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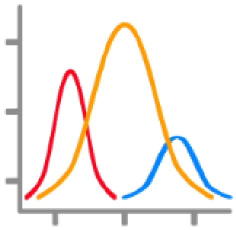
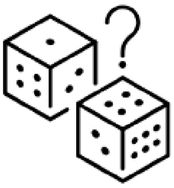
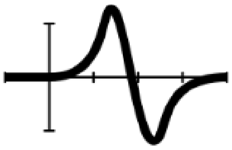




ADVANCE OPTION PRICING QUESTIONS

Consider a European put option with six months to expiration. The stock index is 100, the strike price is 95, the risk-free interest rate is 10% per year, the dividend yield is 5% per year, and the volatility is 20% per year. $S = 100$, $X = 95$, $T = 0.5$, $r = 0.1$, $q = 0.05$, and $\sigma = 0.2$:

$$d_1 = \frac{\ln(100/95) + (0.1 - 0.05 + 0.2^2/2)0.5}{0.2\sqrt{0.5}} = 0.6102$$

$$d_2 = d_1 - 0.2\sqrt{0.5} = 0.4688$$

$$N(d_1) = N(0.6102) = 0.7291 \qquad N(d_2) = N(0.4688) = 0.6804$$

$$N(-d_1) = N(-0.6102) = 0.2709 \qquad N(-d_2) = N(-0.4688) = 0.3196$$

$$p = 95e^{-0.1\times0.5}N(-d_2) - 100e^{-0.05\times0.5}N(-d_1) = 2.4648$$

Consider a European option on the Brent Blend futures with nine months to expiration. The futures price is USD 19, the strike price is USD 19, the risk-free interest rate is 10% per year, and the volatility is 28% per year. $F = 19$, $X = 19$, $T = 0.75$, $r = 0.1$, and $\sigma = 0.28$:

$$d_1 = \frac{\ln(19/19) + (0.28^2/2)0.75}{0.28\sqrt{0.75}} = 0.1212$$

$$d_2 = d_1 - 0.28\sqrt{0.75} = -0.1212$$

$$N(d_1) = N(0.1212) = 0.5483 \qquad N(d_2) = N(-0.1212) = 0.4517$$

$$N(-d_1) = N(-0.1212) = 0.4517 \qquad N(-d_2) = N(0.1212) = 0.5483$$

$$c = e^{-0.1\times0.75}[19N(d_1) - 19N(d_2)] = 1.7011$$

$$p = e^{-0.1\times0.75}[19N(-d_2) - 19N(-d_1)] = 1.7011$$

Consider a European USD-call/EUR-put option with six months to expiration. The USD/EUR exchange rate is 1.56, the strike is 1.6, the domestic risk-free interest rate in EUR is 8% per year, the foreign risk-free interest rate in the United States is 6% per year, and the volatility is 12% per year. $S = 1.56$, $X = 1.6$, $T = 0.5$, $r = 0.06$, $r_f = 0.08$, $\sigma = 0.12$.

$$d_1 = \frac{\ln(1.56/1.6) + (0.06 - 0.08 + 0.12^2/2)0.5}{0.12\sqrt{0.5}} = -0.3738$$

$$d_2 = d_1 - 0.12\sqrt{0.5} = -0.4587$$

$$N(d_1) = N(-0.3738) = 0.3543 \qquad N(d_2) = N(-0.4587) = 0.3232$$

$$c = 1.56e^{-0.08\times0.5}N(d_1) - 1.6e^{-0.06\times0.5}N(d_2) = 0.0291$$

The option premium is thus 0.0291 USD per EUR. Alternatively, the premium can be quoted in EUR per USD $0.0291/1.56^2 = 0.0120$—or as percent of spot, $0.0291/1.56 = 0.0186538$, or 1.8654% of EUR (or the spot price). Hence, if the option has a notional of 100 million EUR, the total option premium is 1,865,384.62 EUR, or $1{,}865{,}384.62 \times 1.56 = 2{,}910{,}000.00$ in USD.

