1. System of equations

System of sentences

System 1	System 2
Dog : black	Dog : black
Cat : orange	Dog : black
Bird : red	Bird : red
complete	redundant
Non-singular	Singular

System 3	System 4
Dog : black	Dog : black
Dog : black	Dog : white
Dog : black	Bird : red
redundant	contradict
Singular	Singular

- Non-singular: the system carries as many pieces of information as sentences meaning that is a complete system

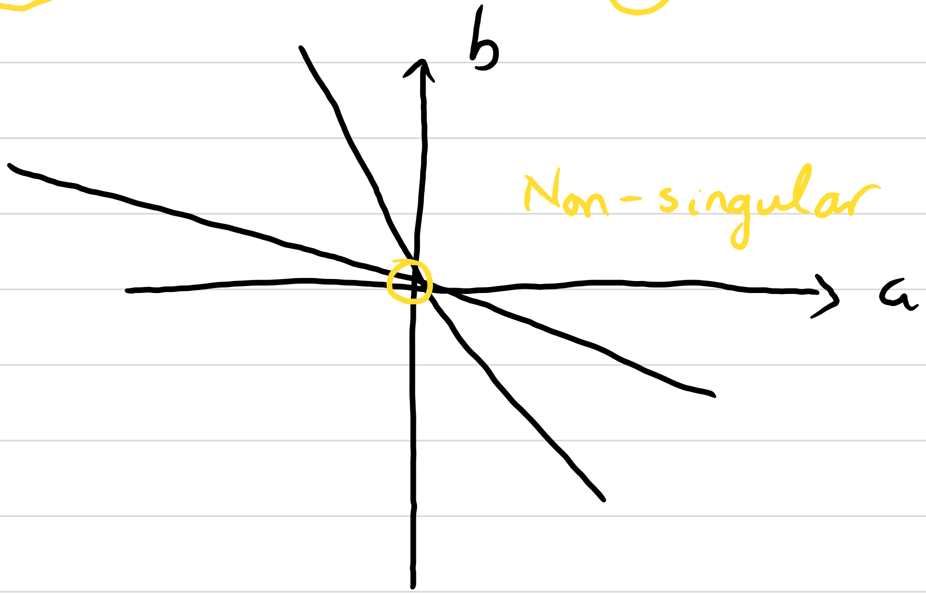
System of equation

Ex:

$$a + b = 10$$
$$a + 2b = 12$$

Tips for determine singularity

$$\begin{aligned} a+b &= 10 \\ a+2b &= 12 \end{aligned} \;\rightarrow\; \text{Set} \; \begin{aligned} a+b &= 0 \\ a+2b &= 0 \end{aligned} \;\rightarrow\; S: \overbrace{(0,0)}^{\text{unique}}$$



Linear dependence and independence

Non-singular	Singular system
$\begin{aligned} a+b&=0 \\ a+2b&=0 \end{aligned}$ $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$	$\begin{aligned} a+b&=0 \\ 2a+2b&=0 \end{aligned}$ $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
- No row is a multiple of the other one	- Second row is a multiple of the first row
- Linear independence	- Linear dependence

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Row 1 + Row 2 = Row 3

$\Rightarrow$ Linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

$(\text{Row } 1 + \text{Row } 3)/2 = \text{Row } 2$

$\Rightarrow$ linearly dependent

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

No relations between rows

$\Rightarrow$ Linearly independent

Determinant

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determinant $= ad - bc$ (det)

If $ad - bc = 0 \Rightarrow$ Singular

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \qquad \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$\det = aei + bfg + cdh - ceg - afh - bdi$

2. Row echelon matrix

$$\begin{cases} 5a + b = 17 \\ 4a - 3b = 6 \end{cases} \to \begin{cases} a + 0.2b = 3.4 \\ b = 2 \end{cases} \to \begin{cases} a = 3 \\ b = 2 \end{cases}$$

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \to \underbrace{\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}}_{\text{Row echelon form}} \to \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{Reduce REF}}$$

