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DeepLearning.AI

Probability and Statistics for Machine Learning and Data Science

Week 3: Sampling and Point Estimates

W3 Lesson 1



DeepLearning.AI

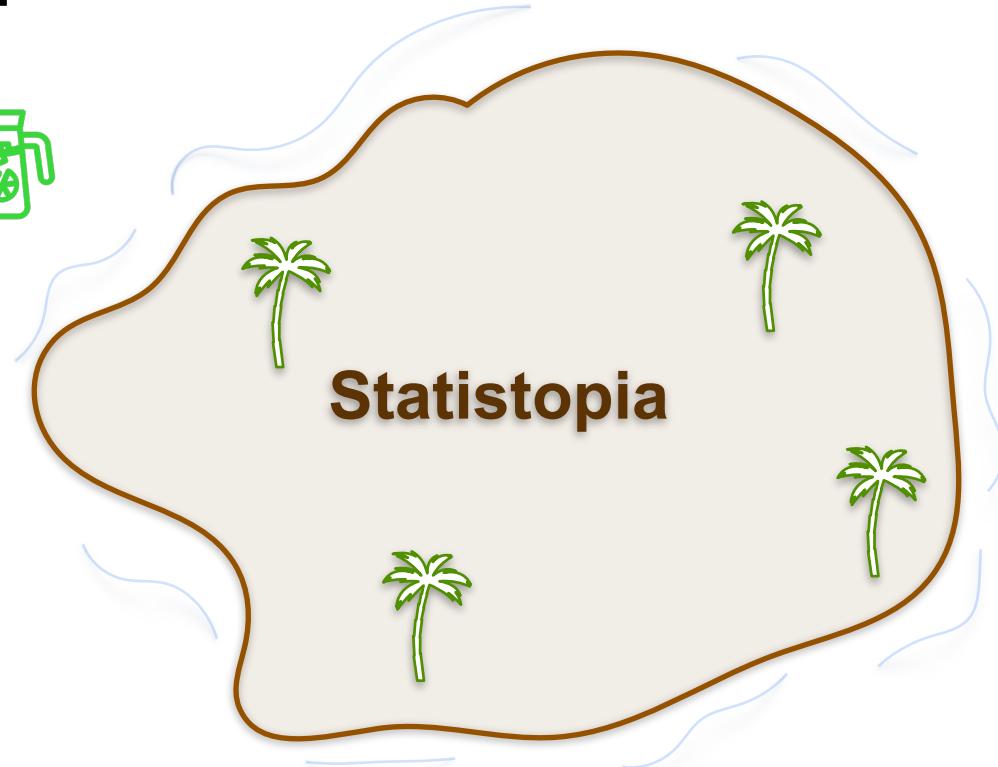
Sample and Population

Population and Sample

Population and Sample



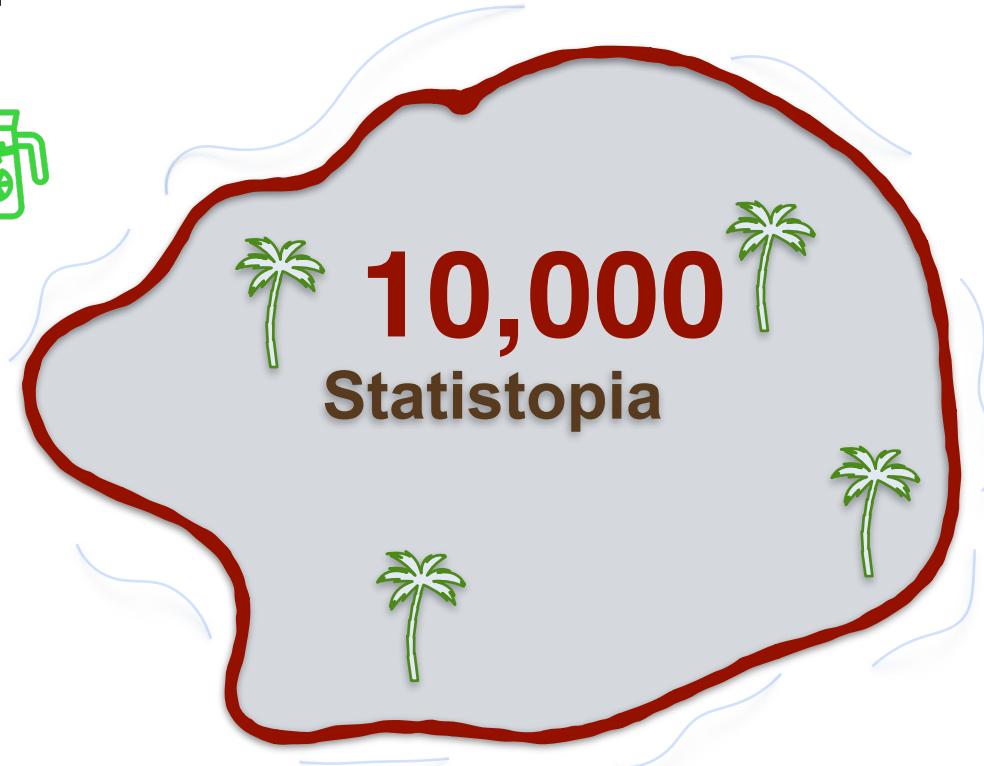
Find the **average height** of
the people living on
Statistopia



Population and Sample



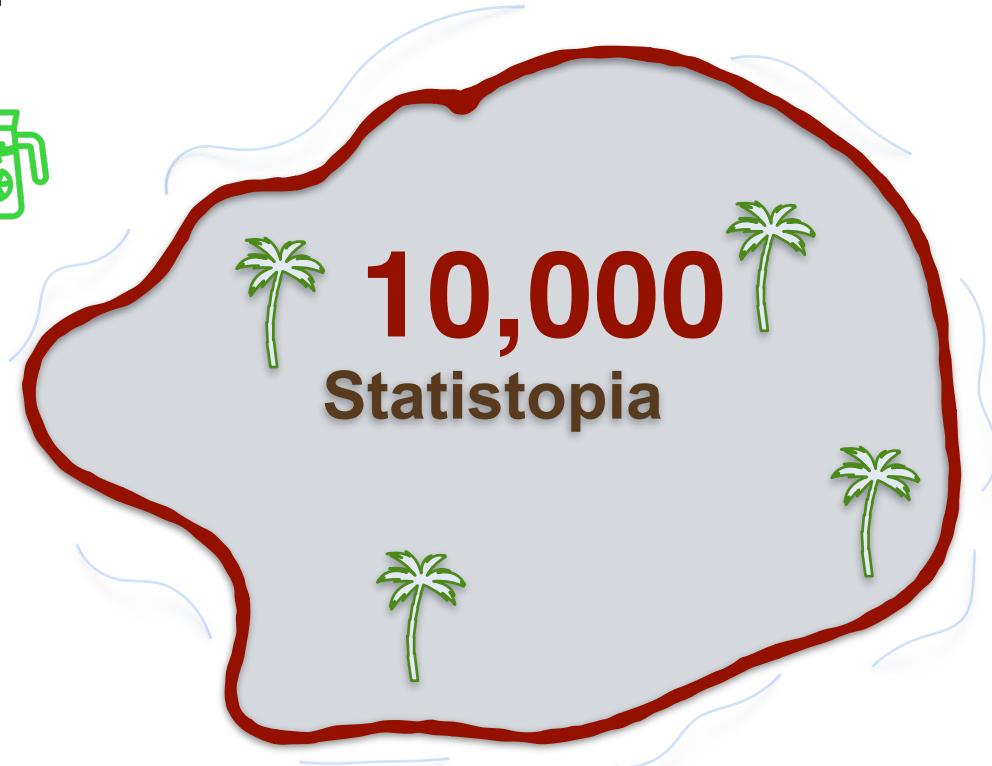
- Ask everyone on the island for their height.
- Divide by the total number



Population and Sample



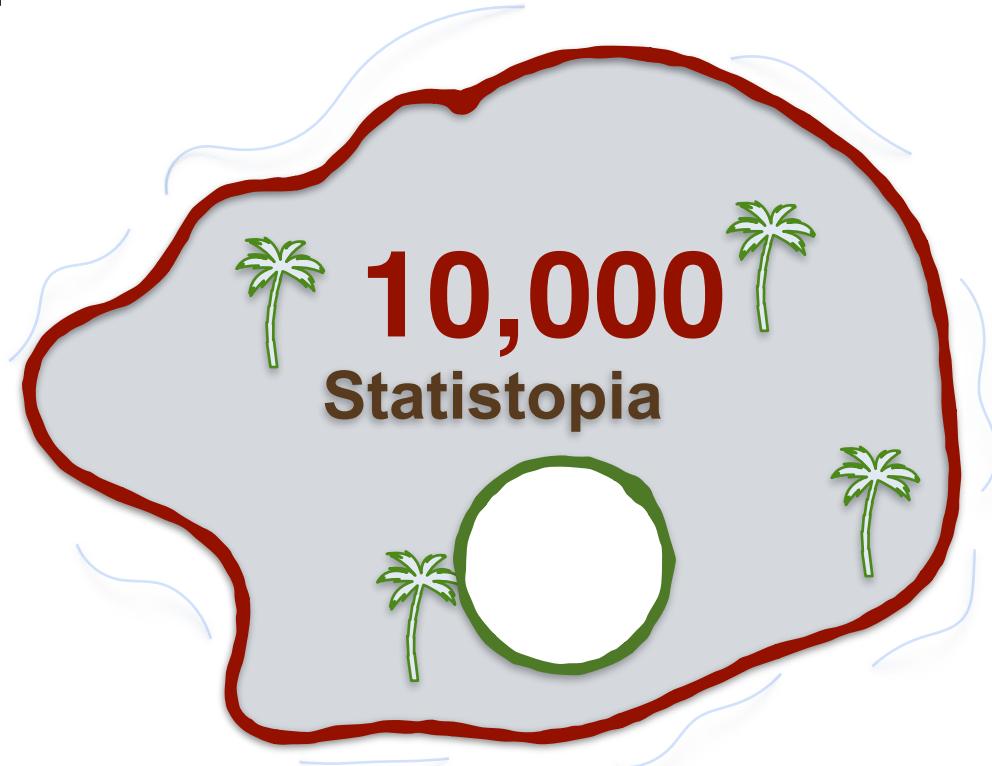
- Ask everyone on the island for their height.
- Divide by the total number



Population and Sample



- Only ask a subset of the group to estimate the average height



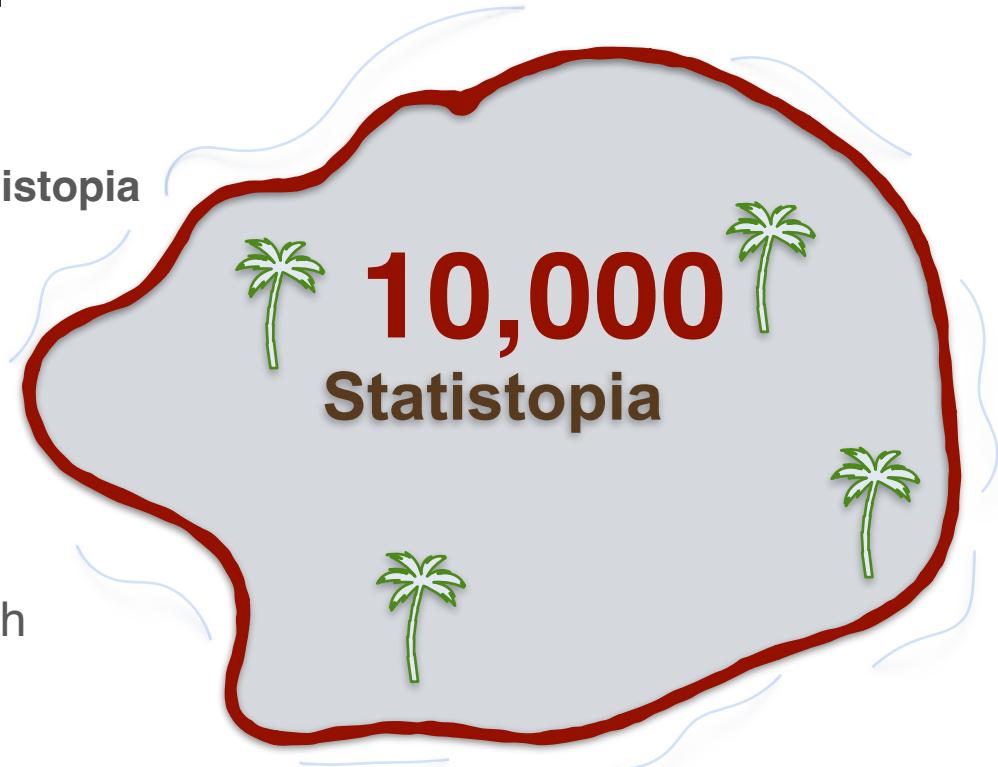
Population and Sample



The people of statistopia

Population:

the entire group of individuals or elements you want to study which share a common behaviour



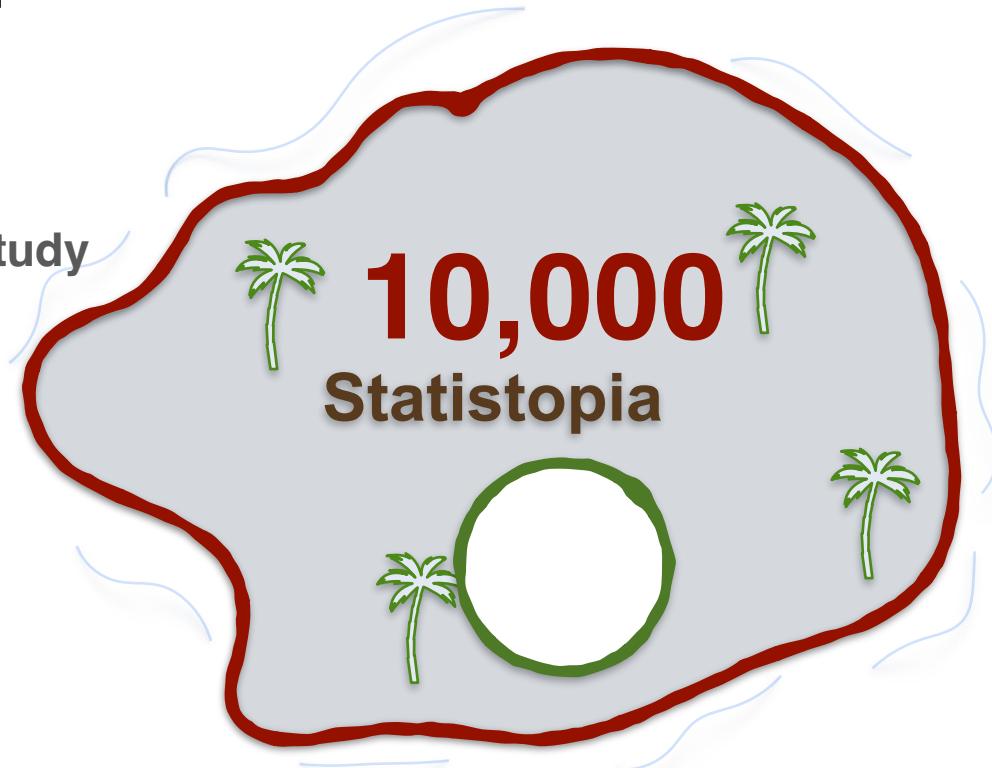
Population and Sample



The people you
select for your study

Sample:

subset of the population you use
to draw conclusions about the
population as a whole



Population and Sample

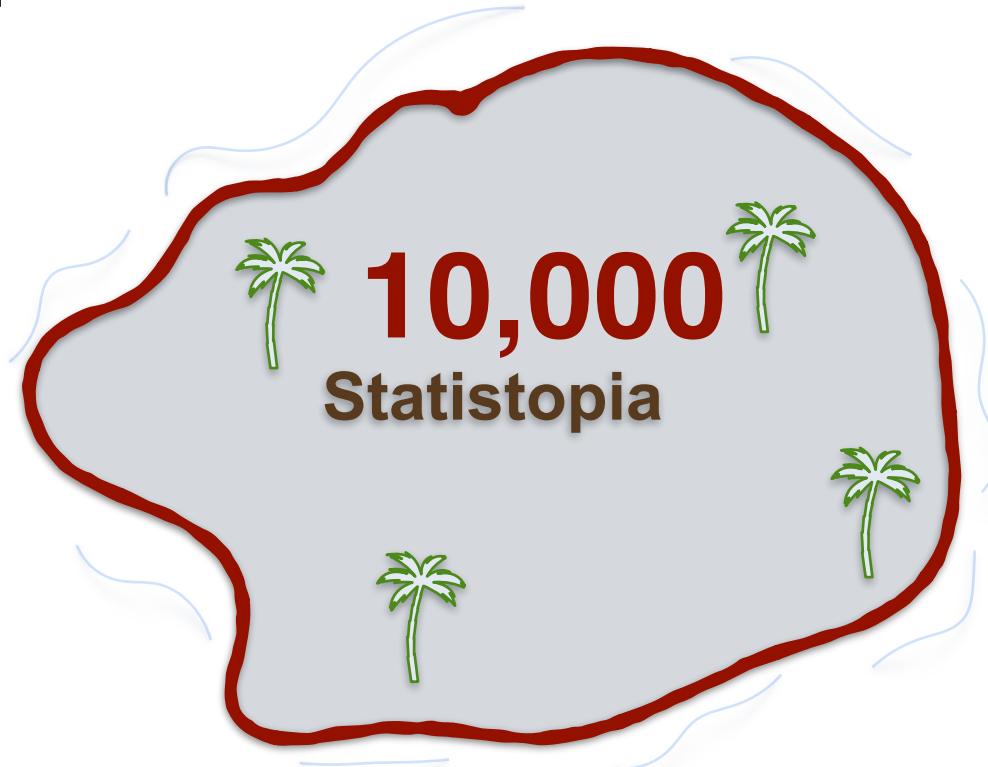


Population Size (N)

10,000

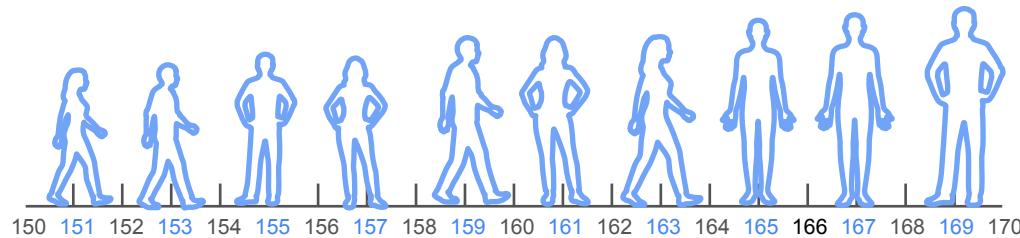
Sample Size (n)

1 - 9,999



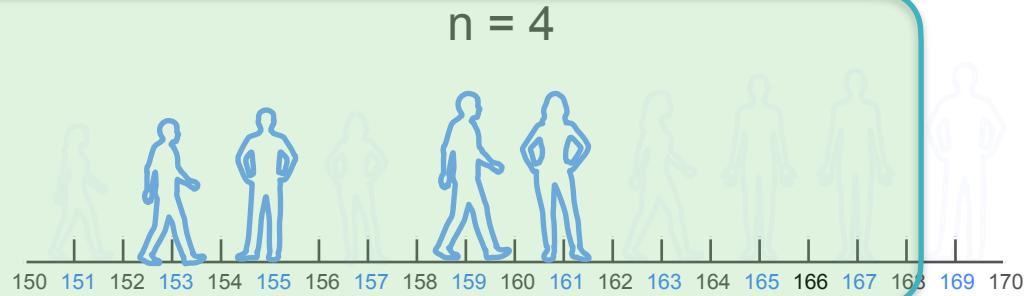
Population and Sample

$N = 10$



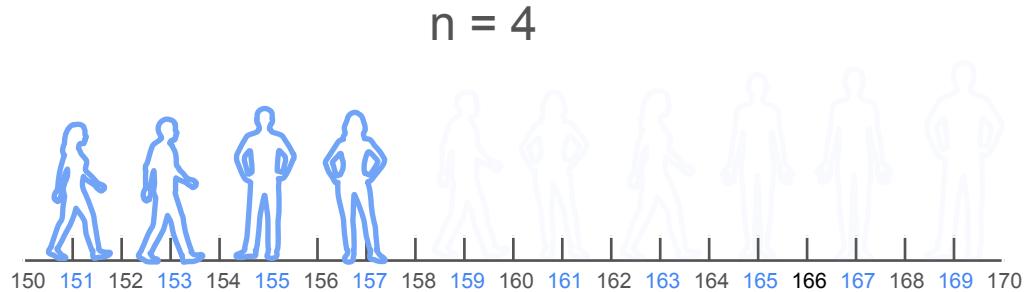
Random Sampling

A



Which is the better sample
to estimate the population
mean height?

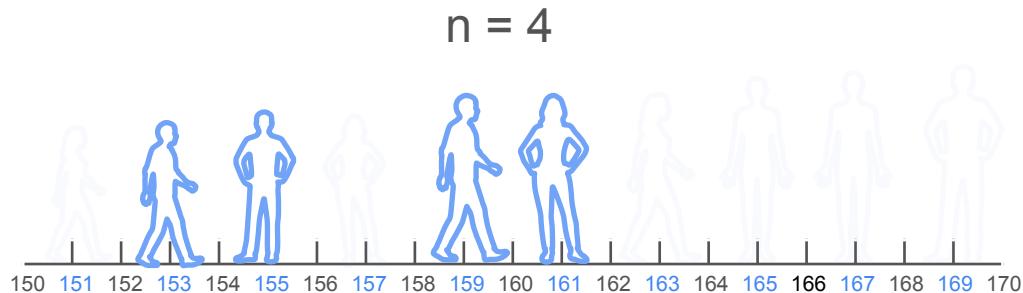
B



Independent Sample

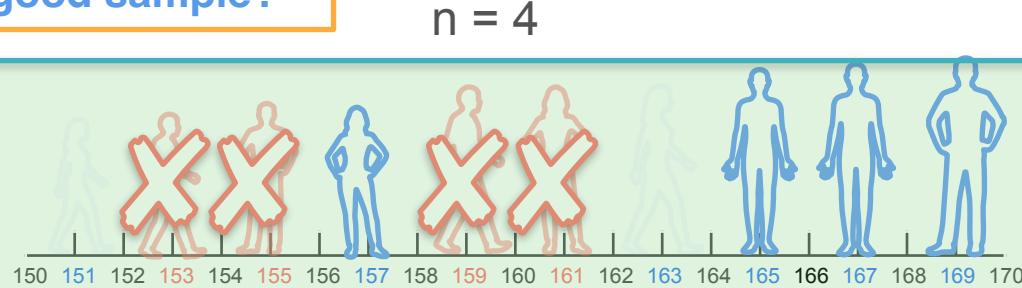
Example 1

1st sample set



Why is sample set two not a good sample?

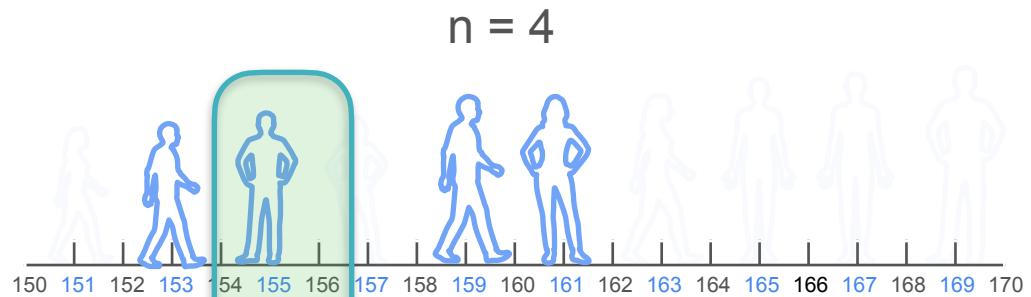
2nd sample set



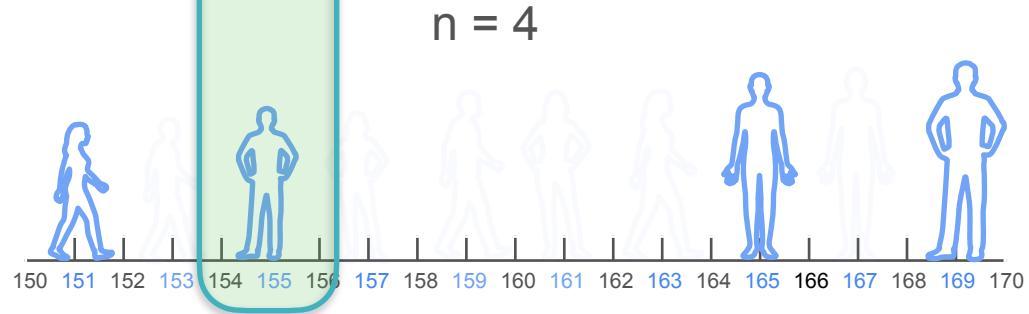
Independent Sample

Example 2

1st sample set



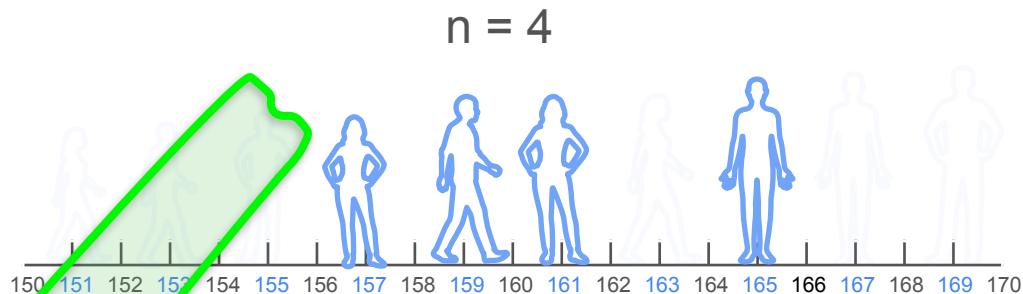
2nd sample set



Identically Distributed Samples

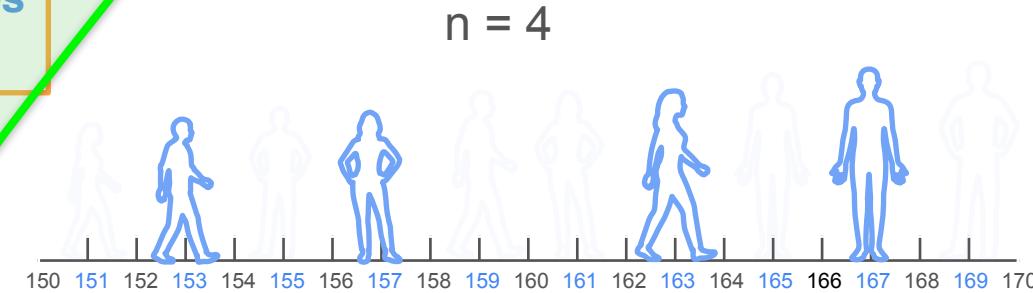
Example 1

A



Which of the following samples
are identically distributed?

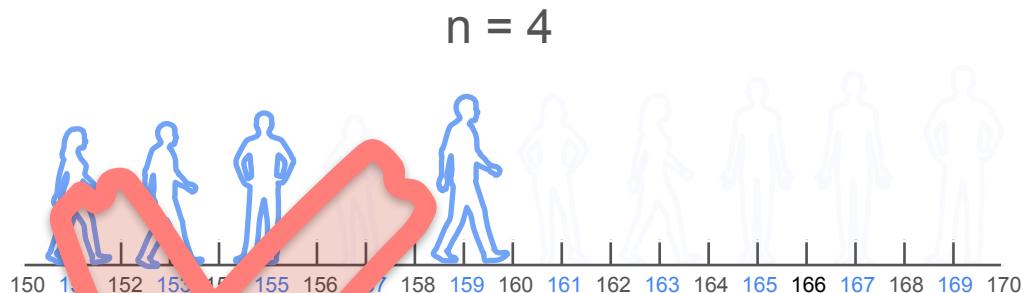
B



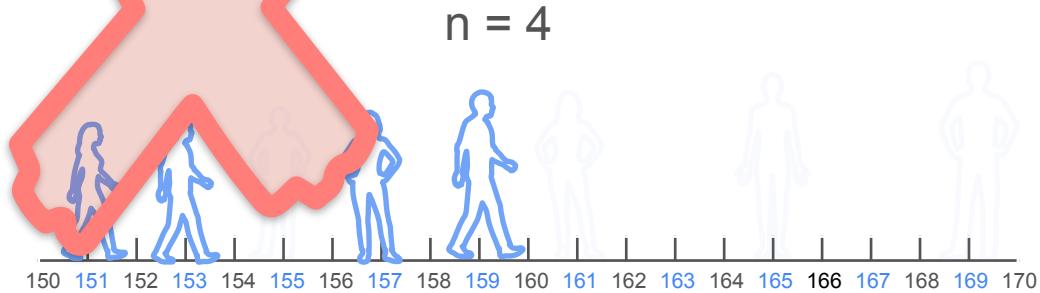
Identically Distributed Samples

Example 2

A



B



The Avocado Toast Trend

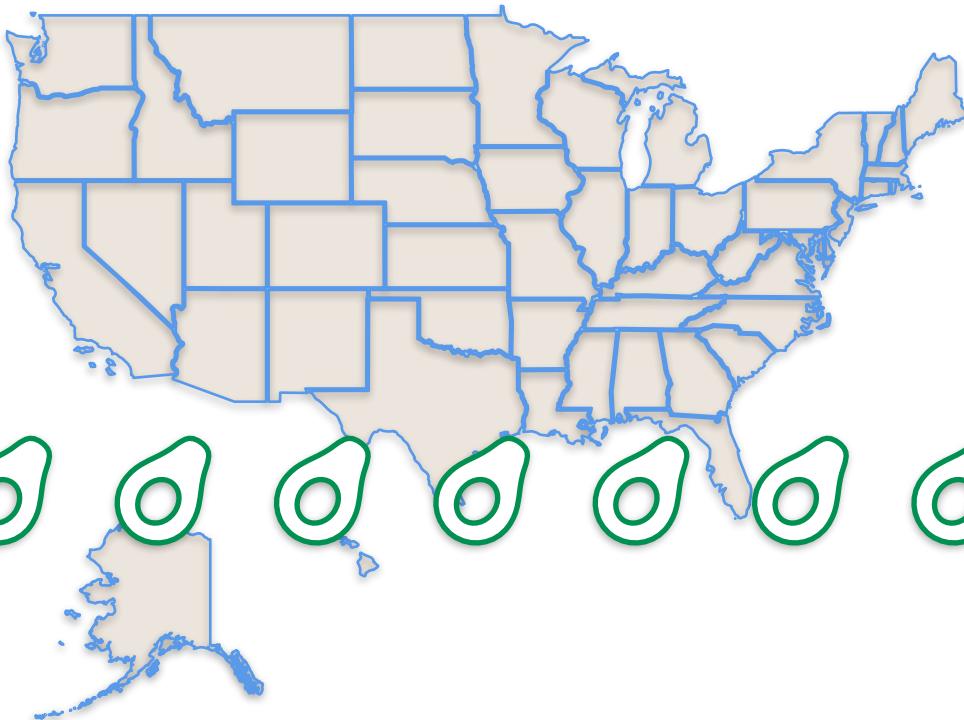


Study the price of avocados
in the United States



What is the population of your study?

The Avocado Toast Trend



All avocados
sold in the US

The Avocado Toast Trend

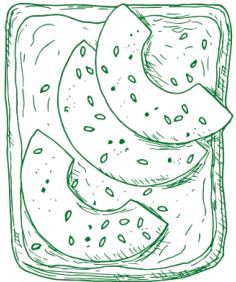


Study the price of avocados
in the United States



What is the sample of your study?

The Avocado Toast Trend



Study the price of avocados
in the United States



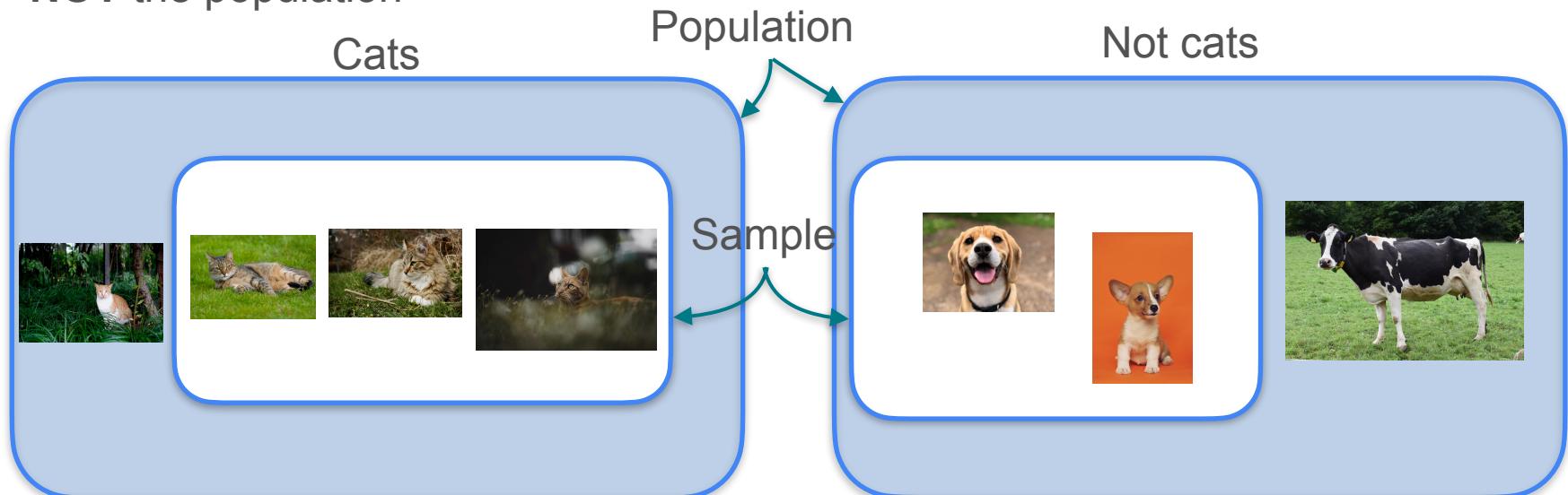
What is the sample of your study?

