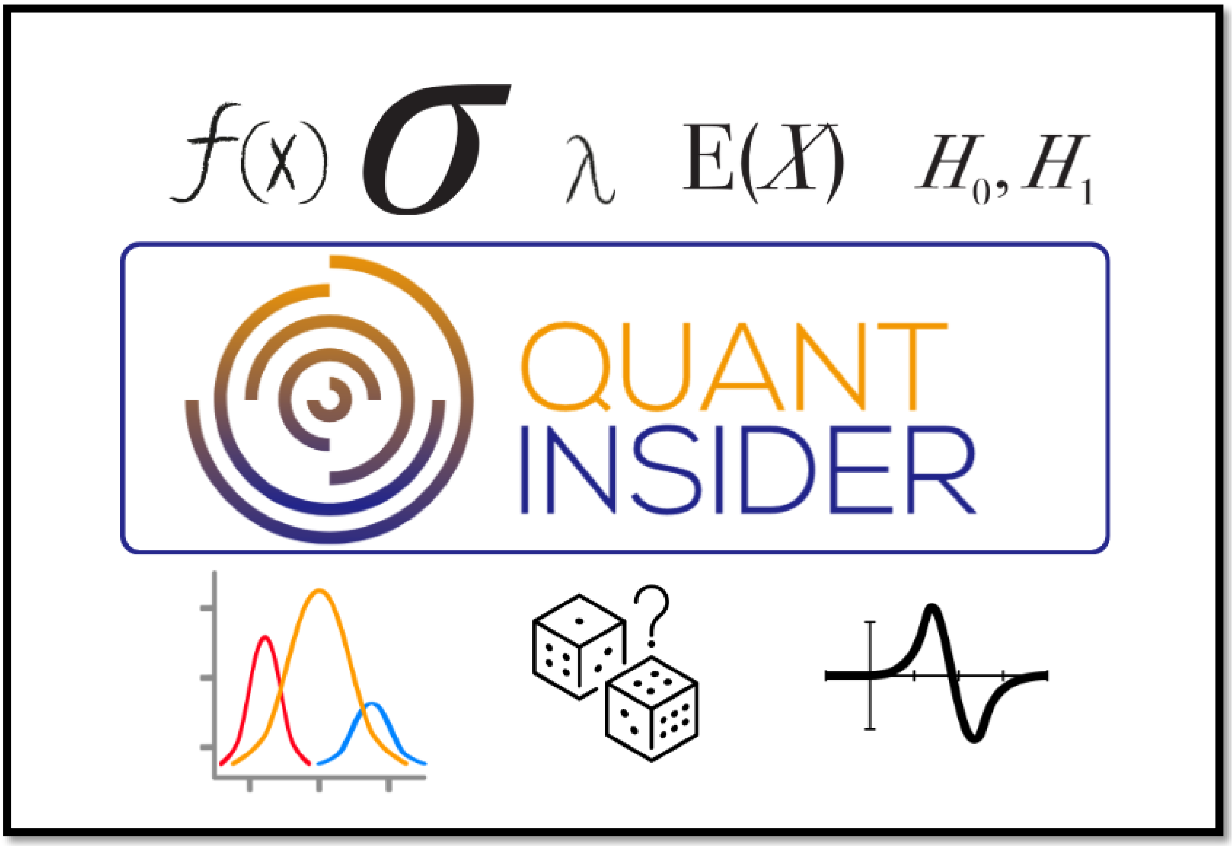

INTEREST RATE DERIVATIVES

Assume a foreign FRA trading at 6% on a deposit starting 90 days from now and expiring 180 days from now with a day count basis of $Act/365$. Further, the spot currency rate is 7, the currency swap for the same period is bid 400 offered 600, and the day count basis in the domestic FRA market is $Act/360$.

$FRA_f = 0.06$, $Spot = 7$, $ShortFXSwap = 600$, $LongFXSwap = 400$, $\tau_1 = 90$, $\tau_2 = 180$, $Basis = 360$, $Basis_f = 365$ yields

$$FRA = \left[\left(\frac{7+\frac{600}{10000}}{7+\frac{400}{10000}}\right)\left(1+0.06\frac{180-90}{365}\right)-1\right]\frac{360}{180-90} = 0.0707$$

Convexity Adjustment Money Market Futures

A Eurodollar futures trades at 95.50 expiring six years from now, and the underlying three-month money deposit expires 90 days later. The money deposit is on a 360-day count basis (money market basis). The expected volatility is 150 basis points per year; the mean reversion speed is 0.04. $F = 96.50$, $t_1 = 6$, $T_2 = 6 + 90/365 = 6.2466$, $\sigma = 0.015$, $\kappa = 0.04$, and Basis = 360.

