Brain Teasers Interview Guide

PUZZLE

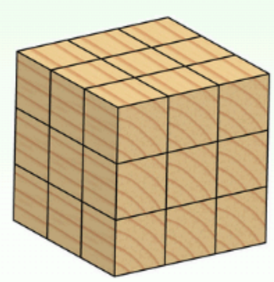
On an island live 13 purple, 15 yellow and 17 maroon chameleons. When two chameleons of different colors meet, they both change into the third color. Is there a sequence of pairwise meetings after which all chameleons have the same color?

SOLUTION

Let <p, y, m> denote a population of p purple, y yellow and m maroon chameleons. Can population <13, 15, 17> can be transformed into <45, 0, 0> or <0, 45, 0> or <0, 0, 45> through a series of pairwise meetings? Define function X(p, y, m) = (0p + 1y + 2m) mod 3. An interesting property of X is that its value does not change after any pairwise meeting because X(p, y, m) = X(p-1, y-1, m+2) = X(p-1, y+2, m-1) = X(p+2, y-1, m-1). Now X(13, 15, 17) equals 1. However, X(45, 0, 0) = X(0, 45, 0) = X(0, 0, 45) = 0. This means that there is no sequence of pairwise meetings after which all chameleons will have identical color.

PUZZLE

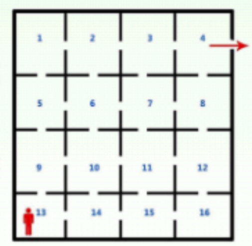
Imagine a 3x3x3 wooden cube. How many cuts do we need to break it into twenty-seven 1x1x1 cubes? A cut may go through multiple wooden pieces.



SOLUTION

The central $1 \times 1 \times 1$ cube has six faces. No cut can reveal two or more of these faces! So six cuts are necessary and sufficient.

PUZZLE



Patient 13 is cured. He wants to pass by all other patients before leaving. But he can't see the same patient twice or he will get sick again. Can he leave from the door in Room 4?

Followup: Instead of a door in room 4, if there were a helicopter in some other room, is it possible to exit the hospital by visiting all other patients but no patient twice?

SOLUTION

If the first few steps in the sequence are $13 \to 9 \to 13 \to 14$, the rest of the sequence is easy to construct :) And it's easy to see that the helicopter problem is also solvable for any square.

PUZZLE

How do we measure forty-five minutes using two identical wires, each of which takes an hour to burn. We have matchsticks with us. The wires burn non-uniformly. So, for example, the two halves of a wire might burn in 10 minute and 50 minutes respectively.

SOLUTION

Light three out of four ends. When two ends meet, light the fourth!

PUZZLE

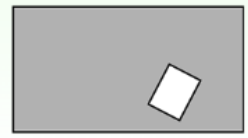
Peter and Cynthia stand at each end of a straight line segment. Peter sends 50 ants towards Cynthia, one after another. Cynthia sends 20 ants towards Peter. All ants travel along the straight line segment. Whenever two ants collide, they simply bounce back and start traveling in the opposite direction. How many ants reach Peter and how many reach Cynthia? How many ant collisions take place?

SOLUTION

Imagine that when two ants meet, they switch identities. So even after a collision, two ants are traveling in two opposite directions. It follows that 20 ants return to Peter while 50 ants reach Cynthia. To calculate the number of ant collisions, imagine that each ant carries a message. So Peter has sent 50 messages to Cynthia, one message per ant. Similarly, Cynthia has sent 20 messages to Peter, one message per ant. Further, imagine that the two ants swap messages when they collide. Then a message always makes forward progress. Each of Cynthia's messages goes through 50 ant collisions. Each of Peter's messages goes through 20 ant collisions. The total number of collisions is 50 times 20, which is 1000.

