

# Space-Time Discontinuous Petrov-Galerkin Finite Elements for Transient Fluid Mechanics

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ENGINEERING & SCIENCES

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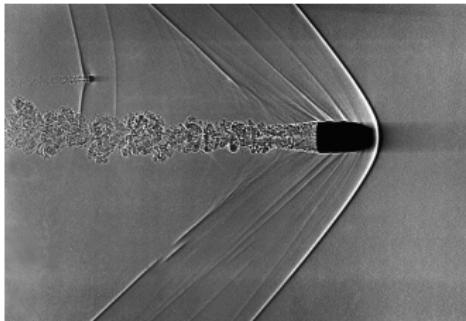
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# Navier-Stokes Equations

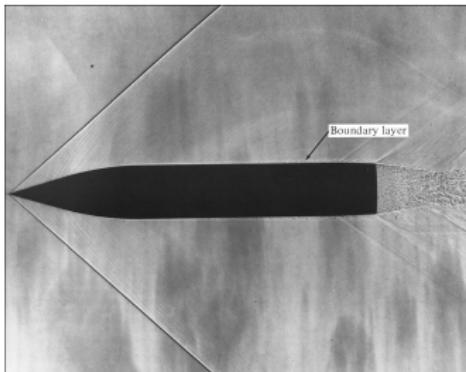
## Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
  - Shocks
  - Boundary layers - resolution needed for drag/load
  - Turbulence (non-localized)
- Stability of numerical schemes
  - Nonlinearity
  - Nature of PDE changes for different flow regimes
  - Coarse/adaptive grids
  - Higher order



Shock

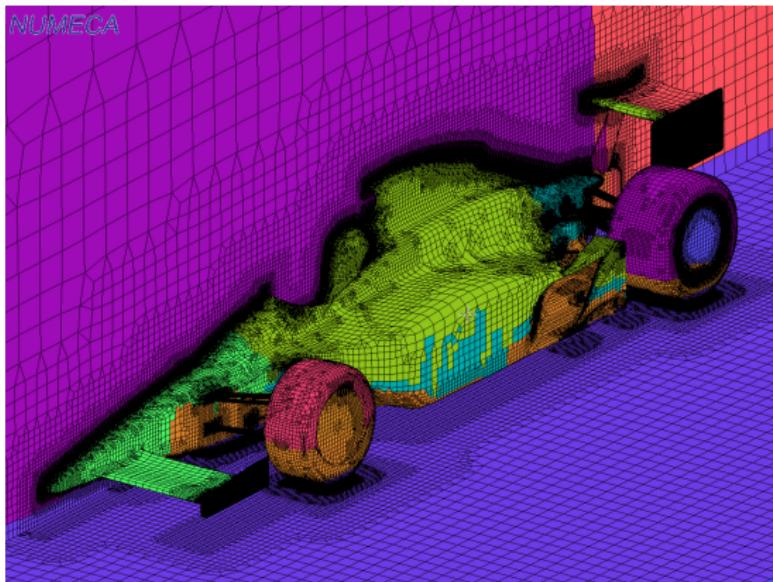


Boundary layer

# Motivation

Initial Mesh Design is Expensive and Time-Consuming

- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC

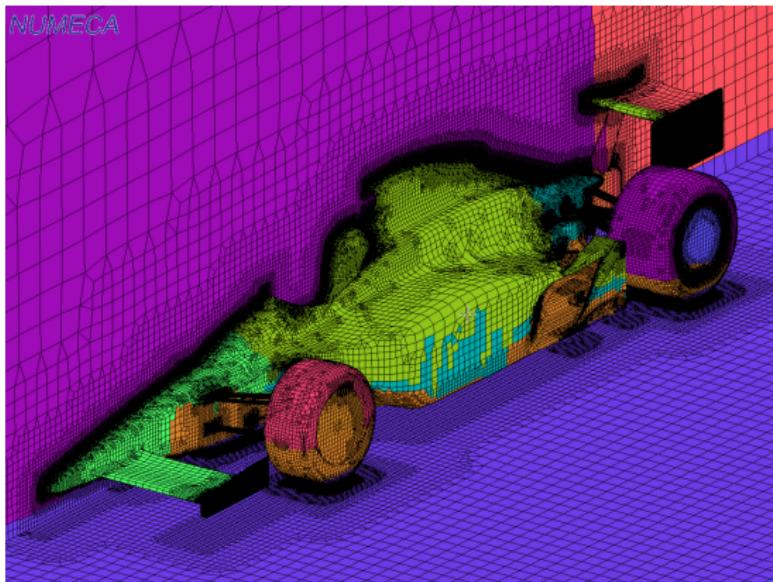


Formula 1 Mesh by Numeca

# Motivation

Initial Mesh Design is Expensive and Time-Consuming

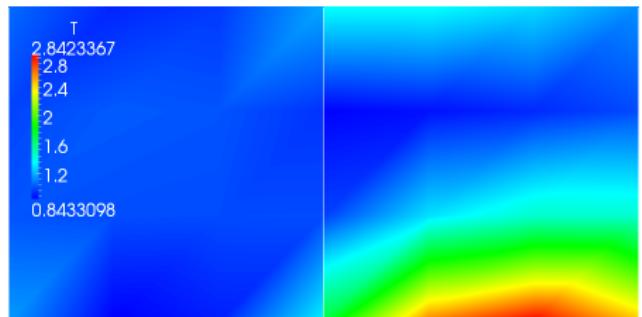
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Engineers often forced to work by trial and error
- Bad in the context of HPC
- We desire an automated computational technology



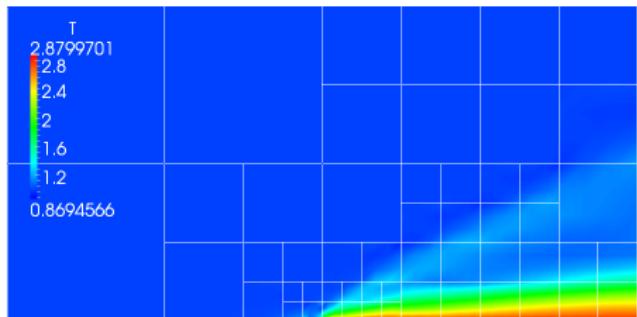
Formula 1 Mesh by Numeca

# DPG on Coarse Meshes

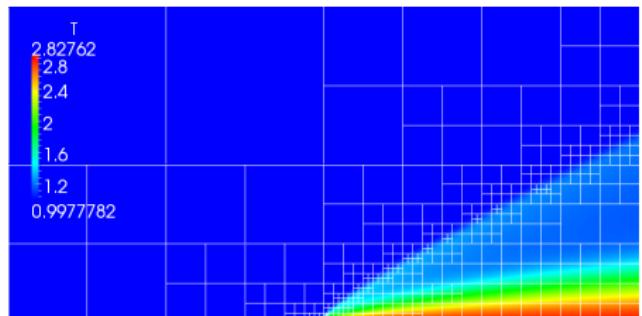
Adaptive Solve of the Carter Plate Problem<sup>1</sup>  $Re = 1000$



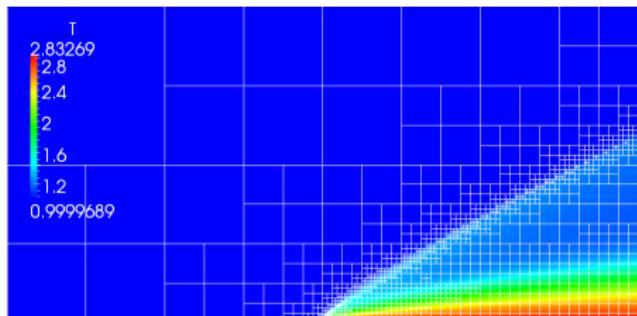
Temperature on Initial Mesh



Temperature after 4 Refinements



Temperature after 8 Refinements



Temperature after 11 Refinements

<sup>1</sup>J.L. Chan. "A DPG Method for Convection-Diffusion Problems". PhD thesis. University of Texas at Austin, 2013.

# Lessons from Other Methods

**Streamline Upwind Petrov-Galerkin:** Adaptively changing the test space can produce a method with better stability.

**Discontinuous Galerkin:** Discontinuous basis functions are a legitimate option for finite element methods.

**Hybridized DG:** Mesh interface unknowns can facilitate static condensation -- reducing the number of DOFs in the global solve.

**Least-Squares FEM:** The finite element method is most powerful in a minimum residual context (i.e. as a Ritz method).

**Space-Time FEM:** Highly adaptive methods should have adaptive time integration. Superior framework for problems with moving boundaries. Requires a method that is both temporally and spatially stable.

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# Overview of DPG

DPG is a Minimum Residual Method

Find  $u \in U$  such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator  $B : U \rightarrow V'$  defined by  $b(u, v) = \langle Bu, v \rangle_{V' \times V}$ .

This gives the operator equation

$$Bu = l \quad \in V'.$$

We wish to minimize the residual  $Bu - l \in V'$ :

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bw_h - l\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|R_V^{-1}(Bw_h - l)\|_V^2 .$$

# Overview of DPG

Petrov-Galerkin with Optimal Test Functions

Taking the Gâteaux derivative to be zero in all directions  $\delta u \in U_h$  gives,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions  $v_{\delta u_h} := R_V^{-1}B\delta u_h$  for each trial function  $\delta u_h$ .

## Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with  $v_{\delta u_h} \in V$  that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

# Overview of DPG

## Mixed Formulation

Identifying the error representation function:

$$\psi := R_V^{-1}(Bu_h - l)$$

allows us to develop an alternative interpretation of DPG.

### DPG as a Mixed Problem

Find  $\psi \in V$ ,  $u_h \in U_h$  such that

$$\begin{aligned} (\psi, \delta v)_V - b(u_h, \delta v) &= -l(\delta v) & \forall \delta v &\in V \\ b(\delta u_h, \psi) &= 0 & \forall \delta u_h &\in U_h \end{aligned}$$

In this unconventional saddle-saint problem, the approximate solution  $u_h$  comes from a finite-dimensional trial space and plays the role of the Lagrange multiplier for the error representation function

## Overview of DPG

DPG is the Most Stable Petrov-Galerkin Method

Babuška's theorem guarantees that *discrete stability and approximability imply convergence*. If bilinear form  $b(u, v)$ , with  $M := \|b\|$  satisfies the discrete inf-sup condition with constant  $\gamma_h$ ,

$$\sup_{v_h \in V_h} \frac{|b(u, v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U ,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U .$$

Optimal test function realize the supremum guaranteeing that  $\gamma_h \geq \gamma$ .

### Energy Norm

If we use the energy norm,  $\|u\|_E := \|Bu\|_{V'}$  in the error estimate, then  $M = \gamma = 1$ . Babuška's theorem implies that the minimum residual method is the most stable Petrov-Galerkin method (assuming exact optimal test functions).

# Overview of DPG

## Other Features

### Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

### Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

### Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \|R_V^{-1}(Bu_h - l)\|_V$$

# Overview of DPG

High Performance Computing

Eliminates human intervention

- Stability
- Robustness
- Adaptivity
- Automaticity
- Compute intensive
- Embarrassingly parallel local solves
- Factorization recyclable
- Low communication
- SPD stiffness matrix
- Multiphysics



Stampede Supercomputer at TACC



Mira Supercomputer at Argonne

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# Locally Conservative DPG

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= \mathbf{0} \\ \nabla \cdot (\beta u - \boldsymbol{\sigma}) &= g \end{aligned}$$

Multiply by test functions and integrate by parts over each element,  $K$ .

$$\begin{aligned} \frac{1}{\epsilon} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle u, \tau_n \rangle_{\partial K} &= 0 \\ -(\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle (\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v \rangle_{\partial K} &= (g, v)_K \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form  $b(u, v) = l(v)$ .

$$\begin{aligned} \frac{1}{\epsilon} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K} \\ - (\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle \hat{t}, v \rangle_{\partial K} &= (g, v)_K \end{aligned}$$

# Locally Conservative DPG

Local Conservation for Convection-Diffusion

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K g,$$

which is equivalent to having  $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$  in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al<sup>2</sup> (also Chang and Nelson<sup>3</sup>), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{t}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K},$$

where  $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_N\}$ .

<sup>2</sup>D. Moro, N.C. Nguyen, and J. Peraire. "A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws". In: *Int. J. Num. Meth. Eng.* (2011).

<sup>3</sup>C.L. Chang and J.J. Nelson. "Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation". In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.

# Locally Conservative DPG

Locally Conservative Saddle Point System

Finding the critical points of  $L(u, \lambda)$ , we get the following equations.

## Locally Conservative Saddle Point System

$$\frac{\partial L(u_h, \lambda)}{\partial u_h} = b(u_h, R_V^{-1}B\delta u_h) - l(R_V^{-1}B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \lambda)}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddle point problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

# Locally Conservative DPG

Optimal Test Functions

For each  $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$ , find  $\mathbf{v}_\mathbf{u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$  such that

$$(\mathbf{v}_\mathbf{u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where  $\mathbf{V}$  becomes  $\mathbf{V}_{p+\Delta p}$  in order to make this computationally tractable.  
 We recently developed this modification to the *robust test norm*<sup>4</sup> which behaves better in the presence of singularities.

## Convection-Diffusion Test Norm

$$\begin{aligned} \|(\mathbf{v}, \boldsymbol{\tau})\|_{\mathbf{V}, \Omega_h}^2 &= \left\| \min \left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}} \right\} \boldsymbol{\tau} \right\|^2 + \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla \mathbf{v}\|^2 \\ &\quad + \|\boldsymbol{\beta} \cdot \nabla \mathbf{v}\|^2 + \epsilon \|\nabla \mathbf{v}\|^2 \quad + \|\mathbf{v}\|^2 \\ &\qquad\qquad\qquad \underbrace{\phantom{\|\mathbf{v}\|^2}}_{\text{No longer necessary}} \end{aligned}$$

<sup>4</sup> J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014), pp. 771–795.

# Locally Conservative DPG

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<sup>4</sup> J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014), pp. 771–795.

# Locally Conservative DPG

## Stability and Robustness Analysis

- We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \mathbf{u} \in \mathbf{U}, p \in Q \\ a(\mathbf{u}, \mathbf{w}) + c(p, \mathbf{w}) = l(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{U} \\ c(q, \mathbf{u}) = g(q) \quad \forall q \in Q \end{cases}$$

where  $a, c, l, g$  denote the appropriate bilinear and linear forms.

- $a(\mathbf{u}, \mathbf{w}) = b(\mathbf{u}, R_V^{-1}B\mathbf{w}) = (R_V^{-1}B\mathbf{u}, R_V^{-1}B\mathbf{w})_V$
- $c(p, \mathbf{w}) = \sum_K \lambda_K \langle \hat{t}, 1_K \rangle_{\partial K}$
- Locally conservative DPG satisfies inf-sup and inf-sup in kernel conditions.
- Robustness is proved by switching to energy norm in Brezzi analysis.
- Full details can be found in *Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems*<sup>5</sup>.

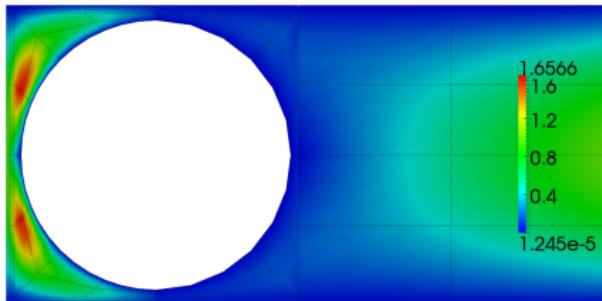
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<sup>5</sup> T.E. Ellis, L.F. Demkowicz, and J.L. Chan. "Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems". In: *Comp. Math. Appl.* 68.11 (2014), pp. 1530–1549.

