

Space-Time Discontinuous Petrov-Galerkin Finite Elements for Transient Computational Fluid Dynamics

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ENGINEERING & SCIENCES

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- Literature Review

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- Local Conservation
- Space-Time DPG

4 Future Work

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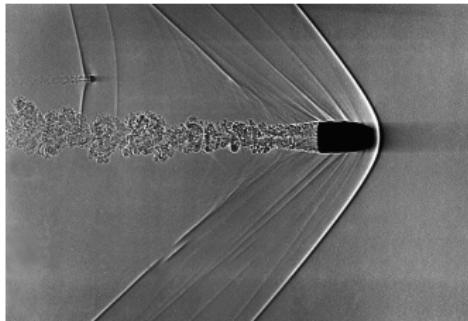
4 Future Work

Navier-Stokes Equations

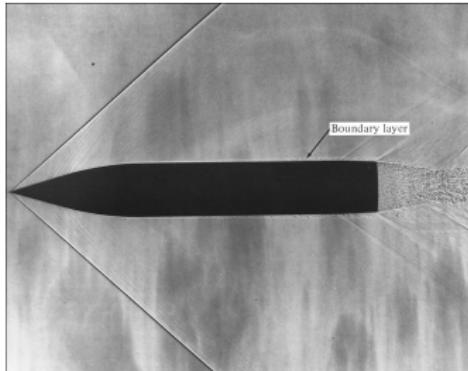
Numerical Challenges

Robust simulation of unsteady fluid dynamics remains a challenging issue.

- Resolving solution features (sharp, localized viscous-scale phenomena)
 - Shocks
 - Boundary layers - resolution needed for drag/load
 - Turbulence (non-localized)
- Nonlinear convergence and uniqueness of solutions
- Stability of numerical schemes
 - Coarse/adaptive grids
 - Higher order



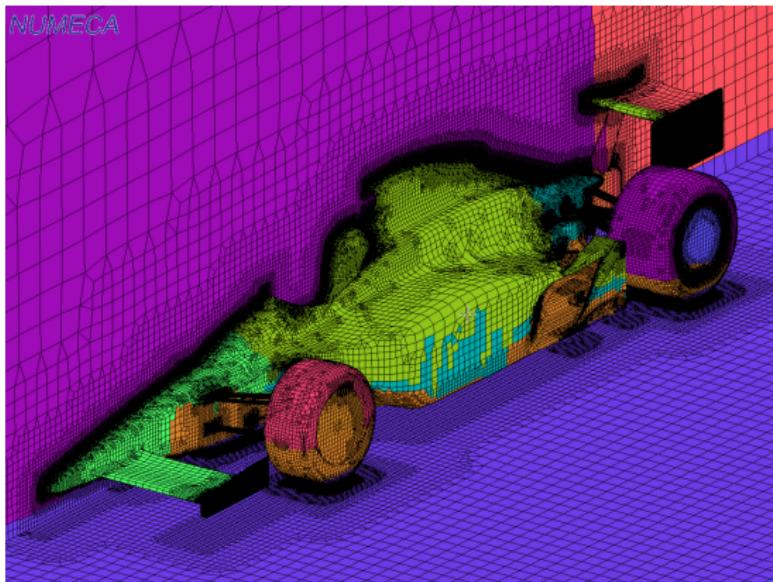
Shock



Motivation

Initial Mesh Design is Expensive and Time-Consuming

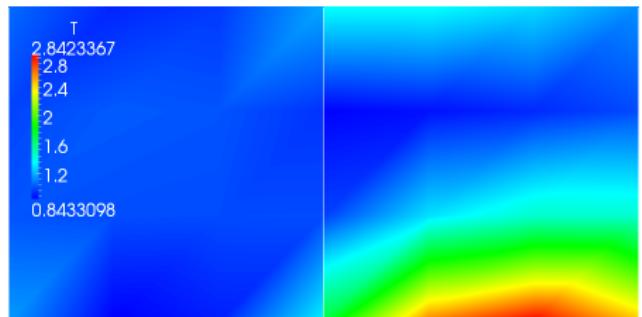
- Surface mesh must accurately represent geometry
- Volume mesh needs sufficient resolution for asymptotic regime
- Boundary layer meshes must respect y^+ guidelines
- Engineers often forced to work by trial and error
- Bad in the context of HPC



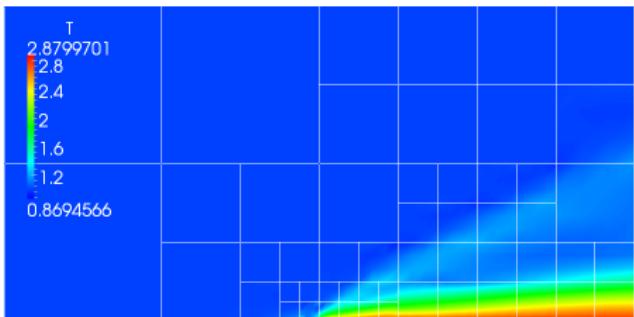
Formula 1 Mesh by Numeca

DPG on Coarse Meshes

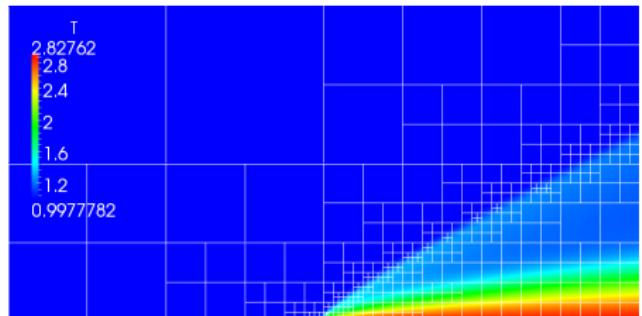
Adaptive Solve of the Carter Plate Problem¹ $Re = 1000$



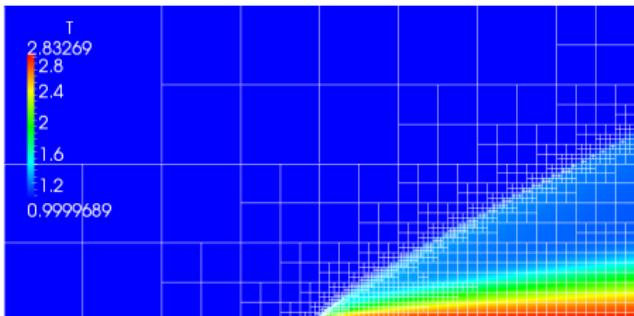
Temperature on Initial Mesh



Temperature after 4 Refinements



Temperature after 8 Refinements



Temperature after 11 Refinements

¹J.L. Chan. "A DPG Method for Convection-Diffusion Problems". PhD thesis. University of Texas at Austin, 2013.

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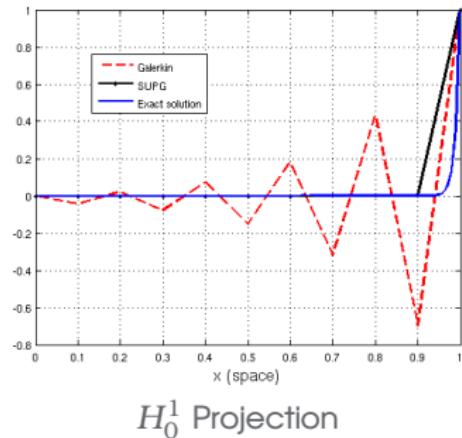
- Local Conservation
- Space-Time DPG

4 Future Work

Stabilized Finite Elements for CFD

Streamline Upwind Petrov-Galerkin

- First successful finite element method for CFD²
- Residual based stabilization preserves consistency
- Upwind biasing of test functions
- Higher order generalizations possible
- Optimal H_0^1 approximation in 1D
- Gave rise to the field of stabilized finite elements
- Major contributors include Hughes, Franca, Johnson, Codina, Tezduyar, and many others
- Variational multiscale may be considered the spiritual successor to SUPG³



²A.N. Brooks and T.J.R. Hughes. "Streamline Upwind/Petrov-Galerkin Formulations for Convection Dominated Flows with Particular Emphasis on the Incompressible Navier-Stokes Equations". In: *Comput. Methods Appl. Mech. Eng.* (Sept. 1990), pp. 199–259

³T.J.R. Hughes et al. "The variational multiscale method -- a paradigm for computational mechanics". In: *Comput. Methods in Appl. Mech. Eng.* 166.1 - 2 (1998). *Advances in Stabilized Methods in Computational Mechanics*, pp. 3 –24

Streamline Upwind Petrov-Galerkin

Two Equivalent Views on Stabilization

Convection-diffusion can be written as

$$Lu = (L_{adv} + L_{diff})u = f.$$

Residual Based Stabilization

$$b_{SUPG}(u, v) = l_{SUPG}(v)$$

where

$$b_{SUPG}(u, v) = b(u, v) + \sum_K \int_K \tau(L_{adv}v)(Lu - f)$$

$$l_{SUPG}(v) = l(v) + \sum_K \int_K \tau(L_{adv}v)f,$$

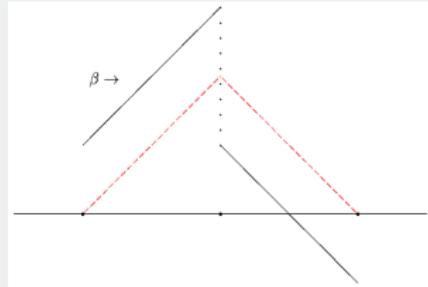
τ is the SUPG stabilization parameter.

Test Space Modification

$$b(u, \tilde{v}) = l(\tilde{v})$$

where

$$\tilde{v} = v + \tau L_{adv}v.$$



Stabilized Finite Elements for CFD

Discontinuous Galerkin

- Combines elements of finite volumes and finite elements
- First proposed for neutron transport⁴
- Early contributors include Babuška, Lions, Nitsche, and Zlámal
- Development for CFD by Cockburn and Shu⁵
- Development for elliptic problems given by Arnold, Brezzi, Cockburn, and Marini⁶
- Nonconforming basis, locally conservative
- Naturally high order, but may require additional stabilization
- Simple to parallelizable
- Other notable contributors include Peraire, Persson, Karniadakis . . .

⁴W.H. Reed and T.R. Hill. *Triangular mesh methods for the neutron transport equation*. Tech. rep. LA-UR-73-479. Los Alamos National Laboratory, 1973.

⁵B. Cockburn and C. Shu. "The Runge-Kutta Discontinuous Galerkin Method for Conservation Laws V: Multidimensional Systems". In: *J. Comp. Phys.* 141.2 (1998), pp. 199–224.

⁶D.N. Arnold et al. "Unified Analysis of Discontinuous Galerkin Methods for Elliptic Problems". In: *SIAM J. Numer. Anal.* 39.5 (2001), pp. 1749–1779.

Stabilized Finite Elements for CFD

Discontinuous Galerkin

Consider 1D convection equation

$$\frac{\partial \beta(x)u}{\partial x} = f, \quad u(0) = u_0.$$

Multiply by test function and integrate by parts over each element

$$K = [x_K, x_{K+1}]$$

$$-\int_K \beta(x)u \frac{\partial v}{\partial x} + \beta uv|_{x_K}^{x_{K+1}} = \int_K fv.$$

Apply upwind flux

$$-\int_K \beta(x)u \frac{\partial v}{\partial x} + \beta(x_{K+1})u(x_{K+1}^-)v(x_{K+1}^-) - \beta(x_K)u(x_K^-)v(x_K^+) = \int_K fv.$$

Hybridized DG (HDG) method introduces trace unknowns which facilitates static condensation, reducing interface unknowns⁷.

⁷ B. Cockburn, J. Gopalakrishnan, and R. Lazarov. "Unified Hybridization of Discontinuous Galerkin, Mixed, and Continuous Galerkin Methods for Second Order Elliptic Problems". In: SIAM J. Numer. Anal. 47.2 (Feb. 2009), pp. 1319–1365.

Space-Time Finite Element Methods

Treat Time as Another Dimension to be Discretized

Space-time methods treat time as just another dimension to be discretized.

