

0.1 Discrete TGV minimisation problem

Let

$$\Omega_h = \{(i, j) \mid i, j \in \mathbb{N}, 1 \leq i \leq N_1, 1 \leq j \leq N_2\}. \quad (1)$$

be ...

An image is a function $\Omega_h \rightarrow \mathbb{R}$.

Let $u \in \mathbb{R}^{N_1 \times N_2}$ be an image, $f \in \mathbb{R}^{N_1 \times N_2}$ the observed image.

The spaces of scalar, vector, and symmetric matrix valued functions are defined as (page 13)

$$U = \{u : \Omega_h \rightarrow \mathbb{R}\}, \quad V = \{u : \Omega_h \rightarrow \mathbb{R}^2\}, \quad W = \{u : \Omega_h \rightarrow \text{Sym}^2(\mathbb{R}^2)\}. \quad (2)$$

where $\text{Sym}^2(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices.

$$\begin{aligned} v \in V \text{ has components } (v)^{(1)} \text{ and } (v)^{(2)} \\ w \in W \text{ has components } (w)^{(11)}, (w)^{(12)}, (w)^{(21)}, (w)^{(22)} \end{aligned} \quad (3)$$

where $(w)^{(12)} = (w)^{(21)}$.

$$\begin{aligned} u, r \in U : \langle u, r \rangle_U &= \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_{i,j} r_{i,j} \\ v, p \in V : \langle v, p \rangle_V &= \langle (v)^{(1)}, (p)^{(1)} \rangle + \langle (v)^{(2)}, (p)^{(2)} \rangle \\ w, q \in W : \langle w, q \rangle_W &= \langle (w)^{(11)}, (q)^{(11)} \rangle + \langle (w)^{(22)}, (q)^{(22)} \rangle + 2 \langle (w)^{(12)}, (q)^{(12)} \rangle \end{aligned} \quad (4)$$

are the scalar products in U, V, W .

The x and y forward finite difference operators are (page 13)

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{for } 1 \leq i < N_1, \\ 0 & \text{for } i = N_1, \end{cases} \quad (5)$$

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{for } 1 \leq j < N_2, \\ 0 & \text{for } j = N_2, \end{cases} \quad (6)$$

and the backward finite difference operators are (page 13)

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{1,j} & \text{if } i = 1, \\ u_{i,j} - u_{i-1,j} & \text{for } 1 < i < N_1, \\ -u_{N_1-1,j} & \text{for } i = N_1, \end{cases} \quad (7)$$

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,1} & \text{if } j = 1, \\ u_{i,j} - u_{i,j-1} & \text{for } 1 < j < N_2, \\ -u_{i,N_2-1} & \text{for } j = N_2, \end{cases} \quad (8)$$

The gradient operator is defined as (page 13)

$$\nabla_h : U \rightarrow V, \quad \nabla_h u = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix}, \quad (9)$$

and the symmetrised gradient operator is defined as (page 14)

$$\mathcal{E}_h : V \rightarrow W, \quad \mathcal{E}_h(v) = \begin{pmatrix} \partial_x^-(v)^{(1)} & \frac{1}{2} (\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}) \\ \frac{1}{2} (\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}) & \partial_y^-(v)^{(2)} \end{pmatrix}, \quad (10)$$

and

$$\text{div}_h : V \rightarrow U, \quad \text{div}_h v = \partial_x^-(v)^{(1)} + \partial_y^-(v)^{(2)}, \quad (11)$$

$$\text{div}_h : W \rightarrow V, \quad \text{div}_h w = \begin{pmatrix} \partial_x^+(w)^{(11)} + \partial_y^+(w)^{(12)} \\ \partial_x^+(w)^{(12)} + \partial_y^+(w)^{(22)} \end{pmatrix}. \quad (12)$$

The discrete ∞ -norms are

$$\begin{aligned} v \in V : \quad \|v\|_\infty &= \max_{(i,j) \in \Omega_h} \left(\left((v)_{i,j}^{(1)} \right)^2 + \left((v)_{i,j}^{(2)} \right)^2 \right)^{1/2}, \\ w \in W : \quad \|w\|_\infty &= \max_{(i,j) \in \Omega_h} \left(\left((w)_{i,j}^{(11)} \right)^2 + \left((w)_{i,j}^{(22)} \right)^2 + 2 \left((w)_{i,j}^{(12)} \right)^2 \right)^{1/2}. \end{aligned} \quad (13)$$

Finally, the TGV minimisation problem is defined as (page ...)

$$\min_{u \in \mathbb{R}^{N_1 \times N_2}} F(u) + \text{TGV}_\alpha^2(u) \quad (14)$$

where the data fidelity/discrepancy function is (page ...)

$$F(u) = \frac{1}{2} \|u - f\|_2^2 = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} (u_{i,j} - f_{i,j})^2 \quad (15)$$

and the regularisation term is (page 14)

$$\text{TGV}_\alpha^2(u) = \max \left\{ \langle u, \text{div}_h v \rangle_U \mid \begin{array}{l} (v, w) \in V \times W, \text{div}_h w = v, \\ \|w\|_\infty \leq \alpha_0, \|v\|_\infty \leq \alpha_1 \end{array} \right\} \quad (16)$$

