0.1 Discrete TGV minimisation problem

Let

$$\Omega_h = \{(i,j) \mid i,j \in \mathbb{N},\ 1 \le i \le N_1,\ 1 \le j \le N_2\}. \tag{1}$$

be ...

An image is a function $\Omega_h \to \mathbb{R}$.

Let $u \in \mathbb{R}^{N_1 \times N_2}$ be an image, $f \in \mathbb{R}^{N_1 \times N_2}$ the observed image.

The spaces of scalar, vector, and symmetric matrix valued functions are defined as (page 13)

$$\begin{aligned} U &= \{u : \Omega_h \to \mathbb{R}\} \\ V &= \{u : \Omega_h \to \mathbb{R}^2\} \\ W &= \{u : \Omega_h \to \mathrm{Sym}^2(\mathbb{R}^2)\} \end{aligned} \tag{2}$$

where $\mathrm{Sym}^2(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices.

We use the notations v and w where

$$\begin{aligned} &\text{v } \in V \text{ has components } (v)^{(1)} \text{ and } (v)^{(2)} \\ &\text{w } \in W \text{ has components } (w)^{(1,1)},\ (w)^{(1,2)},\ (w)^{(2,1)},\ (w)^{(2,2)} \end{aligned} \tag{3}$$

with $(w)^{(1,2)} = (w)^{(2,1)}$.

Use the scalar product notation

$$a, b : \Omega_h \to \mathbb{R} : \quad \langle a, b \rangle = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} a_{i,j} b_{i,j} \tag{4}$$

the scalar products in the spaces V and W are

$$\begin{aligned} v, p \in V : \langle v, p \rangle_V &= \left\langle (v)^{(1)}, (p)^{(1)} \right\rangle + \left\langle (v)^{(2)}, (p)^{(2)} \right\rangle \\ w, q \in W : \langle w, q \rangle_W &= \left\langle (w)^{(1,1)}, (q)^{(1,1)} \right\rangle + \left\langle (w)^{(2,2)}, (q)^{(2,2)} \right\rangle + 2 \left\langle (w)^{(1,2)}, (q)^{(1,2)} \right\rangle \end{aligned} \tag{5}$$

are the scalar products in U, V, W.

The x and y forward finite difference operators are (page 13)

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{for } 1 \le i < N_1 \\ 0 & \text{for } i = N_1 \end{cases} \tag{6}$$

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{for } 1 \le j < N_2 \\ 0 & \text{for } j = N_2 \end{cases} \tag{7}$$

and the backward finite difference operators are (page 13)

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{1,j} & \text{if } i = 1 \\ u_{i,j} - u_{i-1,j} & \text{for } 1 < i < N_1 \\ -u_{N_1-1,j} & \text{for } i = N_1 \end{cases} \tag{8}$$

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,1} & \text{if } j = 1 \\ u_{i,j} - u_{i,j-1} & \text{for } 1 < j < N_2 \\ -u_{i,N_2-1} & \text{for } j = N_2 \end{cases} \tag{9}$$

The gradient operator ∇_h and the symmetrised gradient operator $\mathcal{E}_h$ are defined as (pages 13, 14)

$$\begin{aligned} &\nabla_h : U \to V, \quad \nabla_h u = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix} \\ &\mathcal{E}_h : V \to W, \quad \mathcal{E}_h(v) = \begin{pmatrix} \partial_x^-(v)^{(1)} & \frac{1}{2}\left(\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}\right) \\ \frac{1}{2}\left(\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}\right) & \partial_y^-(v)^{(2)} \end{pmatrix} \end{aligned} \tag{10}$$

and the divergence operators div_h are

$$\begin{aligned} &\mathrm{div}_h : V \to U, \quad \mathrm{div}_h v = \partial_x^-(v)^{(1)} + \partial_y^-(v)^{(2)} \\ &\mathrm{div}_h : W \to V, \quad \mathrm{div}_h w = \begin{pmatrix} \partial_x^+(w)^{(11)} + \partial_y^+(w)^{(12)} \\ \partial_x^+(w)^{(12)} + \partial_y^+(w)^{(22)} \end{pmatrix} \end{aligned} \tag{11}$$

The discrete ∞-norms are

$$\begin{aligned} v \in V: \quad \|v\|_\infty &= \max_{(i,j)\in\Omega_h} \left(\left((v)_{i,j}^{(1)}\right)^2 + \left((v)_{i,j}^{(2)}\right)^2 \right)^{1/2} \\ w \in W: \quad \|w\|_\infty &= \max_{(i,j)\in\Omega_h} \left(\left((w)_{i,j}^{(1,1)}\right)^2 + \left((w)_{i,j}^{(2,2)}\right)^2 + 2\left((w)_{i,j}^{(1,2)}\right)^2 \right)^{1/2} \end{aligned} \tag{12}$$

