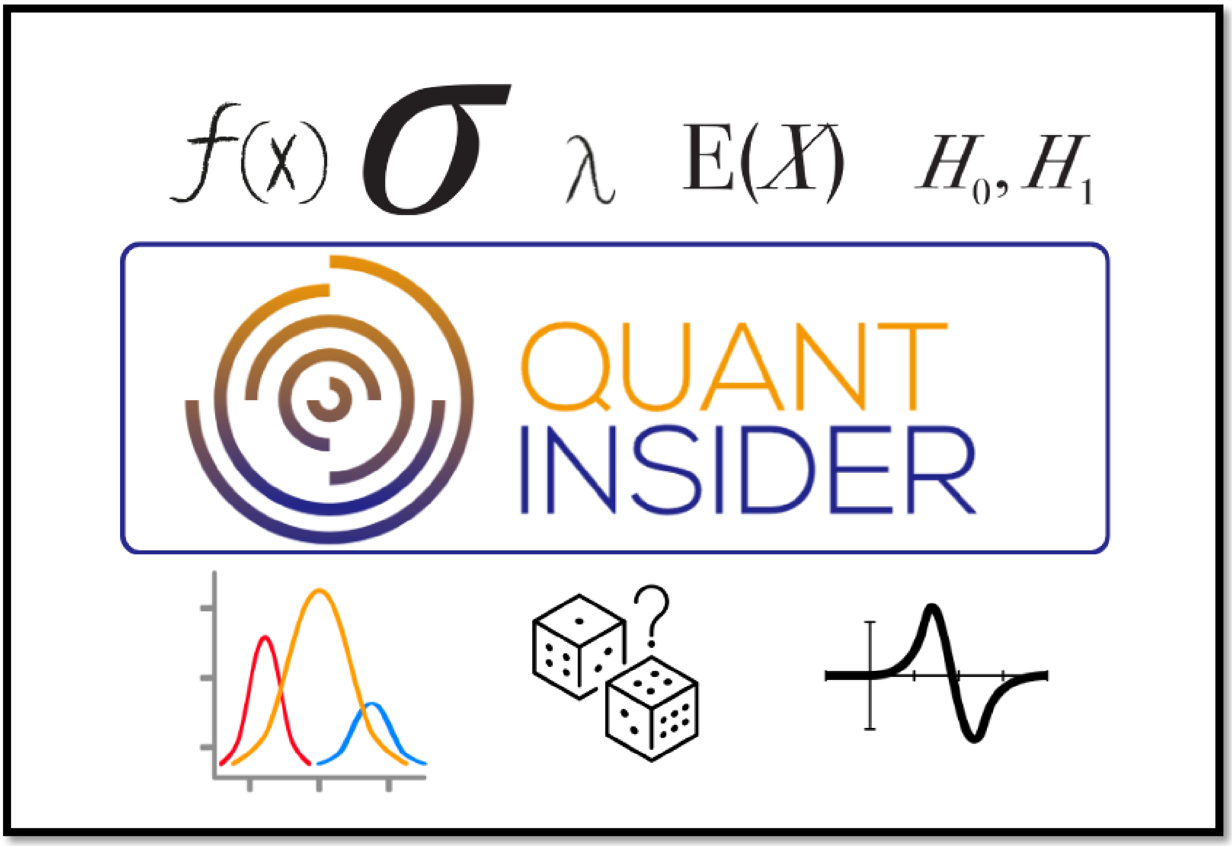


DICE PROBLEMS

Markov Chain Facts

A *Markov chain* is a mathematical model for describing a process that moves in a sequence of steps through a set of states. A finite Markov chain has a finite number of states, $\{s_1, s_2, \dots, s_n\}$. When the process is in state s_i, there is a probability p_{ij} that the process will next be in state s_j. The matrix $P = (p_{ij})$ is called the *transition matrix* for the Markov chain. Note that the rows of the matrix sum to 1.

The ij-th entry of P^k (i.e. the k-th power of the matrix P) gives the probability of the process moving from state i to state j in exactly k steps.

An *absorbing state* is one which the process can never leave once it is entered. An *absorbing chain* is a chain which has at least one absorbing state, and starting in any state of the chain, it is possible to move to an absorbing state. In an absorbing chain, the process will eventually end up in an absorbing state.

Let P be the transition matrix of an absorbing chain. By renumbering the states, we can always rearrange P into *canonical form*:

$$P = \left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)$$

where J is an identity matrix (with 1's on the diagonal and 0's elsewhere) and O is a matrix of all zeros. Q and R are non-negative matrices that arise from the transition probabilities between non-absorbing states.

The series $N = I + Q + Q^2 + Q^3 + \dots$ converges, and $N = (I - Q)^{-1}$. The matrix N gives us important information about the chain, as the following theorem shows.

Theorem 1 *Let P be the transition matrix for an absorbing chain in canonical form. Let $N = (I - Q)^{-1}$. Then:*

- *The ij-th entry of N is the expected number of times that the chain will be in state j after starting in state i.*
- *The sum of the i-th row of N gives the mean number of steps until absorbtion when the chain is started in state i.*
- *The ij-th entry of the matrix $B = NR$ is the probability that, after starting in non-absorbing state i, the process will end up in absorbing state j.*

An *ergodic* chain is one in which it is possible to move from any state to any other state (though not necessarily in a single step).

A *regular* chain is one for which some power of its transition matrix has no zero entries. A regular chain is therefore ergodic, though not all ergodic chains are regular.

Theorem 2 *Suppose P is the transition matrix of an ergodic chain. Then there exists a matrix A such that*

$$\lim_{k \to \infty} \frac{P + P^2 + P^3 + \dots + P^k}{k} = A$$

For regular chains,

$$\lim_{k \to \infty} P^k = A.$$

The matrix A has each row the same vector $\mathbf{a} = (a_1, a_2, \ldots, a_n)$. One way to interpret this is to say that the long-term probability of finding the process in state i does not depend on the initial state of the process.

The components $a_1, a_2, \ldots, a_n$ are all positive. The vector $\mathbf{a}$ is the unique vector such that

$$a_1 + a_2 + \cdots + a_n = 1$$

and

$$\mathbf{a}P = \mathbf{a}$$

For this reason, $\mathbf{a}$ is sometimes called the *fixed point probability vector.*

The following theorem is sometimes called the *Mean First Passage Theorem.*

Theorem 3 *Suppose we have a regular Markov chain, with transition matrix P. Let $E = (e_{ij})$ be a matrix where, for $i \neq j$, e_{ij} is the expected number of steps before the process enters state j for the first time after starting in state i, and e_{ii} is the expected number of steps before the chain re-enters state i. Then*

$$E = (I - Z + JZ')D$$

where $Z = (I - P - A)^{-1}$, $A = \lim_{k\to\infty} P^k$, Z' is the diagonal matrix whose diagonal entries are the same as Z, J is the matrix of all 1's, and D is a diagonal matrix with $D_{ii} = 1/A_{ii}$.

On average, how many times must a pair of 6-sided dice be rolled until all sides appear at least once?

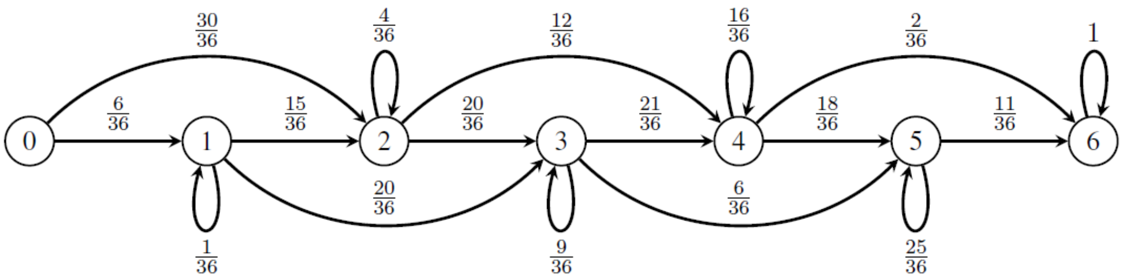
We can solve this by treating the rolling of the dice as a *Markov process*. This means that we view our game as being always in one of a number of *states*, with a fixed probability of moving from one state to each other state in one roll of the dice.

We can define our states by the number of sides we have seen appear so far. Thus, we starts in State 0, and we wish to end up in State 6, reaching some, or all, of States 1, 2, 3, 4 and 5 along the way.

On the very first roll, we will move from State 0 to either State 1 or State 2. We move to State 1 with probability $\frac{6}{36}$, since this happens exactly if we roll "doubles". Otherwise, we move to State 2, so we move to State 2 from State 0 with probability $\frac{30}{36}$.

