NOTE ON CW COMPLEXES

1. F-projective n-space

Definition 1.1. Let F be a division ring and let $n \geq 0$. The quotient set $(F^{n+1} - \{0\})/\sim$, in which $\sim$ is an equivalence relation such that $x \sim y$ iff $x = \lambda y$ for some $\lambda \in F - \{0\}$, is called **F-projective n-space** and is denoted by FP^n.

There are three division rings which would be considered: the reals $\mathbb{R}$, the complexes $\mathbb{C}$ and the quaternions $\mathbb{H}$.

Theorem 1.2. For every $n \geq 0$, $\mathbb{R}P^n$, $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are compact Hausdorff.

2. Attaching Cells

Definition 2.1. Let X and Y be spaces, let A be a closed subset of X and let $f : A \to Y$ be continuous. The space obtained from Y by **attaching** X via f is $(X \coprod Y)/\sim$, in which $\sim$ is the equivalence relation generated by the binary relation $a \sim f(a)$ ($a \in X \coprod Y$). This space is denoted by $X \coprod_f Y$. The map f is called the **attaching map**.

Definition 2.2. The map $\Phi : X \to X \coprod_f Y$, which is the composition of the embedding $X \to X \coprod Y$ and the natural projection $X \coprod Y \to X \coprod_f Y$, is called the **characteristic map**.

Definition 2.3. An n-**cell** $\mathbf{e^n}$ (or simply $\mathbf{e}$) is a homeomorphic copy of the open n-disk $D^n - S^{n-1} \cong \mathbb{R}^n$.

Definition 2.4. Let Y be a Hausdorff space and let $f : S^{n-1} \to Y$ be continuous. Then $D^n \coprod_f Y$ is called the space obtained from Y by **attaching an** n-**cell via f**, and is denoted by Y_f.

Theorem 2.5. (1) $\Phi : (D^n, S^{n-1}) \to (Y_f, Y)$ is a function of pairs.
(2) Y_f is a Hausdorff space, which is compact if Y is.
(3) $\Phi|_{S^{n-1}}$ is the attaching map f.

Definition 2.6. A continuous map $g : (X, A) \to (Y, B)$ is a **relative homeomorphism** if $g|_{X-A} : X - A \to Y - B$ is a homeomorphism.

Theorem 2.7. Let Z be a compact Hausdorff space, let Y be a closed subset of Z and let e be an n-cell in Z with $e \cap Y = \emptyset$. If there is a relative homeomorphism $\Phi : (D^n, S^{n-1}) \to (e \cup Y, Y)$, then $Y_f \cong e \cup Y$.

Theorem 2.8.
(1) For each $n \geq 1$, $\mathbb{R}P^n$ is obtained from $\mathbb{R}P^{n-1}$ by attaching an n-cell; moreover, there is a disjoint union
$$\mathbb{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n.$$
(2) For each $n \geq 1$, $\mathbb{C}P^n$ is obtained from $\mathbb{C}P^{n-1}$ by attaching a $2n$-cell; moreover, there is a disjoint union
$$\mathbb{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}.$$
(3) For each $n \geq 1$, $\mathbb{H}P^n$ is obtained from $\mathbb{H}P^{n-1}$ by attaching a $4n$-cell; moreover, there is a disjoint union
$$\mathbb{H}P^n = e^0 \cup e^4 \cup \cdots \cup e^{4n}.$$

3. Homology and Attaching Cells

Theorem 3.1. Let $n \geq 1$. There there is an exact sequence
$$\cdots \to H_p(S^{n-1}) \xrightarrow{f_*} H_P(Y) \xrightarrow{\iota_*} H_p(Y_f) \to H_{p-1}(S^{n-1}) \to \dots$$
$$\cdots \to H_0(S^{n-1}) \to \mathbb{Z} \oplus H_0(Y) \to H_0(Y_f) \to 0.$$

Corollary 3.2. Suppose $n \geq 2$ and Y is a compact Hausdorff space.
(1) If $p \neq n, n-1$, then
$$H_p(Y) \cong H_p(Y_f).$$
(2) There is an exact sequence
$$0 \to H_n(Y) \xrightarrow{\iota_*} H_n(Y_f) \to \mathbb{Z} \xrightarrow{f_*} H_{n-1}(Y) \to H_{n-1}(Y_f);$$
moreover, the last map is surjective if $n \geq 3$.