- Early contributors include Kaczkowski and Oden
- Satisfies geometric conservation laws⁸
- Tezduyar *et al.*⁹ developed a Galerkin/least-squares method for moving boundaries
- Van der Vegt and Van der Ven developed a popular space-time DG method¹⁰
- Üngör's tent-pitcher algorithm¹¹ decouples space-time elements

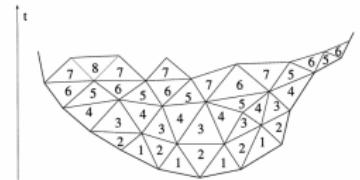


Fig. 2. Ordering of elements for an element-by-element procedure (1DxTIME). Elements with the same number can be solved independently or in parallel; elements with higher numbers depend for the solution on the output of those with lower ones.

⁸M. Lesoinne and C. Farhat. "Geometric conservation laws for flow problems with moving boundaries and deformable meshes, and their impact on aeroelastic computations". In: *Comput. Methods in Appl. Mech. Eng.* 134.1 - 2 (1996), pp. 71 – 90

⁹T.E. Tezduyar, M. Behr, and J. Liou. "A new strategy for finite element computations involving moving boundaries and interfaces – The deforming-spatial-domain/space-time procedure: I. The concept and the preliminary numerical tests". In: *Comput. Methods in Appl. Mech. Eng.* 94.3 (1992), pp. 339 – 351

¹⁰J.W. van der Vegt and H. van der Ven. "Space-Time Discontinuous Galerkin Finite Element Method with Dynamic Grid Motion for Inviscid Compressible Flows: I. General Formulation". In: *J. Comp. Phys.* 182.2 (2002), pp. 546 – 585

¹¹A. Üngör. "Tent-Pitcher: A meshing algorithm for space-time discontinuous Galerkin methods". In: 9th Internat. Meshing Roundtable. 2000, pp. 111–122

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Overview of DPG

DPG is a Minimum Residual Method

Find $u \in U$ such that

$$b(u, v) = l(v) \quad \forall v \in V$$

with operator $B : U \rightarrow V'$ defined by $b(u, v) = \langle Bu, v \rangle_{V' \times V}$.

This gives the operator equation

$$Bu = l \quad \in V'.$$

We wish to minimize the residual $Bu - l \in V'$:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|Bu - l\|_{V'}^2 .$$

Dual norms are not computationally tractable. Inverse Riesz map moves the residual to a more accessible space:

$$u_h = \arg \min_{w_h \in U_h} \frac{1}{2} \|R_V^{-1}(Bu - l)\|_V^2 .$$

Overview of DPG

DPG is a Minimum Residual Method

Taking the Gâteaux derivative to be zero in all directions $\delta u \in U_h$ gives,

$$(R_V^{-1}(Bu_h - l), R_V^{-1}B\delta u)_V = 0, \quad \forall \delta u \in U,$$

which by definition of the Riesz map is equivalent to

$$\langle Bu_h - l, R_V^{-1}B\delta u_h \rangle = 0 \quad \forall \delta u_h \in U_h,$$

with optimal test functions $v_{\delta u_h} := R_V^{-1}B\delta u_h$ for each trial function δu_h .

Resulting Petrov-Galerkin System

This gives a simple bilinear form

$$b(u_h, v_{\delta u_h}) = l(v_{\delta u_h}),$$

with $v_{\delta u_h} \in V$ that solves the auxiliary problem

$$(v_{\delta u_h}, \delta v)_V = \langle R_V v_{\delta u_h}, \delta v \rangle = \langle B\delta u_h, \delta v \rangle = b(\delta u_h, \delta v) \quad \forall \delta v \in V.$$

Overview of DPG

DPG is the Most Stable Petrov-Galerkin Method

Babuška's theorem guarantees that *discrete stability and approximability imply convergence*. If bilinear form $b(u, v)$, with $M := \|b\|$ satisfies the discrete inf-sup condition with constant γ_h ,

$$\sup_{v_h \in V_h} \frac{|b(u, v)|}{\|v_h\|_V} \geq \gamma_h \|u_h\|_U ,$$

then the Galerkin error satisfies the bound

$$\|u_h - u\|_U \leq \frac{M}{\gamma_h} \inf_{w_h \in U_h} \|w_h - u\|_U .$$

Optimal test function realize the supremum guaranteeing that $\gamma_h \geq \gamma$.

Energy Norm

If we use the energy norm, $\|u\|_E := \|Bu\|_{V'}$ in the error estimate, then $M = \gamma = 1$. Babuška's theorem implies that the minimum residual method is the most stable Petrov-Galerkin method (assuming exact optimal test functions).

Overview of DPG¹²

Other Features

Discontinuous Petrov-Galerkin

- Continuous test space produces global solve for optimal test functions
- Discontinuous test space results in an embarrassingly parallel solve

Hermitian Positive Definite Stiffness Matrix

Property of all minimum residual methods

$$b(u_h, v_{\delta u_h}) = (v_{u_h}, v_{\delta u_h})_V = \overline{(v_{\delta u_h}, v_{u_h})_V} = \overline{b(\delta u_h, v_{u_h})}$$

Error Representation Function

Energy norm of Galerkin error (residual) can be computed without exact solution

$$\|u_h - u\|_E = \|B(u_h - u)\|_{V'} = \|Bu_h - l\|_{V'} = \|R_V^{-1}(Bu_h - l)\|_V$$

¹²DPGOverview2.

Overview of DPG

High Performance Computing

Eliminates human intervention

- Stability
- Robustness
- Adaptivity
- Automaticity
- Compute intensive
- Embarrassingly parallel local solves
- Factorization recyclable
- Low communication
- SPD stiffness matrix
- Multiphysics



Stampede Supercomputer at TACC



Mira Supercomputer at Argonne

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Locally Conservative DPG

DPG for Convection-Diffusion

Start with the strong-form PDE.

$$\nabla \cdot (\beta u) - \epsilon \Delta u = g$$

Rewrite as a system of first-order equations.

$$\frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u = \mathbf{0}$$

$$\nabla \cdot (\beta u - \boldsymbol{\sigma}) = g$$

Multiply by test functions and integrate by parts over each element, K .

$$\begin{aligned} \frac{1}{\epsilon} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle u, \tau_n \rangle_{\partial K} &= 0 \\ -(\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle (\beta u - \boldsymbol{\sigma}) \cdot \mathbf{n}, v \rangle_{\partial K} &= (g, v)_K \end{aligned}$$

Use the ultraweak (DPG) formulation to obtain bilinear form $b(u, v) = l(v)$.

$$\begin{aligned} \frac{1}{\epsilon} (\boldsymbol{\sigma}, \boldsymbol{\tau})_K + (u, \nabla \cdot \boldsymbol{\tau})_K - \langle \hat{u}, \tau_n \rangle_{\partial K} \\ - (\beta u - \boldsymbol{\sigma}, \nabla v)_K + \langle \hat{t}, v \rangle_{\partial K} &= (g, v)_K \end{aligned}$$

Locally Conservative DPG

Local Conservation for Convection-Diffusion

The local conservation law in convection diffusion is

$$\int_{\partial K} \hat{t} = \int_K g,$$

which is equivalent to having $\mathbf{v}_K := \{v, \boldsymbol{\tau}\} = \{1_K, \mathbf{0}\}$ in the test space. In general, this is not satisfied by the optimal test functions. Following Moro et al¹³ (also Chang and Nelson¹⁴), we can enforce this condition with Lagrange multipliers:

$$L(u_h, \boldsymbol{\lambda}) = \frac{1}{2} \|R_V^{-1}(Bu_h - l)\|_V^2 - \sum_K \lambda_K \underbrace{\langle Bu_h - l, \mathbf{v}_K \rangle}_{\langle \hat{t}, 1_K \rangle_{\partial K} - \langle g, 1_K \rangle_K},$$

where $\boldsymbol{\lambda} = \{\lambda_1, \dots, \lambda_N\}$.

¹³ D. Moro, N.C. Nguyen, and J. Peraire. "A Hybridized Discontinuous Petrov-Galerkin Scheme for Scalar Conservation Laws". In: *Int. J. Num. Meth. Eng.* (2011).

¹⁴ C.L. Chang and J.J. Nelson. "Least-Squares Finite Element Method for the Stokes Problem with Zero Residual of Mass Conservation". In: *SIAM J. Num. Anal.* 34 (1997), pp. 480–489.

Locally Conservative DPG

Locally Conservative Saddle Point System

Finding the critical points of $L(u, \lambda)$, we get the following equations.

Locally Conservative Saddle Point System

$$\frac{\partial L(u_h, \lambda)}{\partial u_h} = b(u_h, R_V^{-1}B\delta u_h) - l(R_V^{-1}B\delta u_h) - \sum_K \lambda_K b(\delta u_h, \mathbf{v}_K) = 0 \quad \forall \delta u_h \in U_h$$

$$\frac{\partial L(u_h, \lambda)}{\partial \lambda_K} = -b(u_h, \mathbf{v}_K) + l(\mathbf{v}_K) = 0 \quad \forall K$$

A few consequences:

- We've turned our minimization problem into a saddle point problem.
- Only need to find the optimal test function in the orthogonal complement of constants.

Locally Conservative DPG

Optimal Test Functions

For each $\mathbf{u} = \{u, \boldsymbol{\sigma}, \hat{u}, \hat{t}\} \in \mathbf{U}_h$, find $\mathbf{v}_\mathbf{u} = \{v_\mathbf{u}, \boldsymbol{\tau}_\mathbf{u}\} \in \mathbf{V}$ such that

$$(\mathbf{v}_\mathbf{u}, \mathbf{w})_\mathbf{V} = b(\mathbf{u}, \mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{V}$$

where \mathbf{V} becomes $\mathbf{V}_{p+\Delta p}$ in order to make this computationally tractable.

We recently developed this modification to the *robust test norm*¹⁵ which behaves better in the presence of singularities.

Convection-Diffusion Test Norm

$$\begin{aligned} \|(\mathbf{v}, \boldsymbol{\tau})\|_{\mathbf{V}, \Omega_h}^2 &= \left\| \min \left\{ \frac{1}{\sqrt{\epsilon}}, \frac{1}{\sqrt{|K|}} \right\} \boldsymbol{\tau} \right\|^2 + \|\nabla \cdot \boldsymbol{\tau} - \boldsymbol{\beta} \cdot \nabla \mathbf{v}\|^2 \\ &\quad + \|\boldsymbol{\beta} \cdot \nabla \mathbf{v}\|^2 + \epsilon \|\nabla \mathbf{v}\|^2 + \underbrace{\|\mathbf{v}\|^2}_{\text{No longer necessary}} \end{aligned}$$

¹⁵ J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771–795. issn: 0898-1221. doi: <http://dx.doi.org/10.1016/j.camwa.2013.06.010>.

Locally Conservative DPG

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¹⁵ J. Chan et al. "A robust DPG method for convection-dominated diffusion problems II: Adjoint boundary conditions and mesh-dependent test norms". In: *Comp. Math. Appl.* 67.4 (2014). High-order Finite Element Approximation for Partial Differential Equations, pp. 771–795. ISSN: 0898-1221. doi: <http://dx.doi.org/10.1016/j.camwa.2013.06.010>.

Locally Conservative DPG

Stability and Robustness Analysis

- We follow Brezzi's theory for an abstract mixed problem:

$$\begin{cases} \mathbf{u} \in \mathbf{U}, p \in Q \\ a(\mathbf{u}, \mathbf{w}) + c(p, \mathbf{w}) = l(\mathbf{w}) \quad \forall \mathbf{w} \in \mathbf{U} \\ c(q, \mathbf{u}) = g(q) \quad \forall q \in Q \end{cases}$$

where a, c, l, g denote the appropriate bilinear and linear forms.

- $a(\mathbf{u}, \mathbf{w}) = b(\mathbf{u}, R_V^{-1}B\mathbf{w}) = (R_V^{-1}B\mathbf{u}, R_V^{-1}B\mathbf{w})_V$
- $c(p, \mathbf{w}) = \sum_K \lambda_K \langle \hat{t}, 1_K \rangle_{\partial K}$
- Locally conservative DPG satisfies inf-sup and inf-sup in kernel conditions.
- Robustness is proved by switching to energy norm in Brezzi analysis.
- Full details can be found in *Locally Conservative Discontinuous Petrov-Galerkin Finite Elements for Fluid Problems*¹⁶.