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Echelon form

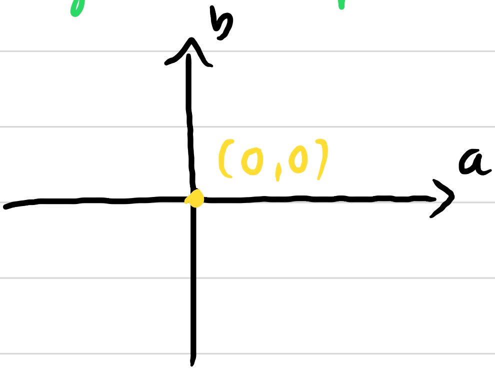
3. Rank

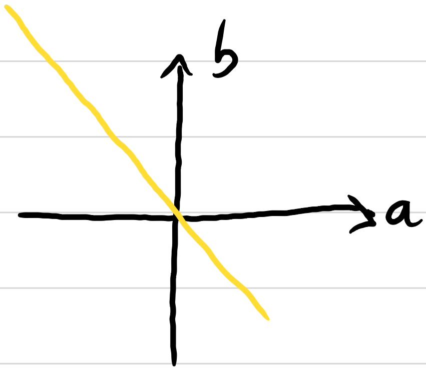
System 1	System 2	System 3
Dog : black Cat : orange	Dog : black Dog : black	Dog Dog
2 sentences	2 sentences	2 sentences
2 pieces of info	1 piece of info	0 pieces of info
Rank = 2	Rank = 1	Rank = 0

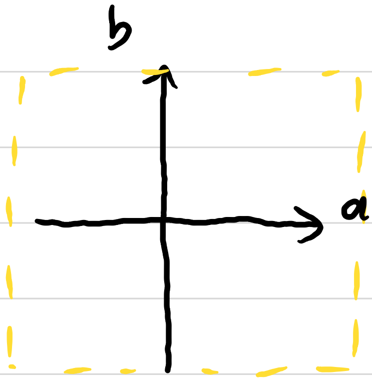
$$a + b = 0 \qquad a + b = 0 \qquad 0a + 0b = 0$$
$$a + 2b = 0 \qquad 2a + 2b = 0 \qquad 0a + 0b = 0$$

Rank = 2 Rank = 1 Rank = 0

Dim of solution space = 0 1 2







Rank = 2 − Dimension of solution space
(for 2 × 2 matrices)

- Non-singular if Rank = # rows

General:
Rank = largest number of linearly independent rows/cols

4. Row echelon form

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \quad \text{Rank } 2$$

$$\begin{bmatrix} 5 & 1 \\ 10 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 0 \end{bmatrix} \quad \text{Rank } 1$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{Rank } 0$$

General

$$\begin{bmatrix} 2 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 3 & * & * \\ 0 & 0 & 0 & -5 & * \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot

Rank = # pivots

Rank = 5

$$\begin{bmatrix} 3 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & -4 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

5. Reduced row echelon form (RREF)

$$\begin{bmatrix} 5 & 1 \\ 4 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Is in REF
- Each pivot is a 1
- Any number above a pivot is 0

$$\begin{bmatrix} 1 & * & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6. Gaussian Elimination algorithm

$$\begin{aligned} 2a - b + c &= 1 \\ 2a + 2b + 4c &= -2 \\ 4a + b &= -1 \end{aligned} \longrightarrow \begin{array}{cc} \left[\begin{array}{cccc} 2 & -1 & 1 & 1 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & -1 \end{array}\right] & \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \end{array}$$

Augmented matrix

$$\begin{aligned} R_1 &\leftarrow \frac{1}{2} R_1 \\ R_2 &\leftarrow R_2 - 2R_1 \\ R_3 &\leftarrow R_3 - 4R_1 \end{aligned} \quad \rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 & -1/2 \\ 0 & 3 & 3 & -3 \\ 0 & 3 & -2 & -3 \end{bmatrix}$$

$$\begin{aligned} R_2 &\leftarrow \frac{1}{3} R_2 \\ R_3 &\leftarrow R_3 - 3R_2 \end{aligned} \quad \rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 & -1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -5 & 0 \end{bmatrix}$$

$$R_3 \leftarrow -\frac{1}{5} R_3 \quad \rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 & -1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Back-substitution

$$\begin{aligned} R_2 &\leftarrow R_2 - R_3 \\ R_1 &\leftarrow R_1 - \frac{1}{2} R_3 \end{aligned} \quad \rightarrow \begin{bmatrix} 1 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R_1 \leftarrow R_1 + \frac{1}{2} R_2 \quad \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \; \text{RREF}$$

