Gorenstein Rings

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May 2018

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Content

Origin

- Grothendieck introduced the notion of Gorenstein variety in algebraic geometry.
- Serre made a remark that rings of finite injective dimension are just Gorenstein rings. The remark can be found in [9].
- Gorenstein rings have now become a popular notion in commutative algebra and given birth to several definitions such as nearly Gorenstein rings or almost Gorenstein rings.

Aim of the Thesis

This thesis aims to

1. present basic results on the minimal injective resolution of a module over a Noetherian ring,
2. introduce Gorenstein rings via Bass number and
3. answer elementary questions when one inspects a type of ring (e.g. Is a subring of a Gorenstein ring Gorenstein?).

Structure of Minimal Injective Resolution

Unless otherwise specified, let R be a Noetherian commutative ring with $1 \neq 0$ and M be an R-module.

Theorem (E. Matlis)

Let E be a nonzero injective R-module. Then we have a direct sum decomposition $E = \oplus_{i \in I} X_i$ in which for each $i \in I$, $X_i \cong \mathrm{E}_R(R/P)$ for some $P \in \mathrm{Spec}(R)$. For each $Q \in \mathrm{Spec}(R)$, we set

$$\Lambda(Q, E) = \{X_i | I \in I, X_i \cong \mathrm{E}_R(R/Q)\}.$$

Definition

Let $i \in \mathbb{Z}$ and $Q \in \mathrm{Spec}(R)$. We set

$$\mu^i(Q, M) = \dim_{R_Q/QR_Q} \mathrm{Ext}^i_{R_Q}(R_Q/QR_Q, M_Q)$$

and call it the ***i-th Bass number of*** *M.*

Structure of Minimal Injective Resolution

Theorem

Let $i \in \mathbb{Z}$, $Q \in \operatorname{Spec}(R)$ *and*

$$0 \to M \xrightarrow{\partial_0} \mathrm{E}_R^0(M) \xrightarrow{\partial_1} \mathrm{E}_R^1(M) \to \cdots \to \mathrm{E}_R^i(M) \xrightarrow{\partial_{i+1}} \mathrm{E}_R^{i+1}(M) \to \cdots$$

be a minimal injective resolution of M*. Then* $\mu^i(Q, M)$ *is equal to the cardinality of the* R*-modules of the form* $\mathrm{E}_R(R/Q)$ *which appear in* $\mathrm{E}_R^i(M)$ *as direct summands, that is* $\mu^i(Q, M) = |\Lambda(Q, \mathrm{E}_R^i(M))|$*. Therefore,*

$$\mathrm{E}_R^i(M) = \oplus_{P \in \operatorname{Spec}(R)} \mathrm{E}_R(R/P)^{\mu^i(P,M)}.$$

A Small Remark

Proposition

If R is local and $\mathrm{id}_R(R) < \infty$, *then*

$$\mathrm{id}_R(R) = \dim(R) = \mathrm{depth}_R(R).$$

Gorenstein Rings

Definition

*Suppose that R is local. R is **Gorenstein** if $\mathrm{id}_R(R) < \infty$. Generally, R is **Gorenstein** if R_P is Gorenstein for each $P \in \mathrm{Spec}(R)$.*

A question naturally arises: Are Gorenstein property and finite injective dimension equivalent? As a matter of fact, we have the following (Bass proved it in [9]).

Proposition

$\mathrm{id}_R(R) < \infty$ *if and only if R is Gorenstein and* $\dim(R) < \infty$.

Gorenstein Rings and Regular Sequence

Proposition

Let $(R, \mathfrak{m})$ be a local ring and $f_1, \ldots, f_t$ be an R-regular sequence. Then R is Gorenstein if and only if $R/(f_1, \ldots, f_t)R$ is Gorenstein.

Proposition (new?)

Let $f_1, \cdots, f_t$ be an R-regular sequence and $f_i \in \mathrm{Jac}(R)$ for all i. Then R is Gorenstein if and only if $R/(f_1, \ldots, f_t)R$ is Gorenstein.