PUZZLE

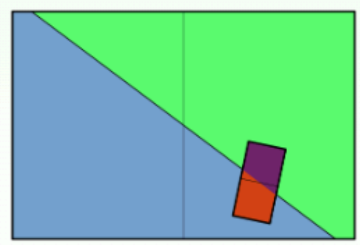
Mary baked a rectangular cake. Merlin secretly carved out a small rectangular piece, ate it and vanished! The remaining cake has to be split evenly between Mary's two kids. How could this be done with only one cut through the cake?



SOLUTION

(by Yuri Robbers): There are two solutions to this problem. The first solution is perhaps a little bit silly, and it only works if the cake is vertically symmetrical. In that case it would be possible to halve the cake horizontally. The top half and the bottom half would then be identical.

A less silly solution is to cut along the line that runs through the center of the original cake, and through the center of the piece that is missing. If the centers happen to coincide, then any line through that joint center will do. Any rectangle is halved by any line through its center. Cutting through the center of the cake therefore halves the cake, while cutting through the center of the missing piece, halves the missing piece.



PUZZLE

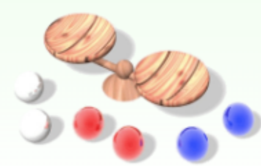
A prison has 1000 cells. Initially, all cells are marked with - signs. From days 1 thru 1000, the jailor toggles marks on some of the cells: from + to - and from - to +. On the i-th day, the signs on cells that are multiples of i get toggled. On the 1001-th day, all cells marked with + signs are opened. Which cells are these?

SOLUTION

A cell is toggled as many times as the number of divisors it has. For example, cell number 24 is toggled on days 1, 2, 3, 4, 6, 8, 12 and 24. Now, divisors come in pairs like 24 = 1*24 = 2*12 = 3*8 = 4*6. So the total number of divisors is even, except when the number is a perfect square, in which case the total number of divisors is odd. Since all cells are initially marked with - signs, only those cells that have an odd number of divisors have + signs eventually. These are cell numbers 1, 4, 9, 16, 25, 36, and so on.

PUZZLE

We have two white, two red and two blue balls. For each color, one ball is heavy and the other is light. All heavy balls weigh the same. All light balls weigh the same. How many weighings on a beam balance are necessary to identify the three heavy balls?



SOLUTION

Two weighings suffice. Weigh one red and one white ball against one blue and the other white ball. If the weights are equal, then the red and the blue differ in weight. A weighing between these two balls allows us to deduce the weights of all other balls. If the red-white combination was heavier than the blue-white combination in the first weighing, then the white ball in the red-white combination is certainly heavy and the other white is light. Now take the red from the red-white combination and the blue from the blue-white combination. Weigh these together against the remaining red and the remaining blue. The only interesting case is when this weighing is "equal". Then, the red from the red-white combination must be heavy and the blue from the blue-white combination must be light.

PUZZLE

You and your opponent shall play a game with three dice: First, your opponent chooses one of the three dice. Next, you choose one of the remaining two dice. The player who throws the higher number with their chosen dice wins. Now, each dice has three distinct numbers between 1 and 9, with pairs of opposite faces being identical. Design the three dice such that you always win! In other words, no matter which dice your opponent chooses, one of the two remaining dice throws a number larger than your opponent, on average.

SOLUTION

A) If the dice were colored pink (1, 6, 8), green (2, 4, 9) and yellow (3, 5, 7), then pink beats yellow, yellow beats green, and green beats pink.

1	2	3	4	5	6	7	8	9

B) If the dice were colored pink (1, 5, 9), green (2, 6, 7) and yellow (3, 4, 8), then green beats pink, pink beats yellow, and yellow beats green.

1	2	3	4	5	6	7	8	9

C) If the dice were colored pink (1, 6, 8), green (2, 3, 9) and yellow (4, 5, 7), then pink beats yellow, yellow beats green, and green beats pink.

1	2	3	4	5	6	7	8	9

Note: Solutions A and B above correspond to rows and columns of a 3x3 magic square!

8	1	6
3	5	7
4	9	2

8	1	6
3	5	7
4	9	2

PUZZLE

How many steps are required to break an m x n bar of chocolate into 1 x 1 pieces? We may break an existing piece of chocolate horizontally or vertically. Stacking of two or more pieces is not allowed.