7. Vector algebra

Norms

$$L1\text{-norm} = |(a,b)_1| = |a + b|$$

$$L2\text{-norm} = |(a,b)_2| = \sqrt{a^2 + b^2}$$

Direction

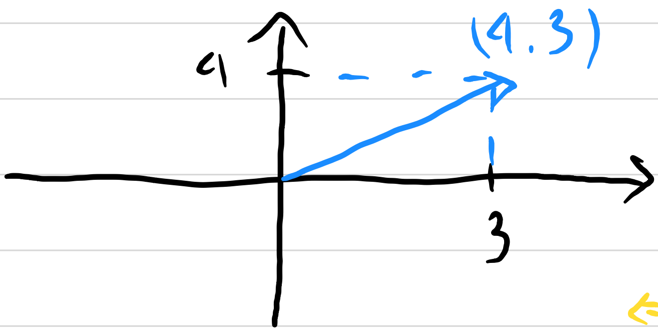
$$\tan(\theta) = \frac{3}{4}$$

$$\theta = \arctan\frac{3}{4} = 0.64$$

Dot product

$$\begin{bmatrix} 2 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} = 28$$

$$(2.3 + 4{,}5 + 1.2 = 28)$$

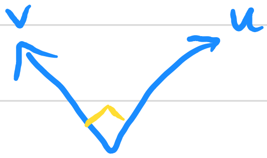


$$\sqrt{4^2 + 3^2} = 25$$

$$L2\text{-norm} = \sqrt{dotproduct(u,u)}$$

$$\Leftrightarrow |u|_2 = \sqrt{\langle u,u \rangle}$$

$$\langle u, u \rangle = |u|^2$$



$$\langle u, v \rangle = |u| \cdot |v| = 0$$

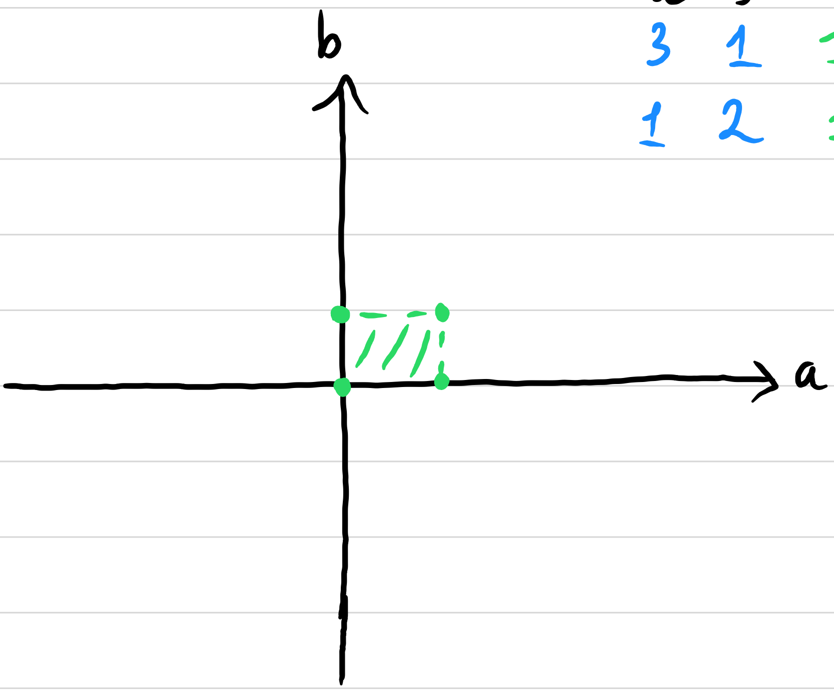
$$\langle u, v \rangle = |u'| \cdot |v|$$