Thus, our question can be stated thus: starting in State 0, what is the expected number of rolls until we reach State 6?

We determine the *transition probabilities*, the probability of transitioning from one state to another in one roll. We can create a diagram like this that shows the probability of moving from one state to each other state in one roll:



To solve the problem, we create a *transition matrix* for this process as follows. We let row 1 represent State 0, row 2 represent state 1, etc. Then the i,j-th entry in the matrix is the probability of transition from the row i state to the row j state in one roll (that is, from state $i-1$ to state $j-1$).

For this process, our transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{5}{12} & \frac{5}{9} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{9} & \frac{5}{9} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{1}{2} & \frac{1}{18} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{36} & \frac{11}{36} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix Q as described in Appendix D is then

$$Q = \begin{pmatrix} 0 & \frac{1}{6} & \frac{5}{6} & 0 & 0 & 0 \\ 0 & \frac{1}{36} & \frac{5}{12} & \frac{5}{9} & 0 & 0 \\ 0 & 0 & \frac{1}{9} & \frac{5}{9} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & \frac{4}{9} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{25}{36} \end{pmatrix}$$

Summing the first row we find the expected number of rolls until all six sides have appeared

$$= 1 + \frac{6}{35} + \frac{57}{56} + \frac{37}{42} + \frac{43}{28} + \frac{461}{154} = \frac{70219}{9240} = 7.59945887445....$$

By looking at the last entry of the first row of powers of the matrix P, we can find the probability of reaching state 6 in a given number of rolls:

rolls	probability of reaching this state in exactly this number of rolls	probability of reaching this state on or before this number of rolls
1	0	0
2	0	0
3	$5/324 \approx 0.015432099$	$5/324 \approx 0.015432099$
4	$575/5832 \approx 0.098593964$	$665/5832 \approx 0.11402606$
5	$22085/139968 \approx 0.15778607$	$38045/139968 \approx 0.27181213$
6	$313675/1889568 \approx 0.16600355$	$1654565/3779136 \approx 0.43781568$
7	$78924505/544195584 \approx 0.14502967$	$317181865/544195584 \approx 0.58284535$
8	$376014275/3265173504 \approx 0.11515905$	$2279105465/3265173504 \approx 0.69800440$
9	$61149474755/705277476864 \approx 0.086702719$	$553436255195/705277476864 \approx 0.78470712$
10	$401672322475/6347497291776 \approx 0.063280424$	$2691299309615/3173748645888 \approx 0.84798754$
11	0.045328994	0.89331653
12	0.032098630	0.92541516
13	0.022567579	0.94798274
14	0.015795289	0.96377803
15	0.011023854	0.97480189
16	0.0076798753	0.98248176
17	0.0053441053	0.98782587
18	0.0037160115	0.99154188
19	0.0025827093	0.99412459
20	0.0017945018	0.99591909
21	0.0012466057	0.99716570
22	0.00086588683	0.99803158
23	0.00060139404	0.99863298
24	0.00041767196	0.99905065

So we see that there is a less than one in a thousand chance that more than 24 rolls would be needed, for instance.

Suppose we roll n dice. What is the expected number of distinct faces that appear?

Let $\mathscr{E}$ be the sought expectation.

I will give three distinct solutions.

Let X be the number of distinct faces appearing in n rolls of a die. Using the **inclusion-exclusion principle**, we have the following probabilities:

$$\begin{aligned}
P(X=1) &= \binom{6}{1}\left(\frac{1}{6}\right)^n \\
P(X=2) &= \binom{6}{2}\left(\left(\frac{2}{6}\right)^n - \binom{2}{1}\left(\frac{1}{6}\right)^n\right) \\
P(X=3) &= \binom{6}{3}\left(\left(\frac{3}{6}\right)^n - \binom{3}{2}\left(\frac{2}{6}\right)^n + \binom{3}{1}\left(\frac{1}{6}\right)^n\right) \\
P(X=4) &= \binom{6}{4}\left(\left(\frac{4}{6}\right)^n - \binom{4}{3}\left(\frac{3}{6}\right)^n + \binom{4}{2}\left(\frac{2}{6}\right)^n - \binom{4}{1}\left(\frac{1}{6}\right)^n\right) \\
P(X=5) &= \binom{6}{5}\left(\left(\frac{5}{6}\right)^n - \binom{5}{4}\left(\frac{4}{6}\right)^n + \binom{5}{3}\left(\frac{3}{6}\right)^n - \binom{5}{2}\left(\frac{2}{6}\right)^n + \binom{5}{1}\left(\frac{1}{6}\right)^n\right) \\
P(X=6) &= \binom{6}{6}\left(\left(\frac{6}{6}\right)^n - \binom{6}{5}\left(\frac{5}{6}\right)^n + \binom{6}{4}\left(\frac{4}{6}\right)^n - \binom{6}{3}\left(\frac{3}{6}\right)^n + \binom{6}{2}\left(\frac{2}{6}\right)^n - \binom{6}{1}\left(\frac{1}{6}\right)^n\right)
\end{aligned}$$

These expressions determine the distribution of the number of distinct faces in n rolls.

To find the expectation, we want

$$\mathscr{E} = \sum_{i=1}^{6} iP(X = i)$$

and, after some chewing, this simplifies to

$$\mathscr{E} = 6 - 6\left(\frac{5}{6}\right)^n.$$

Here's a different approach.

The probability that the j-th roll will yield a face distinct from all previous faces rolled is

$$\frac{6 \cdot 5^{j-1}}{6^j} = \left(\frac{5}{6}\right)^{j-1}$$

since, thinking in reverse, there are 6 faces the j-th roll could be, and then 5^{j-1} ways to roll $j-1$ rolls not including that face, out of a total 6^j ways to roll j dice.

As a result, the expected contribution from the j-th roll to the total number of distinct faces is just the probability that the j-th roll is distinct: the roll contributes 1 with that probability, and 0 otherwise.

Using the additivity of expectation, we thus have

$$\mathscr{E} = \sum_{j=1}^{n} \left(\frac{5}{6}\right)^{j-1} = \frac{6}{5}\left(\sum_{j=0}^{n} \left(\frac{5}{6}\right)^{j} - 1\right) = \frac{6}{5}\left(\frac{1-\left(\frac{5}{6}\right)^{n+1}}{1-\frac{5}{6}} - 1\right) = 6 - 6\left(\frac{5}{6}\right)^n.$$

For a third solution, let X_i be a random variable defined by

$$X_i = \begin{cases} 1 & \text{if the face } i \text{ appears in } n \text{ rolls of a die,} \\ 0 & \text{otherwise.} \end{cases}$$

Let X be the number of distinct faces appearing in n rolls of a die. Then

$$X = X_1 + X_2 + X_3 + \cdots + X_6$$

and so the expected value of X is

$$E(X) = E(X_1) + E(X_2) + \cdots + E(X_6) = 6E(X_1)$$

by symmetry. Now, the probability that a 1 has appeared in n rolls is

$$P(X_1 = 1) = 1 - \left(\frac{5}{6}\right)^n$$

and so

$$E(X_1) = 1 \cdot P(X_1 = 1) = 1 - \left(\frac{5}{6}\right)^n$$

and thus the expected number of distinct faces appearing in n rolls of a die is

$$\mathscr{E} = 6E(X_1) = 6\left(1 - \left(\frac{5}{6}\right)^n\right).$$

Here's a short table of values of $\mathscr{E}$.

n	$\mathscr{E}$
1	1
2	$1.8\overline{3}$
3	$2.52\overline{7}$
4	$3.106\overline{481}$
5	3.588734...
6	3.990612...
7	4.325510...
8	4.604591...
9	4.837159...
10	5.030966...
14	5.532680...
23	5.909430...
27	5.956322...
36	5.991535...
48	5.999050...

Suppose we roll n dice and keep the highest one. What is the distribution of values?

Let's find the probability that the highest number rolled is k. Among the n dice rolled, they must all show k or less. The probability of this occurring is

$$\frac{k^n}{6^n}.$$

However, if $k > 1$, some of these rolls do not actually have any k's. That is, they are made up of only the numbers 1 through $k-1$. The probability of this occurring, for any $k \in \{1, \ldots, n\}$, is

$$\frac{(k-1)^n}{6^n}$$

so the probability that the highest number rolled is k is

$$\frac{k^n - (k-1)^n}{6^n}.$$

So, for instance, the probability that, if 7 dice are rolled, the highest number to turn up will be 3 is

$$\frac{3^7 - 2^7}{6^7} = \frac{2059}{6^7} \approx 0.007355.$$

How many dice must be rolled to have at least a 95% chance of rolling a six? 99%? 99.9%?

