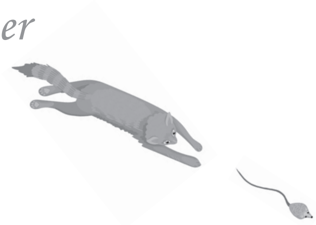
chapter 1



Lewis Carroll's Cats and Rats

Introduction

One of my favorite puzzles is little known, but quite old. It is attributed to Lewis Carroll, the famous author of *Alice in Wonderland* and other wonderful books, but originated much earlier — with a surprising story behind it.

The Puzzle

If 6 cats kill 6 rats in 6 minutes, how many will be needed to kill 100 rats in 50 minutes? (No animals were hurt in preparing this puzzle ...)

Where to Start?

You might need a hint to solve some of the puzzles in this book, however this puzzle seems to be straightforward. All the information that is needed is in the question, in particular the "killing rate" of six rats.

Before you go any further, why don't you try and solve the puzzle right now? What was your answer? Was it 9, 10, 11, 12 or 13?

Solving the Puzzle

Surprisingly, 12, 13 and 14 are *all* correct answers! How is this possible? It all depends on the reasoning. Let's see for ourselves.

- The answer is **12**:

  If 6 cats kill 6 rats in 6 minutes, then 6 cats collectively kill one rat every minute. In 50 minutes, the 6 cats can kill 50 rats. So to kill 100 rats, twice the number of cats is needed — 12 cats.

  or:

  If 6 cats kill 6 rats in 6 minutes, then a group of 3 cats kills a rat in 2 minutes. In 50 minutes, every group of 3 cats can kill 25 rats because $50 \div 2 = 25$. To kill 100 rats in 50 minutes, 4 triplets of cats are needed because $100 \div 25 = 4$. Multiply this by 3, because we're counting triplets of cats, and we get 12 cats.

- The answer is **13**:

  If 6 cats kill 6 rats in 6 minutes, then each cat kills a rat in 6 minutes. In 50 minutes, one cat can kill 8 rats: $50 \div 6 = 8\frac{1}{3}$. Note that the cat will have 2 minutes to spare, since $8 \times 6 = 48$ and you can't kill one-third of a rat. Theoretically, $100 \div 8 = 12\frac{1}{2}$ cats are needed to kill 100 rats, but since there are no "half-cats", the answer in practice is 13. With 13, all 100 rats are killed and if they so wished, the cats could even kill 4 more.

- The answer is **14**:

  If 6 cats kill 6 rats in 6 minutes, then 2 cats kill a rat in 3 minutes. In 50 minutes, every group of 2 cats can kill 16 rats with 2 minutes to spare, because $50 \div 3 = 16\frac{2}{3}$. To kill

100 rats in 50 minutes, 7 pairs of cats are needed because 100 ÷ 16 = 6¼ and there are no “quarter-cats”, so more than 6 pairs are needed. Multiplying this by 2 — two cats a pair — and we get the required number, 14 cats. In fact, if they are really hungry they can kill 112 cats in the allotted time.

So, it’s the cats’ working procedure that determines the solution. If all the cats pounce together on each rat one at a time, or if they split into groups of 3 cats per rat, then 12 cats are needed. If each cat works individually, 13 cats is the answer. If their strategy is to work in pairs — perhaps one of them pins the rat down while the other pounces — the correct solution is 14.

The History of the Puzzle and Related Topics

Lewis Carroll is one of my childhood heroes. I spent hours reading *Alice in Wonderland, Through the Looking Glass* and *The Hunting of the Snark*. Little did I know then that I was “reading” math. Lewis Carroll is the pseudonym of the British mathematician Charles Lutwidge Dodgson. Born in Cheshire in 1832, the son of the Archdeacon of Richmond, he excelled in math, eventually earning him a position as lecturer at Oxford University. But Carroll did not make his fame through math. Rather, it was his literary genius, combined with his impeccable command of nonsense and social status as a profound entertainer for children and adults alike, that made him famous. Carroll used to perform magic tricks, games and displayed an unusual collection of bizarre inventions in front of social acquaintances.