¹⁶ T.E. Ellis, L.F. Demkowicz, and J.L. Chan. "Locally Conservative Discontinuous Petrov-Galerkin Finite Elements For Fluid Problems". In: *Comp. Math. Appl.* (2014).

Numerical Experiments

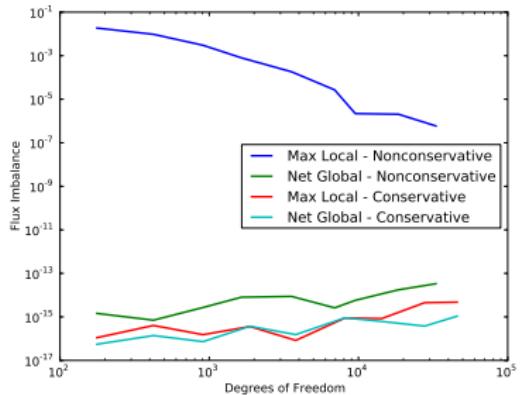
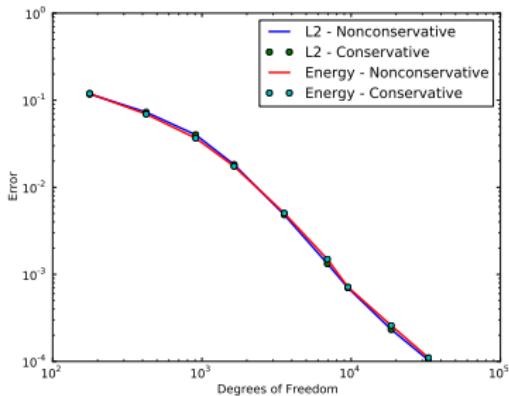
Eriksson-Johnson Problem

On domain $\Omega = [0, 1]^2$, with $\beta = (1, 0)^T$, $f = 0$ and boundary conditions

$$\hat{t} = u_0, \quad \beta_n \leq 0, \quad \hat{u} = 0, \quad \beta_n > 0$$

Separation of variables gives an analytic solution

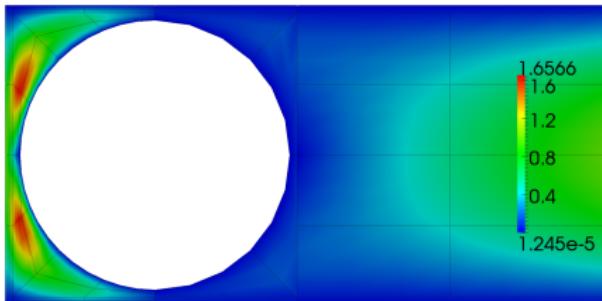
$$u(x, y) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\exp(r_2(x - 1)) - \exp(r_1(x - 1))}{r_1 \exp(-r_2) - r_2 \exp(-r_1)} \cos(n\pi y)$$



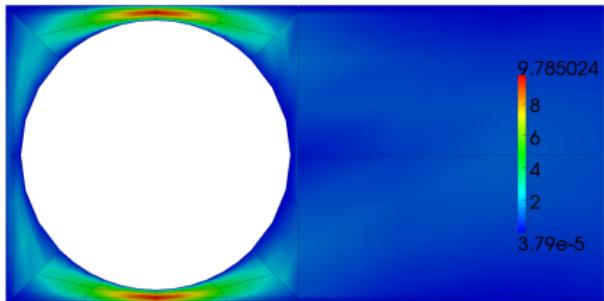
Numerical Experiments

Stokes Flow Around a Cylinder

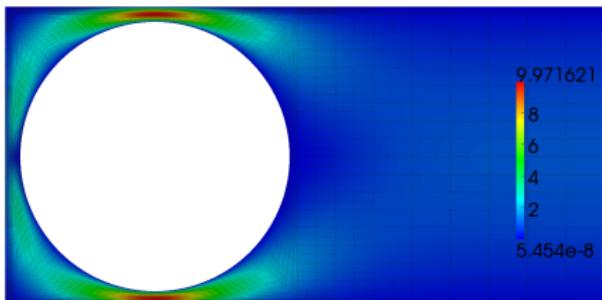
Horizontal Velocity



1 Refinement

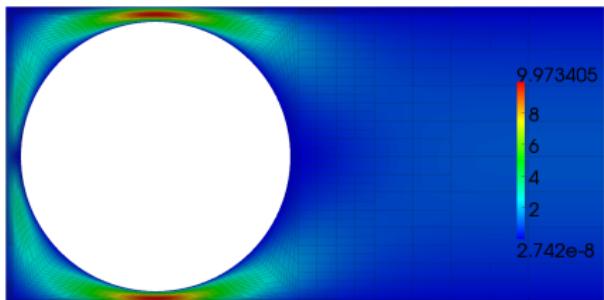


1 Refinement



6 Refinements

Nonconservative



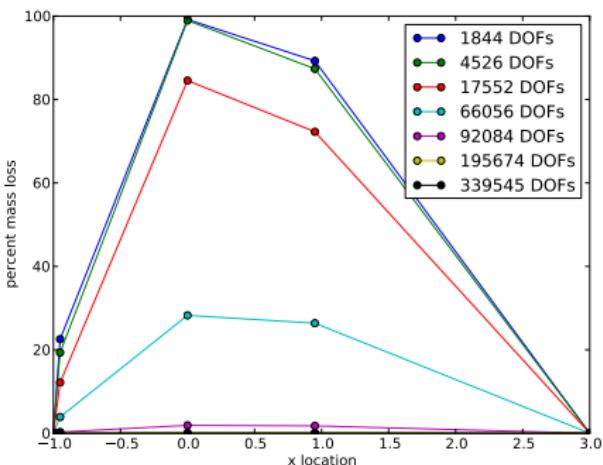
6 Refinements

Conservative

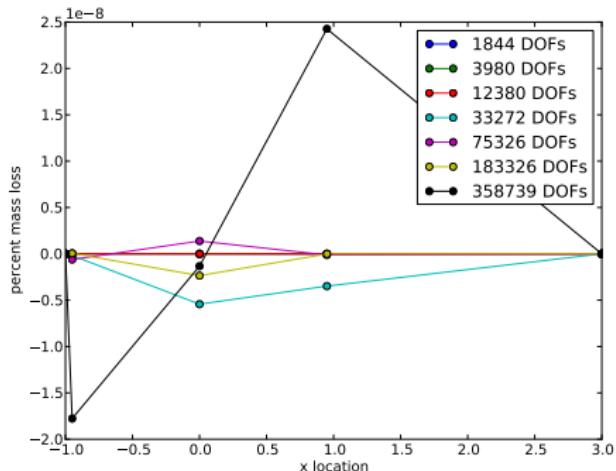
Numerical Experiments

Stokes Flow Around a Cylinder

Percent Mass Loss at $x = [-1, -0.95, 0, 0.95, 3]$



Nonconservative

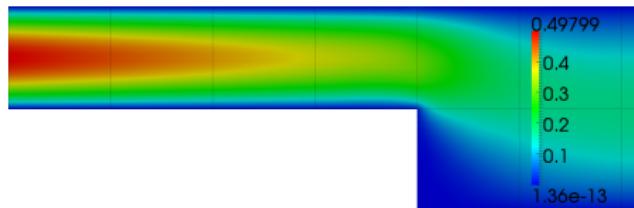


Conservative

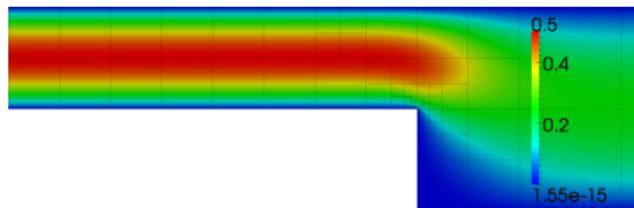
Numerical Experiments

Stokes Flow Over a Backward Facing Step

Horizontal Velocity

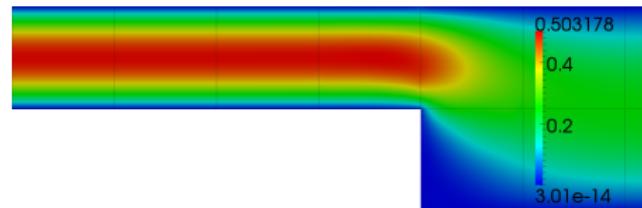


Initial Mesh

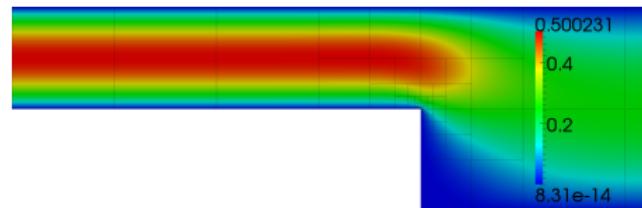


8 Refinements

Nonconservative



Initial Mesh



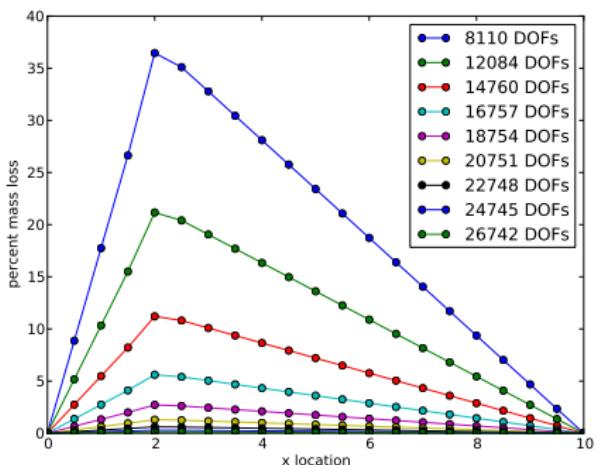
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Conservative

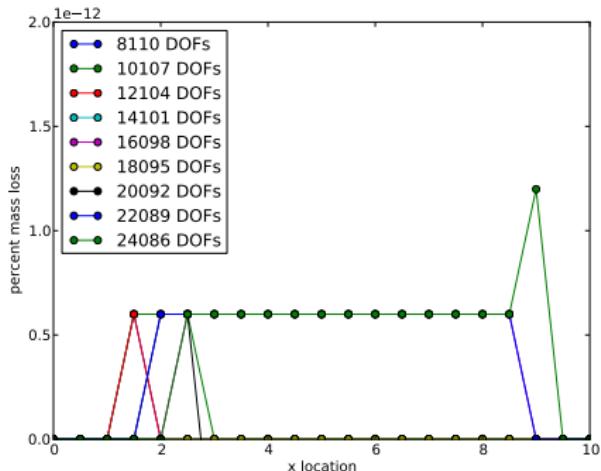
Numerical Experiments

Stokes Flow Over a Backward Facing Step

Percent Mass Loss at $x = [0, 0.5, \dots, 9.5, 10]$



Nonconservative



Conservative

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Space-Time DPG

Motivation

Extends the capabilities of a DPG solver

- Preserves stability and robustness of DPG method
- Unified treatment of space and time
- Local space-time adaptivity (local time stepping)
 - Small solution features require small time step
 - Global time step not limited to smallest element
- Natural framework for moving meshes

More computationally difficult

- Solve requires $d + 1$ dimensions
- Mesh structure more difficult
- Need to differentiate between spatial and temporal boundaries
- Larger global solves than finite difference time stepping

Heat Equation

Simplest Nontrivial Space-Time Problem

Equation is parabolic in space-time.

$$\frac{\partial u}{\partial t} - \epsilon \Delta u = f$$

This is really just a composite of Fourier's law and conservation of energy.

$$\boldsymbol{\sigma} - \epsilon \nabla u = 0$$

$$\frac{\partial u}{\partial t} - \nabla \cdot \boldsymbol{\sigma} = f$$

We can rewrite this in terms of a space-time divergence.

$$\begin{aligned} \frac{1}{\epsilon} \boldsymbol{\sigma} - \nabla u &= 0 \\ \nabla_{xt} \cdot \begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix} &= f \end{aligned}$$

Heat Equation

DPG Formulation

Multiply by test function and integrate by parts over space-time element K.