Remark 3.3. Rotman easily computed homology of $\mathbb{C}P^n$ and $\mathbb{H}P^n$ by induction with above results. Meanwhile, he used more tools just to compute $\mathbb{R}P^2$.

4. CW Complexes

Definition 4.1. Let X be a set and $X = \cup_{j\in J} A_j$ s.t.

(1) Each A_j is a topological space;
(2) For each $j, k \in J$, the topologies of A_j and of A_k agree on $A_j \cap A_k$;
(3) For each $j, k \in J$, the intersection $A_j \cap A_k$ is closed in A_j and A_k.

Then the **weak topology** on X **determined** by $\{A_j | j \in J\}$ is the topology whose closed sets are those subsets F s.t. $F \cap A_j$ is closed in A_j for all $j \in J$.

Notice that A_j is a closed subset of X under weak topology. Hence, if J is finite, there exists only one topology on X satisfying the three conditions and it must be the weak topology.

Definition 4.2. Assume that a topological space X í a disjoint union of cells: $X = \cup\{e | e \in E\}$. For each $k \geq 0$, the k-**skeleton** $X^{(k)}$ if X is defined by

$$X^{(k)} = \cup\{e \in E | \dim(e) \leq k\}.$$

Definition 4.3. A **CW complex** is an ordered triple (X, E, Φ), where X is a Hausdorff space, E is a family of cells in X and $\Phi = \{\Phi_e | e \in E\}$ s.t.

(1) $X = \cup\{e | e \in E\}$ (disjoint union);
(2) For each k-cell $e \in E$, the map $\Phi_e : (D^k, S^{k-1}) \to (e \cup X^{(k-1)}, X^{(k-1)})$ is a relative homeomorphism;
(3) If $e \in E$, then its closure $\bar{e}$ is contained in a finite union of cells of E (this is called **closure finiteness**);
(4) X has the weak topology determined by $\{\bar{e} | e \in E\}$.

In particular, X is called a **CW-space**, (E, Φ) is called a **CW decomposition** of X and $\Phi_e \in \Phi$ is called the **characteristic map** of e.

Theorem 4.4. Let X be a space, and let

$$X^0 \subset X^1 \subset X^2 \subset \dots$$

be a sequence of subsets with $X = \cup_{n\geq 0} X^n$. Assume the following:

(1) X^0 is discrete;

(2) For each $n > 0$, there is an index set A_n and a family of continuous functions $\{f_\alpha^{n-1} : S^{n-1} \to X^{n-1} | \alpha \in A_n\}$ so that
$$X^n = (\coprod_\alpha D^n) \coprod_f X^{n-1},$$
where $f = \coprod f_\alpha^{n-1}$;

(3) X has the weak topology determined by $\{X^n | n \geq 0\}$.

If Φ_α^n denotes the composite
$$D^n \to \coprod_\alpha D^n \to (\coprod_\alpha D^n) \coprod X^{n-1} \to (\coprod_\alpha D^n) \coprod_f X^{n-1},$$
then (X, E, Φ) is a CW complex, where
$$E = X^0 \cup (\cup_{n\geq 1} \{\Phi_\alpha^n(D^n - S^{n-1}) | \alpha \in A_n\})$$
and
$$\Phi = \{\text{constant maps to } X^0\} \cup (\cup_{n \geq 1}\{\Phi_\alpha^n | \alpha \in A_n\}).$$
Notice that the converse of this theorem is also true.