**Avocados sold
in the 4 stores
you selected**

Population and Sample in Machine Learning

Every dataset you work with in machine learning is a sample

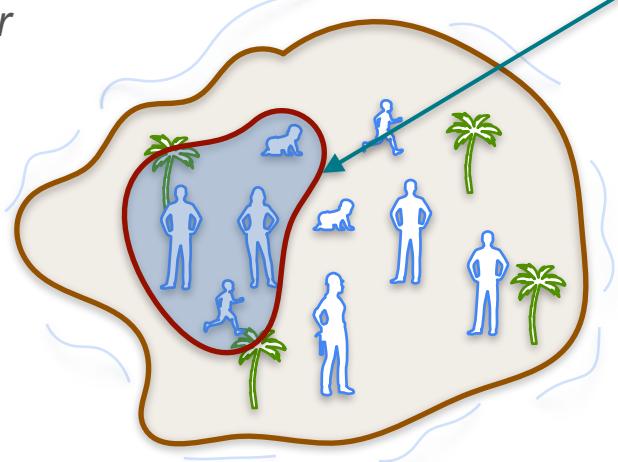
NOT the population



Recap

Population

the entire group of individuals or elements you want to study which share a common behaviour



Sample

subset of the population you use to draw conclusions about the population as a whole

Population Size: N

Sample Size: n

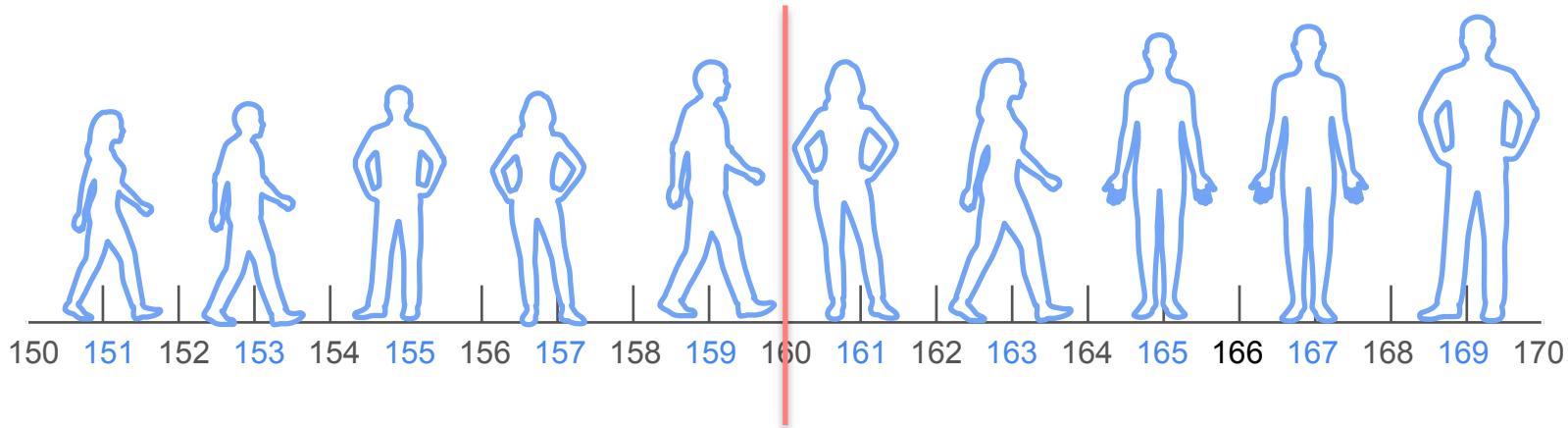


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Sample and Population

Sample Mean

Population and Sample Mean



What is the average height in statistopia?

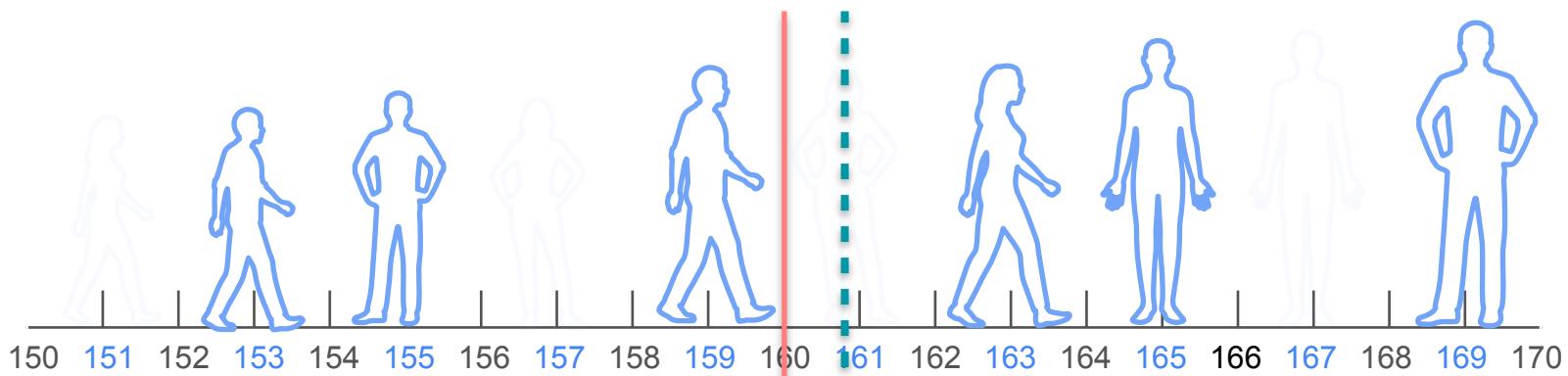
$$\frac{151 + 153 + 155 + 157 + 159 + 161 + 163 + 165 + 167 + 169}{10}$$

$$= \frac{1600}{10} = 160\text{cm}$$

Population mean

μ

Population and Sample Mean



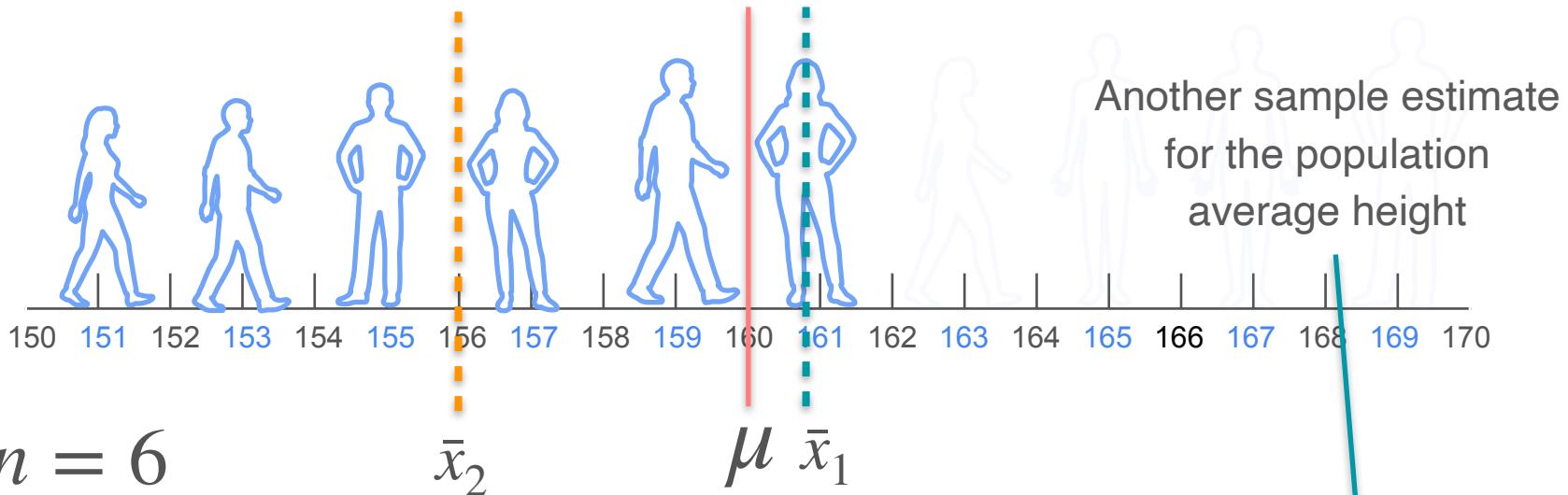
$$n = 6$$

What is the average height in statistopia?

$$\frac{153 + 155 + 159 + 163 + 165 + 169}{6} = \frac{964}{6} = 160.97$$

\bar{x}

Population and Sample Mean



$$n = 6$$

What is the average height in statistopia?

$$\frac{151 + 153 + 155 + 157 + 159 + 161}{6} = \frac{936}{6} = 156\text{cm}$$

Population and Sample Mean

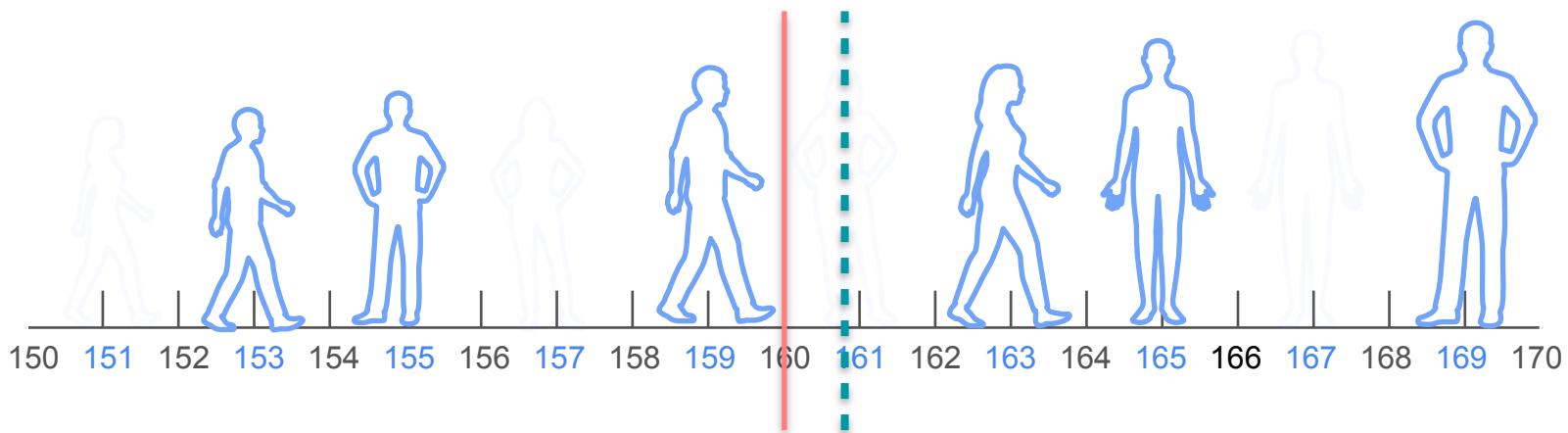
Better estimate of the population mean height

150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170

$$n = 6$$



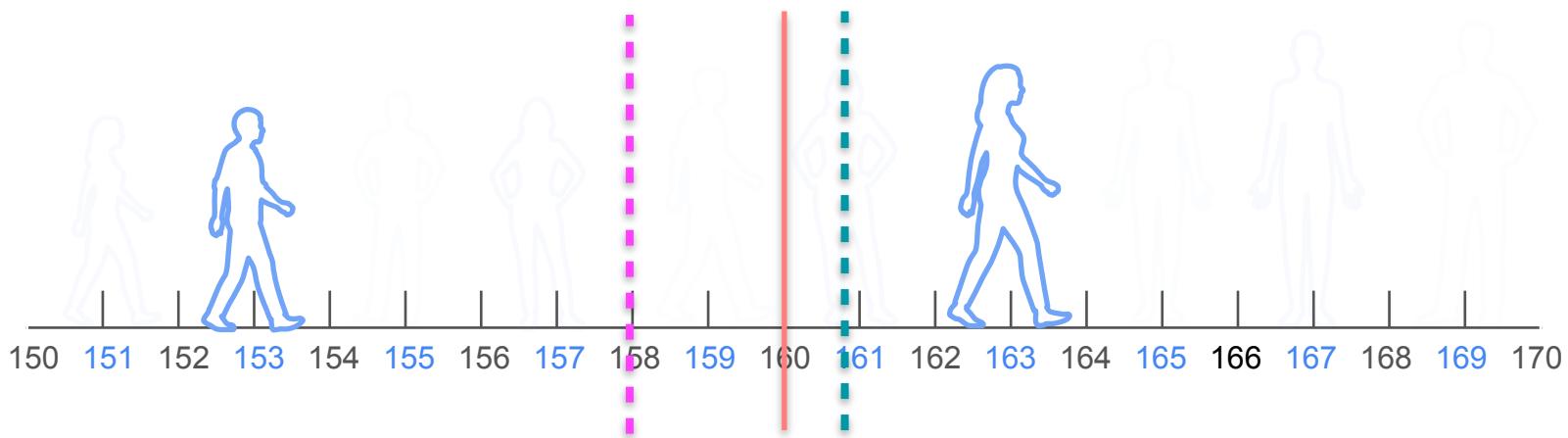
Population and Sample Mean



$$n = 6$$

$$n = 2$$

Population and Sample Mean



$$n = 6$$

$$n = 2$$

What is the average height in statistopia?

$$\frac{153 + 163}{2} = \frac{316}{2} = 158\text{cm}$$



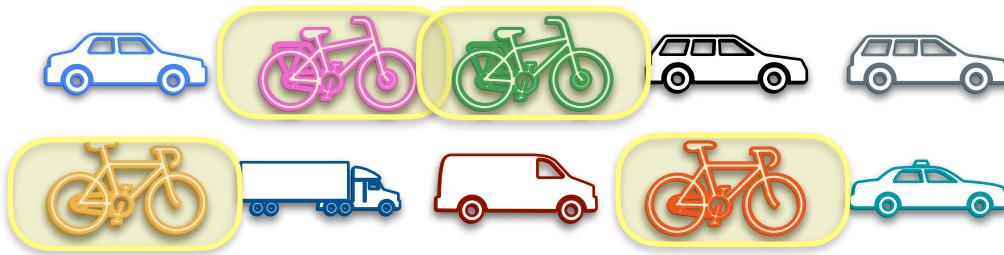
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Sample and Population

Sample Proportion

Proportion

Population size: 10



What proportion of people own a bicycle?

$$p \quad \text{population proportion} = \frac{4}{10} = 0.4 = 40\%$$

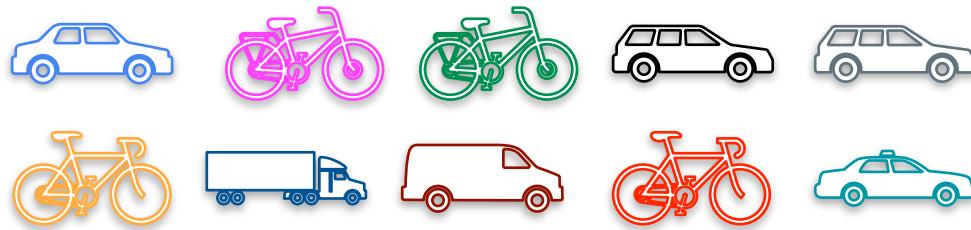
Proportion

population proportion

$$P = \frac{\text{number of items with a given characteristic } (x)}{\text{population } (N)}$$

Proportion

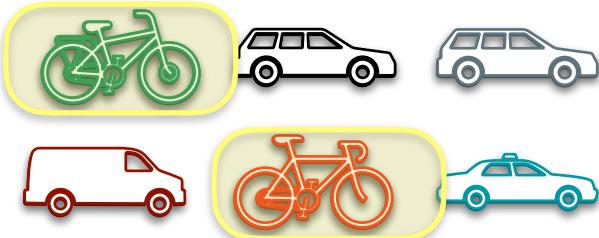
Population size: 10



Sample Proportion

Sample size: 6

estimate of the population proportion



What proportion of people own a bicycle?

$$\hat{p} \text{ sample proportion } = \frac{2}{6} = 0.333 = 33.3\%$$

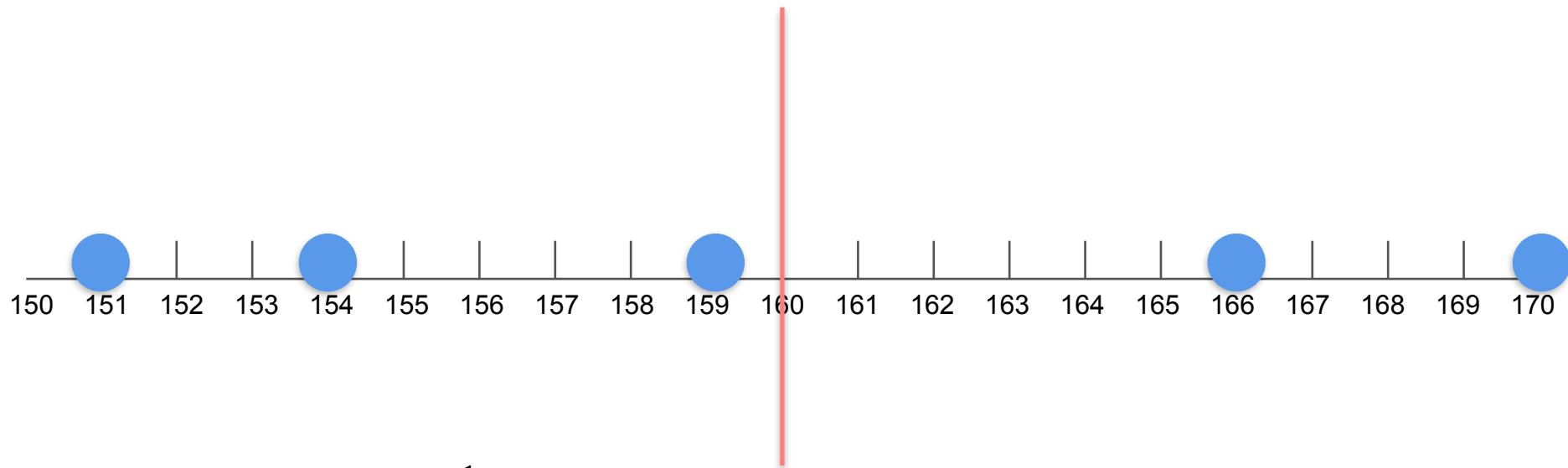


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Sample and Population

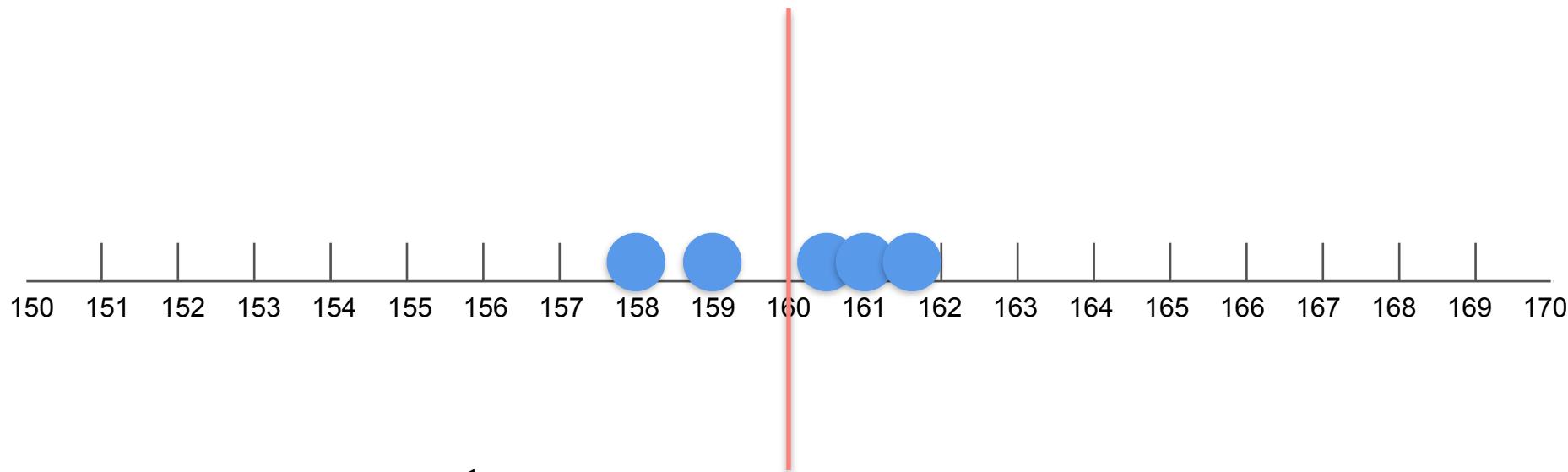
Sample Variance

Sample Variance



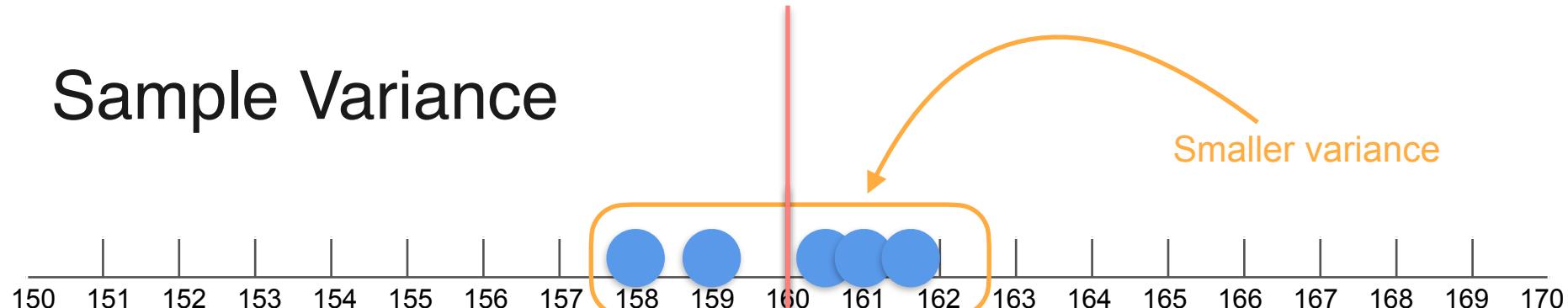
$$\mu = \frac{1}{5} (151 + 154 + 159 + 166 + 170) = 160$$

Sample Variance



$$\mu = \frac{1}{5} (158 + 159 + 160.5 + 161 + 161.5) = 160$$

Sample Variance



$$Var(X) = \sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2$$

population mean

population size

How to estimate population variance with a sample?

Sample Variance

Let's cheat and use
the sample mean

$$Var(X) = \sigma^2 = \frac{1}{N} \sum_{i=1}^N (x_i - \mu)^2 \longrightarrow \widehat{Var(X)} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

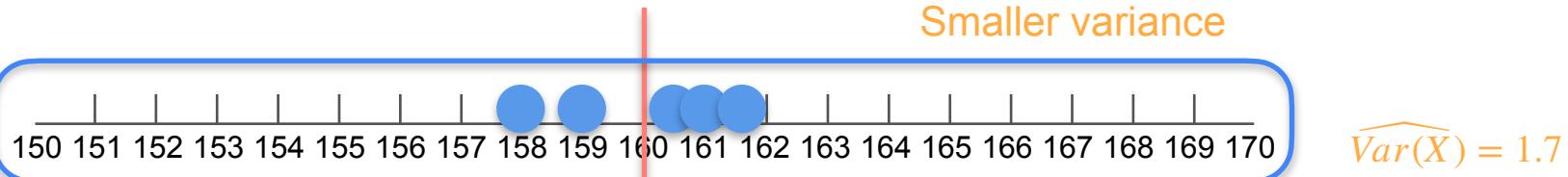
$$\downarrow \qquad \qquad Y = (X - \mu)^2 \qquad \qquad \uparrow$$
$$\mathbb{E}[Y] = \mu_Y = \frac{1}{N} \sum_{i=1}^N y_i \longrightarrow \bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$$

The population mean of Y

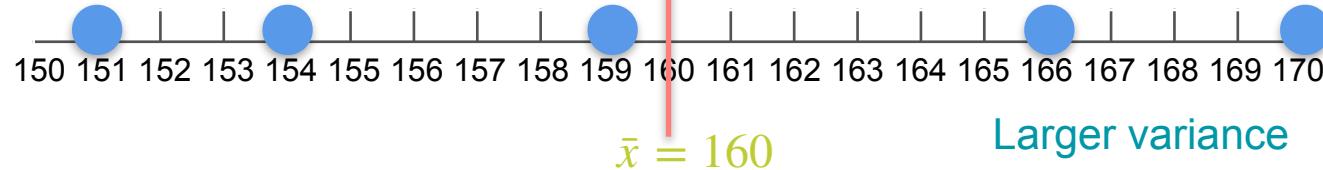
The sample mean of Y

Sample Variance

$$Var(X) = \sigma^2 = \frac{1}{N} \sum (x - \mu)^2 \longrightarrow \widehat{Var(X)} = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$



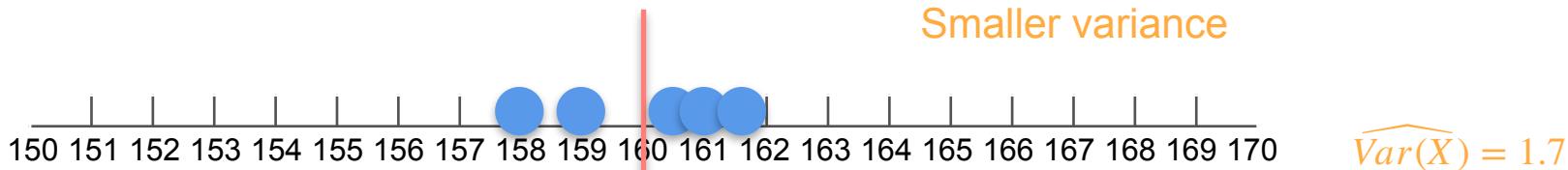
$$\widehat{Var(X)} = \frac{1}{5}((158-160)^2 + (159-160)^2 + (160.5-160)^2 + (161-160)^2 + (161.5-160)^2) = 1.7$$



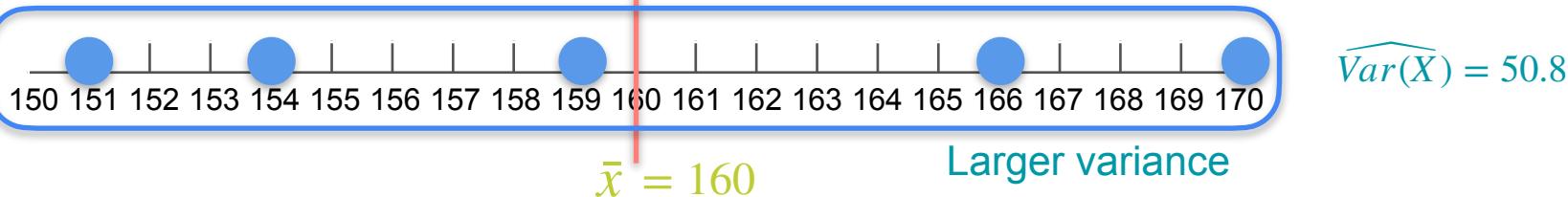
Sample Variance

This equation is “biased”
It underestimates the population variance

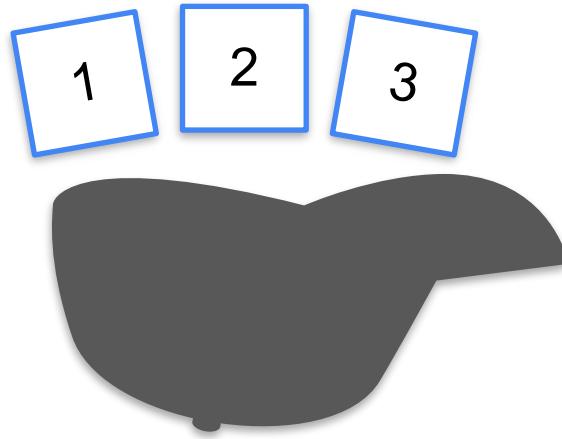
$$Var(X) = \sigma^2 = \frac{1}{N} \sum (x - \mu)^2 \longrightarrow \widehat{Var}(X) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$



$$\widehat{Var}(X) = \frac{1}{5} ((151-160)^2 + (154-160)^2 + (159-160)^2 + (166-160)^2 + (170-160)^2) = 50.8$$



Variance Estimation



$$\mu = \frac{1 + 2 + 3}{3} = \frac{6}{3} = 2$$

Variance Estimation

| | | |
|---|---|---|
| 1 | 2 | 3 |
|---|---|---|

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

| x | $x - \mu$ | $(x - \mu)^2$ |
|-----|-----------|---------------|
| 1 | -1 | 1 |
| 2 | 0 | 0 |
| 3 | 1 | 1 |

$$\frac{\sum (x - \mu)^2}{N} = \frac{2}{3}$$

σ^2
Population variance

Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$$n = 2$$

Samples

| | |
|---|---|
| 1 | 1 |
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 2 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |
| 3 | 3 |

Variance Estimation

1 2 3

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$$n = 2 \text{ Samples}$$
$$\bar{x}$$
$$\widehat{Var}(X) = \frac{\sum (x - \bar{x})^2}{n}$$

| | | |
|---|---|-----|
| 1 | 1 | 1 |
| 1 | 2 | 1.5 |
| 1 | 3 | 2 |
| 2 | 1 | 1.5 |
| 2 | 2 | 2 |
| 2 | 3 | 2.5 |
| 3 | 1 | 2 |
| 3 | 2 | 2.5 |
| 3 | 3 | 3 |

$$\widehat{Var}(X) = \frac{\sum (x - \bar{x})^2}{n}$$

| |
|------|
| 0 |
| 0.25 |
| 1 |
| 0.25 |
| 0 |
| 0.25 |
| 1 |
| 0.25 |
| 0 |

estimated variance

$$= 0.333$$

$$= \frac{1}{3}$$

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$$\begin{matrix} n = 2 \\ \text{Samples} \end{matrix}$$

| | |
|---|---|
| 1 | 1 |
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 2 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |
| 3 | 3 |

$$\bar{x}$$

$$\widehat{Var}(X) = \frac{\sum (x - \bar{x})^2}{n - 1}$$

| |
|------|
| 0 |
| 0.25 |
| 1 |
| 0.25 |
| 0 |
| 0.25 |
| 1 |
| 0.25 |
| 0 |

estimated variance

$$= 0.333$$

$$= \frac{1}{3}$$

Variance Estimation

$$\begin{matrix} 1 & 2 & 3 \end{matrix}$$

$$\mu = 2$$

$$\sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

$$\sigma^2 = \frac{2}{3}$$

$$\begin{matrix} n = 2 \\ \text{Samples} \end{matrix}$$

| | |
|---|---|
| 1 | 1 |
| 1 | 2 |
| 1 | 3 |
| 2 | 1 |
| 2 | 2 |
| 2 | 3 |
| 3 | 1 |
| 3 | 2 |
| 3 | 3 |

$$\bar{x}$$

| |
|-----|
| 1 |
| 1.5 |
| 2 |
| 1.5 |
| 2 |
| 2.5 |
| 2 |
| 2.5 |
| 3 |

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

| |
|-----|
| 0 |
| 0.5 |
| 2 |
| 0.5 |
| 0 |
| 0.5 |
| 2 |
| 0.5 |
| 0 |

estimated variance

$$= 0.667$$

$$= \frac{2}{3}$$

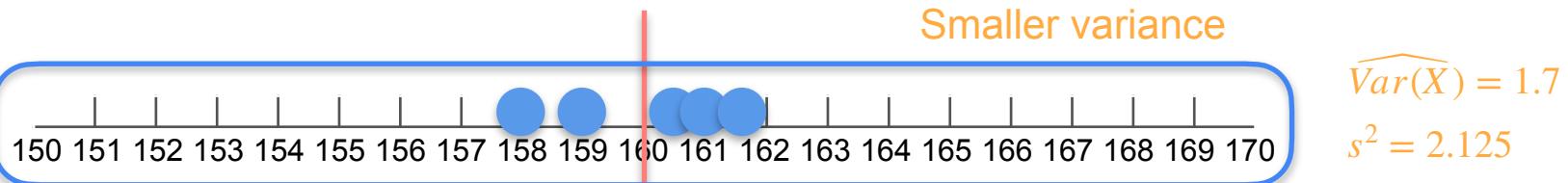
Variance Estimation

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1}$$

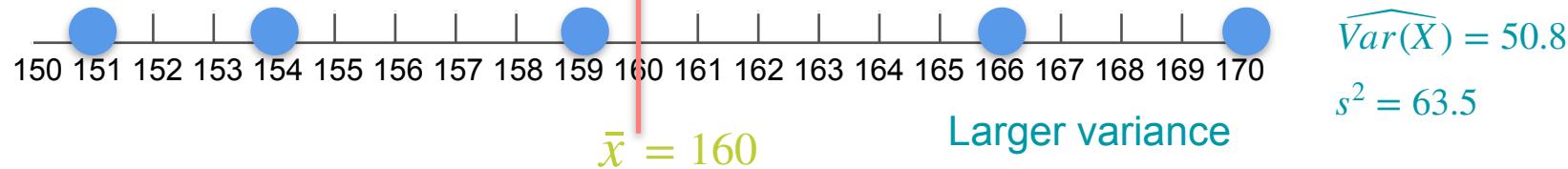
- $n - 1$ fixes bias when all you have is a sample
- As n gets big, the difference matters less
- If it matters, you may have too little data
- Some accepted statistical techniques use n

Sample Variance

$$Var(X) = \sigma^2 = \frac{1}{N} \sum (x - \mu)^2 \longrightarrow s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$



$$s^2 = \frac{1}{5-1} ((158-\bar{x})^2 + (159-\bar{x})^2 + (160.5-\bar{x})^2 + (161-\bar{x})^2 + (161.5-\bar{x})^2)$$



Variance Estimation

Population Variance Formula

$$Var(X) = \sigma^2 = \frac{1}{N} \sum (x - \mu)^2$$

Sample Variance Formula

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n - 1}$$

$$\widehat{\sigma}^2 = \widehat{Var(X)} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}$$

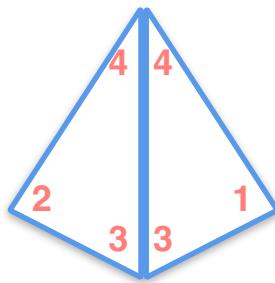


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Sample and Population

Law of Large Numbers

Law of Large Numbers



$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \mu = 2.5 \end{array}$$

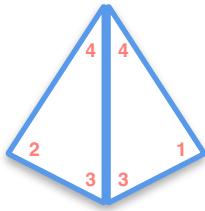
| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1 | 1.5 | 2 | 2.5 |
| 2 | 1.5 | 2 | 2.5 | 3 |
| 3 | 2 | 2.5 | 3 | 3.5 |
| 4 | 2.5 | 3 | 3.5 | 4 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | | | |
| 2 | 2 | | | |
| 3 | 3 | | | |
| 4 | 4 | | | |

Experiment:

Toss the 4-sided dice twice and record the average of your outcomes

Law of Large Numbers



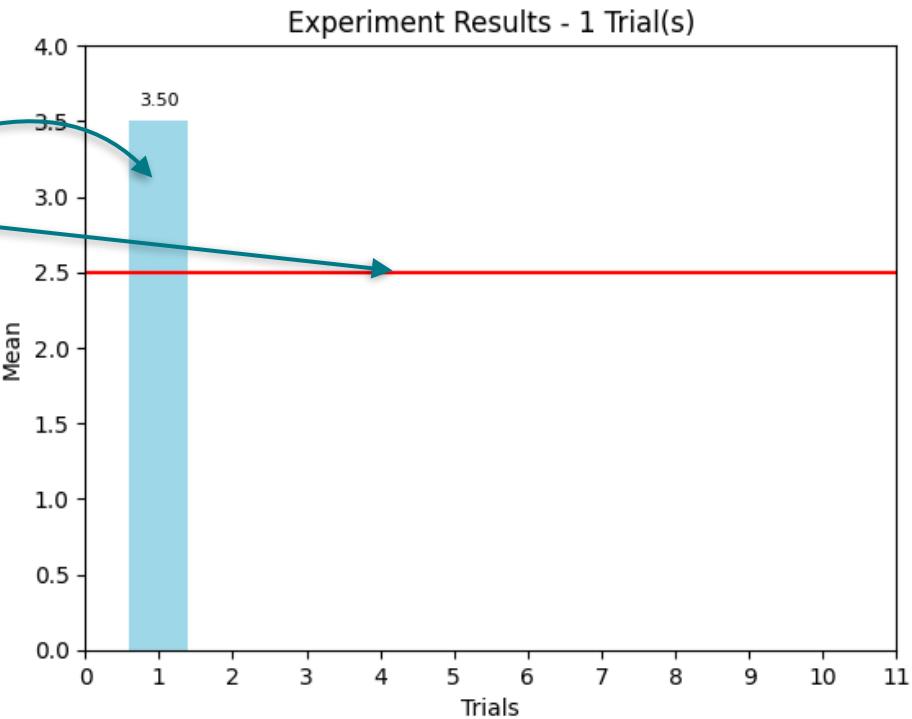
$$1 \ 2 \ 3 \ 4$$
$$\mu = 2.5$$

1 trial

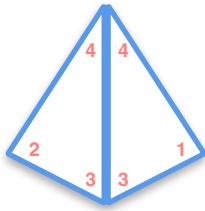
4,3

$$\bar{x}_1$$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |



Law of Large Numbers



1 2 3 4
 $\mu = 2.5$

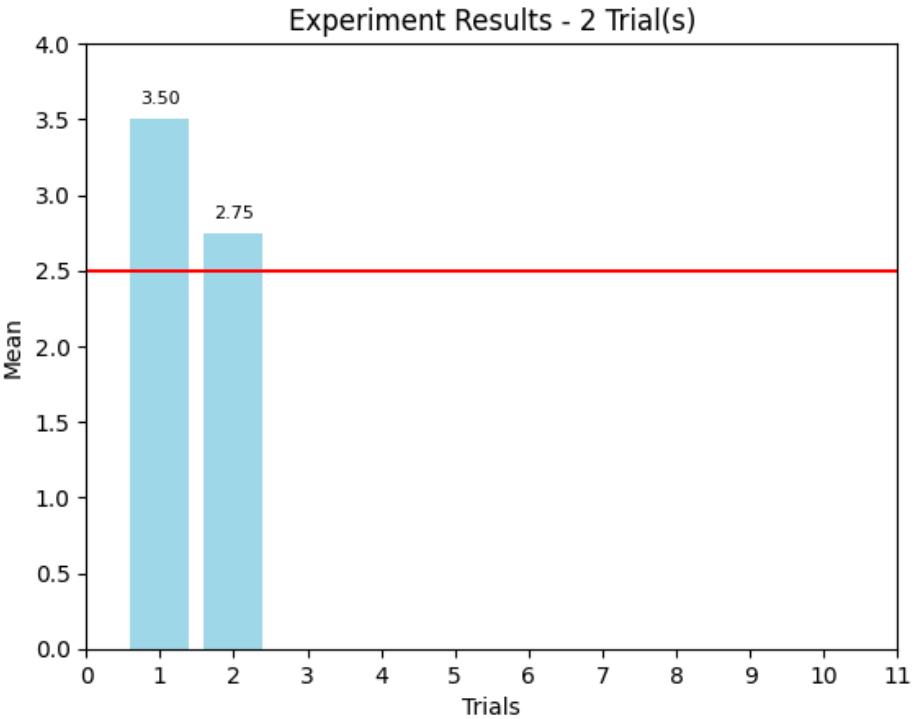
| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

1 trial

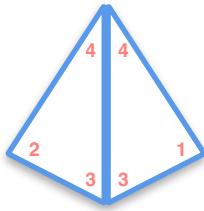
4,3

2 trials

3,4
1,3



Law of Large Numbers



1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

1 trial

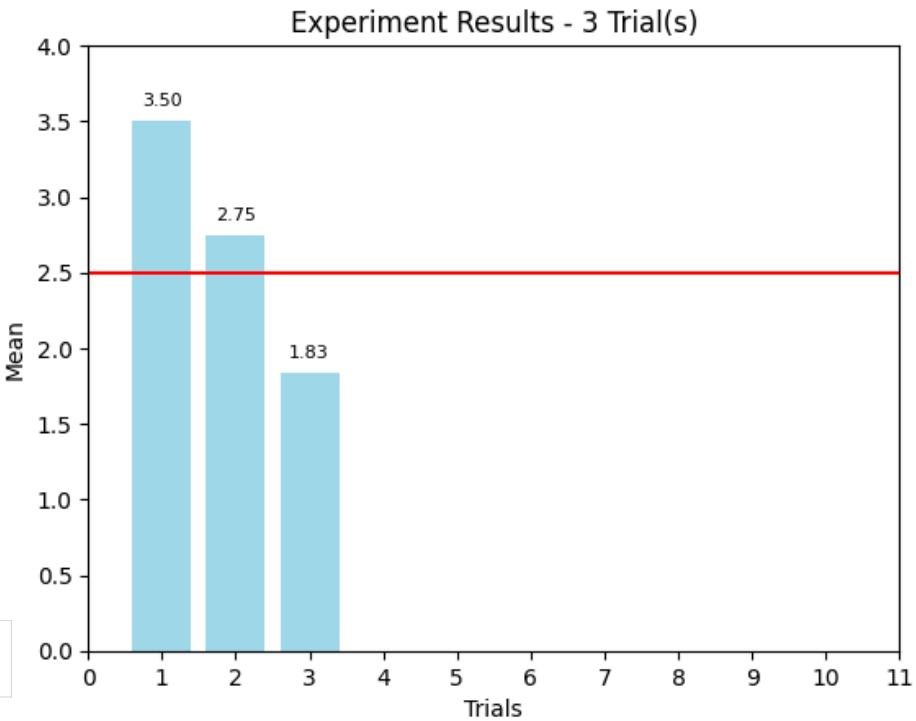
4,3

2 trials

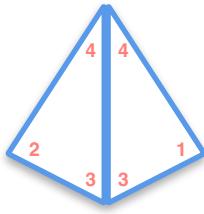
3,4
1,3

3 trials

3,1 1,4 1,1



Law of Large Numbers



1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

2 trials

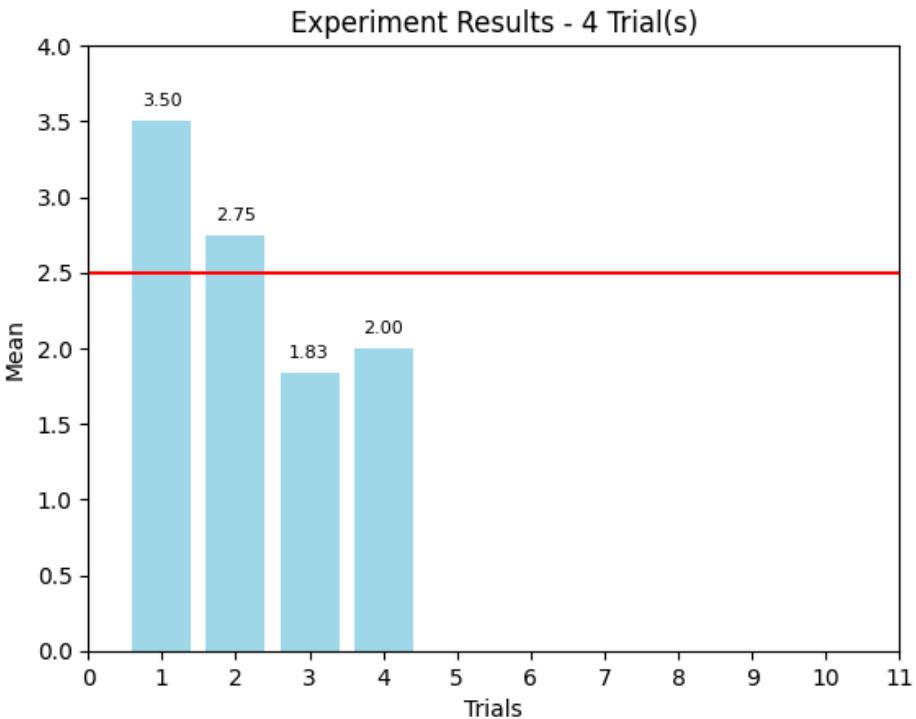
| |
|-----|
| 3,4 |
| 1,3 |

3 trials

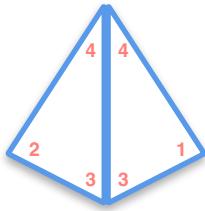
| | | |
|-----|-----|-----|
| 3,1 | 1,4 | 1,1 |
|-----|-----|-----|

4 trials

| | |
|-----|-----|
| 3,1 | 3,1 |
| 1,2 | 3,2 |

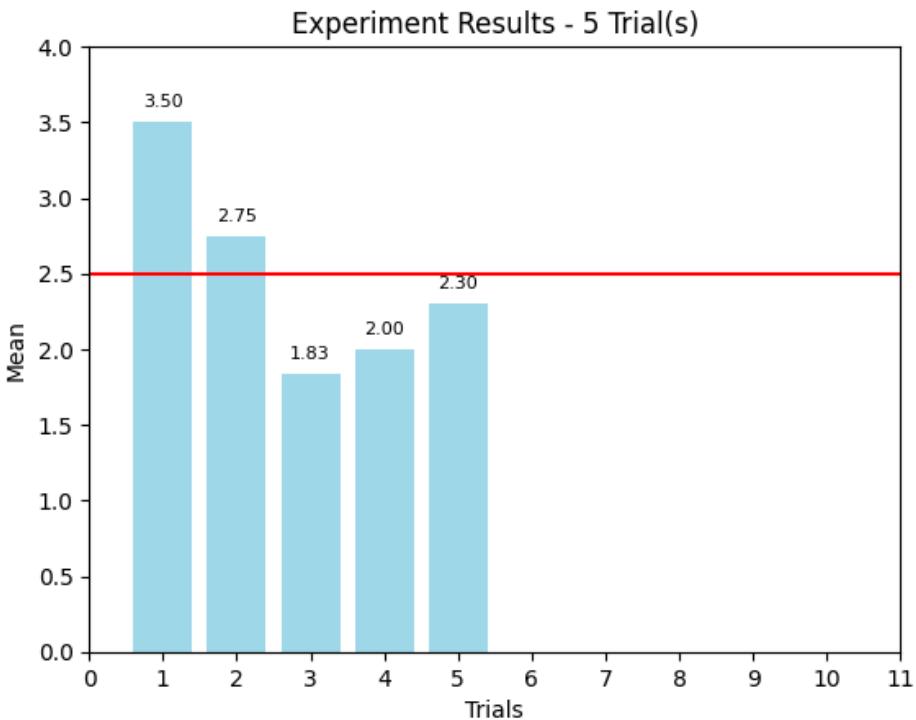


Law of Large Numbers

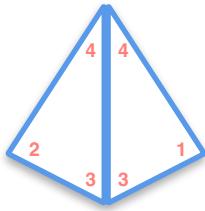


1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

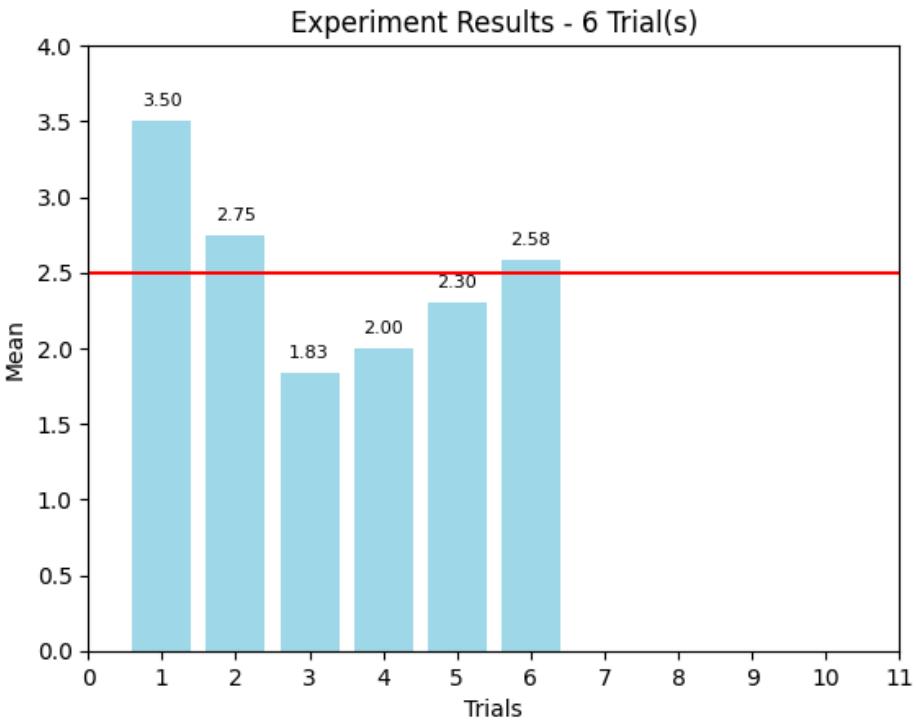


Law of Large Numbers

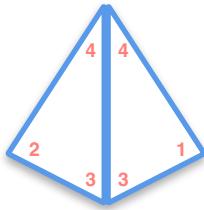


1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

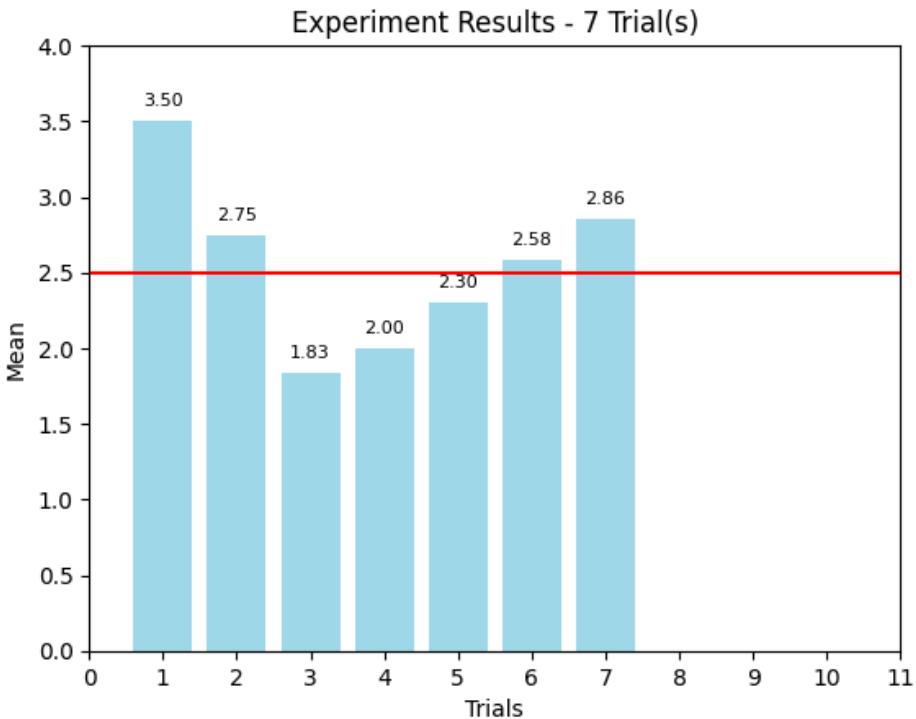


Law of Large Numbers

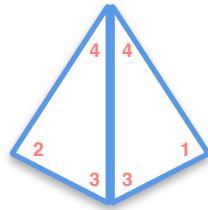


1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
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| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

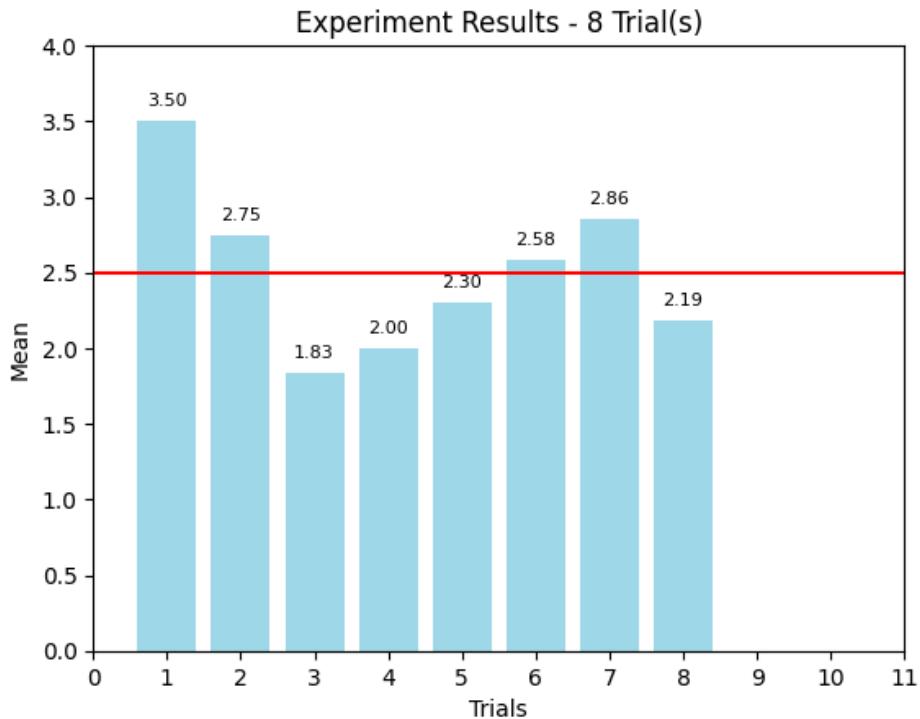


Law of Large Numbers

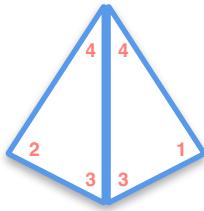


1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

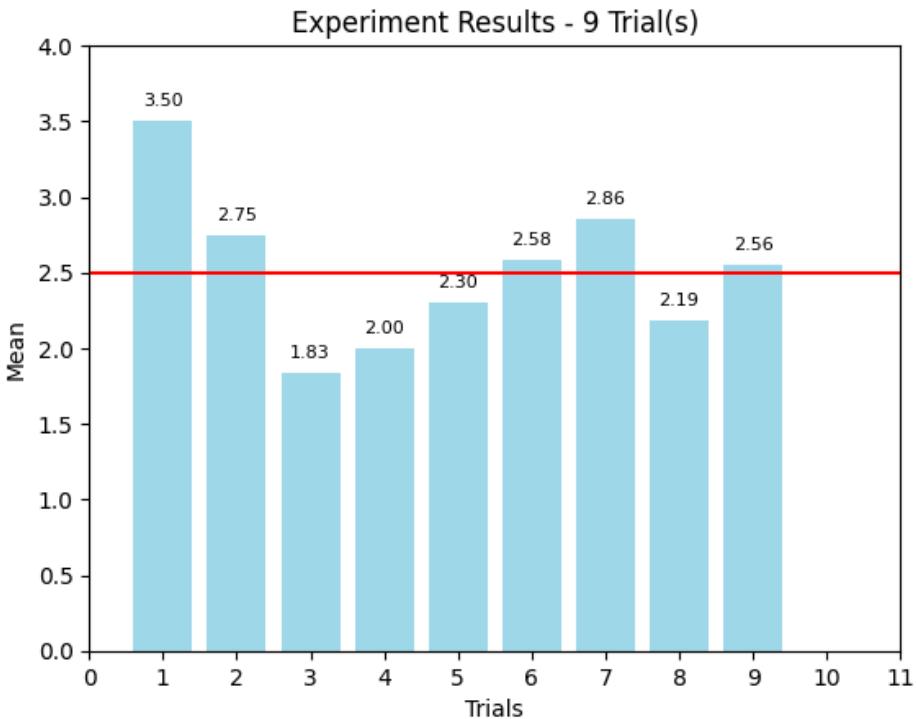


Law of Large Numbers

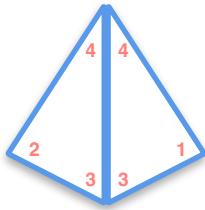


1 2 3 4
 $\mu = 2.5$

| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
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| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |

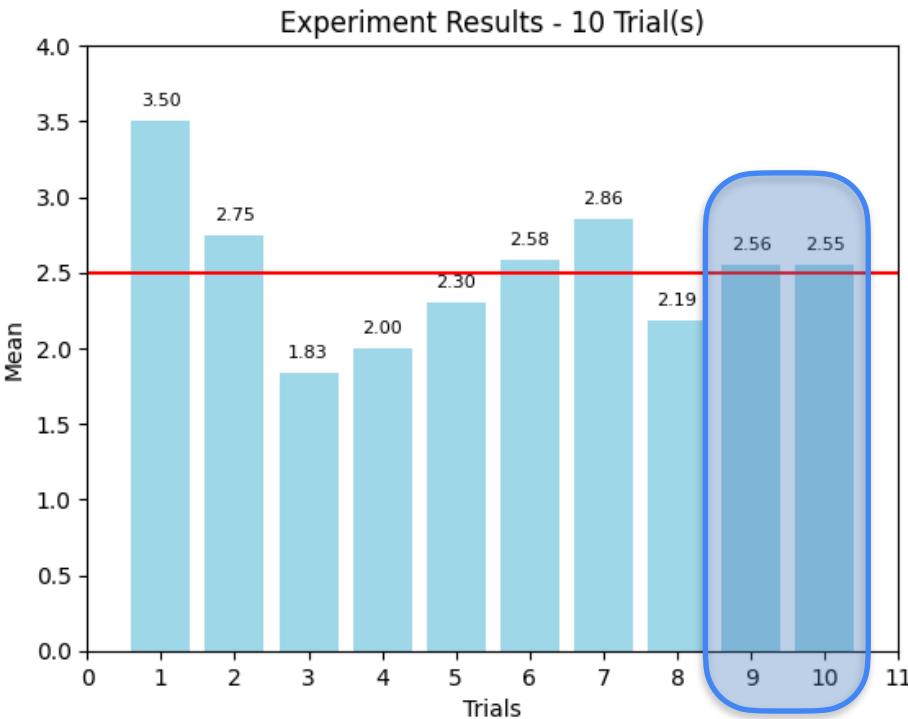


Law of Large Numbers



1 2 3 4
 $\mu = 2.5$

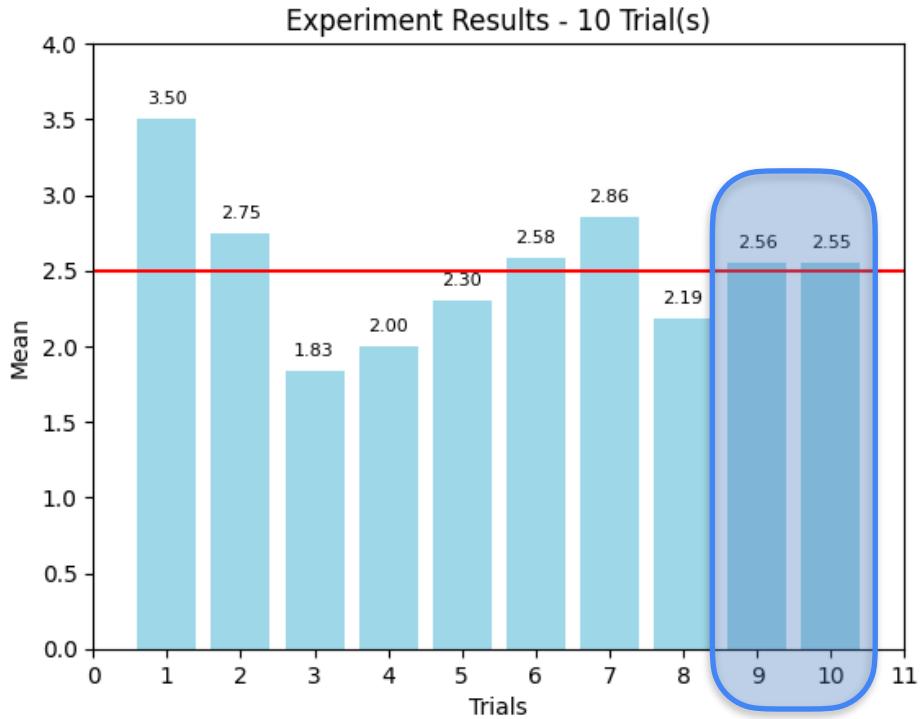
| | 1 | 2 | 3 | 4 |
|---|-----|-----|-----|-----|
| 1 | 1,1 | 1,2 | 1,3 | 1,4 |
| 2 | 2,1 | 2,2 | 2,3 | 2,4 |
| 3 | 3,1 | 3,2 | 3,3 | 3,4 |
| 4 | 4,1 | 4,2 | 4,3 | 4,4 |



Law of Large Numbers

As the sample size increases, the average of the sample will tend to get closer to the average of the entire population.

Law of Large Numbers



Law of Large Numbers

Law of Large Numbers

n : number of samples

X_i : is the i -th random sample from the population.

Each X_i are independent and identically distributed (i.i.d.)

as $n \rightarrow \infty$

$$\frac{X_1 + X_2 + X_3 + \dots + X_n}{n} \longrightarrow \mathbb{E}[X] = \mu_X$$

UNDER CERTAIN CONDITIONS

as $n \rightarrow \infty$

$$\frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathbb{E}[X] = \mu_X$$

Law of Large Numbers

UNDER CERTAIN CONDITIONS

- Sample is randomly drawn.
- Sample size must be sufficiently large.
- Independent observations.



DeepLearning.AI

Sample and Population

**Central Limit Theorem
Discrete Random Variable**

Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$



$$P(T) = 0.5$$

Random variable $\rightarrow X$ number of heads when a coin is flipped n times

$$\text{If } n = 1$$

$$X = 1$$

$$X = 0$$

Discrete Random Variable

Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

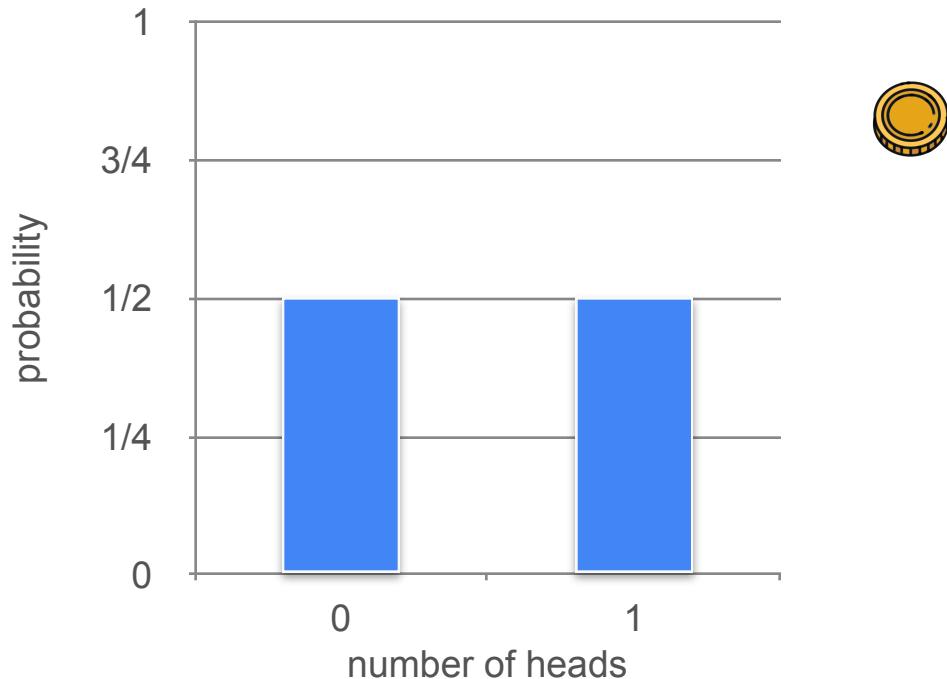


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

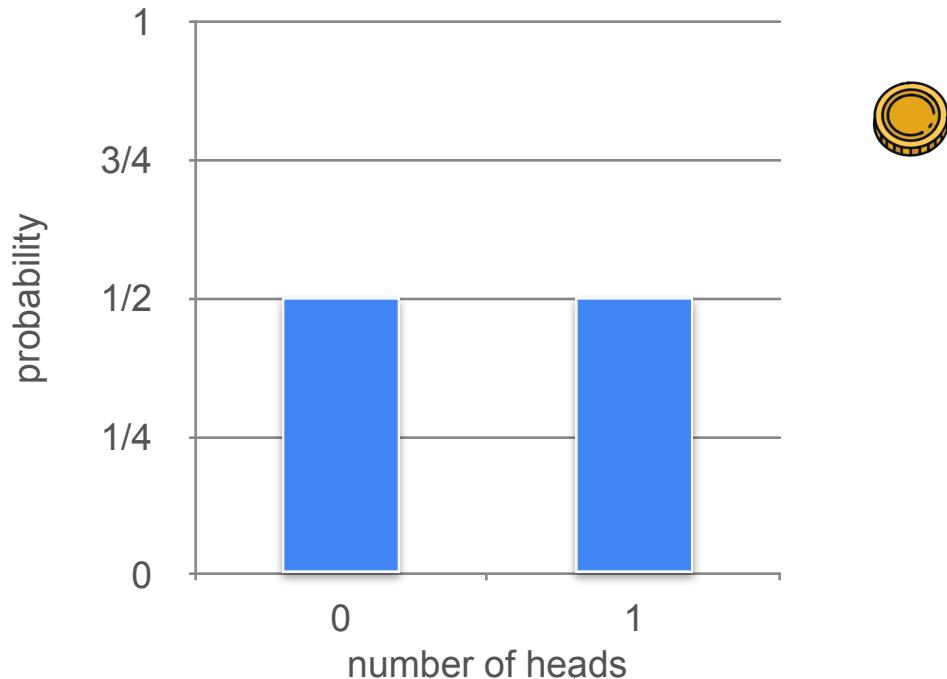


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

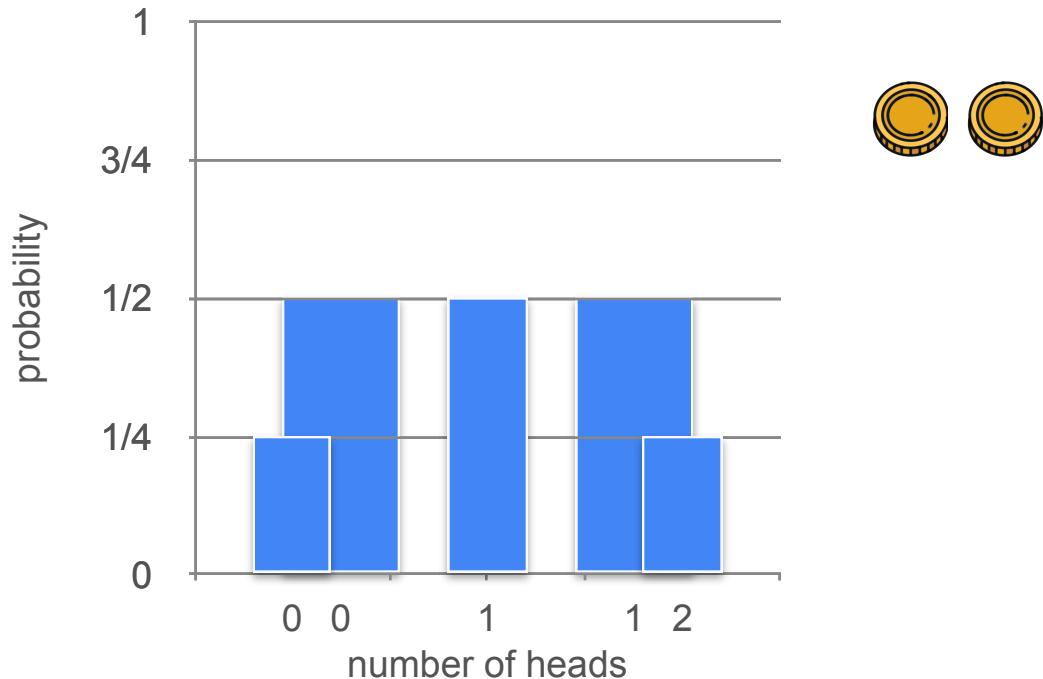


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

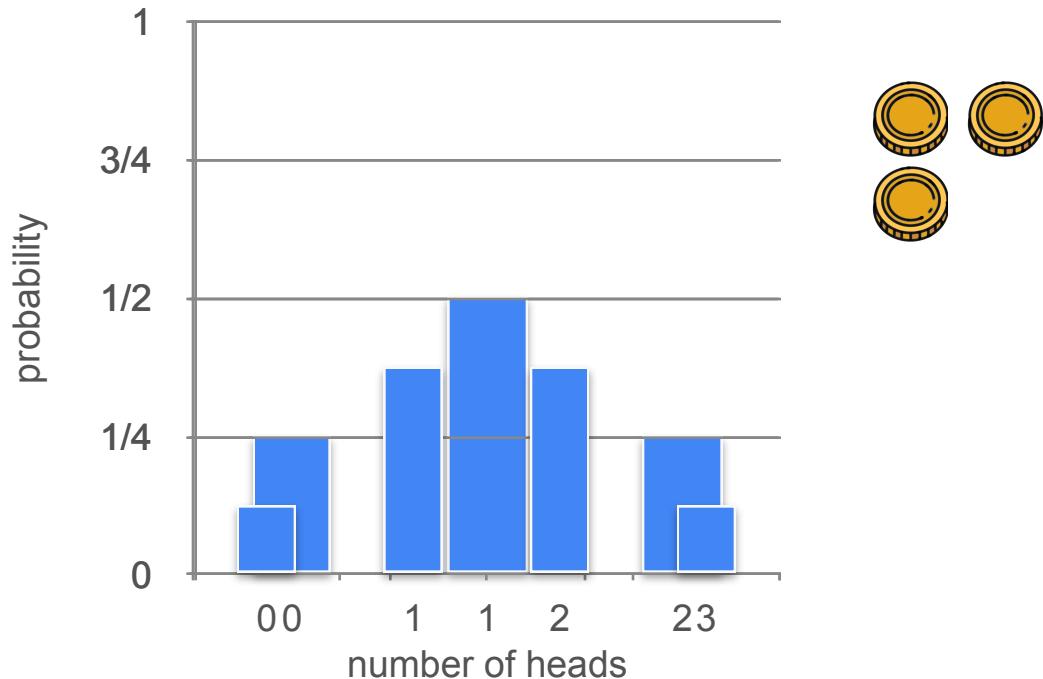


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

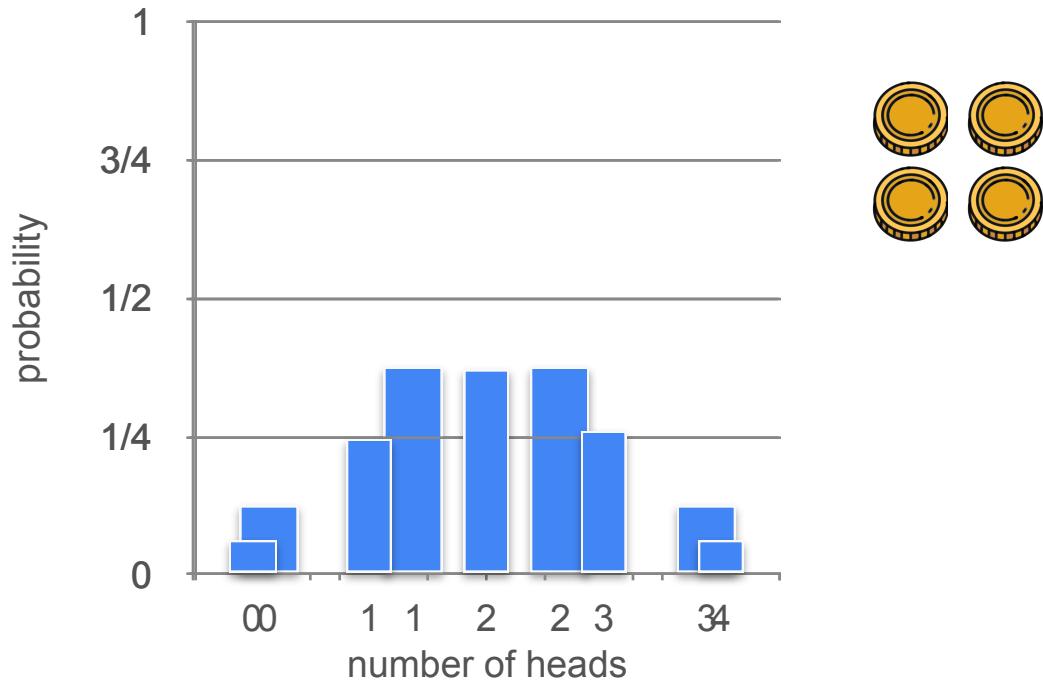


$$P(T) = 0.5$$

$$X = 1$$

$$X = 0$$

What can we say about the probability distribution when the number of coin flips increases?