Black-Scholes-Merton Option Greeks (Partial Derivatives) Summary

Name	Symbol	Derivative	Other name	Formula
Delta call	Δ_{call}	$\frac{\partial c}{\partial S}$	Spot delta	$e^{(b-r)T}N(d_1)$
Delta put	Δ_{put}	$\frac{\partial p}{\partial S}$	Spot delta	$e^{(b-r)T}[N(d_1)-1]$
DdeltaDvol		$\frac{\partial^2 c}{\partial S\partial\sigma}=\frac{\partial^2 p}{\partial S\partial\sigma}$	Vanna, DvegaDspot	$\frac{-e^{(b-r)T}d_2}{\sigma}n(d_1)$
DvannaDvol		$\frac{\partial^3 c}{\partial S\partial\sigma^2}=\frac{\partial^3 p}{\partial S\partial\sigma^2}$		$\text{Vanna}\left(\frac{1}{\sigma}\right)\left(d_1d_2-\frac{d_1}{d_2}-1\right)$
DdeltaDtime call		$-\frac{\partial^2 c}{\partial S\partial T}$	Charm, delta bleed	$-e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}}-\frac{d_2}{2T}\right)+(b-r)N(d_1)\right]$
DdeltaDtime put		$-\frac{\partial^2 p}{\partial S\partial T}$	Charm, delta bleed	$-e^{(b-r)T}\left[n(d_1)\left(\frac{b}{\sigma\sqrt{T}}-\frac{d_2}{2T}\right)-(b-r)N(-d_1)\right]$
Elasticity call	Λ_{call}	$\frac{\partial c}{\partial S}\frac{S}{\text{call}}$	Lambda, leverage	$e^{(b-r)T}N(d_1)\frac{S}{\text{call}}$
Elasticity put	Λ_{put}	$\frac{\partial p}{\partial S}\frac{S}{\text{put}}$	Lambda, leverage	$e^{(b-r)T}[N(d_1)-1]\frac{S}{\text{put}}$
Gamma	Γ	$\frac{\partial^2 c}{\partial S^2}=\frac{\partial^2 p}{\partial S^2}$	Convexity	$\frac{n(d_1)e^{(b-r)T}}{S\sigma\sqrt{T}}$
GammaP	Γ_P	$\frac{S}{100}\frac{\partial^2 c}{\partial S^2}=\frac{S}{100}\frac{\partial^2 p}{\partial S^2}$	Gamma percent	$\frac{S\Gamma}{100}$
DgammaDvol		$\frac{\partial^3 c}{\partial S^2\partial\sigma}=\frac{\partial^3 p}{\partial S^2\partial\sigma}$	Zomma	$\Gamma\left(\frac{d_1d_2-1}{\sigma}\right)$

Name	Symbol	Derivative	Other name	Formula
DdeltaDvar		$\frac{\partial^2 c}{\partial S\partial V}=\frac{\partial^2 p}{\partial S\partial V}$		$-Se^{(b-r)T}n(d_1)\frac{d_2}{2\sigma^2}$
Variance vomma		$\frac{\partial^2 c}{\partial V^2}=\frac{\partial^2 p}{\partial V^2}$		$\frac{Se^{(b-r)T}\sqrt{T}}{4\sigma^3}n(d_1)(d_1d_2-1)$
Variance ultima		$\frac{\partial^3 c}{\partial V^3}=\frac{\partial^3 p}{\partial V^3}$		$\frac{Se^{(b-r)T}\sqrt{T}}{8\sigma^5}n(d_1)[(d_1d_2-1)(d_1d_2-3)-(d_1^2+d_2^2)]$
Theta call	Θ_{call}	$-\frac{\partial c}{\partial T}$	Expected bleed	$-\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}}-(b-r)Se^{(b-r)T}N(d_1)-rXe^{-rT}N(d_2)$
Theta put	Θ_{put}	$-\frac{\partial p}{\partial T}$	Expected bleed	$-\frac{Se^{(b-r)T}n(d_1)\sigma}{2\sqrt{T}}+(b-r)Se^{(b-r)T}N(-d_1)+rXe^{-rT}N(-d_2)$
Driftless theta	θ	$-\frac{\partial c}{\partial T}=-\frac{\partial p}{\partial T}$	Pure bleed ($b=0, r=0$)	$-\frac{Sn(d_1)\sigma}{2\sqrt{T}}$
Rho call	ρ_{call}	$\frac{\partial c}{\partial r}$		$TXe^{-rT}N(d_2)$
Rho call futures option	ρ_{call}	$\frac{\partial c}{\partial r}$		$-Tc$
Rho put	ρ_{put}	$\frac{\partial p}{\partial r}$		$-TXe^{-rT}N(-d_2)$
Rho put futures option	ρ_{put}	$\frac{\partial p}{\partial r}$		$-Tp$