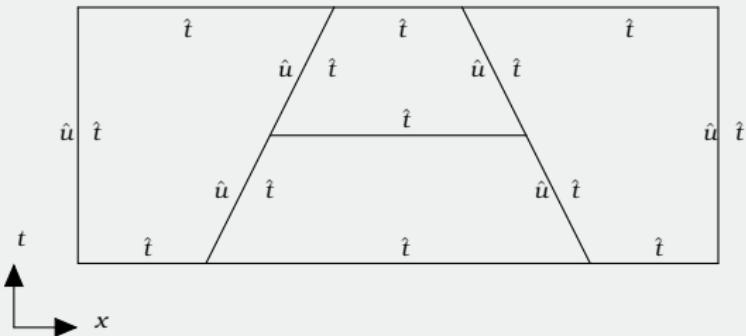
$$\begin{aligned} \left(\frac{1}{\epsilon} \boldsymbol{\sigma}, \boldsymbol{\tau} \right) + (u, \nabla \cdot \boldsymbol{\tau}) - \langle \hat{u}, \boldsymbol{\tau} \cdot \mathbf{n}_x \rangle &= 0 \\ - \left(\begin{pmatrix} -\boldsymbol{\sigma} \\ u \end{pmatrix}, \nabla_{xt} v \right) + \langle \hat{t}, v \rangle &= f \end{aligned}$$

where

$$\hat{u} := \text{tr}(u)$$

$$\hat{t} := \text{tr}(-\boldsymbol{\sigma}) \cdot \mathbf{n}_x + \text{tr}(u) \cdot n_t$$

Support of Trace Variables

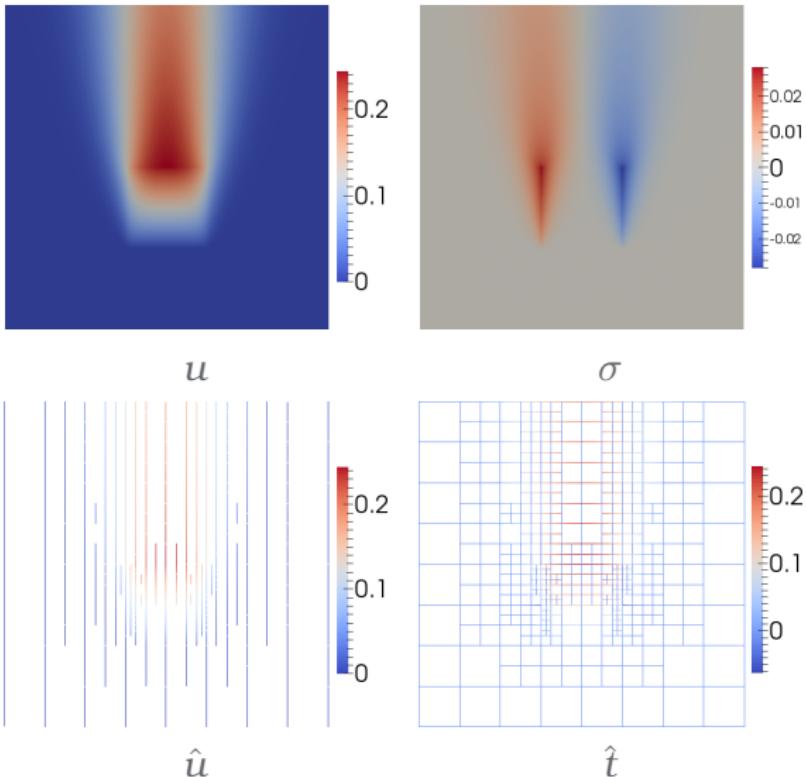


- Trace \hat{u} defined on spatial boundaries
- Flux \hat{t} defined on all boundaries

Heat equation

Pulsed Source Problem

- Initial condition
 $u = 0$.
- Apply unit source
 $x \in [3/8, 5/8]$,
 $t \in [1/4, 1/2]$.
- Should not violate causality
- Space-time adaptivity picks up areas of rapid change.



Compressible Navier-Stokes

Strong Form

The compressible Navier-Stokes equations are

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ \rho \mathbf{u} \\ \rho e_0 \end{bmatrix} + \nabla \cdot \begin{bmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \otimes \mathbf{u} + p \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} e_0 + \mathbf{u} p + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \end{bmatrix} = \begin{bmatrix} f_c \\ \mathbf{f}_m \\ f_e \end{bmatrix},$$

where

$$\mathbb{D} = 2\mu \mathbf{S}^* = 2\mu \left[\frac{1}{2} (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) - \frac{1}{3} \nabla \cdot \mathbf{u} \mathbf{I} \right],$$

$$\mathbf{q} = -C_p \frac{\mu}{Pr} \nabla T,$$

and (assuming an ideal gas EOS)

$$p = \rho R T.$$

Compressible Navier-Stokes

First Order Space-Time Form

Writing this in space-time in terms of ρ , \mathbf{u} , T , \mathbb{D} , and \mathbf{q} :

$$\mathbb{D} - \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) + \frac{2\mu}{3} \nabla \cdot \mathbf{u} \mathbf{I} = 0$$

$$\mathbf{q} + C_p \frac{\mu}{Pr} \nabla T = 0$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix} = f_c$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix} = \mathbf{f}_m$$

$$\nabla_{xt} \cdot \begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \rho R T + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix} = f_e.$$

Compressible Navier-Stokes

DPG Formulation

Multiplying by test functions and integrating by parts:

$$\begin{aligned}
 (\mathbb{D}, \mathbb{S}) + (2\mu \mathbf{u}, \nabla \cdot \mathbb{S}) - \left(\frac{2\mu}{3} \mathbf{u}, \nabla \operatorname{tr} \mathbb{S} \right) - \langle 2\mu \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \rangle + \left\langle \frac{2\mu}{3} \hat{\mathbf{u}}, \mathbb{S} \mathbf{n}_x \right\rangle &= 0 \\
 (\mathbf{q}, \boldsymbol{\tau}) - \left(C_p \frac{\mu}{Pr} T, \nabla \cdot \boldsymbol{\tau} \right) + \left\langle C_p \frac{\mu}{Pr} \hat{T}, \boldsymbol{\tau}_n \right\rangle &= 0 \\
 - \left(\begin{pmatrix} \rho \mathbf{u} \\ \rho \end{pmatrix}, \nabla_{xt} \mathbf{v}_c \right) + \langle \hat{\mathbf{t}}_c, \mathbf{v}_c \rangle &= (f_c, \mathbf{v}_c) \\
 - \left(\begin{pmatrix} \rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbf{I} - \mathbb{D} \\ \rho \mathbf{u} \end{pmatrix}, \nabla_{xt} \mathbf{v}_m \right) + \langle \hat{\mathbf{t}}_m, \mathbf{v}_m \rangle &= (\mathbf{f}_m, \mathbf{v}_m) \\
 - \left(\begin{pmatrix} \rho \mathbf{u} (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) + \mathbf{u} \rho R T + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \\ \rho (C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u}) \end{pmatrix}, \nabla_{xt} \mathbf{v}_e \right) + \langle \hat{\mathbf{t}}_e, \mathbf{v}_e \rangle &= (f_e, \mathbf{v}_e),
 \end{aligned}$$

where $\hat{\mathbf{u}}$ and \hat{T} are spatial traces and $\hat{\mathbf{t}}_c$, $\hat{\mathbf{t}}_m$, and $\hat{\mathbf{t}}_e$ are fluxes.

Compressible Navier-Stokes

Flux and Trace Variables

Spatial traces and fluxes are defined as follows:

$$\hat{\mathbf{u}} = \text{tr}(\mathbf{u})$$

$$\hat{T} = \text{tr}(T)$$

$$\hat{t}_c = \text{tr}(\rho \mathbf{u}) \cdot \mathbf{n}_x + \text{tr}(\rho) n_t$$

$$\hat{\mathbf{t}}_m = \text{tr}(\rho \mathbf{u} \otimes \mathbf{u} + \rho R T \mathbf{I} - \mathbb{D}) \cdot \mathbf{n}_x + \text{tr}(\rho \mathbf{u}) n_t$$

$$\begin{aligned} \hat{t}_e = & \text{tr} \left(\rho \mathbf{u} \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) + \mathbf{u} \rho R T + \mathbf{q} - \mathbf{u} \cdot \mathbb{D} \right) \cdot \mathbf{n}_x \\ & + \text{tr} \left(\rho \left(C_v T + \frac{1}{2} \mathbf{u} \cdot \mathbf{u} \right) \right) n_t. \end{aligned}$$

Linearization

Fluxes, traces, and \mathbf{q} are linear in the above bilinear form, but we need to linearize in ρ , \mathbf{u} , T , and \mathbb{D} (Jacobian and residual not shown here).

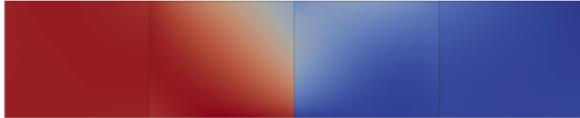
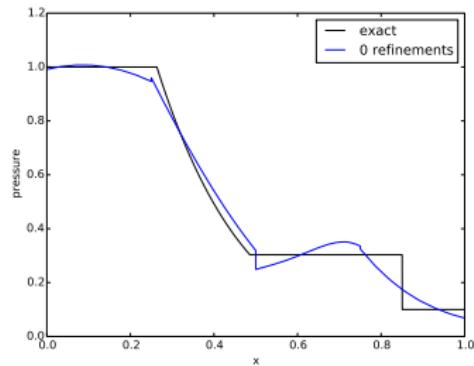
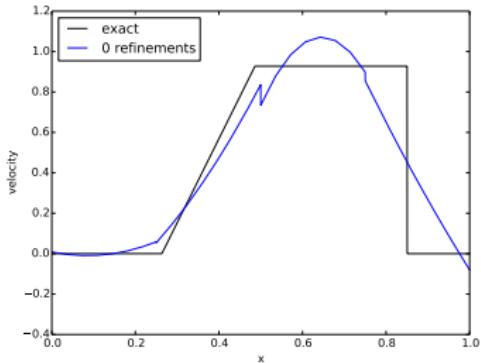
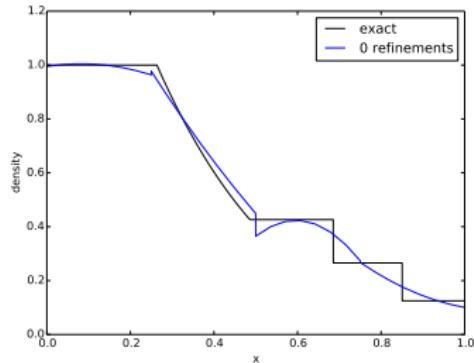
Compressible Navier-Stokes

Test Norm

$$\begin{aligned}
 & \| \nabla \mathbf{v}_m + \nabla v_e \otimes \tilde{\mathbf{u}} \|^2 + \| \nabla v_e \|^2 \\
 & + \left\| -\tilde{\mathbf{u}} \cdot \nabla v_c - \frac{\partial v_c}{\partial t} - \tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla \mathbf{v}_m - R\tilde{T}\nabla \cdot \mathbf{v}_m - \tilde{\mathbf{u}} \cdot \frac{\partial \mathbf{v}_m}{\partial t} \right. \\
 & \quad \left. - C_v \tilde{T} \tilde{\mathbf{u}} \cdot \nabla v_e - \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \tilde{\mathbf{u}} \cdot \nabla v_e - R\tilde{T} \tilde{\mathbf{u}} \nabla v_e - C_v \tilde{T} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| -\tilde{\rho} \nabla v_c - \tilde{\rho} \tilde{\mathbf{u}} \cdot \nabla \mathbf{v}_m - \tilde{\rho} \nabla \mathbf{v}_m \cdot \tilde{\mathbf{u}} - \tilde{\rho} \frac{\partial \mathbf{v}_m}{\partial t} - C_v \tilde{\rho} \tilde{T} \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \cdot \tilde{\mathbf{u}} \nabla v_e \right. \\
 & \quad \left. - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \cdot \nabla v_e \tilde{\mathbf{u}} - \frac{1}{2} \tilde{\rho} \nabla v_e \cdot \tilde{\mathbf{u}} \tilde{\mathbf{u}} - R\tilde{\rho} \tilde{T} \nabla v_e + \tilde{\mathbb{D}} \cdot \nabla v_e - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} - \frac{1}{2} \tilde{\rho} \tilde{\mathbf{u}} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| -R\tilde{\rho} \nabla \cdot \mathbf{v}_m - C_v \tilde{\rho} \tilde{\mathbf{u}} \nabla v_e - R\tilde{\rho} \tilde{\mathbf{u}} \nabla v_e - C_v \tilde{\rho} \frac{\partial v_e}{\partial t} \right\|^2 \\
 & + \left\| \frac{1}{\mu} \mathbb{S} \right\|^2 + \left\| 2\nabla \cdot \mathbb{S} - \frac{2}{3} \nabla \operatorname{tr} \mathbb{S} \right\|^2 + \left\| \frac{Pr}{C_p \mu} \boldsymbol{\tau} \right\|^2 + \left\| \nabla \cdot \boldsymbol{\tau} \right\|^2 \\
 & + \| v_c \|^2 + \| \mathbf{v}_m \|^2 + \| v_e \|^2 .
 \end{aligned}$$