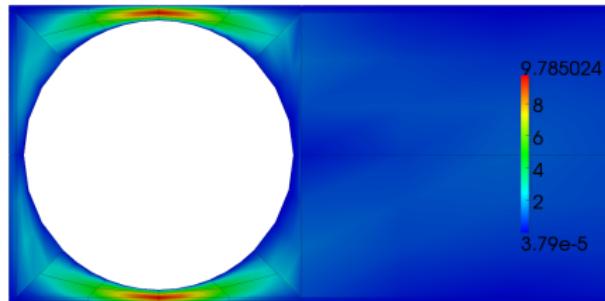
# Numerical Experiments

## Stokes Flow Around a Cylinder

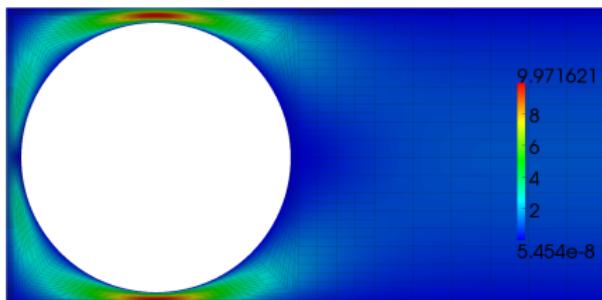
Horizontal Velocity



1 Refinement

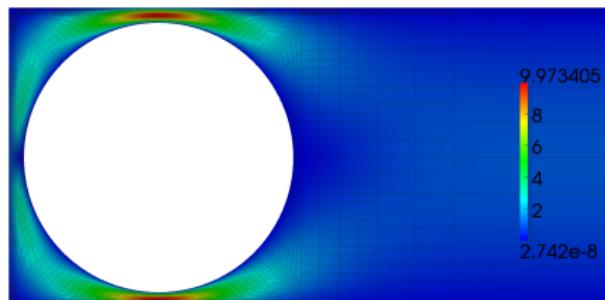


1 Refinement



6 Refinements

Nonconservative



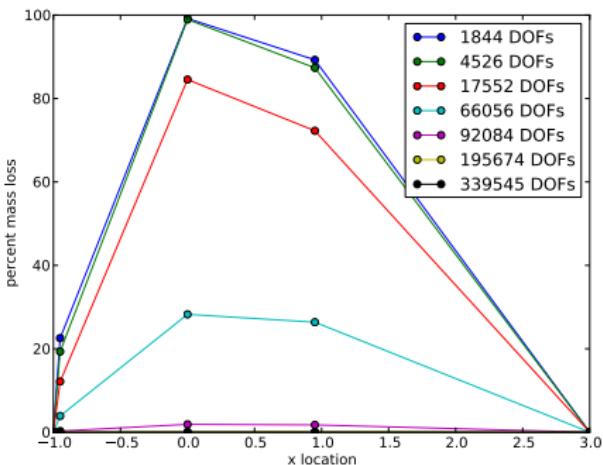
6 Refinements

Conservative

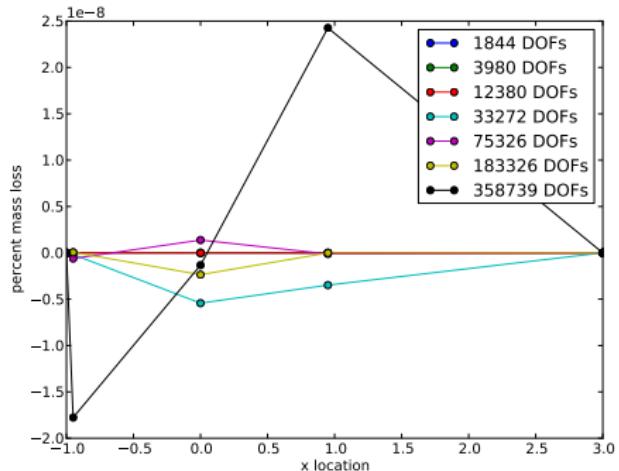
# Numerical Experiments

## Stokes Flow Around a Cylinder

Percent Mass Loss at  $x = [-1, -0.95, 0, 0.95, 3]$



Nonconservative

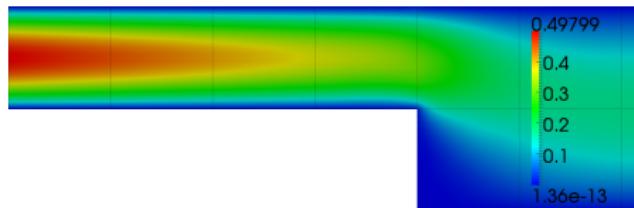


Conservative

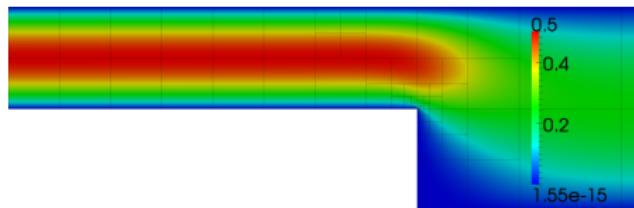
# Numerical Experiments

## Stokes Flow Over a Backward Facing Step

Horizontal Velocity

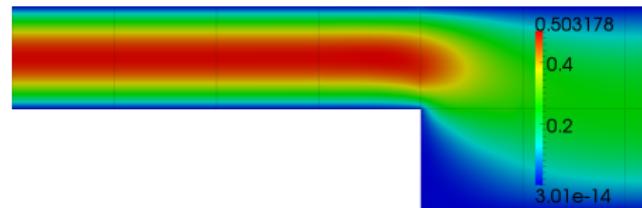


Initial Mesh

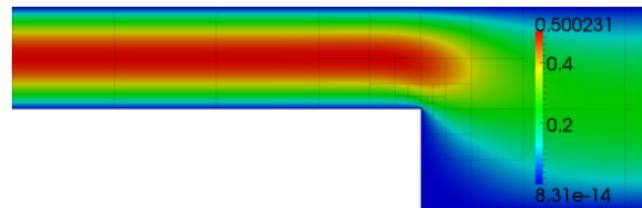


8 Refinements

Nonconservative



Initial Mesh



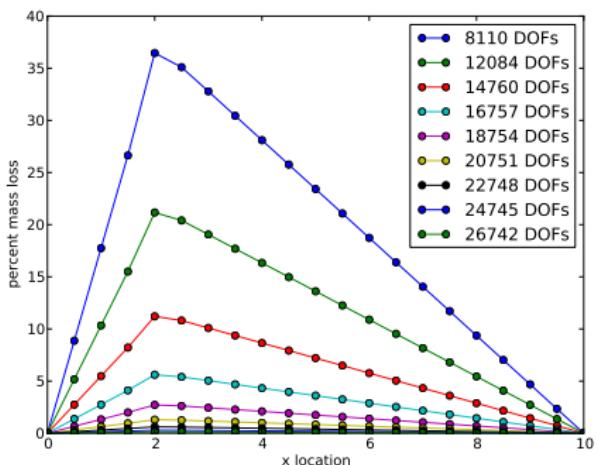
8 Refinements

Conservative

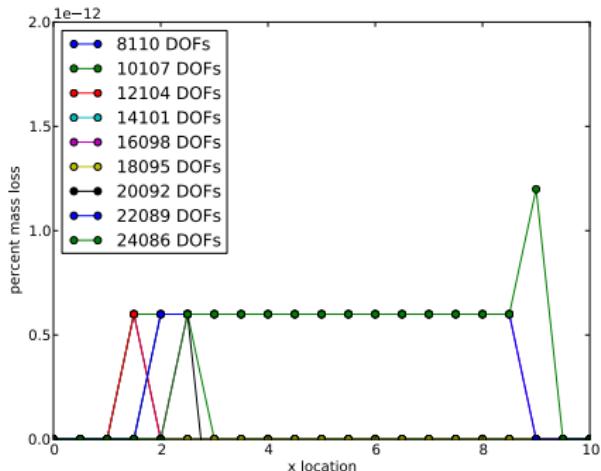
# Numerical Experiments

## Stokes Flow Over a Backward Facing Step

Percent Mass Loss at  $x = [0, 0.5, \dots, 9.5, 10]$



Nonconservative



Conservative

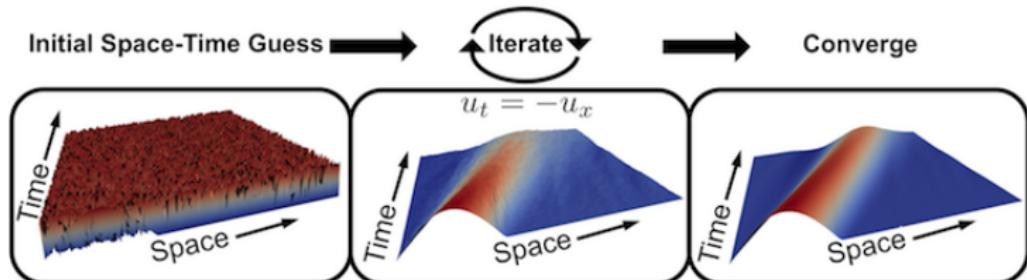
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# Space-Time DPG

Extending DPG to Transient Problems

- Time stepping techniques are not ideally suited to highly adaptive grids
- Space-time FEM proposed as a solution
  - ✓ Unified treatment of space and time
  - ✓ Local space-time adaptivity (local time stepping)
  - ✓ Parallel-in-time integration (space-time multigrid)
  - ✗ Spatially stable FEM methods may not be stable in space-time
  - ✗ Need to support higher dimensional problems
- DPG provides necessary stability and adaptivity



Courtesy of XBraid by LLNL

# Space-Time DPG for Convection-Diffusion

Space-Time Divergence Form

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} + \beta \cdot \nabla u - \epsilon \Delta u = f$$

This is just a composition of a constitutive law and conservation of mass.

$$\sigma - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} + \nabla \cdot (\beta u - \sigma) = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \sigma - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \beta u - \sigma \\ u \end{pmatrix} &= f \end{aligned}$$

# Space-Time DPG for Convection-Diffusion

Ultra-Weak Formulation with Discontinuous Test Functions

Multiply by test function and integrate by parts over space-time element K.

$$\begin{aligned} \left( \frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right)_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle_{\partial K} &= 0 \\ - \left( \begin{pmatrix} \beta u - \boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right)_K + \langle \hat{t}, v \rangle_{\partial K} &= f \end{aligned}$$

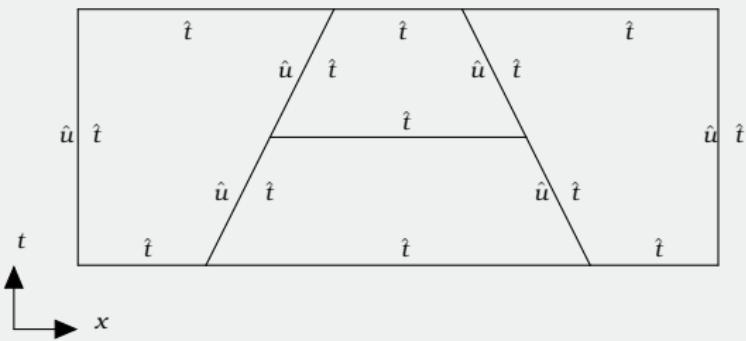
where

$$\hat{u} := \text{tr}(u)$$

$$\begin{aligned} \hat{t} &:= \text{tr}(\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}_x \\ &\quad + \text{tr}(u) \cdot n_t \end{aligned}$$

- Trace  $\hat{u}$  defined on spatial boundaries
- Flux  $\hat{t}$  defined on all boundaries

## Support of Trace Variables



# Space-Time Convection-Diffusion

## $L^2$ Equivalent Norms

Bilinear form with group variables:

$$b((u, \hat{u}), v) = (u, A_h^* v)_{L^2(\Omega_h)} + \langle \hat{u}, [v] \rangle_{\Gamma_h}$$

For conforming  $v^*$  satisfying  $A^* v^* = u$

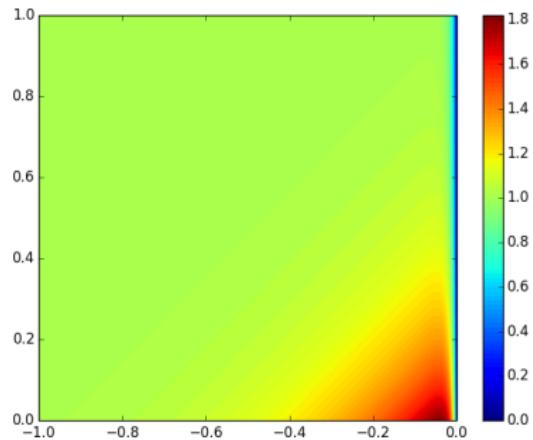
$$\begin{aligned} \|u\|_{L^2(\Omega_h)}^2 &= b(u, v^*) = \frac{b(u, v^*)}{\|v^*\|_V} \|v^*\|_V \\ &\leq \sup_{v^* \neq 0} \frac{|b(u, v^*)|}{\|v^*\|} \|v^*\| = \|u\|_E \|v^*\|_V \end{aligned}$$

Necessary robustness condition:

$$\begin{aligned} \|v^*\|_V &\lesssim \|u\|_{L^2(\Omega_h)} \\ \Rightarrow \|u\|_{L^2(\Omega_h)} &\lesssim \|u\|_E \end{aligned}$$

Analytical Solution

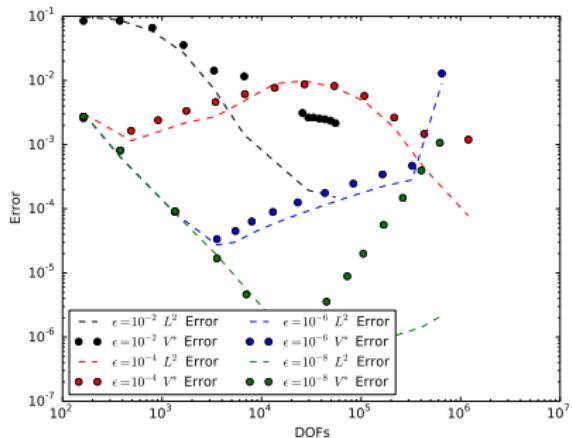
$$\begin{aligned} e^{-lt} (e^{\lambda_1(x-1)} - e^{\lambda_2(x-1)}) \\ + \left(1 - e^{\frac{1}{\epsilon}x}\right) \end{aligned}$$



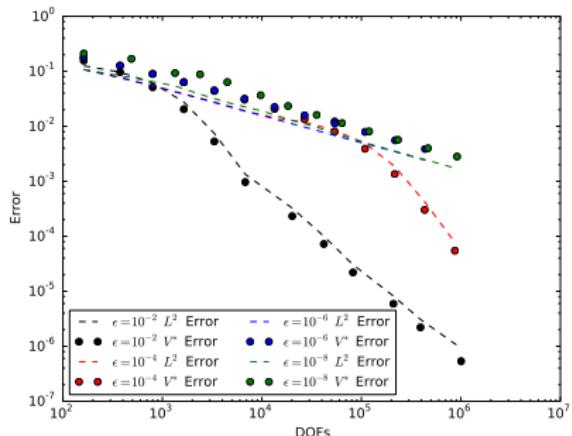
# Space-Time Convection-Diffusion

## $L^2$ Equivalent Norms

A norm should be: bounded by  $\|u\|_{L^2(\Omega_h)}$ , have good conditioning, not produce boundary layers in the optimal test function.



$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &\quad + \left\| \frac{1}{\epsilon} \tau + \nabla v \right\|^2 + \|v\|^2 + \|\tau\|^2 \end{aligned}$$

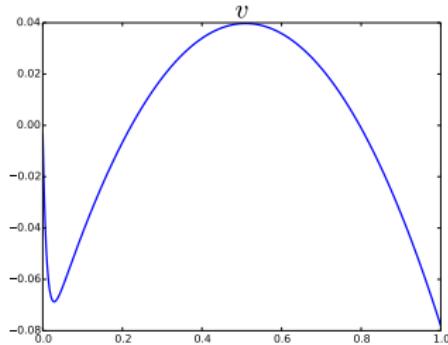


$$\begin{aligned} \|(v, \tau)\|^2 &= \left\| \nabla \cdot \tau - \tilde{\beta} \cdot \nabla_{xt} v \right\|^2 \\ &\quad + \min \left( \frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\tau\|^2 \\ &\quad + \epsilon \|\nabla v\|^2 + \|\beta \cdot \nabla v\|^2 + \|v\|^2 \end{aligned}$$

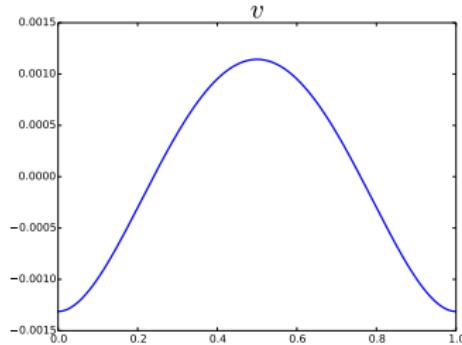
# Steady Convection-Diffusion

## Ideal Optimal Shape Functions

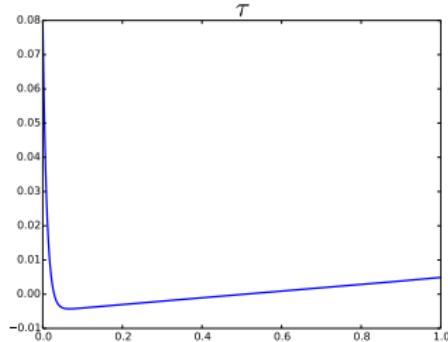
Graph Norm



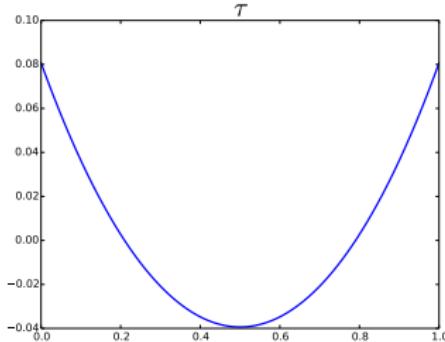
Coupled Robust Norm



$\tau$



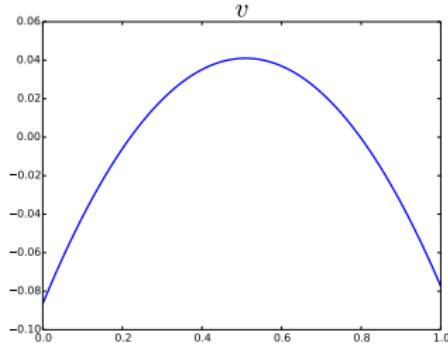
$\tau$



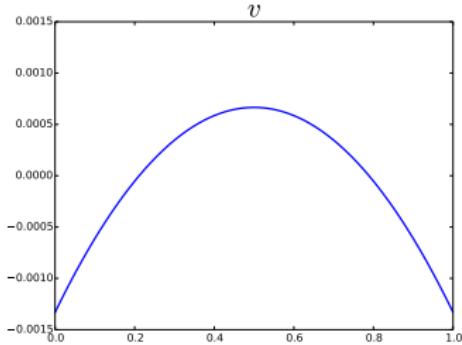
# Steady Convection-Diffusion

Approximated ( $p = 3$ ) Optimal Shape Functions

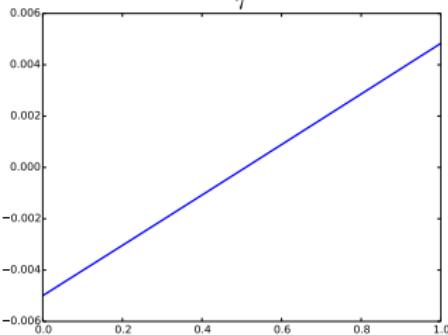
Graph Norm



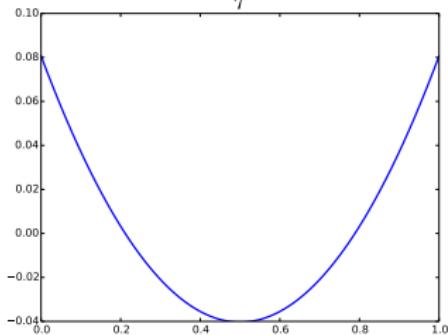
Coupled Robust Norm



$\tau$



$\tau$



# Steady Convection-Diffusion

Two Robust Norms for Steady Convection-Diffusion

The following norms are robust for steady convection-diffusion.