Math, and in particular logic, made its way into Carroll’s *Alice* books. These were first related as stories to the children of Henry Liddell, the dean of Christ Church, Oxford, whom Carroll used to take on short boating trips on the Thames river. Liddell’s middle daughter Alice, whom the wonderland Alice is modeled after, was the one who persuaded Carroll to write down his stories. Alice received an honorary doctorate degree from Columbia University at the age of 80 for "awaking with her girlhood's charm the ingenious fancy of a mathematician familiar with imaginary quantities, stirring him to reveal his complete understanding of the heart of a child." This was in 1932. Such a sentence would probably not be at all appropriate these days, but Alice Liddell had many virtues and achievements of her own right. The Columbia quote also touched on the disturbing reports regarding Lewis Carroll's attraction to young girls. Jenny Wolf has published an excellent article in the *Smithsonian Magazine* regarding Lewis Carroll's "shifting reputation".

The popular math and science writer, Martin Gardner, wrote *The Annotated Alice*, a fantastic literary work uncovering, among other things, the math underlying many of the episodes in the *Alice* books. One of the things Gardner alludes to is Carroll's "pillow problems", puzzles that he tried to solve while lying in bed trying to cope with his insomnia.

> *"Only it is so very lonely here!" Alice said in a melancholy voice; and at the thought of her loneliness two large tears came rolling down her cheeks. "Oh, don't go on like that!" cried the poor Queen, wringing her hands in despair. "Consider what a great girl you are. Consider what a long way you've come today. Consider what o'clock it is. Consider anything, only don't cry!" Alice could not help laughing at this, even in the midst of her tears. "Can you keep from crying by considering things?" she asked. "That's the way it's done," the Queen said with great decision: "Nobody can do two things at once, you know."*
>
> *(Through the Looking Glass)*

Martin Gardner commented on this: "Carroll practiced the White Queen's advice. In his introduction to *Pillow Problems* he speaks of working mathematical problems in his head at night, during wakeful

Fig. 1.1 Lewis Carroll, 1863 photograph by Oscar G. Rejlander

hours, as a kind of mental work-therapy to prevent less wholesome thoughts from tormenting him."

Carroll's compendium of 72 pillow problems was first published in a book, *Curiosa Mathematica: Pillow-problems, thought out during sleepless nights* in 1893, five years before his death. Many of these puzzles are ingenious and some, notoriously difficult. Lewis Carroll published many more puzzles throughout his life, in various publications, and painstaking efforts have been made throughout the years, to locate and republish them. One such puzzle is our Cats and Rats puzzle which was published in 1879 in a puzzle column in *The Monthly Packet*, a magazine for girls, edited by Charlotte Yonge, but it was not Carroll who published it! Robin Wilson and Amirouche Moktefi in their recently published monumental work, *The Mathematical World of Charles L. Dodgson (Lewis Carroll)*, suggest that Carroll came across the puzzle and disliked it. Martin Gardner gives the details in his book, *The Universe in a Handkerchief*. Apparently, these kind of puzzles — *if a does b things in c minutes,* *how many a are needed to do d things in e minutes?* — were very popular and, it was suggested, have a unique solution, similar to the way rate problems were solved, rendering the "proper" solution of 12 cats. Lewis Carroll disliked the absurdity of the problem, so he published in the February 1880 issue of the magazine his "full" solution — those that we gave above — along with a puzzle of his own, mocking the original:

> *"If a cat can kill a rat in a minute, how long would it be killing 60,000 rats? Ah! How long indeed! My private opinion is that the rats would kill the cat."*

No recreational math book is complete without at least one Lewis Carroll reference. Why did he become so famous, both within the math community and the general public? I would argue that it was his mastery of math *and* the arts, especially writing and performing, coupled with the dramatic expansion of pastime opportunities, popular novels and plays of the Victorian era, that brought him his fame. Carroll was arguably the first person to use math for entertainment to the delight of many of the Victorian salons.

Generalizations of the Puzzle

Our minds have been trained to expect one answer for one puzzle. In one of the online puzzle courses that I run, one participant even protested that having more than one correct answer is "against the rules"! Since we make up the rules, there is of course no reason at all to restrict the number of solutions. In fact, in many real-world situations there can be quite a few solutions to a single puzzle.

