PROBABILITY

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Random experiment

Definition

A random experiment is a process leading to two or more possible outcomes, without knowing exactly which outcome will occur.

Example:

- A coin is tossed and the outcome is either a head or a tail.
- The number of persons admitted to a hospital emergency room during any hour cannot be known in advance.
- A customer enters a store and either purchases a shirt or does not.

Basic outcomes and sample space

Definition

- The possible outcomes from a random experiment are called the basic outcomes.
- The set of all basic outcomes is called the sample space, denoted by S.

Example: An investor follows the Dow Jones Industrial index. There are two basic outcomes

O_1 : The index is higher than at yesterday's close;

O_2 : The index is not higher than at yesterday's close.

Therefore, the sample space is

$$S = \{O_1, O_2\}$$

Exercise

Ex 1: Let the left hand toss a coin and the right hand rolls a dice. Determine all basic outcomes and the sample space.

Ex 2: Suppose that two letters are to be selected from A, B, C and arranged in order. Determine all basic outcomes and the sample space.

Event

Definition

An event, E, is any subset of basic outcomes from the sample space. An event occurs if the random experiment results in one of its constituent basic outcomes.

- A basic outcome belonging to the event E is said to be satisfying the event E.
- The empty set $\emptyset$ is called a null event.

Example: Roll a dice. Consider the event A: "The outcome is a even number". Then the basic outcomes which satisfy event A are 2, 4 and 6. Therefore,

$$A = \{2, 4, 6\}$$

Intersection of Events

Definition

Let A and B be two events in the sample space S. Their intersection, denoted by $A \cap B$, is the set of all basic outcomes in S that belong to both A and B. Hence, the intersection $A \cap B$ occurs if and only if both A and B occur.

In particular, If the events A and B have no common basic outcomes,

$$A \cap B = \emptyset,$$

they are called mutually exclusive.

Definition

Let A and B be two events in the sample space, S. Their union, denoted by $A \cup B$, is the set of all basic outcomes in S that belong to at least one of these two events. Hence, the union $A \cup B$ occurs if and only if either A or B or both occur.

In particular, given the n events $E_1, E_2, \ldots, E_n$ is the sample space S, if

$$E_1 \cup E_2 \cup \ldots \cup E_n = S,$$

These events are said to be collectively exhaustive.

Complement

Definition

Let A be an event in the sample space S. The set of basic outcomes of a random experiment belonging to S but not to A is called the complement of A and is denoted by $\overline{A}$.

Note that $\overline{A} = S - A$. Moreover, events A and $\overline{A}$ are mutually exclusive, no basic outcome can belong to both, and collectively exhaustive, every basic outcome must belong to one or the other.

Exercise

Ex 1: We designate four basic outcomes for the Dow Jones Industrial average over two consecutive days:

O_1: The Dow Jones average rises on both days.

O_2: The Dow Jones average rises on the first day but does not rise on the second day.

O_3: The Dow Jones average does not rise on the first day but rises on the second day.

O_4: The Dow Jones average does not rise on either day.

We consider these two events:

A: " The Dow Jones average rises on the first day."

B: " The Dow Jones average rises on the second day."

Find the intersection, union, and complement of A and B. (as a sentence and as a subset)

Exercise

Ex 2: The sample space is defined as follows

$$S = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8, E_9, E_{10}\}$$

Given $A = \{E_3, E_5, E_6, E_{10}\}$ and $\overline{B} = \{E_3, E_4, E_6, E_9\}$.

a. What is the intersection of A and B?

b. What is the union of A and B?

c. Is the union of A and B collectively exhaustive?

Exercise

Ex 3: Florin Frenti operates a small, used car lot that has three Mercedes (M_1, M_2, M_3) and two Toyotas (T_1, T_2). Two customers, Cezara and Anda, come to his lot, and each selects a car. The customers do not know each other, and there is no communication between them. Let the events A and B be defined as follows:

- A: The customers select at least one Toyota.
- B: The customers select two cars of the same model.

a. Identify the sample space.

b. Define event A.

c. Define event B.

d. Find the intersection, union, and complement of A and B.

e. Find the intersection and union A and $\overline{B}$

Classical Probability

Definition

Assume that all outcomes in a sample space are equally likely to occur. Then the probability of an event A is defined by

$$P(A) = \frac{N_A}{N}$$

where

- $N =$ the total number of outcomes in the sample space.
- $N_A =$ the number of outcomes that satisfy the condition of event A.