The robust norm was derived in<sup>6</sup>:

$$\begin{aligned} \|(v, \tau)\|^2 &= \|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 \\ &\quad + \|\nabla \cdot \tau\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\tau\|^2. \end{aligned}$$

The case for the coupled robust norm was made in<sup>7</sup>:

$$\begin{aligned} \|(v, \tau)\|^2 &= \|\beta \cdot \nabla v\|^2 + \epsilon \|\nabla v\|^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|^2 \\ &\quad + \|\nabla \cdot \tau - \beta \cdot \nabla v\|^2 + \min\left(\frac{1}{h^2}, \frac{1}{\epsilon}\right) \|\tau\|^2. \end{aligned}$$

<sup>6</sup> J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014), pp. 771–795.

<sup>7</sup> J.L. Chan. "A DPG Method for Convection-Diffusion Problems". PhD thesis. University of Texas at Austin, 2013.

# Space-Time Convection-Diffusion

Two Robust Norms for Transient Convection-Diffusion

Let  $\tilde{\beta} := \begin{pmatrix} \beta \\ 1 \end{pmatrix}$  and  $\nabla_{xt} v := \begin{pmatrix} \nabla v \\ \frac{\partial v}{\partial t} \end{pmatrix}$ .

The following norms are robust for space-time convection-diffusion.

Robust Norm:

$$\begin{aligned} \|(\boldsymbol{v}, \boldsymbol{\tau})\|^2 &= \left\| \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \epsilon \|\nabla \boldsymbol{v}\|^2 + \min \left( \frac{\epsilon}{h^2}, 1 \right) \|\boldsymbol{v}\|^2 \\ &\quad + \|\nabla \cdot \boldsymbol{\tau}\|^2 + \min \left( \frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2. \end{aligned}$$

Coupled Robust Norm

$$\begin{aligned} \|(\boldsymbol{v}, \boldsymbol{\tau})\|^2 &= \left\| \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \epsilon \|\nabla \boldsymbol{v}\|^2 + \min \left( \frac{\epsilon}{h^2}, 1 \right) \|\boldsymbol{v}\|^2 \\ &\quad + \left\| \nabla \cdot \boldsymbol{\tau} - \tilde{\beta} \cdot \nabla_{xt} \boldsymbol{v} \right\|^2 + \min \left( \frac{1}{h^2}, \frac{1}{\epsilon} \right) \|\boldsymbol{\tau}\|^2. \end{aligned}$$

# Robust Norms for Transient Convection-Diffusion

## Adjoint Operator

Consider the problem with homogeneous boundary conditions

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \tilde{\beta} \cdot \nabla_{xt} u - \nabla \cdot \boldsymbol{\sigma} &= f \\ \beta_n u - \epsilon \frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_- \\ u &= 0 \text{ on } \Gamma_+ \\ u &= u_0 \text{ on } \Gamma_0. \end{aligned}$$

The adjoint operator  $A^*$  is given by

$$A^*(v, \tau) = \left( \frac{1}{\epsilon} \tau + \nabla v, -\tilde{\beta} \cdot \nabla_{xt} v + \nabla \cdot \tau \right).$$

# Robust Norms for Transient Convection-Diffusion

Controlling Different Field Variables

We decompose the continuous adjoint problem  $A^*(\tau, v) = (\sigma, u)$  into

## Continuous part with forcing $u$

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\tau}_1 + \nabla v_1 &= 0 & \boldsymbol{\tau}_1 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ -\tilde{\beta} \cdot \nabla_{xt} v_1 + \nabla \cdot \boldsymbol{\tau}_1 &= u & v_1 &= 0 \text{ on } \Gamma_+ \\ && v_1 &= 0 \text{ on } \Gamma_T \end{aligned}$$

## Continuous part with forcing $\sigma$

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\tau}_2 + \nabla v_2 &= \sigma & \boldsymbol{\tau}_2 \cdot \mathbf{n}_x &= 0 \text{ on } \Gamma_- \\ -\tilde{\beta} \cdot \nabla_{xt} v_2 + \nabla \cdot \boldsymbol{\tau}_2 &= 0 & v_2 &= 0 \text{ on } \Gamma_+ \\ && v_2 &= 0 \text{ on } \Gamma_T \end{aligned}$$

# Robust Norms for Transient Convection-Diffusion

Proved Bounds at Our Disposal

Proofs of these lemmas can be found in<sup>8</sup>.

## Lemma (1)

If  $\nabla \cdot \beta = 0$ , we can bound

$$\|v\|^2 + \epsilon \|\nabla v\|^2 \leq \|u\|^2 + \epsilon \|\sigma\|^2$$

where  $v = v_1 + v_2$ .

## Lemma (2)

If  $\|\nabla \beta - \frac{1}{2} \nabla \cdot \beta \mathbf{I}\|_{L^\infty} \leq C_\beta$ , we can bound

$$\left\| \tilde{\beta} \cdot \nabla_{xt} v_1 \right\| \lesssim \|u\|.$$

<sup>8</sup>T.E. Ellis, J.L. Chan, and L.F. Demkowicz. Robust DPG Methods for Transient Convection-Diffusion. Tech. rep. 15-21. ICES, Oct. 2015.

# Robust Norms for Transient Convection-Diffusion

Control of  $u$

Bound on  $\|(v_1, \tau_1)\|$

$$\text{Lemma (2)} \Rightarrow \|\tilde{\beta} \cdot \nabla_{xt} v_1\| \lesssim \|u\|$$

$$\text{Lemma (2)} \Rightarrow \|\nabla \cdot \tau_1\| \leq \|u\| + \|\tilde{\beta} \cdot \nabla_{xt} v_1\| \lesssim 2\|u\|$$

$$\text{Lemma (2)} \Rightarrow \|\nabla \cdot \tau_1 - \tilde{\beta} \cdot \nabla_{xt} v_1\| = \|u\|$$

$$\text{Lemma (1)} \Rightarrow \|v_1\|^2 + \epsilon \|\nabla v_1\|^2 \leq \|u\|^2$$

$$\text{Lemma (1)} \Rightarrow \frac{1}{\epsilon} \|\tau_1\| = \epsilon \|\nabla v_1\| \leq \|u\|$$

We can guarantee robust control

$$\|(u, 0)\|_{L^2(\Omega_h)} \lesssim \|(u, \sigma)\|_E .$$

# Robust Norms for Transient Convection-Diffusion

Control of  $\sigma$

Bound on  $\|(v_2, \tau_2)\|$

$$\text{Definition} \Rightarrow \left\| \nabla \cdot \tau_2 - \tilde{\beta} \cdot \nabla_{xt} v_2 \right\| = 0 \leq \|\sigma\|$$

$$\text{Lemma (1)} \Rightarrow \|v_2\|^2 + \epsilon \|\nabla v_2\|^2 \leq \epsilon \|\sigma\|^2$$

$$\text{Lemma (1)} \Rightarrow \frac{1}{\epsilon} \|\tau_2\| = \|\sigma\| + \epsilon \|\nabla v_2\| = (1 + \epsilon) \|\sigma\|$$

We have not been able to prove bounds on  $\left\| \tilde{\beta} \cdot \nabla_{xt} v_2 \right\|$  or  $\|\nabla \cdot \tau_2\|$ .

We can **not** guarantee robust control

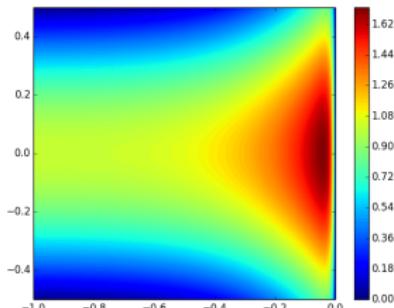
$$\|(0, \sigma)\|_{L^2(\Omega_h)} \not\lesssim \|(u, \sigma)\|_E.$$

# Robust Norms for Transient Convection-Diffusion

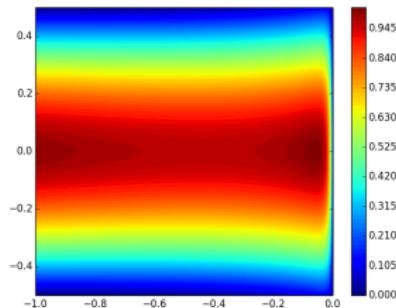
## Transient Analytical Solution

Transient impulse decays to Eriksson-Johnson steady state solution.

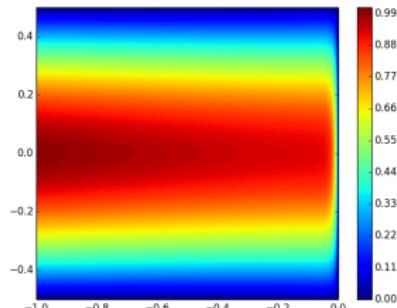
$$u = \exp(-lt) [\exp(\lambda_1 x) - \exp(\lambda_2 x)] + \cos(\pi y) \frac{\exp(s_1 x) - \exp(r_1 x)}{\exp(-s_1) - \exp(-r_1)}$$



$t = 0.0$



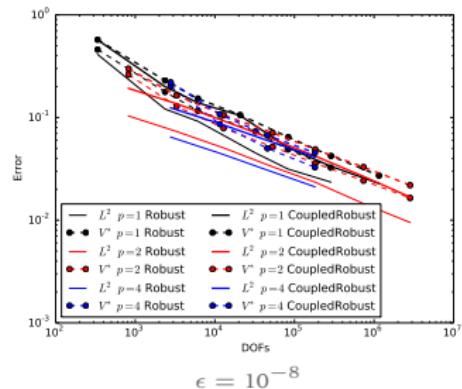
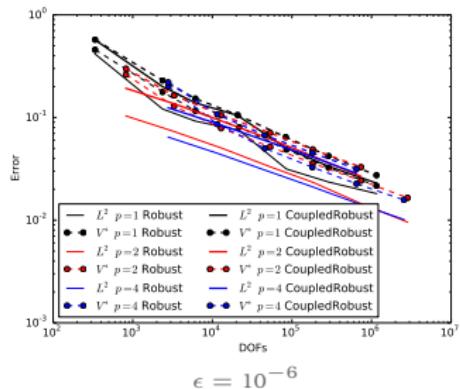
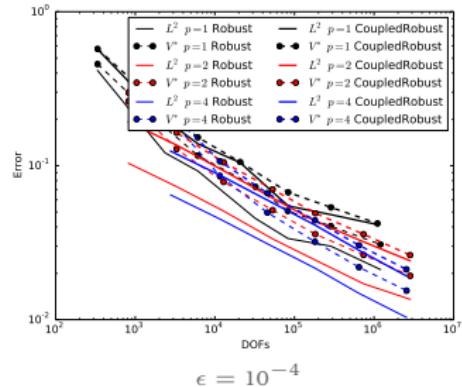
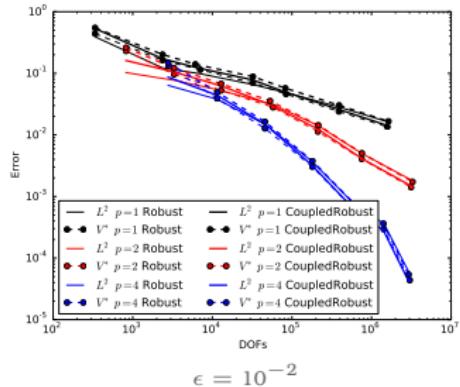
$t = 0.5$



$t = 1.0$

# Robust Norms for Transient Convection-Diffusion

## Robust Convergence to Analytical Solution



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- 6 Space-Time Compressible Navier-Stokes

# Space-Time Incompressible Navier-Stokes

Nonlinear Form

Space-time divergence form:

$$\begin{aligned} \frac{1}{\nu} \mathbb{D} - \nabla \cdot \mathbf{u} &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} \mathbf{u} \otimes \mathbf{u} - \mathbb{D} + p \mathbb{I} \\ \mathbf{u} \end{pmatrix} &= \mathbf{f} \\ \nabla \cdot \mathbf{u} &= 0 \end{aligned}$$

Multiply by  $\mathbb{S} \in \mathbb{H}(\text{div}, Q)$ ,  $\mathbf{v} \in \mathbf{H}_{xt}^1(Q)$ ,  $q \in H^1(Q)$ , and integrate by parts:

$$\begin{aligned} \left( \frac{1}{\nu} \mathbb{D}, \mathbb{S} \right) + (\mathbf{u}, \nabla \cdot \mathbb{S}) - \langle \hat{\mathbf{u}}, \mathbb{S} \cdot \mathbf{n}_x \rangle &= 0 \\ - \left( \begin{pmatrix} \mathbf{u} \otimes \mathbf{u} - \mathbb{D} + p \mathbb{I} \\ \mathbf{u} \end{pmatrix}, \nabla_{xt} \mathbf{v} \right) + \langle \hat{\mathbf{t}}, \mathbf{v} \rangle &= (\mathbf{f}, \mathbf{v}) \\ - (\mathbf{u}, \nabla q) + \langle \hat{\mathbf{u}} \cdot \mathbf{n}, q \rangle &= 0 \end{aligned}$$

# Space-Time Incompressible Navier-Stokes

## Robust Norms

Recall the adjoint and robust norm for convection-diffusion:

$$(\boldsymbol{\sigma}, \frac{1}{\epsilon} \boldsymbol{\tau} + \nabla v) + (u, \nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla v - \frac{\partial v}{\partial t})$$

$$\begin{aligned} \| (v, \boldsymbol{\tau}) \|_{V,K}^2 &:= \left\| \boldsymbol{\beta} \cdot \nabla v + \frac{\partial v}{\partial t} \right\|_K^2 + \epsilon \|\nabla v\|_K^2 + \min\left(\frac{\epsilon}{h^2}, 1\right) \|v\|_K^2 \\ &\quad + \|\nabla \cdot \boldsymbol{\tau}\|_K^2 + \min\left(\frac{1}{\epsilon}, \frac{1}{h^2}\right) \|\boldsymbol{\tau}\|_K^2 \end{aligned}$$

For incompressible Navier-Stokes the adjoint comes from:

$$\begin{aligned} &\left( \Delta \mathbb{D}, \frac{1}{\nu} \mathbb{S} + \nabla \mathbf{v} \right) + \left( \Delta \mathbf{u}, \nabla \cdot \mathbb{S} - \nabla q - \left( \tilde{\mathbf{u}} \cdot \nabla \mathbf{v} + \tilde{\mathbf{u}} \cdot (\nabla \mathbf{v})^T + \frac{\partial \mathbf{v}}{\partial t} \right) \right) \\ &\quad + (p, -\nabla \cdot \mathbf{v}) \end{aligned}$$

# Space-Time Incompressible Navier-Stokes

Norms for Navier-Stokes come from analogy

Convection-Diffusion      Navier-Stokes

$$\begin{aligned}\epsilon &\rightarrow \nu \\ \boldsymbol{\tau} &\rightarrow \mathbb{S} \\ \nabla v &\rightarrow \nabla \mathbf{v} \\ \nabla \cdot \boldsymbol{\tau} &\rightarrow \nabla \cdot \mathbb{S} - \nabla q \\ \boldsymbol{\beta} \cdot \nabla v + \frac{\partial v}{\partial t} &\rightarrow \tilde{\mathbf{u}} \cdot \nabla \mathbf{v} + \tilde{\mathbf{u}} \cdot (\nabla \mathbf{v})^T + \frac{\partial \mathbf{v}}{\partial t}\end{aligned}$$

$$\|(\mathbf{v}, \mathbb{D}, q)\|_{V,K}^2 := \left\| \tilde{\mathbf{u}} \cdot \nabla \mathbf{v} + \tilde{\mathbf{u}} \cdot (\nabla \mathbf{v})^T + \frac{\partial \mathbf{v}}{\partial t} \right\|_K^2 + \nu \|\nabla \mathbf{v}\|_K^2$$

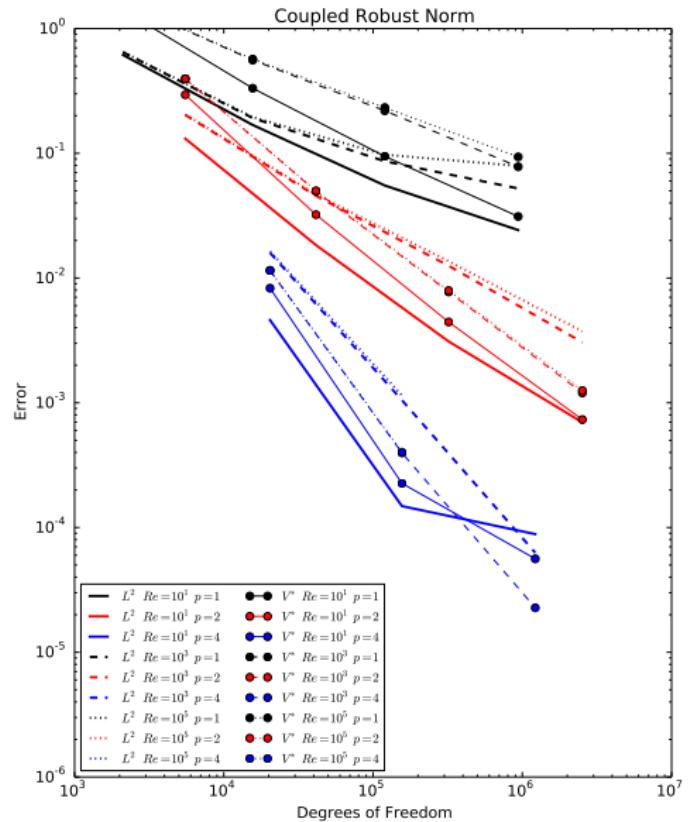
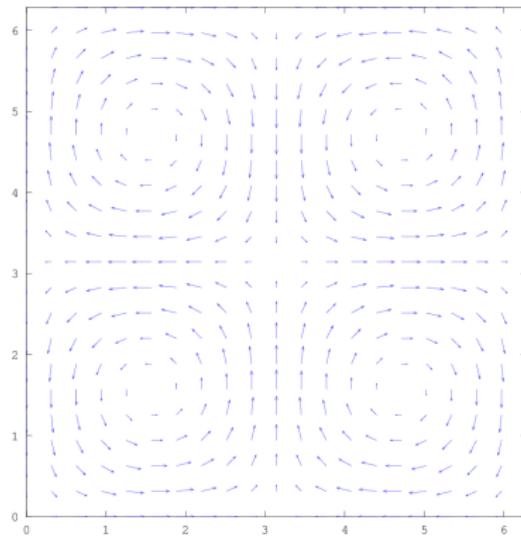
Robust norm:

$$\begin{aligned}&+ \min\left(\frac{\nu}{h^2}, 1\right) \|\mathbf{v}\|_K^2 + \|\nabla \cdot \mathbb{S} - \nabla q\|_K^2 \\ &+ \min\left(\frac{1}{\nu}, \frac{1}{h^2}\right) \|\mathbb{S}\|_K^2 + \|\nabla \cdot \mathbf{v}\|_K^2 + \|q\|_K^2\end{aligned}$$