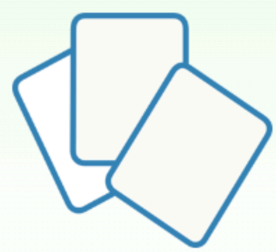
SOLUTION

We need mn - 1 steps. By breaking an existing piece horizontally or vertically, we merely increase the total number of pieces by one. Starting from 1 piece, we need mn - 1 steps to get to mn pieces.

Another way to reach the same conclusion is to focus on "bottom left corners of squares": Keep the chocolate rectangle in front of you and start drawing lines corresponding to cuts. Each cut "exposes" one new bottom left corner of some square. Initially, only one square's bottom left corner is exposed. In the end, all mn squares have their bottom left corners exposed.

PUZZLE

A blind man is handed a deck of 52 cards and told that exactly 10 of these cards are facing up. How can he divide the cards into two piles, not necessarily of equal size, with each pile having the same number of cards facing up?



SOLUTION

If the original pile has c cards with f cards facing up, then the blind man divides them into piles of size f and c - f. Then he flips all cards in the pile with f cards. Let's see why it works for c = 52 and f = 10. The blind man would divide the cards into two piles with 10 and 42 cards each. If there are k face-up cards in the 10-card pile, then there must be 10 - k face-up cards in the 42-card pile (because the total number of face-up cards is 10). So by flipping all cards in the 10-card pile, the number of face-up cards in both piles would become equal to 10 - k.

PUZZLE

Alice and Bob are playing a game. They are teammates, so they will win or lose together. Before the game starts, they can talk to each other and agree on a strategy.

When the game starts, Alice and Bob go into separate soundproof rooms — they cannot communicate with each other in any way. They each flip a coin and note whether it came up Heads or Tails. (No funny business allowed — it has to be an honest coin flip and they have to tell the truth later about how it came out.) Now Alice writes down a guess as to the result of Bob's coin flip; and Bob likewise writes down a guess as to Alice's flip.

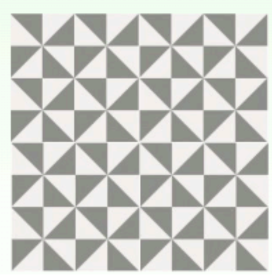
If either or both of the written-down guesses turns out to be correct, then Alice and Bob both win as a team. But if both written-down guesses are wrong, then they both lose.

Can you think of a strategy Alice and Bob can use that is guaranteed to win every time?

SOLUTION

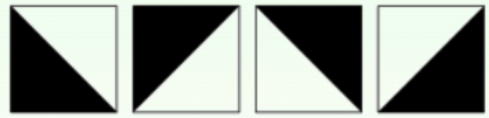
Alice writes down whatever her coin turns out to be. Bob writes down the opposite of whatever his coin turns out to be. Explanation: Alice is guessing that the coin tosses turned out to be identical. Bob is guessing that the two coin tosses turned out to be different.

PUZZLE



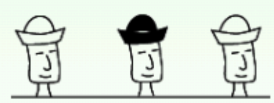
An 8x8 square grid has to be covered with isosceles triangular tiles with two tiles per square. Tiles come in two colors: black and white. Such tilings are called Truchet tilings. A tiling is said to be "fine" if no two tiles sharing an edge have the same color. How many "fine" Truchet tilings are there?

SOLUTION



Let us solve the problem for an m by n grid. We have four choices for the top-left square as shown in the figure. For any of these choices, there are only two ways to tile the square to the right of this square. For example, if the tiles in the figure are denoted by A, B, C, D, then, A or B may be followed only by A or B, both choices being valid. Similarly, C or D may be followed by only C or D. Continuing this way, there are exactly 2^{n+1} ways to tile the top-most row. Now, let us tile the row below the top-most row. There are two choices for the left-most square. For each of these two choices, there is exactly one choice for the square to its right, and so on. In other words, the entire row is determined by the row above and the left-most tile in that row. So for an m x n grid, there are 2^{m+n} possible tilings.