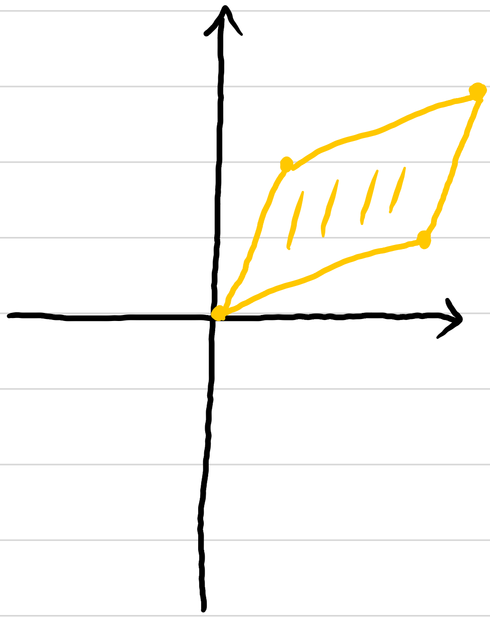
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 12 \end{bmatrix}$$

$$3 \times 3 \qquad 3 \times 1$$

8. Linear transformations

$$\begin{array}{cc} a & b \\ 3 & 1 \\ 1 & 2 \end{array} \quad \begin{array}{c} 1 \\ 1 \end{array} = \begin{array}{c} 4 \\ 3 \end{array}$$





$$\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{bmatrix}$$

2×2 $\quad$ 2×2 $\quad$ 2×2

$$\begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -2 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 9 & 21 & -6 \\ 1 & -3 & 8 & -4 \end{bmatrix}$$

$2 \times \underline{3}$ $\quad$ $\underline{3} \times 4$ $\quad$ 2×4

Identity matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- When conduct linear transformation on identity matrix, it will send each point precisely to itself

Matrix inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\uparrow$$

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$$

$$\Rightarrow \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 1 \qquad a = 2/5$$

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 0 \qquad b = -1/5$$

$$\Rightarrow$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix} = 1 \qquad c = -1/5$$

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix} = 0 \qquad d = 3/5$$

Formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $\det A = 0$, no inverse matrix

9. Rank of linear transformation

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Rank 2

Dimension 2

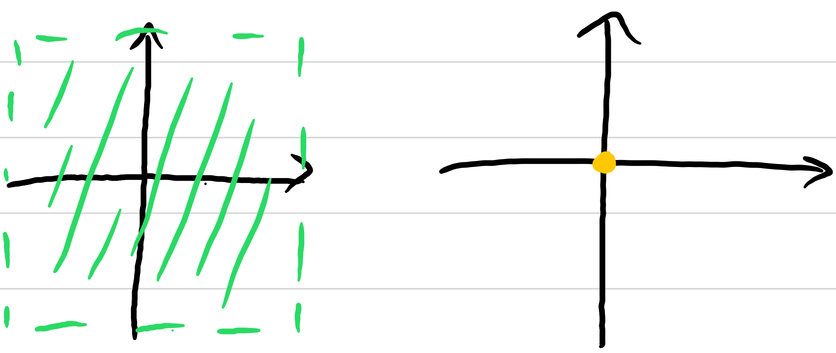
$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Rank 1

