0.1 Discrete TGV minimisation problem

Let

$$\Omega_h = \{(i,j) \mid i,j \in \mathbb{N},\ 1 \le i \le N_1,\ 1 \le j \le N_2\}. \tag{1}$$

be ...

An image is a function $\Omega_h \to \mathbb{R}$.

Let $u \in \mathbb{R}^{N_1 \times N_2}$ be an image, $f \in \mathbb{R}^{N_1 \times N_2}$ the observed image.

The spaces of scalar, vector, and symmetric matrix valued functions are defined as (page 13)

$$U = \{u : \Omega_h \to \mathbb{R}\},\ V = \{u : \Omega_h \to \mathbb{R}^2\},\ W = \{u : \Omega_h \to \mathrm{Sym}^2(\mathbb{R}^2)\}. \tag{2}$$

where $\mathrm{Sym}^2(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices.

$$\begin{aligned} &\text{v} \in V \text{ has components } (v)^1 \text{ and } (v)^2 \\ &\text{w} \in W \text{ has components } (w)^{11}, (w)^{12} = (w)^{21},\ (w)^{22}. \end{aligned} \tag{3}$$

$$\begin{aligned} u, r \in U &: \langle u, r \rangle_U = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_{i,j} r_{i,j} \\ v, p \in V &: \langle v, p \rangle_V = \left\langle (v)^1, (p)^1 \right\rangle + \left\langle (v)^2, (p)^2 \right\rangle \\ w, q \in W &: \langle w, q \rangle_W = \left\langle (w)^{11}, (q)^{11} \right\rangle + \left\langle (w)^{22}, (q)^{22} \right\rangle + 2 \left\langle (w)^{12}, (q)^{12} \right\rangle \end{aligned} \tag{4}$$

are the scalar products in U, V, W.

The x and y forward finite difference operators are (page 13)

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{for } 1 \le i < N_1, \\ 0 & \text{for } i = N_1, \end{cases} \tag{5}$$

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{for } 1 \le j < N_2, \\ 0 & \text{for } j = N_2, \end{cases} \tag{6}$$

and the backward finite difference operators are (page 13)

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{1,j} & \text{if } i = 1, \\ u_{i,j} - u_{i-1,j} & \text{for } 1 < i < N_1, \\ -u_{N_1 - 1,j} & \text{for } i = N_1, \end{cases} \tag{7}$$

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,1} & \text{if } j = 1, \\ u_{i,j} - u_{i,j-1} & \text{for } 1 < j < N_2, \\ -u_{i,N_2-1} & \text{for } j = N_2, \end{cases} \tag{8}$$

The gradient operator is defined as (page 13)

$$\nabla_h : U \to V, \quad \nabla_h u = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix}, \tag{9}$$

and the symmetrised gradient operator is defined as (page 14)

$$\mathcal{E}_h : V \to W, \quad \mathcal{E}_h(v) = \begin{pmatrix} \partial_x^-(v)^1 & \frac{1}{2}\left(\partial_y^-(v)^1 + \partial_x^-(v)^2\right) \\ \frac{1}{2}\left(\partial_y^-(v)^1 + \partial_x^-(v)^2\right) & \partial_y^-(v)^2 \end{pmatrix}, \tag{10}$$

and

$$\mathrm{div}_h : V \to U, \quad \mathrm{div}_h v = \partial_x^-(v)^1 + \partial_y^-(v)^2, \tag{11}$$

$$\mathrm{div}_h : W \to V, \quad \mathrm{div}_h w = \begin{pmatrix} \partial_x^+(w)^{11} + \partial_y^+(w)^{12} \\ \partial_x^+(w)^{12} + \partial_y^+(w)^{22} \end{pmatrix}. \tag{12}$$

