NOTES ON COHEN-MACAULAY RINGS

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Definition 1. A Noetherian local ring R is called **Cohen-Macaulay** if $\operatorname{depth}(R) = \dim(R)$. Generally, a Noetherian ring R is **Cohen-Macaulay** if R_P is Cohen-Macaulay for all $P \in \operatorname{Spec}(R)$ (or equivalently, for all $P \in \operatorname{Max}(R)$).

Example 1.
(1) Every 0-dimensional Noetherian ring is Cohen-Macaulay such as $k[x, y]/(x^2, xy, y^2)$.
(2) Every 1-dimensional Noetherian reduced ring is Cohen-Macaulay since in reduced rings, the set of all zero-divisors is the union of minimal prime ideals. Particularly, 1-dimensional Noetherian domains are Cohen-Macaulay.
(3) $R = k[x, y]/(x^2, xy)$ is not Cohen-Macaulay since $\dim(R) = 1$ and $\operatorname{depth}(R) = 0$. This is because by definition of depth, we have for a local ring R, $\operatorname{depth}(R) = 0$ iff the maximal ideal $\mathfrak{m} \in \operatorname{Ass}_R(R)$.

Proposition 1. Let R be a Noetherian local ring and $f \in \mathfrak{m}$ be regular. Then R/fR is Cohen-Macaulay iff R is Cohen-Macaulay.

Proof. $\dim(R/fR) = \dim(R) - 1$ and $\operatorname{depth}(R/fR) = \operatorname{depth}(R) - 1$. $\square$

This is an important property since in many results, we can reduce R to R/fR, a ring of smaller dimension.

Proposition 2. Let R be a Cohen-Macaulay Noetherian local ring of dimension n and let $P \in \operatorname{Spec}(R)$. Then R_P is Cohen-Macaulay and

$$\dim(R) = \dim(R/P) + \dim(R_P).$$

Hence R is equidimensional and $\operatorname{Ass}_R(R) = \operatorname{Min}(R) = \{Q \in \operatorname{Spec}(R) | \dim(R/Q) = n\}$.

Proof. It suffices to show that every maximal chain of prime ideals

$$Q = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{n-1} \subsetneq Q_n = \mathfrak{m}$$

has the same length n. Suppose $n > 0$ and the claim holds for $n - 1$. Let $P = Q_{n-1}$. Then $\dim(R/P) = 1$ and R_P is Cohen-Macaulay with $\dim(R_P) = n - 1$. Therefore the chain from Q_0 to P has the same length $n - 1$. $\square$

Proposition 3. Let R be a Noetherian local ring and $f \in \mathfrak{m}$. Then $\dim(R/fR) = dim(R) - 1$ iff $f \notin Q$ for all $Q \in \operatorname{Spec}(R)$ s.t. $\dim(R/Q) = \dim(R)$.

Proof. If $f \in Q$ for some $Q \in \mathrm{Spec}(R)$ s.t. $\dim(R/Q) = \dim(R)$, then $Q \in \mathrm{Spec}(R) \cap V(fR) = \mathrm{Spec}(R/fR)$. Therefore $\dim(R/fR) \geq \dim(R)$. Conversely if $\dim(R/fR) = \dim(R)$, then we have a chain of prime ideals s.t. f is inside the minimal prime ideal. □

Theorem 1. Let R be a Noetherian local ring of dimension n. TFAE.

(1) R is Cohen-Macaulay.

(2) Every system of parameters of R is a regular sequence.

(3) There is a system of parameters of R that forms a regular sequence.

Proof. It suffices to show (1) $\implies$ (2). Let $(f_1, ..., f_n)$ be a system of parameters of R. Then $f_1 \notin Q$ for all $Q \in \mathrm{Spec}(R)$ s.t. $\dim(R/Q) = \dim(R)$. By Proposition 2, $f + 1 \notin Q$ for all $Q \in \mathrm{Ass}(R)$, that is f_1 is regular. Then $\dim(R/f_1R) = \dim(R) - 1 = \mathrm{depth}(R/f_1R)$. By induction we have the complete proof. □

Theorem 2. A regular Noetherian local ring R is Cohen-Macaulay.

Proof. Let $(f_1, ..., f_n)$ be a regular system of parameters of R. Then for all $1 \leq t \leq n$, we have $\dim(R/(f_1, ..., f_t)) = n - t$ and its maximal ideal is generated by $n - t$ elements $(f_{t+1}, ..., f_n)$. Hence $R/(f_1, ..., f_t)$ is regular and hence an integral domain. Therefore f_{t+1} is a nonzero-divisor of this ring for all t. □

Theorem 3. Let R be a Noetherian ring. Then $R[x]$ is Cohen-Macaulay iff R is Cohen-Macaulay.