Dimension 1

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Rank 0

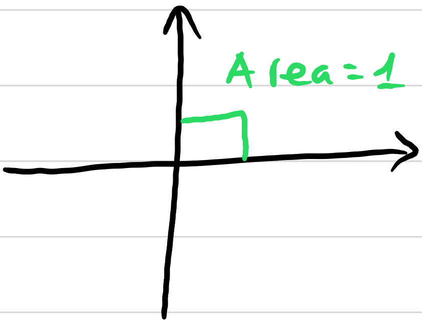


Dimension 0

10. Determinant as an area

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\det = 5$$



If $\det = 0 \Rightarrow$ Area $= 0$

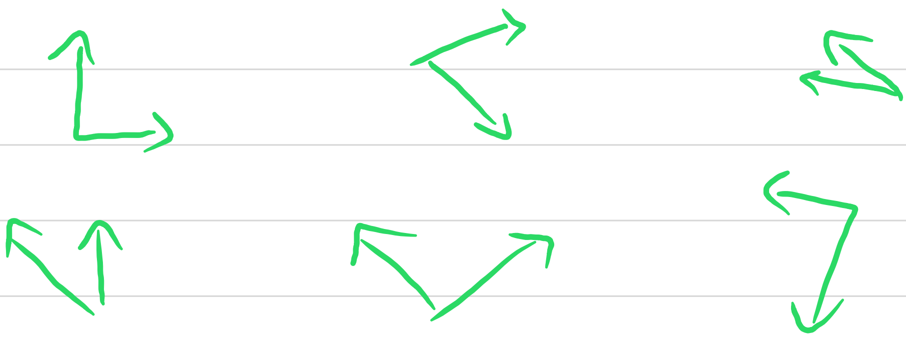
11. Determinant of a product

$$\det(AB) = \det A \cdot \det B$$

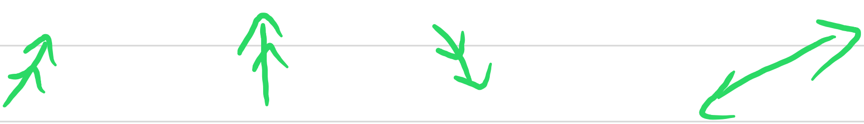
12. Determinant of inverses

$$\det A^{-1} = (\det A)^{-1}$$

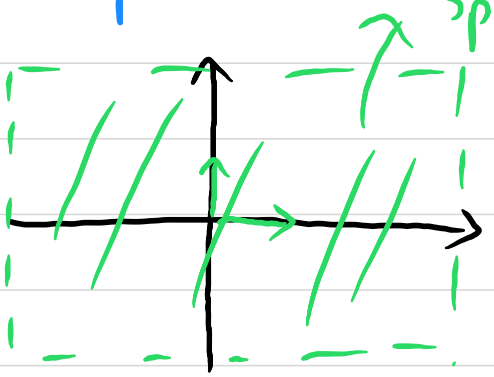
13. Bases (plural of basis)