Suppose we roll n dice. The probability that none of them turn up six is

$$\left(\frac{5}{6}\right)^n$$

and so the probability that at least one is a six is

$$1-\left(\frac{5}{6}\right)^n.$$

To have a 95% chance of rolling a six, we need

$$1-\left(\frac{5}{6}\right)^n \geq 0.95$$

which yields

$$n \geq \frac{\log 0.05}{\log(5/6)} = 16.43\ldots > 16.$$

Hence, $n \geq 17$ will give at least a 95% chance of rolling at least one six. Since $\log(0.01)/\log(5/6) = 25.2585\ldots$, 26 dice are needed to have a 99% chance of rolling at least one six. Similarly, since $\log(0.001)/\log(5/6) = 37.8877\ldots$, 38 dice are needed for a 99.9% chance.

Show that the probability of rolling a sum of 9 *with a pair of* 5*-sided dice is the same as rolling a sum of* 9 *with a pair of* 10*-sided dice. Are there other examples of this phenomenon? Can we prove there are infinitely many such?*

Here, by an m-sided die, we mean a die with sides $1, 2, \ldots, m$ all with equal probability of being thrown.

Since

$$\left(\frac{1}{5}\left(x+x^2+x^3+x^4+x^5\right)\right)^2 = \frac{1}{25}x^{10}+\frac{2}{25}x^9+\frac{3}{25}x^8+\frac{4}{25}x^7+\frac{1}{5}x^6+\frac{4}{25}x^5+\frac{3}{25}x^4+\frac{2}{25}x^3+\frac{1}{25}x^2$$

and

$$\left(\frac{1}{10}\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}\right)\right)^2$$

$$\frac{1}{100}x^{20}+\frac{1}{50}x^{19}+\cdots+\frac{9}{100}x^{10}+\frac{2}{25}x^9+\frac{7}{100}x^8+\cdots++\frac{1}{100}x^2$$

we may conclude that the probability of rolling 9 with a pair of 5-sided dice is the same as with a pair of 10-sided dice.

There are lots of examples. Here is a short table of some:

sides 1	sides 2	sum
5	10	9
5	15	10
10	20	17
10	30	19
15	30	25
13	65	26
15	45	28
17	68	33

Are there infinitely many such examples? We have the following theorem.

Theorem 1 *Let m be a positive integer. Then $m-1$ is divisible by 8 or the square of an odd prime if and only if there exist positive integers s_1 and s_2, $s_1 < s_2$, such that the probability of rolling a sum of m with a pair of s_1-sided dice is the same as with a pair of s_2-sided dice.*

Proof: Let m be a positive integer. Suppose there exist s_1 and s_2 as described in the theorem. From the nature of the probability distributions of sums of a pair of dice, we can conclude that

$$\frac{m-1}{s^2} = \frac{2s_1 - m + 1}{s_1^2}$$

or equivalently,

$$m = 1 + \frac{2s_1 s_2^2}{s_1^2 + s_2^2}.$$

Let $r = \gcd(s_1, s_2)$, $\hat{s}_1 = \frac{s_1}{r}$, and $\hat{s}_2 = \frac{s_2}{r}$. Then

$$m = 1 + \frac{2r\hat{s}_1\hat{s}_2^2}{\hat{s}_1^2 + \hat{s}_2^2}.$$

Let $R = m - 1$. Then we have

$$\hat{s}_1^2 R + \hat{s}_2^2 R = 2r\hat{s}_1\hat{s}_2^2.$$

Note that, since $s_1 < s_2$, we cannot have $\hat{s}_2 = 1$ and so there exists a prime p that divides $\hat{s}_2$. Hence, p^2 divides R. If $p = 2$, we conclude that 8 divides R. Hence $m-1$ is either divisible by 8 or by the square of an odd prime.

Now, suppose m is a positive integer. Let $R = m - 1$. Suppose R is divisible by the square of an odd prime or by 8. If R is divisible by an odd prime, let p be that prime; else, let $p = 2$.

Then let $\hat{s}_2 = p$ and $\hat{s}_1 = 1$. Let

$$r = \frac{1+p^2}{2}\frac{R}{p^2}.$$

Then

$$R = \frac{2r\hat{s}_1\hat{s}_2^2}{\hat{s}_1^2 + \hat{s}_2^2}.$$

Let

$$s_1 = \frac{(1+p^2)R}{2p^2} \text{ and } s_2 = \frac{(1+p^2)R}{2p}.$$

Then

$$m = 1 + \frac{2s_1 s_2^2}{s_1^2 + s_2^2}$$

and so the probability of rolling a sum of m with a pair of s_1-sided dice is the same as with a pair of s_2-sided dice. ■

Thus, the infinite sequence of such sums begins $9, 10, 17, 19, 25, 26, 28, 33, 37, 41, 46, 49, 50, \ldots$.

Questions: Is there a nice way to characterize the numbers of sides for which there exist another number of sides for dice yielding equal sum probabilities? The sequence begins

$$5, 10, 13, 15, 17, 20, 25, 26, 30, 35, 37, 39, 40, 41, \ldots.$$

Suppose we roll n dice and sum the highest 3. What is the probability that the sum is 18?

In order for the sum to be 18, there must be at least three 6's among the n dice. So, we could calculate probability that there are 3,4,5,. . . ,n 6's among the n dice. The sum of these probabilities would be the probability of rolling 18. Since n could be much greater than 3, an easier way to solve this problem is to calculate the probability that the sum is *not* 18, and then subtract this probability from 1. To get a sum that is not 18, there must be 0, 1 or 2 6's among the n dice. We calculate the probability of each occurrence:

zero 6's: the probability is $\dfrac{5^n}{6^n}$

one 6: the probability is $\dfrac{n5^{n-1}}{6^n}$

two 6's: the probability is $\dfrac{\binom{n}{2}5^{n-2}}{6^n}$

Hence, the probability of rolling a sum of 18 is

$$1-\left(\frac{5^n}{6^n}+\frac{n5^{n-1}}{6^n}+\frac{\binom{n}{2}5^{n-2}}{6^n}\right)=1-\left(\frac{5}{6}\right)^n\left(1+\frac{9}{50}n+\frac{1}{50}n^2\right)=p(n)$$

say. Then, for example, $p(1)=p(2)=0$, $p(3)=1/216$, $p(4)=7/432$, and $p(5)=23/648$.

A die is rolled once; call the result N. Then N dice are rolled once and summed. What is the distribution of the sum? What is the expected value of the sum? What is the most likely value?

What the heck, take it one more step: roll a die; call the result N. Roll N dice once and sum them; call the result M. Roll M dice once and sum. What's the distribution of the sum, expected value, most likely value?

Since each of the possible values $\{1, 2, 3, 4, 5, 6\}$ of N are equally likely, we can calculate the distribution by summing the individual distributions of the sum of 1, 2, 3, 4, 5, and 6 dice, each weighted by $\frac{1}{6}$. We can do this using polynomial generating functions. Let

$$p=\frac{1}{6}(x+x^2+x^3+x^4+x^5+x^6).$$

Then the distribution of the sum is given by the coefficients of the polynomial

$$D = \sum_{i=1}^{6} \frac{1}{6} p^i$$

$$= \frac{1}{279936}x^{36} + \frac{1}{46656}x^{35} + \frac{7}{93312}x^{34} + \frac{7}{34992}x^{33} + \frac{7}{15552}x^{32} + \frac{7}{7776}x^{31} + \frac{77}{46656}x^{30} + \frac{131}{46656}x^{29} + \frac{139}{31104}x^{28} + \frac{469}{69984}x^{27} + \frac{889}{93312}x^{26} + \frac{301}{23328}x^{25} + \frac{4697}{279936}x^{24} + \frac{245}{11664}x^{23} + \frac{263}{10368}x^{22} + \frac{691}{23328}x^{21} + \frac{1043}{31104}x^{20} + \frac{287}{7776}x^{19} + \frac{11207}{279936}x^{18} + \frac{497}{11664}x^{17} + \frac{4151}{93312}x^{16} + \frac{3193}{69984}x^{15} + \frac{1433}{31104}x^{14} + \frac{119}{2592}x^{13} + \frac{749}{15552}x^{12} + \frac{2275}{46656}x^{11} + \frac{749}{15552}x^{10} + \frac{3269}{69984}x^{9} + \frac{4169}{93312}x^{8} + \frac{493}{11664}x^{7} + \frac{16807}{279936}x^{6} + \frac{2401}{46656}x^{5} + \frac{343}{7776}x^{4} + \frac{49}{1296}x^{3} + \frac{7}{216}x^{2} + \frac{1}{36}x.$$