Why and when do we get more than one answer to these and other puzzles? The crux of the matter is this: when some of the information is missing, there *may* be more than one solution to a puzzle. The missing information in the Cats and Rats puzzle is the cats' methodology — we do not know the cats' method of rat hunting. If they all pounce on each rat separately, or if three cats tackle each rat, then 12 cats are sufficient. 13 cats will solve the puzzle if each cat hunts down one rat, and if two cats pounce on one poor rat, then 14 cats are required.

What other puzzles can be classified as "missing information" puzzles? Here is another example — this time a "real-life" puzzle. Suppose you want to find the shortest distance between the corner of 5th Avenue and West 32nd Street in Manhattan to the corner of Madison Avenue and East 34th Street. For simplicity we take the length of an East–West street as one unit, and that of a North–South street as ½ unit because they are approximately a half length of the East–West streets as can be seen in Fig. 1.2 (this approximation is not far from the truth).

Most people who are in a "school geometry" frame of mind will use the Pythagorean theorem to calculate the distance and argue

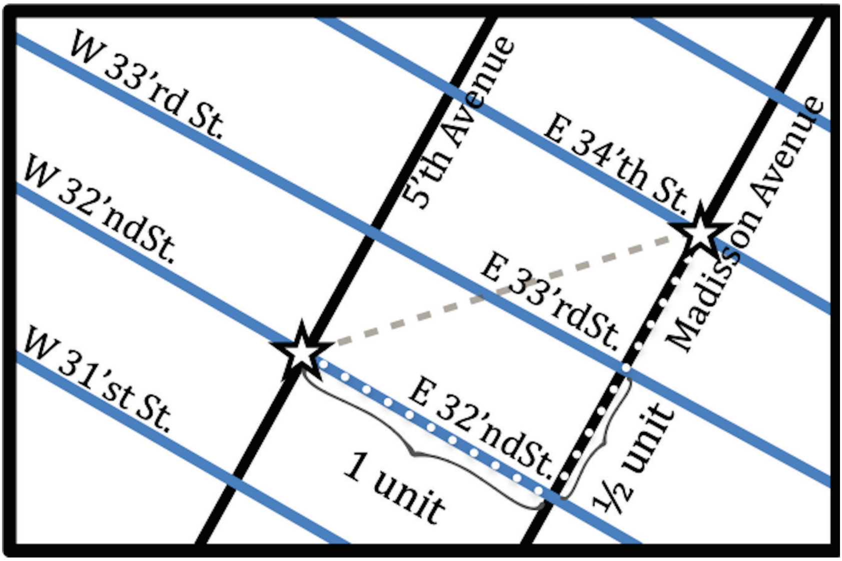


Fig. 1.2 The Manhattan route puzzle

for a shortest route of “the square root of 2” units. Using math terminology, this is the *hypotenuse* of an imaginary *right-angled triangle* whose *catheti (legs)* are:

- along 32nd Street between 5th Avenue and Madison Avenue = 1 unit long.
- along Madison Avenue between East 32nd Street and East 34th Street = two ½ units = 1 unit long.

According to the theorem, the length of the hypotenuse is $\sqrt{1^2+1^2}=\sqrt{2}=1.414\ldots$, nearly 1½ units long. This distance is shown as the dashed line in Fig. 1.2.

If you happen to be a pedestrian or a taxi driver for that matter, you’re most likely to look for the shortest, *most practical* way to drive or walk between the two points. Since you don’t walk or drive through buildings, you might argue that there are three “shortest routes”, all three, 2 units long. The dotted lines in Fig. 1.2 show one of these three routes — going a block east on 32nd Street and then two blocks north on Madison Avenue. The “taxicab” calculation (as it is typically called) is straightforward. We just sum the street lengths of the different routes between the two points: 1 + ½ + ½ = 2.