PUZZLE



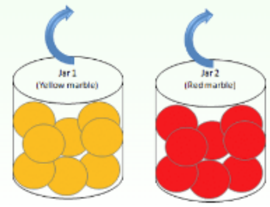
Three wizards are seated at a circular room. A magician shall make hats appear on their heads, one hat per wizard. Hats are either black or white, chosen uniformly at random. A wizard cannot see his own hat. At the sound of a bell, all wizards react simultaneously. A wizard reacts by either announcing a color or keeping quiet. If at least one wizard makes an announcement and if all the announcements are correct, the wizards have collectively won the game! Wizards are allowed to confer beforehand to devise a strategy. On average, can they win more than half the times the game is played?

SOLUTION

A wizard should keep quiet if he sees one black and one white hat. If he sees two black hats, he should announce white. If he sees two white hats, he should announce black. With this strategy, the probability of winning the game is 3/4. Why? The game is lost only when all hats are white or all hats are black. These events occur with probability 1/8 each.

PUZZLE

How would you divide 50 black and 50 white marbles into two piles, not necessarily of same size, so that the probability of picking a white marble as follows is maximized: we first pick one of the piles uniformly at random, then we pick a marble in that pile uniformly at random?



SOLUTION

Put one white marble into one pile and the rest of the marbles into the second pile. The overall probability of drawing a white marble are close to 75%.

Formally proving that putting one white in one pile is the optimal solution requires some work. Let us analyze the general case with w white marbles and b black marbles, where both w and b are positive integers. If the two piles are equi-sized, then no matter how we distribute the balls, the probability of drawing a white is $\frac{w}{w+b}$. This configuration is always superior to putting all balls in one pile and making the other pile empty. We are now left with configurations with unequal pile sizes, with each pile having at least one marble each. Divide the marbles arbitrarily into two piles with U and V marbles each, with U < V. We now change the configuration in three steps. With each step, the overall probability of drawing a white marble increases.

Step 1: Repeatedly swap one black marble from the first pile with one white marble from the second pile. One such swap increases the overall probability of drawing a white marble by $\frac{1}{2U} - \frac{1}{2V}$. If w ≥ U, the smaller pile has only white marbles, and we skip Step 2 and go directly to Step 3. However, if w < U, the larger pile has only black marbles and we execute Step 2.

Step 2 (executed only if w < U) Repeatedly transfer a black marble in the smaller pile to the larger pile. Clearly, this operation increases the overall probability of drawing a white marble. At the end of this step, the smaller pile has only white marbles.

Step 3: At the end of Steps 1 and 2, the smaller pile has only white marbles. Repeatedly move one white marble from the smaller pile to the larger pile, leaving only one white marble behind.

For the resulting configuration, the overall probability of drawing a white marble is $\frac{1}{2} + \frac{1}{2}\frac{w-1}{w+b-1}$, which simplifies to $\frac{w+(w+b-2)}{w+(w+b-2)+b}$, which exceeds $\frac{w}{w+b}$ when w + b > 2, because $\frac{w+w'}{w+w'+b} > \frac{w}{w+b}$ for any w' > 0. So the optimal configuration has one white marble in one pile and all others in the second pile. The only exception is the case w = b = 1 in which both marbles may be in the same pile.

PUZZLE

N women stand in a queue to take seats in an auditorium. Seating is pre-assigned. However, the first woman is an absent-minded professor who chooses any of the N seats at random. Subsequent women in the queue behave as follows: if the seat assigned to her is available, she takes it. Otherwise, she chooses an unoccupied seat at random. What are the chances that the last woman in the queue shall get the seat assigned to her?