Central Limit Theorem (CLT) - Example 1



$$P(H) = 0.5$$

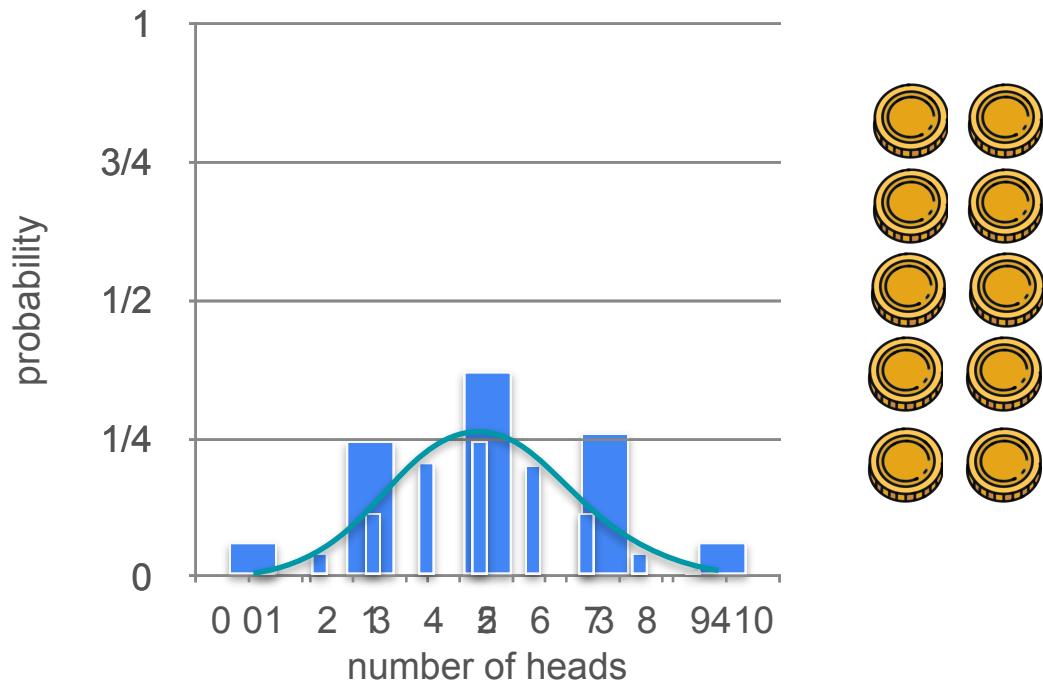


$$P(T) = 0.5$$

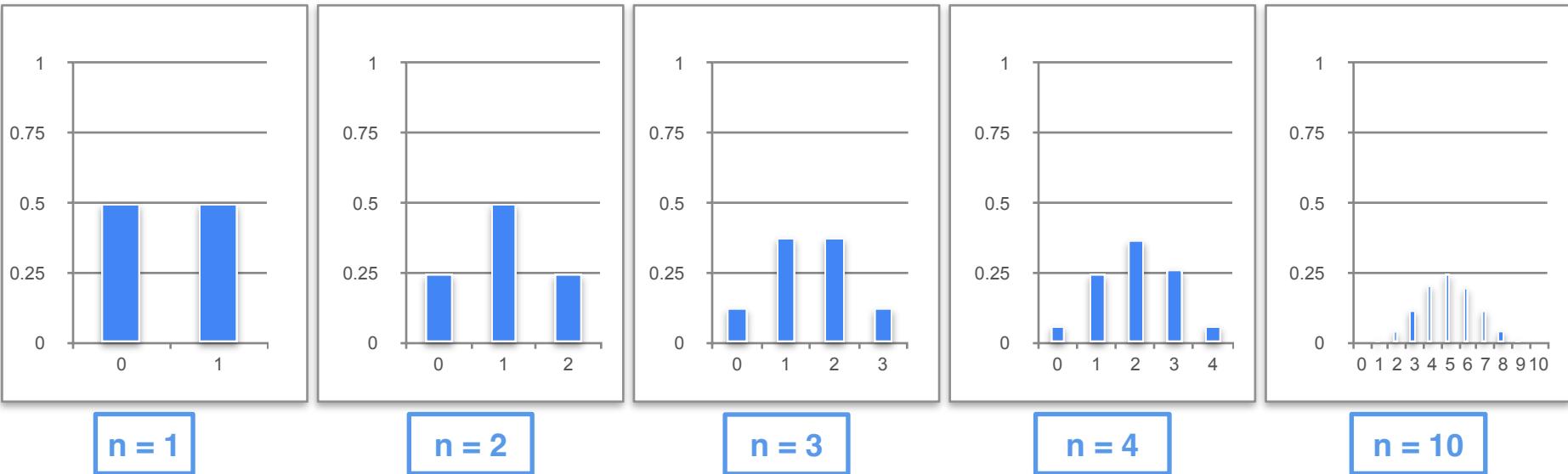
$$X = 1$$

$$X = 0$$

What can we say about the distribution when the number of coins we flip increases?



Central Limit Theorem (CLT) - Example 1



As n increases, the probability distribution becomes closer to a Gaussian distribution

Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5$$



$$\mathbf{P}(T) = 0.5$$

Random variable



X number of heads when a coin is flipped n times

$$\mu = np = n\mathbf{P}(H)$$



$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$



Central Limit Theorem (CLT) - Example 1



$$\mathbf{P}(H) = 0.5 \quad \mathbf{P}(T) = 0.5$$

$$\mu = np = n\mathbf{P}(H)$$
Two gold-colored coins, one showing 'H' and the other showing 'T'.

$$\sigma^2 = np(1 - p) = n\mathbf{P}(H)\mathbf{P}(T)$$
Two gold-colored coins, one showing 'H' and the other showing 'T'.



$$n = 1$$

$$\mu = np$$



$$\mu = 1 \times 0.5 = 0.5$$

$$\sigma^2 = np(1 - p)$$

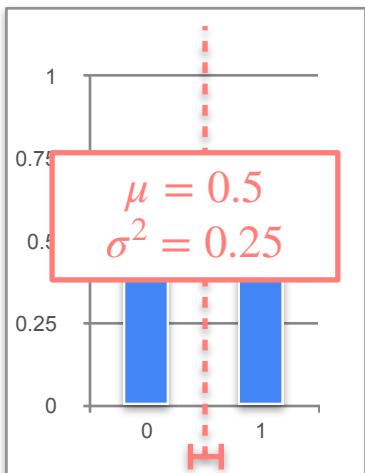


$$\sigma^2 = (1 \times 0.5)(0.5) = 0.25$$

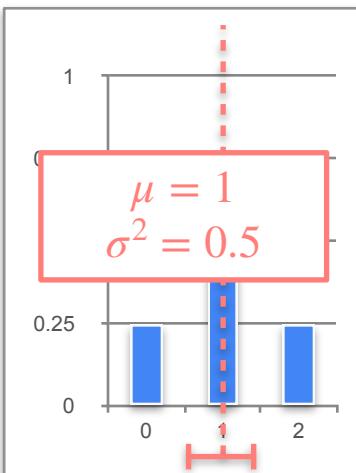
Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

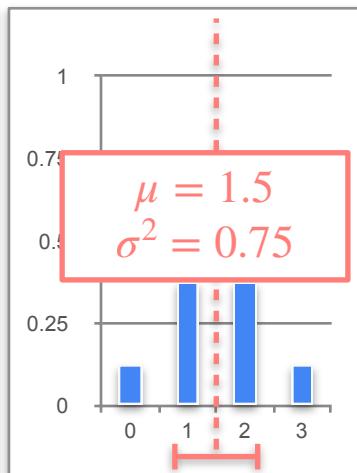
$$\sigma^2 = np(1 - p)$$



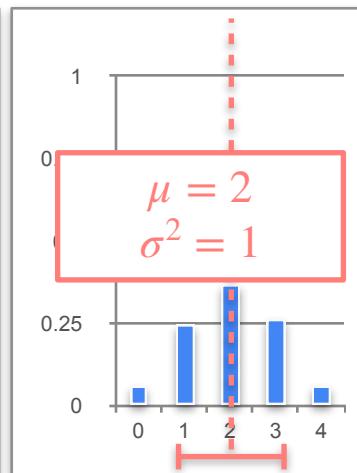
$n = 1$



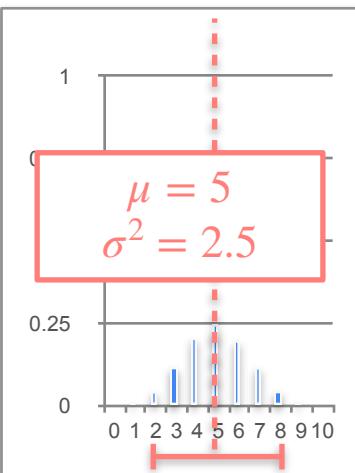
$n = 2$



$n = 3$



$n = 4$



$n = 10$

As n increases, the probability distribution becomes closer to a gaussian distribution

Central Limit Theorem (CLT) - Example 1

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

$$\begin{aligned}\mu &= 0.5 \\ \sigma^2 &= 0.25\end{aligned}$$

$$\begin{aligned}\mu &= 1 \\ \sigma^2 &= 0.5\end{aligned}$$

$$\begin{aligned}\mu &= 1.5 \\ \sigma^2 &= 0.75\end{aligned}$$

$$\begin{aligned}\mu &= 2 \\ \sigma^2 &= 1\end{aligned}$$

$$\begin{aligned}\mu &= 5 \\ \sigma^2 &= 2.5\end{aligned}$$

as n become sufficiently large we will get a normal distribution with parameters

$$\mu = np$$

$$\sigma^2 = np(1 - p)$$

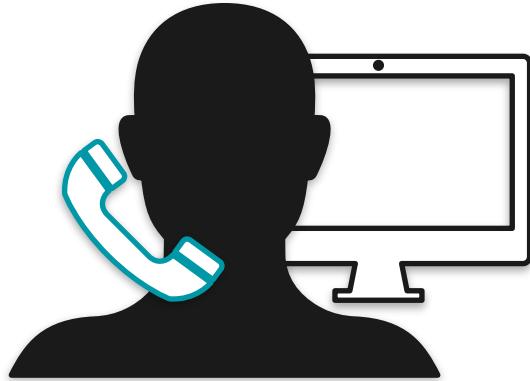


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Sample and Population

**Central Limit Theorem
Continuous Random Variable**

Uniform Distribution: Motivation



You're calling a tech support line. They can answer any time between zero and 15 minutes and if they don't answer in this time, the line is disconnected.

X = "Wait time for a called to be answered "

$$X \sim \mathcal{U}(0,15)$$

Central Limit Theorem (CLT) - Example 2

$$n = 1$$

$$Y_1 = \frac{X_1}{1}$$

Record the average of all n experiments

$$n = 2$$

$$Y_2 = \frac{X_1 + X_2}{2}$$

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$

$$n = 3$$

$$Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

⋮

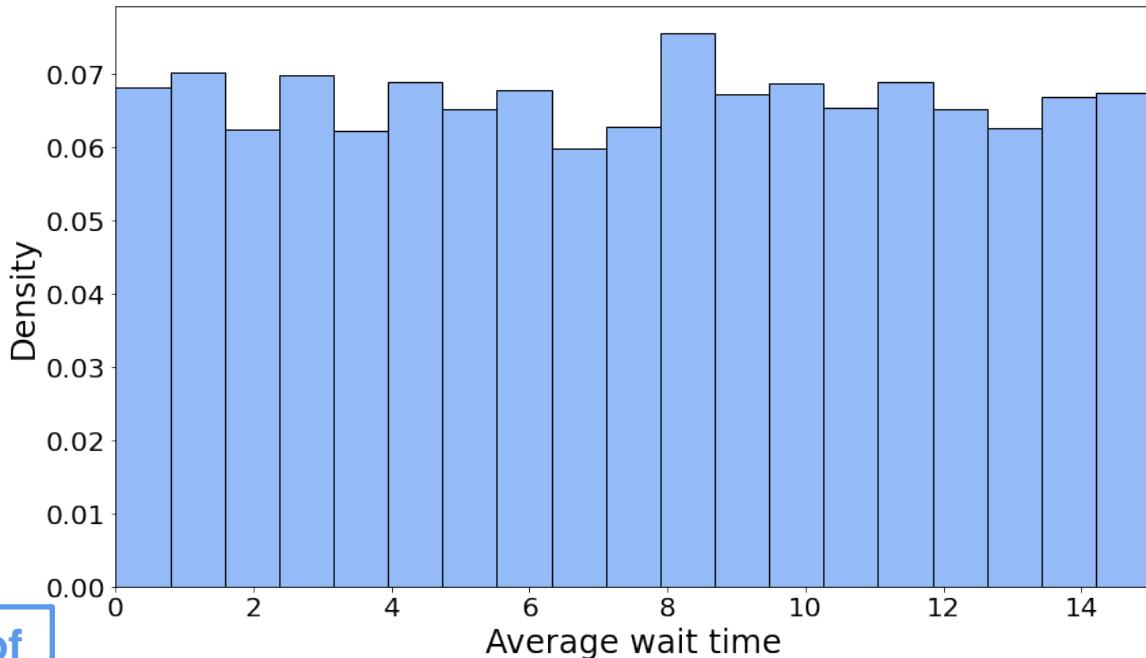
⋮

Can we say anything about the distribution of this average?

Central Limit Theorem (CLT) - Example 2

$$n = 1 \quad Y_1 = \frac{X_1}{1}$$

Create many samples of Y_1 so you can get a pretty histogram



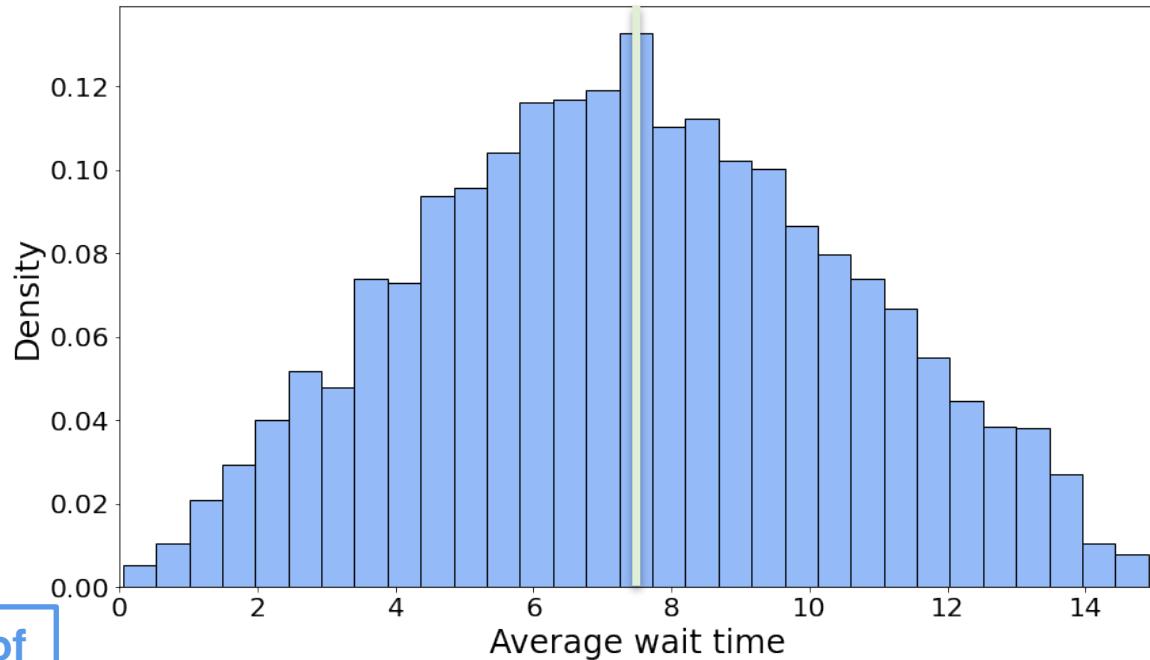
What happens to the distribution of these averages as n increases?

Central Limit Theorem (CLT) - Example 2

$$n = 2 \quad Y_2 = \frac{X_1 + X_2}{2}$$

Create many samples of Y_2 so you can get a pretty histogram

What happens to the distribution of these averages as n increases

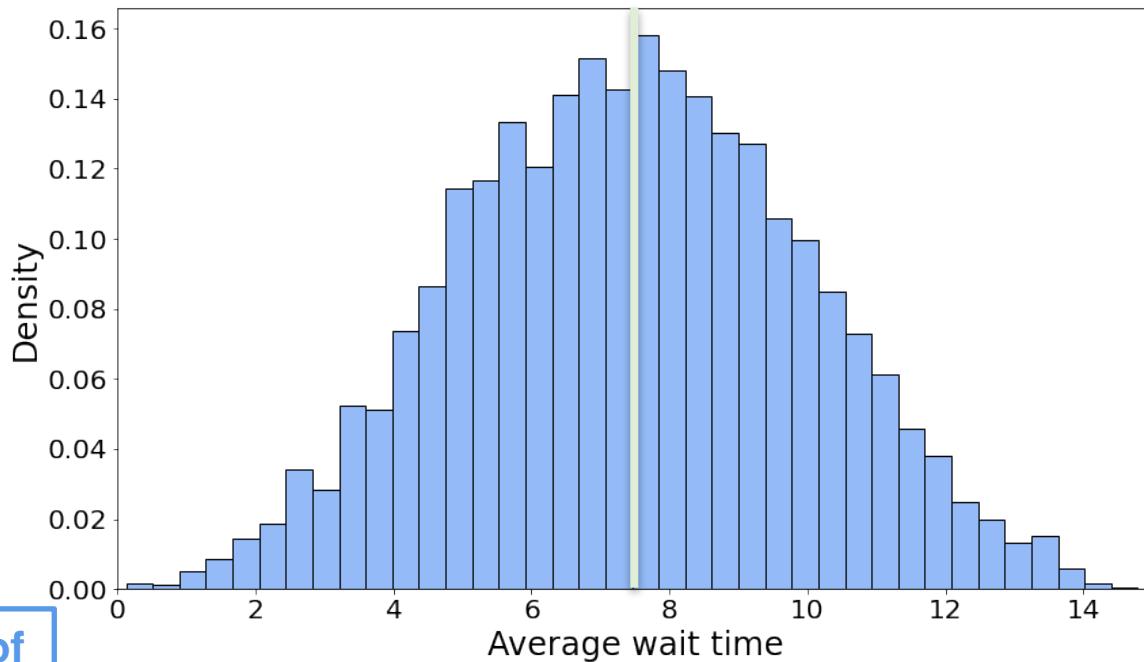


Central Limit Theorem (CLT) - Example 2

$$n = 3 \quad Y_3 = \frac{X_1 + X_2 + X_3}{3}$$

Create many samples of Y_3 so you can get a pretty histogram

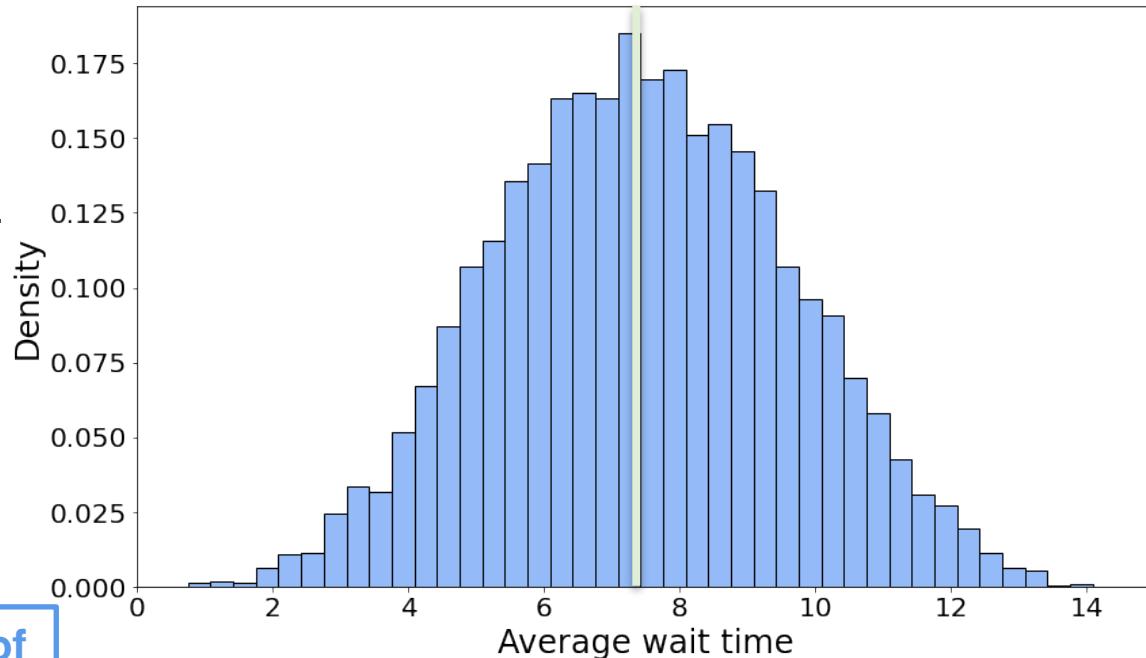
What happens to the distribution of these averages as n increases



Central Limit Theorem (CLT) - Example 2

$$n = 4 \quad Y_4 = \frac{X_1 + X_2 + X_3 + X_4}{4}$$

Create many samples of Y_4 so you can get a pretty histogram

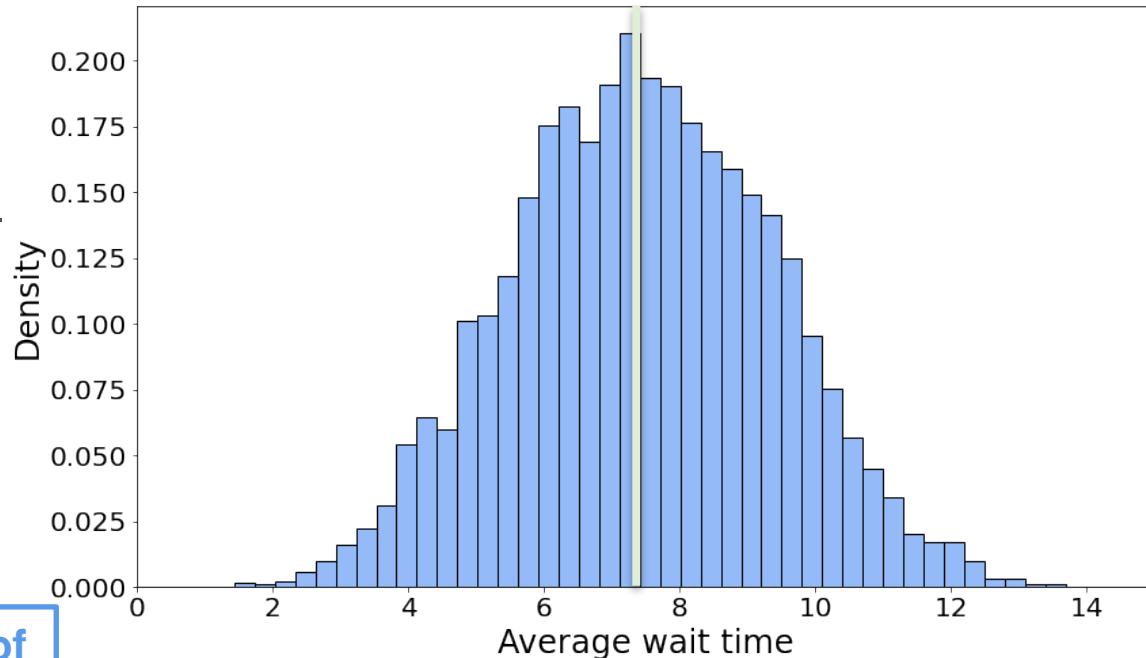


What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

$$n = 5 \quad Y_5 = \frac{X_1 + X_2 + X_3 + X_4 + X_5}{5}$$

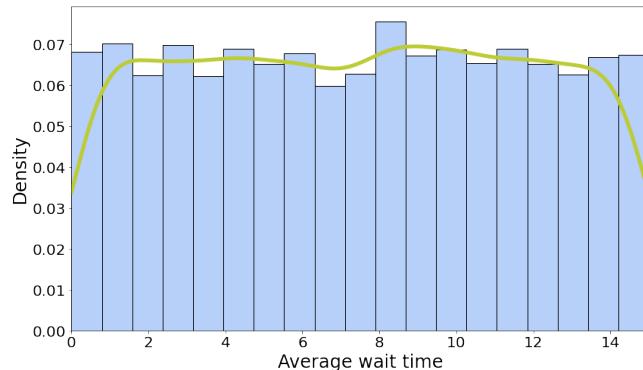
Create many samples of Y_5 so you can get a pretty histogram



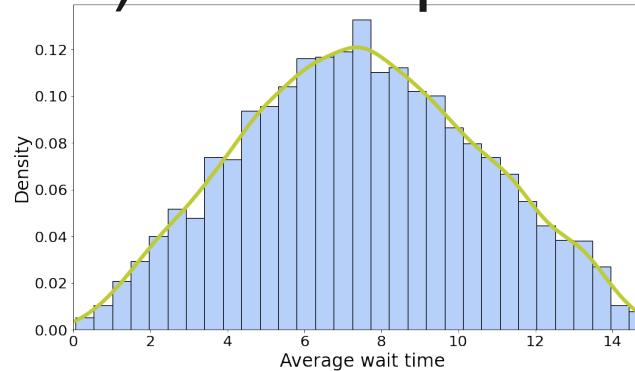
What happens to the distribution of these averages as n increases

Central Limit Theorem (CLT) - Example 2

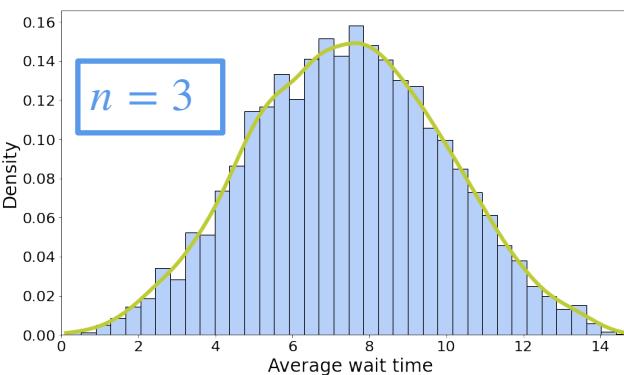
$n = 1$



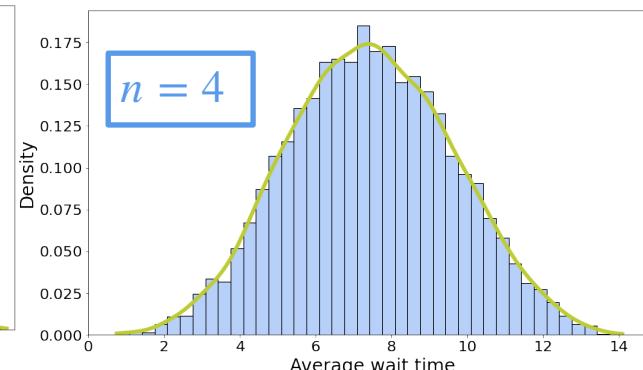
$n = 2$



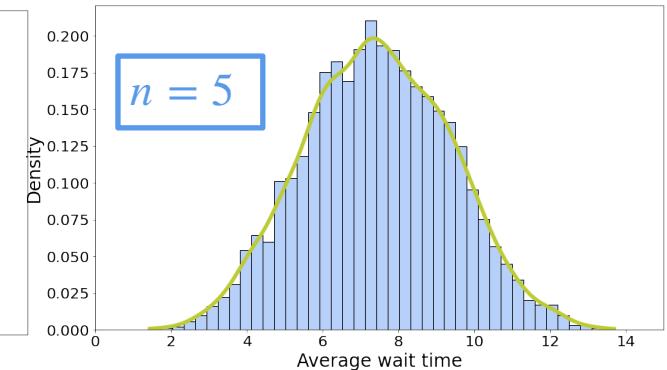
$n = 3$



$n = 4$



$n = 5$



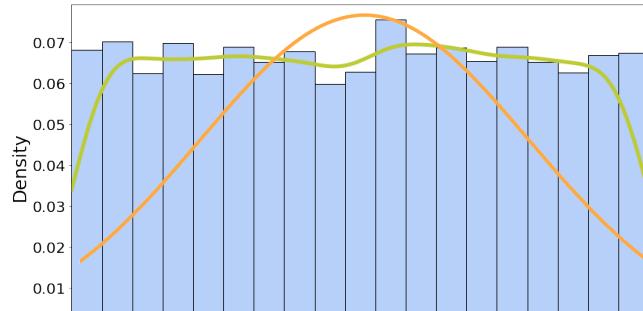
Central Limit Theorem (CLT) - Example 2

$$\mathbb{E}[Y_n] = \mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n X_i \right] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[X_i] = \frac{1}{n} n \mathbb{E}[X] = \mathbb{E}[X] = 7.5$$

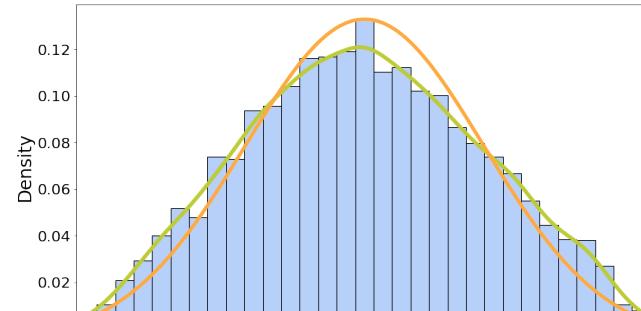
$$\begin{aligned} Var[Y_n] &= Var \left(\frac{1}{n} \sum_{i=1}^n X_i \right) = \frac{1}{n^2} \sum_{i=1}^n Var(X_i) \\ &= \frac{1}{n^2} n Var(X) = \frac{Var(X)}{n} = \frac{18.75}{n} \end{aligned}$$

Central Limit Theorem (CLT) - Example 2

$n = 1$

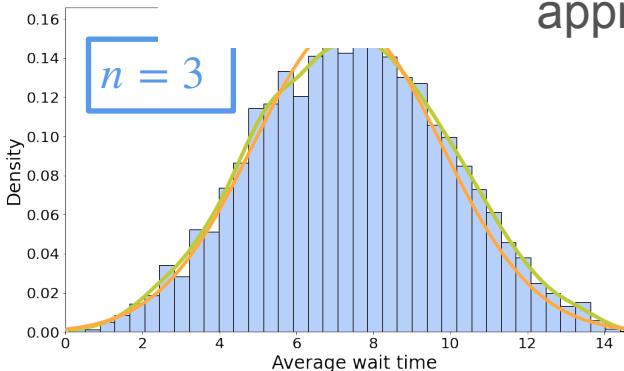


$n = 2$

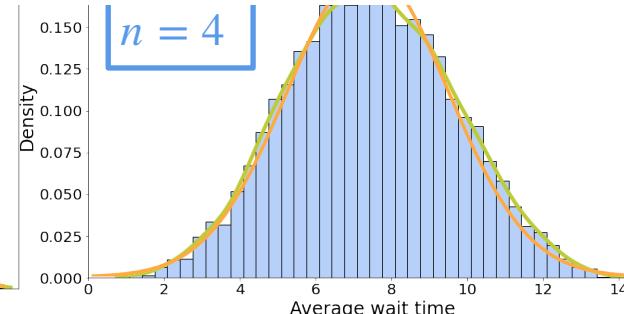


When you average a large enough number of variables, the distribution will approximately follow a normal distribution