To get the expected value E, we must calculate

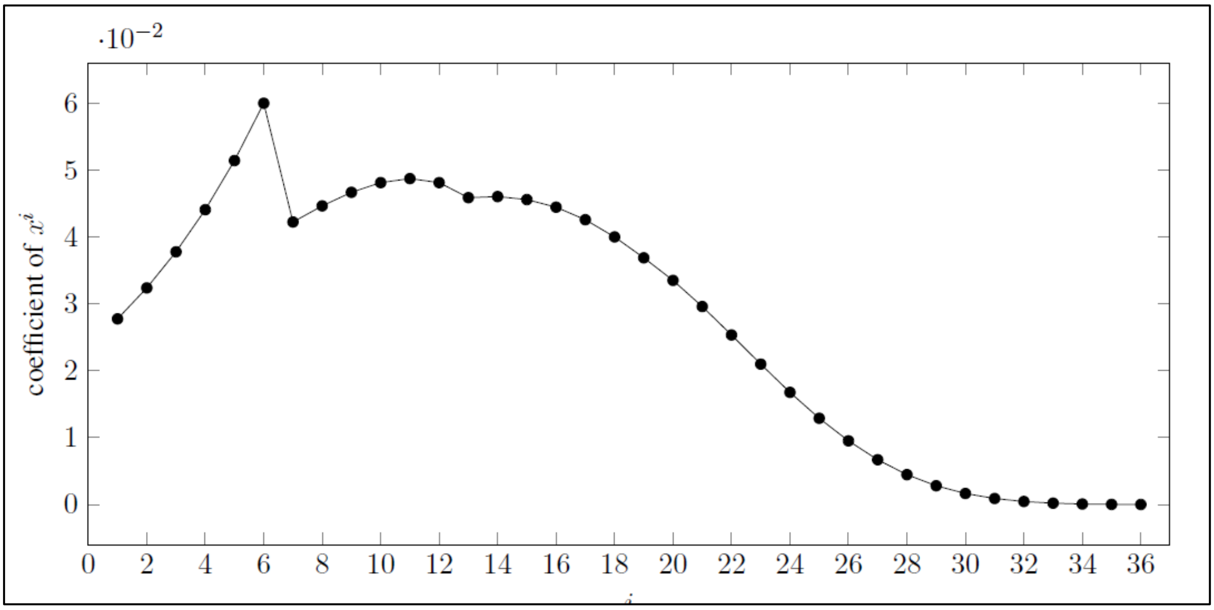
$$E = \sum_{i=1}^{36} i d_i$$

where $D = \sum_{i=1}^{36} d_i x^i$. This works out to $E = \frac{49}{4} = \left(\frac{7}{2}\right)^2 = 12.25$.

More simply, one can calculate the expected value of the sum as follows, using the fact that the expected value of a single roll is 3.5:

$$E = \frac{1}{6}(3.5 + 2 \times 3.5 + 3 \times 3.5 + \cdots + 6 \times 3.5) = 12.25.$$

Since $\sum_{i=1}^{11} d_i = \frac{1255}{2592} < \frac{1}{2}$, and $\sum_{i=1}^{12} d_i = \frac{8279}{15552} > \frac{1}{2}$, we can say that the median value is between 11 and 12.



You can see from the plot of the coefficients of D that 6 is the most likely value. It is perhaps a bit surprising that there are three "local maxima" in the plot, at $i = 6, 11$, and 14.

Okay, now lets do one more step.

After rolling the dice, getting a sum of N, and then rolling N dice, the sum distribution is

$$D_1 = \sum_{i=1}^{6} \frac{1}{6} p^i$$

as above. The coefficient of x^i in D_1 then gives us the probability that the sum of i. Hence if we call the sum M and then roll M dice once, the sum distribution is given by

$$D_2 = \sum_{i=1}^{36} D_1(i) p^i$$

where $D_1(i)$ is the coefficient on x^i in D_1.

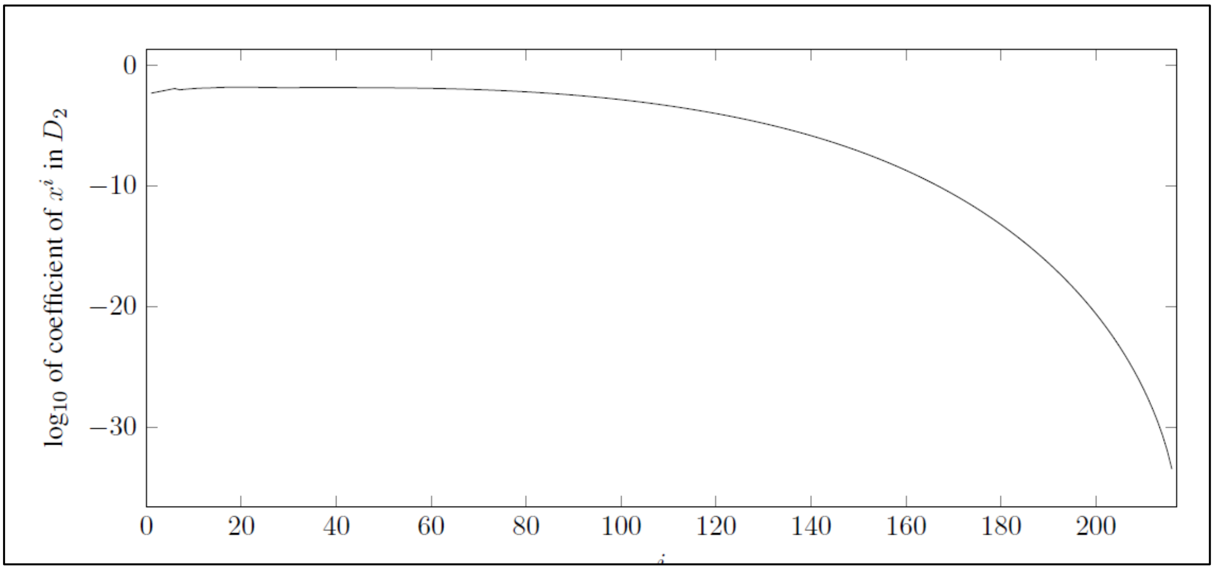
Now, D_2 is a degree 216 polynomial with massive rational coefficients, so there is little point in printing it here. Let $D_2(i)$ be the coefficient on x^i in D_2.

We can find the expected value of the sum as

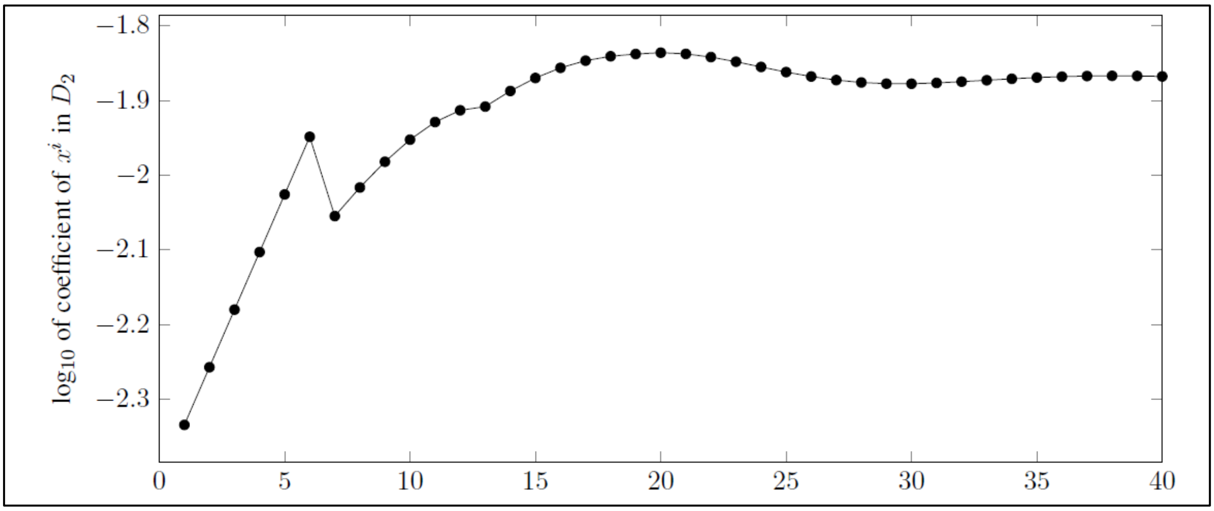
$$\sum_{i=1}^{216} i D_2(i) = \frac{343}{8} = \left(\frac{7}{2}\right)^3 = 42.875.$$

Since $\sum_{i=1}^{40} D_2(i) < \frac{1}{2}$, and $\sum_{i=1}^{41} D_2(i) > \frac{1}{2}$, we can say that the median sum is between 40 and 41.