Definition 4.5. Let (X, E, Φ) be a CW complex. If $E' \subset E$, define
$$|E'| = \cup\{e | e \in E'\} \subset X,$$
and define $\Phi' = \{\Phi_e | e \in E'\}$. Call $(|E'|, E', \Phi')$ a **CW subcomplex** if $\bar{e} \subset |E'|$ for every $e \in E'$. Note that a CW subcomplex is itself a CW complex.

Theorem 4.6. Let (X, E) be a CW complex and E' be a subset of E. Then $|E'|$ is a CW subcomplex if and only if $|E'|$ is closed.

Corollary 4.7. Let (X, E) be a CW complex, $n > 0$ and $E' \subset E$. Then

(1) $|E'| \cup X^{(n-1)}$ is closed in X;
(2) Every n-skeleton $X^{(n)}$ is closed in X;
(3) Every n-cell e is open in $X^{(n)}$;
(4) $X^{(n)} - X^{(n-1)}$ is open in $X^{(n)}$.

Theorem 4.8. Let X be a CW complex.

(1) Every path component of X is a CW subcomplex;
(2) The path components of X are closed and open;
(3) The path components of X are the components of X;
(4) X is connected if and only if X is path connected.

Theorem 4.9. Every CW complex X is locally path connected.

Theorem 4.10. Every CW complex X is paracompact and perfectly normal (T_6).

Theorem 4.11. If (X, E, Φ) is a CW complex, (Y, E', Φ') is its CW subcomplex and $v : X \to X/Y$ be the natural map, then X/Y is a CW complex constructed as follows:

$$(X/Y)^0 = \{v(e)|e \in E_0 - E'_0\} \cup \{v(Y)\},$$
$$(X/Y)^n = \{v(e)|e \in E_n - E'_n\}$$

for $n > 0$. The characteristic map of $v(e)$ is the composite $v\Phi_e$.

Theorem 4.12. (**Homotopy Extension Theorem**) Let X be a CW complex, Y be a CW subcomplex and Z be a space. For every continuous $f : X \to Z$ and every homotopy $h : Y \times [0, 1] \to Z$ with $h(y, 0) = f(y)$ for all $y \in Y$, there exists a (extending) homotopy $H : X \times [0, 1] \to Z$ with

$$H(y, t) = h(y, t) \text{ for all } (y, t) \in Y \times [0, 1]$$

and

$$H(x, 0) = f(x) \text{ for all } x \in X.$$

5. Cellular Homology

Definition 5.1. A **filtration** of a topological space X is a sequence of subspaces $\{X_n|n \in \mathbb{Z}\}$ with $X^n \subset X^{n+1}$ for all n. A filtration is **cellular** if:

1. $H_p(X^n, X^{n-1}) = 0$ for all $p \neq n$;
2. For every $m \geq 0$ and every continuous $\sigma : \Delta^m \to X$, there is an integer n with $Im(\sigma) \subset X^n$.

Definition 5.2. A **cellular space** is a topological space with a cellular filtration. If X and Y are cellular spaces, then a **cellular map** is a continuous function $f : X \to Y$ with $f(X^n) \subset Y^n$ for all $n \in \mathbb{Z}$.

Definition 5.3. If X is a cellular space and $k \geq 0$, define

$$W_k(X) = H_k(X^k, X^{k-1})$$

and define $d_k : W_k(X) \to W_{k-1}(X)$ to be the composite (of connecting homomorphism and inclusion ???)

$$H_k(X^k, X^{k-1}) \to H_{k-1}(X^{k-1}) \to H_{k-1}(X^{k-1}, X^{k-2}).$$

Lemma 5.4. $(W_*(X), d)$ is a chain complex, called the **cellular chain complex**.

Definition 5.5. Let X be a CW complex with CW subcomplex Y. Define

$$X_Y^k = X^{(k)} \cup Y.$$

$W_*(X, Y)$ is the chain complex determined by the filtration of X by X_Y^k, that is $W_k(X, Y) = H_k(X_Y^k, X_Y^{k-1})$.

Theorem 5.6. Let X be a CW complex with CW subcomplex Y.