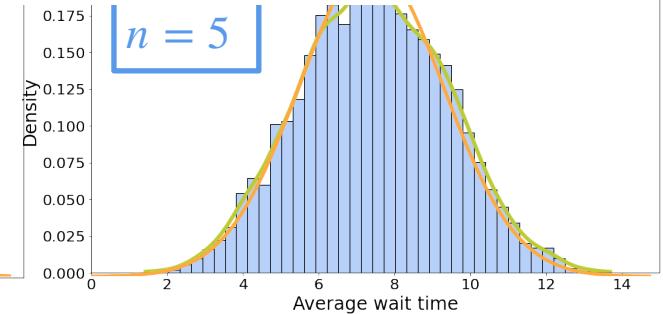
$n = 3$



$n = 4$

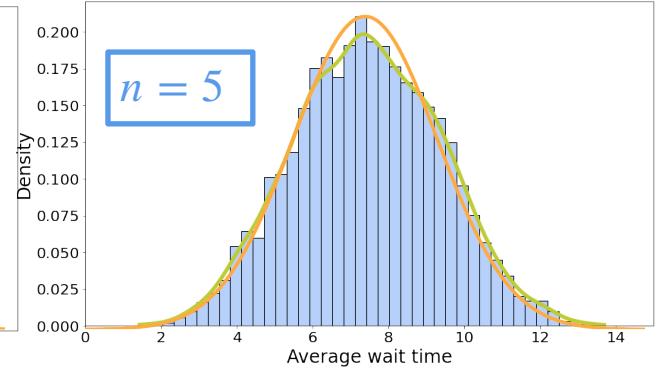
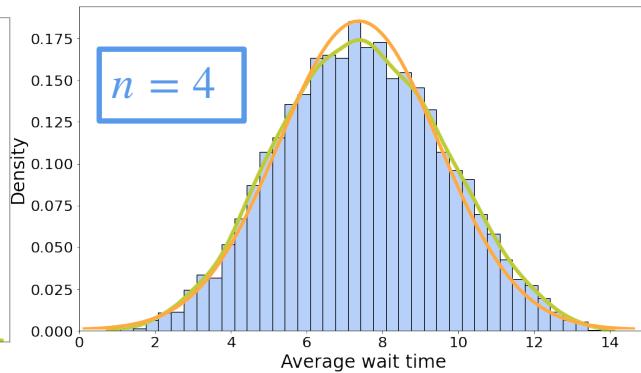
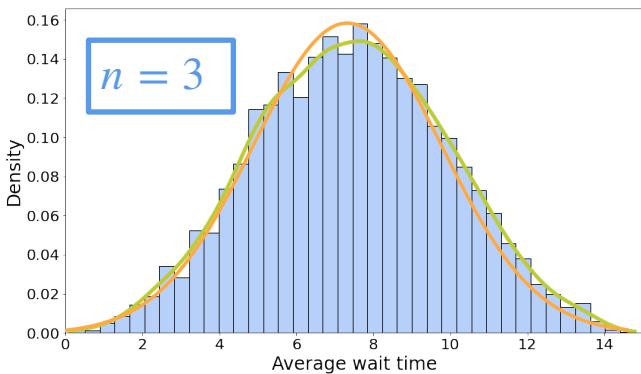


$n = 5$



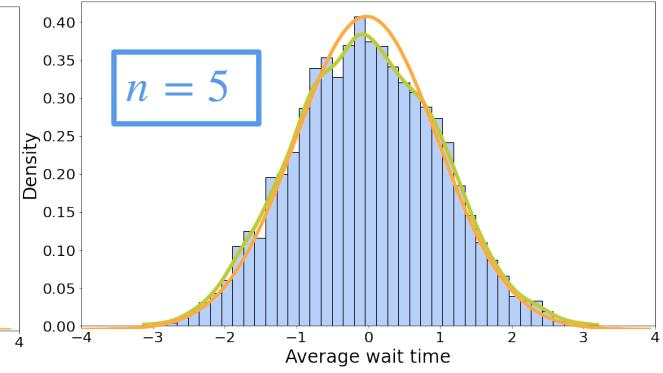
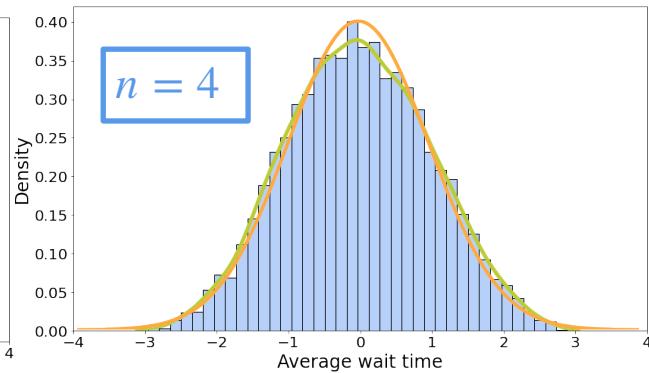
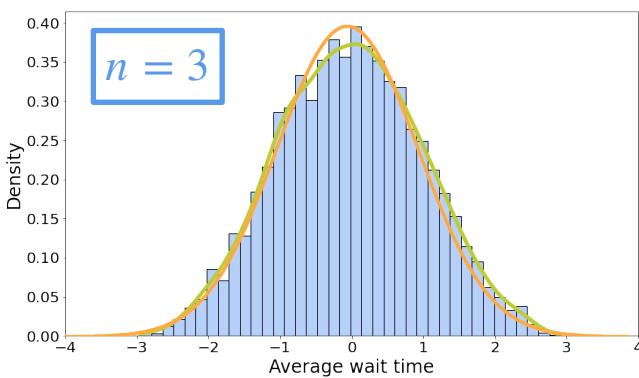
Central Limit Theorem (CLT) - Example 2

$$\frac{Y_n - 7.5}{\sqrt{18.75/n}}$$



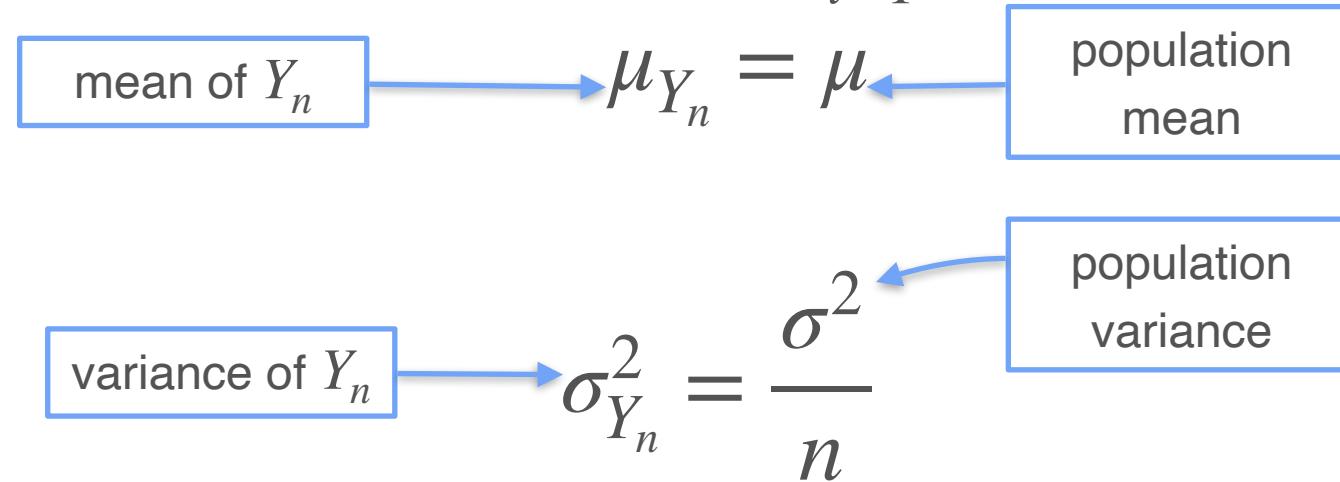
Central Limit Theorem (CLT) - Example 2

$$\frac{Y_n - 7.5}{\sqrt{18.75/n}} \xrightarrow{n \uparrow} \mathcal{N}(0,1)$$



Central Limit Theorem (CLT) - Example 2

$$Y_n = \frac{1}{n} \sum_{i=1}^n X_i$$



Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

$$\text{As } n \rightarrow \infty \quad \frac{1}{\cancel{\sqrt{n}}} \left(\frac{\sum_{i=1}^n X_i - \cancel{\frac{1}{n} n \mathbb{E}[X]}}{\sigma_X} \right) \cancel{\sqrt{n}} \sim \mathcal{N}(0, 1^2)$$

Central Limit Theorem (CLT) - Formal Definition

$$\text{As } n \rightarrow \infty \quad \frac{\frac{1}{n} \sum_{i=1}^n X_i - \mathbb{E}[X]}{\sigma_X} \sqrt{n} \sim \mathcal{N}(0, 1^2)$$

$$\text{As } n \rightarrow \infty \quad \frac{\sum_{i=1}^n X_i - n\mathbb{E}[X]}{\sqrt{n}\sigma_X} \sim \mathcal{N}(0, 1^2)$$

W3 Lesson 2



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Point Estimation

Point Estimation

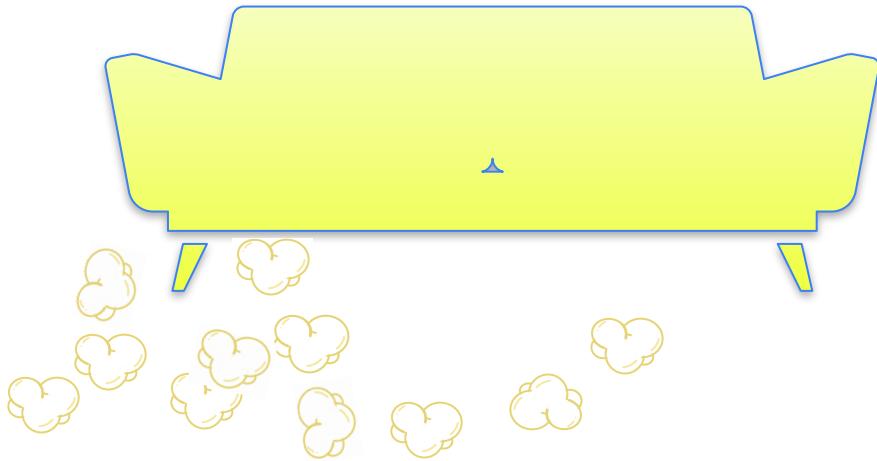


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Point Estimation

**Maximum Likelihood
Estimation: Motivation**

There's Popcorn on the Floor. What Happened?



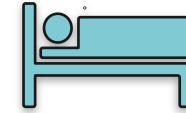
Movies



Board Games



Nap

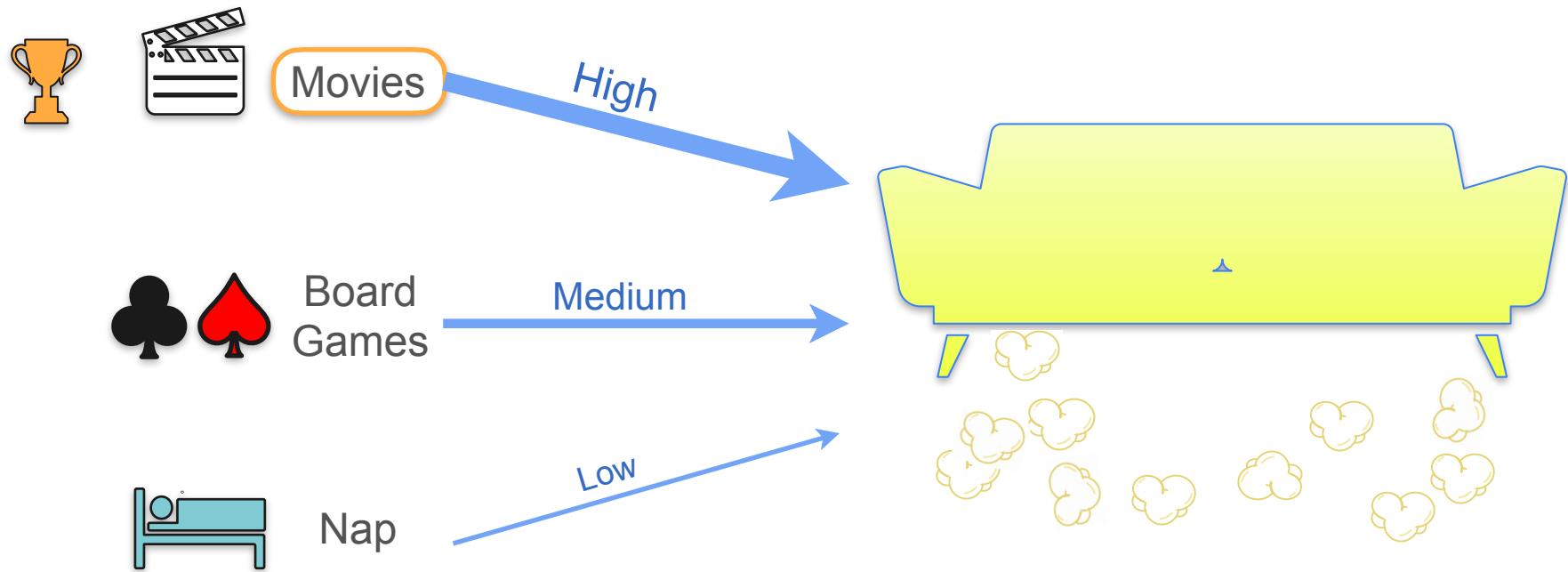


Quiz

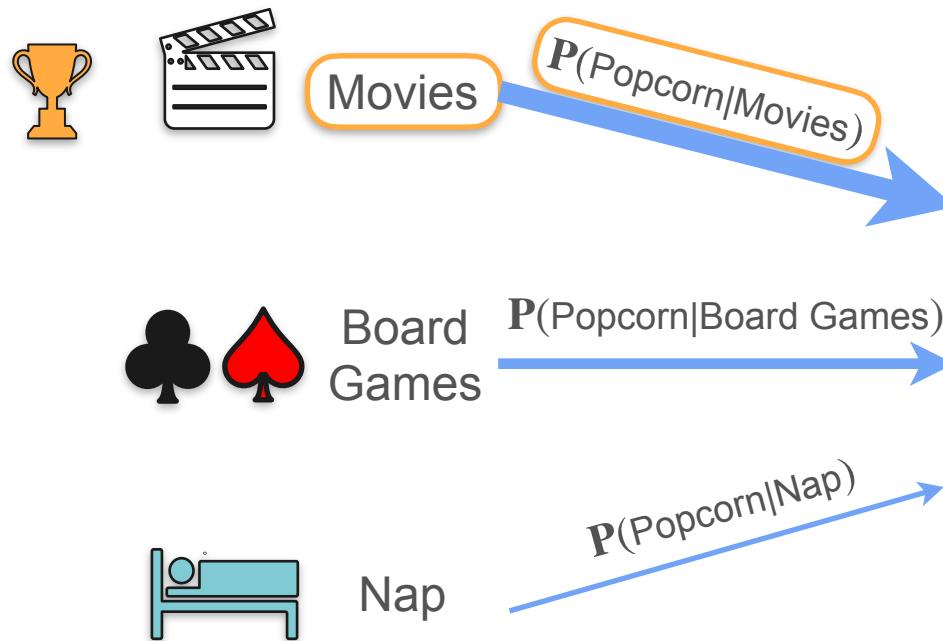
What do you think happened?

- A. People were watching a movie
- B. People were playing boardgames
- C. People were taking a nap

There's Popcorn on the Floor. What Happened?



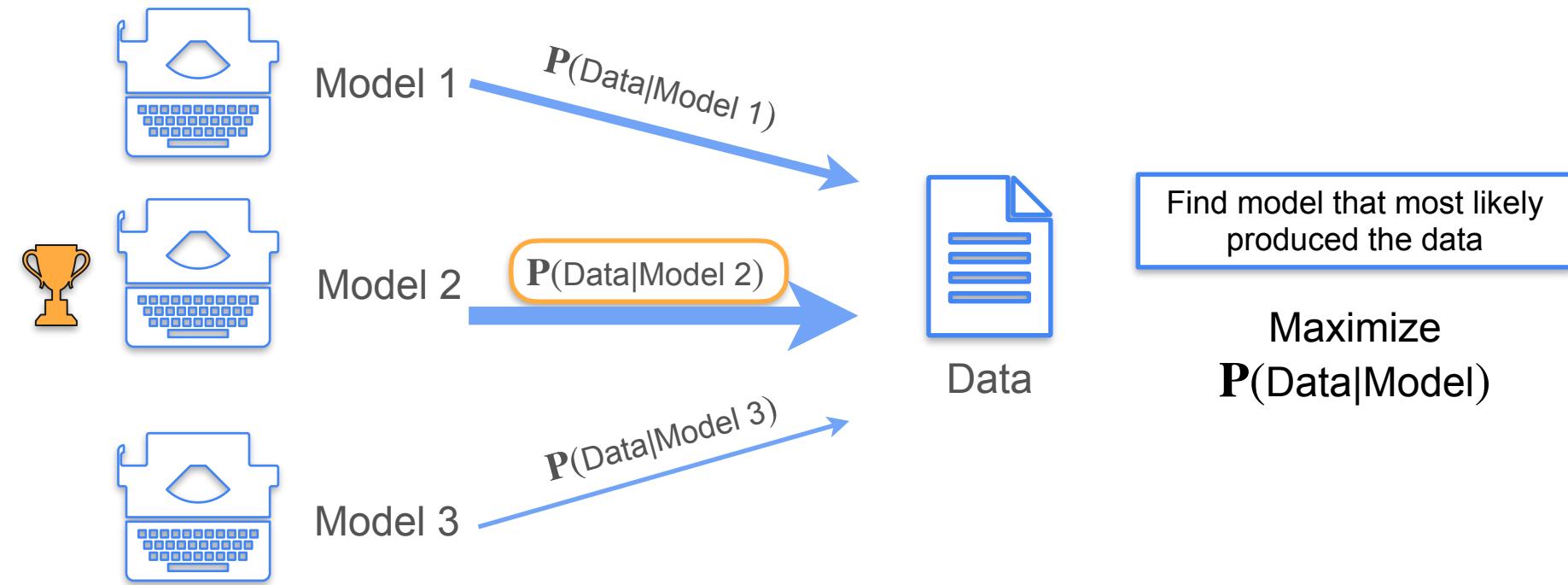
There's Popcorn on the Floor. What Happened?



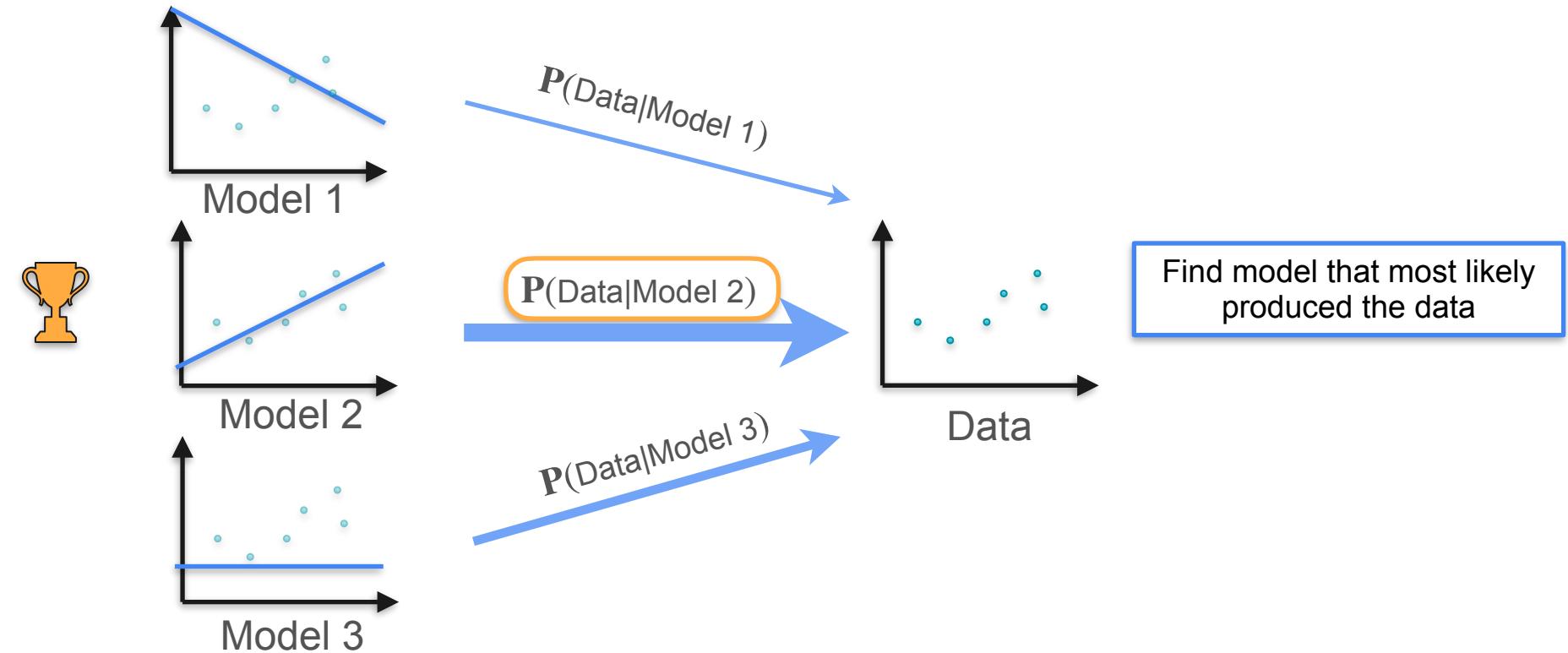
Find scenario that most likely leads to popcorn on the floor

Maximum Likelihood

Maximum Likelihood



Example: Linear Regression





DeepLearning.AI

Point Estimation

MLE: Bernoulli Example

Maximum Likelihood: Bernoulli Example



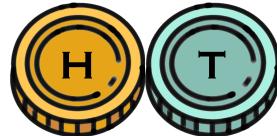
Coin 1



$$P(H) = 0.7$$

$$P(T) = 0.3$$

Coin 2



$$P(H) = 0.5$$

$$P(T) = 0.5$$

Coin 3



$$P(H) = 0.3$$

$$P(T) = 0.7$$

Maximum Likelihood: Bernoulli Example



| | | | | | | | | | | |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|--------------|
| Coin 1 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.7 | 0.3 | 0.3 = 0.0051 |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|--------------|

| | | | | | | | | | | |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|--------------|
| Coin 2 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 | 0.5 = 0.0010 |
|---------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|--------------|

| | | | | | | | | | | |
|---------------|-----|-----|-----|-----|-----|-----|-----|------|-----|---------------|
| Coin 3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.3 | 0.30 | 0.7 | 0.7 = 0.00003 |
|---------------|-----|-----|-----|-----|-----|-----|-----|------|-----|---------------|

Maximum Likelihood: Bernoulli Example



Coin 1
 $P(H) = 0.7$

$$P(8H, 2T | C_1) = 0.0051$$



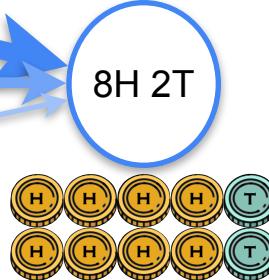
Coin 2
 $P(H) = 0.5$

$$P(8H, 2T | C_2) = 0.0010$$



Coin 3
 $P(H) = 0.3$

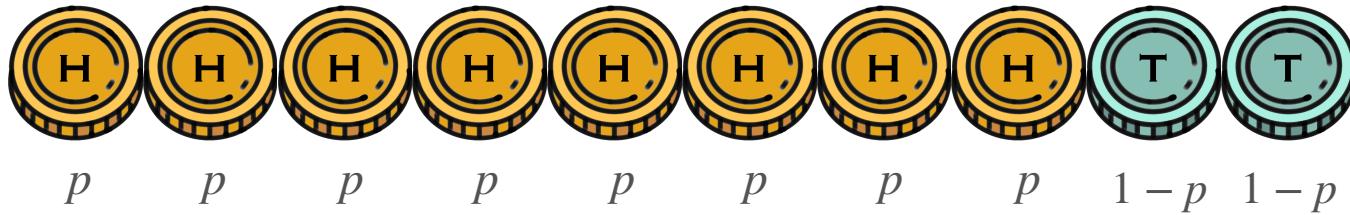
$$P(8H, 2T | C_3) = 0.00003$$



Find coin that most likely produced the 8 heads and 2 tails

Maximize
 $P(8H, 2T | \text{Coin})$

Maximum Likelihood: Bernoulli Example



Can you do any better?

$$p = \mathbf{P}(H)$$

$$p^8(1 - p)^2$$

You want p that maximizes the chances of seeing 8H

Maximum Likelihood: Bernoulli Example



$$p = \mathbf{P}(H) \quad \text{Likelihood} \quad L(p; 8H) = p^8(1-p)^2 \quad \text{Function of } p$$

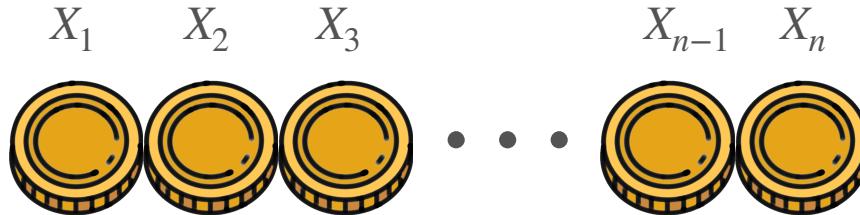
You want p that maximizes the chances of seeing 8H

$$\text{Log-likelihood} \quad \ell(p; 8H) = \log((p^8(1-p)^2)) = 8\log(p) + 2\log(1-p)$$

$$\frac{d}{dp} (8\log(p) + 2\log(1-p)) = \frac{8}{p} + \frac{2}{1-p}(-1) = 0 \rightarrow \hat{p} = \frac{8}{10}$$

Maximum Likelihood: Bernoulli Example

n coins
 k heads



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

Likelihood

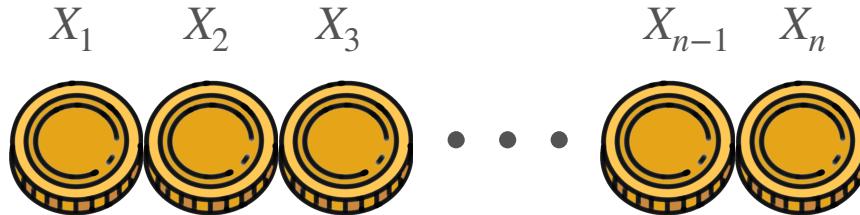
$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i}$$

If $x_i = 1$, $p^{[x_i]}(1-p)^{[1-x_i]} = p$
If $x_i = 0$, $p^{[x_i]}(1-p)^{[1-x_i]} = (1-p)$

$$\sum_{i=1}^n x_i = \# \text{ heads}$$

$$n - \sum_{i=1}^n x_i = \# \text{ tails}$$

Maximum Likelihood: Bernoulli Example



$$\mathbf{X} = (X_1, \dots, X_n)$$

$$X_i \stackrel{i.i.d}{\sim} \text{Bernoulli}(p)$$

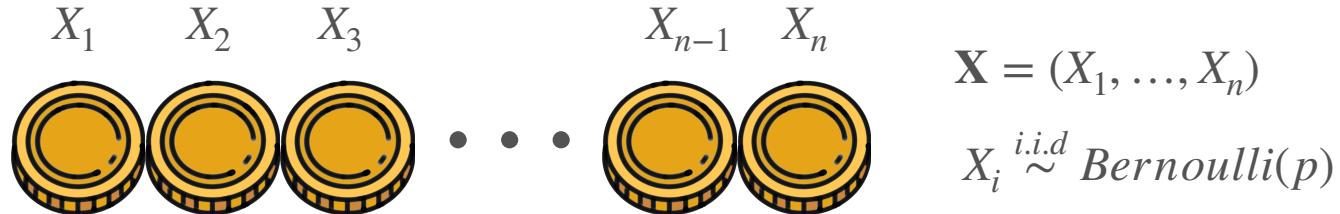
Likelihood

$$L(p; \mathbf{x}) = P_p(\mathbf{X} = \mathbf{x}) = \prod_{i=1}^n p_{X_i}(x_i) = \prod_{i=1}^n p^{x_i} (1-p)^{1-x_i} = p^{\left(\sum_{i=1}^n x_i\right)} (1-p)^{\left(n - \sum_{i=1}^n x_i\right)}$$

Log-likelihood

$$\ell(p; \mathbf{x}) = \log \left((p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left(\sum_{i=1}^n x_i \right) \log(p) + \left(n - \sum_{i=1}^n x_i \right) \log(1-p)$$

Maximum Likelihood: Bernoulli Example



$$\ell(p; \mathbf{x}) = \log \left((p^{\sum_{i=1}^n x_i} (1-p)^{n - \sum_{i=1}^n x_i}) \right) = \left(\sum_{i=1}^n x_i \right) \log(p) + \left(n - \sum_{i=1}^n x_i \right) \log(1-p)$$

Find the maximum!

$$\begin{aligned} \frac{d}{dp} \ell(p; \mathbf{x}) &= \frac{d}{dp} \left(\left(\sum_{i=1}^n x_i \right) \log(p) + \left(n - \sum_{i=1}^n x_i \right) \log(1-p) \right) \\ &= \frac{\sum_{i=1}^n x_i}{p} + \frac{n - \sum_{i=1}^n x_i}{1-p} (-1) = 0 \end{aligned} \quad \rightarrow \quad \hat{p} = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$$



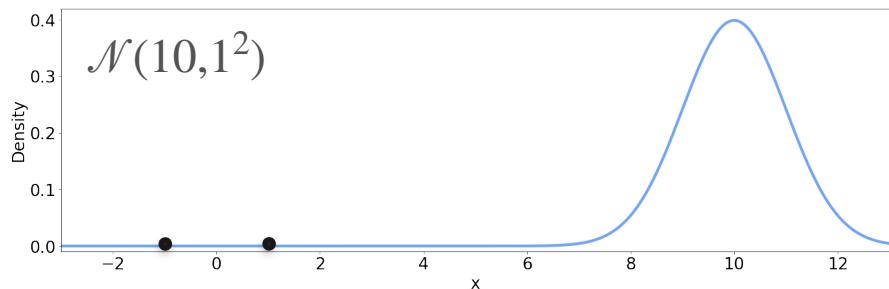
DeepLearning.AI

Point Estimation

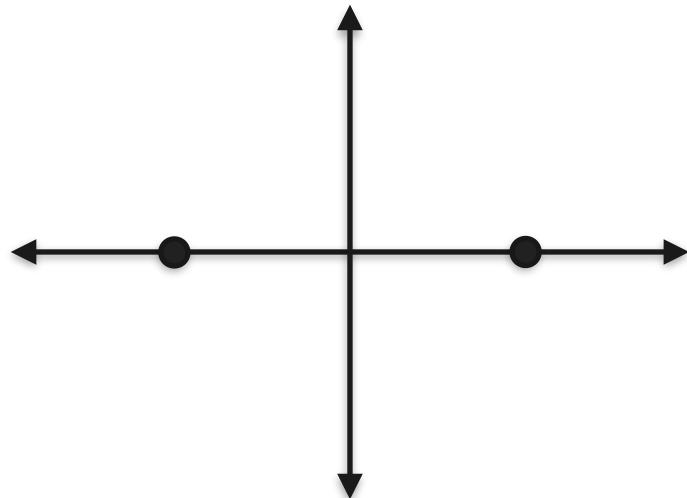
MLE: Gaussian Example

Maximum Likelihood: Gaussian Example

Candidates

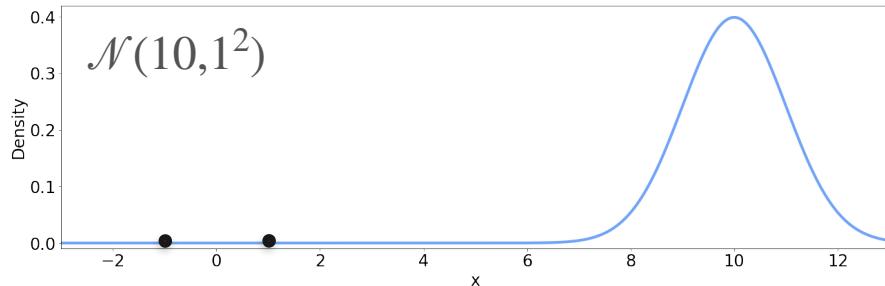


Observations

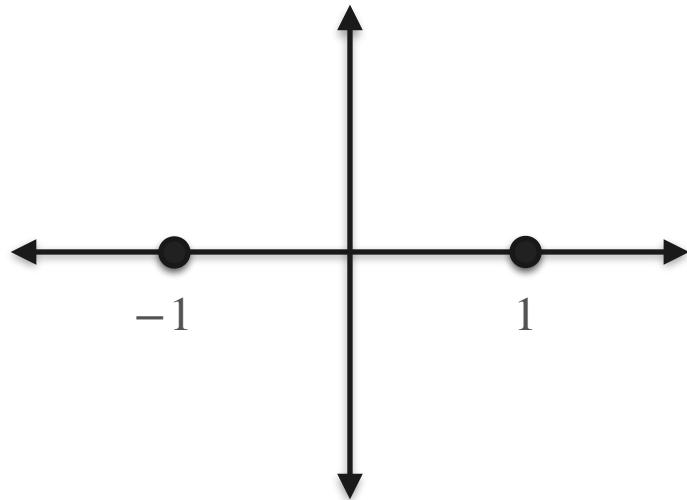


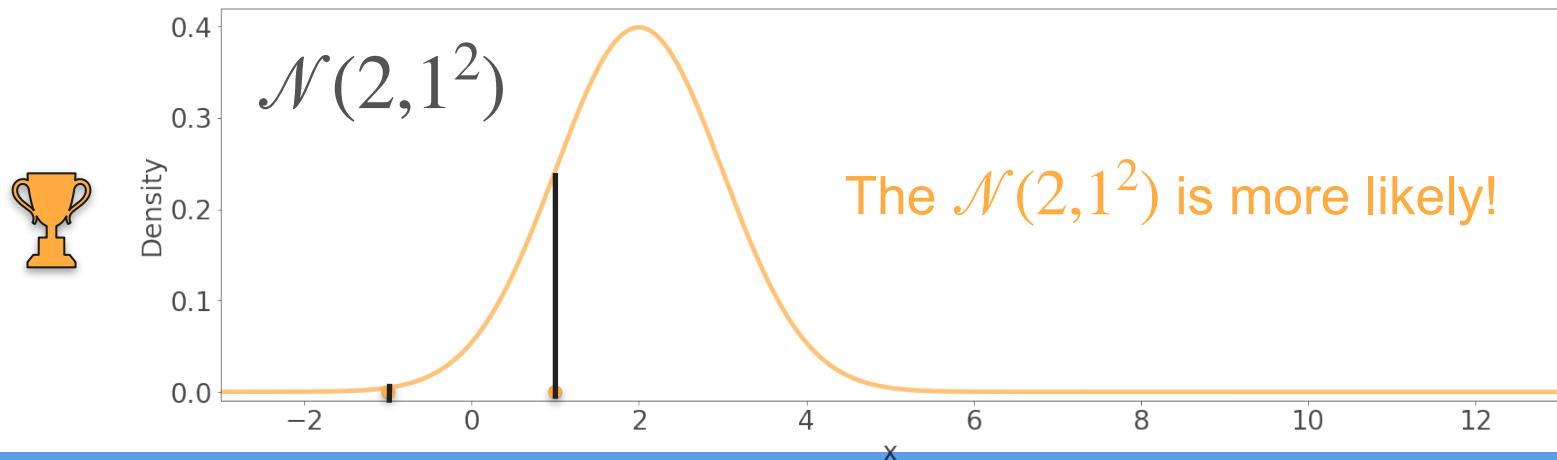
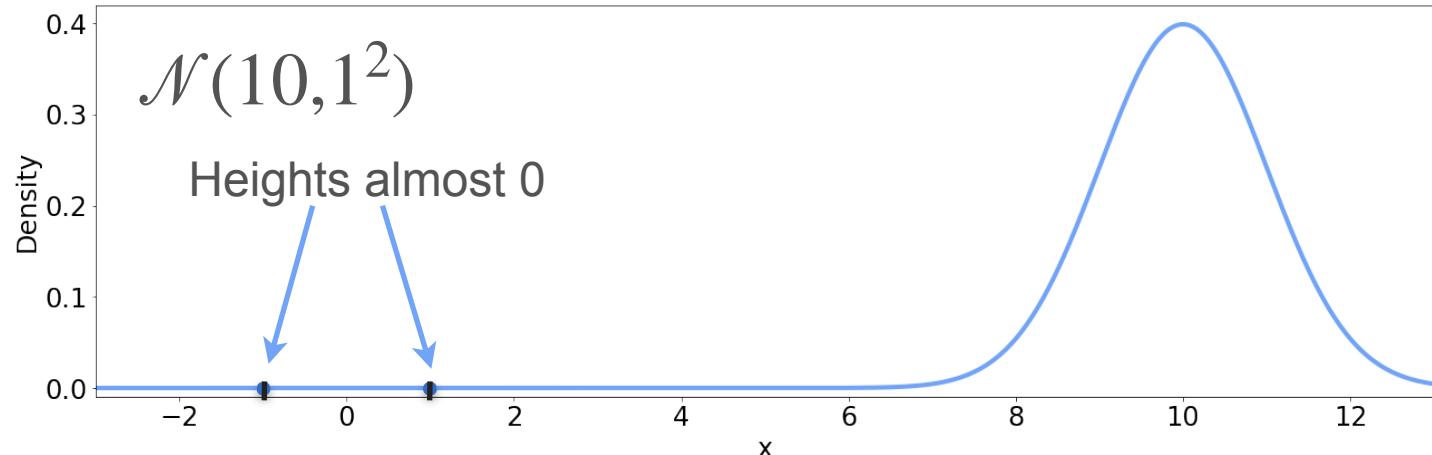
Maximum Likelihood: Gaussian Example

