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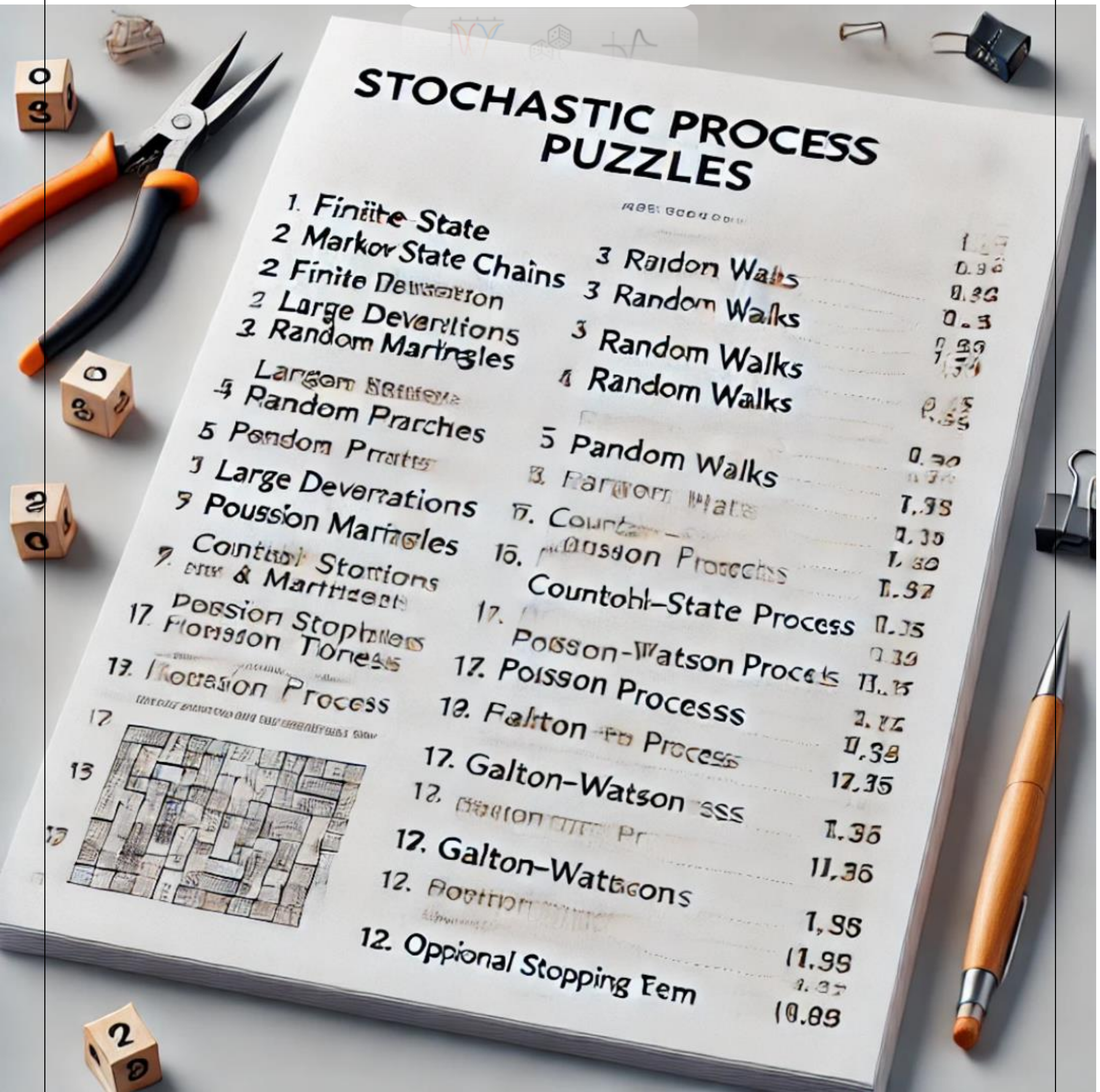

Finite-State Markov Chains

Puzzle 1: **Martingale Process**

Problem: You have a fair coin, and you start with $100. Every time you flip the coin, you win $1 if it's heads and lose $1 if it's tails. What is the probability that you'll reach $200 before going bankrupt?

