0.1 Discrete TGV minimisation problem

Let

$$\Omega_h = \{(i,j) \mid i,j \in \mathbb{N},\ 1 \le i \le N_1,\ 1 \le j \le N_2\}. \tag{1}$$

be ...

An image is a function $\Omega_h \to \mathbb{R}$.

Let $u \in \mathbb{R}^{N_1 \times N_2}$ be an image, $f \in \mathbb{R}^{N_1 \times N_2}$ the observed image.

The spaces of scalar, vector, and symmetric matrix valued functions are defined as (page 13)

$$U = \{u : \Omega_h \to \mathbb{R}\},\ V = \{u : \Omega_h \to \mathbb{R}^2\},\ W = \{u : \Omega_h \to \mathrm{Sym}^2(\mathbb{R}^2)\}. \tag{2}$$

where $\mathrm{Sym}^2(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices.

$$\begin{aligned}&\text{v} \in V \text{ has components } (v)^1 \text{ and } (v)^2\\ &\text{w} \in W \text{ has components } (w)^{11}, (w)^{12} = (w)^{21},\ (w)^{22}.\end{aligned} \tag{3}$$

$$\begin{aligned}&u, r \in U : \langle u, r\rangle_U = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2} u_{i,j} r_{i,j}\\ &v, p \in V : \langle v, p\rangle_V = \left\langle (v)^1, (p)^1 \right\rangle + \left\langle (v)^2, (p)^2 \right\rangle\\ &w, q \in W : \langle w, q\rangle_W = \left\langle (w)^{11}, (q)^{11} \right\rangle + \left\langle (w)^{22}, (q)^{22} \right\rangle + 2\left\langle (w)^{12}, (q)^{12} \right\rangle\end{aligned} \tag{4}$$

are the scalar products in U, V, W.

The x and y forward finite difference operators are (page 13)

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{for } 1 \le i < N_1,\\ 0 & \text{for } i = N_1,\end{cases} \tag{5}$$

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{for } 1 \le j < N_2,\\ 0 & \text{for } j = N_2,\end{cases} \tag{6}$$

and the backward finite difference operators are (page 13)

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{1,j} & \text{if } i = 1,\\ u_{i,j} - u_{i-1,j} & \text{for } 1 < i < N_1,\\ -u_{N_1-1,j} & \text{for } i = N_1,\end{cases} \tag{7}$$

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,1} & \text{if } j = 1, \\ u_{i,j} - u_{i,j-1} & \text{for } 1 < j < N_2, \\ -u_{i,N_2-1} & \text{for } j = N_2, \end{cases} \tag{8}$$

The gradient operator is defined as (page 13)

$$\nabla_h : U \to V, \quad \nabla_h u = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix}, \tag{9}$$

and the symmetrised gradient operator is defined as (page 14)

$$\mathcal{E}_h : V \to W, \quad \mathcal{E}_h(v) = \begin{pmatrix} \partial_x^-(v)^1 & \frac{1}{2}\left(\partial_y^-(v)^1 + \partial_x^-(v)^2\right) \\ \frac{1}{2}\left(\partial_y^-(v)^1 + \partial_x^-(v)^2\right) & \partial_y^-(v)^2 \end{pmatrix}, \tag{10}$$

and

$$\operatorname{div}_h : V \to U, \quad \operatorname{div}_h v = \partial_x^-(v)^1 + \partial_y^-(v)^2, \tag{11}$$

$$\operatorname{div}_h : W \to V, \quad \operatorname{div}_h w = \begin{pmatrix} \partial_x^+(w)^{11} + \partial_y^+(w)^{12} \\ \partial_x^+(w)^{12} + \partial_y^+(w)^{22} \end{pmatrix}. \tag{12}$$

The discrete ∞-norms are

$$\begin{aligned} v \in V: \quad & \|v\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((v)_{i,j}^1\right)^2 + \left((v)_{i,j}^2\right)^2 \right)^{1/2}, \\ w \in W: \quad & \|w\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((w)_{i,j}^{11}\right)^2 + \left((w)_{i,j}^{22}\right)^2 + 2\left((w)_{i,j}^{12}\right)^2 \right)^{1/2}. \end{aligned} \tag{13}$$

Finally, the TGV minimisation problem is defined as (page ...)

$$\min_{u \in \mathbb{R}^{N_1 \times N_2}} F(u) + \mathrm{TGV}_\alpha^2(u) \tag{14}$$

where the data fidelity/discrepancy function is (page ...)

$$F(u) = \frac{1}{2}||u - f||_2^2 \tag{15}$$

and the regularisation term is (page 14)

$$\mathrm{TGV}_\alpha^2(u) = \max \left\{ \langle u, \operatorname{div}_h v \rangle_U \;\middle|\; \begin{array}{l} (v,w) \in V \times W,\ \operatorname{div}_h w = v, \\ \|w\|_\infty \le \alpha_0,\ \|v\|_\infty \le \alpha_1 \end{array} \right\} \tag{16}$$