Here's a plot of the distribution:



Here's a plot showing just the coefficients of x^i for small values of i. There are local maxima at $i = 6$, $i = 20$ (the absolute max), and $i = 38$, and a local minimum at $i = 7$ and $i = 30$.



Show that the probability of rolling doubles with a non-fair ("fixed") die is greater than with a fair die.

For a fair, n-sided die, the probability of rolling doubles with it is $n \times \frac{1}{n^2} = \frac{1}{n}$. Suppose we have a "fixed" n-sided die, with probabilities $p_1, ..., p_n$ of rolling sides 1 through n respectively. The probability of rolling doubles with this die is

$$p_1^2 + \cdots + p_n^2.$$

We want to show that this is greater than $\frac{1}{n}$. A nice trick is to let

$$\epsilon_i = p_i - \frac{1}{n} \text{ for } i = 1, ..., n.$$

Then

$$p_1^2 + \cdots + p_n^2 = (\epsilon_1 + \frac{1}{n})^2 + \cdots + (\epsilon_n + \frac{1}{n})^2 = \epsilon_1^2 + \cdots + \epsilon_n^2 + \frac{2}{n}(\epsilon_1 + \cdots + \epsilon_n) + \frac{1}{n}.$$

Now, since $p_1 + \cdots + p_n = 1$, we can conclude that $\epsilon_1 + \cdots + \epsilon_n = 0$. Hence,

$$p_1^2 + \cdots + p_n^2 = \epsilon_1^2 + \cdots + \epsilon_n^2 + \frac{1}{n} > \frac{1}{n}$$

precisely when not all the ϵ_i's are zero, i.e. when the die is "fixed".

Find a pair of 6-sided dice, labelled with positive integers differently from the standard dice, so that the sum probabilities are the same as for a pair of standard dice.

Number one die with sides 1,2,2,3,3,4 and one with 1,3,4,5,6,8. Rolling these two dice gives the same sum probabilities as two normal six-sided dice.

A natural question is: how can we find such dice? One way is to consider the polynomial

$$(x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

This factors as

$$x^2(1 + x)^2(1 + x + x^2)^2(1 - x + x^2)^2.$$

We can group this factorization as

$$\begin{aligned}&\left(x(1 + x)(1 + x + x^2)\right)\left(x(1 + x)(1 + x + x^2)(1 - x + x^2)^2\right)\\ &= (x + 2x^2 + 2x^3 + x^4)(x + x^3 + x^4 + x^5 + x^6 + x^8).\end{aligned}$$

This yields the "weird" dice (1,2,2,3,3,4) and (1,3,4,5,6,8). These dice are known as *Sicherman* dice, named for George Sicherman who communicated with Martin Gardner about them in the 1970s.

Is it possible to have two non-fair n-sided dice, with sides numbered 1 through n, with the property that their sum probabilities are the same as for two fair n-sided dice?

Another way of asking the question is: suppose you are given two n-sided dice that exhibit the property that when rolled, the resulting sum, as a random variable, has the same probability distribution as for two fair n-sided dice; can you then conclude that the two given dice are fair? This question was asked by Lewis Robertson, Rae Michael Shortt and Stephen Landry in [2]. Their answer is surprising: you *can* sometimes, depending on the value of n. Specifically, if n is 1,2,3,4,5,6,7,8,9,11 or 13, then two n-sided dice whose sum "acts fair" are, in fact, fair. If n is any other value, then there exist pairs of n-sided dice which are not fair, yet have "fair" sums.

The smallest example, with $n = 10$, gives dice with the approximate probabilities (see [Rob 2] for the exact values)

$$(0.07236, 0.14472, 0.1, 0.055279, 0.127639, 0.127639, 0.055279, 0.1, 0.14472, 0.07236)$$

and

$$(0.13847, 0, 0.2241, 0, 0.13847, 0.13847, 0, 0.2241, 0, 0.13847).$$

It's clear that these dice are not fair, yet the sum probabilities for them are the same as for two fair 10-sided dice.

Find every six-sided die with sides numbered from the set {1,2,3,4,5,6} such that rolling the die twice and summing the values yields all values between 2 and 12 (inclusive). For instance, the die numbered 1,2,4,5,6,6 is one such die. Consider the sum probabilities of these dice. Do any of them give sum probabilities that are "more uniform" than the sum probabilities for a standard die? What if we renumber two dice differently - can we get a uniform (or more uniform than standard) sum probability?

The numbers 1, 2, 5 and 6 must always be among the numbers on the die, else sums of 2, 3, 11 and 12 would not be possible. In order to get a sum of 5, either 3 or 4 must be on the die also. The last place on the die can be any value in {1,2,3,4,5,6}. Hence there are 11 dice with the required property. Listed with their corresponding error, they are:

1,2,4,5,6,6	0.0232884399551066
1,2,4,5,5,6	0.0325476992143659
1,2,4,4,5,6	0.0294612794612795
1,2,3,5,6,6	0.0232884399551066
1,2,3,5,5,6	0.026374859708193
1,2,3,4,5,6	0.0217452300785634
1,2,3,3,5,6	0.0294612794612795
1,2,2,4,5,6	0.026374859708193
1,2,2,3,5,6	0.0325476992143659
1,1,2,4,5,6	0.0232884399551066
1,1,2,3,5,6	0.0232884399551066

The error here is the sum of the square of the difference between $1/11$ and the actual probability of rolling each of the sums 2 through 12 (the probability we would have for each sum if we had a uniform distribution). That is, if p_i is the probability of rolling a sum of i with this die, then the error is

$$\sum_{i=2}^{12}\left(p_i-\frac{1}{11}\right)^2.$$

Note that the standard die gives the smallest error (i.e., the closest to uniform sum).

If we renumber two dice differently, many more cases are possible. One pair of dice are 1,3,4,5,6,6 and 1,2,2,5,6,6. These two dice give all sum values between 2 and 12, with an error (as above) of 0.018658810325477, more uniform than the standard dice. The best dice for near-uniformity are 1,2,3,4,5,6 and 1,1,1,6,6,6 which yield all the sums from 2 to 12 with near equal probability: the probability of rolling 7 is 1/6 and all other sums are 1/12. The error is 5/792, or about 0.00631.

If we roll a standard die twice and sum, the probability that the sum is prime is $\frac{15}{36}=\frac{5}{12}$. *If we renumber the faces of the die, with all faces being different, what is the largest probability of a prime sum that can be achieved?*

To get prime sums other than 2, we need to have both even and odd faces. If three faces are even and three are odd, then there will be 18 odd sums out of the 36 possible combinations. If two faces are odd and four are even, there will be 16 odd sums, and if one face is odd and five are even, then there will be 10 odd sums. So, the maximum number of prime sums out of 36 is 19 (if all odd sums are prime, and the sum 2 is achievable). This is achieved with the die $\{1,2,3,4,9,10\}$ which yields the sum set $\{2,3,4,5,6,7,8,10,11,12,13,14,18,19,20\}$ in which all odd numbers are prime. Thus, this die has a $\frac{19}{36}=0.52\overline{7}$ probability of throwing a prime sum when rolled twice, compared to $\frac{5}{12}=0.41\overline{6}$ for a standard die.

Show that you cannot have a pair of dice with more than two sides that only gives sums that are Fibonacci numbers.

Here we consider each die to have distinct integer faces (i.e., no face is repeated), but we do not need to assume that there is no face common to both dice.

Let's start with the two-sided case, and we'll see this leads easily to the greater-than-two sides case.

Suppose we have two-sided dice with sides $\{r, s\}$,$\{t, u\}$ with sums that are all Fibonacci numbers. We can subtract r from the first die's faces, and add r to the second to get the dice $\{0, s-r\}, \{t+r, u+r\}$ with the same sumset. Since the first die has a zero, and all sums are Fibonacci, we can relabel the dice as $\{0, x\}, \{F_a, F_b\}$, where F_n is the n-th Fibonacci number (e.g., $F_1 = 1, F_2 = 1, F_3 = 2$, etc.) We may assume $F_a < F_b$.