Finally, the TGV minimisation problem is defined as (page ...)

$$\min_{u \in \mathbb{R}^{N_1 \times N_2}} F(u) + \mathrm{TGV}_\alpha^2(u) \tag{13}$$

where the data fidelity/discrepancy function is (page ...)

$$F(u) = \frac{1}{2}||u - f||_2^2 = \frac{1}{2}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(u_{i,j} - f_{i,j})^2 \tag{14}$$

and the regularisation term is (page 14)

$$\mathrm{TGV}_\alpha^2(u) = \max\left\{ \langle u, \mathrm{div}_h v\rangle_U \;\middle|\; \begin{array}{l} (v,w) \in V \times W,\ \mathrm{div}_h w = v, \\ \|w\|_\infty \le \alpha_0,\ \|v\|_\infty \le \alpha_1 \end{array} \right\} \tag{15}$$

In [14, 34] it is demonstrated that the TGV functional can be equivalently written as

$$\mathrm{TGV}_{\alpha}^{2}(u) = \min_{w \in BD(\Omega)} \alpha_1 |Du - w|(\Omega) + \alpha_0 |\mathcal{E} w|(\Omega), \tag{16}$$

0.2 Numerical algorithm for TGV minimisation problem

Algorithm 1 PDHG algorithm for image denoising with fixed regularisation parameter-map $\mathbf{\Lambda}$ (adapted from Kofler et al., 2023 using the implementation by Shote, 2024)

Input: $L = \|[\mathbf{I}, \nabla]^{\mathrm{T}}\|_2$, $\tau = \mathrm{sigmoid}(10)/L$, $\sigma = \mathrm{sigmoid}(10)/L$, $\theta = \mathrm{sigmoid}(10)$, noisy image $\mathbf{x}_0$
Output: reconstructed image $\hat{\mathbf{x}}$
$\bar{\mathbf{x}}_0 = \mathbf{x}_0$
$\mathbf{p}_0 = \mathbf{x}_0$
$\mathbf{q}_0 = \mathbf{0}$
for $k < T$ **do**
  $\mathbf{p}_{k+1} = \left(\mathbf{p}_k + \sigma(\bar{\mathbf{x}}_k - \mathbf{x}_0)\right)/(1+\sigma)$
  $\mathbf{q}_{k+1} = \mathrm{clip}_{\mathbf{\Lambda}}\left(\mathbf{q}_k + \sigma \nabla \bar{\mathbf{x}}_k\right)$
  $\mathbf{x}_{k+1} = \mathbf{x}_k - \tau \mathbf{p}_{k+1} - \tau \nabla^{\mathrm{T}} \mathbf{q}_{k+1}$
  $\bar{\mathbf{x}}_{k+1} = \mathbf{x}_{k+1} + \theta(\mathbf{x}_{k+1} - \mathbf{x}_k)$
end for
$\hat{\mathbf{x}} = \mathbf{x}_T$

Algorithm 2 Solve $\min_{u \in U} F_h(u) + \mathrm{TGV}_\alpha^2(u)$

Input: $f \in U$, $\alpha_0, \alpha_1 > 0$
Choose $\sigma > 0$, $\tau > 0$ such that $\sigma\tau\frac{1}{2}(17 + \sqrt{33}) \leq 1$. Here we choose $\sigma = \tau = 0.29$
$u^{[0]} = f$
$p^{[0]} = \mathbf{0}_V$
$v^{[0]} = \mathbf{0}_V$
$w^{[0]} = \mathbf{0}_W$
$\bar{u}^{[0]} = u^{[0]}$
$\bar{p}^{[0]} = p^{[0]}$
for $n = 0, 1, 2, \ldots$ **do**
    $v^{[n+1]} = \mathcal{P}_{\alpha_1}\left(v^{[n]} + \sigma(\nabla_h \bar{u}^{[n]} - \bar{p}^{[n]})\right)$
    $w^{[n+1]} = \mathcal{P}_{\alpha_0}\left(w^{[n]} + \sigma\mathcal{E}_h(\bar{p}^{[n]})\right)$
    $u^{[n+1]} = \left(u^{[n]} + \tau(\mathrm{div}_h v^{[n+1]} + f)\right)/(1+\tau)$
    $p^{[n+1]} = p^{[n]} + \tau(v^{[n+1]} + \mathrm{div}_h w^{[n+1]})$
    $\bar{u}^{[n+1]} = 2u^{[n+1]} - u^{[n]}$
    $\bar{p}^{[n+1]} = 2p^{[n+1]} - p^{[n]}$
end for
Return $u^{[N]}$ for some large N.