Phi call	Φ_{call}	$\frac{\partial c}{\partial q}$	Rho-2	$-TSe^{(b-r)T}N(d_1)$
Phi put	Φput	$\frac{\partial p}{\partial q}$	Rho-2	$TSe^{(b-r)T}N(-d_1)$
Carry rho call		$\frac{\partial c}{\partial b}$		$TSe^{(b-r)T}N(d_1)$
Carry rho put		$\frac{\partial p}{\partial b}$		$-TSe^{(b-r)T}N(-d_1)$
Zeta call	ζ_{call}		In-the-money prob.	$N(d_2)$
Zeta put	ζ_{put}		In-the-money prob.	$N(-d_2)$
DzetaDvol call		$\frac{\partial \zeta_{\text{call}}}{\partial \sigma}$		$-n(d_2)\left(\frac{d_1}{\sigma}\right)$
DZetaDVol put		$\frac{\partial \zeta_{\text{put}}}{\partial \sigma}$		$n(d_2)\left(\frac{d_1}{\sigma}\right)$
DZetaDTime call		$-\frac{\partial \zeta_{\text{call}}}{\partial T}$		$n(d_2)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right)$
DZetaDTime put		$-\frac{\partial \zeta_{\text{put}}}{\partial T}$		$-n(d_2)\left(\frac{b}{\sigma\sqrt{T}} - \frac{d_1}{2T}\right)$
Strike delta call		$\frac{\partial c}{\partial X}$	Discounted probability	$-e^{-rT}N(d_2)$
Strike delta put		$\frac{\partial p}{\partial X}$	Discounted probability	$e^{-rT}N(-d_2)$
Strike gamma		$\frac{\partial^2 c}{\partial X^2} = \frac{\partial^2 p}{\partial X^2}$	RND	$\frac{n(d_2)e^{-rT}}{X\sigma\sqrt{T}}$

Consider a futures option with six months to expiration. The futures price is 105, the strike price is 100, the risk-free interest rate is 10% per year, and the volatility is 36% per year. Thus, $S = 105$, $X = 100$, $T = 0.5$, $r = 0.1$, $b = 0$, and $\sigma = 0.36$.

$$d_1 = \frac{\ln(105/100) + (0 + 0.36^2/2)0.5}{0.36\sqrt{0.5}} = 0.3189$$

$$N(d_1) = N(0.3189) = 0.6251$$

$$\Delta_{\text{call}} = e^{(0-0.1)0.5}N(d_1) = 0.5946$$

$$\Delta_{\text{put}} = e^{(0-0.1)0.5}[N(d_1) - 1] = -0.3566$$

Consider a commodity option with two years to expiration. The commodity price is 90, the strike price is 40, the risk-free interest rate is 3% per year, the cost-of-carry is 9% per year, and the volatility is 20%. What is the delta of a call option? $S = 90$, $X = 40$, $T = 2$, $r = 0.03$, $b = 0.09$, and $\sigma = 0.2$.

$$d_1 = \frac{\ln(90/40) + (0.09 + 0.2^2/2)2}{0.2\sqrt{2}} = 3.6449$$

$$N(d_1) = N(3.6449) = 0.9999$$

$$\Delta_{\text{call}} = e^{(0.09-0.03)2}N(3.6449) = 1.1273$$

The delta of this option is about 112.73%, which implies that the call price will increase (decrease) with about 1.13 dollars if the spot price increases (decreases) by one dollar.

What should the strike be for a delta-neutral straddle with nine months to maturity, the risk-free rate 10%, volatility 60%, for a stock trading at 100? $S = 100$, $T = 0.75$, $r = 0.1$, $b = 0.1$, and $\sigma = 0.6$.

$$X_{\text{call}} = X_{\text{put}} = 100e^{(0.1+0.6^2/2)0.75} = 123.3678$$

What should the strike be for a three-month call stock index option to get a delta of 0.25, the risk-free rate 7%, dividend yield 3%, and volatility 50%, and with the stock index trading at 1800? $S = 1800$, $T = 0.25$, $r = 0.07$, $b = 0.07 - 0.03 = 0.04$, $\sigma = 0.5$, and thus

$$N^{-1}(\Delta_{\text{call}}e^{(r-b)T}) = N^{-1}(0.25e^{(0.07-0.04)0.25}) = -0.6686$$

$$X_{\text{call}} = 1800 \times \exp[0.6686 \times 0.5\sqrt{0.25} + (0.04 + 0.5^2/2)0.25] = 2217.0587$$

That is, to get a delta of 0.25, we need to set the strike to 2217.0587.