Candidates

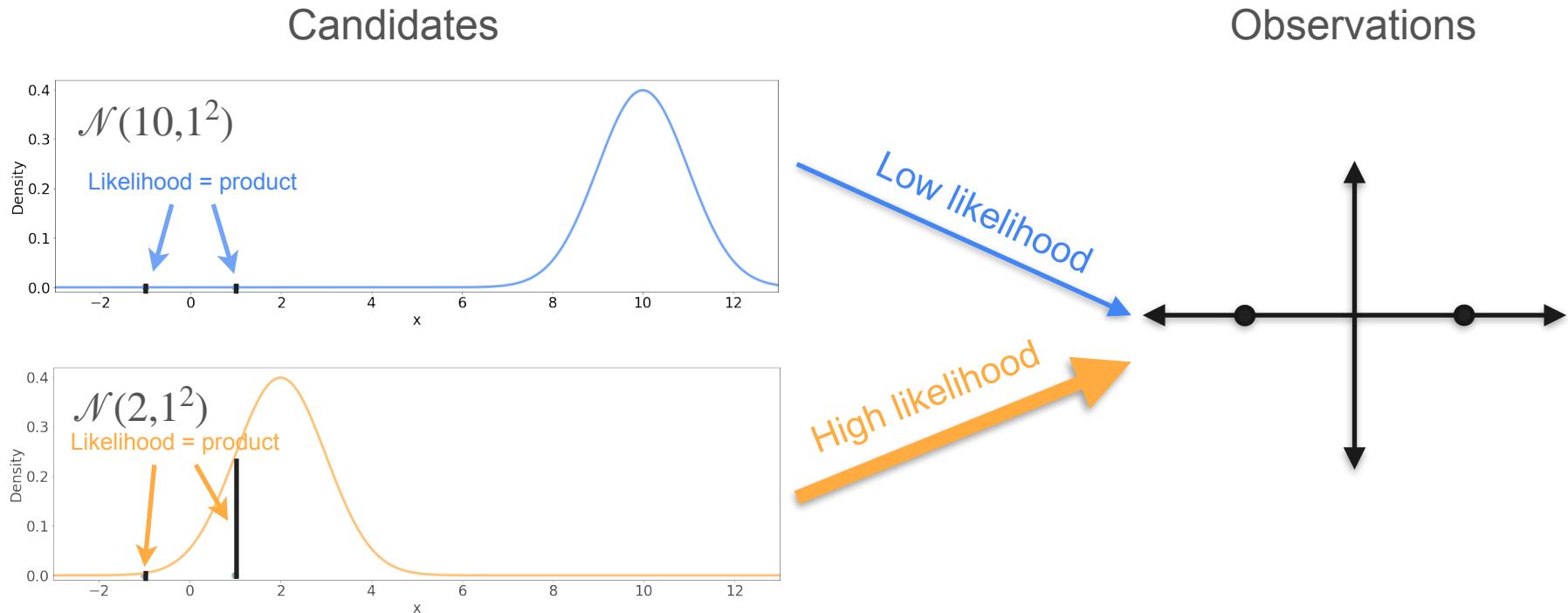


Observations





Maximum Likelihood: Gaussian Example



Gaussians With Three Different Means

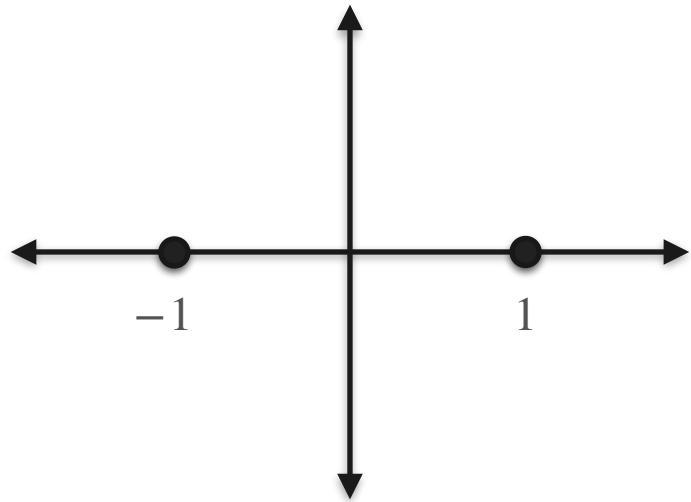
Candidates

$$\mathcal{N}(-1, 1^2)$$

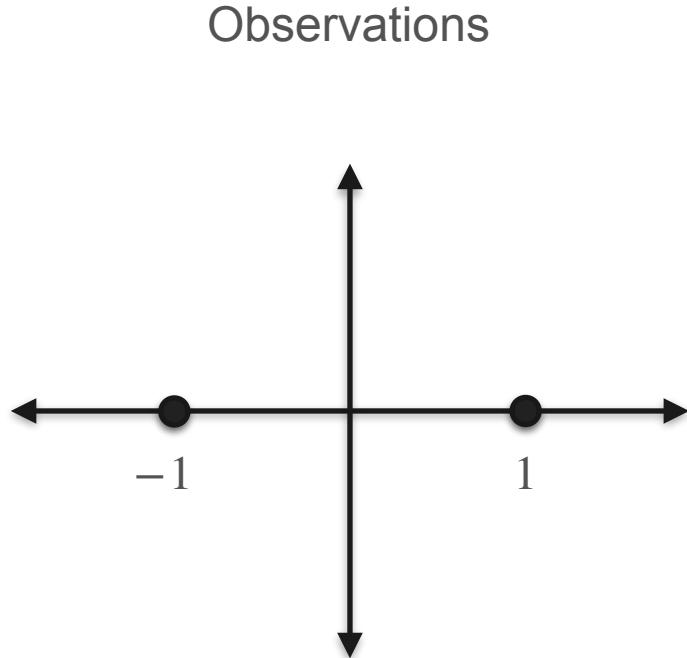
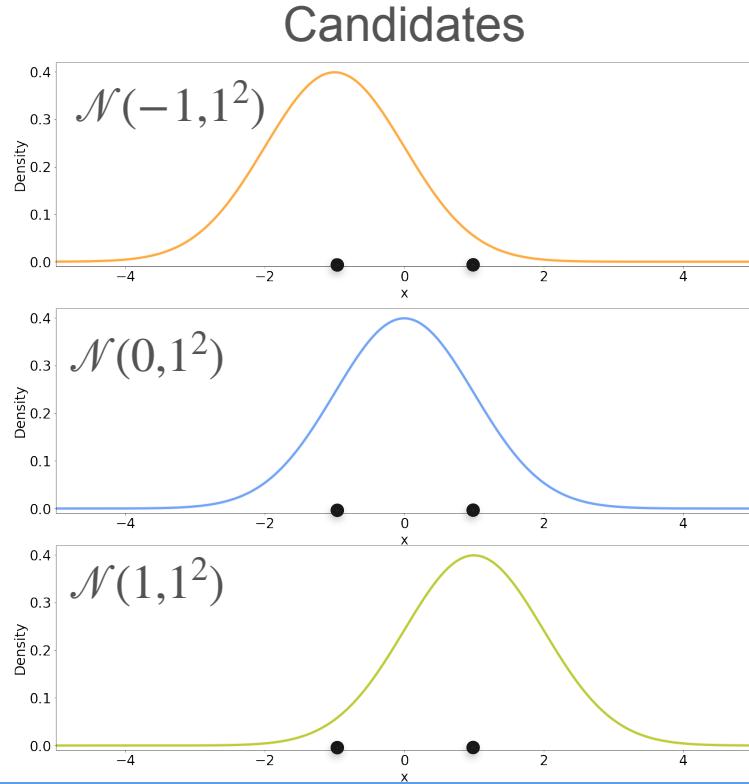
$$\mathcal{N}(0, 1^2)$$

$$\mathcal{N}(1, 1^2)$$

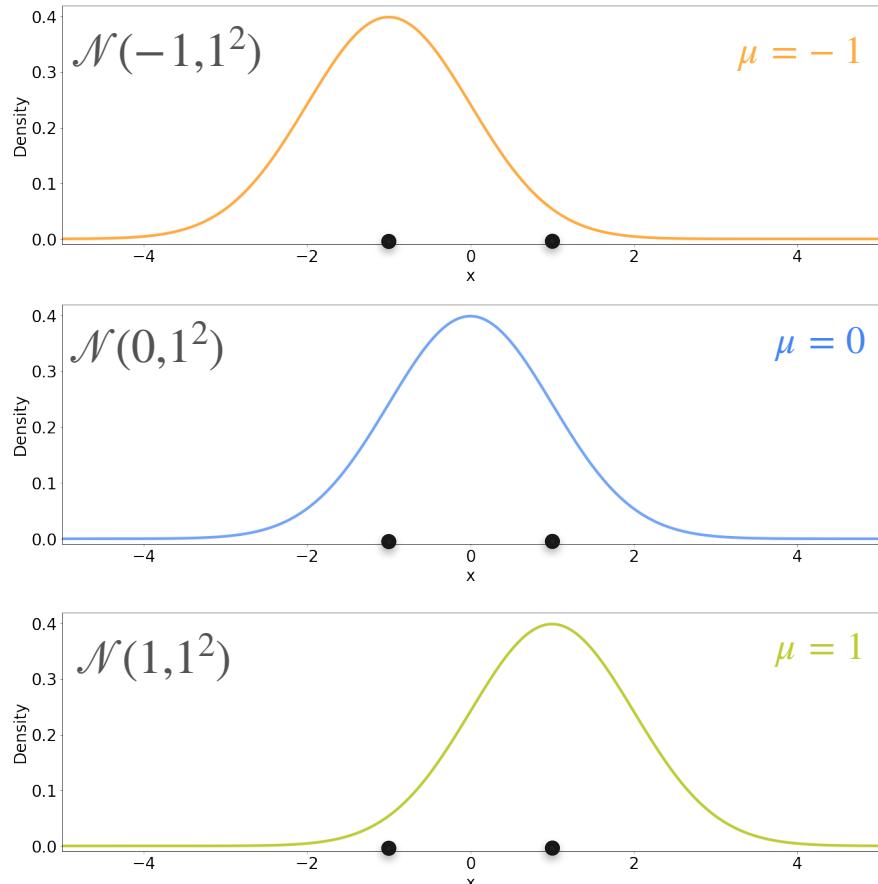
Observations



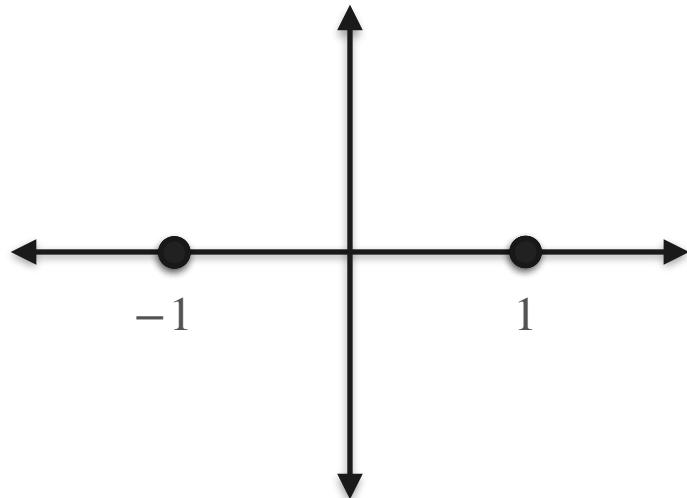
Gaussians With Three Different Means



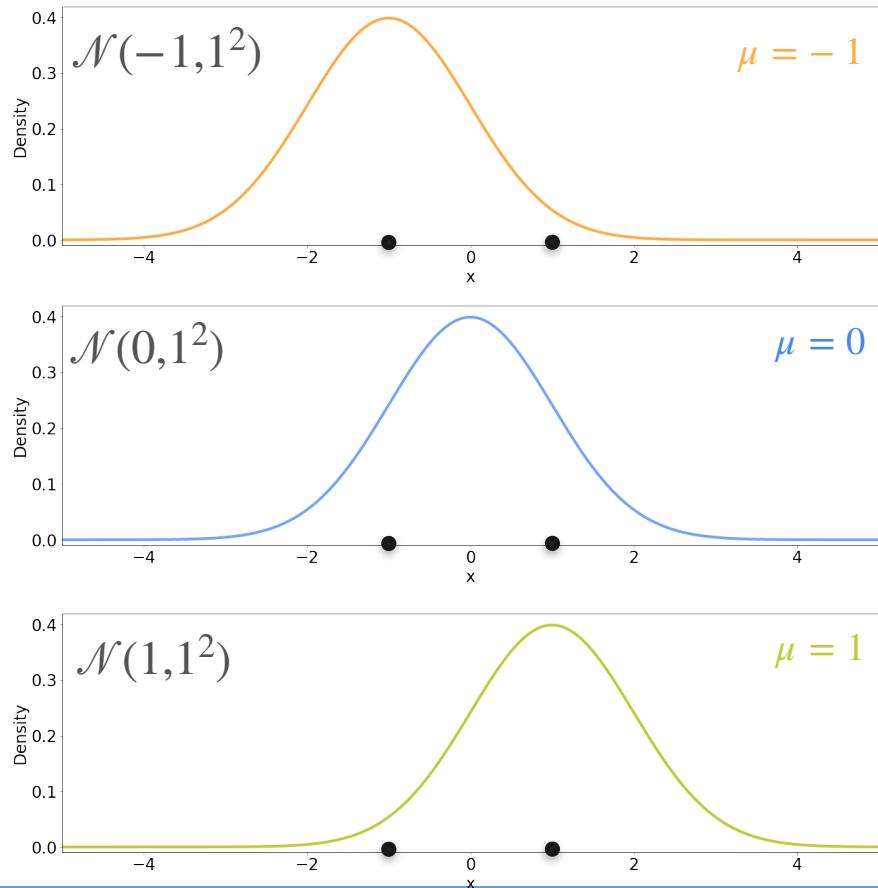
Candidates



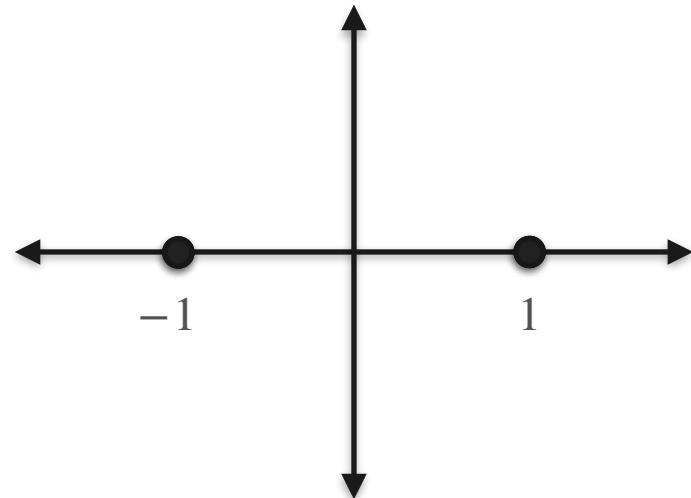
Observations



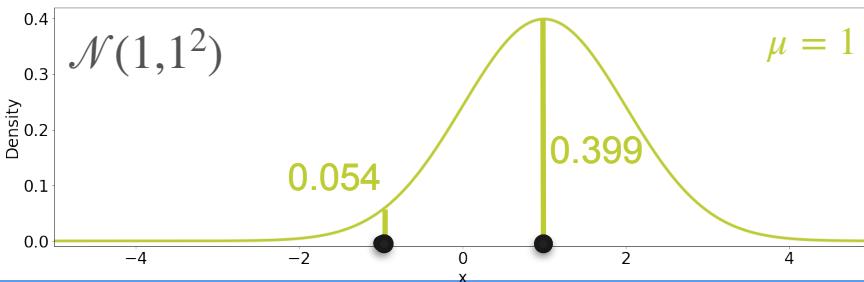
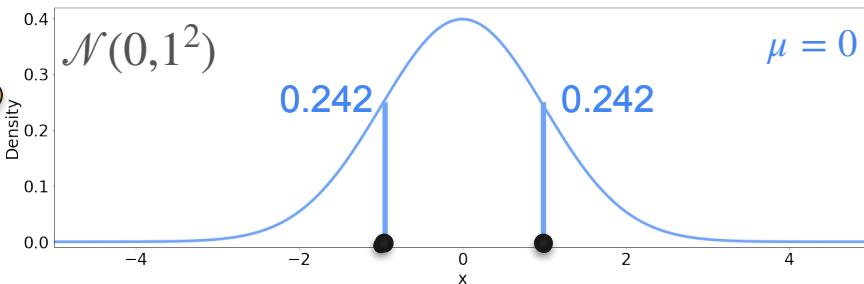
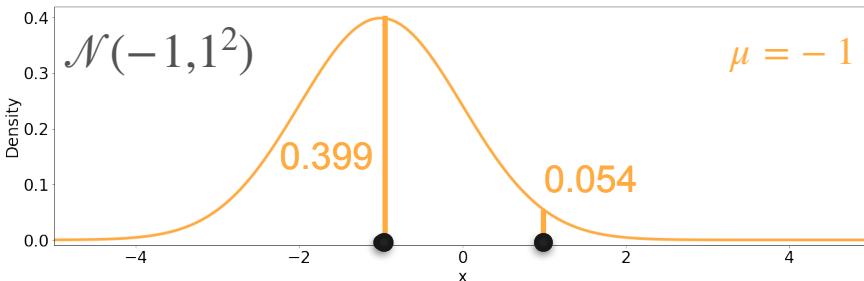
Candidates



Observations



Candidates



Observations

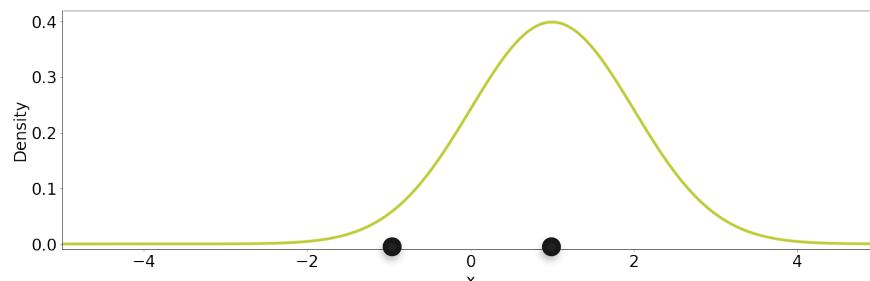
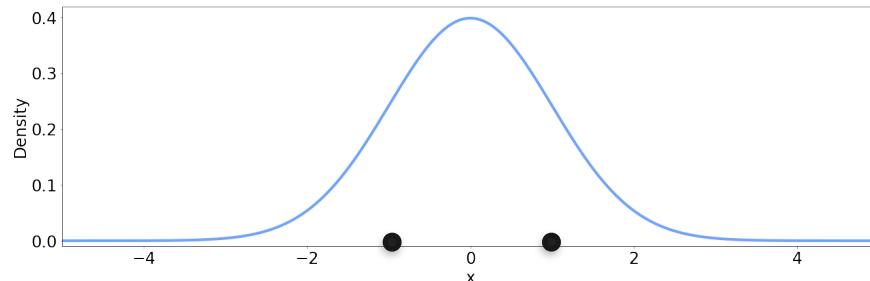
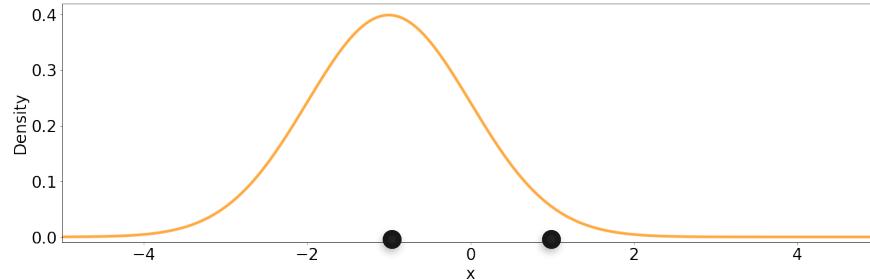
= 0.022

The $\mathcal{N}(0, 1)$ is more likely!

= 0.059

= 0.022

Candidates

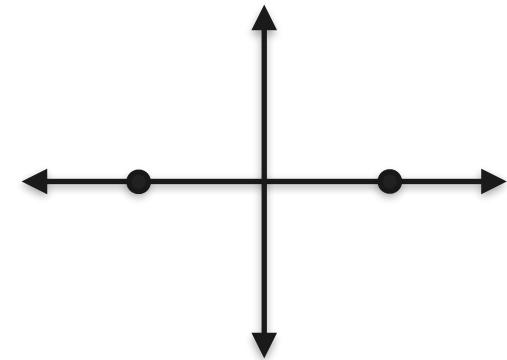


Likelihood = 0.022

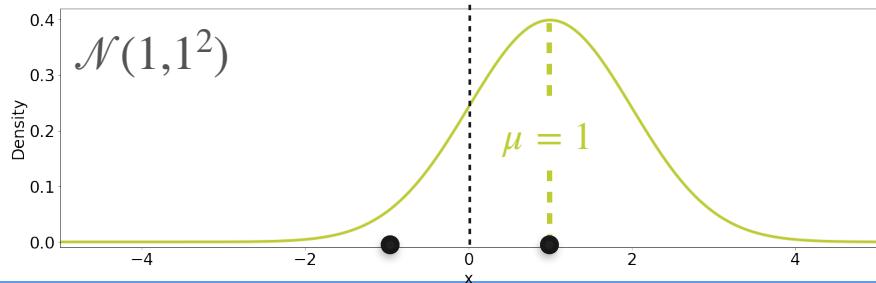
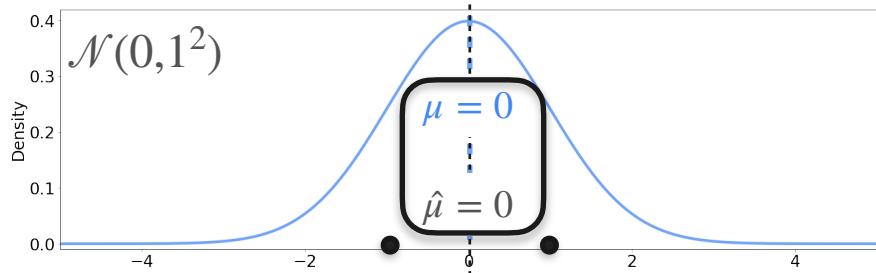
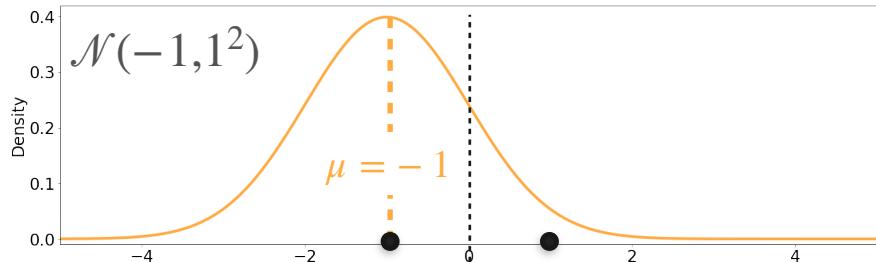
Likelihood = 0.059

Likelihood = 0.022

Observations



Candidates



The best distribution is the one where
the **mean** of the distribution is the
mean of the sample

Gaussians With Three Different Variance

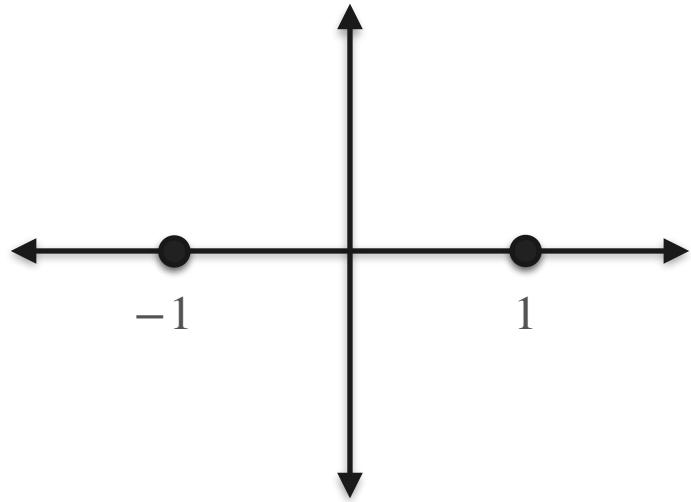
Candidates

$$\mathcal{N}(0,0.5^2)$$

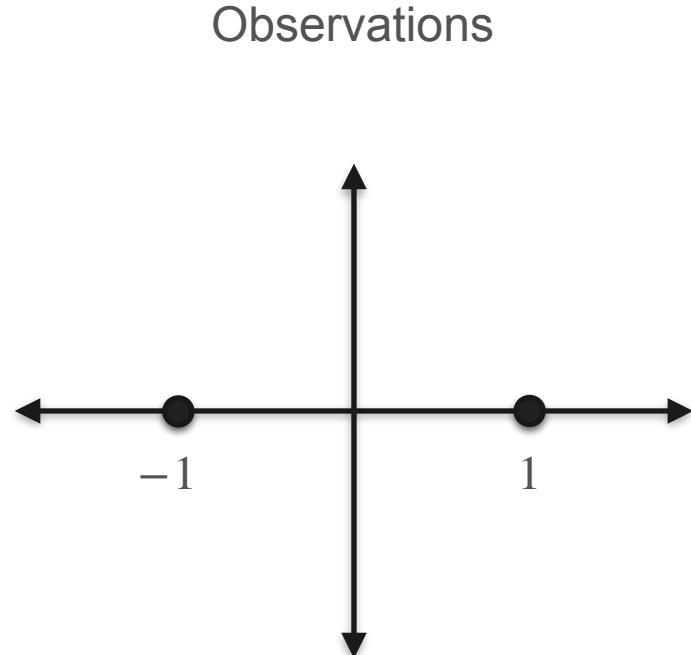
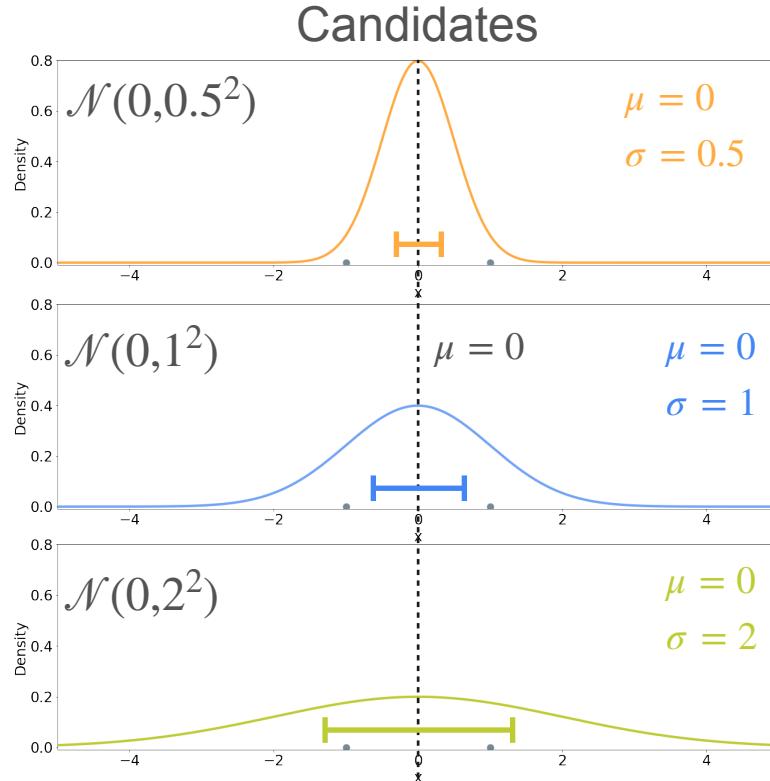
$$\mathcal{N}(0,1^2)$$

$$\mathcal{N}(0,2^2)$$

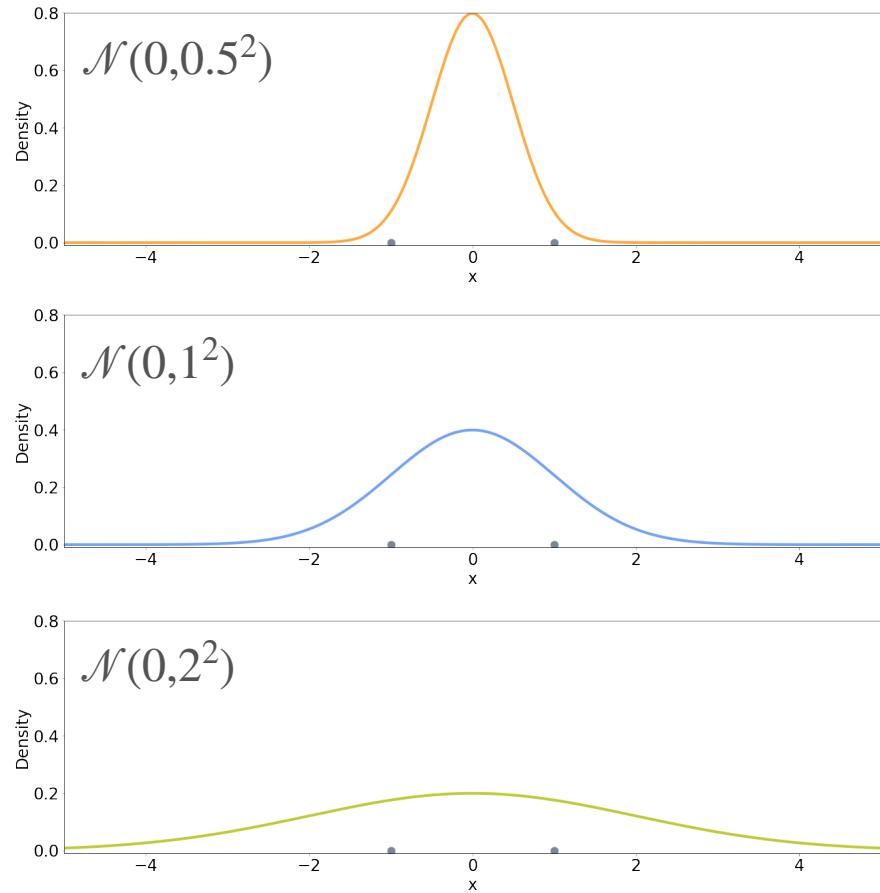
Observations



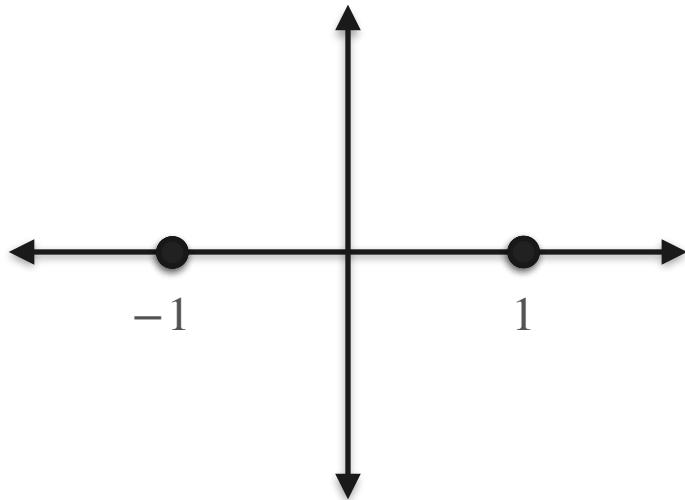
Gaussians With Three Different Variance



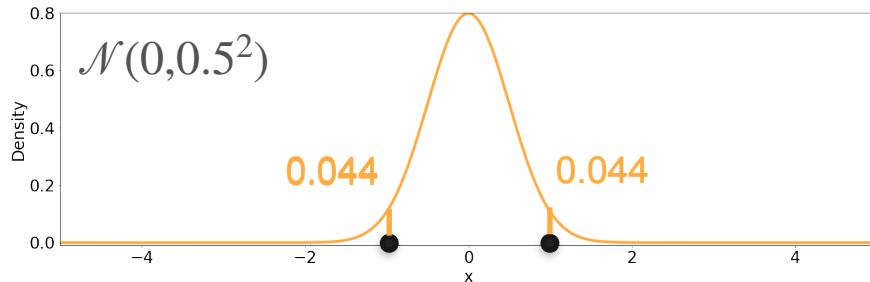
Candidates



Observations



Candidates

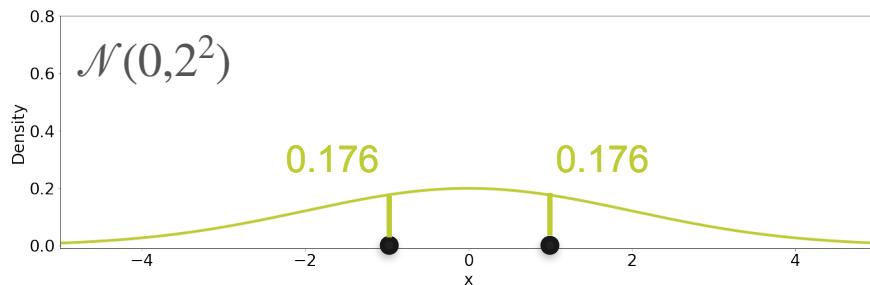
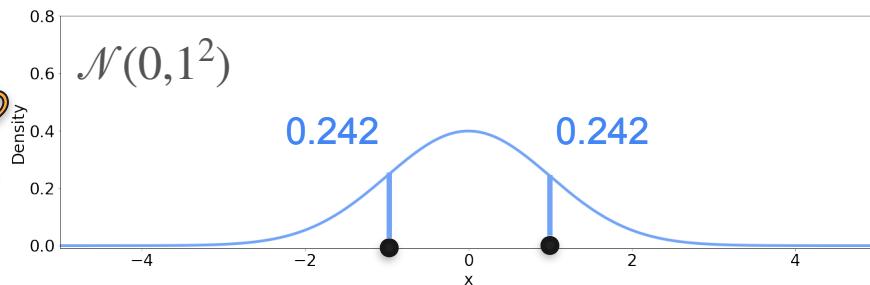


Observations

= 0.002

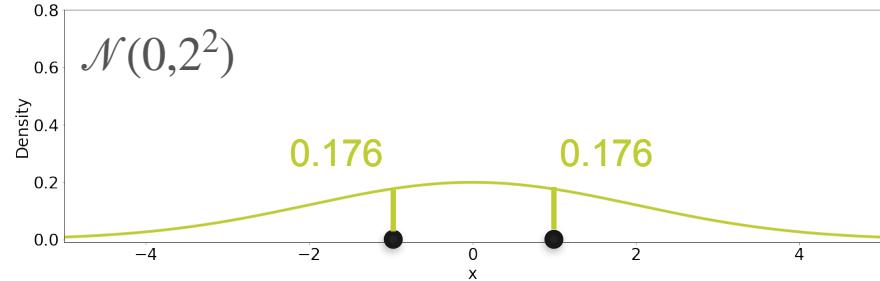
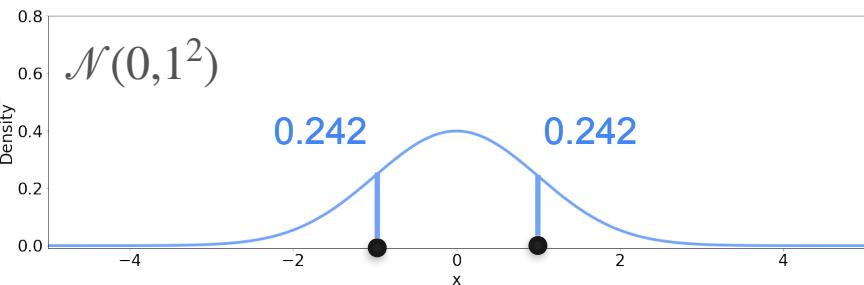
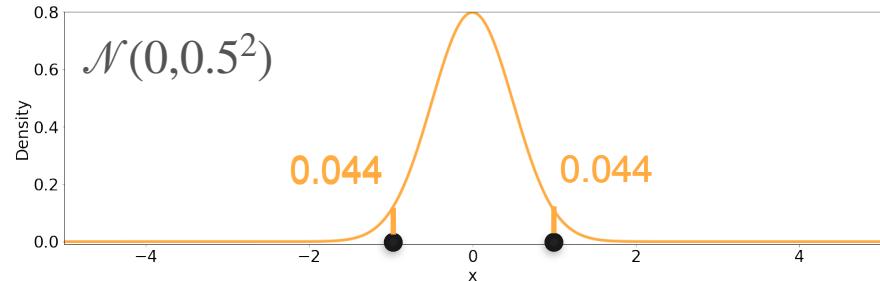
The $\mathcal{N}(0, 1^2)$ is more likely!