(1) The filtration of X by X_Y^k is a cellular filtration.
(2) There are isomorphisms for all $k \geq 0$,

$$H_k(W_*(X, Y)) \cong H_k(X, Y).$$

Theorem 5.7. Let (X, E) be a CW complex with CW subcomplex (Y, E'). For each $k \geq 0$,

$$W_k(X, Y) \cong \mathbb{Z}^{r_k}$$

in which r_k is the number of k-cells in $E - E'$ (possibly infinite).

Definition 5.8. A CW complex (X, E, Φ) is **finite** if E is a finite set.

Theorem 5.9. If (X, E) is a CW complex, then X is compact if and only if E is finite for every CW decomposition E.

Theorem 5.10. (Corollary of Theorem 5.7) If X is a compact CW complex of dimension m, then

(1) $H_p(X)$ is f.g. for every $p \geq 0$;
(2) $H_p(X) = 0$ for all $p > m$;
(3) $H_m(X)$ is free abelian.

Remark 5.11. With Theorem 5.7, we can compute $H_k(\mathbb{C}P^n)$ and $H_k(\mathbb{H}P^n)$ instantly. In the case of $\mathbb{R}P^n$, we need to inspect the differentiations explicitly.

Theorem 5.12. If n is odd, then

$$H_p(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } p = 0, n \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \text{ is odd and } 0 < p < n \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, then

$$H_p(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } p = 0 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } p \text{ is odd and } 0 < p < n \\ 0 & \text{otherwise} \end{cases}$$

Idea: Let $v : S^n \to \mathbb{R}P^n$ be the usual identificaion, which identifies the antipodal points. We have the commutative diagram

$$\begin{tikzcd} W_k(S^n) \arrow[r, "d_k"] \arrow[d, "v_*"] & W_{k-1}(S^n) \arrow[d, "v_*"] \\ W_k(\mathbb{R}P^n) \arrow[r, "D_k"] & W_{k-1}(\mathbb{R}P^n) \end{tikzcd}$$

We computed $W_k(\mathbb{R}P^n) = \mathbb{Z}$, hence we only care about a generator a. We can prove that $D_k(a) = \pm(1 + (-1)^k)a$. Let W_k denote $W_k(\mathbb{R}P^n)$, we have

$$\begin{tikzcd} 0 \arrow[r] & W_n \arrow[r, "D_n"] \arrow[d, equal] & W_{n-1} \arrow[r] \arrow[d, equal] & \dots \arrow[r] & W_3 \arrow[r, "D_3"] \arrow[d, equal] & W_2 \arrow[r, "D_2"] \arrow[d, equal] & W_1 \arrow[r, "D_1"] \arrow[d, equal] & W_0 \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "1+(-1)^n"] & \mathbb{Z} \arrow[r] & \dots \arrow[r] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "2"] & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Theorem 5.13. For $i = 1, 2$, let X_i be a CW complex with CW subcomplex Y_i; let $f : (X_1, Y_1) \to (X_2, Y_2)$ be a continuous map of pairs that induces a homeomorphism $\bar{f} : X_1/Y_1 \to X_2 \to Y_2$. Then f induces isomorphisms for all $k \geq 0$

$$H_k(X_1, Y_1) \cong H_k(X_2, Y_2).$$

Corollary 5.14. (Excision) If X is a CW complex and Y_1, Y_2 are CW subcomplexes with $X = Y_1 \cup Y_2$, then

$$H_k(Y_1, Y_1 \cap Y_2) \cong H_k(X, Y_2).$$

Definition 5.15. Let (X, E) be a finite CW complex and let α_i denote the number of i-cells in E. The **Euler characteristics** of (X, E) is

$$\chi(X) = \sum_{i \geq 0} (-1)^i \alpha_i.$$

Example 5.16. $\chi(\mathbb{C}P^n) = \chi(\mathbb{H}P^n) = n + 1$, $\chi(\mathbb{R}P^n) = (1 + (-1)^n)/2$.