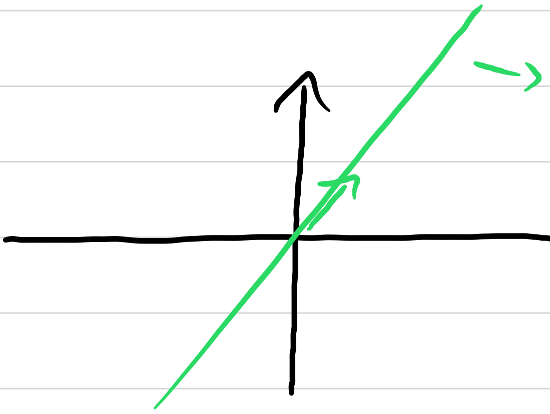


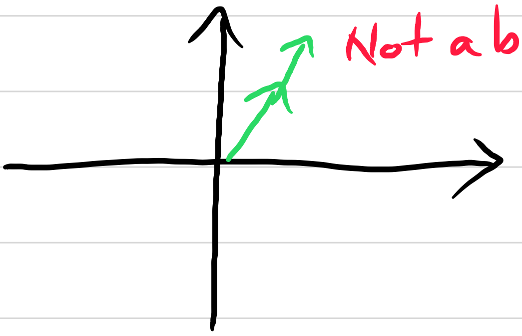
Non bases



14. Spans







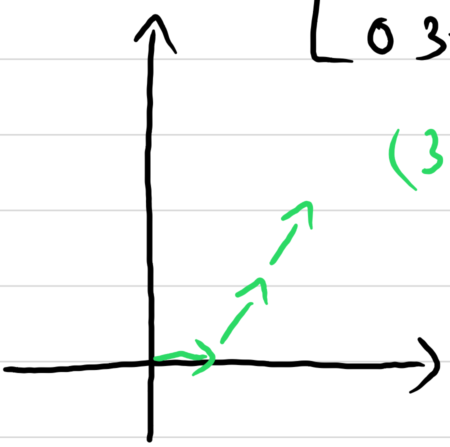
Not a basic

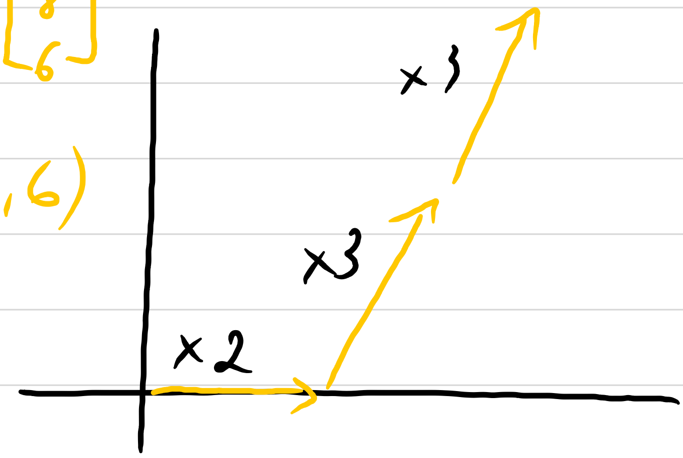
A basic is a minimal spanning set

15. Eigenbases

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$(3,2) \to (8,6)$





Eigenvector

Eigenvalue

$$A v_1 = \lambda_1 v_1$$

eigenvalue eigenvector

Example :

+ Find eigen value of $A = \begin{pmatrix} 9 & 4 \\ 4 & 3 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (9-\lambda)(3-\lambda) - 16 = 0$$

$$\Rightarrow \lambda = 11$$
$$\lambda = 1$$

+ Find eigenvectors

If $\lambda = 1$,

$$A - 1I = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = \frac{-1}{2} y$$
$$\Rightarrow v = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

If $\lambda = 11$,

$$\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = 2y$$
$$\Rightarrow v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

16. Dimensionality Reduction

- Reduce dimensions/columns of dataset

Projections

$y = x$

x	y
1.0	1.0
1.2	1.6
-0.5	0.2
-1.3	-0.6

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Norm of 1

Final coordinates

$(1 + 1)/\sqrt{2}$

$(1.2 + 1.6)/\sqrt{2}$

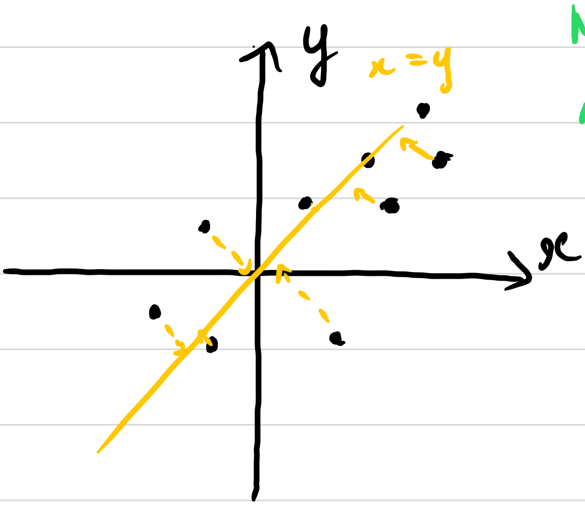
$(-0.5 + 0.2)/\sqrt{2}$

$(-1.3 - 0.6)/\sqrt{2}$

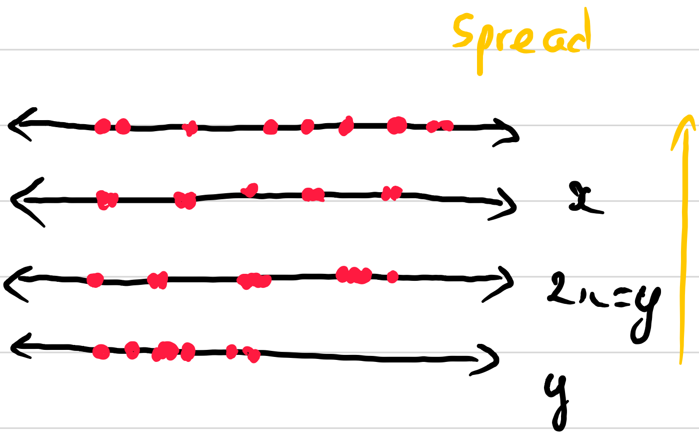
To project matrix A onto a vector v

$$A_p = A \frac{v}{\|v\|_2}$$

$$r \times 1 \qquad r \times c \qquad c \times 1$$



More spread
more info



Covariance matrix

$$A - \mu = \begin{pmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{pmatrix}$$