Hint: This is a classic problem that involves understanding the Martingale process and the concept of ruin probabilities.

Puzzle 2: **Markov Chain Stock Prediction**

Problem: Consider a stock that can be in one of three states: Bull, Bear, or Stable. The transitions between these states follow a Markov chain with the following probabilities:

- From Bull to Bear: 0.3
- From Bull to Stable: 0.4
- From Bull to Bull: 0.3
- From Bear to Stable: 0.2
- From Bear to Bull: 0.4
- From Bear to Bear: 0.4
- From Stable to Bull: 0.5
- From Stable to Bear: 0.3
- From Stable to Stable: 0.2

Question: If the stock is currently in the Bull state, what is the probability that it will be in the Stable state after two days?

Hint: Use the transition matrix and the properties of Markov chains to find the solution.

Puzzle 3: Mean Reversion Strategy

Problem: You are analyzing a stock that historically shows a mean-reverting behavior. The price Pt at time t follows the equation:

$$P_{t+1} = \mu + \phi(P_t - \mu) + \epsilon_t$$

where μ is the long-term mean, ϕ is a constant, and ϵt is a white noise term with a mean of 0. How would you construct a trading strategy based on this information, and what would be the key risks involved?

Hint: Consider how you would exploit deviations from the mean and when the strategy might fail.

Puzzle 4: **High-Frequency Trading Execution**

Problem: You need to execute a large order in a market without significantly moving the price. The market impact model suggests that the cost C of executing q shares is given by:

$$C(q) = \alpha q + \beta q^2$$

where α and β are constants. How would you split the order to minimize the cost, and how does your strategy change as the size of the order increases?

Hint: Consider breaking down the problem using calculus or optimization techniques.

Puzzle 5: Stochastic Process and Trading Algorithm

- **Problem:** You are designing a trading algorithm that follows a simple stochastic process. The process is defined as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where St is the stock price, μ is the drift, σ is the volatility, and dWt is a Wiener process. How would you simulate the stock prices over time, and how would you use this simulation to backtest your trading strategy?

Hint: Consider numerical methods for simulating stochastic differential equations (SDEs) like the Euler-Maruyama method.

RANDOM WALKS, LARGE DEVIATIONS, AND MARTINGALES

Puzzle 1: **Random Walk and Trading Strategy**

Problem: You are developing a trading strategy based on a stock that follows a random walk. The price P_t at time t is modeled as $P_t = P_{t-1} + X_t$ where X_t are IID random variables with a mean of zero. Suppose you start with \$100, and you want to double your money before losing half of it. What is the probability that you will reach \$200 before your account drops to \$50?

Hint: This is a classic gambler's ruin problem with a driftless random walk.

Puzzle 2: **Large Deviations in Option Pricing**

Problem: Consider a stock whose returns follow a normal distribution with mean μ and variance σ^2. You are asked to price a deep out-of-the-money European call option using a large deviations approach. The strike price K is much higher than the current stock price S_0. Estimate the price of the option using the asymptotic behavior of large deviations.

Hint: Consider the exponential decay of probabilities for large deviations and apply the Chernoff bound.

Puzzle 3: **Martingale and Stopping Time**

Problem: Assume you are trading a stock with price S_t that follows a geometric Brownian motion. You adopt a strategy where you stop trading once the stock price hits a certain threshold T. If S_t is a martingale, what is the expected value of the stopping time T? How does this change if you know that S_t has drift?

Hint: Use martingale properties and Wald's identity to solve this.