# Space-Time Incompressible Navier-Stokes

## Taylor-Green Vortex

$$\mathbf{u} = e^{-\frac{2}{Re}t} \begin{pmatrix} \sin x \cos y \\ -\cos x \sin y \end{pmatrix}$$



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# Space-Time Compressible Navier-Stokes

First Order System with Primitive Variables

Assuming Stokes hypothesis, ideal gas law, and constant viscosity:

$$\frac{1}{\mu} \mathbb{D} - \nabla \mathbf{u} = 0$$

$$\frac{Pr}{C_p \mu} \mathbf{q} + \nabla T = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I} - (\mathbb{D} + \mathbb{D}^T - \frac{2}{3} \text{tr}(\mathbb{D}) \mathbb{I}) \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \rho R T \mathbf{u} + \mathbf{q} - \mathbf{u} \cdot (\mathbb{D} + \mathbb{D}^T - \frac{2}{3} \text{tr}(\mathbb{D}) \mathbb{I}) \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e$$

# Space-Time Compressible Navier-Stokes

Compact Notation

Conserved quantities

$$C_c := \rho$$

$$\mathbf{C}_m := \rho \mathbf{u}$$

$$C_e := \rho(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u})$$

Euler fluxes

$$\mathbf{F}_c := \rho \mathbf{u}$$

$$\mathbb{F}_m := \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbb{I}$$

$$\mathbf{F}_e := \rho \mathbf{u} \left( C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \rho R T \mathbf{u}$$

Viscous fluxes

$$\mathbf{K}_c := \mathbf{0}$$

$$\mathbb{K}_m := \mathbb{D} + \mathbb{D}^T - \frac{2}{3} \text{tr}(\mathbb{D}) \mathbb{I}$$

$$\mathbf{K}_e := -\mathbf{q} + \mathbf{u} \cdot \left( \mathbb{D} + \mathbb{D}^T - \frac{2}{3} \text{tr}(\mathbb{D}) \mathbb{I} \right)$$

Viscous variables

$$\mathbb{M}_{\mathbb{D}} := \mathbb{D}$$

$$\mathbf{M}_q := \frac{Pr}{C_p} \mathbf{q}$$

$$\mathbf{G}_{\mathbb{D}} := 2\mathbf{u}$$

$$G_q := -T$$

Use change of variables to get conservation or entropy variables.

# Space-Time Compressible Navier-Stokes

Conservation Variables (Popular for Time-Stepping)

Change of variables:

$$\rho = \rho$$

$$\mathbf{m} = \rho \mathbf{u}$$

$$E = \rho \left( C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right)$$

Euler fluxes:

$$\mathbf{F}_c^c = \mathbf{m}$$

$$\mathbb{F}_m^c = \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} + (\gamma - 1) \left( E - \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho} \right) \mathbb{I}$$

$$\mathbf{F}_e^c = \gamma E \frac{\mathbf{m}}{\rho} - (\gamma - 1) \frac{\mathbf{m} \cdot \mathbf{m}}{2\rho^2} \mathbf{m}$$

# Space-Time Compressible Navier-Stokes

Entropy Variables (Symmetrize the Bubnov-Galerkin Stiffness Matrix)

Change of variables:

$$V_c = \frac{-E + (E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}) \left( \gamma + 1 - \ln \left[ \frac{(\gamma-1)(E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m})}{\rho^\gamma} \right] \right)}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

$$\mathbf{V}_m = \frac{\mathbf{m}}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

$$V_e = \frac{-\rho}{E - \frac{1}{2\rho} \mathbf{m} \cdot \mathbf{m}}$$

Euler fluxes:

$$\mathbf{F}_c^e = \left[ \frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[ \frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \mathbf{V}_m$$

$$\mathbb{F}_m^e = \left[ \frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[ \frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \left( -\frac{\mathbf{V}_m \otimes \mathbf{V}_m}{V_e} + (\gamma - 1) \mathbb{I} \right)$$

$$\mathbf{F}_e^e = \left[ \frac{\gamma - 1}{(-V_e)^\gamma} \right]^{\frac{1}{\gamma-1}} \exp \left[ \frac{-\gamma + V_c - \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m}{\gamma - 1} \right] \frac{\mathbf{V}_m}{V_e} \left( \frac{1}{2V_e} \mathbf{V}_m \cdot \mathbf{V}_m - \gamma \right)$$

# Space-Time Compressible Navier-Stokes

Define Group Variables

Group terms

$$C := \{C_c, \mathbf{C}_m, C_e\}$$

$$F := \{\mathbf{F}_c, \mathbb{F}_m, \mathbf{F}_e\}$$

$$K := \{\mathbf{K}_c, \mathbb{K}_m, \mathbf{K}_e\}$$

$$M := \{\mathbb{M}_{\mathbb{D}}, \mathbf{M}_{\mathbf{q}}\}$$

$$G := \{\mathbf{G}_{\mathbb{D}}, G_{\mathbf{q}}\}$$

$$f := \{f_c, \mathbf{f}_m, f_e\}$$

Group variables

$$W := \{\rho, \mathbf{u}, T\}$$

$$\hat{W} := \{2\hat{\mathbf{u}}, -\hat{T}\}$$

$$\Sigma := \{\mathbb{D}, \mathbf{q}\}$$

$$\hat{t} := \{\hat{t}_e, \hat{\mathbf{t}}_m, , \hat{t}_e\}$$

$$\Psi := \{\mathbb{S}, \tau\}$$

$$V := \{v_c, \mathbf{v}_m, , v_e\}$$

Navier-Stokes variational formulation is

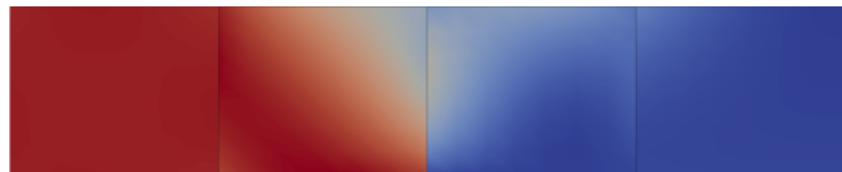
$$\left( \frac{1}{\mu} M, \Psi \right) + (G, \nabla \cdot \Psi) - \langle \hat{W}, \Psi \cdot \mathbf{n}_x \rangle = 0$$

$$- \left( \begin{pmatrix} F - K \\ C \end{pmatrix}, \nabla_{xt} V \right) + \langle \hat{t}, V \rangle = (f, V)$$

# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

Mesh 1



Primitive Variables



Conservation Variables

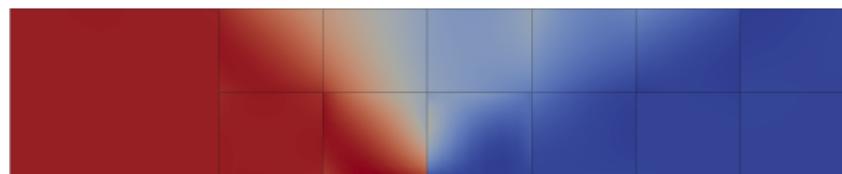


Entropy Variables

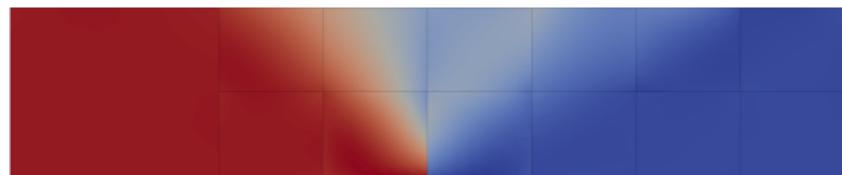
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

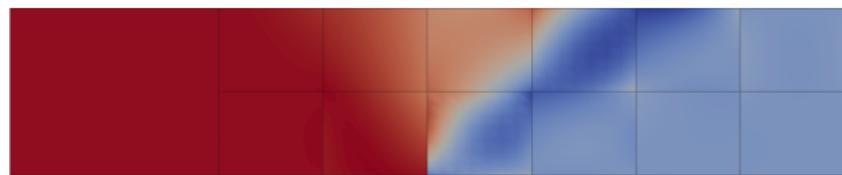
Mesh 2



Primitive Variables



Conservation Variables

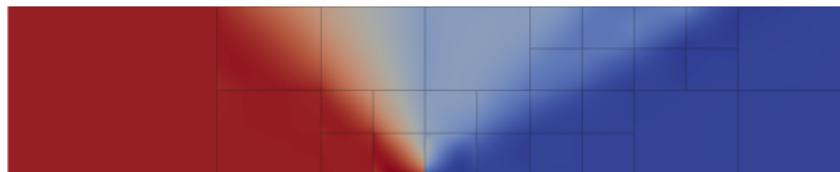


Entropy Variables

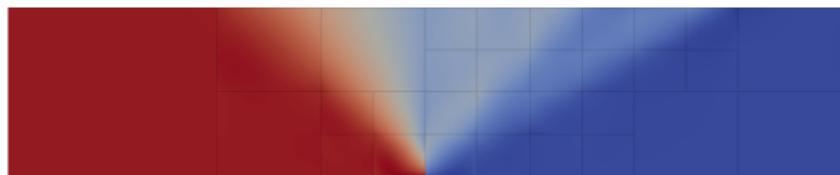
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

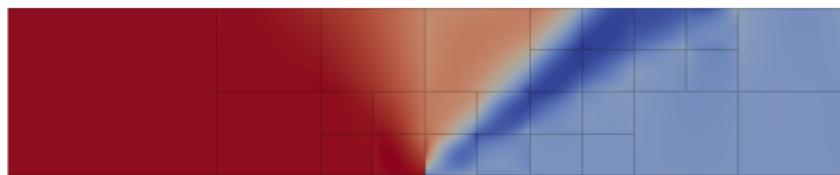
Mesh 3



Primitive Variables



Conservation Variables

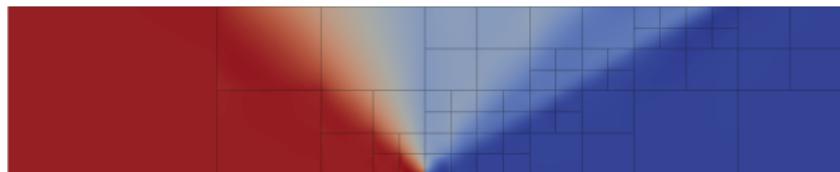


Entropy Variables

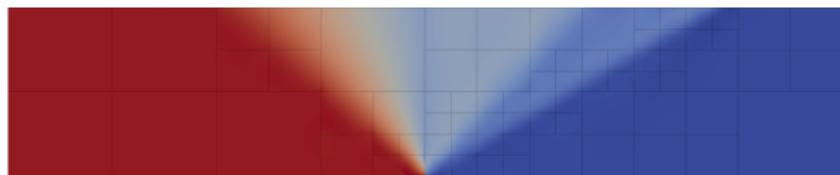
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

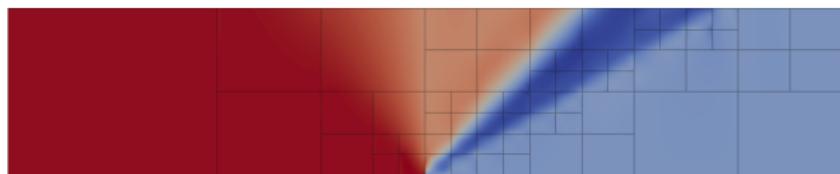
Mesh 4



Primitive Variables



Conservation Variables

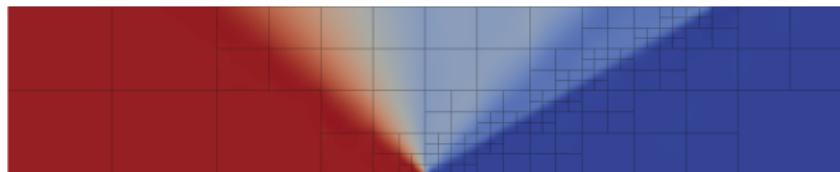


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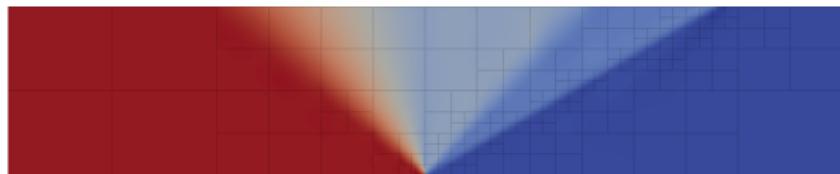
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

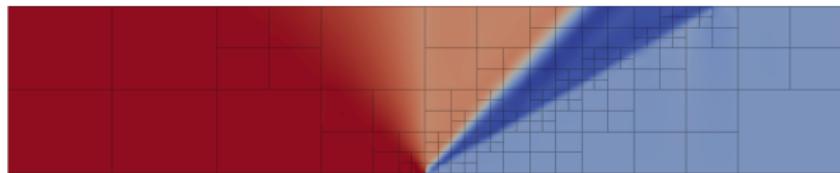
Mesh 5



Primitive Variables



Conservation Variables

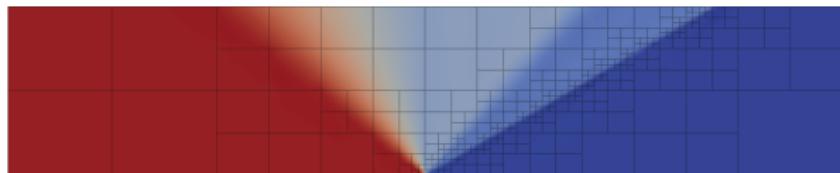


Entropy Variables

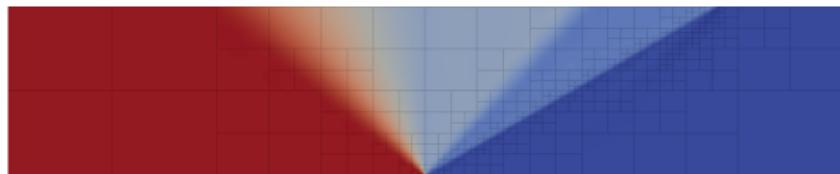
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

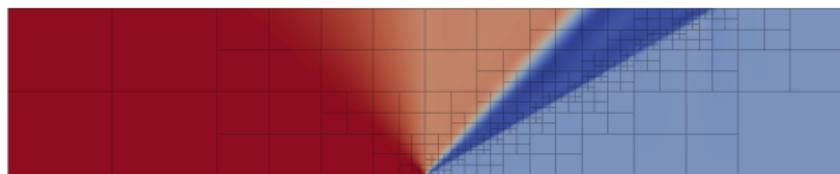
Mesh 6



Primitive Variables



Conservation Variables

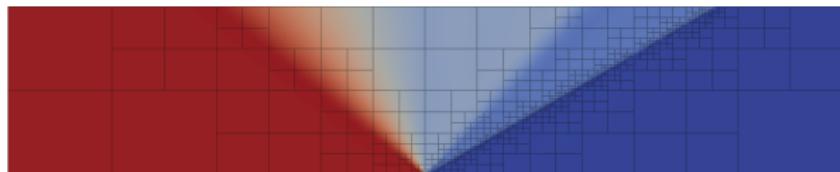


Entropy Variables

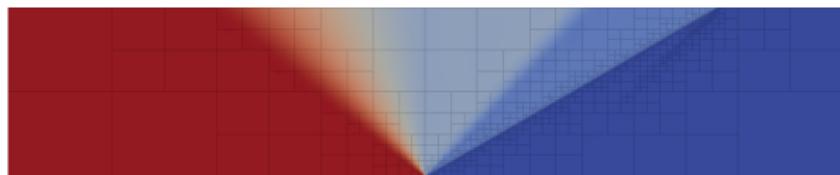
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

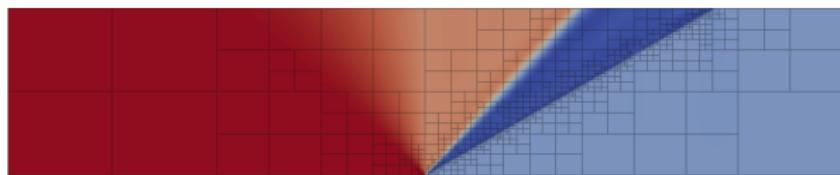
Mesh 7



Primitive Variables



Conservation Variables

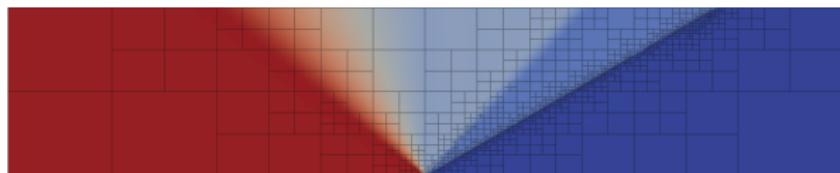


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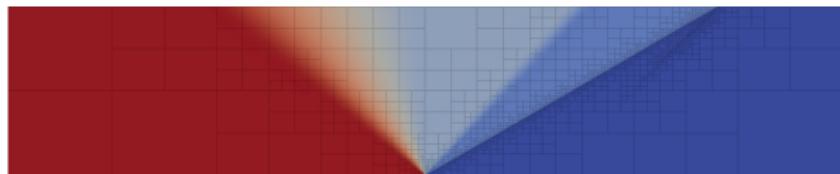
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