SOLUTION

The chances are one in two. Proof by induction. The base case, N = 2, is immediate. In general, the absent-minded professor might choose her assigned seat with probability 1/N, in which case the last woman surely gets her assigned seat. The absent-minded professor might choose the last woman's seat with probability 1/N, in which case the last woman definitely loses her seat. In each of the remaining N-2 cases, by induction, the chances of the last woman getting her assigned seat are one in two. Combining these cases, we see that the overall probability is $\frac{1}{N}(1 + 0 + \frac{1}{2}(N-2))$.

PUZZLE

Consider five holes in a line. One of them is occupied by a fox. Each night, the fox moves to a neighboring hole, either to the left or to the right. Each morning, you get to inspect a hole of your choice. What strategy would ensure that the fox is eventually caught?

SOLUTION

Let holes be numbered 1 thru 5. Inspecting the holes in any of the following sequences suffices:

2, 3, 4, 2, 3, 4

2, 3, 4, 4, 3, 2

4, 3, 2, 2, 3, 4

4, 3, 2, 4, 3, 2

Explanation for sequence 2, 3, 4, 2, 3, 4: Let F denote the set of holes where the fox might be hiding. On any morning, the fox is either in an even numbered hole or an odd numbered hole. So on the first morning, either F = {1, 3, 5} or F = {2, 4}. If F = {2, 4}, then the following sequence of inspections suffices to catch the fox: 2, 3, 4. However, if the fox was not caught, then F must have equalled {1, 3, 5} on the first morning, so F must equal {2, 4} on the fourth morning. Therefore, repeating the sequence 2, 3, 4 from the fourth day onwards would suffice to catch the fox.

Another explanation to convince us that the sequence 2, 3, 4, 2, 3, 4 suffices to catch the fox: Let F denote the set of holes where the fox might be hiding. Initially, F = {1, 2, 3, 4, 5}. If the fox is not in hole 2, it must move so that F = {2, 3, 4, 5}. If the fox is not in hole 3, it must move so that to F = {1, 3, 4, 5}. If the fox is not in hole 4, it must move so that F = {2, 4}. If the fox is not in hole 2, it must move so that F = {3, 5}. If the fox is not in hole 3, it must move so that F = {4}.

PUZZLE

A magician has a bag with B black balls and W white balls. He also has a large supply of white balls and black balls outside the bag.

At every step, the magician removes two balls from the bag chosen uniformly at random and tosses them away. If the colors of these two balls were identical, he puts one white ball in the bag, otherwise he puts a black ball in the bag.

Given B and W, what are the chances that the last ball left in the bag is white?

SOLUTION

The last ball is black iff B is odd.

(Boring) Explanation: Instead of black and white balls, let's imagine black and white warriors. At successive time steps, two warriors — chosen arbitrarily — fight with each other.

1. When two black warriors fight, they kill each other and a new white warrior comes into being magically :)
2. When two white warriors fight, exactly one dies; the other survives.
3. When a white warrior and a black warrior fight, the white warrior dies; black survives.

As time progresses,

(A) What's happening to black warriors? They are getting knocked out in pairs! If B is even, all black warriors die. If B is odd, exactly one black warrior survives.

(B) What's happening to white warriors? Either a white warrior dies or a new white warrior appears. But new white warriors appear exactly $\lfloor B/2 \rfloor$ times. Other than that, white warriors are simply dying one by one. Do any white warriors eventually survive? That depends on outcome of (A) and the special ability possessed by black warriors to eliminate white warriors when they fight. Therefore, if a black warrior survives (which happens when B is odd), no white warrior survives and vice versa.

Slick Explanation: Let 0 denote the color white. Let 1 denote the color black. At any step, the color of the replacement ball is determined by the XOR of 0's and 1's corresponding to the colors of the two balls drawn from the bag. Overall, the magician is simply computing the XOR over all 1's and 0's corresponding to black and white balls in the bag. How magical is that? :)

PUZZLE

Three out of six lookalike balls are heavy. The other three are light. How many weighings on a beam balance are necessary to identify the heavy balls?

SOLUTION

Three weighings suffice. There are two different techniques for solving the problem. Let the balls be numbered 1 thru 6.