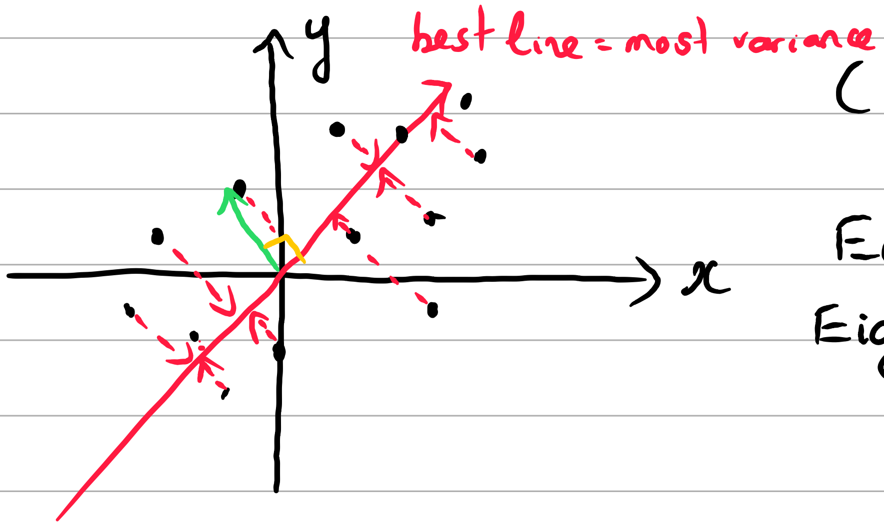
$$C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

PCA

Projections

Eigen values / eigen vectors $\longrightarrow$ PCA

Covariance Matrix



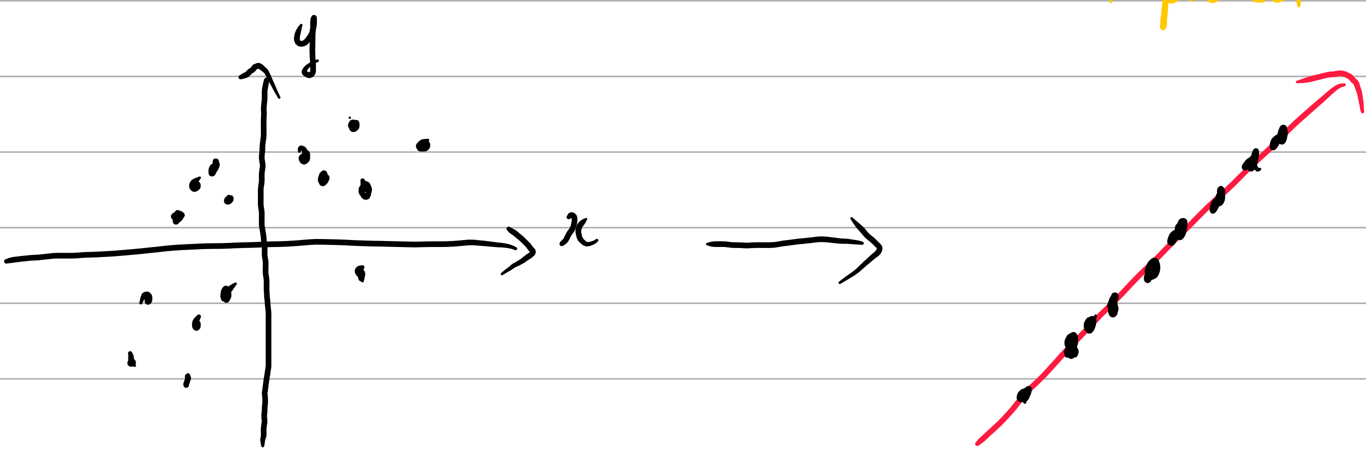
$$C = \begin{pmatrix} 9 & 4 \\ 4 & 3 \end{pmatrix}$$

