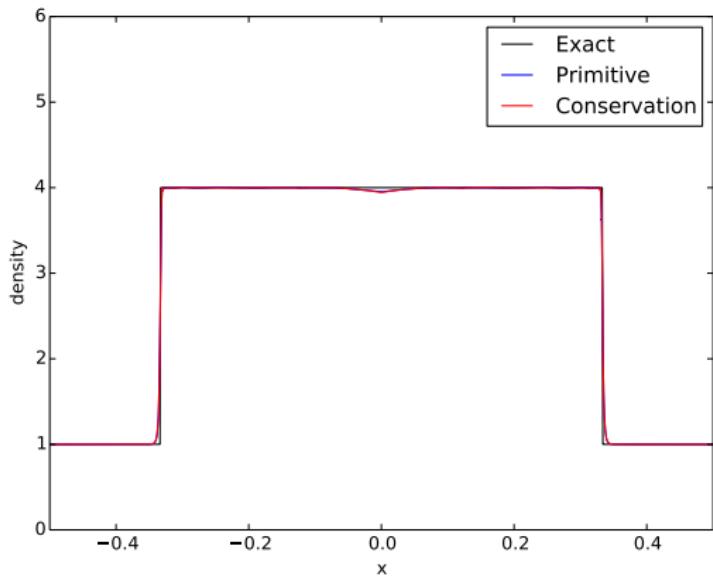
Sod Shock Tube with  $\mu = 10^{-5}$



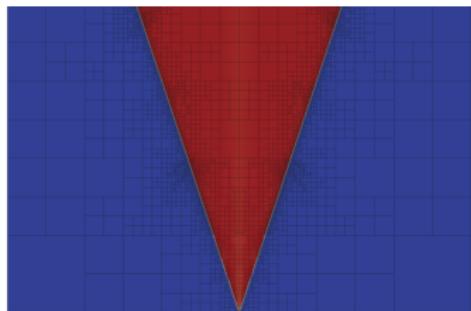
# Compressible Navier-Stokes

Noh Implosion with  $\mu = 10^{-3}$

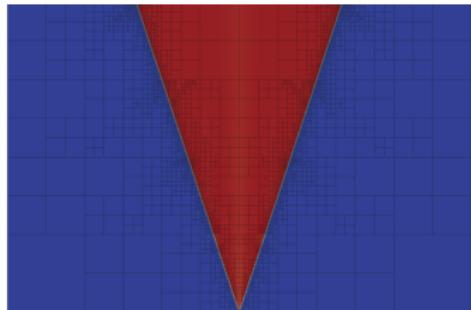
Infinitely strong shock propagation.



Sequence of 4 Time Slabs  
Primitive Variables



Conservation Variables



# Related Research

## Past and Present Topics in DPG Research

- Multiphysics
  - Heat conduction (Poisson and Heat equation)
  - Wave problems (Helmholtz and Maxwell)
  - Linear elasticity and plate problems
  - Convection-Diffusion, Stokes, incompressible Navier-Stokes, compressible Navier-Stokes, Euler
- Natively nonlinear DPG
- DPG for non-Hilbert  $L^p$  spaces
- Local conservation
- Iterative solvers
- Entropy scaling for physically meaningful test norms
- General polyhedral elements

# Thank You!

## Recommended References

- ▶ L.F. Demkowicz and J. Gopalakrishnan. "Recent Developments in Discontinuous Galerkin Finite Element Methods for Partial Differential Equations (eds. X. Feng, O. Karakashian, Y. Xing)". In: vol. 157. IMA Volumes in Mathematics and its Applications, 2014. Chap. An Overview of the DPG Method, pp. 149–180.
- ▶ T.E. Ellis, L.F. Demkowicz, and J.L. Chan. "Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems". In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530 –1549.
- ▶ N.V. Roberts. "Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods". In: *Comp. Math. Appl.* 68.11 (2014). Minimum Residual and Least Squares Finite Element Methods, pp. 1581 –1604.
- ▶ L.F. Demkowicz and N. Heuer. "Robust DPG Method for Convection-Dominated Diffusion Problems". In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
- ▶ J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771 –795.
- ▶ N. Roberts, T. Bui-Thanh, and L. Demkowicz. "The DPG method for the Stokes problem". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 966 –995.