1. Weigh (1,2) vs (4,5), then (2,3) vs (5,6), then (3,1) vs (6,4).
2. All weighings involve one ball on each side of the beam balance. First weigh 1 against 2. If these are equal, then weigh 1 against 3, otherwise weigh 3 against 4. The reader may work out what the third weighing should be.

PUZZLE

The first box has two red balls. The second box has two green balls. The third box has one red and one green ball. Boxes are labeled but all labels are wrong! You are allowed to open one box, pick one of its balls at random, see its color and put it back into the box (you do not get to know the color of the other ball). How many such operations are necessary to correctly label the boxes?

SOLUTION

Since no label is correct, we have to distinguish between two cases: (GR is labeled GG, GG is labeled RR, RR is labeled GR) or (GR is labeled RR, RR is labeled GG, GG is labeled GR). So pick one ball at random from the box labeled GR.

PUZZLE

Nine schoolgirls are to be arranged in three rows and three columns on four different days so that any pair of schoolgirls is in the same row on exactly one of the four days.

SOLUTION

Let A, B, C, …, I denote the schoolgirls. Each color represents a row of schoolgirls:

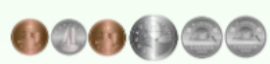
A	B	C
D	E	F
G	H	I

A	B	C
D	E	F
G	H	I

A	B	C
D	E	F
G	H	I

A	B	C
D	E	F
G	H	I

PUZZLE



30 coins of arbitrary denominations are laid out in a row. Simran and Tavleen alternately pick one of the two coins at the ends of the row so as to pick up as much money as possible. If Simran makes the first move, could Tavleen ever collect more money than Simran, if Simran makes the optimal choices?

SOLUTION

When the total number of coins is even, the first player could pick all coins in odd-numbered positions or all coins in even-numbered positions, whichever set is larger in value.

PUZZLE

Four honest and hard-working computer engineers are sipping coffee at Starbucks. They wish to compute their average salary. However, nobody is willing to reveal an iota of information about his/her own salary to anybody else. How do they do it?

SOLUTION

The first engineer picks a random k-digit integer for some large k, adds his salary to it and writes the sum on a chit. The chit is passed around. When it returns to the first engineer, he subtracts the k-digit integer.

PUZZLE

Alice writes two distinct real numbers between 0 and 1 on two chits of paper and places them in two different envelopes. Bob selects one of the envelopes randomly to inspect it. He then has to declare whether the number he sees is the bigger or smaller of the two. Is there any way he can expect to be correct more than half the times Alice plays this game with him?

SOLUTION

Let the number revealed to Bob be p. Then Bob should say "bigger" with probability p, "smaller" otherwise. If the other number is q, then the probability of winning is ½ + ½ |p - q|.

PUZZLE

A traveler has to pass tests of increasing difficulty to meet an Eastern mystical master. For the first test, he meets a pair of twins at a fork in the road: one path leads to the jungle, the other to the mystic. One of the twins always says the truth, the other always lies. What yes/no question should you ask one of the twins to determine the path that goes to the mystic? For the second test, the traveler encounters a second fork in the road. Again, one path leads to the jungle, the other to the mystic. This time, there are three look-alike brothers: one always tells the truth, the second always lies but the third sometimes tells the truth and sometimes lies. What two Yes/No questions should the traveler ask two of the brothers to determine the path to the mystic? Each question is answered by only the brother it is posed to. For the second question, the traveler may choose the brother and the question depending upon the answer to the first question.

SOLUTION

The two-brother problem may be solved in two different ways:

1. The traveler may ask, "Would your brother agree that the road on the left leads to the mystic?" The answer is guaranteed to be 'Yes' if and only if the road on the right leads to the mystic.

2. The traveler may ask, "If I were to ask you whether the road on the left leads to the mystic, would you say 'Yes'?" The 'if i were to ask you' construct makes both twins tell the truth. So if the answer is 'Yes', the road on the left indeed leads to the mystic, otherwise the road on the right does so.