Puzzle 4: **Sequential Hypothesis Testing in High-Frequency Trading**

Problem: In an HFT setup, you receive noisy signals $Y_1, Y_2, \ldots$ about a market indicator, which can be in one of two states: Bullish (H=1) or Bearish (H=0). You use sequential hypothesis testing to decide as quickly as possible whether the market is Bullish or Bearish. Describe how you would set up this problem and calculate the thresholds for deciding between the two states, balancing the trade-off between decision speed and accuracy.

Hint: This puzzle involves applying sequential analysis and the Neyman-Pearson lemma to find optimal thresholds.

Puzzle 5: **Mean Reversion Trading and Random Walks**

Problem: A stock's price follows a random walk with a slight mean reversion tendency modeled as $P_{t+1}=P_t+\theta(\mu-P_t)+\epsilon_t$, where ϵ_t is white noise. How would you design a trading strategy to exploit the mean reversion? What are the risks if the mean reversion parameter θ is small?

Hint: Analyze the long-term behavior of the stock price using properties of random walks and mean reversion.

POISSON PROCESSES

Puzzle 1: Market Order Arrivals

Problem:
A high-frequency trading (HFT) firm observes market orders arriving according to a Poisson process with a rate λ of 50 orders per minute. Assume the orders are for a particular stock.

Question:

1. What is the probability that exactly 3 orders arrive in a 1-minute window?
2. What is the expected time until the first market order arrives?
3. If a trader is only interested in the 10th market order, what is the expected time until the trader sees this order?

Puzzle 2: Order Flow Splitting

Problem:
An HFT firm splits its order flow between two trading strategies. Orders arrive according to a Poisson process with rate $\lambda=100$ per minute. Each order is routed to Strategy A with probability 0.6 and to Strategy B with probability 0.4.

Question:

1. What is the distribution of the number of orders processed by Strategy A in a 5-minute interval?
2. What is the probability that Strategy A processes more than 300 orders in a 5-minute interval?

Puzzle 3: Non-Homogeneous Trading Volume

Problem:
A trader models the arrival of trades during the trading day as a non-homogeneous Poisson process with a rate function $\lambda(t)=50+20$ where t is in hours from the market open (0 to 6 hours).

Question:

1. What is the expected number of trades during the first 3 hours of the trading day?
2. If the trader wants to execute an order when the expected number of trades reaches 200, at what time should they expect this to happen?

Puzzle 4: Combining Market Data Sources

Problem:
A trading algorithm combines data from two independent sources. The arrival of data updates from Source 1 follows a Poisson process with a rate of $\lambda1=30$ per minute, and from Source 2 with $\lambda2=20$ per minute.

Question:

1. What is the rate of the combined data update arrivals?
2. What is the probability that the first update comes from Source 1?

Puzzle 5: Traffic Flow at a Toll Booth

Problem:
Cars arrive at a toll booth according to a Poisson process with a rate $\lambda=10$ cars per hour.

Question:

1. What is the probability that no cars arrive in the first 15 minutes?
2. What is the probability that exactly 2 cars arrive in the first 30 minutes?
3. If the toll booth is open for 2 hours, what is the expected number of cars that will arrive during this period?

Puzzle 6: Customer Arrivals at a Bank

Problem:
Customers arrive at a bank according to a Poisson process with a rate $\lambda=12$ customers per hour.

Question:

1. What is the probability that the bank sees at least one customer in the next 5 minutes?
2. If the bank opens at 9:00 AM, what is the probability that the first customer arrives between 9:10 AM and 9:15 AM?
3. What is the expected number of customers arriving between 9:30 AM and 10:00 AM?

Puzzle 7: Call Center Inquiries

Problem:
A call center receives inquiries according to a Poisson process with a rate $\lambda=18$ inquiries per hour.

Question:

1. What is the probability that exactly 4 inquiries are received in the next 10 minutes?
2. If 6 inquiries have been received in the first 20 minutes, what is the probability that fewer than 10 inquiries will be received in the first 30 minutes?
3. What is the probability that the time until the first inquiry is more than 5 minutes?