The discrete ∞-norms are

$$\begin{aligned} v \in V: \quad & \|v\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((v)^1_{i,j}\right)^2 + \left((v)^2_{i,j}\right)^2 \right)^{1/2}, \\ w \in W: \quad & \|w\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((w)^{11}_{i,j}\right)^2 + \left((w)^{22}_{i,j}\right)^2 + 2\left((w)^{12}_{i,j}\right)^2 \right)^{1/2}. \end{aligned} \tag{13}$$

Finally, the TGV minimisation problem is defined as (page ...)

$$\min_{u\in\mathbb{R}^{N_1\times N_2}} F(u) + \mathrm{TGV}^2_\alpha(u) \tag{14}$$

where the data fidelity/discrepancy function is (page ...)

$$F(u) = \frac{1}{2}||u - f||_2^2 = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(u_{i,j} - f_{i,j})^2 \tag{15}$$

and the regularisation term is (page 14)

$$\mathrm{TGV}^2_\alpha(u) = \max\left\{ \langle u, \mathrm{div}_h v\rangle_U \;\middle|\; \begin{array}{l} (v,w) \in V\times W,\ \mathrm{div}_h w = v, \\ \|w\|_\infty \le \alpha_0,\ \|v\|_\infty \le \alpha_1 \end{array} \right\} \tag{16}$$

0.2 Numerical algorithm for TGV minimisation problem

Algorithm 1 Solve $\min_{u \in U} F_h(u) + \mathrm{TGV}_\alpha^2(u)$

1: Choose $\sigma > 0$, $\tau > 0$ such that $\sigma\tau\frac{1}{2}(17 + \sqrt{33}) \le 1$.
2: Choose $(u^0, p^0) \in U \times V$, $(v^0, w^0) \in V \times W$ and set $\bar{u}^0 = u^0$, $\bar{p}^0 = p^0$.
3: **for** $n = 0, 1, 2, \ldots$ **do**
4:
$$\begin{cases} v^{n+1} = \mathcal{P}_{\alpha_1}\left(v^n + \sigma(\nabla_h \bar{u}^n - \bar{p}^n)\right), \\ w^{n+1} = \mathcal{P}_{\alpha_0}\left(w^n + \sigma\mathcal{E}_h(\bar{p}^n)\right), \\ u^{n+1} = \left(u^n + \tau(\mathrm{div}_h v^{n+1} + f)\right)/(1+\tau), \\ p^{n+1} = p^n + \tau(v^{n+1} + \mathrm{div}_h w^{n+1}), \\ \bar{u}^{n+1} = 2u^{n+1} - u^n, \\ \bar{p}^{n+1} = 2p^{n+1} - p^n. \end{cases}$$
5: **end for**
6: Return u^N for some large N.

where

$$\begin{aligned} \mathcal{P}_{\alpha_1}(v) &= \frac{v}{\max\left(1, \frac{|v|}{\alpha_1}\right)}, \quad v \in V \\ \mathcal{P}_{\alpha_0}(w) &= \frac{w}{\max\left(1, \frac{|w|}{\alpha_0}\right)}, \quad w \in W, \end{aligned} \tag{17}$$

$$\begin{aligned} v \in V: \quad |v|_{i,j} &= \left(\left((v)_{i,j}^1\right)^2 + \left((v)_{i,j}^2\right)^2\right)^{1/2}, \\ w \in W: \quad |w|_{i,j} &= \left(\left((w)_{i,j}^{11}\right)^2 + \left((w)_{i,j}^{22}\right)^2 + 2\left((w)_{i,j}^{12}\right)^2\right)^{1/2}. \end{aligned} \tag{18}$$

This algorithm is used to solve the minimisation problem 14. It is taken from the paper Bredies, 2014, page ....

References

Bredies, K. (2014). Recovering piecewise smooth multichannel images by minimization of convex functionals with total generalized variation penalty. In A. Bruhn, T. Pock, & X.-C. Tai (Eds.), *Efficient algorithms for global optimization methods in computer vision* (pp. 44–77). Springer Berlin Heidelberg.