SVMs for Structured Output

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SVM*struct*

Tsochantaridis Hofmann Joachims Altun 04

Extending SVMs

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- SVM = parametric function

$$f(\mathbf{x}|w) = \text{sign}\left(\langle w, \Psi(\mathbf{x})\rangle + b\right)$$

  - ***arbitrary input***
  - ***binary output***

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Today:

➡ how to handle ***arbitrary output***

➡ (while keeping the computation efficient)

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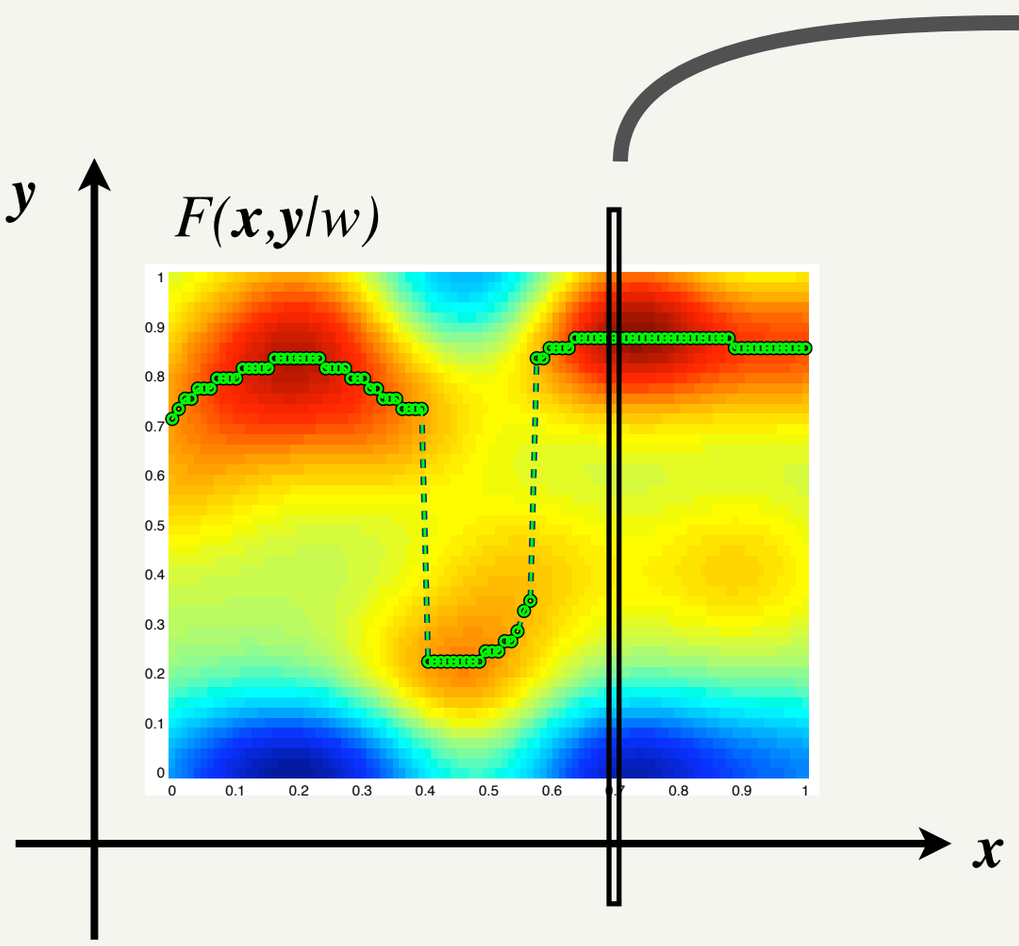
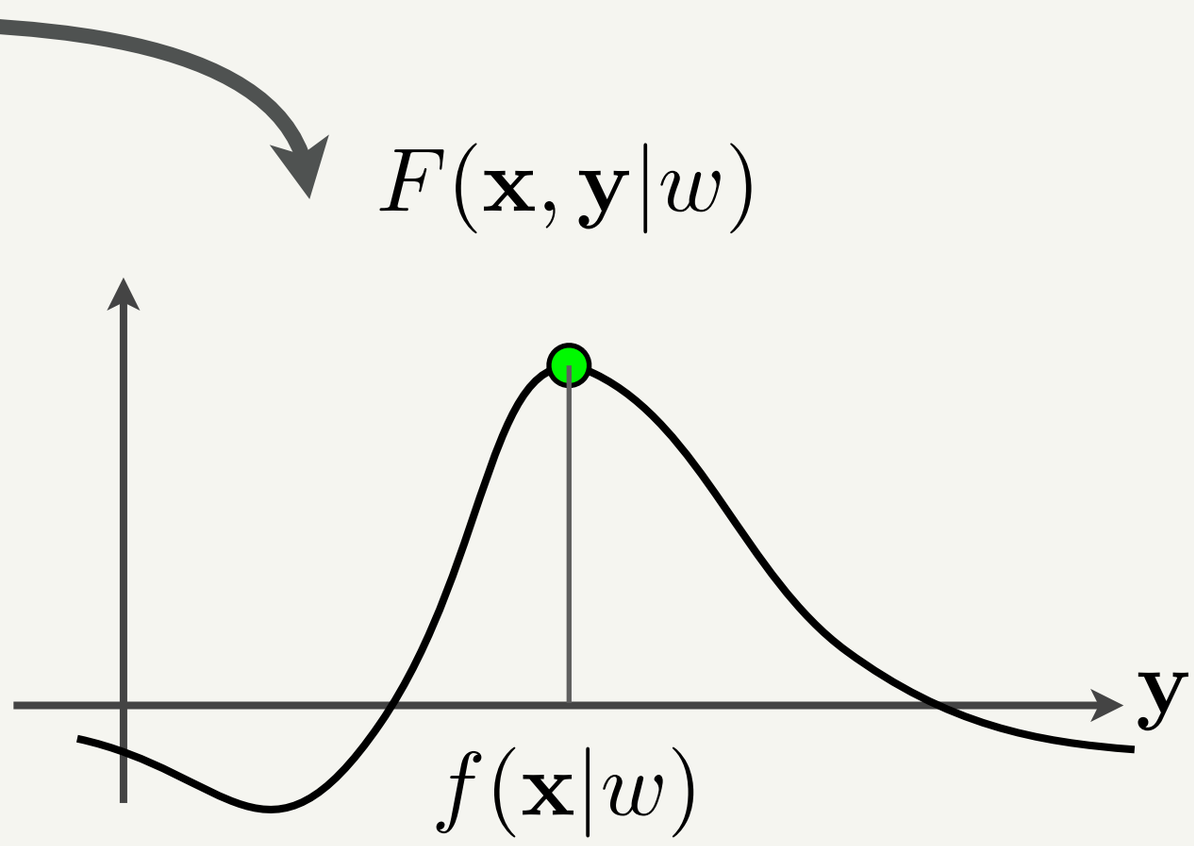
Example: real function