Example

A small computer store has three Hewlett-Packard and two Dell computers in stock. Suppose that Susan Spencer comes to purchase two computers. Any computer on the shelf is equally likely to be selected. What is the probability that Susan will purchase one Hewlett-Packard and one Dell computer?

The sample space S is

$$S = \{H_1D_1, H_1D_2, H_2D_1, H_2D_2, H_3D_1, H_3D_2, H_1H_3, H_2H_3, D_1D_2\}.$$

If A is the event "one Hewlett- Packard and one Dell computer" then

$$A = \{H_1D_1, H_1D_2, H_2D_1, H_2D_2, H_3D_1, H_3D_2\}$$

Hence the probability of event A is

$$P(A) = \frac{N_A}{N} = \frac{6}{10}$$

Ordering, Permutation and Combination

Formulas

- The total number of possible ways of arranging x objects in order is given by
$$x.(x-1).(x-2)\ldots 2.1 = x!$$
- The total number of possible ways of selecting and arranging in order x objects from n is given by
$$P_x^n = \frac{n!}{(n-x)!}$$
- The total number of possible ways of selecting x objects from n is given by
$$C_x^n = \frac{n!}{x!(n-x)!}$$

Example

A small computer store has three Hewlett-Packard and two Dell computers in stock. Suppose that Susan Spencer comes to purchase two computers. Any computer on the shelf is equally likely to be selected. What is the probability that Susan will purchase one Hewlett-Packard and one Dell computer?

Sample space S: 2 computers

$$N = C_2^5 = 10$$

The event A: one Hewlett-Packard and one Dell computer.

$$N_A = C_1^3 . C_1^2 = 6$$

Then

$$P(A) = \frac{N_A}{N} = \frac{6}{10}$$

Exercise

Ex 1: A personnel officer has 20 candidates to fill 3 similar positions. 15 candidates are men, and 5 are women. If, in fact, every combination of candidates is equally likely to be chosen,

a. What is the probability that no women will be hired?

b. What is the probability that no men will be hired?

c. What is the probability that at least one woman will be hired?

Exercise

Ex 2: You are 1 of 7 female candidates auditioning for 2 parts—the heroine and her best friend—in a play. Before the auditions you know nothing of the other candidates, and you assume all candidates have equal chances for the parts.

a. Find the probability that you will be chosen to play the heroine.

b. Find the probability that you will be chosen to play 1 of the 2 parts.

Exercise

Ex 3: There are 3 customers entering a bank with 6 service counters. Calculate the probability to:

a. All 3 customers came to the same counter.

b. Each person goes to a different counter.

c. Two of the three people went to a counter.

Relative Frequency Probability

Definition

The relative frequency probability is the limit of the proportion of times that event A occurs in a large number of trials, n,

$$P(A) = \lim_{n \to +\infty} \frac{n_A}{n} \approx \frac{n_A}{n}$$

where

- n_A is the number of A outcomes.
- n is the number of trials.

Example

Example: There were 54,345 households in City X and that 31,496 had incomes above \$75,000. We computed the probability for event A, "family income greater than \$75,000" as follows:

$$P(A) = \frac{n_A}{n} = \frac{31,496}{54,345} = 0.58 = 58\%$$

Properties

Properties

i. If A is any event in the sample space S, then

$$0 \leq P(A) \leq 1.$$

ii. Let A be an event in S and let O_i denote the basic outcomes. Then

$$P(A) = \sum_{A} P(O_i)$$

iii. $P(S) = 1, P(\emptyset) = 0$

Complement Rule

Rule

Let A be an event and $\overline{A}$ its complement. Then the complement rule is as follows:

$$P(\overline{A}) = 1 - P(A)$$

Addition Rule

Rule

Let A and B be two events. Using the addition rule, the probability of their union is as follows:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

As a corollary,

- $P(A \cap \overline{B}) = P(A) - P(A \cap B)$
- $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Example

Example 1: The probability of A is 0.60, the probability of B is 0.45, and the probability of both is 0.30. What is the probability of either A and B?