Gorenstein Rings and Bass Number

Theorem

Let $(R, \mathfrak{m})$ be a local ring and set $d = \dim(R)$. TFAE:

1. *R is Gorenstein.*
2. $\mu^i(\mathfrak{m}, R) = 0$ *for some* $i > d$.
3. $\mu^i(\mathfrak{m}, R) = 0$ *for every* $i > d$.
4. $\mu^i(\mathfrak{m}, R) = \begin{cases} 1 & \text{if } i = d \\ 0 & \text{otherwise} \end{cases}.$
5. $\mu^i(Q, R) = \begin{cases} 1 & \text{if } i = \dim(R_Q) \\ 0 & \text{otherwise} \end{cases}$ *for every* $i \in \mathbb{Z}$ *and* $Q \in \operatorname{Spec}(R)$.
6. $\mu^i(\mathfrak{m}, R) = 0$ *for every* $i < d$ *and* $\mu^d(\mathfrak{m}, R) = 1$.

Gorenstein Rings of Zero Krull Dimension

Theorem

Let $(R, \mathfrak{m})$ *be local and suppose that* $\dim(R) = 0$ *(equivalently,* $l_R(R) < \infty$*). TFAE:*

1. *R is Gorenstein.*
2. *0 is irreducible in R, that is if $0 = I \cap J$ for some ideals I and J of R, then $I = 0$ or $J = 0$.*
3. $l_R((0 :_R \mathfrak{m})) = 1$.

Gorenstein Rings and Other Types of Ring

Definition

Suppose $(R, \mathfrak{m})$ *is local.* R *is* ***regular*** *if* $\mathfrak{m}$ *can be generated by* $\dim(R)$ *elements.*
Generally, R *is* ***regular*** *if* R_P *is a regular for each* $P \in \operatorname{Spec}(R)$.

Definition

Suppose R *is local.* R *is* ***Cohen-Macaulay*** *if* $\operatorname{depth}_R(R) = \dim(R)$.
Generally, R *is* ***Cohen-Macaulay*** *if* R_P *is Cohen-Macaulay for each* $P \in \operatorname{Spec}(R)$.

Proposition

regular rings $\subset$ *Gorenstein rings* $\subset$ *Cohen-Macaulay rings.*

Examples

Example

Let k be a field.

1. *The formal power series ring $k[[x_1, \ldots, x_n]]$ and the polynomial ring $k[x_1, \ldots, x_n]$ are regular.*
2. $k[[x, y]]/(xy)$ *is Gorenstein but not regular.*
3. $k[[x, y]]/(x^2, xy, y^2)$ *is Cohen-Macaulay but not Gorenstein.*
4. $k[[x, y]]/(x^2, xy)$ *is not even Cohen-Macaulay.*
5. *A quotient of a Gorenstein ring is not necessarily Gorenstein:* $k[[x, y, z]]/(x^3 - z^2, y^2 - xz, z^3)$ *is Gorenstein but* $k[[x, y, z]]/(x, y, z)^2$ *is not.*
6. *A subring of a Gorenstein ring is not necessarily Gorenstein: for* $a \geq 3$, $k[[t^a, t^{a+1}, \ldots, t^{2a-2}]]$ *is Gorenstein while* $k[[t^a, t^{a+1}, \ldots, t^{2a-1}]]$ *is not.*

Examples I

Example

7. *Finite direct product of Gorenstein rings is Gorenstein.*
8. *Nagata ([13]) constructed the following ring. Let* $R = k[x_1, x_2, \ldots]$. *We set* $I_1 = \{1\}$ *and* $I_n = \{1 + n(n-1)/2, \ldots, n(n+1)/2\}$ *for each* $n \geq 2$. *Let* $P_i = (x_j | j \in I_i)$ *be prime ideals of* R. *Set* $S = R \setminus \cup_{i \geq 1} P_i$. *Then the ring* $S^{-1}R$ *is Noetherian and has infinite Krull dimension. It is in fact regular and hence it is Gorenstein and has infinite injective dimension.*

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