So this puzzle has two “correct” answers: 2 or $\sqrt{2}$. The missing information in the puzzle is *topological.* Very broadly speaking, topology is the study of the shape of things. What is the topology of the puzzle? We just don’t know! If we regard Manhattan as just one flat slab without streets or buildings, the topology is *planar* and we use *planar geometry* — the kind of geometry usually taught in school to calculate “as the crow flies” distances. But for a pedestrian or taxi driver, the problem should probably be solved within the framework of what’s known as *taxicab geometry*.

Planar geometry has been known for millennia. As its name suggests, planar geometry problems occur in two-dimensional space. A typical, unambiguous, planar geometry “real-life” problem could be to calculate the shortest distance when driving from one spot to another on a flat, empty slab of asphalt — perhaps an unmarked parking lot. The answer would be a straight line. In planar geometry there is only one straight line between two points, so there is only one "shortest" route for this problem.

Planar geometry itself is part of a larger geometrical system known as *Euclidean geometry,* that includes three-dimensional solid geometry as well. This system was first described in *The Elements*, a series of 13 math textbooks, compiled by the 3rd century BCE Greek mathematician, Euclid of Alexandria. *The Elements* was the first rigorous attempt to lay the foundations of mathematical proof, containing much of the geometry and number theory known to man. Beginning with a set of five *axioms,* self-evident, unproved assumptions, the books devise a huge number of *theorems,* conclusions that arise from, and can be proved by these "accepted" truths. *The Elements* was so thorough and popular, that it was used as a textbook, in some places, even up to the 20th century! Over a thousand editions of *The Elements* have been published and it has been translated into many different languages.

The five planar geometry axioms, the Euclidean axioms or postulates, are usually taught in school. *Wolfram's Mathworld* defines these as:

1. A straight-line segment can be drawn joining any two points.
2. Any straight-line segment can be extended indefinitely in a straight line.
3. Given any straight-line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough.

The fifth axiom is equivalent to what is known as the parallel postulate, and since it is not as “self-evident” as the other four axioms, mathematicians tried, time after time, to dispose of it by trying to deduce it from the other axioms. It was only in 1823 that both the Hungary mathematician, János Bolyai, and the Russian mathematician, Nikolai Lobachevsky, independently realized that not only is the parallel postulate essential for the completeness of Euclidean geometry, but that, by disposing of it, new “non-Euclidean geometric systems” arise. The 19th and 20th centuries witnessed the birth of elliptical and hyperbolic geometries — and many more.

Taxicab geometry was introduced by the 19th century Russian-born German mathematician, Hermann Minkowski, and was further developed into a system complete with its own axioms by Donald R. Byrkit in 1971. Distances in taxicab geometry are non-Euclidean. They are calculated only along horizontal and vertical segments on a Cartesian grid. Interestingly, the popular navigation app Waze™ shows both the planar distance, and the taxicab distance between two points of interest.

So far, we have seen two puzzles which give rise to different solutions, because some of the information in the puzzles is missing. This gives rise to the question of whether *all* puzzles and math problems are “missing information” puzzles that can be solved differently depending on how they are interpreted. We might be tempted to include some instances of algebraic equations. Quadratic equations, for example, have two solutions, so one might suggest that they are also problems with more than one solution because of missing information. But even if at first glance we can’t explain why, we can feel that this is not the same genre as Lewis Carroll’s Cats and Rats puzzle, or the Manhattan distance puzzle. In quadratic equations and other problems in algebra, everything is perfectly defined. The number of solutions is a manifestation of the equations themselves, independent of outside circumstances or interpretations. Conversely, "missing information" puzzles have different solutions precisely because of their interpretation.

Here are two other puzzles that might be considered "missing information" puzzles. You may have heard of them — they're quite popular — but their origins are not known.

- There are three on-off switches outside an attic, one of them controls an incandescent light bulb in the attic, but you don't know which one. You are allowed only one visit to the attic. How do you find the correct switch?
- Draw four straight lines that go through the middle of all the points on the following grid (Fig. 1.3) without lifting your pencil.