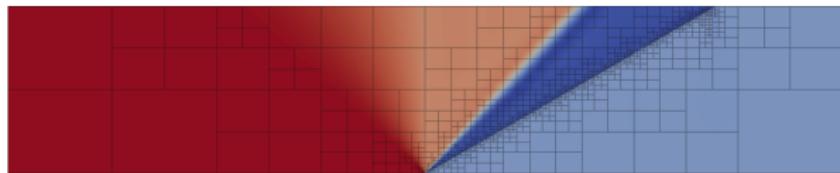
Mesh 8



Primitive Variables



Conservation Variables

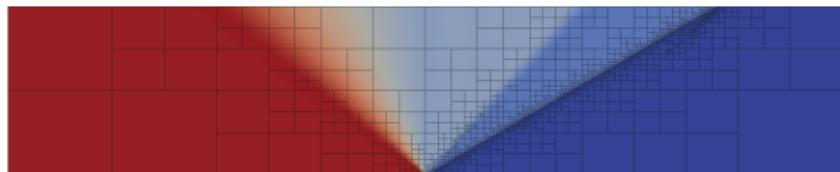


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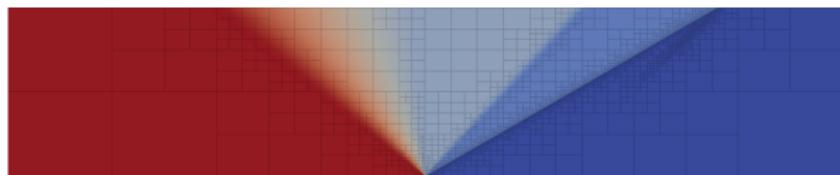
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Sod Shock Tube with  $\mu = 10^{-5}$

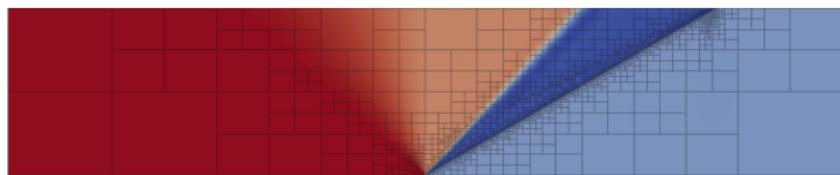
Mesh 9



Primitive Variables



Conservation Variables

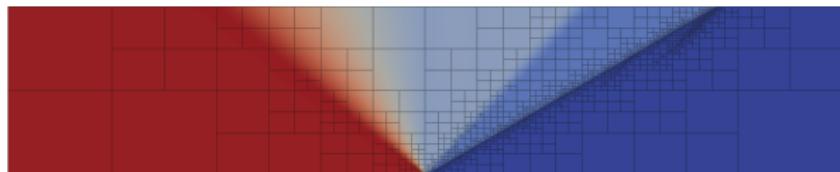


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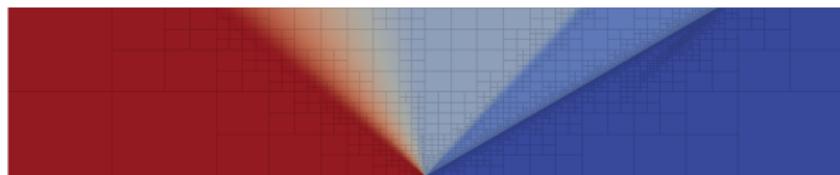
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Sod Shock Tube with  $\mu = 10^{-5}$

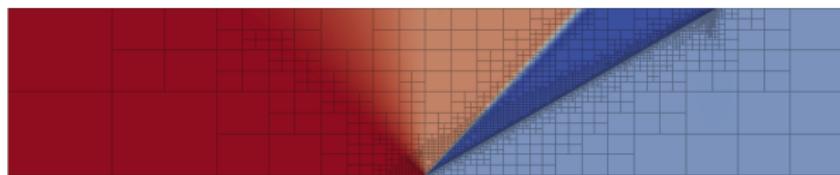
Mesh 10



Primitive Variables



Conservation Variables

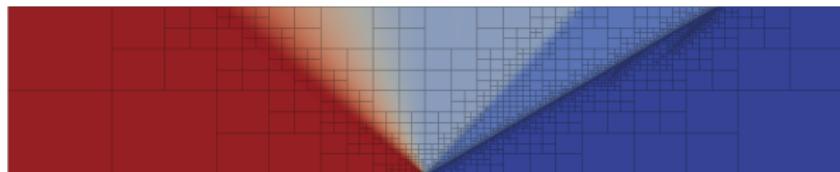


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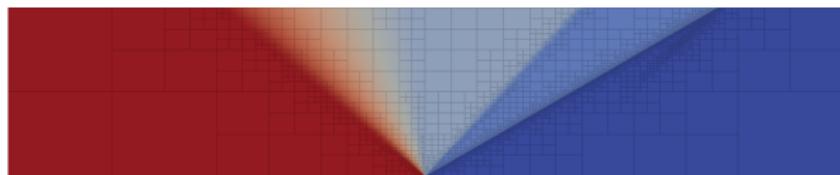
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

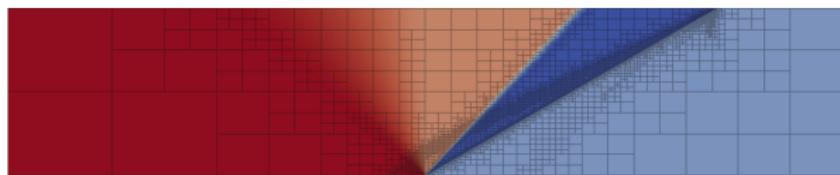
Mesh 11



Primitive Variables



Conservation Variables

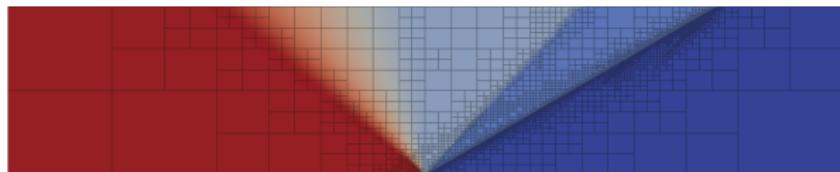


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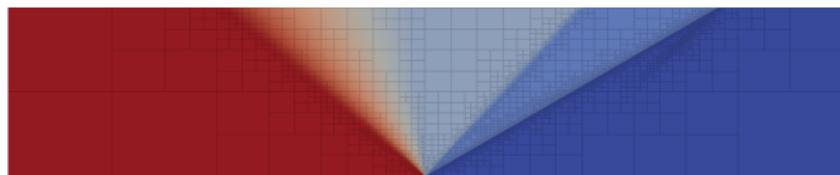
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

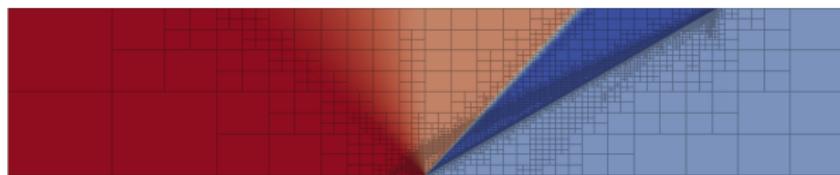
Mesh 12



Primitive Variables



Conservation Variables

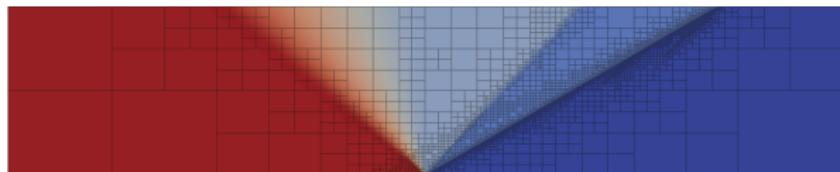


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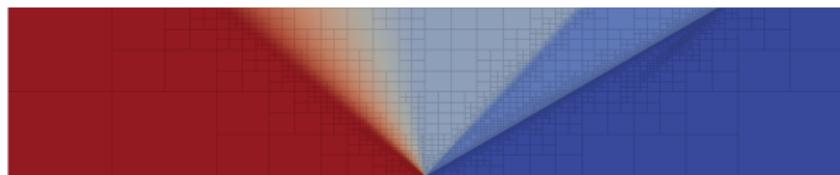
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

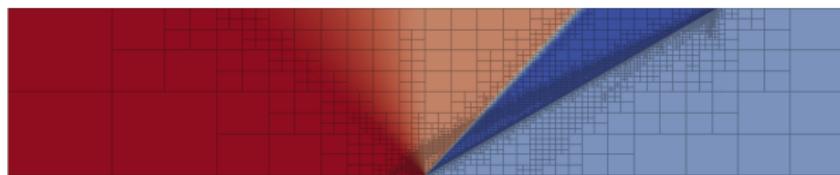
Mesh 13



Primitive Variables



Conservation Variables

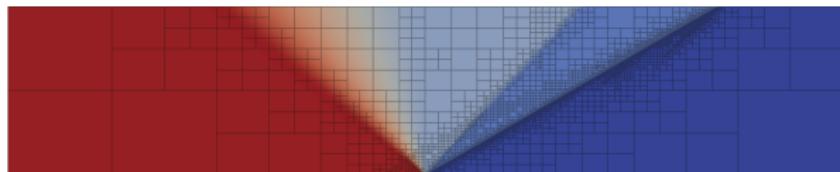


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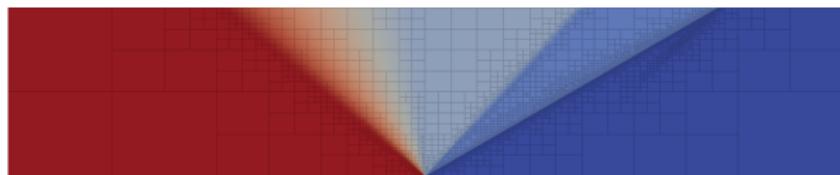
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

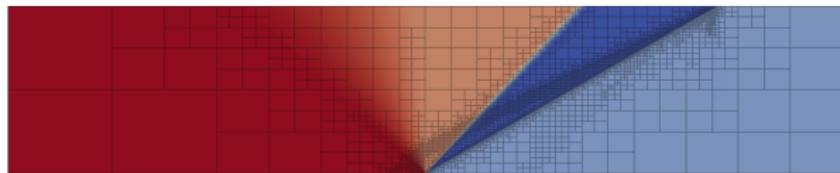
Mesh 14



Primitive Variables



Conservation Variables

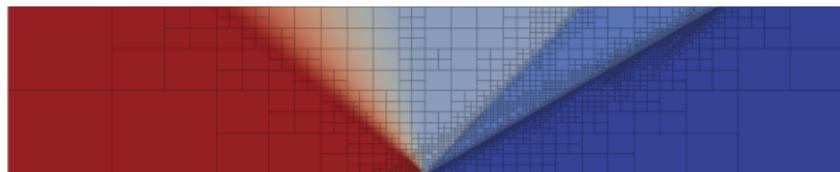


Entropy Variables

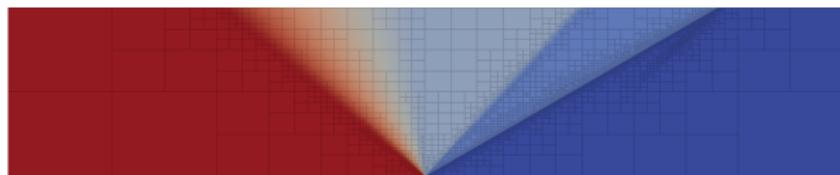
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$

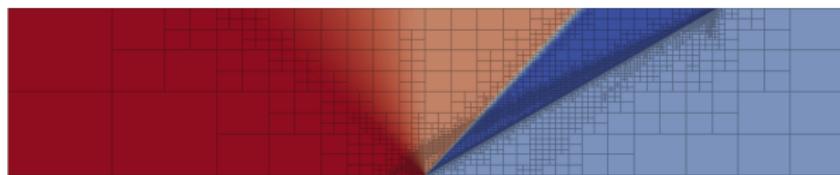
Mesh 15



Primitive Variables



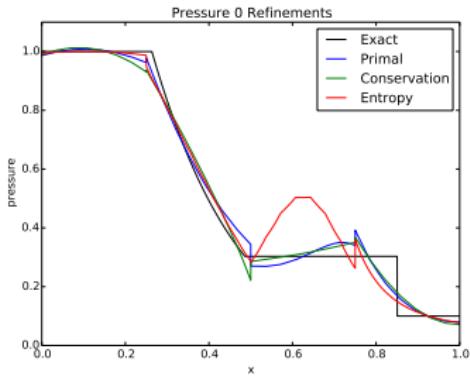
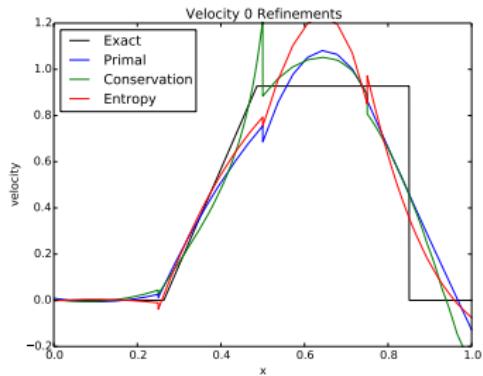
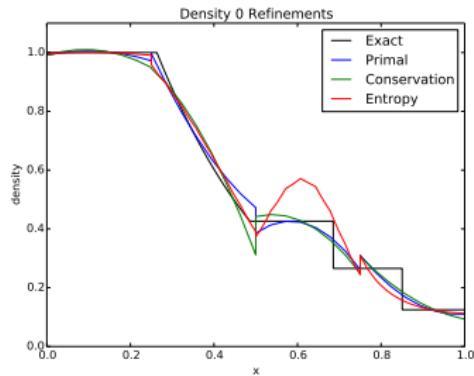
Conservation Variables



Entropy Variables

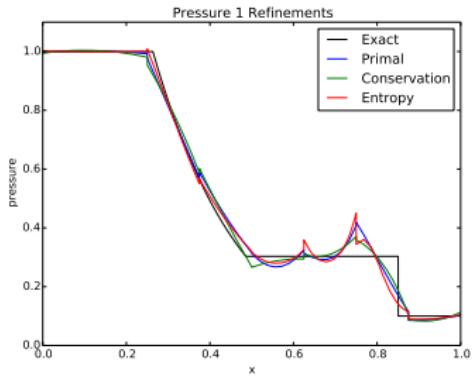
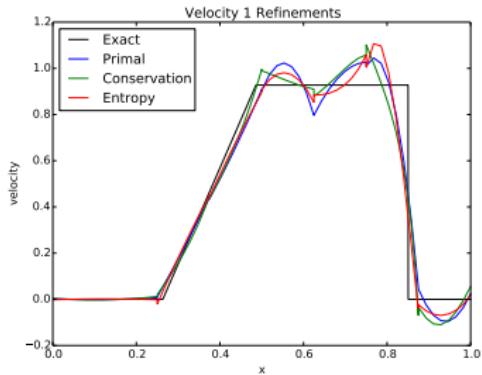
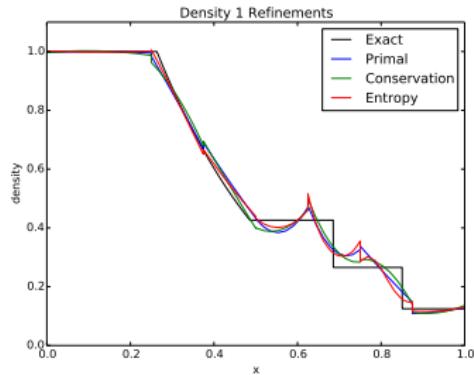
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



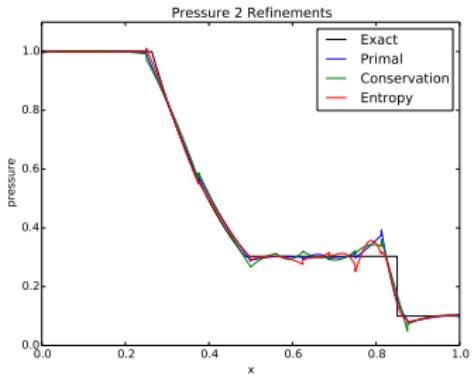
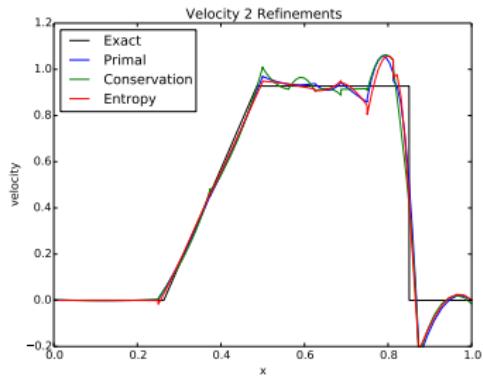
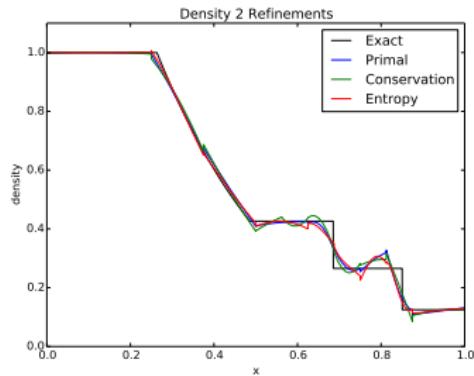
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