Consider a put option on a stock trading at 90, with three months to maturity, strike 80, three month risk-free interest rate of 5%, and

volatility of 20%. What is the DdeltaDvol? $S = 90$, $X = 80$, $T = 0.25$, $r = 0.05$, $b = 0.05$, $\sigma = 0.2$, thus

$$d_1 = \frac{\ln(90/80) + (0.05 + 0.2^2/2)0.25}{0.2\sqrt{0.25}} = 1.3528$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.3528 - 0.2\sqrt{0.25} = 1.2528$$

$$n(d_1) = n(1.3528) = 0.1598$$

$$\frac{\partial^2 p}{\partial S \partial \sigma} = \frac{-e^{(0.05-0.05)0.25}1.2528}{0.2}0.1598 = -1.0008$$

If the volatility increases from 20% to 21%, the delta of the put will thus decrease by about one percentage point $\frac{-1.0008}{100}$. Similarly, if the volatility decreases from 20% to 19%, delta will increase by

one percent point. Alternatively, this shows that the options vega will decrease by -0.0100 if the stock price increases by one.

Consider a European put option on a futures currently priced at 105. The strike price is 90, the time to expiration is three months, the risk-free interest is 14% per year, and the volatility is 24% per year. What is the charm of the option? $S = 105$, $X = 90$, $T = 0.25$, $r = 0.14$, $b = 0$, $\sigma = 0.24$, and thus

$$d_1 = \frac{\ln(105/90) + 0.25 \times 0.24^2/2}{0.24\sqrt{0.25}} = 1.3446$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.3446 - 0.24\sqrt{0.25} = 1.2246$$

$$n(d_1) = n(1.3446) = 0.1616$$

$$N(-d_1) = N(-1.3446) = 0.0894$$

$$-\frac{\partial \Delta_{\text{put}}}{\partial T} = -e^{(0-0.14)0.25}\left[0.1616\left(\frac{0}{0.24\sqrt{0.25}} - \frac{1.2246}{2 \times 0.25}\right) - (0 - 0.14)0.0894\right] = 0.3700$$

The DdeltaDtime for one day is thereby $\frac{0.3700}{365} = 0.0010$.

Consider a stock option with strike 500, risk-free rate 8%, and volatility 40%. For what time and stock price does gamma have a saddle point, and what is the gamma at this point? With $X = 500$, $r = 0.08$, $b = 0.08$, and $\sigma = 0.4$, the saddle time must be (in number of years)

$$T_S = \frac{1}{2(0.4^2 + 2 \times 0.08 - 0.08)} = 2.0833$$

the saddle point stock price must be

$$S_{\bar{\Gamma}} = 500e^{(-0.08-3\times0.4^2/2)2.0833} = 256.7086$$

and the gamma at this point is

$$\Gamma_S = \Gamma(256.7086, 2.0833) = \frac{\sqrt{\frac{e}{\pi}}\sqrt{\frac{2\times0.08-0.08}{0.4^2} + 1}}{500} = 0.0023$$

That is, for the saddle point $(S = 256.7086, T = 2.0833)$, we have a gamma of 0.0023.

Consider a put option on a futures contract trading at 100, with three months to expiration, strike 80, three-month risk-free interest rate of 5%, and volatility of 26%. What is the DgammaDvol/zomma and also the zommaP? Letting $S = 100$, $X = 80$, $T = 0.25$, $r = 0.05$, $b = 0$, $\sigma = 0.26$, we have

$$d_1 = \frac{\ln(100/80) + 0.26^2/2 \times 0.25}{0.26\sqrt{0.25}} = 1.7815$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.7815 - 0.26\sqrt{0.25} = 1.6515$$

$$n(d_1) = n(1.7815) = 0.0816$$

$$\Gamma_{\text{call,put}} = \frac{n(d_1)e^{(0-0.05)0.25}}{100 \times 0.26\sqrt{0.25}} = 0.0062$$

$$\text{DgammaDvol}_{\text{call, put}} = \frac{\partial\Gamma}{\partial\sigma} = 0.0062\left(\frac{1.7815 \times 1.6515 - 1}{0.26}\right) = 0.0463$$

Assume a option on a futures contract trading at 80, with a strike price of 65. The risk-free rate is 5% and the volatility is 30%. For what time does this option have its maximum vega, *ceteris paribus*? $S = 80$, $X = 65$, $r = 0.05$, $b = 0$, $\sigma = 0.3$.

$$T = \frac{2\left[1 + \sqrt{1 + (8 \times 0.05 \times \frac{1}{0.3^2} + 1)\ln(80/65)^2}\right]}{8 \times 0.05 + 0.3^2} = 8.6171$$

Consider a stock option with nine months to expiration. The stock price is 55, the strike price is 60, the risk-free interest rate is 10% per year, and the volatility is 30% per year. Moreover, the gamma of the option is 0.0278 (as we calculated in the gamma example above). What is the vega? $S = 55$, $X = 60$, $T = 0.75$, $r = 0.1$, $b = 0.1$, $\sigma = 0.3$, $\Gamma = 0.0278$, and

$$\text{Vega} = 0.0278 \times 0.3 \times 55^2 \times 0.75 = 18.9358$$

To look at vega for a one-percent-point move in implied volatility, we need to divide it by 100, so the vega of the option is 0.1894.