Let's write $F_c = x + F_a$ and $F_d = x + F_b$.

Then $F_b + F_c = F_a + F_d$.

Suppose $b < c$. Then $F_b + F_c \leq F_{c-1} + F_c = F_{c+1} \leq F_d < F_d + F_a$, a contradiction.

Suppose $b > c$. Then $F_b + F_c \geq F_b + F_{b-1} = F_{b+1} \leq F_d < F_d + F_a$, a contradiction.

Hence, $b = c$, and so we have, simply, $F_b = F_a + x$ and $F_d = F_b + x$.

Then $x \geq F_{b-1}$, since otherwise we'd have $F_b + x < F_b + F_{b-1} = F_{b+1} \leq F_d$.

On the other hand, $F_b = F_a + x$ implies $x < F_b$. Thus, $x = F_{b-1}$. (Note we are using the fact that $F_a > 0$ here).

Thus, if we have two-sided dice with sums that are Fibonacci, they must be "equivalent" to the dice $\{0, F_b - 1\}, \{F_{b-2}, F_b\}$ for some integer $b > 1$. By "equivalent", I mean any dice derived from these by adding an integer α to all faces of one die and subtracting α from all the faces of the other.

So we can have dice like $\{0, 3\}, \{2, 5\}$, or $\{0, 8\}, \{5, 13\}$.

Now, if we have more than two sides, then all non-zero faces of the die with the zero on it would have to be identical (in these cases, our x above standing for any non-zero face), something we are not allowing here. Hence, dice with three or more sides whose sums are all Fibonacci are impossible.

In the previous problem, we find out that the game is not fair. Are there sum targets for player A and player B that would make the game fair? What about using a different number of dice, or allowing targets to include more than one sum?

Let p be the probability of player A rolling their target sum in one roll, and r be the probability of player B rolling their target sum in one roll. In order for the game to be fair, we require

$$\frac{1}{2} = \frac{p}{p + r - pr}.$$

so $r = \dfrac{p}{1-p}$.

Let's say $p = \frac{k}{s^2}$ and $r = \frac{m}{s^2}$ where s is the number of sides of the dice. Note k and m are at most s, and at most one of them may be equal to s, since $r \neq p$. Then

$$\frac{k}{s^2 - k} = \frac{m}{s^2}$$

and so $ks^2 = ms^2 - km$. Hence, s^2 divides km. But this is impossible, since km is not zero, and $km \leq s(s-1) < s^2$.

Thus, there is no choice of target sums that would make this game fair.

So that doesn't work.

But, we can note that if $p = \frac{9}{36}$, then $\frac{p}{1-p} = \frac{12}{36}$. Then we can note that the probability of throwing a sum of 4 or 5 with two dice is $\frac{9}{36}$, while the probability of throwing a sum of $8, 9$ or 10 with two dice is $\frac{12}{36}$. Thus, the game is fair if the first player's target is a sum of 4 or 5, and the second player's target is a sum of $8, 9$ or 10.

In the same way, the game is also fair if the first player's target is a sum of $8, 9$ or 10, while the second player's target is $5, 6$ or 7.

If the players throw *three* dice, and player A's target is a sum of 4 or 8 while player B's target is a sum of 11, then the game is fair. In this situation, we'd have $p = \frac{24}{216} = \frac{1}{9}$ and $r = \frac{27}{216} = \frac{1}{8} = \frac{p}{1-p}$.

Yahtzee *There are many probability questions we may ask with regard to the game of Yahtzee. For starters, what is the probability of rolling, in a single roll,*

a) Yahtzee

b) Four of a kind (but not Yahtzee)

c) A full house

d) Three of a kind (but not Yahtzee, four of a kind or full house)

e) A long straight

f) A small straight

These questions aren't *too* tricky, so I'll just give the probabilities here:

(a) Yahtzee: $\dfrac{6}{6^5} = \dfrac{1}{1296} \approx 0.07716\%$

(b) Four of a kind (but not Yahtzee): $\dfrac{\binom{5}{4} \cdot 6 \cdot 5}{6^5} = \dfrac{25}{1296} \approx 1.929\%$

(c) A full house: $\dfrac{\binom{5}{3} \cdot 6 \cdot 5}{6^5} = \dfrac{25}{648} \approx 3.858\%$

(d) Three of a kind (but not Yahtzee, four of a kind or full house) :

$$\frac{\binom{5}{3} \cdot 6 \cdot 5 \cdot 4}{6^5} = \frac{25}{162} \approx 15.432\%$$

(e) A long straight: $\dfrac{2 \cdot 5!}{6^5} = \dfrac{5}{162} \approx 3.086\%$

(f) A small straight (but not a long straight):

$$\frac{\frac{5!}{2!} \cdot 4 + 2\left(\frac{5!}{2!} \cdot 4 + 5!\right)}{6^5} = \frac{10}{81} \approx 12.346\%$$

. **More Yahtzee** *What is the probability of getting Yahtzee, assuming that we are trying just to get Yahtzee, we make reasonable choices about which dice to re-roll, and we have three rolls? That is, if we're in the situation where all we have left to get in a game of Yahtzee is Yahtzee, so all other outcomes are irrelevant.*

This is quite a bit trickier than the previous questions on Yahtzee. The difficulty here lies in the large number of ways that one can reach Yahtzee: roll it on the first roll; roll four of a kind on the first roll and then roll the correct face on the remaining die, etc. One way to calculate the probability is to treat the game as a *Markov chain* (see Appendix D for general information on Markov chains).

We consider ourselves in one of five *states* after each of the three rolls. We will say that we are in state b if we have b common dice among the five. For example, if a roll yields 12456, we'll be in state 1; if a roll yields 11125, we'll be in state 3. Now, the goal in Yahtzee is to try to get to state 5 in three rolls (or fewer). Each roll gives us a chance to change from our initial state to a better, or equal, state. We can determine the probabilities of changing from state i to state j. Denote this probability by $P_{i,j}$. Let the 0 state refer to the initial state before rolling. Then we have the following probability matrix:

$$P = (P_{i,j}) = \begin{pmatrix} 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & \frac{120}{1296} & \frac{900}{1296} & \frac{250}{1296} & \frac{25}{1296} & \frac{1}{1296} \\ 0 & 0 & \frac{120}{216} & \frac{80}{216} & \frac{15}{216} & \frac{1}{216} \\ 0 & 0 & 0 & \frac{25}{36} & \frac{10}{36} & \frac{1}{36} \\ 0 & 0 & 0 & 0 & \frac{5}{6} & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.20}$$

The one representing $P_{5,5}$ indicates that if we reach yahtzee, state 5, before the third roll, we simply stay in that state. Now, the probability of being in state 5 after 3 rolls is given by

$$\sum P_{0,i_1} P_{i_1,i_2} P_{i_2,5} = (M^3)_{1,5}$$

where the sum is over all triples (i_1, i_2, i_3) with $0 \le i_j \le 5$. Calculating M^3 gives us the probability

$$\frac{2783176}{6^{10}} = \frac{347897}{7558272} \approx 0.04603.$$

Since $\frac{347897}{7558272} = \frac{1}{21.7256026...}$, a player will get Yahtzee about once out of every twenty two attempts.

Suppose we play a game with a die where we roll and sum our rolls. We can stop any time and take the sum as our score, but if we roll a face we've rolled before then we lose everything. What strategy will maximize our expected score?

We want to roll until the expected sum after rolling is less than our current sum. Let C be our current sum and S be the set of faces that have been rolled already. Then we should stop if

$$\frac{|S|}{6} \cdot 0 + \sum_{i \notin S} \frac{1}{6}(C + i) < C.$$

Using the fact that $\sum_{i \in S} i = C$, this inequality simplifies to

$$C(|S| + 1) > 21.$$

After our first roll, $|S| = 1$ and $C < 6$ so $C(|S| + 1) \le 12$. Hence we should roll again. After our second roll, $|S| = 2$, so we should stop if $C > 7$. After our third roll, $|S| = 3$, so we should stop if

$C > \frac{21}{4}$, that is, if $C \geq 6$. However, if we have made it to our third roll, C must be at least 6, and so we should stop at this point.

Thus: Roll twice. If the second roll is not the same as the first, and the sum is less than 7, roll again and stop; otherwise, stop.

With this strategy, the expected score is $\dfrac{223}{36} = 6.19\bar{4}$ and the game ends with a zero score with probability $\dfrac{5}{18} = 0.2\bar{7}$.