The unadjusted implied futures yield is $100 - 95.50 = 4.50$—that is, 4.5%. What is the implied convexity adjusted yield?

$$Z = 0.015^2\left(\frac{1-e^{-2\times 0.04\times 6}}{2\times 0.04}\right)\left(\frac{1-e^{-0.04(6.2466-6)}}{0.04}\right)^2$$

$$+\frac{0.015^2}{2\times 0.04^3}(1-e^{-0.04(6.2466-6)})(1-e^{-0.04\times 6}) = 0.00085$$

$$\text{ConvexityBias} = (1-e^{-0.00085})\left(100-95.5+100\times\frac{360}{90}\right) = 0.3437$$

$$F_{Adj} = 95.5 + 0.3437 = 95.8437$$

This gives us an implied forward rate of $100 - 95.8437 = 4.1563$—that is, 4.1563%, which is 34 basis points lower than the unadjusted futures rate due to the convexity adjustment.

Modified Duration and BPV

Consider a bond with duration 6.43, yield 5.28%, clean price 103.02, and accrued interest 1.48. What is the BPV of the bond?

$$BPV = -\frac{6.43}{1+0.0528}(103.02+1.48) = -638.24$$

Thus, for every 1-basis-point increase in yield, the bond will decrease 638.24 dollars for every 1 million in notional.

Price and Yield Volatility in Money Market Futures

Consider a Eurodollar futures with price 94.53 and price volatility 1.5%. What is the equivalent yield volatility? $F = 94.53$, $\sigma_F = 0.015$, and $y = 5.47\%$.

$$\sigma_y = 0.015\frac{94.53}{100 - 94.53} = 25.92\%$$

Caps and Floors

What is the value of a caplet on a 182-day forward rate, with six months to expiration and a notional principle of 100 million? The six-month forward rate is 8% ($\frac{Act}{360}$ basis), the strike is 8%, the risk-free interest rate is 7%, and the volatility of the forward rate is 28% per year. $Basis = 360$, $\tau = 182$, $F = 0.08$, $X = 0.08$, $T = 0.5$, $r = 0.07$, and $\sigma = 0.28$, which yields

$$d_1 = \frac{\ln(0.08/0.08) + (0.28^2/2)0.5}{0.28\sqrt{0.5}} = 0.0990 \qquad d_2 = d_1 - 0.28\sqrt{0.5} = -0.0990$$

$$N(d_1) = N(0.0990) = 0.5394 \qquad N(d_2) = N(-0.0990) = 0.4606$$

$$\text{Caplet value} = \frac{100{,}000{,}000 \times \frac{182}{360}}{\left(1 + 0.08\frac{182}{360}\right)} e^{-0.07\times 0.5}[0.08N(d_1) - 0.08N(d_2)] = 295.995$$

Swaptions

Consider a two-year payer swaption on a four-year swap with semiannual compounding. The forward swap rate of 7% starts two years from now and ends six years from now. The strike is 7.5%, the risk-free interest rate is 6%, and the volatility of the forward starting swap rate is 20% per year. $t_1 = 4$, $m = 2$, $F = 0.07$, $X = 0.075$, $T = 2$, $r = 0.06$, and $\sigma = 0.2$.

$$d_1 = \frac{\ln(0.07/0.075) + (0.2^2/2)2}{0.2\sqrt{2}} = -0.1025 \qquad d_2 = d_1 - 0.2\sqrt{2} = -0.3853$$

$$N(d_1) = N(-0.1025) = 0.4592 \qquad N(d_2) = N(-0.3853) = 0.3500$$

$$c = e^{-0.06\times 2}[0.07N(d_1) - 0.075N(d_2)] = 0.5227\%$$

With a semiannual forward swap rate, the up-front value of the payer swaption in percent of the notional is

$$c \times \left[\frac{1 - \frac{1}{\left(1+\frac{0.07}{2}\right)^{4\times 2}}}{0.07}\right] = 1.7964\%$$

Convexity Adjustments

Consider a derivative instrument with a single payment five years from now that is based on the notional principal times the yield of a standard four-year swap with annual payments. The forward yield of the four-year swap, starting five years in the future and ending nine years in the future, is 7%. The volatility of the forward swap yield is 18%. What is the convexity adjustment of the swap yield? The value of the fixed side of the swap with annual yield is equal to the value of a bond where the coupon is equal to the forward swap rate/yield y_f:

$$P = \frac{c}{1+y_F} + \frac{c}{(1+y_F)^2} + \frac{c}{(1+y_F)^3} + \frac{1+c}{(1+y_F)^4}$$