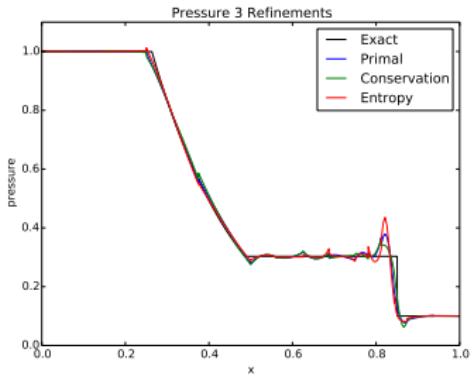
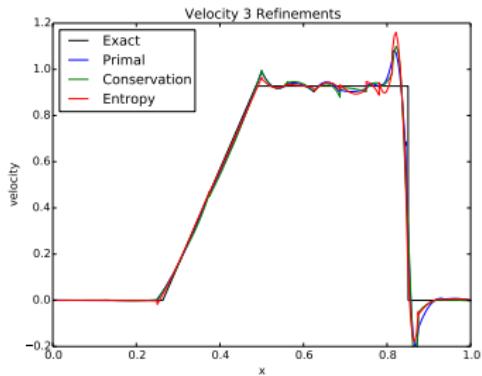
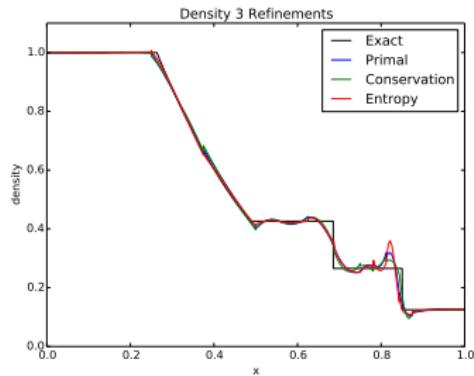
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



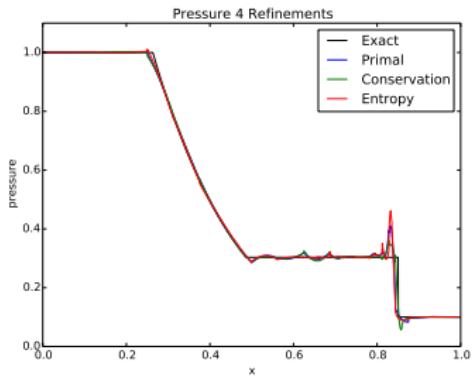
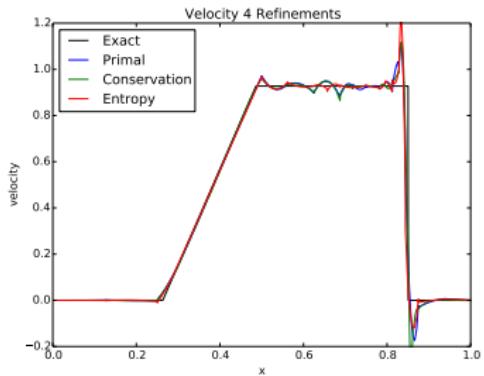
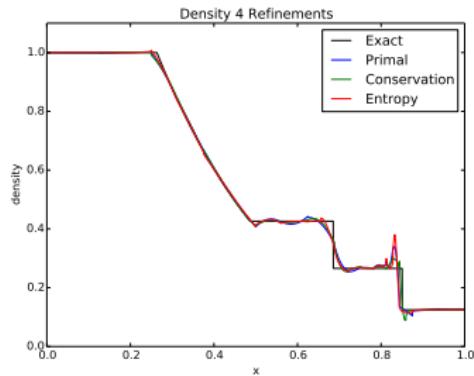
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



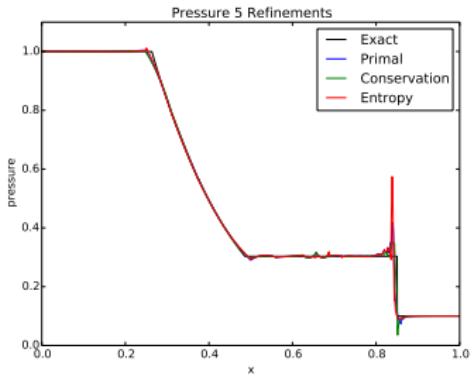
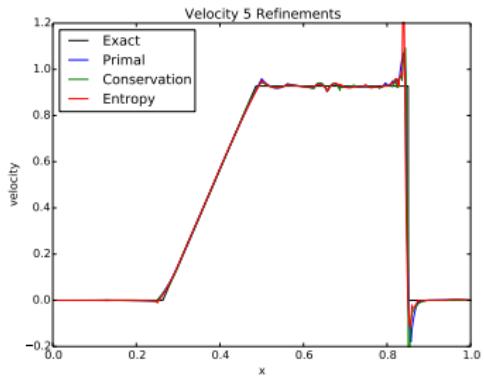
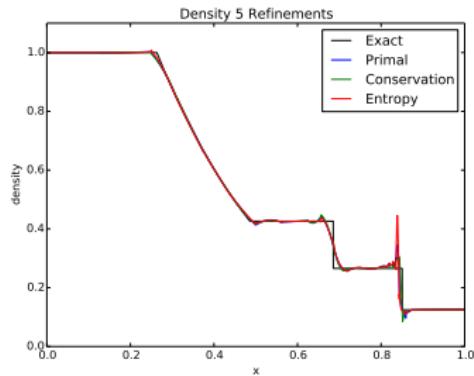
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



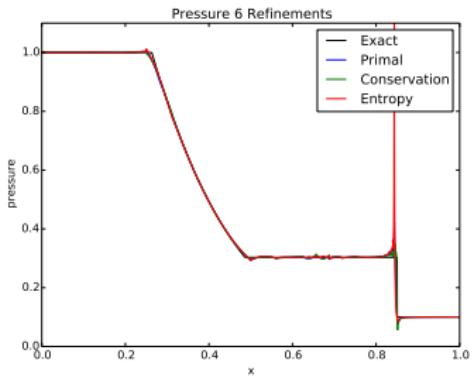
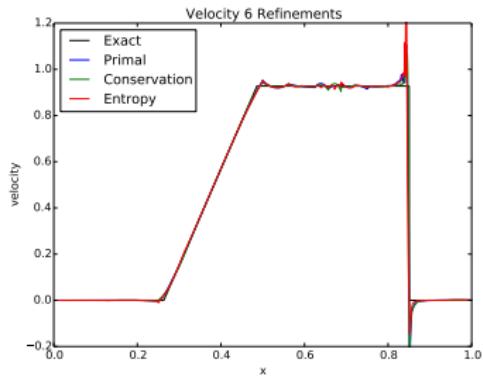
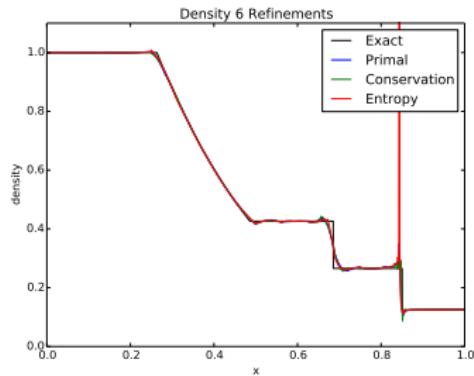
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



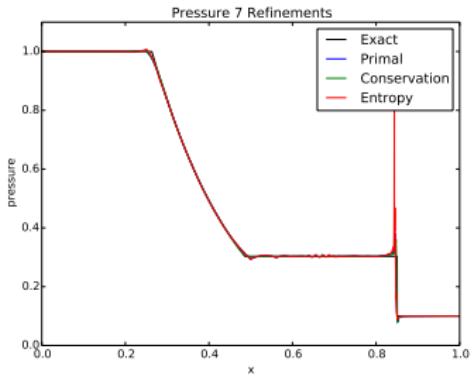
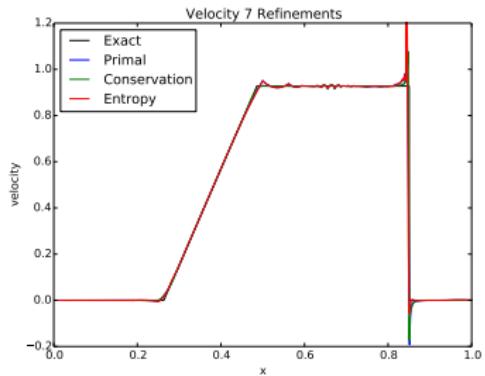
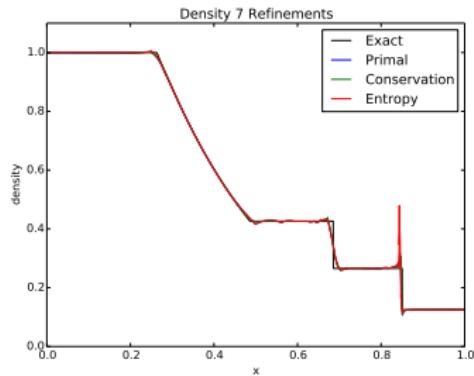
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



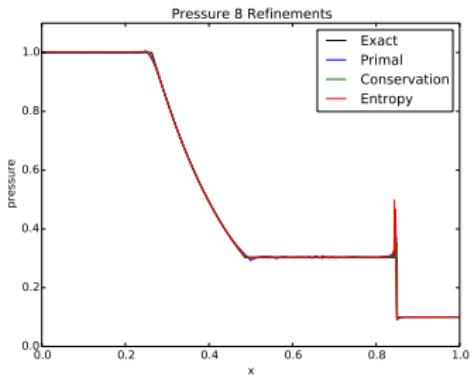
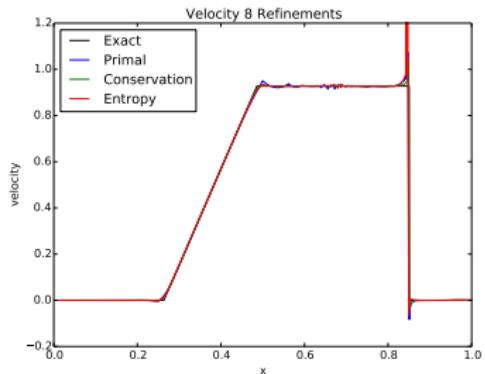
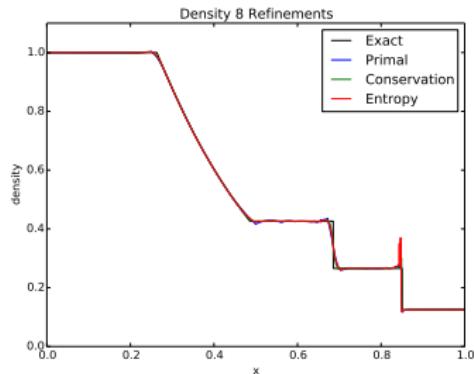
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



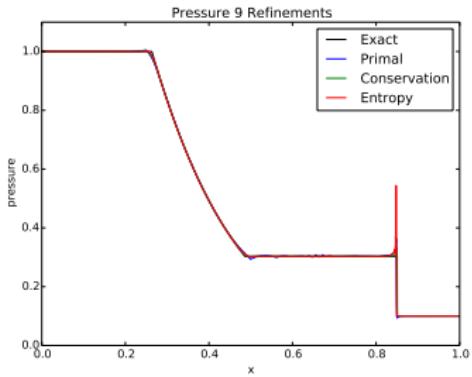
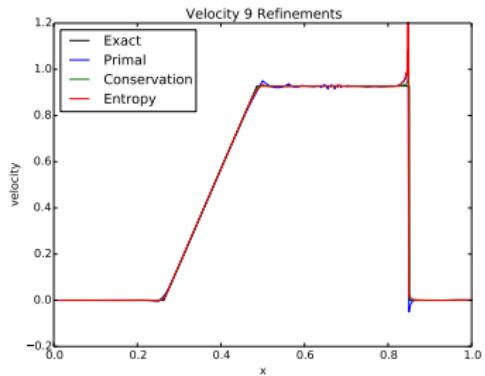
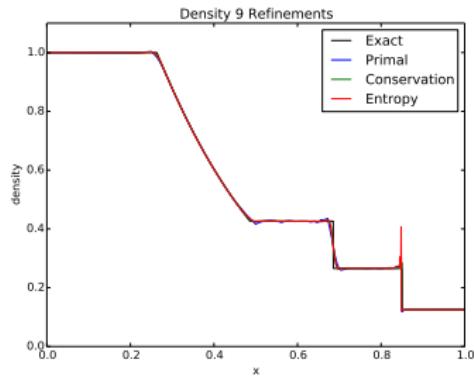
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



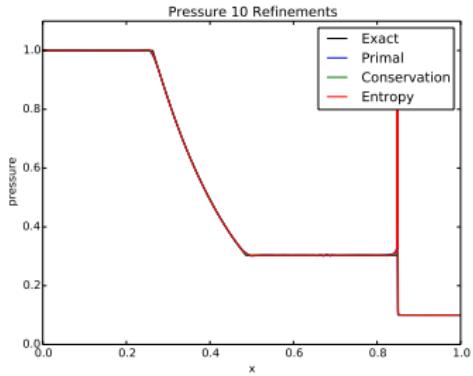
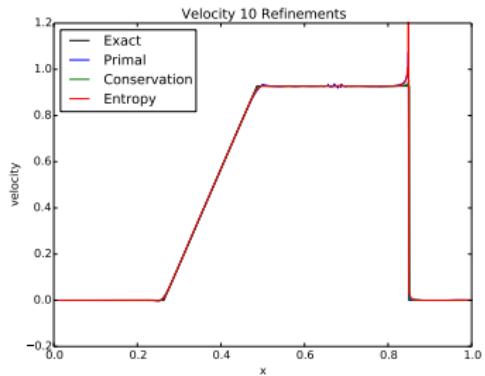
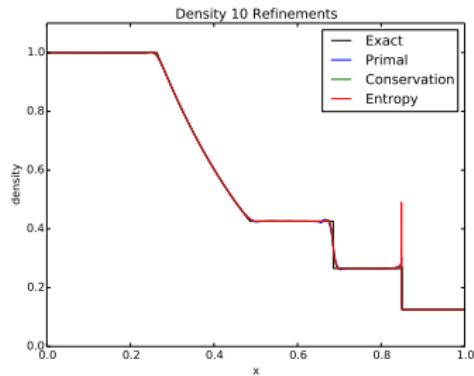
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



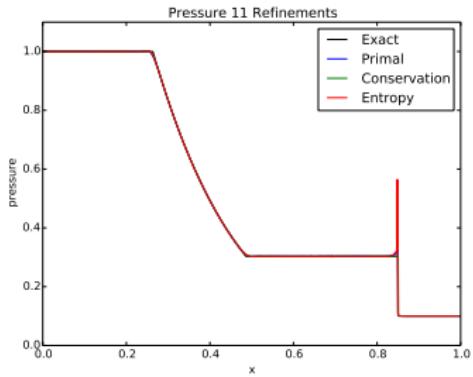
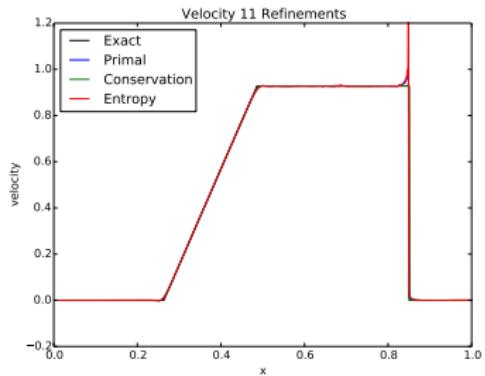
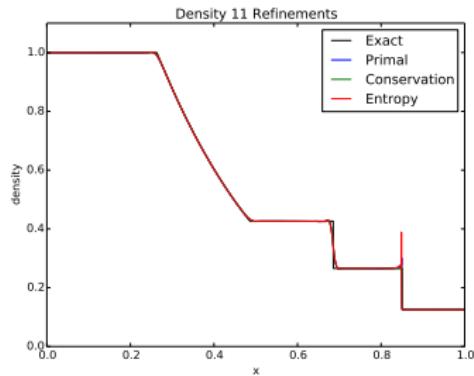
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



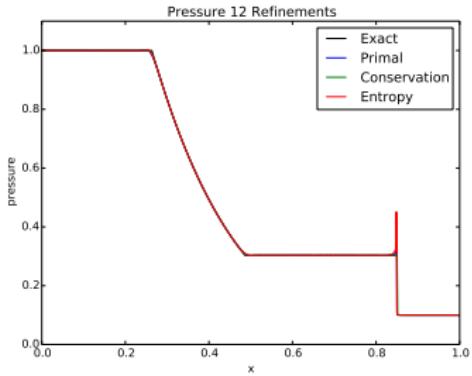
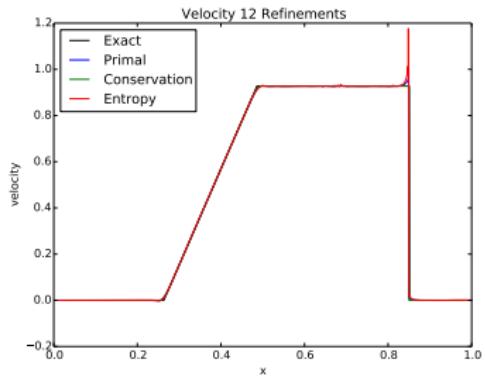
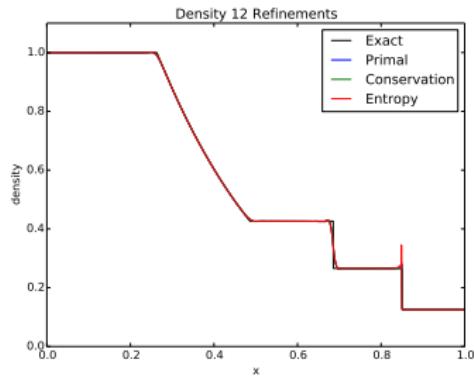
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