Consider a put option on a stock index currently trading at 733, with six months to expiration, a strike price of 453, a dividend yield of 7.68%, a volatility of the index of 28%, and a risk-free rate of 10.68%. What is the sensitivity of the option to a one-percentage-point change in the dividend yield? With $S = 733$, $X = 453$, $T = 0.5$, $r = 0.1068$, $b = 0.1068 - 0.0768 = 0.03$, and $\sigma = 0.28$, we get

$$d_1 = \frac{\ln(733/453) + (0.03 + 0.28^2/2)0.5}{0.28\sqrt{0.5}} = 2.6055$$

$$N(-d_1) = N(-2.6055) = 0.0046$$

Let us assume that we own the portfolio shown in Table below. How will the portfolio react to shifts in the term structure of volatility? To calculate *NWV,* we need estimates of future volatility of volatilities with different maturities, and correlation coefficients between different volatilities. We chose a reference volatility equal to the volatility of the option with the longest time to maturity, that is, 120-day volatility today, 119-day volatility tomorrow, and so on. Assume we have calculated the following historical volatilities of volatilities: 6.5 percentage points 30-day, 5.5 percentage points 60-day, 4.0 percentage points 120-day, and correlation coefficients of 0.65 between 30-day volatility and the reference volatility (120-day), 0.85 for the 60-day volatility and, naturally, 1.0 between 120-day volatility and the reference volatility. It follows that II/1 = 6.5, = 5.5, 1P3 -= 4.0, 'PR = 4.0, *PLR* = 0.65, *,o2,R* = 0.85, *,o3,R* = 1.0.

Option Portfolio

$(S = 100, \sigma = 0.25, r = 0.1, b = 0.1)$

Days to maturity	120	60	60	30
Strike	105.00	85.00	100.00	100.00
Call price	4.99	16.53	4.88	3.27
Vega	22.86	3.11	15.81	11.31
Number of contracts	450	100	−400	−300
Volatility of volatility	4.00%	5.50%	5.50%	6.50%
Correlation coefficients	1.00	0.85	0.85	0.65

Espen Gaarder Haug, "Opportunities and Perils of Using Option Sensitivities," *Journal of Financial Engineering*, vol. 2, no. 3, September 1993. Used by permission.

$$
\begin{aligned}
NWV &= \sum_{T=1}^{m} \sum_{i=1}^{n} Q_{i,T}\, \text{vega}_{i,T} \frac{\Psi_T}{\Psi_R} \rho_{\sigma(T),\sigma(R)} \\
&= -300 \times 11.31 \times \frac{6.5}{4.0} \times 0.65 - 400 \times 15.81 \times \frac{5.5}{4.0} \times 0.85 \\
&\quad + 100 \times 3.11 \times \frac{5.5}{4.0} \times 0.85 + 450 \times 22.86 \times \frac{4.0}{4.0} \times 1.0 \\
&= -324.55
\end{aligned}
$$

Hence, for each percentage point rise in the reference volatility (120-day), we will lose approximately $325.

Consider a futures option with three months to expiration. The stock price is 60, the strike price is 60, the risk-free interest rate is 6% per year, and the volatility is 30% per year. Assuming we rebalance the hedge 20 times, what is the standard deviation in P&L? $S = 60$, $X = 60$, $T = 0.25$, $r = 0.06$, $b = 0$, $\sigma = 0.3$, and $N = 20$. Given these parameters, the vega of the option is

$$d_1 = \frac{\ln(60/60) + (0.3^2/2)0.25}{0.3\sqrt{0.25}} = 0.0750$$

$$n(d_1) = n(0.0750) = 0.3978$$

$$\text{vega} = 60e^{(0-0.06)0.25} \times 0.3978\sqrt{0.25} = 11.7570$$

The standard deviation in terms of dollars is then given by

$$\sigma_{\text{P\&L}} \approx \sqrt{\frac{\pi}{4}} \times 11.7570 \times \frac{0.3}{\sqrt{20}} = 0.6990$$

The one standard deviation in our expected profit and loss is thus ± 0.6990 dollars. This is independent of a call or a put option.