(Same as previous game, but with two dice.) Suppose we play a game with two dice where we roll and sum our rolls. We can stop any time and take the sum as our score, but if we roll a sum we've rolled before then we lose everything. What strategy will maximize our expected score?

Let C be our current score, and S be the set of sums rolled so far, so

$$C = \sum_{i \in S} i.$$

Let $p(i)$ be the probability of rolling a sum of i with a single roll of two dice.

Then, to maximize our expected score, we should stop rolling if the expected score after rolling again is less than our current score. That is,

$$\left(\sum_{i \in S} p(i)\right) \cdot 0 + \sum_{i \notin S} p(i)(C+i) < C.$$

This simplifies to

$$\sum_{i \in S} p(i)(C+i) > 7. \tag{3.21}$$

When does this occur?

After a single roll r, the stopping condition is

$$2rp(r) > 7.$$

Since the left-hand side maxes out at $r = 7$, with $2rp(r) = \frac{7}{3}$, we should always roll at least twice.

One can check as well that, after two rolls, the left-hand side of (3.21) is at most $\frac{247}{36} \approx 6.861 < 7$, so we should always roll at least three times (if we can).

After rolling three times, there are many S for which we should stop. We should stop after three rolls if any of the following are true:

- $\{6,7\}, \{7,8\}, \{7,9\}, \{7,10\}$, or $\{8,9\} \subset S$
- $2 \notin S$ and $\{6,8\}, \{6,9\}, \{6,10\}, \{7,11\}, \{7,12\}, \{8,10\}, \{8,11\} \subset S$
- current score is 28 or greater and $S \neq \{5,11,12\}$
- $S \in \{\{3,9,10\}, \{4,5,7\}, \{4,5,8\}, \{4,5,9\}, \{4,6,11\}, \{4,8,12\}, \{4,9,10\}, \{4,9,11\}, \{4,9,12\}, \{4,10,11\}, \{5,6,11\}, \{5,6,12\}, \{5,8,12\}, \{5,9,10\}, \{5,9,11\}, \{5,9,12\}, \{5,10,11\}, \{5,10,12\}\}$

After rolling four times, we should stop unless one of the following is true:

(a) $\{2, 3, 4\} \subset S$ and $7 \notin S$

(b) S equals one of $\{2, 3, 5, 6\}$, $\{2, 3, 5, 11\}$, $\{2, 3, 5, 12\}$, $\{2, 3, 10, 12\}$, $\{2, 3, 11, 12\}$, $\{2, 4, 11, 12\}$.

If we are lucky enough to roll five times, we should stop. With $|S| = 5$, $\sum_{i \in S} p(i)(C + i)$ is minimal when $S = \{2, 3, 4, 5, 12\}$, with the sum being $\frac{169}{18} \approx 9.389 > 7$.

If we apply this optimal strategy, the expected score will be $\frac{513389}{34992} \approx 14.671611$.

With this strategy, the smallest non-zero score that can be attained is 15, and the largest possible score is 39. The probability of zero is $\dfrac{217495}{629856} \approx 0.345309$, and the most likely non-zero score is 21, occuring with probability $\frac{10109}{139968} \approx 0.0722237$.

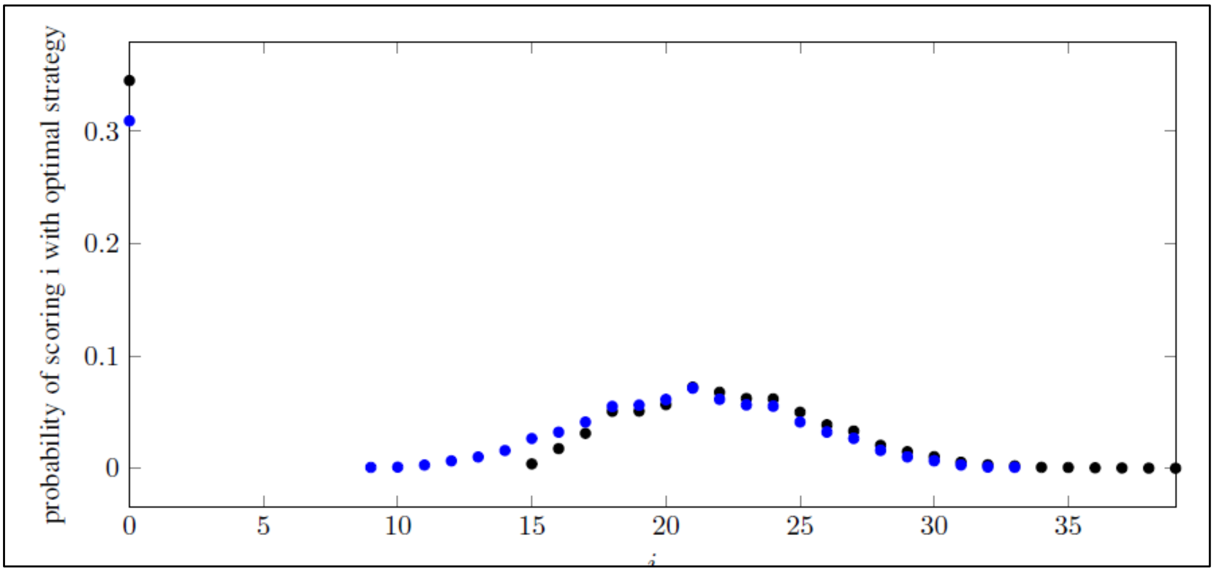
In the optimum strategy described above, the stopping condition was

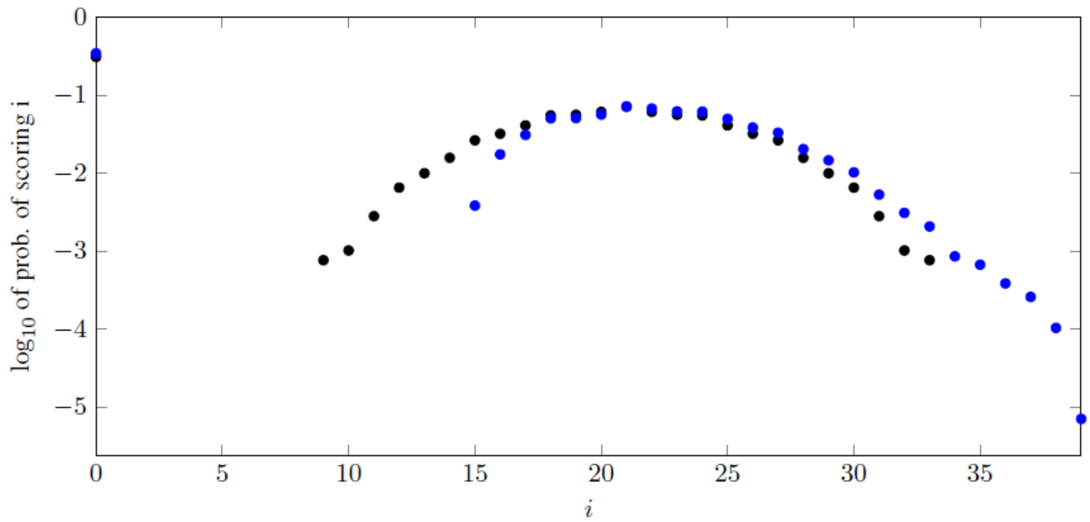
$$\sum_{i \in S} p(i)(C + i) > 7.$$

The greater than symbol can be replaced with greater-than-or-equals and the resulting strategy will yield the same expected value, but a very slightly different score distribution.

Interestingly, instead of using this complex strategy, a very good and simple strategy is to always stop after three rolls. With this simple strategy, the expected score is $\dfrac{6265}{432} \approx 14.502315$, not that much less than the optimal strategy. The range of possible non-zero scores is 9 to 33. The probability of scoring zero is $\dfrac{401}{1296} \approx 0.309414$ and the most likely non-zero score is 21, as it is with the complex strategy, and this occurs with probability $\dfrac{277}{3888} \approx 0.0712449$.

The plots below show the score distribution for the "> 7" strategy in black and the "stop after three rolls" strategy in blue; the second plot has the values on a logarithmic scale.





The simple strategy can be made quite a bit better while keeping it simple by adding a score condition. If the strategy is to stop rolling if we have rolled at least three times *and* our score is 16 or greater, experiments show the expected value is close to 14.6! If we want a strategy based purely on the score, the best strategy appears to be stopping when the score is at least 18, which yields an expected value just over 14.3.

Suppose we play a game with a die where we roll and sum our rolls. We can stop any time and take the sum as our score, but if we roll the same face twice in a row we lose everything. What strategy will maximize our expected score?