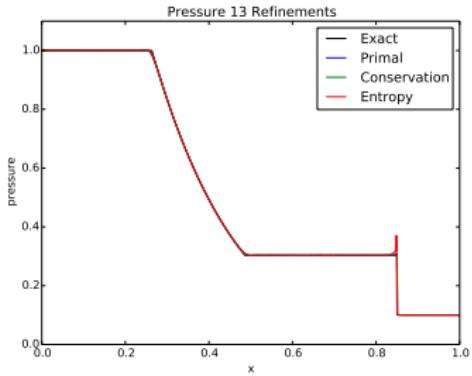
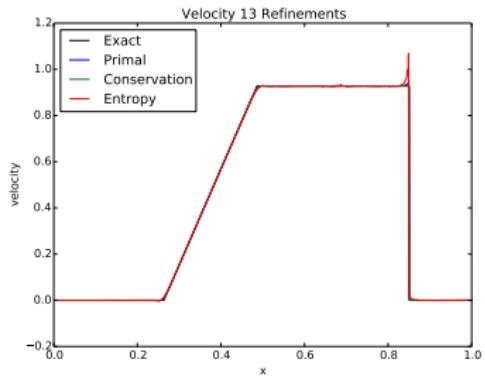
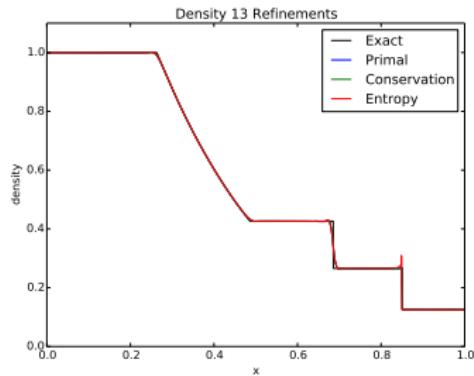
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



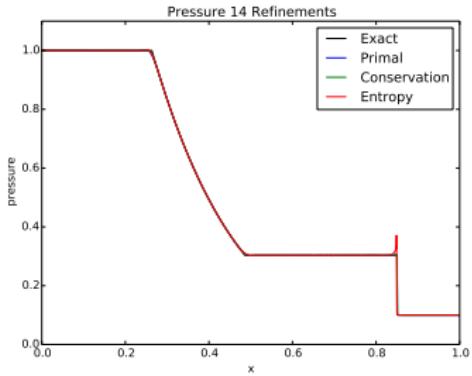
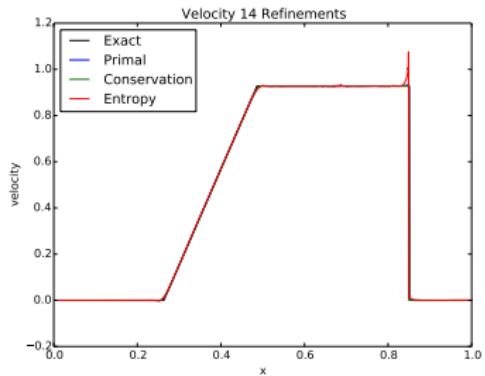
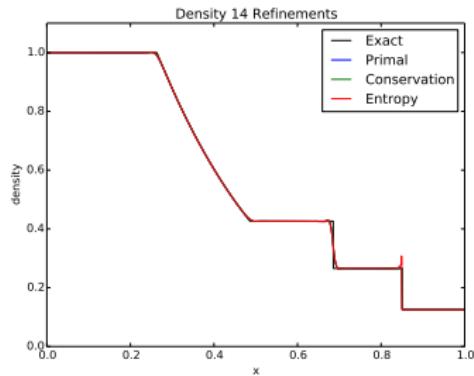
# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



# Space-Time Compressible Navier-Stokes

Sod Shock Tube with  $\mu = 10^{-5}$



# Space-Time Compressible Navier-Stokes

Entropy Scaled Test Norms

Let  $W$ ,  $U$ , and  $V$  be the set of primitive, conservation, and entropy variables.

The entropy function

$$H = -\rho \log(p\rho^{-\gamma})$$

provides a natural residual for the Navier-Stokes equations.

$A_0 = H_{UU} = V_{,U}$  is known as the symmetrizer and  $(U, A_0 U)$  provides a natural metric for the Euler equations.

In primitive variables:

$$(u, A_0 U) = (U_{,W} W, V_{,U} U_{,W} W) = (W, U_{,W}^T V_{,U} U_{,W} W) = (W, A_0(W) W)$$

where

$$A_0(W) = U_{,W}^T V_{,U} U_{,W} = \begin{bmatrix} \frac{\gamma-1}{\rho} & 0 & 0 \\ 0 & \frac{\rho}{C_v T} & 0 \\ 0 & 0 & \frac{\rho}{T^2} \end{bmatrix}$$

# Space-Time Compressible Navier-Stokes

Entropy Scaled Test Norms

Bilinear form with group variables:

$$b((W, \hat{W}), v) = (W, A_h^* v)_{L^2(\Omega_h)} + \langle \hat{W}, [v] \rangle_{\Gamma_h}$$

For conforming  $v^*$  satisfying  $A^* v^* = A_0 W$

$$\begin{aligned} \left\| A_0^{\frac{1}{2}} W \right\|^2 &= \frac{b(W, v^*)}{\|v^*\|_V} \|v^*\|_V \\ &\leq \sup_{v^* \neq 0} \frac{|b(W, v^*)|}{\|v^*\|} \|v^*\| = \|W\|_E \|v^*\|_V . \end{aligned}$$

Necessary robustness condition:

$$\begin{aligned} \|v^*\|_V &\lesssim \left\| A_0^{\frac{1}{2}} W \right\|_{L^2(\Omega_h)} \\ \Rightarrow \left\| A_0^{\frac{1}{2}} W \right\|_{L^2(\Omega_h)} &\lesssim \|W\|_E \end{aligned}$$

# Space-Time Compressible Navier-Stokes

Entropy Scaled Test Norms

We load our adjoint equations with  $A_0 W$ :

$$\begin{aligned} \frac{1}{\mu} M^*(\Psi) + K^*(\nabla V) &= 0 \\ - \begin{pmatrix} F^* \\ C^* \end{pmatrix} (\nabla_{xt} V) + G^*(\nabla \Psi) &= A_0 W \end{aligned}$$

This leads to the entropy scaled robust norm:

$$\begin{aligned} \|(V, \Psi)\|_{V,K}^2 &:= \left\| A_0^{-\frac{1}{2}} (F^* + C^*) \right\|_K^2 + \mu \left\| A_0^{-\frac{1}{2}} K^* \right\|_K^2 \\ &\quad + \min \left( \frac{\mu}{h^2}, 1 \right) \left\| A_0^{-\frac{1}{2}} V \right\|_K^2 + \left\| A_0^{-\frac{1}{2}} G^* \right\|_K^2 \\ &\quad + \min \left( \frac{1}{\mu}, \frac{1}{h^2} \right) \left\| A_0^{-\frac{1}{2}} M^* \right\|_K^2 \end{aligned}$$

# Space-Time Compressible Navier-Stokes

Entropy Scaled Test Norms

We load our adjoint equations with  $A_0 W$ :

$$\frac{1}{\mu} M^*(\Psi) + K^*(\nabla V) = 0$$

$$-\begin{pmatrix} F^* \\ C^* \end{pmatrix} (\nabla_{xt} V) + G^*(\nabla \Psi) = A_0 W$$

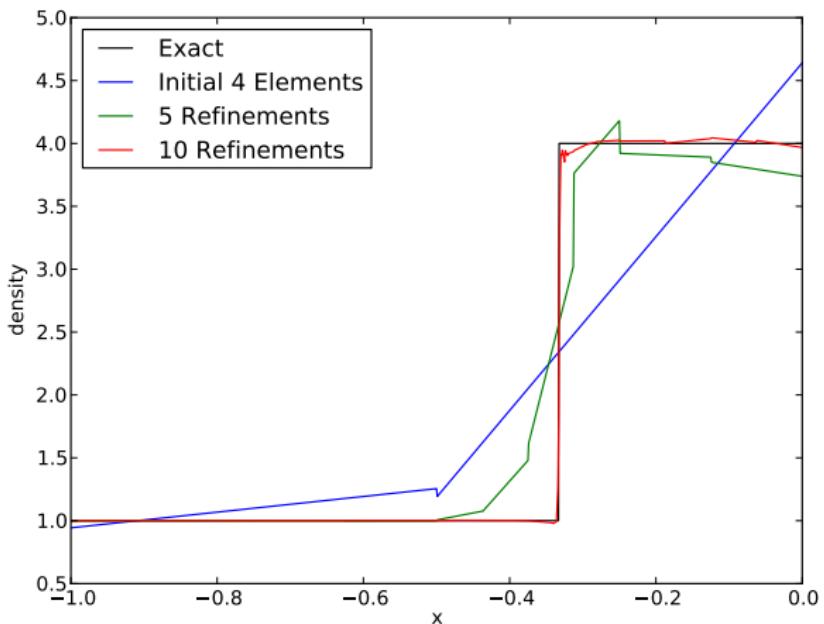
This leads to the entropy scaled robust norm:

$$\begin{aligned} \|(V, \Psi)\|_{V,K}^2 &:= \left\| A_0^{-\frac{1}{2}} (F^* + C^*) \right\|_K^2 + \mu \left\| A_0^{-\frac{1}{2}} K^* \right\|_K^2 \\ &\quad + \min\left(\frac{\mu}{h^2}, 1\right) \left\| A_0^{-\frac{1}{2}} V \right\|_K^2 + \left\| A_0^{-\frac{1}{2}} G^* \right\|_K^2 \\ &\quad + \min\left(\frac{1}{\mu}, \frac{1}{h^2}\right) \left\| A_0^{-\frac{1}{2}} M^* \right\|_K^2 \end{aligned}$$

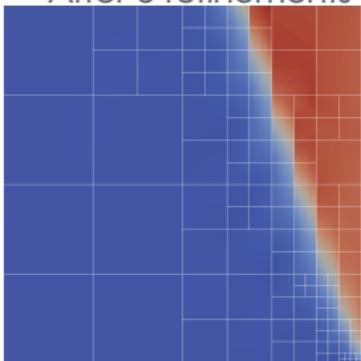
Numerical results were disappointing.

# Space-Time Compressible Navier-Stokes

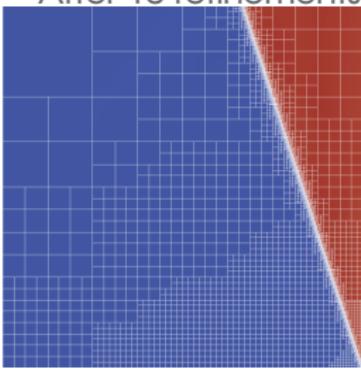
Noh Implosion with Primitive Variables, Robust Norm,  $\mu = 10^{-3}$



After 5 refinements



After 10 refinements



# Space-Time Compressible Navier-Stokes

Piston with  $\mu = 10^{-2}$

$$\hat{t}_c = \sqrt{2}(-\rho u + \rho)$$

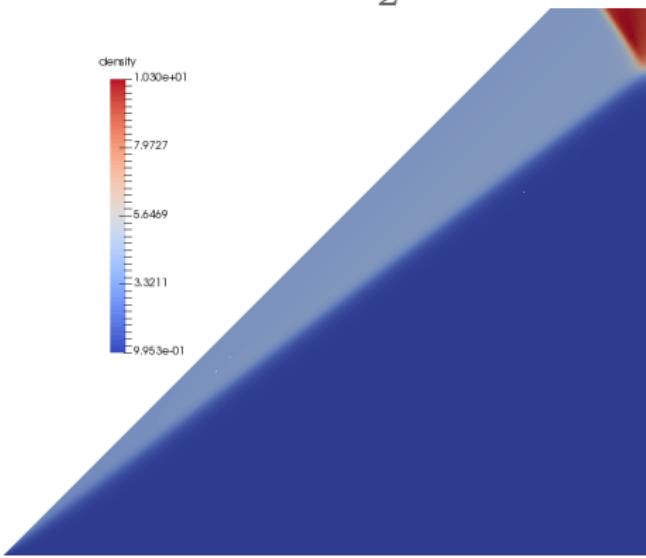
$$\hat{u} = 1$$

$$\hat{t}_m = \sqrt{2}(-\rho u^2 - \rho RT + \rho u)$$

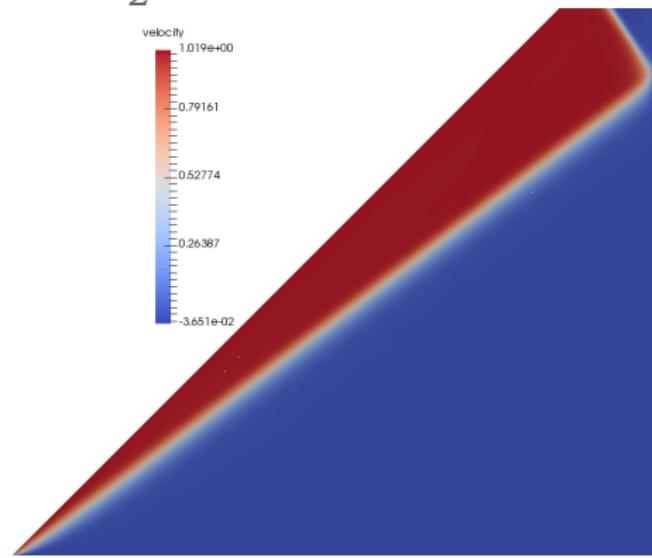
$$\hat{t}_c = 0$$

$$\hat{t}_e = \sqrt{2}(-\rho u(C_v T + \frac{1}{2}u^2) - u\rho RT + \rho(C_v T + \frac{1}{2}u^2))$$

$$\hat{t}_m - \hat{t}_e = 0$$



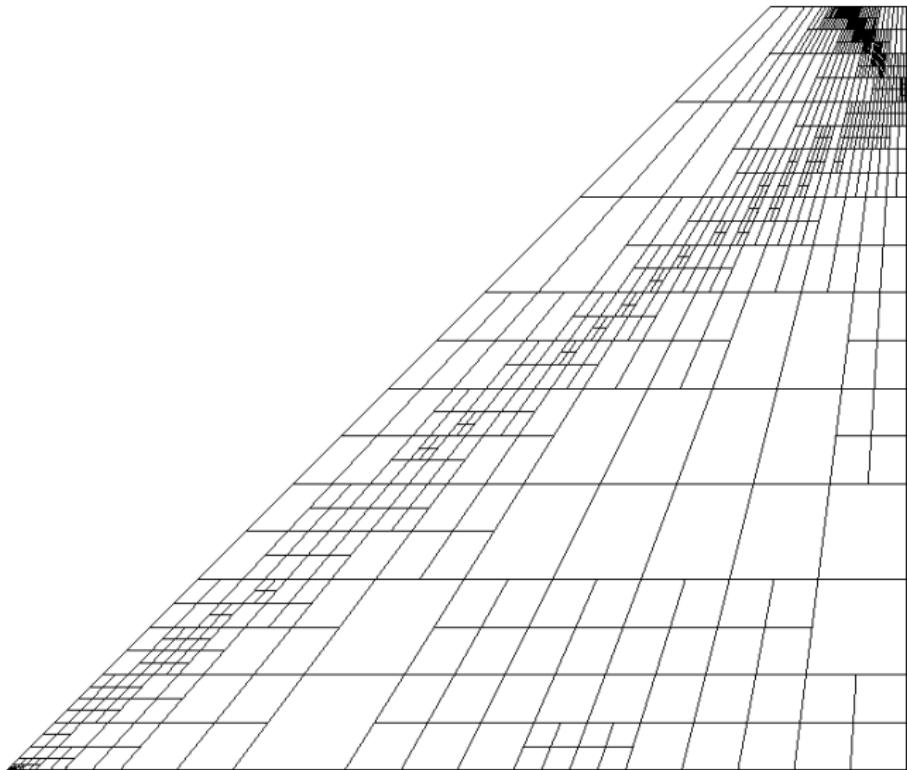
Density



Velocity

# Space-Time Compressible Navier-Stokes

Piston with  $\mu = 10^{-2}$



Mesh after 8 adaptive refinements

# Thank You!

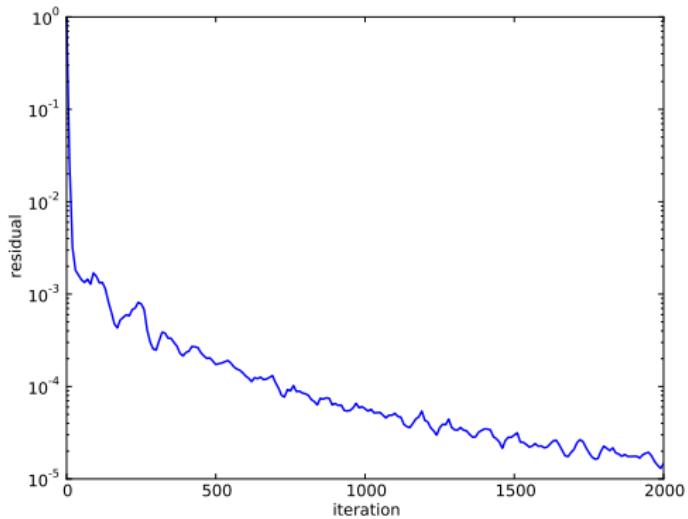
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- ▶ D. Moro, N.C. Nguyen, and J. Peraire. "A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws". In: *Int. J. Num. Meth. Eng.* (2011).
- ▶ C.L. Chang and J.J. Nelson. "Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation". In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.
- ▶ J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014), pp. 771–795.
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- ▶ T.E. Ellis, J.L. Chan, and L.F. Demkowicz. *Robust DPG Methods for Transient Convection-Diffusion*. Tech. rep. 15-21. ICES, Oct. 2015.
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- ▶ N.V. Roberts. "Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods". In: *Comp. Math. Appl.* 68.11 (2014), pp. 1581 –1604.
- ▶ L.F. Demkowicz and N. Heuer. "Robust DPG Method for Convection-Dominated Diffusion Problems". In: *SIAM J. Numer. Anal.* 51.5 (2013), pp. 1514–2537.
- ▶ N. Roberts, T. Bui-Thanh, and L. Demkowicz. "The DPG method for the Stokes problem". In: *Comp. Math. Appl.* 67.4 (2014), pp. 966 –995.