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.6 + 0.45 - 0.3 \\ &= 0,75 \end{aligned}$$

Example 2: The probability of A is 0.40, the probability of B is 0.45, and the probability of either is 0.85. What is the probability of both A and B?

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.4 + 0.45 - 0.85 \\ &= 0 \end{aligned}$$

Exercise

Ex 1: A cell phone company found that 65% of all customers want text messaging on their phones, 70% want photo capability, and 40% want both.

a. What is the probability that a customer will want at least one of these?

b. What is the probability that a customer will want text messaging but not photo capability?

c. What is the probability that a customer will only want photo capability?

Ex 2: A customer intends to buy a box of products by randomly taking out 4 products at the same time from the box to check. If there is no more than 1 defective product, then buy a box of products. Calculate the probability that the customer will buy the box. Given that the product box has 20 products, of which 5 are defective.

Example

Example: In the case, there are 3 people including 2 men and 1 woman suspected of being the murderer: Peter, Jack and Mary. Let A denote an event: "Peter is the murderer". It is very easy to see that the probability

$$P(A) = \frac{N_A}{N} = \frac{1}{3}.$$

However, after careful investigation, the police concluded that the murderer must be a man.

Can the probability of "Peter is the killer" change?

Conditional Probability

Definition

Let A and B be two events. The conditional probability of event A, given that event B has occurred, is denoted by $P(A|B)$ and is found to be as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{provided that} \quad P(B) > 0$$

Similarly,

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{provided that} \quad P(A) > 0$$

Multiplication Rule

Rule

Let A and B be two events. Using the multiplication rule of probabilities, the probability of their intersection can be found as

$$P(A \cap B) = P(A|B).P(B)$$

and also as

$$P(A \cap B) = P(B|A).P(A)$$

Exercise

Ex 1: A cell phone company found that 65% of all customers want text messaging on their phones, 70% want photo capability, and 40% want both.

a. Suppose a person wants photo capability. What is the probability that this person will want text messaging?

b. What is the probability that a person who wants text messaging also wants photo capability?

Exercise

Ex 2: A device has 2 parts with probabilities of broken of the first part and second are 0.1; 0.2. The probability that both parts are broken is 0.04. Find probability to:

a. There is at least one part that works well.

b. Both parts work well.

c. Only first part works well.

d. Only one part works well.

e. Part 1 works fine if part 2 is broken.

f. First part works fine if only one part is broken.

Exercise

Ex 3: A lawn-care service makes telephone solicitations, seeking customers for the coming season. A review of the records indicates that 15% of these solicitations produce new customers and that, of these new customers, 80% had used some rival service in the previous year. It is also estimated that, of all solicitation calls made, 60% are to people who had used a rival service the previous year. What is the probability that a call to a person who had used a rival service the previous year will produce a new customer for the lawncare service?

Exercise

Ex 4: An editor may use all, some, or none of three possible strategies to enhance the sales of a book:

a. An expensive prepublication promotion

b. An expensive cover design

c. A bonus for sales representatives who meet predetermined sales levels

In the past, these three strategies have been applied simultaneously to only 2% of the company's books. Twenty percent of the books have had expensive cover designs, and, of these, 80% have had expensive prepublication promotion. A rival editor learns that a new book is to have both an expensive prepublication promotion and an expensive cover design and now wants to know how likely it is that a bonus scheme for sales representatives will be introduced. Compute the probability of interest to the rival editor.

Statistical Independence

Definition

Let A and B be two events. These events are said to be statistically independent if and only if

$$P(A \cap B) = P(A).P(B)$$

It follows that

$$P(A|B) = P(A)(P(B) > 0)$$

$$P(B|A) = P(B)(P(A) > 0)$$

Example

Example 1: (*Check by formula*) Suppose that women obtain 54% of all bachelor's degrees in a particular country and that 20% of all bachelor's degrees are in business. Also, 8% of all bachelor's degrees go to women majoring in business. Are the events "the bachelor's degree holder is a woman" and "the bachelor's degree is in business" statistically independent?