In [14, 34] it is demonstrated that the TGV functional can be equivalently written as

$$\text{TGV}_\alpha^2(u) = \min_{w \in BD(\Omega)} \alpha_1 |Du - w|(\Omega) + \alpha_0 |\mathcal{E}w|(\Omega), \quad (17)$$

0.2 Numerical algorithm for TGV minimisation problem

Algorithm 1 Solve $\min_{u \in U} F_h(u) + \text{TGV}_\alpha^2(u)$

- 1: Choose $\sigma > 0, \tau > 0$ such that $\sigma\tau\frac{1}{2}(17 + \sqrt{33}) \leq 1$.
 - 2: Choose $(u^{[0]}, p^{[0]}) \in U \times V, (v^{[0]}, w^{[0]}) \in V \times W$ and set $\bar{u}^{[0]} = u^{[0]}, \bar{p}^{[0]} = p^{[0]}$.
 - 3: **for** $n = 0, 1, 2, \dots$ **do**
 - 4:
$$\begin{cases} v^{[n+1]} = \mathcal{P}_{\alpha_1}(v^{[n]} + \sigma(\nabla_h \bar{u}^{[n]} - \bar{p}^{[n]})), \\ w^{[n+1]} = \mathcal{P}_{\alpha_0}(w^{[n]} + \sigma \mathcal{E}_h(\bar{p}^{[n]})), \\ u^{[n+1]} = (u^{[n]} + \tau(\text{div}_h v^{[n+1]} + f)) / (1 + \tau), \\ p^{[n+1]} = p^{[n]} + \tau(v^{[n+1]} + \text{div}_h w^{[n+1]}), \\ \bar{u}^{[n+1]} = 2u^{[n+1]} - u^{[n]}, \\ \bar{p}^{[n+1]} = 2p^{[n+1]} - p^{[n]}. \end{cases}$$
 - 5: **end for**
 - 6: Return $u^{[N]}$ for some large N .
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where

$$\begin{aligned} \mathcal{P}_{\alpha_1}(v) &= \frac{v}{\max\left(1, \frac{|v|}{\alpha_1}\right)}, \quad v \in V \\ \mathcal{P}_{\alpha_0}(w) &= \frac{w}{\max\left(1, \frac{|w|}{\alpha_0}\right)}, \quad w \in W, \end{aligned} \tag{18}$$

$$\begin{aligned} v \in V : \quad |v|_{i,j} &= \left(\left((v)_{i,j}^{(1)} \right)^2 + \left((v)_{i,j}^{(2)} \right)^2 \right)^{1/2}, \\ w \in W : \quad |w|_{i,j} &= \left(\left((w)_{i,j}^{(11)} \right)^2 + \left((w)_{i,j}^{(22)} \right)^2 + 2 \left((w)_{i,j}^{(12)} \right)^2 \right)^{1/2}. \end{aligned} \tag{19}$$

This algorithm is used to solve the minimisation problem 14. It is taken from the paper Bredies, 2014, page

0.3 Why Projection is dividing by matrix instead of a scalar

$$((v_{ij}^{(1)})^2 + (v_{ij}^{(2)})^2)^{1/2} \leq 1 \quad \forall i, j \tag{20}$$

Tc là ta có $n \times n$ constraints, các constraints này chỉ áp dụng cho 1 vị trí v_{ij} , independent vì các vị trí khác. Do đó: bài toán optimisation ln có thể chia thành n^2 bài toán nh:

$$u_{ij} = \arg \min_{t_{ij} \leq 1} \|v_{ij} - a_{ij}\| \quad \forall i, j \tag{21}$$

Denote $v_{ij} = x$, $a_{ij} = y$, $u_{ij} = z$:

$$z = \arg \min_{\|x\|_2 \leq 1} \|x - y\| \quad x, y, z \in \mathbb{R}^2 \quad (22)$$

Thus

$$z = \frac{y}{\|y\|_2} \quad (23)$$

Áp dụng ngc li cho u và a :

$$u = \left(\frac{a_{ij}}{\|a_{ij}\|_2} \right) \text{ if } \|y\|_2 > 1 \quad (24)$$

$$u = \left(\frac{a_{ij}}{\|a_{ij}\|_2} \right) \text{ if } \|y\|_2 > 1 \quad (25)$$

...

$$= \frac{y}{\max(1, \|y\|_2)} \quad (26)$$

Áp dụng cho u và a , ta có:

$$u_{ij} = \frac{a_{ij}}{\max(1, \|a_{ij}\|_2)} \quad (27)$$

References

- Bredies, K. (2014). Recovering piecewise smooth multichannel images by minimization of convex functionals with total generalized variation penalty. In A. Bruhn, T. Pock, & X.-C. Tai (Eds.), *Efficient algorithms for global optimization methods in computer vision* (pp. 44–77). Springer Berlin Heidelberg.