The partial derivative of the swap with respect to the yield is

$$\frac{\partial P}{\partial y_F} = -\frac{c}{(1+y_F)^2} - \frac{2c}{(1+y_F)^3} - \frac{3c}{(1+y_F)^4} - \frac{4(1+c)}{(1+y_F)^5}$$

$$= -\frac{0.07}{(1+0.07)^2} - \frac{2\times 0.07}{(1+0.07)^3} - \frac{3\times 0.07}{(1+0.07)^4} - \frac{4(1+0.07)}{(1+0.07)^5} = -3.3872,$$

and the second partial derivative with respect to the forward swap rate is

$$\frac{\partial^2 P}{\partial y_F^2} = \frac{2c}{(1+y_F)^3} + \frac{6c}{(1+y_F)^4} + \frac{12c}{(1+y_F)^5} + \frac{20(1+c)}{(1+y_F)^6}$$

$$= \frac{2\times 0.07}{(1+0.07)^3} + \frac{6\times 0.07}{(1+0.07)^4} + \frac{12\times 0.07}{(1+0.07)^5} + \frac{20(1+0.07)}{(1+0.07)^6} = 15.2933$$

From Price to Yield Volatility in Bonds

Consider a government bond where the implied price volatility is 9%. The bond has a duration of six years and a yield to maturity of 8%. What is the equivalent yield volatility of the bond?

$$\sigma_y = \frac{0.09}{0.08\frac{6}{1+0.08}} = 20.25\%$$

The Vasicek Model

Consider a European call option on a zero-coupon bond. Time to expiration is two years, the strike price is 92, the volatility is 3%, the mean-reverting level is 9%, and the mean reverting rate is 0.05. The face value of the bond is 100 with time to maturity three years and initial risk-free rate of 8%. $F = 100$, $X = 92$, $T = 2$, $\tau = 3$, $\theta = 0.09$, $\kappa = 0.05$, $r = 0.08$, and $\sigma = 0.03$.

$$B(t,T) = B(0,2) = \frac{1 - e^{-0.05(2-0)}}{0.05} = 1.9032$$

$$B(T,\tau) = B(2,3) = \frac{1 - e^{-0.05(3-2)}}{0.05} = 0.9754$$

$$B(t,\tau) = B(0,3) = \frac{1 - e^{-0.05(3-0)}}{0.05} = 2.7858$$

$$A(t,T) = A(0,2) = \exp\left[\frac{(B(0,2) - 2 + 0)(0.05^2 \times 0.09 - 0.03^2/2)}{0.05^2} - \frac{0.03^2 B(0,2)^2}{4 \times 0.05}\right] = 0.9924$$

$$A(t,\tau) = A(0,3) = \exp\left[\frac{(B(0,3) - 3 + 0)(0.05^2 \times 0.09 - 0.03^2/2)}{0.05^2} - \frac{0.03^2 B(0,3)^2}{4 \times 0.05}\right] = 0.9845$$

$$P(t,T) = P(0,2) = A(0,2)e^{-B(0,2)0.08} = 0.8523$$

$$P(t,\tau) = P(0,3) = A(0,3)e^{-B(0,3)0.08} = 0.7878$$

$$\sigma_P = B(2,3)\sqrt{\frac{0.03^2(1 - e^{-2\times 0.05\times 2})}{2 \times 0.05}} = 0.0394$$

$$h = \frac{1}{\sigma_P}\ln\left[\frac{P(0,3)}{P(0,2)92}\right] + \frac{\sigma_P}{2} = 0.1394$$

The call value for one USD in face value is

$$c = P(0,3)N(h) - 92P(0,2)N(h - \sigma_P) = 0.0143$$

With a face value of 100, the call value is 1.43 USD (100×0.0143)

Jamshidian's Approach for Coupon Bonds

Consider a European call option on a coupon bond. Time to expiration is four years, the strike price 99.5, the volatility is 3%, the mean-reverting level is 10%, and the mean-reverting rate is 0.05. The face value of the bond is 100, and it pays a semiannual coupon of 4. Time to maturity is seven years, and the risk-free rate is initially 9%.

First find the rate $\hat{r}$ that makes the value of the coupon bond equal to the strike price at the option's expiration. Trial and error gives $\hat{r} = 8.0050\%$. To find the value of the option, we have to determine the value of six different options:

1. A four-year option with strike price 3.8427 on a 4.5-year zero-coupon bond with a face value of four
2. A four-year option with strike price 3.6910 on a five-year zero-coupon bond with a face value of four
3. A four-year option with strike price 3.5452 on a 5.5-year zero-coupon bond with a face value of four
4. A four-year option with strike price 3.4055 on a six-year zero-coupon bond with a face value of four
5. A four-year option with strike price 3.2717 on a 6.5-year zero-coupon bond with a face value of four
6. A four-year option with strike price 81.7440 on a seven-year zero-coupon bond with a face value of 104.

The value of the six options are, respectively, 0.0256, 0.0493, 0.0713, 0.0917, 0.1105, and 3.3219. This gives a total value of 3.6703.