If the last face rolled is r and our current sum is S, then the expected value of our score if we roll again is

$$\frac{1}{6} \cdot 0 + \frac{5}{6}S + \frac{1}{6}\left(\sum_{i=1, i\neq r}^{6} i\right).$$

If this is less than S, we should not roll. When is it less than S? It depends on S and r. Specifically, if S is greater than the sum of all faces other than r, we should not roll. In other words, if

$$r + S > 21$$

then rolling will not, on average, increase our score, and so we should stop. With this strategy, the expected score is about 8.7, with zero scores occurring about 56% of the time. Generalizing to m-sided dice, we should stop if the current sum plus the last roll exceeds the sum of all faces of the die.

Suppose we play a game with a die where we roll and add our rolls to our total when the face that appears has not occurred before, and subtract it from our total if it has.

For example, if we rolled the sequence $1, 3, 4, 3$*, our corresponding totals would be* $1, 4, 8, 5$.

We can stop any time and take the total as our score. What strategy should we employ to maximize our expected score?

The optimal strategy need only consider what faces have already appeared.

If A is the set of distinct faces which have already appeared, then the expected change C in the total on the next roll of the die is

$$C = -\frac{1}{6} \sum_{\substack{1 \le i \le 6 \\ i \in A}} i + \frac{1}{6} \sum_{\substack{1 \le i \le 6 \\ i \notin A}} i.$$

If this is negative, we should stop rolling, since on the next roll we expect to decrease our total, and any further rolling only makes the situation worse.

Now, C is negative if the sum of the distinct faces thrown is 11 or more, and positive otherwise. Hence, to maximize the expected value of our score, we should keep rolling until the sum of distinct faces thrown is 11 or more.

For example, if we roll $1, 3, 5, 1, 6$, then we should stop, with a score of 16.

Experimentally, we can find that this strategy yields an expected score of about 8.7.

What is the probability that, if we roll two dice, the product of the faces will start with the digit '1'? What if we roll three dice, or, ten dice? What is going on?

When we roll two dice, the possible products that begin with the digit '1' are 1, 10, 12, 15, 16 and 18, and these occur with probability $\frac{1}{36}$, $\frac{1}{18}$, $\frac{1}{9}$, $\frac{1}{18}$, $\frac{1}{36}$, and $\frac{1}{18}$, respectively. Hence, the probability of the product of two dice starting with the digit '1' is $\frac{1}{3} = 0.\overline{3}$.

With three dice, the probability is $\frac{65}{216} = 0.300\overline{925}$.

Here's a table with the probabilities for various numbers of dice.

1	$\frac{1}{6}$	$0.1\overline{6}$
2	$\frac{1}{3}$	$0.\overline{3}$
3	$\frac{65}{216}$	$0.300\overline{925}$
4	$\frac{379}{1296}$	0.2924382716...
5	$\frac{2317}{7776}$	0.2979681069...
6	$\frac{193}{648}$	0.2978395061...
7	$\frac{41977}{139968}$	0.2999042638...
8	$\frac{28123}{93312}$	0.3013867455...
9	$\frac{3043945}{10077696}$	0.3020477101...
10	$\frac{18271529}{60466176}$	0.3021776836...

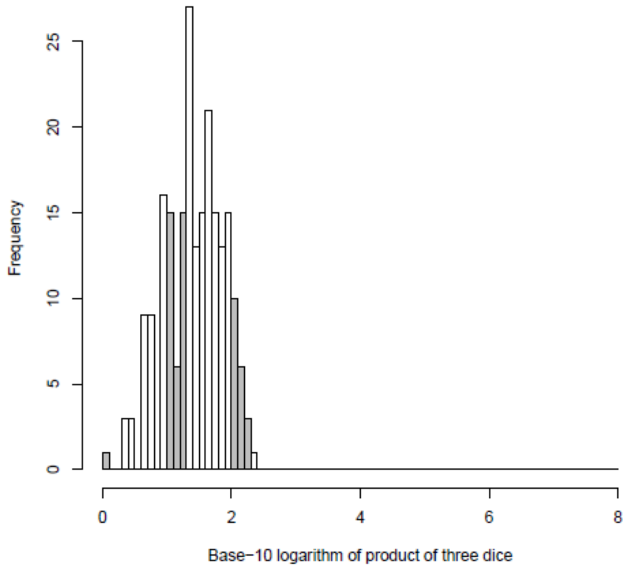
It seems that, if we roll more than one die, the probability is about 0.3. Why is this?

If a positive real number begins with a '1' digit, then the base-10 logarithm of the number will have a fractional part less than $\log_{10} 2 = 0.301029995...$

If, instead of considering the product of m rolled dice, we consider the base-10 logarithm of the product, then this can be viewed as a sum of values chosen with equal likelihood from the set $\{0, \log_{10} 2, \log_{10} 3, \ldots, \log_{10} m\}$. By the Central Limit Theorem, the distribution of these sums will tend toward a Gaussian distribution as m goes to infinity.

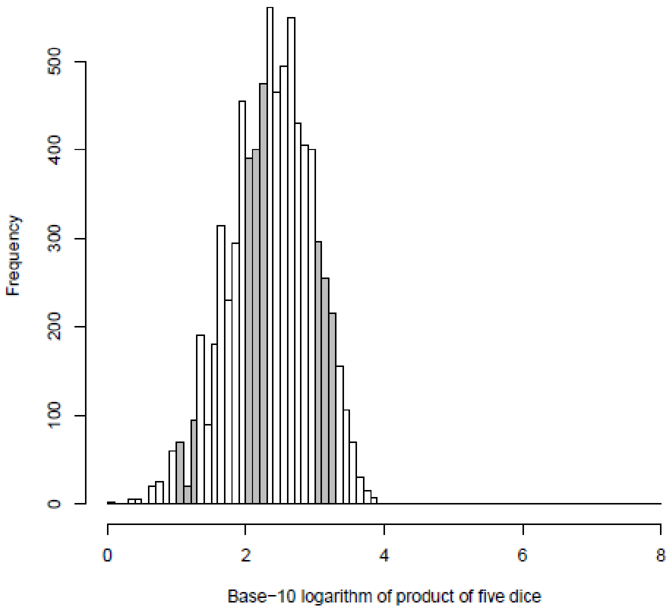
We can make some histograms to see this process.

If we consider all rolls of three dice, and take the base-10 logarithm of the product on each roll, we get the following histogram.

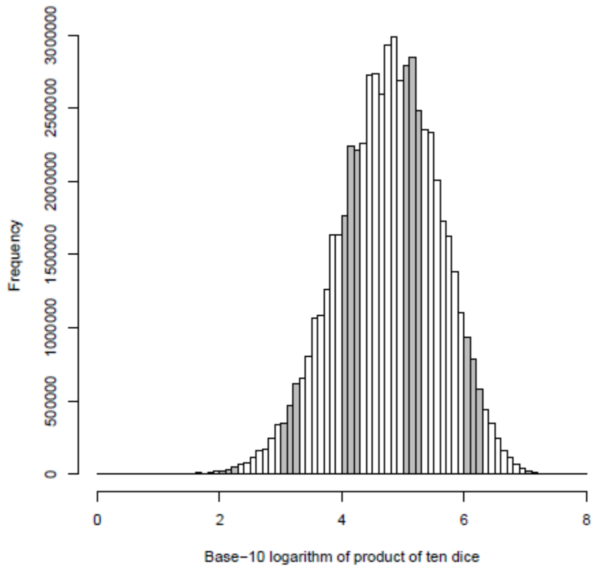


The greyed portions of the histogram represent those rolls whose products begin with a digit '1'.

Here is the same thing, using five dice.



And here it is using ten dice.



In each histogram, the greyed bit corresponds to products with a base-10 logarithm with fractional part less than $\log_{10} 2$, i.e., products that start with the digit '1'.

As the number of dice tends to infinity, the distribution becomes more and more similar to a normal distribution. Meanwhile, the variance increases, so the "spread" of the distribution covers more and more integers - the distribution of the products covers more orders of magnitude. As a result, the number of grey intervals in the histograms will grow to infinity as well. One can be convinced, then, that, as the number of dice tends to infinity, the greyed portion of the histogram tends to $\log_{10} 2 = 0.301029995...$.

(For a formal argument, it would be sufficient to show that a normal distribution is uniformly distributed modulo 1 as the variance goes to infinity.)

This is an example of what is often call *Benford's Law*, that certain distributions of numbers tend to have a probability of a leading '1' digit of around $\log_{10} 2$.