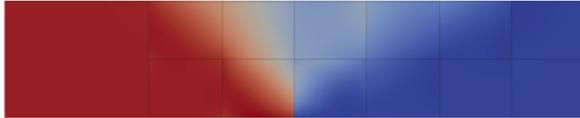
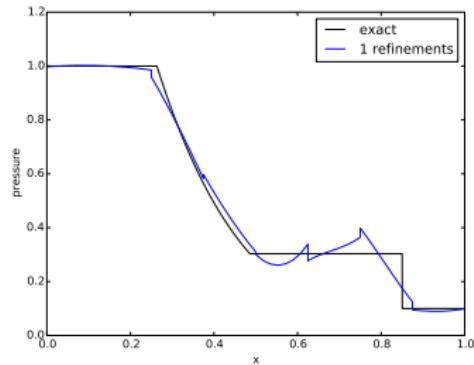
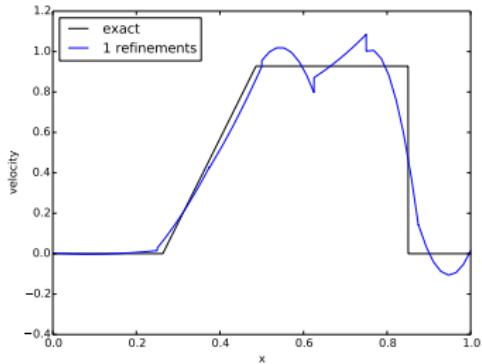
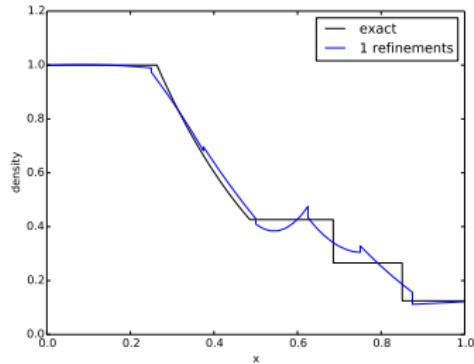
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



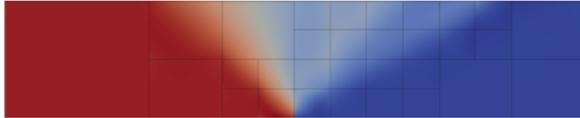
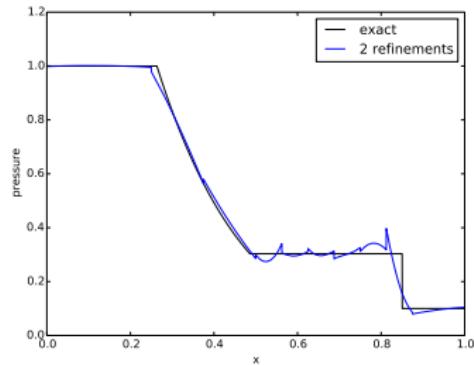
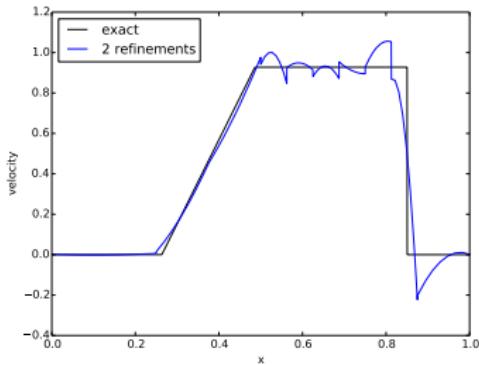
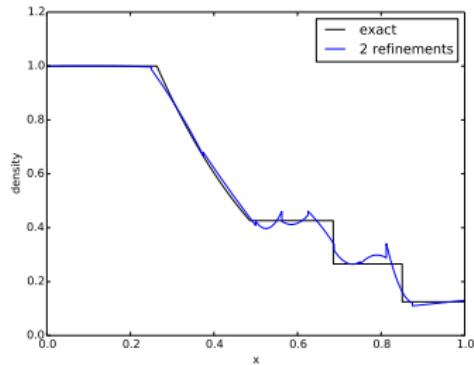
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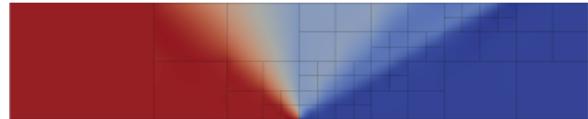
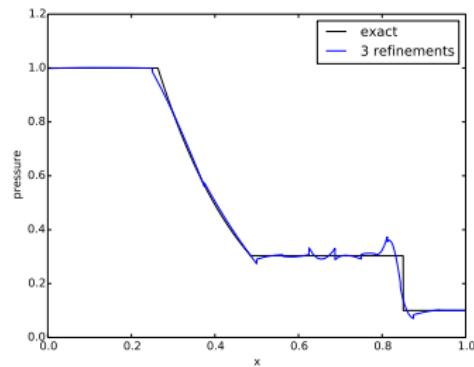
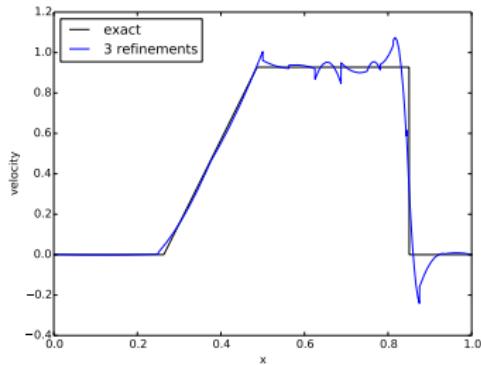
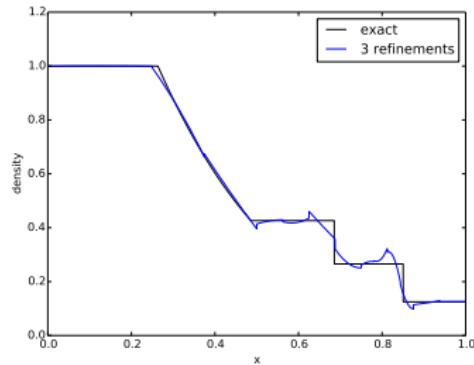
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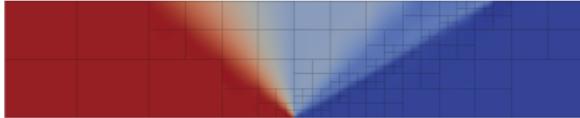
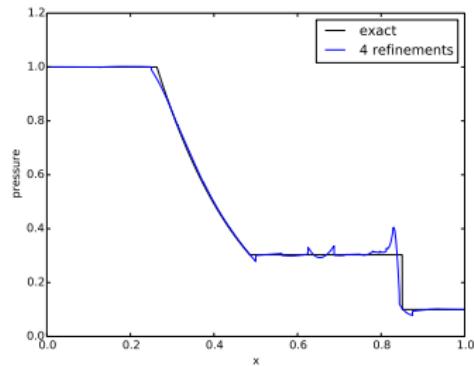
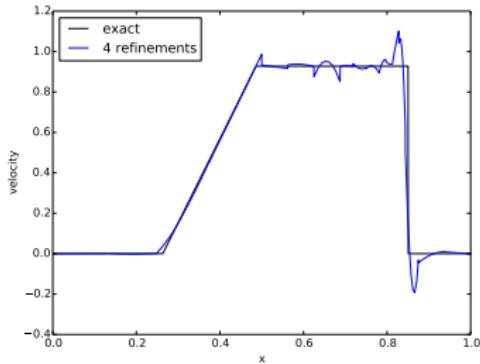
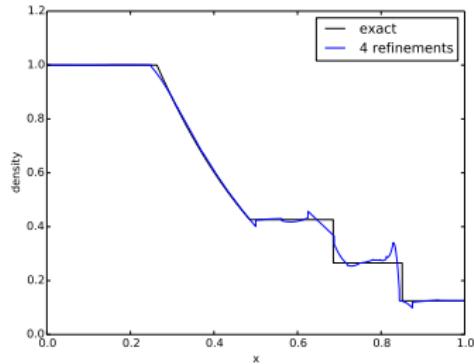
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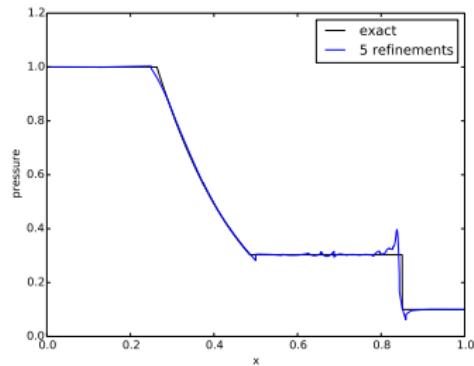
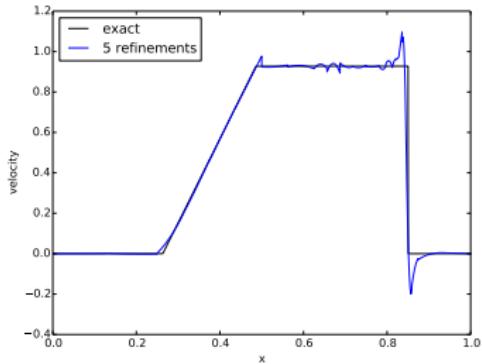
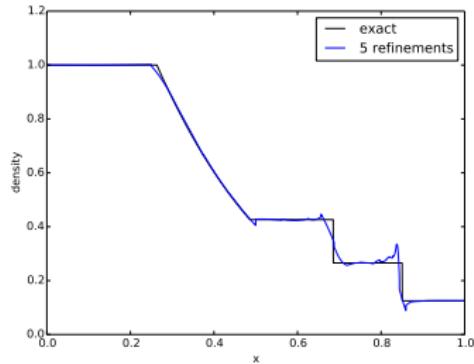
Compressible Navier-Stokes

