0.1 Discrete TGV minimisation problem

Let

$$\Omega_h = \{(i,j) \mid i,j \in \mathbb{N},\ 1 \le i \le N_1,\ 1 \le j \le N_2\}. \tag{1}$$

be ...

An image is a function $\Omega_h \to \mathbb{R}$.

Let $u \in \mathbb{R}^{N_1 \times N_2}$ be an image, $f \in \mathbb{R}^{N_1 \times N_2}$ the observed image.

The spaces of scalar, vector, and symmetric matrix valued functions are defined as (page 13)

$$U = \{u : \Omega_h \to \mathbb{R}\},\ V = \{u : \Omega_h \to \mathbb{R}^2\},\ W = \{u : \Omega_h \to \mathrm{Sym}^2(\mathbb{R}^2)\}. \tag{2}$$

where $\mathrm{Sym}^2(\mathbb{R}^2)$ is the space of symmetric 2×2 matrices.

$$\begin{aligned} &\text{v} \in V \text{ has components } (v)^{(1)} \text{ and } (v)^{(2)} \\ &\text{w} \in W \text{ has components } (w)^{(11)},\ (w)^{(12)},\ (w)^{(21)},\ (w)^{(22)} \\ &\qquad\qquad\qquad\qquad \text{where } (w)^{(12)} = (w)^{(21)}. \end{aligned} \tag{3}$$

$$\begin{aligned} &u, r \in U : \langle u, r \rangle_U = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_{i,j} r_{i,j} \\ &v, p \in V : \langle v, p \rangle_V = \left\langle (v)^{(1)}, (p)^{(1)} \right\rangle + \left\langle (v)^{(2)}, (p)^{(2)} \right\rangle \\ &w, q \in W : \langle w, q \rangle_W = \left\langle (w)^{(11)}, (q)^{(11)} \right\rangle + \left\langle (w)^{(22)}, (q)^{(22)} \right\rangle + 2 \left\langle (w)^{(12)}, (q)^{(12)} \right\rangle \end{aligned} \tag{4}$$

are the scalar products in U, V, W.

The x and y forward finite difference operators are (page 13)

$$(\partial_x^+ u)_{i,j} = \begin{cases} u_{i+1,j} - u_{i,j} & \text{for } 1 \le i < N_1, \\ 0 & \text{for } i = N_1, \end{cases} \tag{5}$$

$$(\partial_y^+ u)_{i,j} = \begin{cases} u_{i,j+1} - u_{i,j} & \text{for } 1 \le j < N_2, \\ 0 & \text{for } j = N_2, \end{cases} \tag{6}$$

and the backward finite difference operators are (page 13)

$$(\partial_x^- u)_{i,j} = \begin{cases} u_{1,j} & \text{if } i = 1, \\ u_{i,j} - u_{i-1,j} & \text{for } 1 < i < N_1, \\ -u_{N_1-1,j} & \text{for } i = N_1, \end{cases} \tag{7}$$

$$(\partial_y^- u)_{i,j} = \begin{cases} u_{i,1} & \text{if } j = 1, \\ u_{i,j} - u_{i,j-1} & \text{for } 1 < j < N_2, \\ -u_{i,N_2-1} & \text{for } j = N_2, \end{cases} \tag{8}$$

The gradient operator is defined as (page 13)

$$\nabla_h : U \to V, \quad \nabla_h u = \begin{pmatrix} \partial_x^+ u \\ \partial_y^+ u \end{pmatrix}, \tag{9}$$

and the symmetrised gradient operator is defined as (page 14)

$$\mathcal{E}_h : V \to W, \quad \mathcal{E}_h(v) = \begin{pmatrix} \partial_x^-(v)^{(1)} & \frac{1}{2}\left(\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}\right) \\ \frac{1}{2}\left(\partial_y^-(v)^{(1)} + \partial_x^-(v)^{(2)}\right) & \partial_y^-(v)^{(2)} \end{pmatrix}, \tag{10}$$

and

$$\mathrm{div}_h : V \to U, \quad \mathrm{div}_h v = \partial_x^-(v)^{(1)} + \partial_y^-(v)^{(2)}, \tag{11}$$

$$\mathrm{div}_h : W \to V, \quad \mathrm{div}_h w = \begin{pmatrix} \partial_x^+(w)^{(11)} + \partial_y^+(w)^{(12)} \\ \partial_x^+(w)^{(12)} + \partial_y^+(w)^{(22)} \end{pmatrix}. \tag{12}$$

The discrete ∞-norms are

$$\begin{aligned} v \in V: \quad & \|v\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((v)_{i,j}^{(1)}\right)^2 + \left((v)_{i,j}^{(2)}\right)^2 \right)^{1/2}, \\ w \in W: \quad & \|w\|_\infty = \max_{(i,j)\in\Omega_h} \left(\left((w)_{i,j}^{(11)}\right)^2 + \left((w)_{i,j}^{(22)}\right)^2 + 2\left((w)_{i,j}^{(12)}\right)^2 \right)^{1/2}. \end{aligned} \tag{13}$$

Finally, the TGV minimisation problem is defined as (page ...)

$$\min_{u\in\mathbb{R}^{N_1\times N_2}} F(u) + \mathrm{TGV}_\alpha^2(u) \tag{14}$$

where the data fidelity/discrepancy function is (page ...)

$$F(u) = \frac{1}{2}||u - f||_2^2 = \sum_{i=1}^{N_1}\sum_{j=1}^{N_2}(u_{i,j} - f_{i,j})^2 \tag{15}$$

and the regularisation term is (page 14)

$$\mathrm{TGV}_\alpha^2(u) = \max \left\{ \langle u, \mathrm{div}_h v\rangle_U \;\middle|\; \begin{array}{l} (v,w) \in V \times W,\ \mathrm{div}_h w = v, \\ \|w\|_\infty \le \alpha_0,\ \|v\|_\infty \le \alpha_1 \end{array} \right\} \tag{16}$$

In [14, 34] it is demonstrated that the TGV functional can be equivalently written as

$$\mathrm{TGV}_\alpha^2(u) = \min_{w\in BD(\Omega)} \alpha_1|Du - w|(\Omega) + \alpha_0|\mathcal{E}w|(\Omega), \tag{17}$$

0.2 Numerical algorithm for TGV minimisation problem

Algorithm 1 Solve $\min_{u \in U} F_h(u) + \mathrm{TGV}_\alpha^2(u)$

Choose $\sigma > 0$, $\tau > 0$ such that $\sigma\tau\frac{1}{2}(17 + \sqrt{33}) \leq 1$.