COUNTABLE-STATE MARKOV CHAINS

Puzzle 1: Trading Strategy and Market State Transitions

Problem:
A trading strategy is modeled using a countable-state Markov chain where the states represent different market conditions (e.g., bullish, bearish, neutral). Transitions between these states occur based on market signals.

Question:

1. Given that the market starts in a neutral state, what is the probability that it enters a bullish state for the first time after exactly 5 transitions?
2. If the market is currently bullish, what is the expected number of transitions before it returns to a neutral state?

Hint:

- Use the concept of first-passage-time probabilities to solve the first part. Consider the probabilities of transitions and the structure of the Markov chain to determine the likelihood of moving from the neutral to the bullish state.
- For the second part, consider the recurrence properties of the Markov chain, and think about how to calculate the expected time for the market to return to a specific state.

Puzzle 2: Queueing System in High-Frequency Trading

Problem:
In a high-frequency trading system, the number of orders in the queue is modeled as a birth-death Markov chain. Arrivals to the system (births) occur with probability p per unit time, and departures (deaths) occur with probability q.

Question:

1. If the system starts empty, what is the steady-state probability that there are exactly 3 orders in the queue?
2. What condition on p and q ensures that the queue remains stable (i.e., it does not grow indefinitely)?

Hint:

- To find the steady-state probability, use the relationship between the birth and death rates and the steady-state probabilities of the Markov chain.

- Stability of the queue depends on the balance between arrival and departure rates. Consider the long-term behavior of the system as the number of orders in the queue increases.

Puzzle 3: Reversibility in a Trading Model

Problem:
A trading algorithm is designed to adjust its positions based on the historical behavior of a Markov chain model of price movements. The algorithm assumes the chain is reversible.

Question:

1. Verify whether the Markov chain describing the price movements is reversible.

2. If the chain is reversible, explain how the concept of reversibility can simplify the analysis of long-term expected returns.

Hint:

- To verify reversibility, check if the transition probabilities satisfy the detailed balance equations. This involves showing that the transition probabilities in both directions between any two states satisfy a specific condition related to the steady-state distribution.

- Reversibility implies certain symmetries in the system that can be leveraged to simplify the calculation of expected returns and other long-term metrics in the trading model.

Puzzle 4: Customer Support Center

Problem:
A customer support center is modeled as a Markov chain where the states represent the number of customers in the queue. The system is designed such that there can be an infinite number of customers in the queue. Arrivals occur with probability λ and service completions occur with probability μ\muμ in each time step.

Question:

1. What is the probability that the queue will ever be empty after starting with exactly one customer in the queue?

2. If the system starts with zero customers, what is the probability that the queue will grow indefinitely?

Hint:

- To solve the first question, think about the concept of recurrence and the conditions under which the queue can return to the empty state.
- For the second question, consider the conditions for the system to be transient, where the probability of returning to the empty state is less than 1.

Puzzle 5: Reliability of a Computer Network

Problem:
A computer network's reliability is modeled as a Markov chain with states representing the number of functioning nodes. Nodes can fail with probability p or be repaired with probability q in each time step.

Question:

1. What is the expected time until the entire network fails (i.e., all nodes are non-functional) starting from a state where all nodes are functional?
2. If the network starts with one node failed, what is the probability that it will reach a state where all nodes are functioning before all nodes fail?

Hint:

- For the first question, consider the concept of the expected first-passage time to the absorbing state (complete failure) from the initial state.
- For the second question, think about the transitions between states and how they affect the probability of reaching full functionality versus complete failure.

Puzzle 6: Epidemic Spread in a Population

Problem:
The spread of an epidemic in a population is modeled as a Markov chain. The states represent the number of infected individuals. In each time step, an infected individual can either recover with probability r or infect a new individual with probability s.

Question:

1. What is the probability that the epidemic dies out (i.e., the number of infected individuals reaches zero) if the process starts with a single infected individual?

2. If the epidemic starts with two infected individuals, what is the expected number of time steps before there are no infected individuals left?