Solutions to the three-brother problem:

1. With 4 questions: We first ask each brother, "Is one of the other two inconsistent, i.e., sometimes tells the truth and sometimes lies?" The truthful brother would say 'Yes', the lying brother would say 'No'. The third brother might say 'Yes' or 'No. Interestingly, if the answers form the multi-set {'Yes', 'Yes', 'No'}, then the 'No' answer was certainly given by the lying brother. If the answers form the multi-set {'Yes', 'No', 'No'}, then the 'Yes' answer was certainly given by the truthful brother. No other multi-set of answers is possible. So the fourth question may be posed appropriately to either the truthful brother or the lying brother depending upon which multi-set was observed.

2. With 3 questions: We ask the same question to all three, "Would your consistent brother (who always tells the truth or always tells lies, other than you) agree that the road on the left leads to the mystic?" In response, the consistent brothers would both say 'Yes' or both say 'No'. The inconsistent brother may say either 'Yes' or 'No'. So the majority vote allows us determine which road leads to the mystic.

3. With 2 questions: The key idea is to utilize the first question to identify a consistent brother (who always speaks the truth or always lies). The second question is posed to the brother just identified — at this point, the problem is identical to the 2-brother problem. Let us label the brothers as #1, #2 and #3. We shall use X to denote the brother whom we shall ask the second question. Here are some possibilities for the first question. In each case, if the answer is 'Yes', set X to #3, otherwise set X to #2. Claim: X is consistent!
    1. (devised by Ovidiu Gheorghioiu at Google) Ask #1 whether the following logical proposition is true: ("You always tell the truth" == "#2 gives random answers").
    2. Ask #1, "Is #2 more likely to give the correct answer than #3?"
    3. Ask #1, "If I were to ask you 'Is #2 consistent?', would you say 'Yes'?" Note that the 'if I were to ask you' construct makes consistent brothers truthful!

    The second question is posed to X (see the solution to the 2-brother problem). Either of the following suffices:
    1. "Would your consistent brother agree that the road on the left leads to the mystic?"
    2. "If I were to ask you whether the road on the left leads to the mystic, would you say 'Yes'?"

PUZZLE

Shankar chooses a number between 1 and 10,000. Geeta has to guess the chosen number as quickly as possible. Shankar will let Geeta know whether her guess is smaller than, larger than or equal to the number. The caveat is that Geeta loses the game if her guess is larger than Shankar's chosen number two or more times. (A) How many guesses are necessary? (B) What if Shankar is allowed to pick an arbitrarily large positive number?

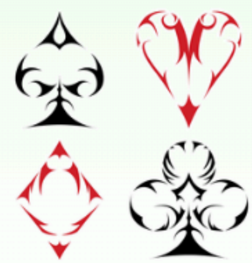
SOLUTION

(A) When Shankar chooses a number between 1 and n, Geeta should start guessing $\sqrt{n}$, $2\sqrt{n}$, $3\sqrt{n}$, $4\sqrt{n}$, and so on. The first time her guess exceeds Shankar's number, the range of numbers has been narrowed down to $\sqrt{n}$ numbers; she then starts guessing sequentially in that range.

(B) If Shankar guesses an arbitrarily large number n, then Geeta may guess 1, 4, 9, 16, 25, and so on, to discover k such that Shankar's number lies between k^2 and $(k+1)^2$. Then guess k^2+1, k^2+2, k^2+3 and so on. On the whole, this requires $O(\sqrt{n})$ steps.

PUZZLE

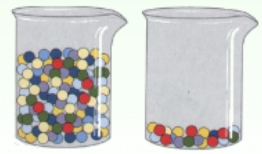
Alice repeatedly draws a card randomly, without replacement, from a pack of fifty-two cards. Bob has a one-time privilege to raise his hand just before a card is about to be drawn. Bob must execute his privilege before the last card is drawn. If the card drawn is Red just after Bob raises his hand, Bob wins; otherwise he loses. Is there any way for Bob to be correct more than half the times he plays this game with Alice?