where $\mathbf{0}_V \in V$ and $\mathbf{0}_W \in W$ are the zeros matrices in V and W respectively, and $\mathcal{P}_{\alpha_1}$ and $\mathcal{P}_{\alpha_0}$ are the projection operators defined as

$$
\begin{aligned}
\mathcal{P}_{\alpha_1}(v) &= \frac{v}{\max\left(\mathbf{1}, \frac{1}{\alpha_1}|v|\right)}, \quad v \in V \\
\mathcal{P}_{\alpha_0}(w) &= \frac{w}{\max\left(\mathbf{1}, \frac{1}{\alpha_0}|w|\right)}, \quad w \in W
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
|v|_{i,j} &= \left(\left((v)_{i,j}^{(1)}\right)^2 + \left((v)_{i,j}^{(2)}\right)^2\right)^{1/2} \\
|w|_{i,j} &= \left(\left((w)_{i,j}^{(11)}\right)^2 + \left((w)_{i,j}^{(22)}\right)^2 + 2\left((w)_{i,j}^{(12)}\right)^2\right)^{1/2}
\end{aligned}
\tag{18}
$$

Here $\mathbf{1}$ is a matrix of ones. Note that the division is element-wise. Denote the matrix underneath as a. The matrix under the division is 2D whereas the matrix above the division is 3D and 4D. Therefore each element $v_{i,j}^{(k)}$ and $w_{i,j}^{(k,h)}$ of the matrices v and w is divided by the corresponding element $a_{i,j}$ of the matrix under the division.

This algorithm is used to solve the minimisation problem 13. It is taken from the paper Bredies, 2014, page ....

0.3 Why Projection is dividing by matrix instead of a scalar

Given matrix a, we need to find matrix u such that

$$u = \arg\min_{||v||_\infty \le 1} ||v-a||_V \tag{19}$$

Solution:
Recall that the $L_{1,2}$-norm is defined as

$$||v||_V = \sum_{i,j} \left(\left(v_{i,j}^{(1)}\right)^2 + \left(v_{i,j}^{(2)}\right)^2 \right)^{1/2} \tag{20}$$

Constraints:

$$||v||_\infty = \max_{(i,j)\in\Omega_h} \left(\left(v_{i,j}^{(1)}\right)^2 + \left(v_{i,j}^{(2)}\right)^2 \right)^{1/2} \le 1 \qquad \forall i,j \tag{21}$$

or equivalently

$$\left(\left(v_{i,j}^{(1)}\right)^2 + \left(v_{i,j}^{(2)}\right)^2 \right)^{1/2} \le 1 \qquad \forall i,j \tag{22}$$

Let $t_{i,j} = \sqrt{(v_{i,j}^{(1)})^2 + (v_{i,j}^{(2)})^2}$. Then the constraints can be written as

$$t_{i,j} \le 1 \qquad \forall i,j \tag{23}$$

We have n^2 constraints. Each constraint is applied to a specific element $v_{i,j}$ and is independent of the others. Therefore, the solution can be found by solving n^2 independent problems:

$$u_{i,j} = \arg\min_{t_{i,j}\le 1} ||v_{i,j} - a_{i,j}|| \quad \forall i,j \tag{24}$$

Denote $v_{i,j} = x$, $a_{i,j} = y$, $u_{i,j} = z$:

$$z = \arg\min_{||x||_2 \le 1} ||x-y|| \qquad x,y,z \in \mathbb{R}^2 \tag{25}$$

Thus

$$\begin{aligned} z &= \begin{cases} \frac{y}{||y||_2} & \text{if } ||y||_2 > 1 \\ y & \text{if } ||y||_2 < 1 \end{cases} \\ &= \frac{y}{\max(\mathbf{1}, ||y||_2)} \end{aligned} \tag{26}$$

Applying to u and a, we have

$$u_{i,j} = \frac{a_{i,j}}{\max(1, ||a_{i,j}||_2)} \tag{27}$$

or in matrix notation,

$$u = \frac{a}{\max(\mathbf{1}, ||a||_\infty)} \tag{28}$$

where $\mathbf{1}$ is a matrix of ones.
TODO: $||a||_\infty$ is a scalar, but we need a matriasdasdx?

References

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Shote, D. (2024). Pdhg algorithm implementation for dynamic image denoising. https://github.com/koflera/LearningRegularizationParameterMaps/blob/7537667e81adf3bdebbab8ebdc98fd631b586c03/networks/dyn_img_primal_dual_nn.py