Without going through the calculations, we can say the call or put option[3] value with these parameters is 3.5337. This gives us a standard deviation in percent of option value of $0.6990/3.5337 = 19.78\%$. We could alternatively have calculated this using formula (6.6):

$$\frac{\sigma_{\text{P\&L}}}{c} \approx \sqrt{\frac{\pi}{4 \times 20}} = 19.82\%$$

As pointed out earlier, stochastic volatility, jumps, and other real market effects will in general increase the standard deviation of the profit and loss from the discrete hedge.

Consider an option that we delta hedge once a week. Assume the expected volatility of the asset is 50%, the risk-free rate is 5%, and the expected growth rate of the stock is 20%. What is the volatility we should use in the BSM formula to value the option? $\Delta t = \frac{1}{52}$, $\sigma = 0.5$, $r = 0.05$, and $\mu = 0.2$ yields

$$\hat{\sigma} = \sigma \left(1 + \frac{\frac{1}{52}}{2 \times 0.5^2}(0.2 - 0.05)(0.05 - 0.2 - 0.5^2) \right) = 49.88\%$$

An adjustment of 0.12 percentage points to the volatility may seem trivial. In a competitive options market, this may still be of economic consequence.

Assume the transaction costs are 0.1% of the stock value, that we rebalance the hedge daily, and that the expected volatility of the stock is 30%. What volatility should be used to price plain vanilla options? With $\sigma = 0.3$, $\kappa = 0.001$, and $\Delta t = \frac{1}{365}$, the volatility we should use for long options is

$$\sigma_{\text{long}} = 0.3\left(1 - \frac{0.001}{0.3}\sqrt{\frac{8}{\pi\frac{1}{365}}}\right)^{\frac{1}{2}} = 28.43\%,$$

while for short options we should use

$$\sigma_{\text{short}} = 0.3\left(1 + \frac{0.001}{0.3}\sqrt{\frac{8}{\pi\frac{1}{365}}}\right)^{\frac{1}{2}} = 31.49\%$$

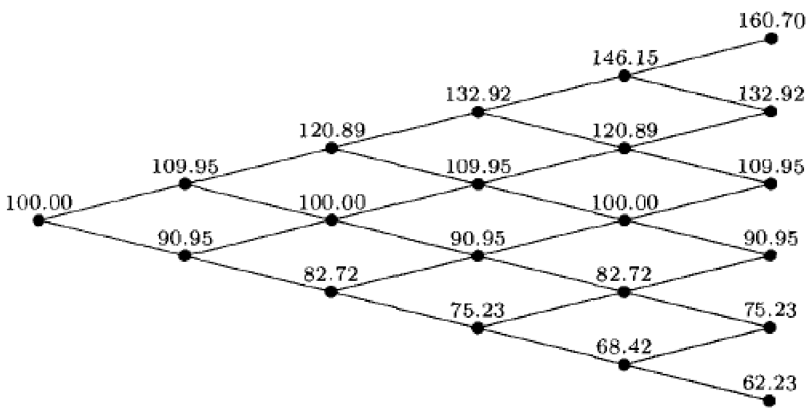
These volatilities are then to be used as input in the BSM formula.

Consider an American stock put option with six months to expiration. The stock price is 100, the strike price is 95, the risk-free interest rate is 8%, and the volatility is 30%. The option is priced in a binomial tree with five time steps. $S = 100$, $X = 95$, $T = 0.5$, $r = b = 0.08$, $\sigma = 0.3$, and $n = 5$.

$$\Delta t = \frac{0.5}{5} = 0.1$$

$$u = e^{0.3\sqrt{0.1}} = 1.0995 \qquad d = e^{-0.3\sqrt{0.1}} = 0.9095$$

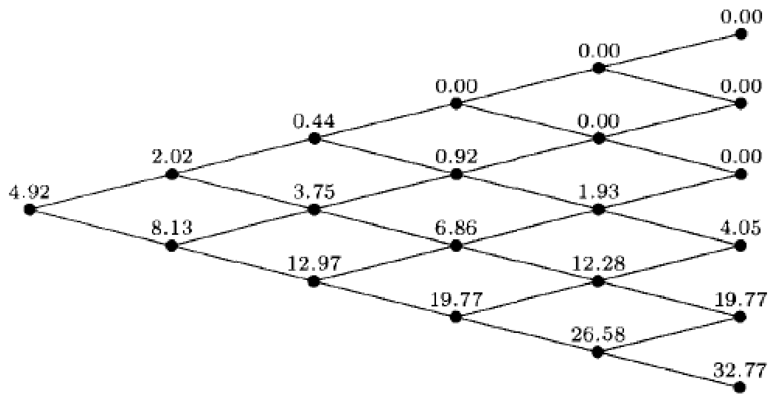
$$p = \frac{e^{0.08\times 0.1} - 0.9095}{1.0995 - 0.9095} = 0.5186$$