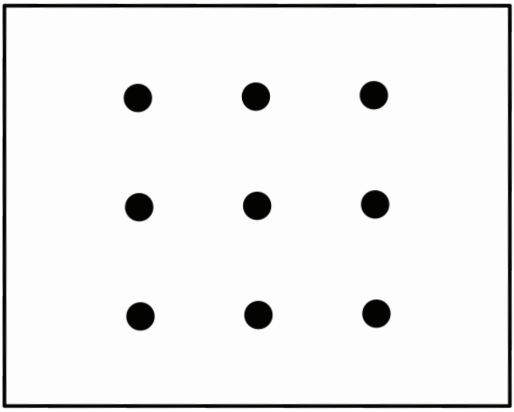


Fig. 1.3 The nine dots puzzle

In the earlier puzzles in this chapter at least one solution easily pops to mind, so much so, that you don't even question whether you have understood the puzzle correctly. Contrariwise, in these two puzzles, it seems at first as though there is no solution at all! The information that is given seems just not enough to solve the puzzle. A lot of the time, when confronted with puzzles like these, people realize that information must be missing, so they tend to ask questions to get that information. In the attic light bulb puzzle, a question might be: can you see the attic from where the switches are? In the nine dots puzzle, people might ask: what do you mean by straight lines? However, in cases like these, the missing information is not usually what it seems and you need to look “outside the problem” for at least one solution. These kinds of puzzles are also known as puzzles with solutions that are “out-of-the-box”.

The attic light bulb puzzle has one popular solution. Turn on two of the switches, wait a few minutes, turn off one of the two switches and then go into the attic. If the light is on, then the switch that is currently in the “ON” position controls the light in the attic. If the light is off but the light bulb is warm, then the switch that was turned “ON” and then “OFF” controls the light. If the light bulb is cold, the remaining switch is the solution.

As often happens in “missing information” puzzles, I have also come across some more bizarre solutions. Mathematician and puzzler, Peter Winkler, suggests that for those who can’t reach the light bulb but have a lot of time, to turn on one switch, wait two months, turn it off and turn on another switch and then visit the attic. The first switch controls the light if the light bulb is burnt out. Of course, this solution assumes that incandescent light bulbs are still in use — in many places nowadays they have been replaced by eco-friendly alternatives. The solution also assumes that you can visibly see when a light bulb is burnt out.

The solution to the nine dots puzzle literally *does* require out-of-the-box thinking, as can be seen in Fig. 1.4.

What *was* the missing information in these two puzzles? In the attic light bulb puzzle, it was the fact that you are allowed to use physics (i.e. feel the light bulb). In the nine dots puzzle it was that you are not confined to the nine-dot grid. But are these pieces of information missing? Only in the minds of the solver — there is nothing in the puzzles to suggest that!

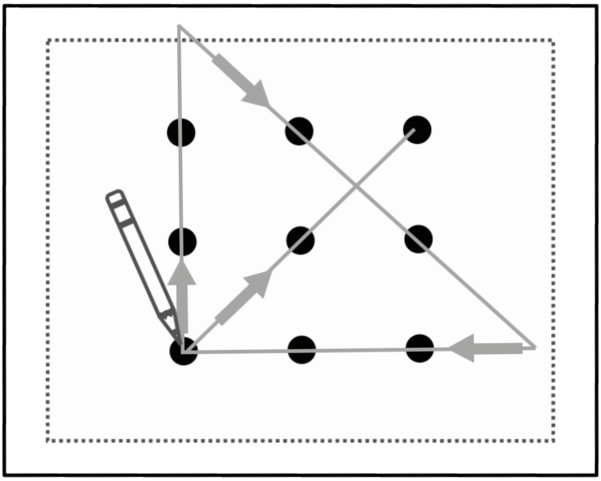


Fig. 1.4 Solution to the nine dots puzzle

Recap

Lewis Carroll's Cats and Rats puzzle is famous for having more than one solution — not because this is mathematically imperative, but because the puzzle can be interpreted in different ways due to some "missing information". There are many types of "missing information" puzzles. We saw three examples. In the first example, the Cats and Rats puzzle, the missing information was *methodological* — what was the cats' method for killing the rats? In the second example, the Manhattan route puzzle, the missing information was *topological* — what was the geometrical layout of the puzzle? In the attic light bulb and nine dots puzzles, the missing information was completely outside the *assumed* sphere of the puzzle; you are not confined to the puzzle's grid or border (the nine dots puzzle) or to math (the attic light bulb puzzle).