# Scaling Issues

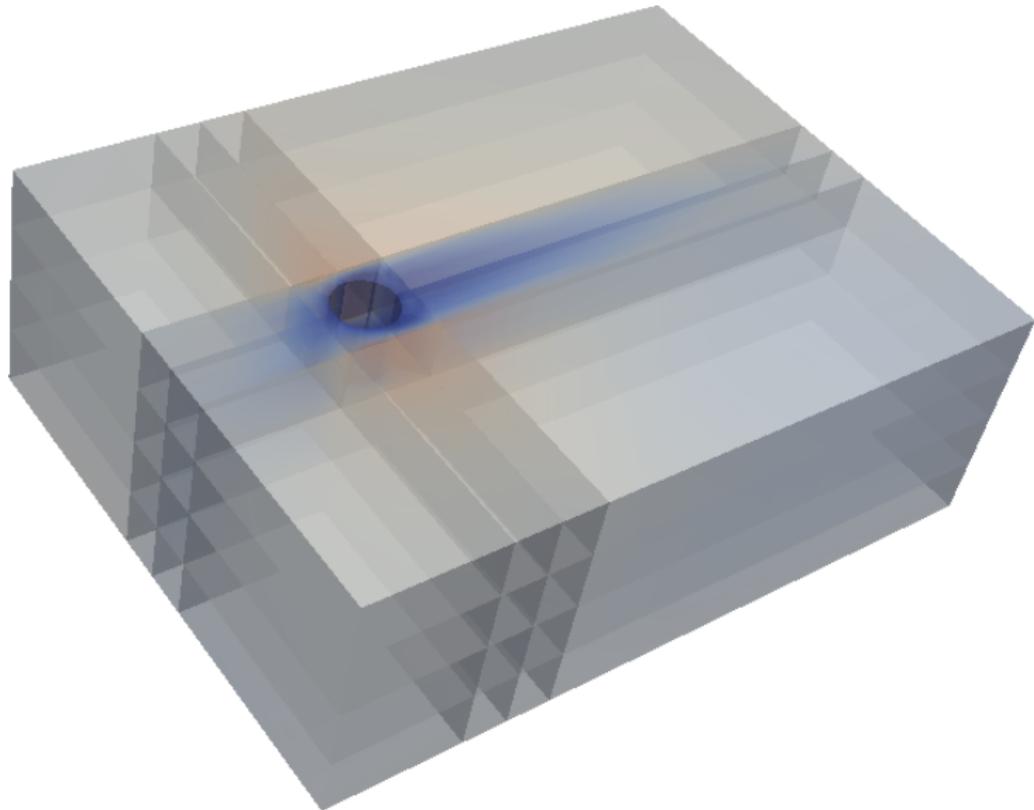
## Multigrid and Convection-Diffusion

- Convection-diffusion,  
 $\epsilon = 10^{-2}$ ,  $64 \times 64$  mesh
- Conjugate gradient
- Geometric multigrid preconditioner
- Multiplicative V-cycle
- Overlapping additive Schwarz smoother
- Hierarchy of  $p$ -coarsening followed by  $h$ -coarsening



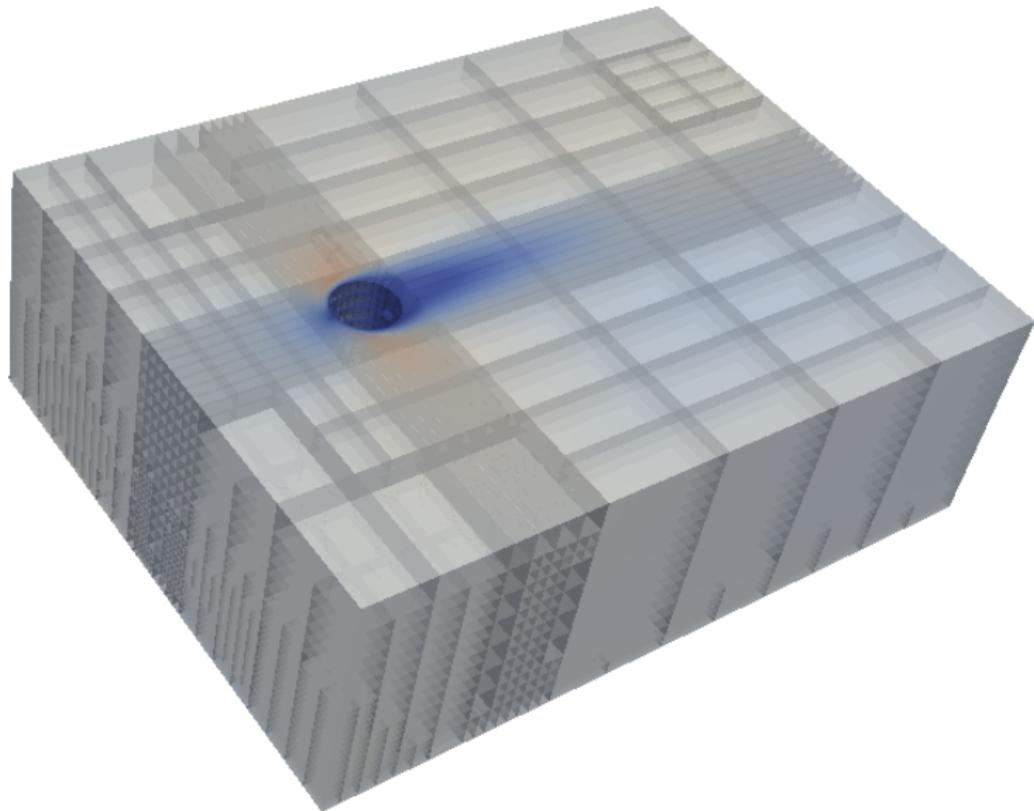
# Scaling Issues

Incompressible Flow Over a Cylinder, Initial Mesh



# Scaling Issues

Incompressible Flow Over a Cylinder, 4 Refinements



# Scaling Issues

Solve Times and Strong Scaling

## Transient Flow Over a Cylinder

Ref	Elems	DOFs	1 Node	4 Nodes		32 Nodes	
			Time	Time	Scaling vs 1	Time	Scaling vs 4
0	80	31304	1772	453	3.91	451	1.01
1	605	225908	8190	3574	2.29	717	4.98
2	3013	1081598	32008	12076	2.65	2648	4.56
3	9726	3429384		28744		6319	4.54
4	11742	4144674				8510	

Computations on Lonestar, 1 node = 24 processors

32008 seconds = 8.8 hours

28744 seconds = 8.0 hours

8510 seconds = 2.4 hours

# Scaling Issues

Solve Times and Strong Scaling

## Taylor-Green Vortex

Ref	Elems	DOFs	1 Node	4 Nodes	
			Time	Time	Scaling vs 1
0	60	21302	331	140	2.35
1	312	108410	945	290	3.25
2	2020	691834	4880	1363	3.58
3	9244	3043024		6171	

Computations on Lonestar, 1 node = 24 processors

4880 seconds = 1.4 hours

6171 seconds = 1.7 hours

# Scaling Issues

## Space-Time Slabs

Assumptions:

- The maximum required spatial resolution is much finer than the required temporal resolution.
- Regions requiring high spatial resolution are concentrated in relatively compact parts of the domain.
- Only isotropic refinements are permitted.
- The number of time slabs is a power of 2.

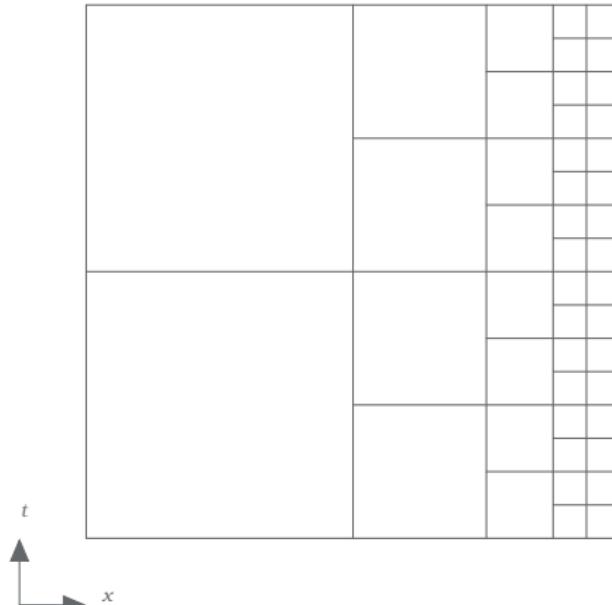
Test problem: convection diffusion with exact solution

$$u = 1 - e^{\frac{x}{\epsilon}}$$

on space-time domain  $[-1, 0] \times [0, 1]$ .

# Scaling Issues

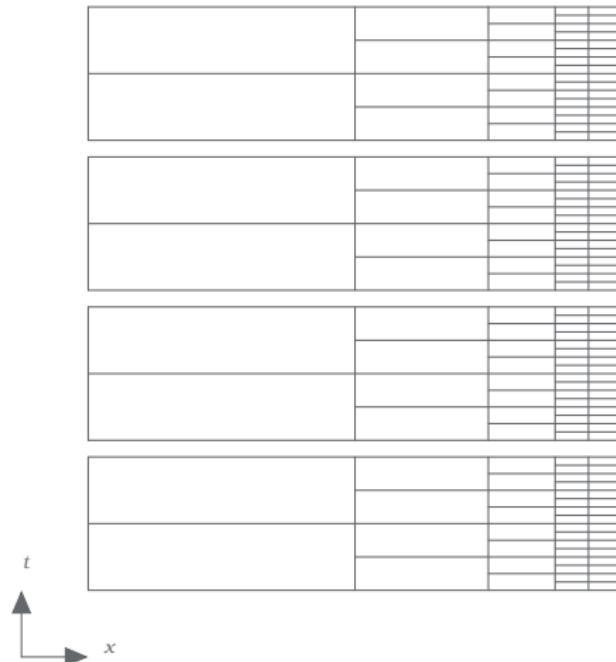
## Space-Time Slabs



Single Slab Strategy

# Scaling Issues

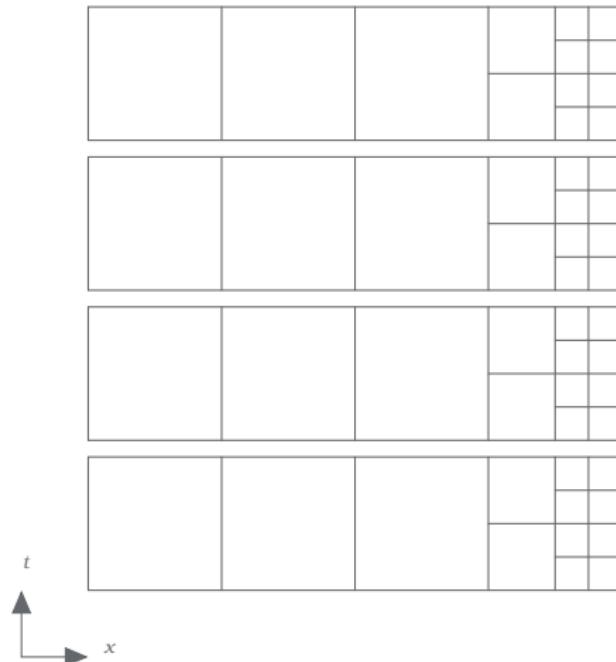
## Space-Time Slabs



Naive Time Slab Strategy

# Scaling Issues

## Space-Time Slabs

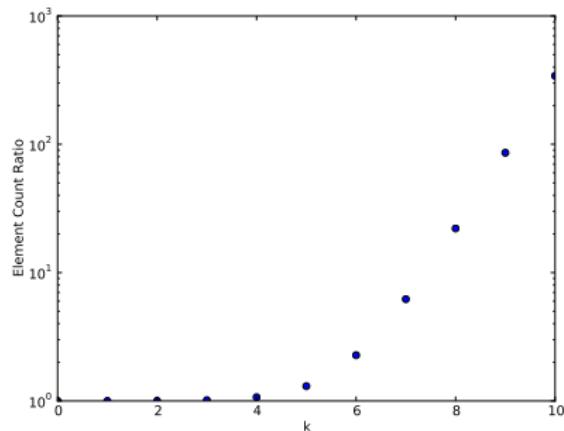


Smarter Time Slab Strategy

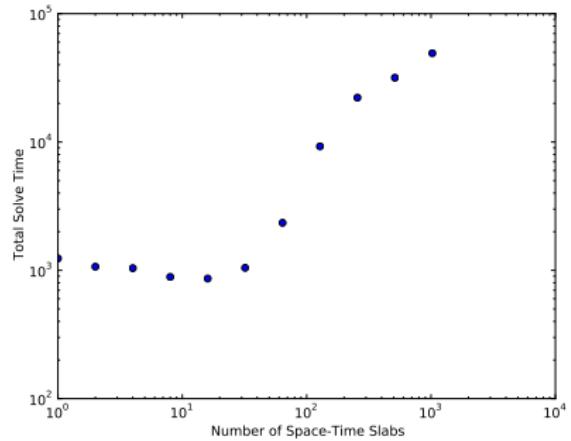
# Scaling Issues

## Space-Time Slabs

Number of time slabs =  $2^k$ .



Ratio of total element counts



Total solve time using smart time slabs

- Without anisotropic refinements, time slabs don't significantly speed up computations.
- Time slabs could be useful for memory constrained problems.

# Camellia: DPG for the Masses

Overview

## Design Goal

Make DPG research and experimentation as simple as possible, while maintaining computational efficiency and scalability.

Built on Trilinos (Teuchos, Intrepid, Shards, Epetra, etc).

Mature support for:

- Rapid specification of DPG variational forms, inner products, etc.
- Distributed computation of stiffness matrix
- 1D - 3D geometries
- Curvilinear elements
- $h$ - and  $p$ -refinements (anisotropic in  $h$ )

Experimental support for:

- Space-time computations
- Iterative solvers (tested up to 32,768 cores)

# Convection-Diffusion in Three Slides

## Building the Bilinear Form

```

VarFactory vf;
//fields:
VarPtr u = vf.fieldVar("u", L2);
VarPtr sigma = vf.fieldVar("sigma", VECTOR_L2);

// traces:
VarPtr u_hat = vf.traceVar("u_hat");
VarPtr t_n = vf.fluxVar("t_n");

// test:
VarPtr v = vf.testVar("v", HGRAD);
VarPtr tau = vf.testVar("tau", HDIV);

double eps = .01;
FunctionPtr beta_x = Function::constant(1);
FunctionPtr beta_y = Function::constant(2);
FunctionPtr beta = Function::vectorize(beta_x, beta_y);

BFPtr bf = Teuchos::rcp( new BF(vf) );

bf->addTerm((1/eps) * sigma, tau);
bf->addTerm(u, tau->div());
bf->addTerm(-u_hat, tau->dot_normal());

bf->addTerm(sigma - beta * u, v->grad());
bf->addTerm(t_n, v);

RHSPtr rhs = RHS::rhs();

```

Find  $u \in L^2(\Omega_h)$ ,  $\sigma \in \mathbf{L}^2(\Omega_h)$ ,  
 $\hat{u} \in H^{\frac{1}{2}}(\Gamma_h)$ ,  $\hat{t}_n \in H^{-\frac{1}{2}}(\Gamma_h)$   
such that

$$\begin{aligned} \frac{1}{\epsilon} (\sigma, \tau) + (u, \nabla \cdot \tau) - \langle \hat{u}, \tau \cdot \mathbf{n} \rangle \\ - (\beta u - \sigma, \nabla v) + \langle \hat{t}_n, v \rangle = (f, v) \end{aligned}$$

for all  $v \in H^1(K)$ ,  $\tau \in \mathbf{H}(\text{div}, K)$ .

where  $\epsilon = 10^{-2}$ ,  $\beta = (1, 2)^T$  and

$$f = 0.$$

# Convection-Diffusion in Three Slides

## Boundary Conditions and Mesh

```

int k = 2;
int delta_k = 2;
MeshPtr mesh = MeshFactory::quadMesh(bf, k+1, delta_k);
BCPtr bc = BC::bc();

SpatialFilterPtr y_equals_one = SpatialFilter::matchingY(1.0);
SpatialFilterPtr y_equals_zero = SpatialFilter::matchingY(0);
SpatialFilterPtr x_equals_one = SpatialFilter::matchingX(1.0);
SpatialFilterPtr x_equals_zero = SpatialFilter::matchingX(0.0);

FunctionPtr zero = Function::zero();
FunctionPtr x = Function::xn(1);
FunctionPtr y = Function::yn(1);
bc->addDirichlet(t_n, y_equals_zero, -2 * (1-x));
bc->addDirichlet(t_n, x_equals_zero, -1 * (1-y));
bc->addDirichlet(u_hat, y_equals_one, zero);
bc->addDirichlet(u_hat, x_equals_one, zero);

```

Create a square mesh  $[0, 1] \times [0, 1]$  with boundary conditions

- $\hat{t}_n = 2x - 2$  on  $y = 0$
- $\hat{t}_n = x - 1$  on  $x = 0$
- $\hat{u} = 0$  on  $y = 1$
- $\hat{u} = 0$  on  $x = 1$

## Note

- Can subclass `SpatialFilter` to match any geometry
- Adding new mesh readers is straightforward

# Convection-Diffusion in Three Slides

## Test Norm, Solving, and Adaptivity

```
IPPtr ip = bf->graphNorm();

SolutionPtr soln = Solution::solution(mesh, bc, rhs, ip);

double threshold = 0.20;
RefinementStrategy refStrategy(soln, threshold);

int numRefs = 10;

ostringstream refName;
refName << "ConvectionDiffusion";
HDF5Exporter exporter(mesh, refName.str());

for (int refIndex=0; refIndex < numRefs; refIndex++) {
    soln->solve();

    double energyError = soln->energyErrorTotal();
    cout << "After " << refIndex << "_refinements, _energy_error_is_" << energyError << endl;

    exporter.exportSolution(soln, vf, refIndex);

    if (refIndex != numRefs)
        refStrategy.refine();
}
```

# Convection-Diffusion in Three Slides

Computed Solution