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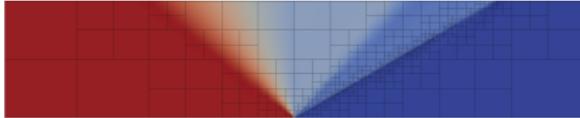
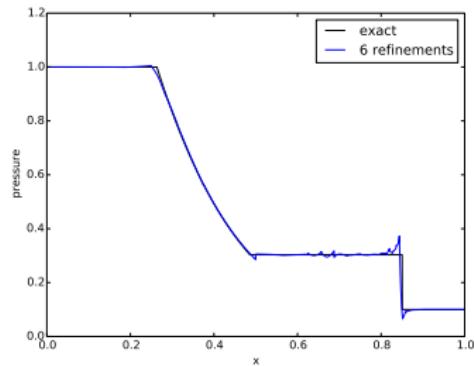
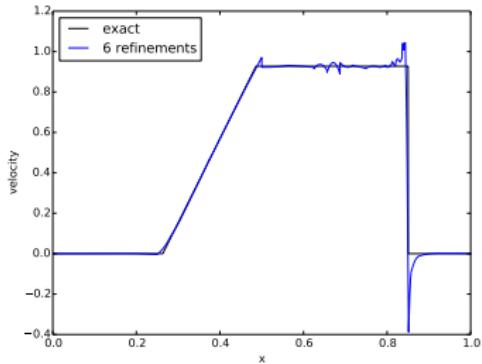
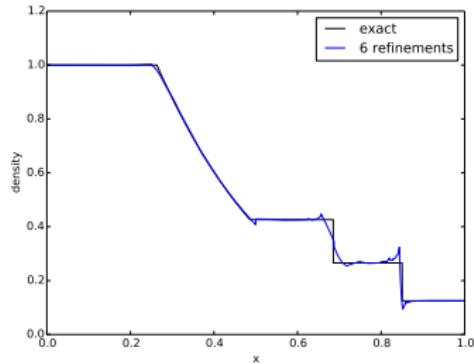
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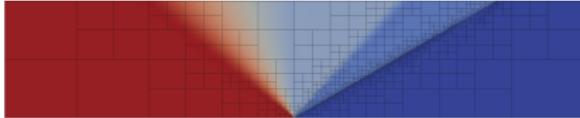
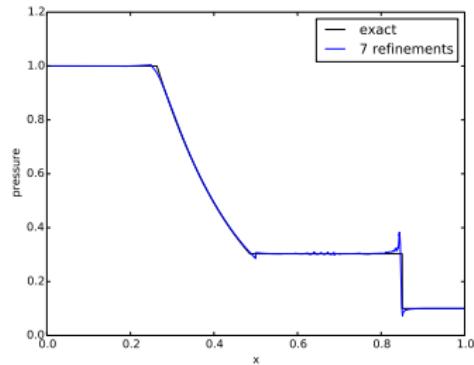
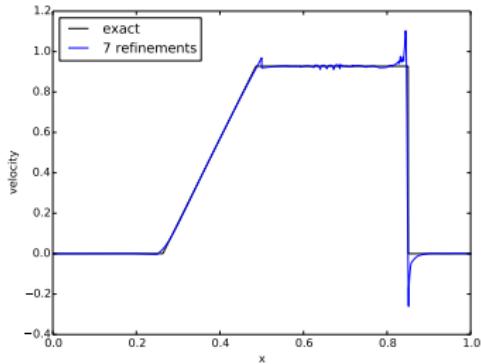
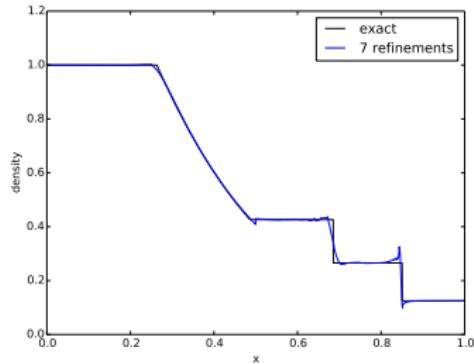
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



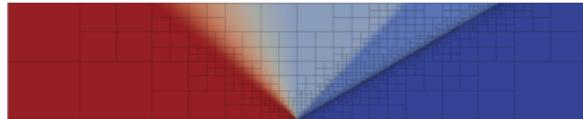
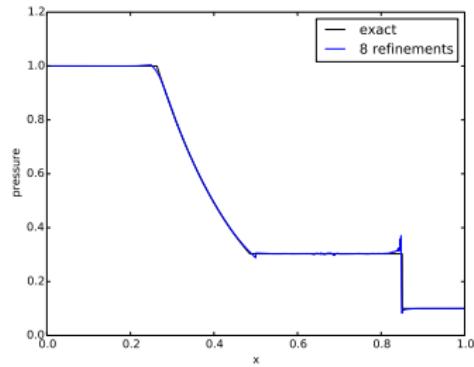
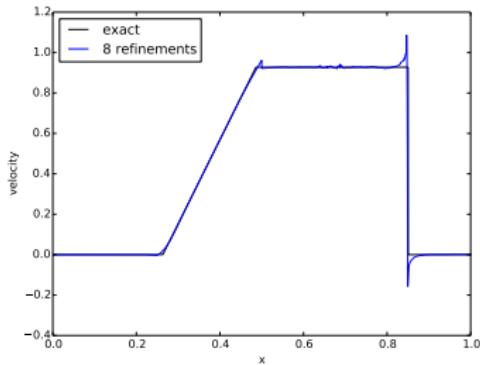
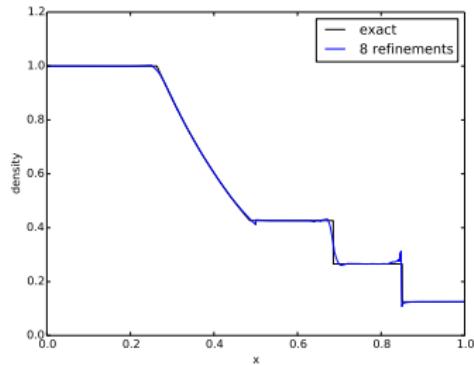
Compressible Navier-Stokes

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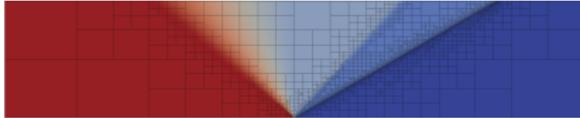
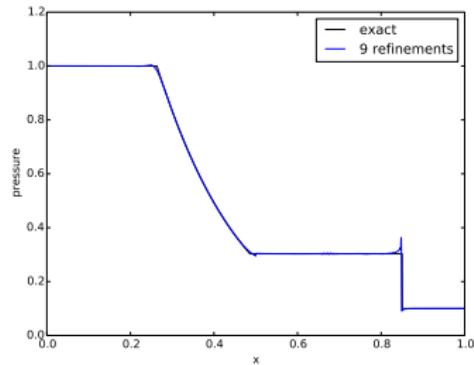
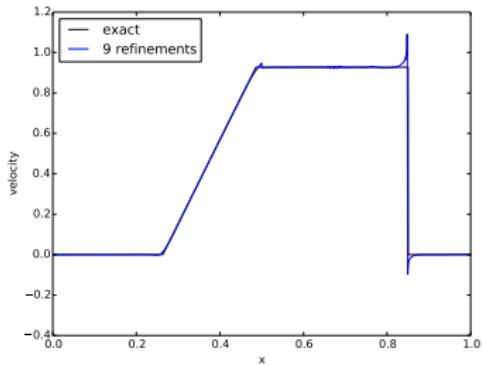
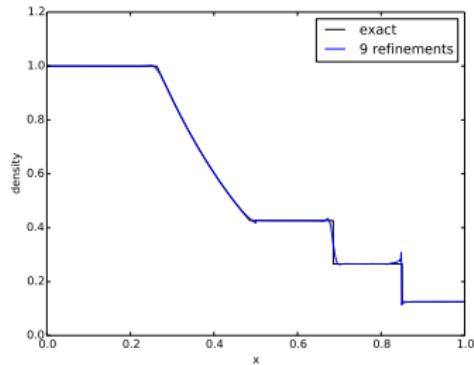
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



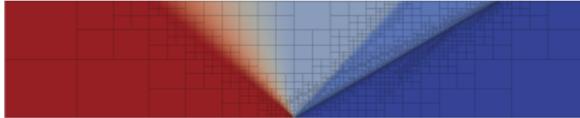
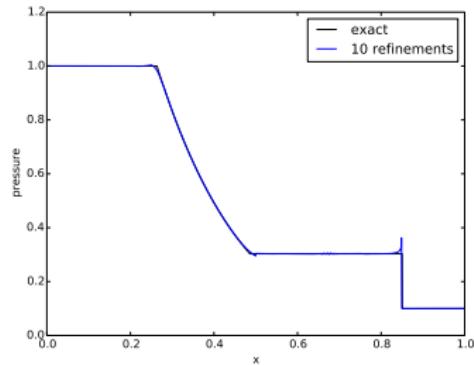
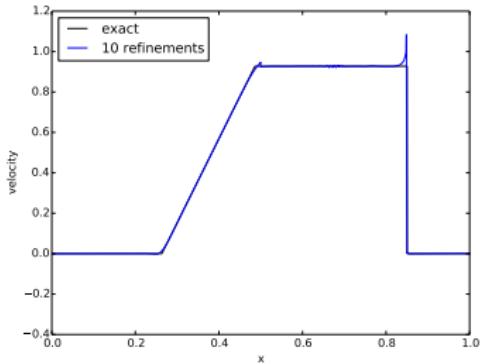
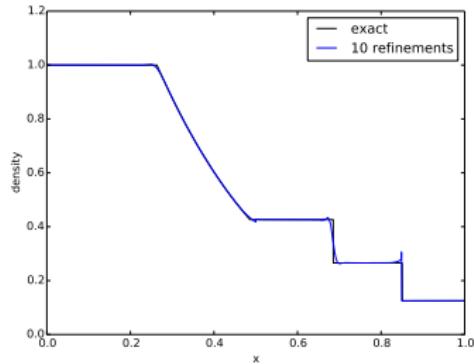
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



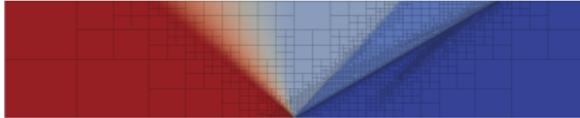
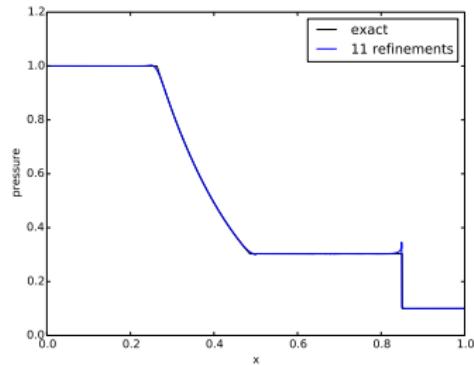
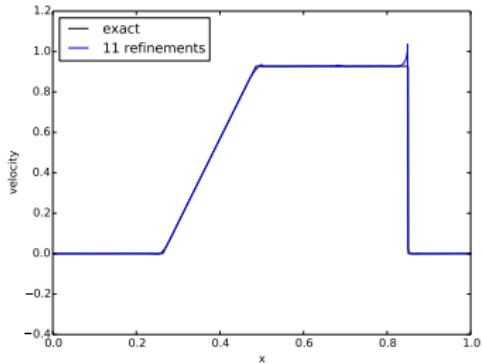
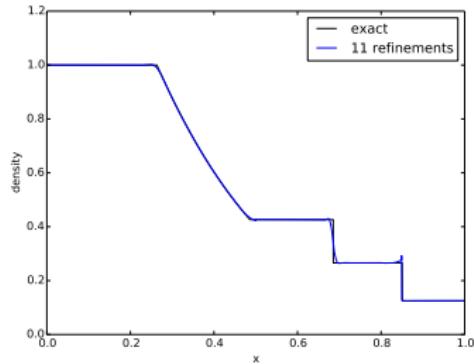
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



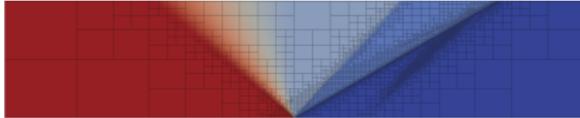
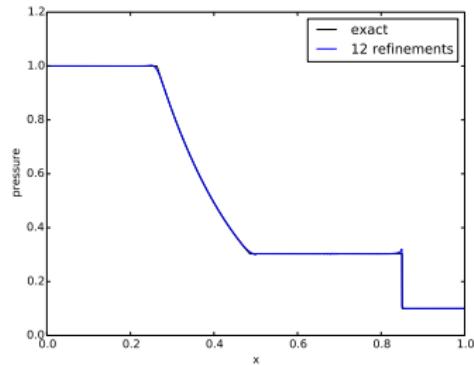
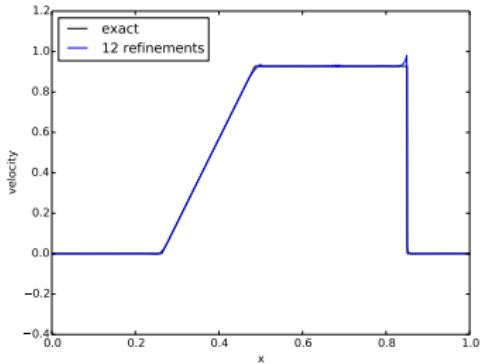
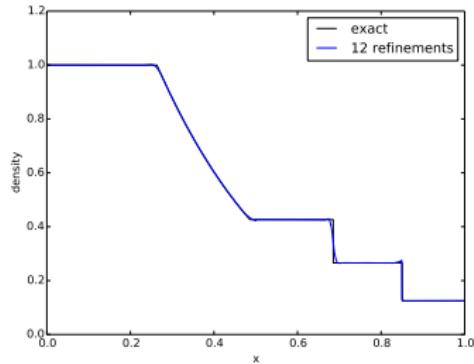
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