Along the way, we briefly introduced a few concepts that you might want to remember. Here are some short, informal definitions:

- *Out-of-the-box puzzles* — puzzles where the solution can be found outside the perceived boundaries of the puzzle.
- *Pillow problem* — a problem or puzzle that you "sleep on". The term was coined by the British mathematician, Charles Lutwidge Dodgson (aka Lewis Carroll), the author of the *Alice in Wonderland* books.
- *Planar geometry* — the study of shapes in the plane.
- *Pythagorean theorem* — the square of the lengths of the legs in a right triangle is equal to the square of the hypotenuse.
- *Taxicab geometry* — the study of shapes on a grid.
- *Topology* — the study of the properties that are preserved through certain, continuous deformations of objects, such as bending, stretching and twisting, but not puncturing or cutting.

Challenge Yourself!

1. This is one of Lewis Carroll's most famous pillow problems. A bag contains one counter, known to be either white or black (with equal probability). A white counter is put in, the bag shaken, and a counter drawn out, which proves to be white. What is now the chance of drawing a white counter?

2. British puzzler, Henry Dudeney, proposed the following "missing information" puzzle: if you add the square of Tom's age to Mary's age, the sum is 62; but if you add the square of Mary's age to Tom's age, the result is 176. What are the ages of Tom and Mary? Apart from solving the puzzle, can you tell what the missing information is, and how you can get around it?

3. If you add my age to yours, you'll get 66. My age is your age in reverse. How old are we?

4. There are two different values of π — the ratio between the circumference of a circle and its diameter — depending on whether we are discussing a planar geometry circle or a taxicab circle. Find both of them!

5. Place the numbers 1–6 in the circles on the "magic" triangle in Fig. 1.5, so that the sum along each of the edges is the same. There are four possible solutions. What constraint would you add so that the solution is unique?

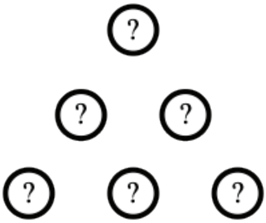


Fig. 1.5 Magic triangle puzzle

6. It takes six minutes to boil an egg. How long does it take to boil three eggs?

Solutions

1. Two thirds. There are three possible cases, in two of them the counter in the bag is white:
   (a) The initial counter remained in the bag and is white, and the one removed was the white counter that was put in.
   (b) The initial counter remained in the bag and is black, and the one removed was the white counter that was put in.
   (c) The white counter that was put in is now the one in the bag and the counter removed was white.

2. Tom's age is 7, Mary's age is 13. One way of solving this is by using algebra — but that does get difficult. An easier way is by understanding the missing information; there are two things missing that we can infer:
   (a) We can assume that the ages will be whole numbers
   (b) We can assume that the ages will be between 0 and 120 (more or less ...)

   This makes things much easier. Tom's age has to be smaller than 8 because $8^2 = 64$ which is larger than the sum of the square of Tom's age plus Mary's age (given as 62).

   Since the square of Mary's age plus Tom's age is 176, Mary's age has to be at least 8 (otherwise Tom is older than 112) but less than 14 ($14^2 = 196$). Trial and error on the four possible ages of Mary gives the answer.

3. There are four possible solutions to this. Our ages can be 6 and 60, 51 and 15, 33 and 33, or 42 and 24.

4. The typical (planar) value of π — the ratio between the circumference of a circle and its diameter — is 3.14159... .

   To find the value of taxicab π, we first need to define a taxicab circle, and then divide its circumference by its diameter. A circle is the collection of points at an equidistance from its center. An example can help. Figure 1.6 shows a taxicab circle with a radius of three units. Funnily enough, it's the shape of a diamond! The circumference — the distance along the perimeter of the circle until returning to the initial point is 24 units. Dividing that by the diameter (twice the radius = 6 units), gives $\pi = 4$. You can see for yourself that changing the radius will change the circumference proportionally, so the value of $\pi = 4$ is indeed constant.