Choose $(u^{[0]}, p^{[0]}) \in U \times V$, $(v^{[0]}, w^{[0]}) \in V \times W$ and set $\bar{u}^{[0]} = u^{[0]}$, $\bar{p}^{[0]} = p^{[0]}$.

for $n = 0, 1, 2, \ldots$ **do**

$$\begin{cases} v^{[n+1]} = \mathcal{P}_{\alpha_1}\left(v^{[n]} + \sigma(\nabla_h \bar{u}^{[n]} - \bar{p}^{[n]})\right), \\ w^{[n+1]} = \mathcal{P}_{\alpha_0}\left(w^{[n]} + \sigma\mathcal{E}_h(\bar{p}^{[n]})\right), \\ u^{[n+1]} = \left(u^{[n]} + \tau(\mathrm{div}_h v^{[n+1]} + f)\right)/(1+\tau), \\ p^{[n+1]} = p^{[n]} + \tau(v^{[n+1]} + \mathrm{div}_h w^{[n+1]}), \\ \bar{u}^{[n+1]} = 2u^{[n+1]} - u^{[n]}, \\ \bar{p}^{[n+1]} = 2p^{[n+1]} - p^{[n]}. \end{cases}$$

end for

Return $u^{[N]}$ for some large N.

where

$$\begin{aligned} \mathcal{P}_{\alpha_1}(v) &= \frac{v}{\max\left(1, \frac{|v|}{\alpha_1}\right)}, \quad v \in V \\ \mathcal{P}_{\alpha_0}(w) &= \frac{w}{\max\left(1, \frac{|w|}{\alpha_0}\right)}, \quad w \in W, \end{aligned} \tag{18}$$

$$\begin{aligned} v \in V: \quad |v|_{i,j} &= \left(\left((v)_{i,j}^{(1)}\right)^2 + \left((v)_{i,j}^{(2)}\right)^2\right)^{1/2}, \\ w \in W: \quad |w|_{i,j} &= \left(\left((w)_{i,j}^{(11)}\right)^2 + \left((w)_{i,j}^{(22)}\right)^2 + 2\left((w)_{i,j}^{(12)}\right)^2\right)^{1/2}. \end{aligned} \tag{19}$$

This algorithm is used to solve the minimisation problem 14. It is taken from the paper Bredies, 2014, page ....

0.3 Why Projection is dividing by matrix instead of a scalar

Given matrix a, we need to find matrix u such that

$$u = \arg\min_{||v||_\infty \leq 1} ||v - a||_V \tag{20}$$

Solution:

$$||v||_V = \sum_{i,j} \left(\left(v_{ij}^{(1)}\right)^2 + \left(v_{ij}^{(2)}\right)^2 \right)^{1/2} \tag{21}$$

Constraints:

$$||v||_\infty \leq 1 \tag{22}$$

or equivalently

$$\max_{i,j} \left(\left(v_{ij}^{(1)}\right)^2 + \left(v_{ij}^{(2)}\right)^2 \right)^{1/2} \leq 1 \qquad \forall i, j \tag{23}$$

$$u = \frac{a}{\max(1, ||a||_\infty)} \tag{24}$$

$$((v_{ij}^{(1)})^2 + (v_{ij}^{(2)})^2)^{1/2} \leq 1 \quad \forall i, j \tag{25}$$

Tc là ta có $n \times n$ constraints, các constraints này ch áp dng cho 1 v trí v_{ij}, independent vi các v trí khác. Do ó: bài toán optimisation ln có th chia thành n^2 bài toán nh:

$$u_{ij} = \arg \min_{t_{ij} \leq 1} ||v_{ij} - a_{ij}|| \quad \forall i, j \tag{26}$$

Denote $v_{ij} = x$, $a_{ij} = y$, $u_{ij} = z$:

$$z = \arg \min_{||x||_2 \leq 1} ||x - y|| \qquad\qquad x, y, z \in \mathbb{R}^2 \tag{27}$$

Thus

$$z = \frac{y}{||y||_2} \tag{28}$$

Áp dng ngc li cho u và a:

$$u = \left(\frac{a_{ij}}{||a_{ij}||_2} \right) \text{if} ||y||_2 > 1 \tag{29}$$

$$u = \left(\frac{a_{ij}}{||a_{ij}||_2} \right) \text{if} ||y||_2 > 1 \tag{30}$$

...

$$= \frac{y}{\max(1, ||y||_2)} \tag{31}$$

Ap dung cho u va a, ta co:

$$u_{ij} = \frac{a_{ij}}{\max(1, ||a_{ij}||_2)} \tag{32}$$

References

Bredies, K. (2014). Recovering piecewise smooth multichannel images by minimization of convex functionals with total generalized variation penalty. In A. Bruhn, T. Pock, & X.-C. Tai (Eds.), *Efficient algorithms for global optimization methods in computer vision* (pp. 44–77). Springer Berlin Heidelberg.