Eigenvalues $11 > 1$

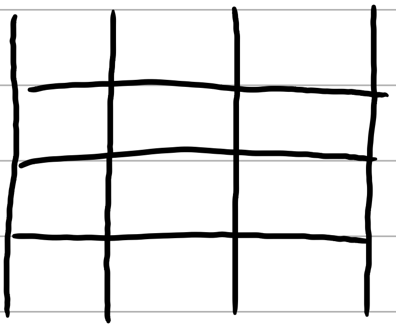
Eigen vectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

best line to project

discard



Large table
(9 features)



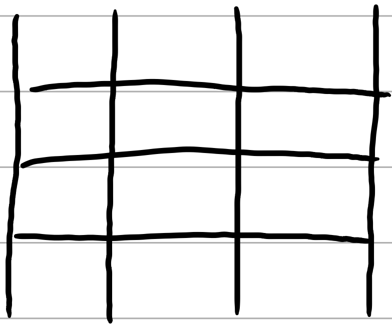
$C = (\quad)$

$\rightarrow$

$$\begin{array}{cc} v_1 & \lambda_1 \\ v_2 & \lambda_2 \\ \vdots & \vdots \\ v_9 & \lambda_9 \end{array}$$

Big

Small



$$\left(\frac{v_1}{\|v_1\|_2} \quad \frac{v_2}{\|v_2\|_2}\right) \rightarrow$$

2 columns

PCA Mathematical formulation

n observations of 5 variables $(x_1, x_2, x_3, x_4, x_5) \rightarrow$ 2 variables

① Create matrix

5 variables

$X = (\quad)$

② Center the data

$X - \mu = (\quad)$

③ Calculate cov matrix

$$C = \frac{1}{n-1}(A-\mu)^T(A-\mu)$$

④ Calculate eigens

$$\begin{array}{cc} \lambda_1 & v_1 \\ \lambda_2 & v_2 \\ \vdots & \vdots \end{array}$$

Big

Small

⑤ Create Projection matrix

$$V = \begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}$$

⑥ Project centered data

$$X_{PCA} = (X - \mu) V \quad \text{(2 columns)}$$

17. Discrete Dynamical Systems

$$\begin{array}{c|ccc} & \text{Sunny} & \text{Cloudy} & \text{Rainy} \\ \hline \text{Sunny} & 0.8 & 0.45 & 0.3 \\ & + & + & + \\ \text{Cloudy} & 0.15 & 0.35 & 0.4 \\ & + & + & + \\ \text{Rainy} & 0.05 & 0.2 & 0.3 \\ & \| & \| & \| \\ & 1 & 1 & 1 \end{array}$$

- All positive values
- Columns add to 1

↓

Markov matrix

Transition matrix P

If predict long-run probabilities, we will have :

$$\begin{pmatrix} 0,6665 \\ 0,2223 \\ 0,1112 \end{pmatrix} \rightarrow X_\infty : \text{equilibrium vector}$$

(eigenvector of P)

$(3-\lambda)(1-\lambda) = 0 \Rightarrow \lambda = 3$

$\lambda = 1$

$* \ \lambda = 1$

$$\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} x_1 \\ x_2 \end{matrix} \Rightarrow \begin{matrix} 3x_1 = x_1 \\ -2x_1 + x_2 = x_2 \end{matrix}$$

$$\Rightarrow \begin{matrix} 2x_1 = 0 \\ -2x_1 = 0 \end{matrix} \quad \text{OK}$$

$$\begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{matrix} 3x_1 = 3x_1 \\ -2x_1 + x_2 = 3x_2 \end{matrix}$$

$0 = 0$

$2x_1 + 2x_2 = 0$

$2(x_1 + x_2) = 0$