= 0.059



= 0.031

Candidates



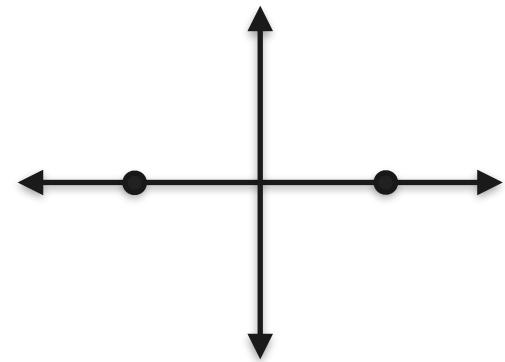
Observations

Likelihood = 0.002

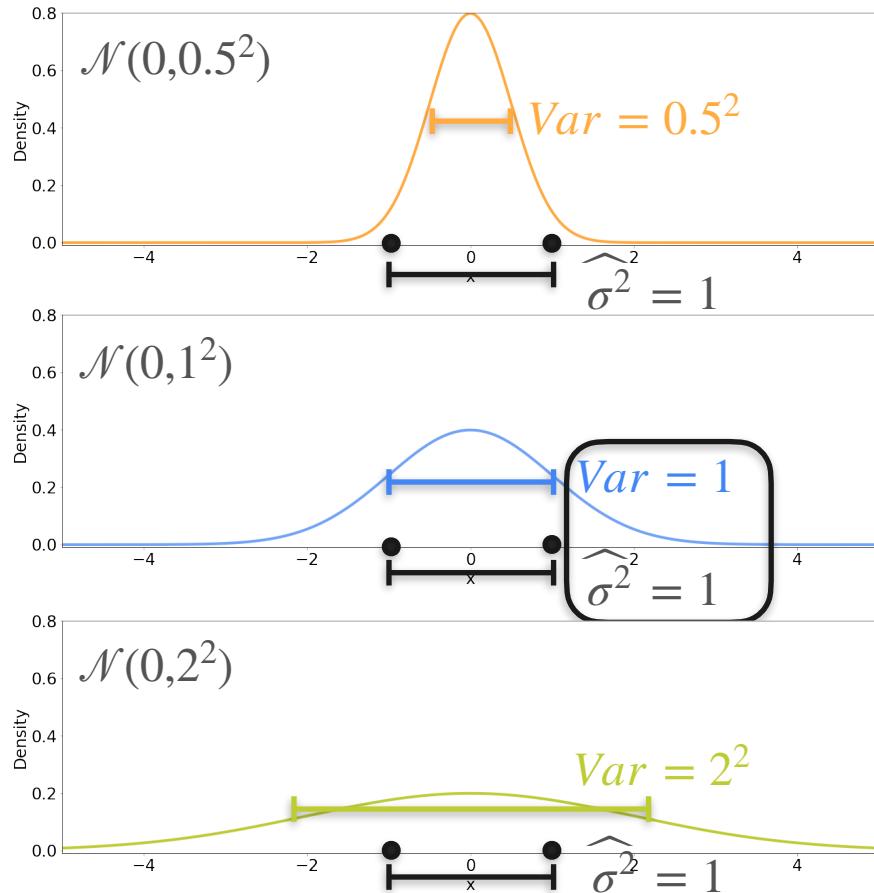
Likelihood = 0.059

Likelihood = 0.031

Observations



Candidates



Observations

Variance of the observations

$$\widehat{\sigma}^2 = \frac{1}{2} ((0 - 1)^2 + (0 + 1)^2) = 1$$

The best distribution is the one where the **variance** of the distribution is the **variance** of the sample

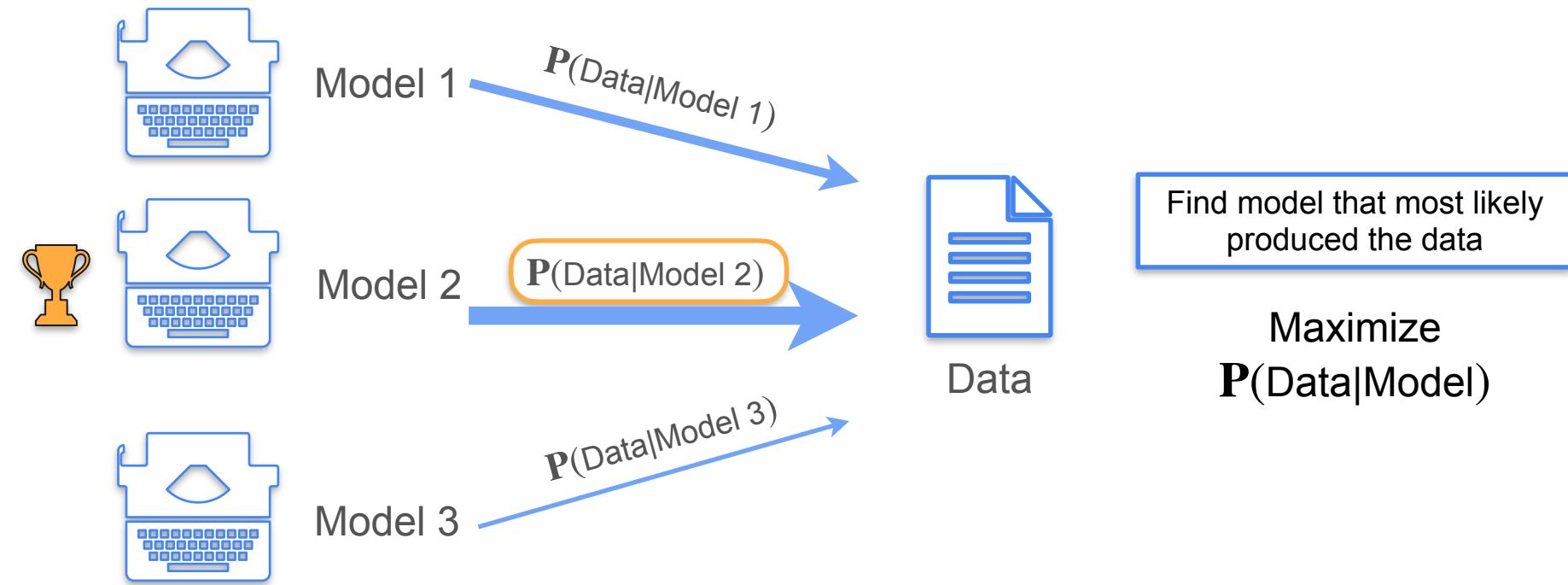


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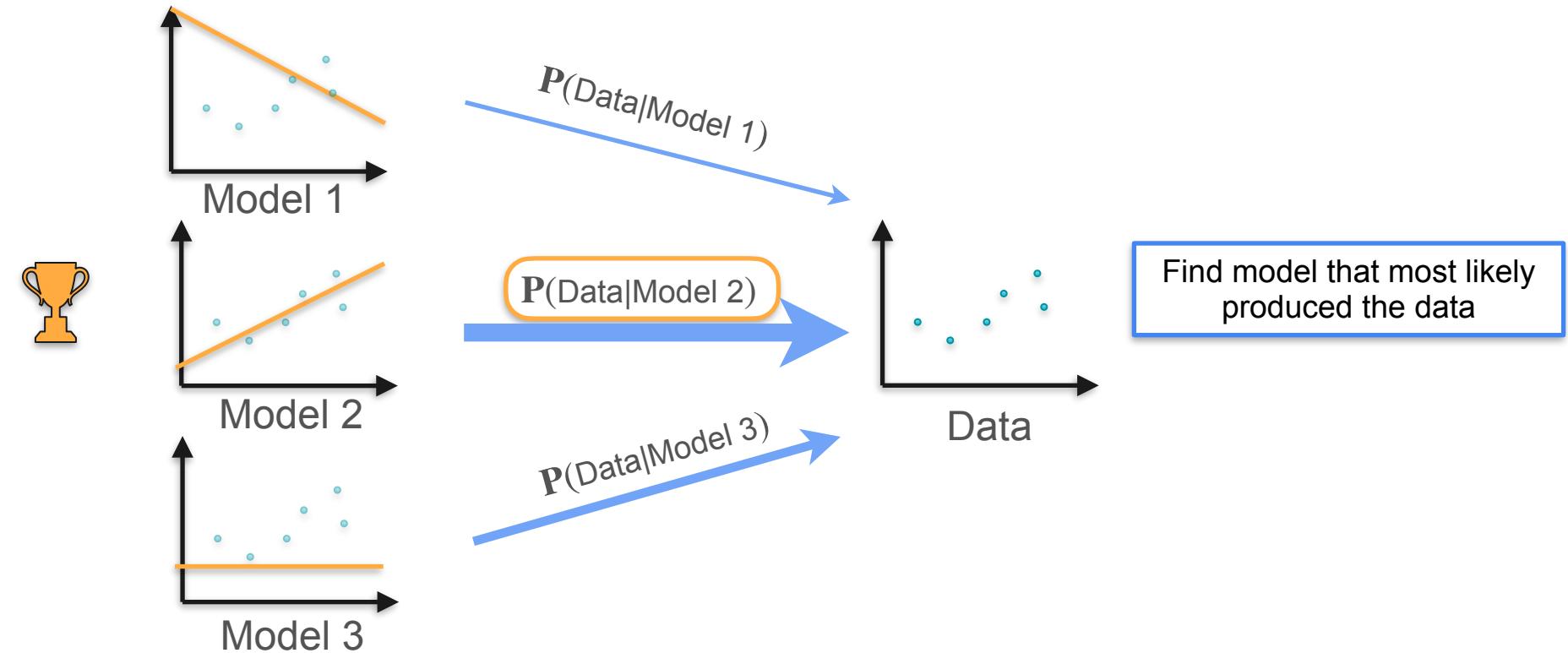
Point Estimation

MLE: Linear Regression

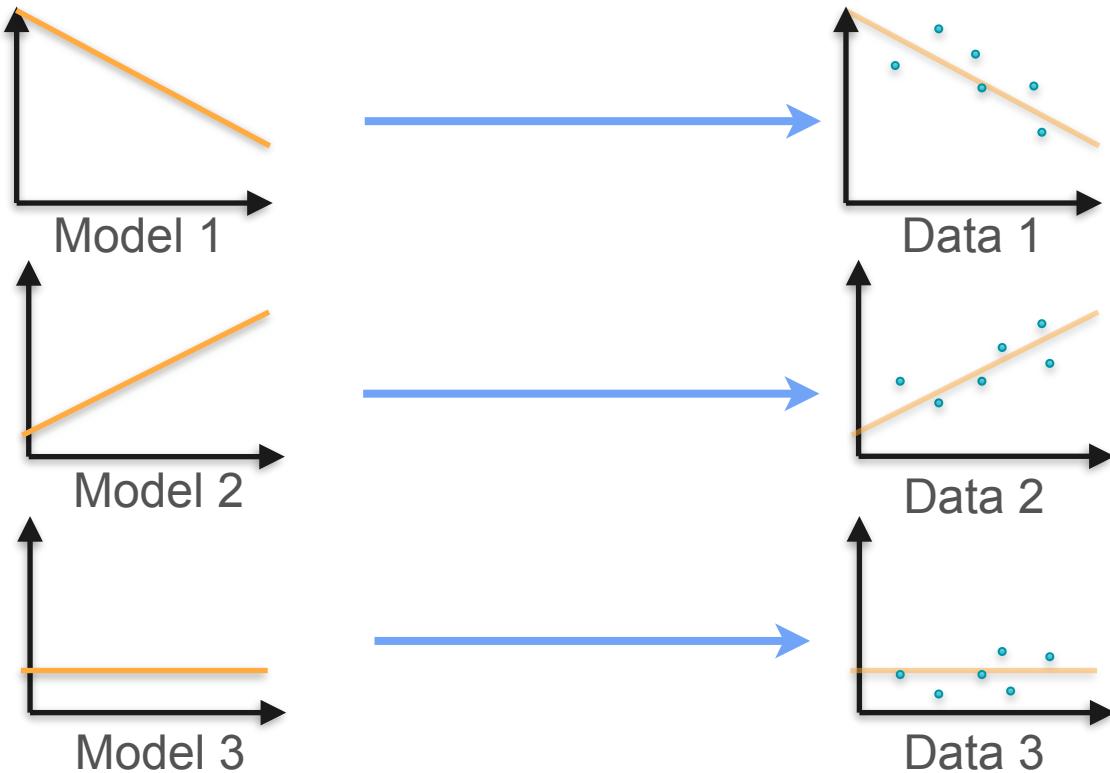
Maximum Likelihood



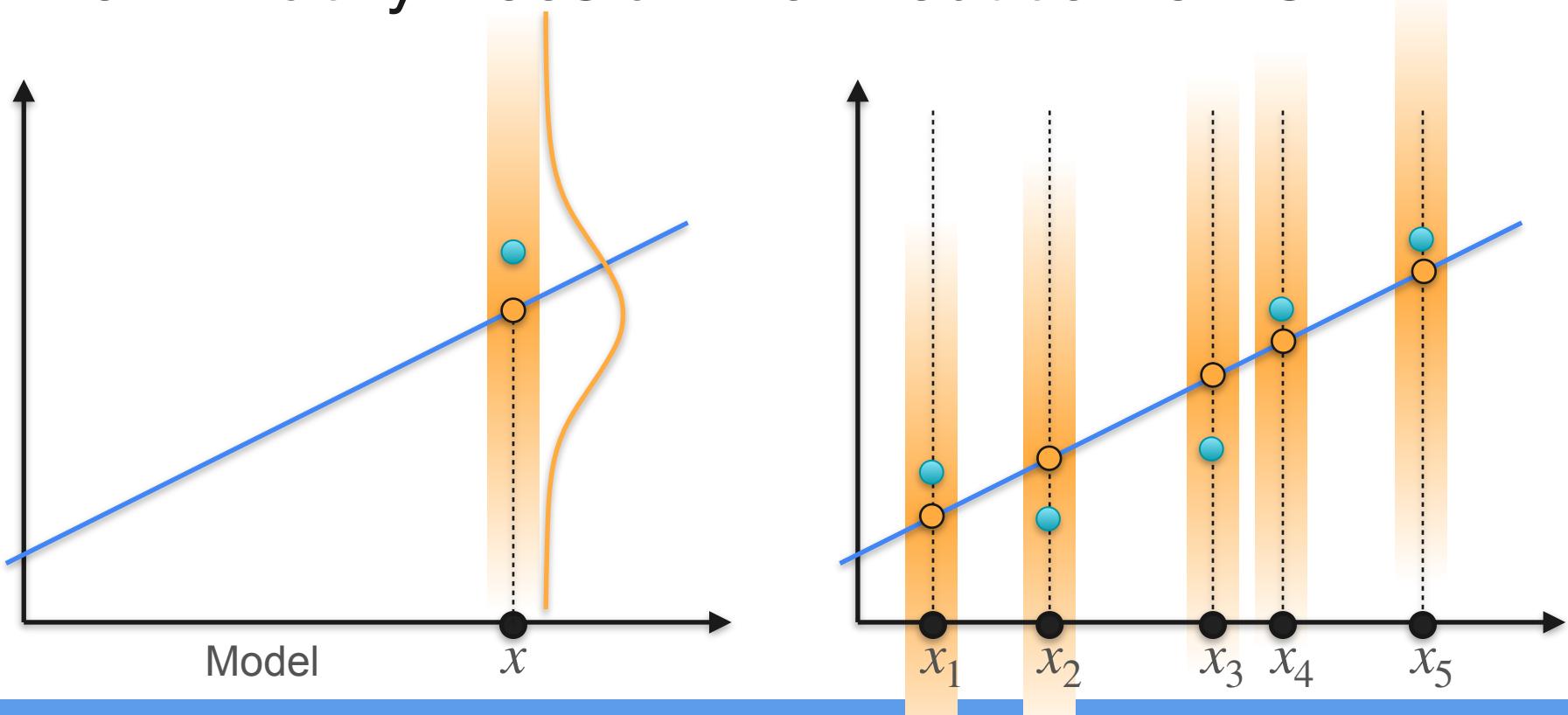
Example: Linear Regression



How Exactly Does a Line Produce Points?

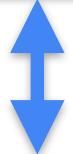


How Exactly Does a Line Produce Points?



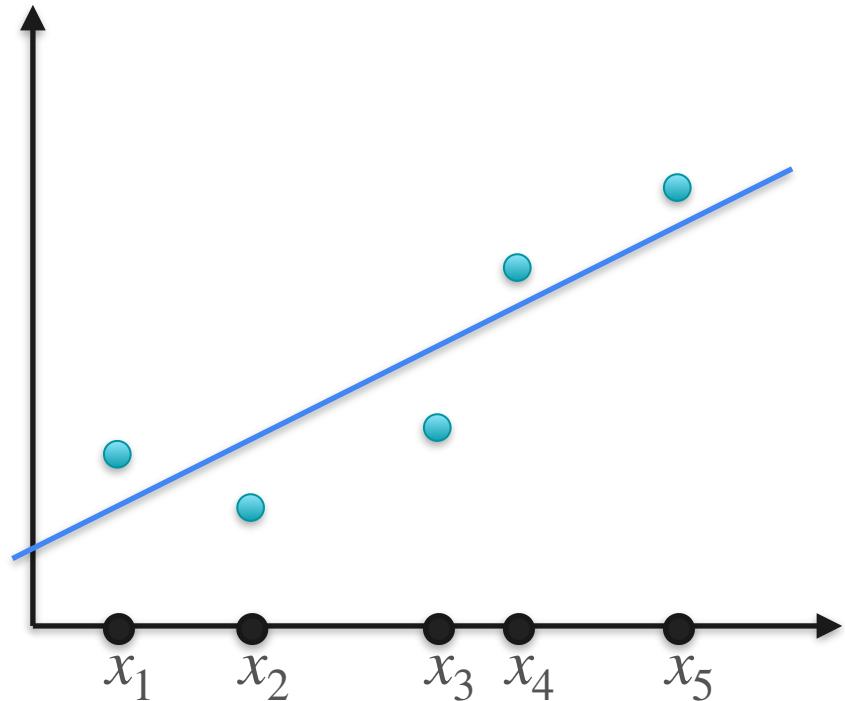
Linear Regression

Line that best produced the points



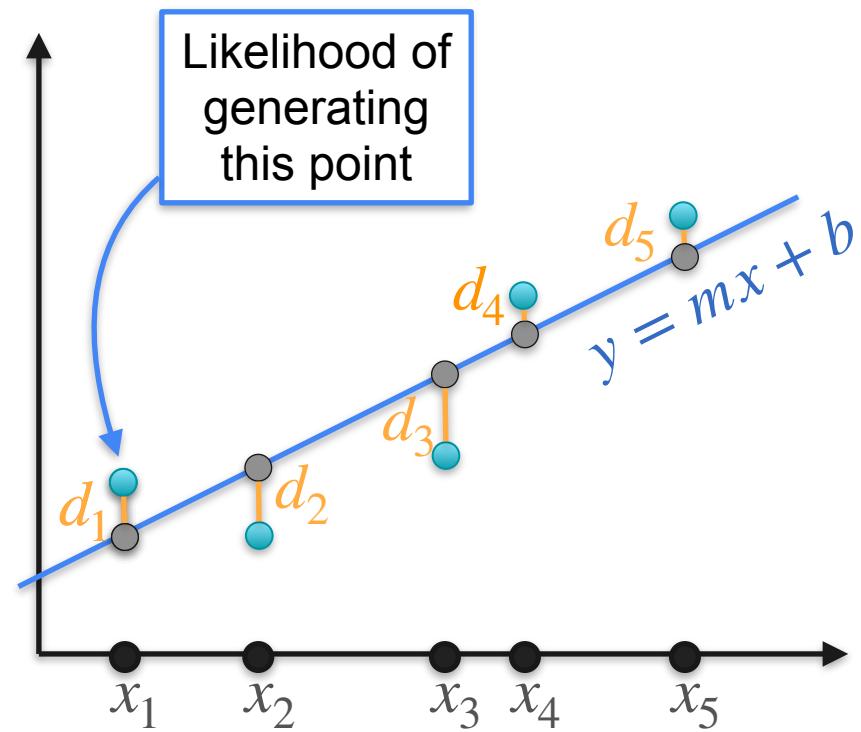
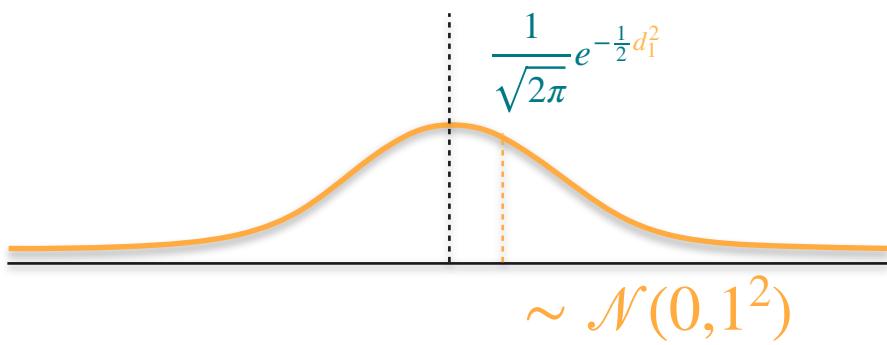
How?

Line that best fits the data
(linear regression)



Linear Regression and Likelihood

Likelihood:



Linear Regression and Likelihood

Likelihood:

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

Maximize

$$e^{-\frac{1}{2}d_1^2} \cdot e^{-\frac{1}{2}d_2^2} \cdot e^{-\frac{1}{2}d_3^2} \cdot e^{-\frac{1}{2}d_4^2} \cdot e^{-\frac{1}{2}d_5^2}$$

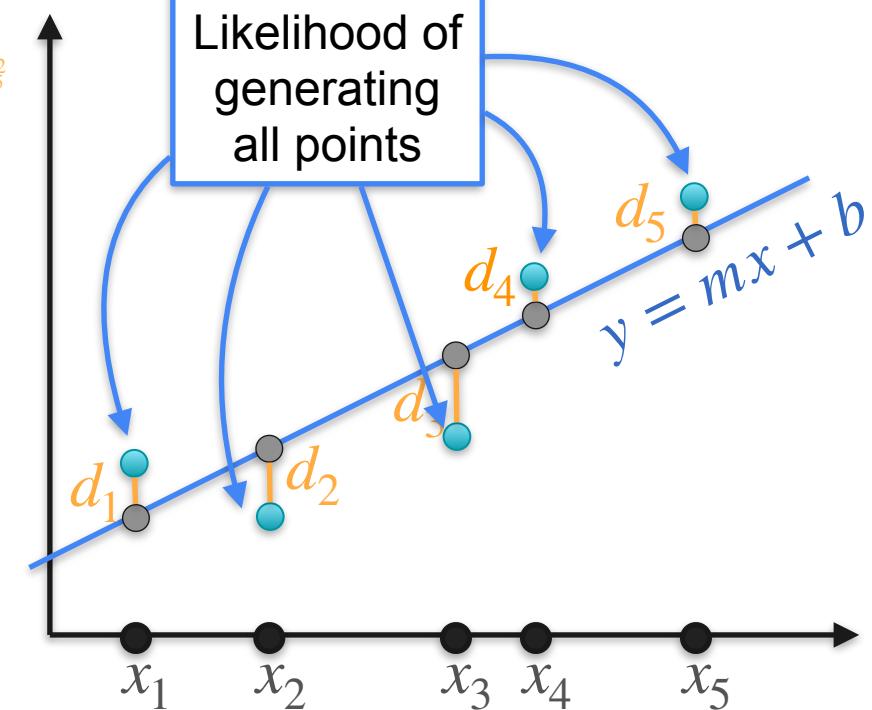
$$\cancel{e^{-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)}}$$

Minimize

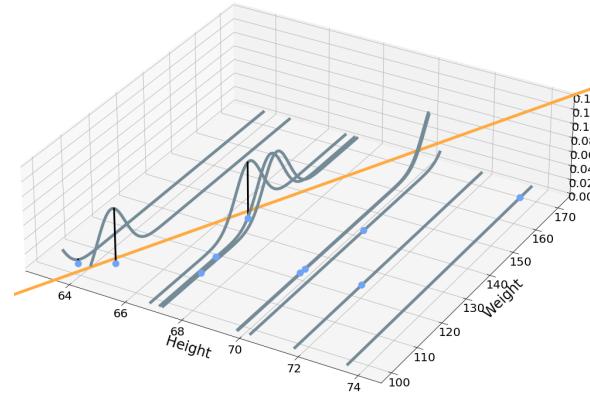
$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

Linear regression!

Least squares error!

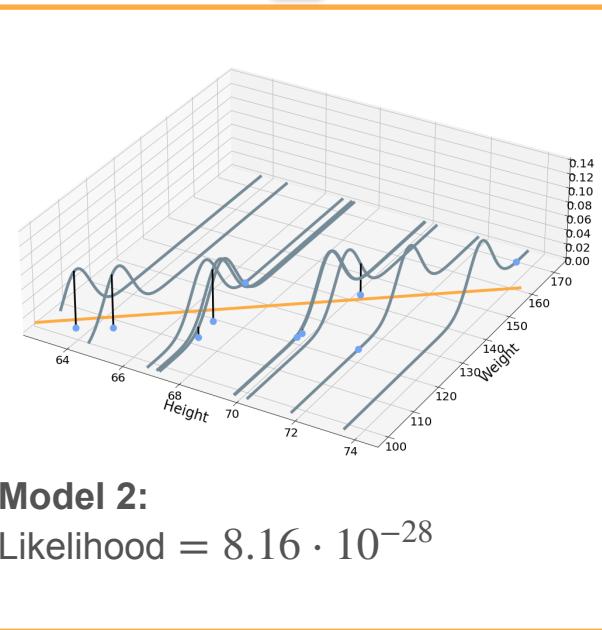


Picking the Right Model



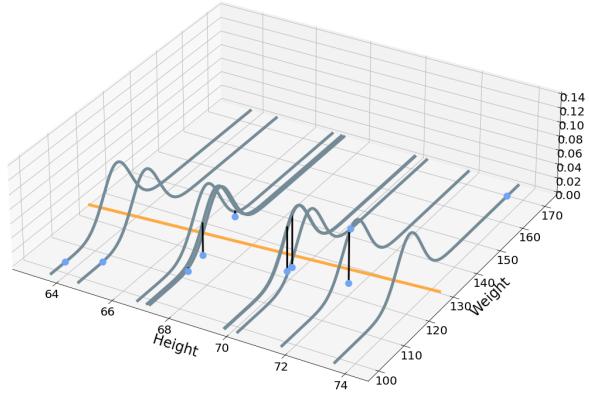
Model 1:

$$\text{Likelihood} = 4.91 \cdot 10^{-260}$$



Model 2:

$$\text{Likelihood} = 8.16 \cdot 10^{-28}$$



Model 3:

$$\text{Likelihood} = 3.48 \cdot 10^{-49}$$

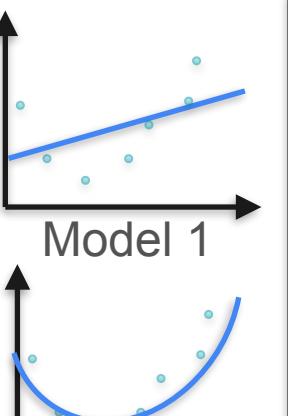
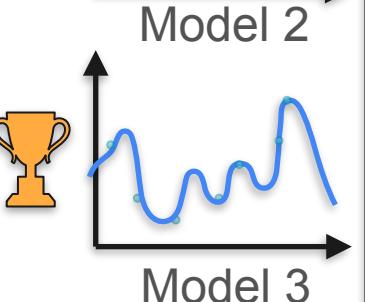


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Point Estimation

Regularization

Example: Polynomial Regression

| | Loss | Equation | Penalty | New loss |
|---|------|---|--|----------|
|  Data | 10 | $y = 4x + 3$ | $L_2 = 4^2 = 16$ | 26 |
|  Model 2 | 2 | $y = 2x^2 - 4x + 5$ | $L_2 = 2^2 + (-4)^2 = 20$ | 22 |
|  Model 3 | 0.1 | $y = 4x^{10} - 9x^8 - 2x^6 + 3x^5 - 6x^4 - 10x + 4$ | $L_2 = 4^2 + (-9)^2 + (-2)^2 + 3^2 + (-6)^2 + (-10)^2 = 246$ | 246.1 |

Regularization Term

Model: $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

Log-loss: $\ell\ell$

L2 Regularization Error: $a_n^2 + a_{n-1}^2 + \dots + a_1^2$

Regularization parameter: λ

Regularized error: $+ (\text{L2 Regularization Error})$

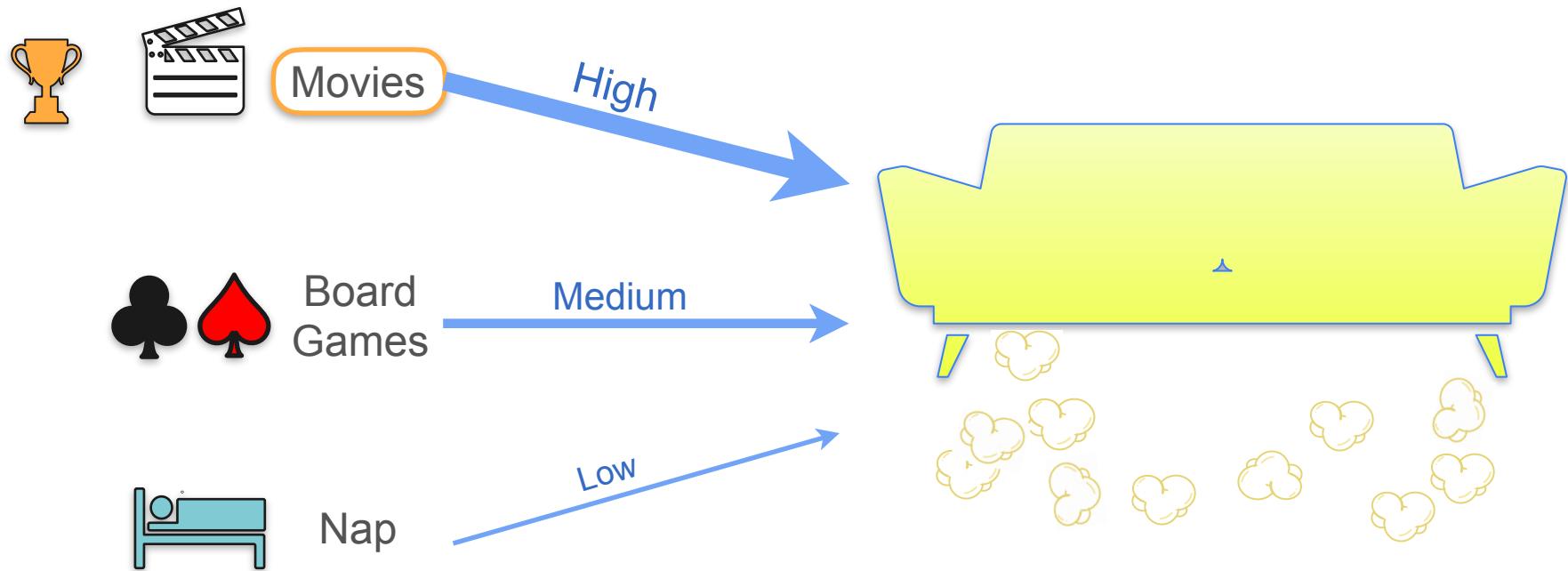


DeepLearning.AI

Point Estimation

Back to “Bayesics”

There's Popcorn on the Floor. What Happened?



There's Popcorn on the Floor. What Happened?