Hint:

- To solve the first question, consider the absorbing state (no infected individuals) and whether the system is likely to reach this state given the transition probabilities.
- For the second question, use the concept of the expected time to absorption in a Markov chain, and think about how the number of initial infected individuals affects this expectation.

Puzzle 7: High-Frequency Trading (HFT) Latency Modeling

Problem:
An HFT firm models the latency of its trades using a countable-state Markov chain, where each state represents a specific latency range in milliseconds (e.g., state 0 for 0-1 ms, state 1 for 1-2 ms, etc.). Transitions between these states occur based on network conditions and are influenced by factors such as server load and network congestion.

Question:

1. Derive the steady-state distribution for the latency states if the transition probabilities between states are given by $P_{i,i+1}=\alpha$ (probability of increasing latency) and $P_{i+1,i}=\beta$ (probability of decreasing latency), with $\alpha+\beta=1$ and $\alpha\neq\beta$.
2. Analyze the impact on the steady-state distribution if the firm optimizes its network infrastructure, reducing the probability of increasing latency (α\alphaα) by 20%.

Hint:

- For the first part, use the detailed balance equations to set up a system of linear equations for the steady-state probabilities, and solve them under the given constraints.
- For the second part, consider how the change in α affects the steady-state distribution and whether the system becomes more or less stable (i.e., how latency is distributed across states).

Puzzle 8: Coin Flipping Game

Problem:
You are playing a game where you repeatedly flip a fair coin. You start with a score of 0, and each time you flip heads, your score increases by 1, while each time you flip tails, your score decreases by 1. The game continues indefinitely unless your score reaches either 3 or -3-3-3, at which point the game ends.

Question:

1. What is the probability that the game ends with a score of 3, starting from a score of 0?
2. On average, how many flips will it take for the game to end, starting from a score of 0?

Hint:

- For the first question, model the problem as a countable-state Markov chain with states $-2,-1,0,1,2$ and absorbing states -3 and 3. Use the concept of absorption probabilities to find the probability of reaching the state 3 before -3.
- For the second question, use the concept of expected first-passage times. Set up a system of equations to determine the expected number of flips needed to reach either 3 or -3 from each intermediate state, particularly focusing on the starting state 0.

Galton-Watson process

Puzzle 1: The Family Fortune Dilemma

You inherit a family fortune that grows based on a Galton-Watson process. Each year, your fortune multiplies according to a distribution μ, where the average growth rate mmm is just over 1. However, there's a risk that your fortune might vanish entirely due to market crashes.

Question: What is the probability that your family fortune will eventually be wiped out? How can you ensure that your fortune survives for generations?

Hint: Consider the smallest root of the generating function f(s)=s and how the average growth rate mmm influences the probability of extinction.

Puzzle 2: The Critical Market Model

Imagine you are managing a market model that operates on a critical Galton-Watson process with m=1. The model is on the edge—neither expanding nor shrinking on average. However, the market has inherent risks that could cause the model to collapse.

Question: What is the fate of your market model? Will it eventually collapse, and if so, what's the underlying probability?

Hint: Delve into the nature of the critical process where m=1. Consider how the process's equilibrium state impacts the likelihood of eventual extinction.

Puzzle 3: The Extinction Bet

You enter a high-stakes betting game where each round represents a generation in a Galton-Watson process. The game's outcome is modeled by a probability distribution where the average payoff per round is slightly more than breaking even, but there's a chance of losing everything.

Question: How would you calculate the likelihood of eventually losing all your money? What strategies might you employ to maximize your chances of continuing to play?

Hint: Focus on the extinction probability and the role of the generating function in predicting whether the game will end in complete loss or continued play.

Puzzle 4: The Second Moment Gambit

You're tasked with assessing the risk of a new trading strategy that can be modeled by a supercritical Galton-Watson process. Your risk assessment hinges on the second moment of the number of trades Z1 in the first round.