Proof. Let $P \in \mathrm{Max}(R[x])$ and set $Q = P \cap R \in \mathrm{Max}(R)$. Since $R \backslash Q \subseteq R[x] \backslash P$, we have $R[x]_P = R_Q[x]_P$. So we can assume R is local with maximal ideal Q. The ring $R[x]/Q[x] = (R/Q)[x]$ is a PID, so $P = (Q, f)$ for some polynomial f. Since $R[x]$ is a free R-module, if $(x_1, ..., x_n)$ is an R-regular sequence, it is also an $R[x]$-regular sequence. Moreover, let $I = (x_1, ..., x_n)$, then f is a nonzero-divisor of $R[x]/I[x] = (R/I)[x]$ because of the same argument . Therefore $\mathrm{depth}(P) \geq 1 + \mathrm{depth}(Q)$.

On the other hand, since $P/Q[x] = (f)$, by KPIT, $\dim(P) - \dim(Q) \leq \dim(P) - \dim(Q[x]) \leq 1$. Since $\dim(Q) = \mathrm{depth}(Q)$, we have $\dim(P) = \mathrm{depth}(P)$ and hence $R[x]$ is Cohen-Macaulay. □

Now we move on to Cohen-Macaulay property and its relation to Noether Normalization Theorem.

Theorem 4. (**Noether Normalization Theorem**) Let R be an $\mathbb{N}$-graded ring and finitely generated as a k-algebra. If $(x_1, ..., x_)$ is a homogeneous system of parameters for R, then $x_1, ..., x_d$ are algebraically independent over k, and R is finitely generated as a $k[x_1, ..., x_d]$-module.

Theorem 5. Let R be an $\mathbb{N}$-graded ring and finitely generated as a k-algebra. If $(x_1, ..., x_)$ is a homogeneous system of parameters for R, then R is Cohen-Macaulay if and only if it is free over $k[x_1, ..., x_d]$.

Proof. By Hilbert's syzygy theorem, R has finite projective dimension over $A = k[x_1, ..., x_d]$. By Auslander-Buchsbaum theorem,

$$\mathrm{depth}(R) + \mathrm{pd}_A(R) = \mathrm{depth}(A).$$

Hence R is Cohen-Macaulay if and only if $pd_A(R) = 0$, i.e. R is a projective A module. Since R is a finitely generated graded A-module, it is projective if and only if it is free (Quillen-Suslin Theorem). $\square$

Example 2. (1) $R = k[x^d, x^{d-1}y, ..., xy^{d-1}, y^d]$. Since x^d, y^d is a system of parameters for R, by NNT, R is finitely generated over $A = k[x^d, y^d]$. It is straightforward to check that $1, x^iy^j$ in which $1 \leq i, j \leq d-1$ is a basis of R over A using module d. Therefore R is Cohen-Macaulay.

(2) $R = k[x^4, x^3y, xy^3, y^4], A = k[x^4, y^4]$. The monomials $1, x^3y, xy^3, x^6y^2, x^2y^6$ are a minimal generating set for R as an A-module. But $y^4(x^6y^2) = x^4(x^2y^6)$, hence R is not Cohen-Macaulay.

Theorem 6. Let T be a polynomial ring over a field k and G a finite grou acting on T by degree preserving k-algebra automorphisms. If $|G|$ is invertible in k, then T^G is Cohen-Macaulay.

Proof. We can well define the map $\rho : T \to T^G$ as follows

$$\rho(t) = \frac{1}{|G|} \sum_{g \in G} gt.$$

We have $\rho(t) = t$ for all $t \in T^G$ and hence ρ is a T^G-module homomorphism. We also have the embedding $T^G \to T$. So the map ρ splits and hence T^G is a direct summand of T as a T^G-module.

Let x be a homogeneous system of parameters for T^G . Since G is finite, by Noether Theorem, T is an integral extension of T^G, so x is a system of parameters for T as well. The ring T is Cohen-Macaulay, so x is a regular sequence on T. But then it is also a regular sequence on its direct summand T^G . $\square$

Example 3. In the first part of Example 2, we are actually dealing with R^G in which $G = \langle a | a^n = 1 \rangle$ that acts as follows: $ax = ax, ay = ay$.