Let A denote the event "the bachelor's degree holder is a woman" and B denote the event "the bachelor's degree is in business." We then have the following:

$$P(A) = 0.54, P(B) = 0.20, P(A \cap B) = 0.08$$

Since

$$P(A)P(B) = (0.54)(0.20) = 0.108 \neq 0.08 = P(A \cap B)$$

these events are not independent.

Example

Example 2: (*Check by the nature of a random experiment*) If we toss a fair coin two or more times, the probability of a head is the same for each toss and is not influenced by the outcome of the previous toss.

Exercise

A quality-control manager found that 30% of workrelated problems occurred on Mondays and that 20% occurred in the last hour of a day's shift. It was also found that 4% of worker-related problems occurred in the last hour of Monday's shift. a. What is the probability that a worker-related problem that occurs on a Monday does not occur in the last hour of the day's shift? b. Are the events "problem occurs on Monday" and "problem occurs in the last hour of the day's shift" statistically independent?

Total Probability Rule

Rule

Let $E_1, E_2, \ldots E_n$ be a system of events which are both mutually exclusive and collectively exhaustive. Let A be an event. Then

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \ldots + P(E_n)P(A|E_n)$$

In particular, A and $\overline{A}$ are both mutually exclusive and collectively exhaustive. For any event B, we have

$$P(B) = P(A)P(B|A) + P(\overline{A})P(B|\overline{A})$$

Bayes' Theorem

Theorem

Let $E_1, E_2, \ldots E_n$ be a system of events which are both mutually exclusive and collectively exhaustive. Let A be an event. Then

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(A)}$$

or

$$P(E_i|A) = \frac{P(A|E_i)P(E_i)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + \ldots + P(E_n)P(A|E_n)}$$

In particular,

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} = \frac{P(A)P(B|A)}{P(A)P(B|A) + P(\overline{A})P(B|\overline{A})}$$

Solution steps for total probability and Bayes' Theorem

Step 1: Define the events from the problem.

Step 2: Define the probabilities and conditional probabilities for the events defined in Step 1.

Step 3: Find the system of events which is both mutually exclusive and collectively exhaustive (compute the complement if needed).

Step 4: Apply the formula.

Exercise

Ex 1: A stock market analyst examined the prospects of the shares of a large number of corporations. It turned out that 25% performed much better than the market average, 25%, much worse, and the remaining 50%, about the same as the average. Forty percent of the stocks that turned out to do much better than the market were rated good buys by the analyst, as were 20% of those that did about as well as the market and 10% of those that did much worse.

a. What is the probability that a stock will be rated a good buy by the analyst?

b. What is the probability that a stock rated a good buy by the analyst performed much better than the average?

Exercise

Ex 2: Given that 10% of the people who come into the showroom and talk to a salesperson will purchase a car. To increase the chances of success, you propose to offer a free dinner with a salesperson. The project is conducted, and 40% of the people who purchased cars had a free dinner. In addition, 10% of the people who did not purchase cars had a free dinner.

a. What is the probability that a person will have a free dinner?

b. What is the probability that a person who accept a free dinner will purchase a car?

c. What is the probability that a person who does not accept a free dinner will purchase a car?

Exercise

Ex 3: Disease X was present in 5 percent of the population, and the testing technique's accuracy rate is given as follows:

- 90% of infected people have a positive result.
- 95% of uninfected people have a negative result.

What is the probability that a person who has a positive result will be infected?

Exercise

Ex 4: The Watts New Lightbulb Corporation ships large consignments of lightbulbs to big industrial users. When the production process is functioning correctly, which is 90% of the time, 10% of all bulbs produced are defective. However, the process is susceptible to an occasional malfunction, leading to a defective rate of 50%.

a. If a defective bulb is found, what is the probability that the process is functioning correctly?

b. If a nondefective bulb is found, what is the probability that the process is operating correctly?

Exercise

Ex 5: There are 3 identical boxes. The first box contains 10 products including 4 defective products, the second box contains 15 products including 5 defective products, the third box contains 20 products, including 5 defective products. Take a box at random and from there take a product at random. Find the probability of getting the defective product.