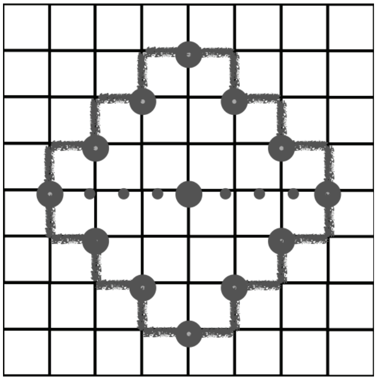


Fig. 1.6 A taxicab circle

5. Figure 1.7 shows four possible solutions, up to symmetry. A constraint might be — find the solution that has the sides with the lowest/highest sum.

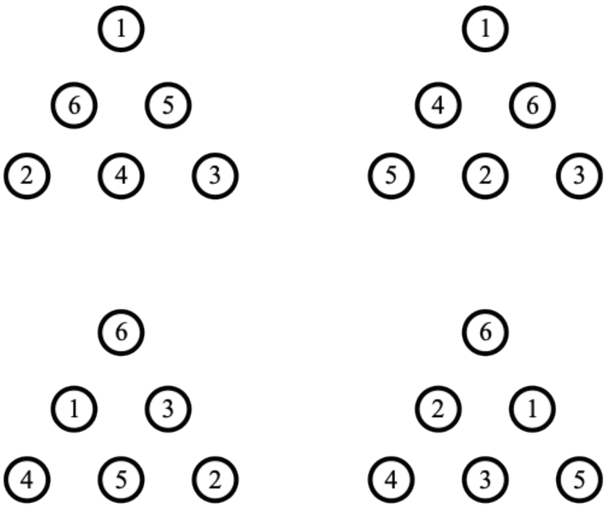


Fig. 1.7 Magic triangle puzzle solutions

6. This is a classic “missing information” puzzle. It all depends on how many saucepans and burner rings you have! If you have one saucepan that holds all three eggs or three saucepans that hold one egg each and three burner rings, or two saucepans (one that holds one egg and the other, two eggs) and two burner rings, then you can boil all three eggs in six minutes. If you have two saucepans that hold only one egg each and two burner rings, or at least one saucepan that holds only two eggs and one burner ring, then it will take you twelve minutes. (Oded Margalit pointed out to me that this is assuming that each egg has to boil six minutes continuously. If we can remove and replace eggs as we like, then we can boil two eggs on two burners in just nine minutes — boil eggs 1 and 2 for three minutes, then boil eggs 1 and 3 for three minutes, and finally, boil eggs 2 and 3 for three minutes.) If you have just one saucepan that holds one egg and one burner ring, then you'll have to wait a whole eighteen minutes to boil all three eggs.

Bibliography and Further Reading

Byrkit, D. R. (1964). Taxicab geometry—a non-Euclidean geometry of lattice points, *The Mathematics Teacher*, **64**(5), pp. 418–422.

Carroll, L., Gardner, M., Burstein, M., & Tenniel, J. (2015). *The Annotated Alice: Alice's Adventures in Wonderland & Through the Looking-glass,* 150th Anniversary Deluxe Edition. W. W. Norton & Company, USA.

Gardner, M. (1996). *The Universe in a Handkerchief: Lewis Carroll's Mathematical Recreations, Games, Puzzles, and Word Plays*. Copernicus, New York.

Levitin, A., & Levitin, M. (2011). *Algorithmic Puzzles*. Oxford University Press, Oxford.

Weisstein, E. W. (n.d.), Euclid's Postulates, from *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/EuclidsPostulates.html

Weisstein, E. W. (n.d.), Topology, from *MathWorld*—A Wolfram Web Resource. https://mathworld.wolfram.com/Topology.html

Wilson, R., & Moktefi, A. (Eds.) (2019). *The Mathematical World of Charles L. Dodgson (Lewis Carroll)*. Oxford University Press, Oxford.

Winkler, P. (2003). *Mathematical Puzzles: A Connoisseur's Collection*. CRC Press, USA.

Wolf, J. (2010). Lewis Carroll's Shifting Reputation, *Smithsonian Magazine.* https://www.smithsonianmag.com/arts-culture/lewis-carrolls-shifting-reputation-9432378/