SOLUTION

Imagine that whenever Bob raises his hand, Alice draws the *last* card in the pile (instead of the next card). This process is equivalent to the original process as far as the color of the card being drawn is concerned. So the probability that the card drawn is red is half. In other words, Bob can never be correct more than half the times he plays this game with Alice.

PUZZLE



There are two urns with one ball each. Each of subsequent n-2 balls is placed into one of these urns, with probability proportional to the number of balls already in that urn. What is the expected number of balls in the urn with fewer balls?

SOLUTION

To analyze the problem, let us first understand card shuffling as follows. We start with 1 card. The second card moves to one of two "slots" chosen uniformly at random: either to the left or to the right of the card already in the shuffle. In general, the i-th card chooses one out of i slots. When all n cards are in the shuffle, we have a random permutation of cards. Now, let us see how the process of populating two urns with n balls, as outlined in the puzzle, is related to card shuffling. Imagine that the first two balls are joined with a rod. These correspond to the first card in the pile. The third ball (which corresponds to the second card) has two choices. In general, the i-th ball corresponds to card i-1 and has i-1 choices. Now, the first card could be in any position uniformly at random. This means that all configurations with a balls in the first pile and b balls in the second pile, with a + b = n, are equally likely. In other words, the following configurations of balls in the two urns are all equally likely: (1, n-1), (2, n-2), (3, n-3), ..., (n-3, 3), (n-2, 2) and (n-1, 1). It turns out that the average number of balls in the smaller urn is (n+1) / 4 if n is odd, and n^2 / (4n - 4) when n is even. As n increases, both quantities converge to n/4.

PUZZLE

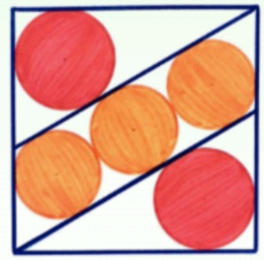
An enemy spy has poisoned one out of 1000 barrels of wine in a king's cellar. Even a sip of the poisoned wine has potency to kill. However, the effects of the poison show only in 30 days. The king asks the royal jailor to identify the poisoned barrel within 30 days. What is the least number of prisoners the jailor must employ to identify the poisoned barrel?

SOLUTION

The jailor prepares 10 mixtures labeled 0, 1, 2, ..., 9. Wine from the i-th barrel is added to those mixtures that correspond to 1's in the binary representation of i. Thus it takes exactly 10 prisoners to find out the poisoned barrel.

PUZZLE

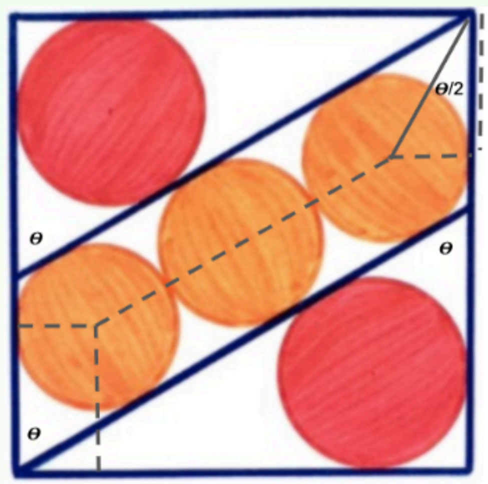
Five circles in a square. Total red area is 24. What is the total orange area?



SOLUTION

First, let's explore a tedious, algebraic approach to computing Θ by measuring the sides of the square in two ways: horizontally and vertically. Let the radius of the orange circle be 1. Horizontal length of the square is 2 + 4 sin Θ. Vertical length of square is 2 cot (Θ / 2) + 4 cos Θ. These must be equal, so

$$2 + 4 \sin \Theta = 2 \cot (\Theta / 2) + 4 \cos \Theta$$



We may simplify the equation by applying formulas for half-angles to get a degree-4 equation in sin Θ. Turns out that Θ equals 60 degrees (!) Wow. Can we derive this in a more elegant way?