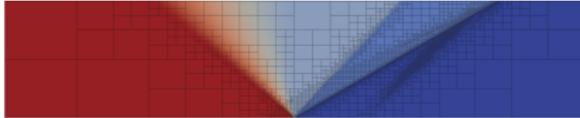
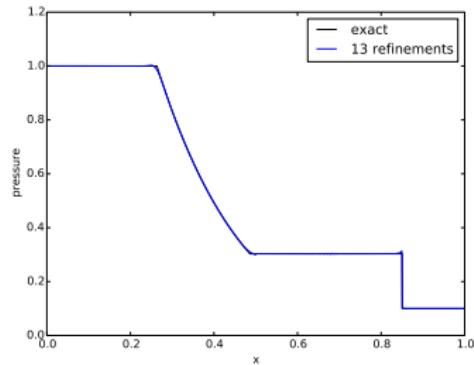
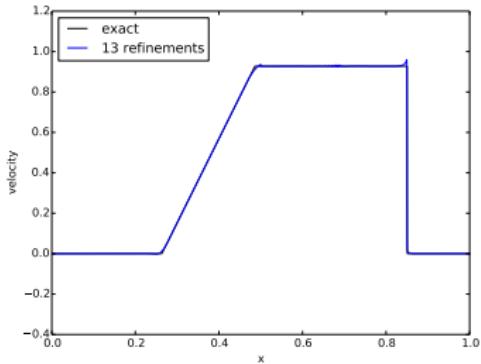
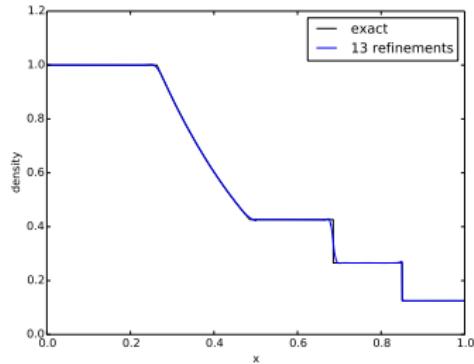
Compressible Navier-Stokes

Sod Shock Tube with $\mu = 10^{-5}$



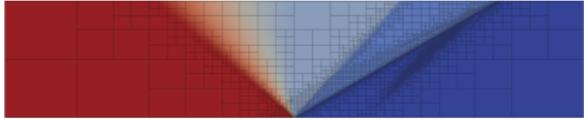
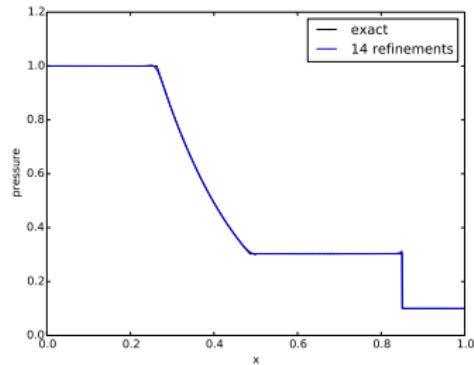
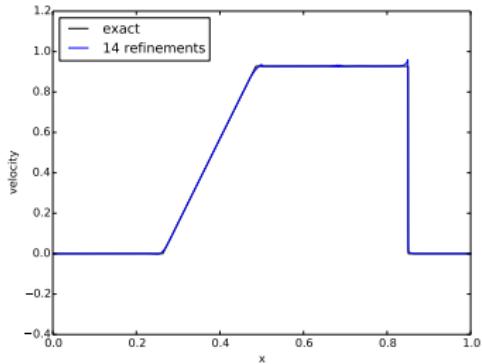
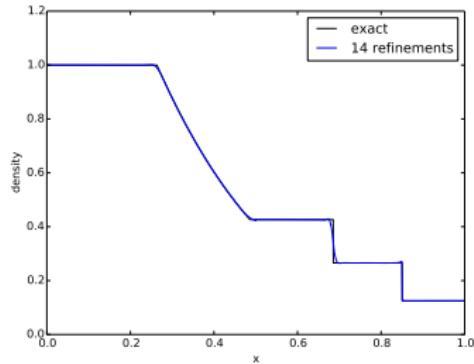
Compressible Navier-Stokes

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Compressible Navier-Stokes

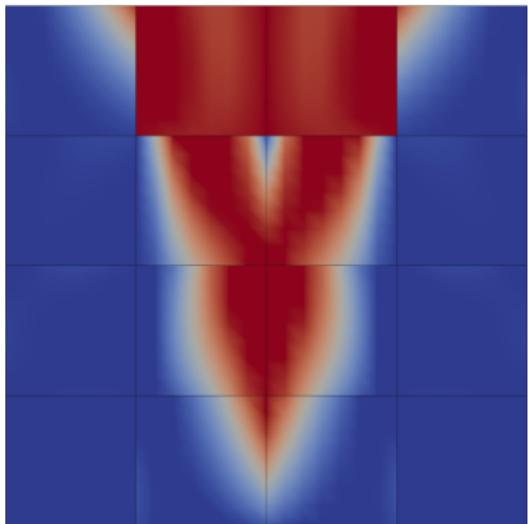
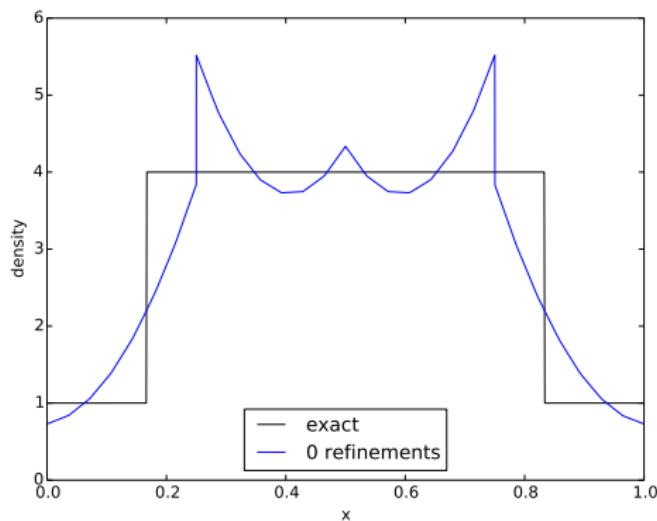
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Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

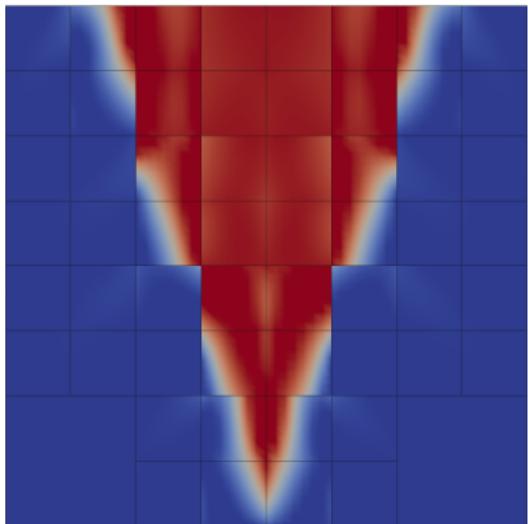
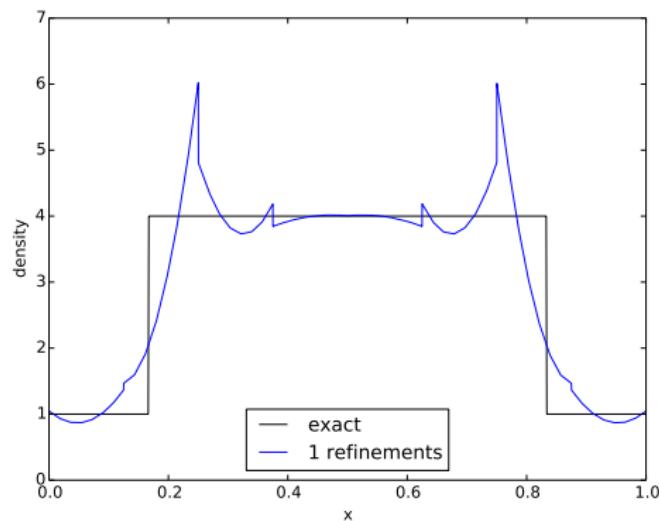
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

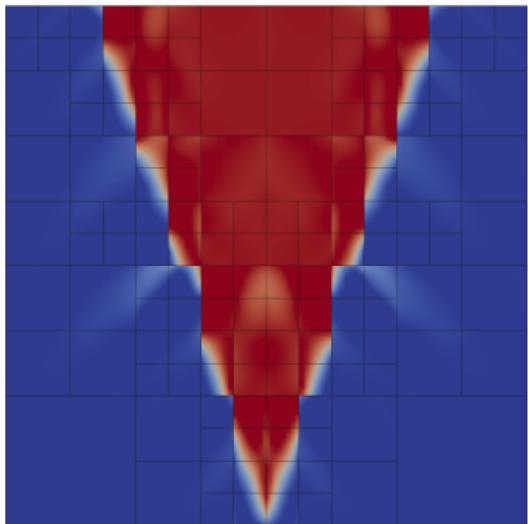
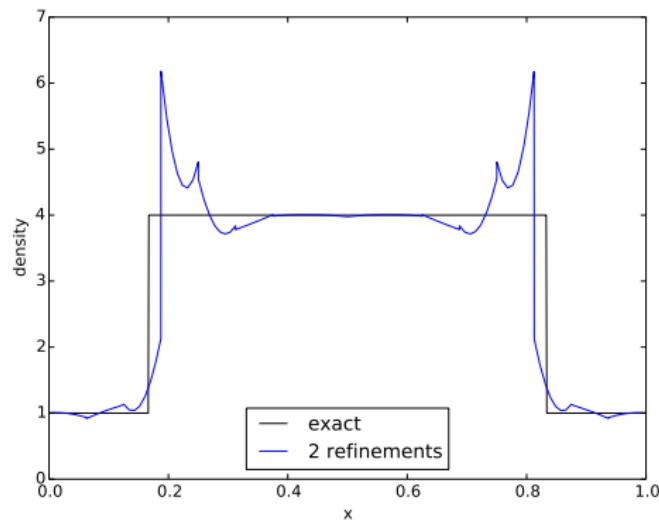
Infinitely strong shock propagation.



Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

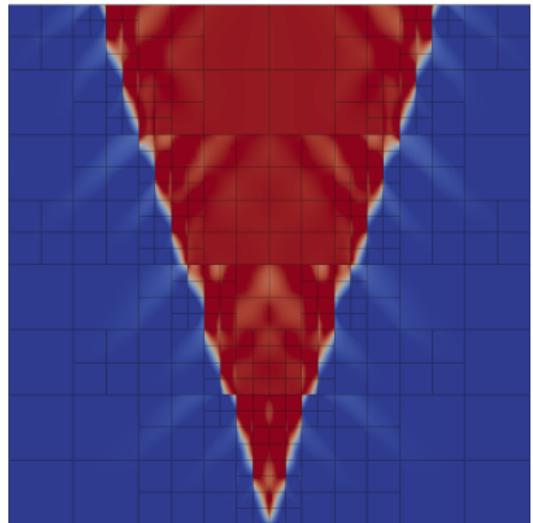
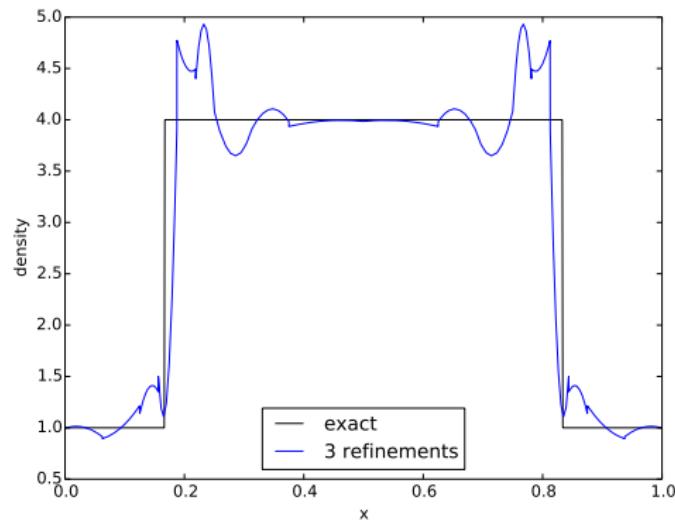
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Compressible Navier-Stokes

Noh Implosion with $\mu = 10^{-3}$

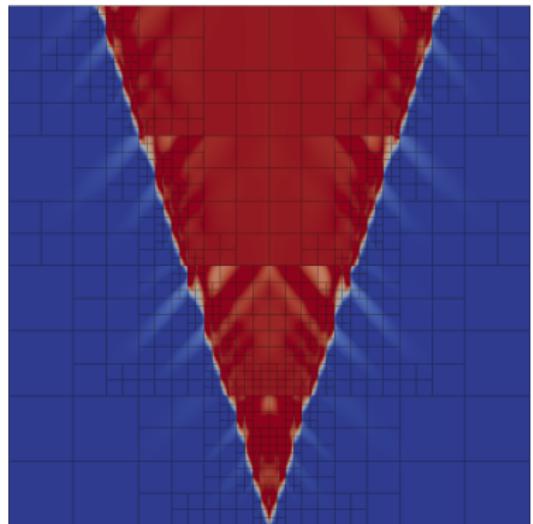
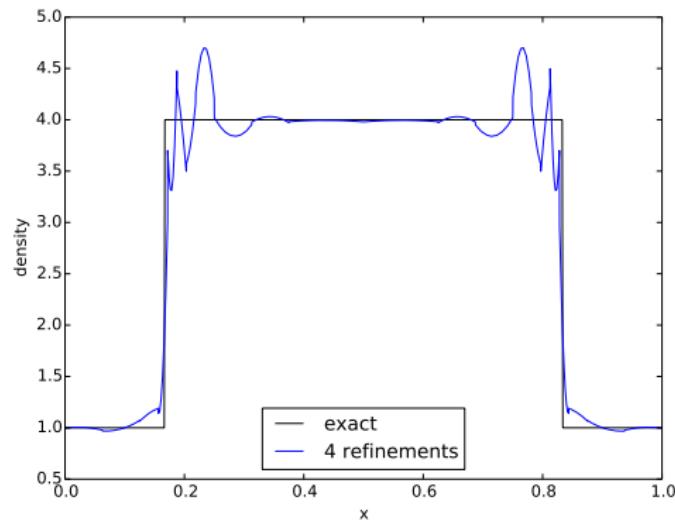
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Compressible Navier-Stokes

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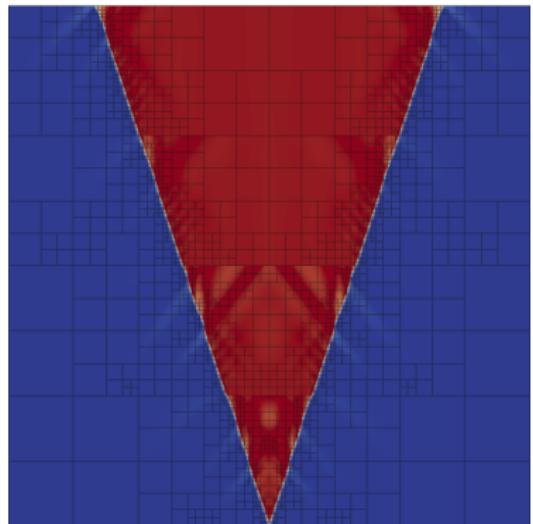
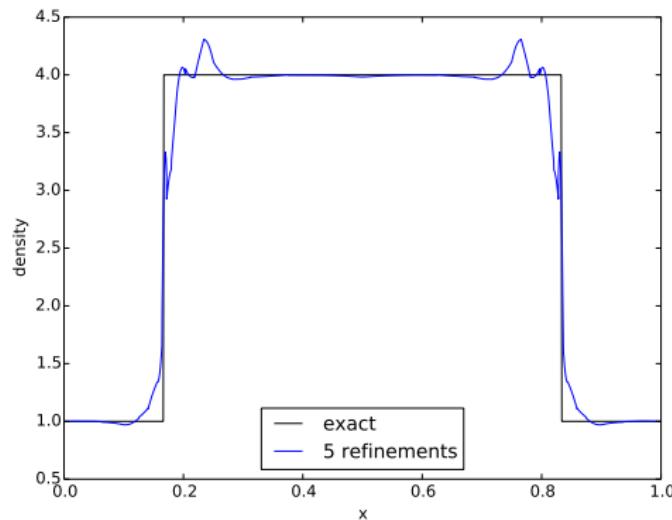
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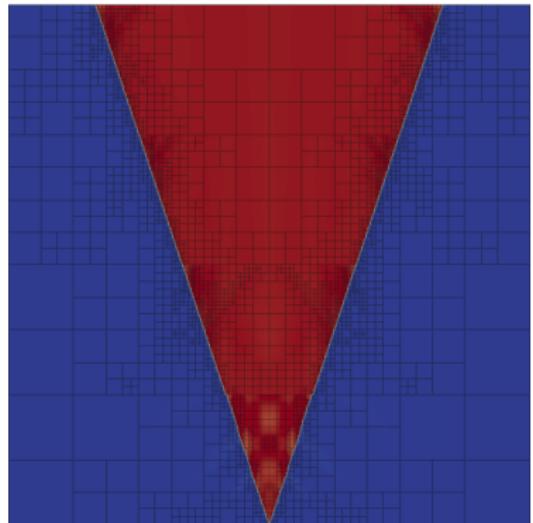
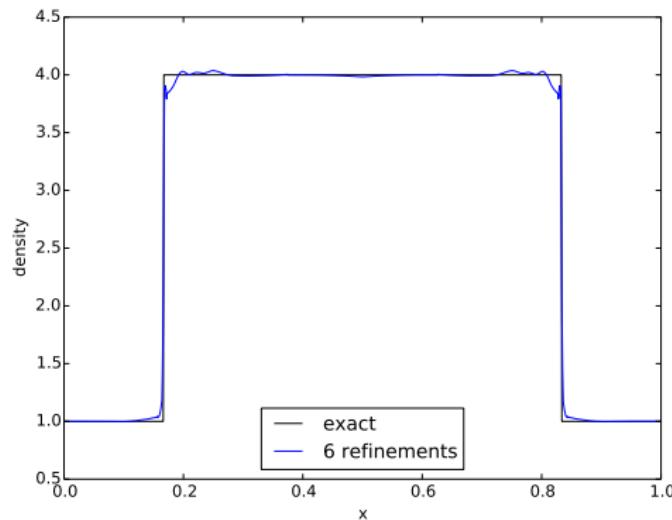
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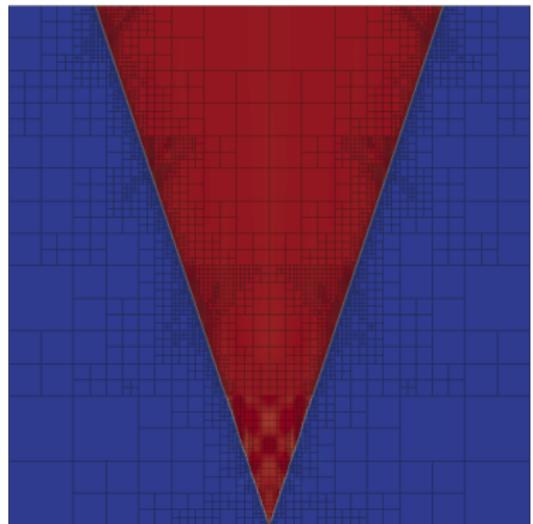
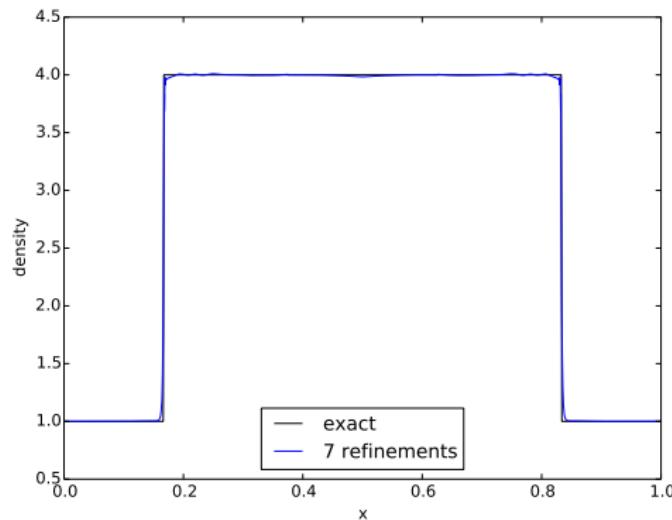
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Compressible Navier-Stokes

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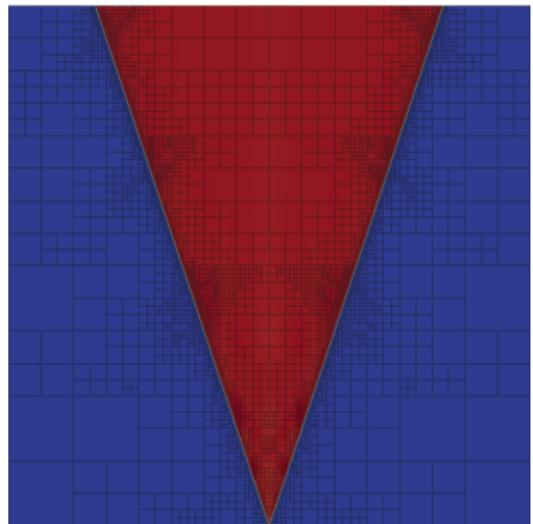
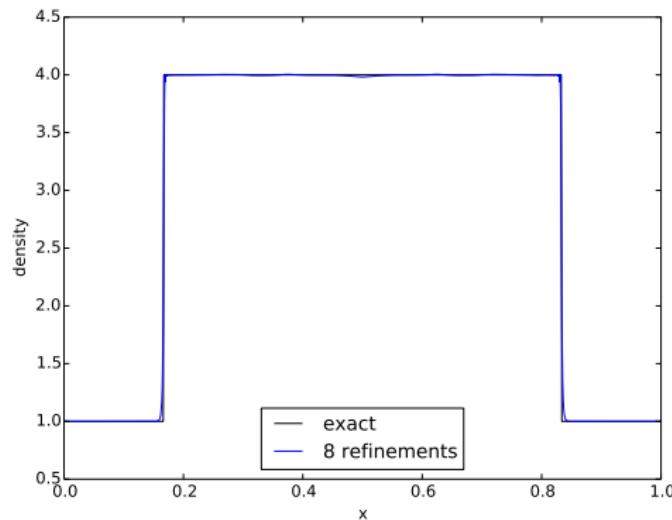


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Future Work

Area A: Applicable Mathematics

- Robustness analysis for space-time convection-diffusion
- Explore positivity preserving techniques for compressible Navier-Stokes

Area B: Scientific Computation

- Support development of Camellia¹⁷
 - Development and verification of 2D space-time simulations.
 - Implement time slabs to decrease solve size
 - Contribute to auxiliary features
- Run parallel simulations on HPC systems at TACC and ANL

Area C: Modeling and Applications

- Revisit Carter plate solve for higher Reynolds numbers
- Incompressible Taylor-Green vortex problem
- Incompressible vortex shedding problems
- Possibly investigate compressible Sedov and Noh problems

¹⁷N.V. Roberts. "Camellia: A Software Framework for Discontinuous Petrov-Galerkin Methods". In: *Comp. Math. Appl.* (2014), submitted.

Thank You!

