Auslander-Buchsbaum Theorem

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Content

Auslander-Buchsbaum Theorem

Theorem

Let (R, m) be a Noetherian local ring and let M be a non-zero finitely generated R-module. If $pd_R M < \infty$, then

$$depth R = pd_R M + depth_R M$$

Motivation

The formula is a powerful tool.

Theorem

Every regular local ring is a UFD.

- In 1958, Nagata proved in [2] that if every regular local ring of dimension 3 is a UFD, then every regular local ring is a UFD.
- In 1959, Auslander and Buchsbaum proved in [3] that every regular local ring of dimension 3 is a UFD, using the Auslander-Buchsbaum Theorem.

Proof Sketch of Auslander-Buchsbaum Theorem

Let $t = depthR, n = pd_R M$ and $s = depth_R M$

Claim

1. *If $n = 0$, then $t = n + s$.*
2. *If $t = 0$, then $t = n + s$.*

Proof:

1. If $n = 0$, then $t = depthR = depth_R(M) = n + s$ since M is free.
2. If $t = 0$, then $m \in Ass_R(R)$ which implies $n = 0$ as $n < \infty$.

$\square$

Proof Sketch of Auslander-Buchsbaum Theorem

Back to the Theorem, we will proceed by induction on $t + n$.

- $t + n = 0$: then $n = 0$.
- Suppose for every Noetherian local ring (R_0, m_0) and nonzero finitely generated R_0-module M_0 such that $depth R_0 + pd_{R_0}(M_0) < n + t$, we have

$$depth R_0 = pd_{R_0}(M_0) + depth_{R_0}(M_0)$$

- $t + n \geq 1$:
  1. $t = 0$.
  2. $n = 0$.
  3. $t > 0, n > 0, s > 0$.
  4. $t > 0, n > 0, s = 0$.

Proof Sketch of Auslander-Buchsbaum Theorem

3. $t > 0, n > 0, s > 0$: $depth R > 0 \implies m \notin Ass_R(R)$. Similarly $m \notin Ass_R(M)$. Choose an element

$$f \in m \setminus \bigcup_{Ass_R(M) \cup Ass_R(R)} P$$

then f is both R-regular and M-regular.
Since

$$depth(R/fR) + pd_{R/fR}(M/fM) = (t-1) + n < t + n$$

then by the induction hypothesis,

$$\begin{aligned} depth(R/fR) &= pd_{R/fR}(M/fM) + depth_{R/fR}(M/fM) \\ \Rightarrow (depth R - 1) &= pd_R(M) + (depth_R(M) - 1) \\ \Rightarrow depth R &= pd_R(M) + depth_R(M) \\ \Rightarrow t &= n + s \end{aligned} \tag{1}$$

Proof Sketch of Auslander-Buchsbaum Theorem

4. $t > 0, n > 0, s = 0$: Let the short exact sequence

$$0 \to L \to F_0 \to M \to 0$$

be the initial part of the minimal free resolution of M.
Since

$$depth R + pd_R(L) = t + (n-1) < t + n$$

then by the induction hypothesis,

$$\begin{aligned} depth(R) &= pd_R(L) + depth_R(L) \\ \Rightarrow depth R &= (pd_R(M) - 1) + (depth_R(M) + 1) \\ \Rightarrow depth R &= pd_R(M) + depth_R(M) \\ \Rightarrow t &= n + s \end{aligned} \tag{2}$$

The proof is complete.

Application

Exercise

Let R be a Cohen-Macaulay local ring of dimension d. Then R is a regular local ring if and only if every Cohen-Macaulay R-module of dimension d is free.

Proof:

1. ($\Rightarrow$) Let M be a Cohen-Macaulay R-module of dimension d. Since R is a RLR, we deduce $pd_R M \leq gldim R < \infty$ (Serre's Theorem).
   So we can apply the Auslander-Buchsbaum theorem:

$$pd_R M = depth R - depth_R M = d - d = 0$$

Hence, M is free.

$\square$

Application

2. ($\Leftarrow$) By Serre's Theorem, we only need to prove $pd_R(M) < \infty$ for all finitely generated R-module M.
   - Assume $depth_R(M) = depth R$. We have the inequality

     $$d = depth R = depth_R(M) \leq dim_R(M) \leq dim R = d$$

     We deduce $dim_R(M) = depth_R(M) = d$. Due to the hypothesis, M is free.
   - Otherwise $depth_R(M) < depth R$, consider the exact sequence

     $$0 \to K_0 \to R^{n_0} \to M \to 0$$

     By Depth Lemma, $depth_R(K_0) = depth_R(M) + 1$. If necessary, we repeat the process until we receive $depth_R(K_s) = depth(R)$, which would imply K_s is free with the first situation. Then

     $$0 \to K_s \to R^{n_s} \to \ldots \to R^{n_0} \to M \to 0$$

     is a free resolution of M. So $pd_R(M) \leq s < \infty$

References

[1] Shiro Goto, *Homological Methods in Commutative Algebra*, Graduate Lecture Series, VIASM 2017.

[2] Nagata, Masayoshi, *A General Theory of Algebraic Geometry over Dedekind Domains II*, Am. J. Math., 80, 382-420 (1958).

[3] Auslander, Maurice; Buchsbaum, D. A. *Unique Factorization in Regular Local Rings*, PNAS, 45 (5), 733-734 (1959).