Question: Under what conditions on Z1 will your trading strategy have a strictly positive chance of long-term survival? How does this relate to the volatility (second moment) of your trades?

Hint: Think about the Kesten-Stigum theorem and the importance of the second moment E [Z1^2] in ensuring a positive survival probability. Consider how controlling the volatility can impact the strategy's longevity.

Puzzle 5: **The Mysterious Forest Growth**

You find yourself in an enchanted forest where trees grow in a peculiar way. Each tree starts as a single seed and has the ability to sprout a random number of new seeds in the next season. The number of seeds each tree produces follows a certain mysterious pattern known only to the forest. As you walk deeper into the forest, you notice that some trees seem to disappear altogether after a few seasons, while others flourish and spread across the landscape.

Question: If you were to track the growth of these trees over many seasons, how would you determine whether the entire forest will eventually vanish or thrive indefinitely? What factors would you need to consider to predict the long-term fate of the forest?

Hint: Think about the average number of seeds each tree produces and how this affects the overall growth of the forest. Consider what happens if the trees collectively produce fewer seeds on average versus producing more. Explore the balance between growth and extinction in this mysterious environment.

Optional Stopping theorem

Puzzle 1: The Casino Conundrum

You enter a casino with a plan to use a strategy based on a martingale betting system. You start with a certain amount of money and plan to keep doubling your bet until you win. Your goal is to end up with a profit, but the casino has a rule that forces you to stop after a certain number of bets or if your money runs out.

Question: Can you guarantee a profit with this strategy? What does the Optional Stopping Theorem say about your expected outcome?

Hint: Consider the properties of a martingale and how the stopping time imposed by the casino rules affects your expected winnings. Think about whether the expectation of your final fortune can differ from your initial amount.

Puzzle 2: The Trader's Dilemma

As a quant trader, you develop a trading strategy where you place trades until a certain profit level is reached or a loss threshold forces you to stop. The strategy's profitability follows a supermartingale pattern, where the expected value of your portfolio decreases over time.

Question: Given the Optional Stopping Theorem, what can you say about the expected value of your portfolio at the stopping time? Is there a way to ensure that you stop with a profit?

Hint: Analyze the implications of the supermartingale property on your trading strategy. Consider how the stopping time influences the expected value of your portfolio and whether it can exceed your initial capital.

Puzzle 3: The Risky Gambit

You are managing a risky investment portfolio where the value of your holdings follows a stochastic process modeled as a submartingale. You decide to liquidate your entire portfolio once it reaches a certain high value or drops to a specific low threshold.

Question: What does the Optional Stopping Theorem tell you about the expected value of your portfolio at the time of liquidation? How does the submartingale nature of the process influence your decision to sell?

Hint: Consider how the submartingale property affects the expected value at the stopping time. Think about the conditions under which you might expect the portfolio to reach the higher threshold before hitting the lower one.

Puzzle 4: The Optimal Exit Strategy

You are developing an algorithm to automatically exit a trade when a certain condition is met, such as a specific profit target or stop-loss level. The price movements of the asset you are trading are modeled as a martingale. Your goal is to maximize your expected returns by choosing an optimal stopping time.

Question: How should you design your stopping rule to maximize expected returns? What role does the Optional Stopping Theorem play in determining whether your stopping rule is effective?

Hint: Explore how the martingale property influences the choice of an optimal stopping time. Consider whether any stopping rule can increase the expected value beyond the initial value.

Puzzle 5: The Last Chance Bet

You are playing a game where you can continue betting until you either lose everything or double your current wealth. The game follows a stochastic process modeled as a martingale with bounded increments. You decide to play until you either achieve your goal or hit the losing threshold.

Question: What is the expected outcome if you continue betting indefinitely? How does the Optional Stopping Theorem apply to your decision to stop playing?

Hint: Consider the impact of bounded increments on the martingale. Use the Optional Stopping Theorem to understand the long-term expectation of your wealth given your stopping rule.