First, we start at the end of the tree to see if it is optimal to exercise the option $\max[X - S, 0]$. For example, in the end node with asset price 62.23, it is naturally optimal to exercise the put option: $\max[95 - 62.23, 0] = 32.77$, while at, for example, the end node with asset price

100.95, it is not optimal to exercise the option $\max[95 - 100.95, 0] = 0$. After checking for optimal exercise at each end node, we can now easily find the value of the American put option by standard backward induction (rolling back through the tree), where we check at each node if it is optimal with early exercise:

$$P_{j,i} = \max\{X - Su^i d^{j-i}, e^{-r\Delta t}[pP_{j+1,i+1} + (1-p)P_{j+1,i}]\}$$



The value of the American put option is therefore approximately 4.92.

Consider an at-the-money-forward stock call option with three months to expiration. The stock price is 59, the strike price is 60, the risk-free interest rate is 6.7% per year, and the market price of the option is 2.82. $S = 59$, $X = 60$, $T = 0.25$, $r = b = 0.067$, and $c_m = 2.82$. What is the implied volatility?

$$\sigma \approx \frac{2.82\sqrt{2\pi}}{59e^{(0.067-0.067)0.25}\sqrt{0.25}} = 23.96\%$$

For comparison, the exact implied volatility is 23.99%.

Consider a put option with six months to expiration. The futures price is 108, the strike price is 100, the risk-free interest rate is 10.50% per year, and the market price of the put option is 5.08. $S = 108$, $X = 100$, $T = 0.5$, $r = 0.105$, $b = 0$, and $p_m = 5.08$. What is the implied volatility?

$$\sigma \approx \frac{\sqrt{2\pi}}{108e^{(0-0.105)0.5} + 100e^{-0.105\times0.5}} \left\{ 5.08 - \frac{100e^{-0.105\times0.5} - 108e^{(0-0.105)0.5}}{2} + \left[\left(5.08 - \frac{100e^{-0.105\times0.5} - 108e^{(0-0.105)0.5}}{2} \right)^2 - \frac{(100e^{-0.105\times0.5} - 108e^{(0-0.105)0.5})^2}{\pi} \right]^{\frac{1}{2}} \right\} / \sqrt{0.5} \approx 29.90\%$$

For comparison, the exact implied volatility is 30.00%.

Assume a 10-hour trading day. What is the probability that we will have a high or low price in the last hour of the day, assuming zero drift and normally distributed returns? Intuition may suggest to you 1/10=10%, but the right answer is: $T = 10$, $\tau = 9$, yielding

$$\frac{2}{\pi}\arctan\left(\frac{\sqrt{10-9}}{\sqrt{9}}\right) = 0.205.$$

The probability of a new high or low in the last hour is thus 20.5%, *assuming* the asset follows a Brownian motion without drift.

(The above question is based on **Probability of High or Low, the Arctangent Rule)**

Consider a European option on the Brent Blend forward that expires in 12 months, with 9 months to expiration. The forward price is USD 19, the strike price is USD 19, the risk-free interest rate is 10% per year, and the volatility is 28% per year. $F = 19$, $X = 19$, $T = 0.75$, $T_f = 1$, $r = 0.1$, and $\sigma = 0.28$. Thus:

$$d_1 = \frac{\ln(19/19) + (0.28^2/2)0.75}{0.28\sqrt{0.75}} = 0.1212$$

$$d_2 = d_1 - 0.28\sqrt{0.75} = -0.1212$$

$$N(d_1) = N(0.1212) = 0.5483 \qquad N(d_2) = N(-0.1212) = 0.4517$$

$$N(-d_1) = N(-0.1212) = 0.4517 \qquad N(-d_2) = N(0.1212) = 0.5483$$

$$c = e^{-0.1\times 1}[19N(d_1) - 19N(d_2)] = 1.6591$$

$$p = e^{-0.1\times 1}[19N(-d_2) - 19N(-d_1)] = 1.6591$$

Consider a call on a quarterly electricity swap, with six months to maturity. The start of the delivery period is 17 days after the option expires, and the delivery period is 2208 hours, or 92 days. The swap/forward trades at 33 EUR/MwH, and the strike is 35 EUR/MwH. The number of fixings in the delivery period is 92. The risk-free rate from now until the beginning of the delivery period is 5%. The daily compounding swap rate starting at the beginning of the delivery period and ending at the end of the delivery period is 5%. The volatility of the swap is 18%. What is the option value? $T = 0.5$, $T_b = 0.5 + 17/365 = 0.5466$, $r_b = 0.05$, $r_j = 0.05$, $j = 365$, $n = 92$, and $\sigma = 0.18$ yields