$P(\text{Movies})$

Movies



$P(\text{Movies}) \gg P(\text{Contest})$



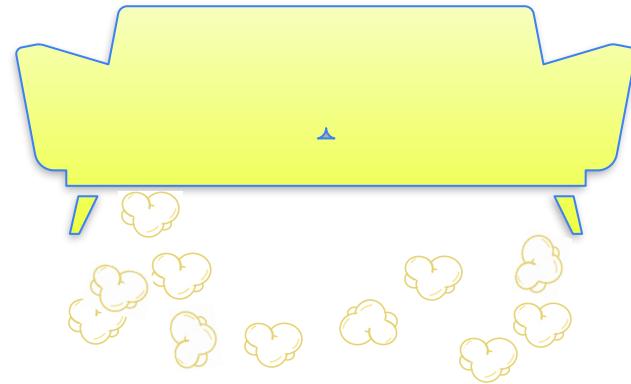
$P(\text{Contest})$

Popcorn
throwing
contest



$$P(\text{Popcorn}|\text{Movies}) < P(\text{Popcorn}|\text{Contest})$$

$$P(\text{Popcorn} \cap \text{Movie}) > P(\text{Popcorn} \cap \text{Contest})$$



$$P(A | B)P(B) = P(A \cap B)$$



DeepLearning.AI

Point Estimation

**Frequentist vs Bayesian
Statistics**

Frequentists vs. Bayesians

Frequentist



0.8

Bayesian



Belief (prior)

0.52



Frequentist Vs. Bayesian Statistics

Frequentists

- Probabilities represent long term frequency of events
- Concept of Likelihood
- Goal: Find the model that most likely generated the observed data

Bayesians

- Probabilities represent the degree of belief (or certainty)
- Concept of Prior
- Goal: update prior belief based on observations

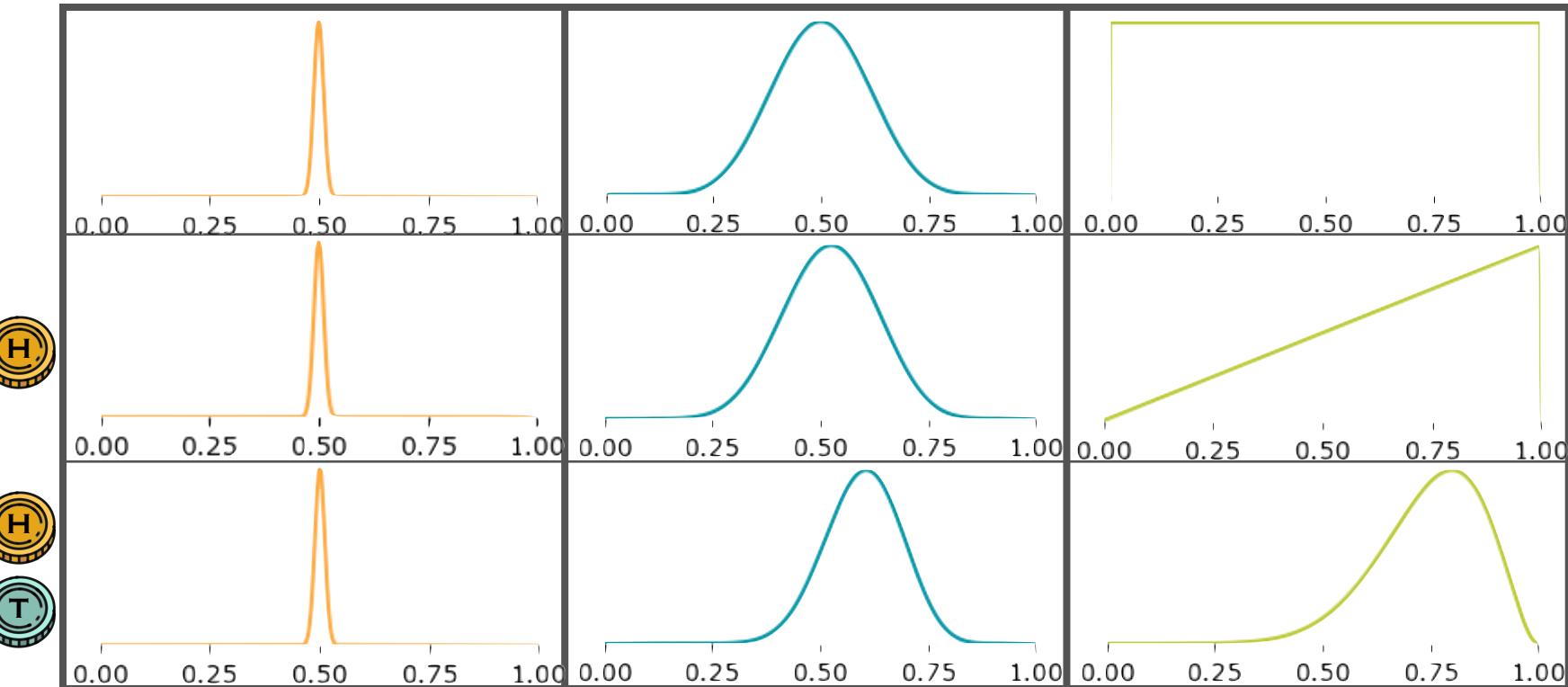


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Point Estimation

Bayesian Statistics MAP

Updating Your Beliefs



Maximum A Posteriori (MAP)

1 value for the parameter?



Choose the one with highest probability

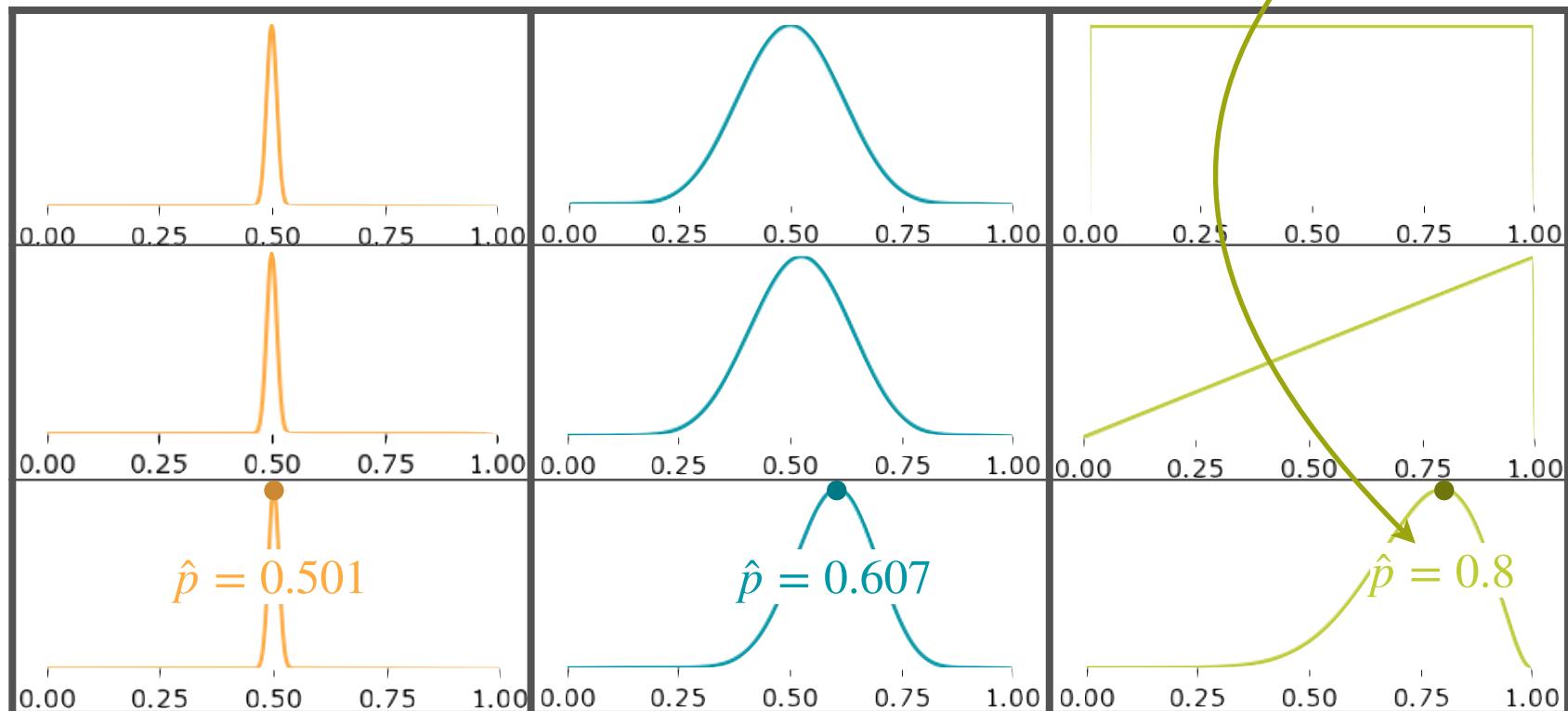


Mode of the updated belief

Posterior

Maximum A Posteriori (MAP)

Same result as frequentist!





DeepLearning.AI

Point Estimation

**Bayesian Statistics
Updating Priors**

Bayesian Statistics

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

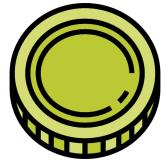
The diagram illustrates the Bayesian formula with annotations:

- "Posterior" Belief that A will happen after considering evidence B: $P(A | B)$ (orange arrow pointing to the formula)
- Likelihood of evidence B appearing, given A happened: $P(B | A)$ (purple arrow pointing to the numerator)
- "Prior" Belief that A will happen, before considering evidence B: $P(A)$ (blue arrow pointing to the numerator)
- Probability of evidence B in any circumstances: $P(B)$ (green arrow pointing to the denominator)
- $P(B | A)P(A) + P(B | A')P(A')$: The denominator of the formula, shown as a curved arrow pointing from the bottom right towards the denominator.

A : an event you are trying to predict (you are offered a job)

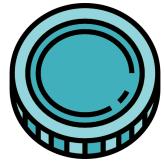
B : another event, or evidence, that helps refine your prediction (you were asked to a follow-up phone call)

Bayesian Statistics



“Fair”

$$P(H) = 0.5$$



“Biased”

$$P(H) = 0.8$$



“Mystery”

Either Fair or Biased

$$p_{Y|X=1}(0.5)$$

$$= \frac{p_{X|Y=0.5}(1) \quad p_Y(0.5)}{p_X(1)} =$$

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

Event to Predict

$A \rightarrow Y$ takes some value

Y : odds of H for your coin

$$Y = \begin{cases} 0.5 & \text{if coin is fair} \\ 0.8 & \text{if coin is biased} \end{cases}$$

Evidence

$B \rightarrow X$ take some value

X : result of coin flip

$$X = \begin{cases} 0 & \text{if } T \\ 1 & \text{if } H \end{cases}$$

Priors

$$\begin{aligned} P(Y = 0.5) &= 0.75 \\ P(Y = 0.80) &= 0.25 \end{aligned}$$

$x = 1$

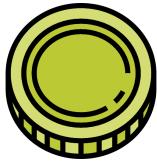
0.5 0.75

0.5 0.75 0.8 0.25

$$\begin{aligned} P(Y = 0.5 | X = 1) &= 0.652 \\ P(Y = 0.8 | X = 1) &= 0.348 \end{aligned}$$

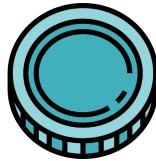
Posterior

Bayesian Statistics



“Fair”

$$P(H) = 0.5$$



“Biased”

$$P(H) = 0.8$$



“Mystery”

Either Fair or Biased

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

Event to Predict

$A \rightarrow Y$ takes some value

Y : odds of H for your coin

$$Y = \begin{cases} 0.5 & \text{if coin is fair} \\ 0.8 & \text{if coin is biased} \end{cases}$$

$$P(Y = 0.5) = 0.75$$

$$P(Y = 0.80) = 0.25$$

Evidence

$B \rightarrow X$ take some value

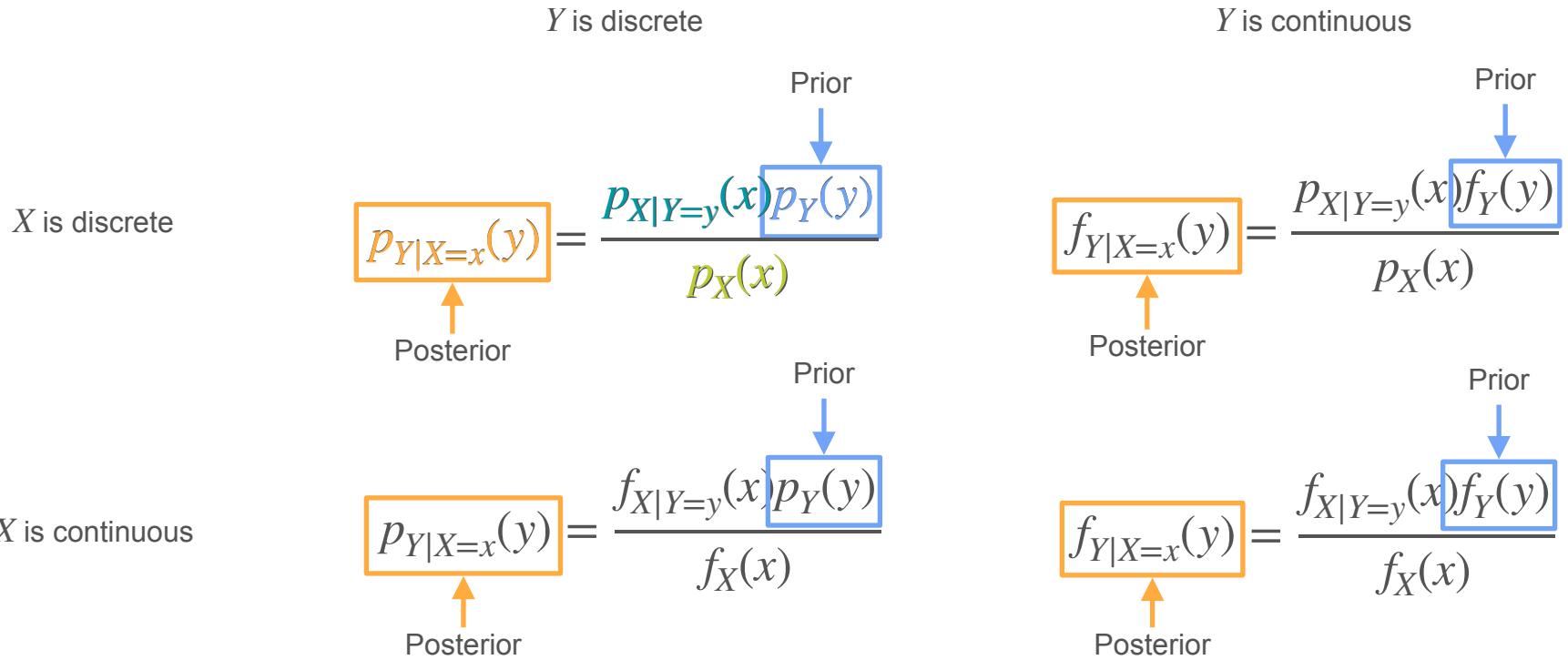
X : result of coin flip

$$X = \begin{cases} 0 & \text{if } T \\ 1 & \text{if } H \end{cases}$$

$$x = 1$$

$$p_{Y|X=x}(y) = \frac{p_{X|Y=y}(x) p_Y(y)}{p_X(x)}$$

Bayesian Statistics



Bayesian Statistics

X is discrete

Θ is discrete

$$p_{\Theta|X=x}(\theta) = \frac{p_{X|\Theta=\theta}(x)p_{\Theta}(\theta)}{p_X(x)}$$

X is continuous

Θ is continuous

$$f_{\Theta|X=x}(\theta) = \frac{p_{X|\Theta=\theta}(x)f_{\Theta}(\theta)}{f_X(x)}$$

$$p_{\Theta|X=x}(\theta) = \frac{f_{X|\Theta=\theta}(x)p_{\Theta}(\theta)}{f_X(x)}$$

$$f_{\Theta|X=x}(\theta) = \frac{f_{X|\Theta=\theta}(x)f_{\Theta}(\theta)}{f_X(x)}$$

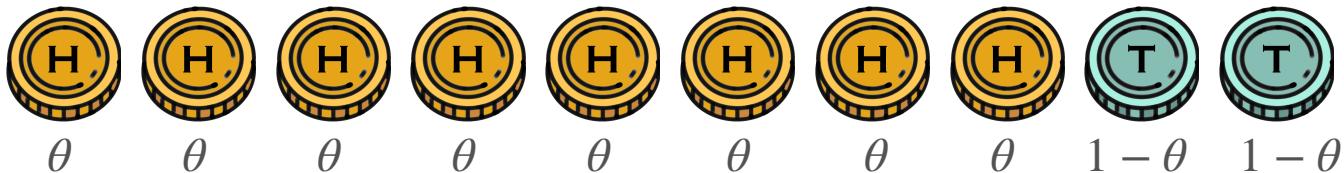


DeepLearning.AI

Point Estimation

**Bayesian Statistics
Full Worked Example**

Bayesian Statistics: Bernoulli Example



$$\Theta = P(H)$$

$$\mathbf{X} = (X_1, X_2, \dots, X_{10})$$

$X_i = 1$ if H , 0 if T

$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta)$$

$$f_{\Theta|\mathbf{X}=\mathbf{x}}(\theta) = \frac{p_{\mathbf{X}|\Theta=\theta}(\mathbf{x}) f_{\Theta}(\theta)}{p_{\mathbf{X}}(\mathbf{x})}$$

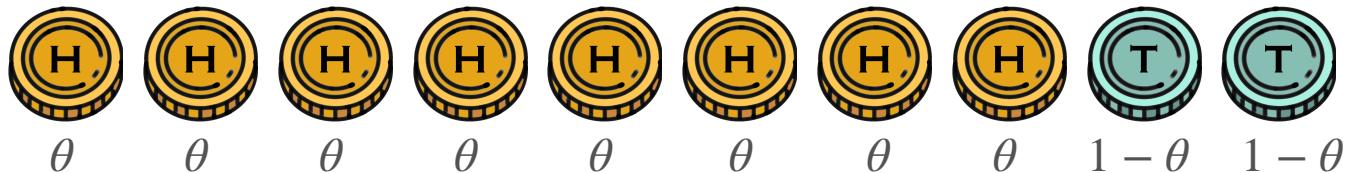
Θ is a continuous random variable

\mathbf{X} is a discrete random variable

Use this version of Bayes' Theorem

$$p_{\mathbf{X}|\Theta=\theta}(1,1,\dots,1,0,0) = p(X_1 = 1, X_2 = 1, \dots, X_8 = 1, X_9 = 0, X_{10} = 0 | \Theta = \theta) = \theta^8(1 - \theta)^2$$

Bayesian Statistics: Bernoulli Example



$$X_i | \Theta = \theta \sim \text{Bernoulli}(\theta)$$

$$\Theta \sim \text{Uniform}(0,1)$$

$$f_{\Theta|X=x}(\theta) = \frac{p_{\mathbf{X}|\Theta=\theta}(\mathbf{x})f_{\Theta}(\theta)}{p_{\mathbf{X}}(\mathbf{x})}$$

$$p_{\mathbf{X}|\Theta=\theta}(1,1,\dots,1,0,0) = \theta^8(1-\theta)^2 \quad f_{\Theta}(\theta) = 1, \quad 0 \leq \theta \leq 1$$

$$f_{\Theta|X=x}(\theta) = \frac{\theta^8(1-\theta)^2}{\text{constant}} = \frac{1}{\text{constant}}\theta^8(1-\theta)^2$$

This is a Beta Distribution, it's possible to calculate this constant, but it's unnecessary

Bayesian Statistics: Bernoulli Example

$$f_{\Theta|X=x}(\theta) = \frac{\theta^8(1-\theta)^2 1}{\text{constant}}$$



$$f_{\Theta}(\theta) = 1$$

$$f_{\Theta|X=x}(\theta) = \frac{1}{\text{constant}} \theta^8(1-\theta)^2$$

Bayesian Statistics: MAP Estimator

θ : a model of the coin
where $P(H) = \theta$

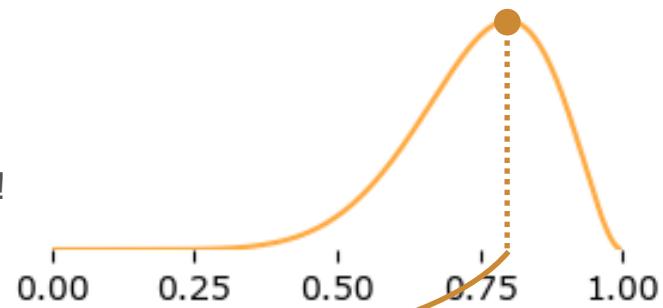
$$f_{\Theta|X=x}(\theta) \propto \theta^8(1-\theta)^2 [1] \quad \text{Likelihood of the model}$$

\uparrow
 $P(\text{data} | \text{model})$

Maximizing this is all MLE does!

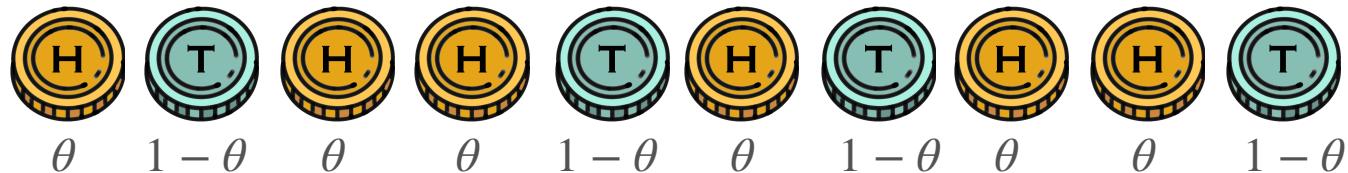
Maximum a Posteriori (MAP)

$$\hat{\theta} = \arg \max_{\theta} f_{\Theta|X=x}(\theta) = \frac{8}{10}$$



Non informative prior: MAP = MLE

Bayesian Statistics: Bernoulli Example Continued



$$f_{\Theta|X=x}(\theta) = \theta^8(1 - \theta)^2$$

$$p_{X|\Theta=\theta}(1,1,0,\dots,1,1,0) = \theta^6(1 - \theta)^4$$

New prior!

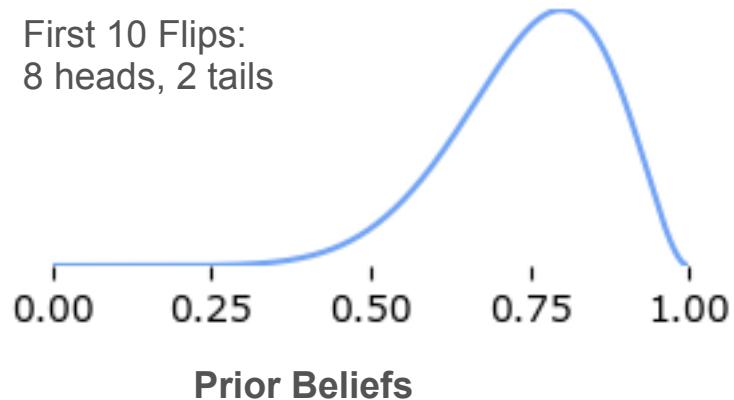
Repeat process

$$f_{\Theta|X=x}(\theta) = \frac{\text{constant}}{\theta^{14}(1 - \theta)^6}$$

Bayesian Statistics: Bernoulli Example Continued

$$f_{\Theta|X=x}(\theta) = \frac{\theta^6(1-\theta)^4 \theta^8(1-\theta)^2}{\text{constant}}$$

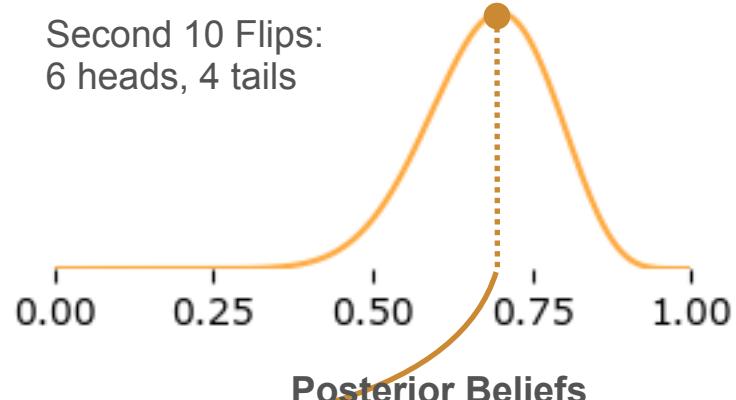
First 10 Flips:
8 heads, 2 tails



$$f_{\Theta}(\theta) = \frac{1}{\text{constant}} \theta^8(1-\theta)^2$$



Second 10 Flips:
6 heads, 4 tails



MAP
 $\hat{\theta} = 0.7$ ←
14 out of 20 heads
Same result as Frequentist

$$f_{\Theta|X=x}(\theta) = \frac{1}{\text{constant}} \theta^{14}(1-\theta)^6$$

Bayesian Statistics: Final Summary



- Bayesians update prior beliefs
- MAP with uninformative priors is just MLE
- With enough data, MLE and MAP estimates usually converge
- Good for instances when you have limited data or strong prior beliefs
- Wrong priors, wrong conclusions

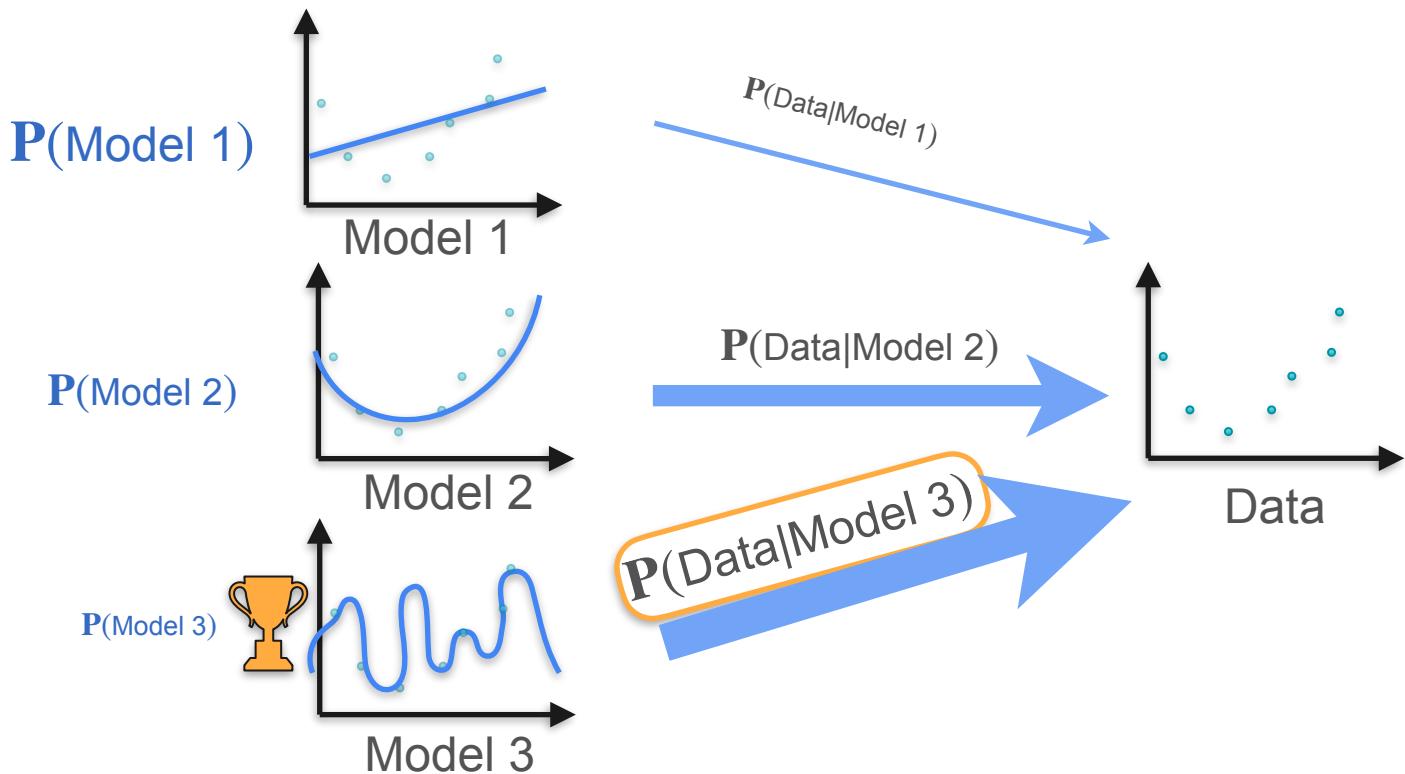


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Point Estimation

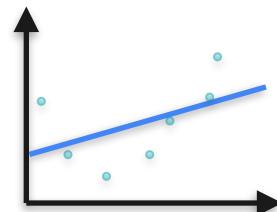
**Relationship between MAP,
MLE, and Regularization**

Example: Polynomial Regression

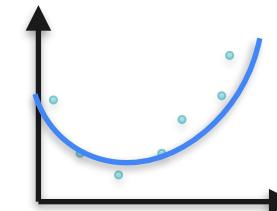


Example: Polynomial Regression

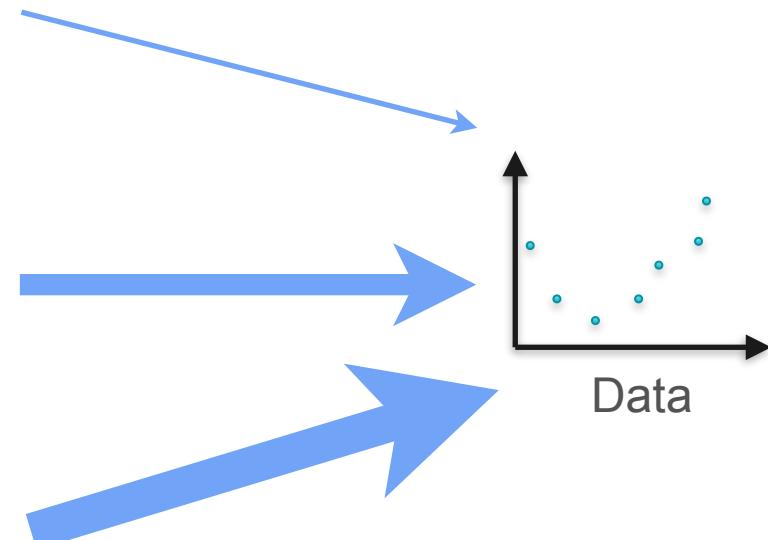
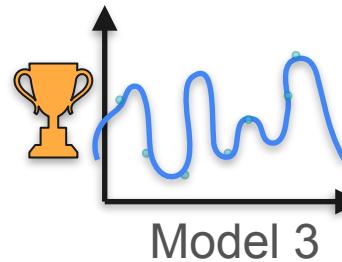
$P(\text{Model 1}) P(\text{Data}|\text{Model 1})$



$P(\text{Model 2}) P(\text{Data}|\text{Model 2})$



$P(\text{Model 3}) P(\text{Data}|\text{Model 3})$



Maximum likelihood
with Bayes

Polynomial regression
with regularization

$P(\text{Data}|\text{Model})$

$P(\text{Model})$

?

Log-loss

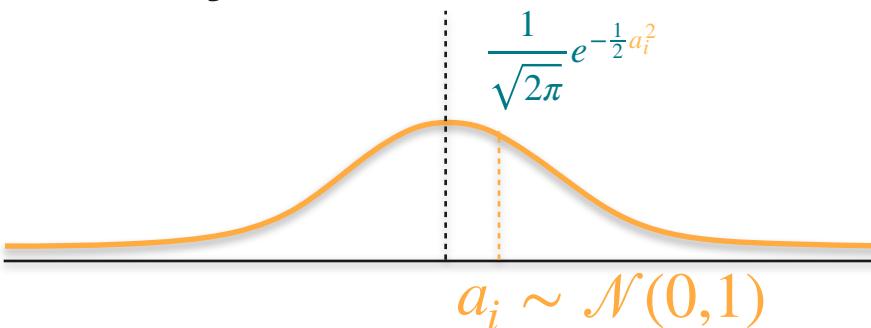
+

Regularization term

What Is the Probability of a Model?

$$P(\text{Model 1}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2}$$

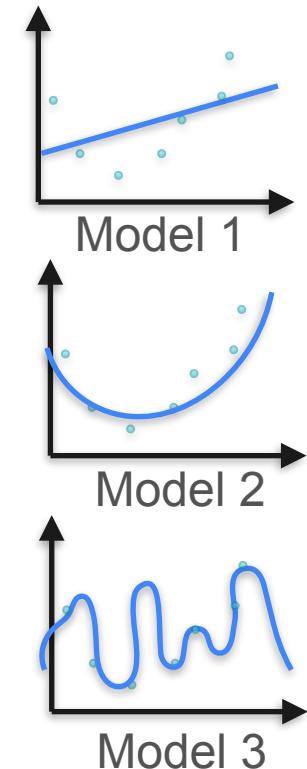
$a_1x + b$



$$P(\text{Model 2}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_2^2}$$

$a_1x^2 + a_2x + b$

$$P(\text{Model 3}) = \prod_{i=1}^{10} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}a_i^2}$$

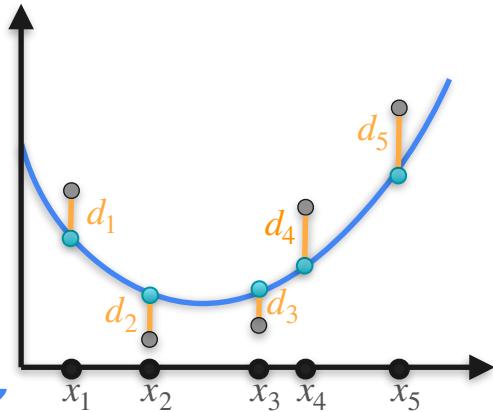


Bayes and Regularization

$P(\text{Data}|\text{Model})$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_1^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_3^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_4^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_5^2}$$

$P(\text{Model})$



Maximize

$$-\frac{1}{2}(d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2)$$

log

log

log

+

$$-\frac{1}{2}(a_1^2 + a_2^2)$$

Minimize

$$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + a_1^2 + a_2^2$$

Regularization

P(Model 1)

$$\text{Minimize } x_1^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Model 2)

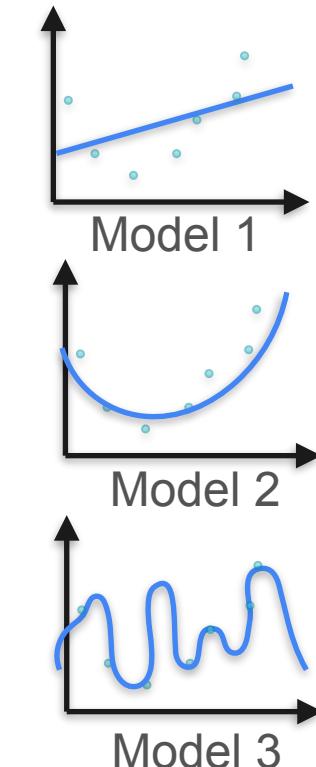
$$\text{Minimize } x_1^2 + x_2^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

P(Model 3)

$$\text{Minimize } x_1^2 + \dots + x_{10}^2 + d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2$$

New Loss

P(Data|Model 1)





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Point Estimation

Conclusion