$$d_1 = \frac{\ln(33/35) + 0.5 \times 0.18^2/2}{0.18\sqrt{0.5}} = -0.3987$$

$$d_2 = -0.3987 - 0.18\sqrt{0.5} = -0.5259$$

$$N(d_1) = N(-0.3987) = 0.3451 \qquad N(d_2) = N(-0.5259) = 0.2995$$

$$c = \frac{\left(1 - \frac{1}{(1+0.05/365)^{92}}\right)}{0.05} \frac{365}{92} e^{-0.05\times 0.5466}[33N(d_1) - 35N(d_2)] = 0.8761$$

To find the value of an option on one swap/forward contract, we need to multiply by the number of delivery hours. This yields a price of $2208 \times 0.8761 = 1{,}934.37$ EUR. Alternatively, we could have found the option value using the approximation (10.8), using time from now to the middle of delivery period $T_m = 0.5 + 17/365 + 92/2/365 = 0.6726$ and assuming the rate from now to the end of the delivery period is $r_e \approx 0.05$:

$$c \approx e^{-0.05\times 0.6260}[33N(d_1) - 35N(d_2)] = 0.8761$$

At four-decimals accuracy, the approximation evidently gives the same result as the more accurate formula.

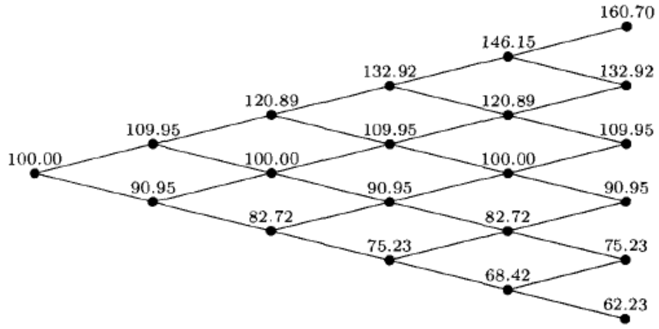
Cox-Ross-Rubinstein American Binomial Tree

Consider an American stock put option with six months to expiration. The stock price is 100, the strike price is 95, the risk-free interest rate is 8%, and the volatility is 30%. The option is priced in a binomial tree with five time steps. $S = 100$, $X = 95$, $T = 0.5$, $r = b = 0.08$, $\sigma = 0.3$, and $n = 5$.

$$\Delta t = \frac{0.5}{5} = 0.1$$

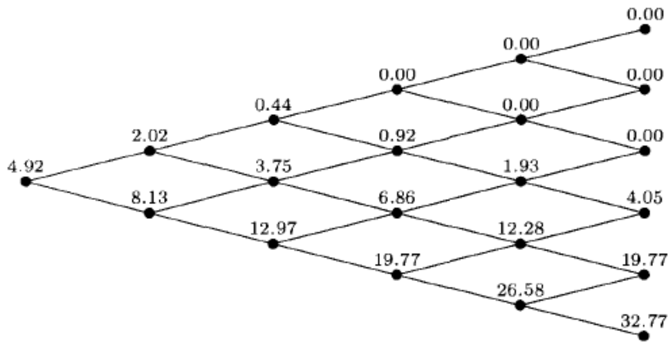
$$u = e^{0.3\sqrt{0.1}} = 1.0995 \qquad d = e^{-0.3\sqrt{0.1}} = 0.9095$$

$$p = \frac{e^{0.08\times 0.1} - 0.9095}{1.0995 - 0.9095} = 0.5186$$



First, we start at the end of the tree to see if it is optimal to exercise the option $\max[X - S, 0]$. For example, in the end node with asset price 62.23, it is naturally optimal to exercise the put option: $\max[95 - 62.23, 0] = 32.77$, while at, for example, the end node with asset price 100.95, it is not optimal to exercise the option $\max[95 - 100.95, 0] = 0$. After checking for optimal exercise at each end node, we can now easily find the value of the American put option by standard backward induction (rolling back through the tree), where we check at each node if it is optimal with early exercise:

$$P_{j,i} = \max\{X - Su^i d^{j-i}, e^{-r\Delta t}[pP_{j+1,i+1} + (1-p)P_{j+1,i}]\}$$



The value of the American put option is therefore approximately 4.92.