- Encode $f(\mathbf{x}|w) : \mathbb{R} \rightarrow \mathbb{R}$
- $F(\mathbf{x}, \mathbf{y}|w) : \mathbb{R}^2 \rightarrow \mathbb{R}$





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- **Problem:** Find w that minimizes $R(w)$.

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- Necessary and sufficient condition:
  For each point $\mathbf{x}_i$ the maximum is reached at $\mathbf{y}_i$

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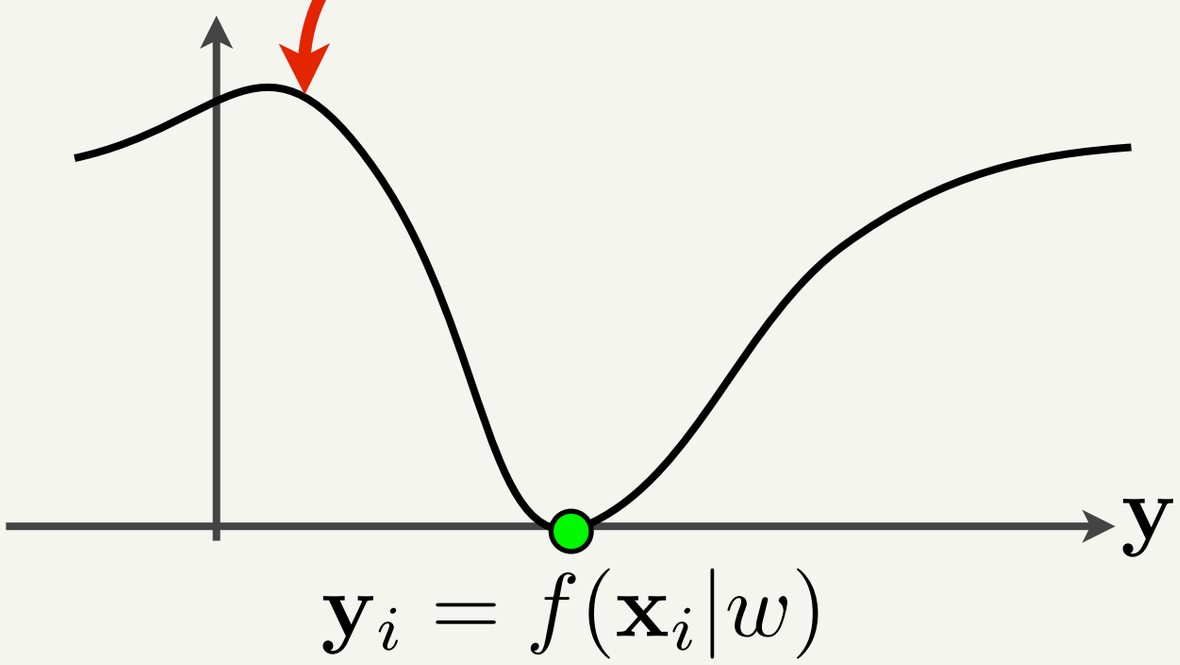

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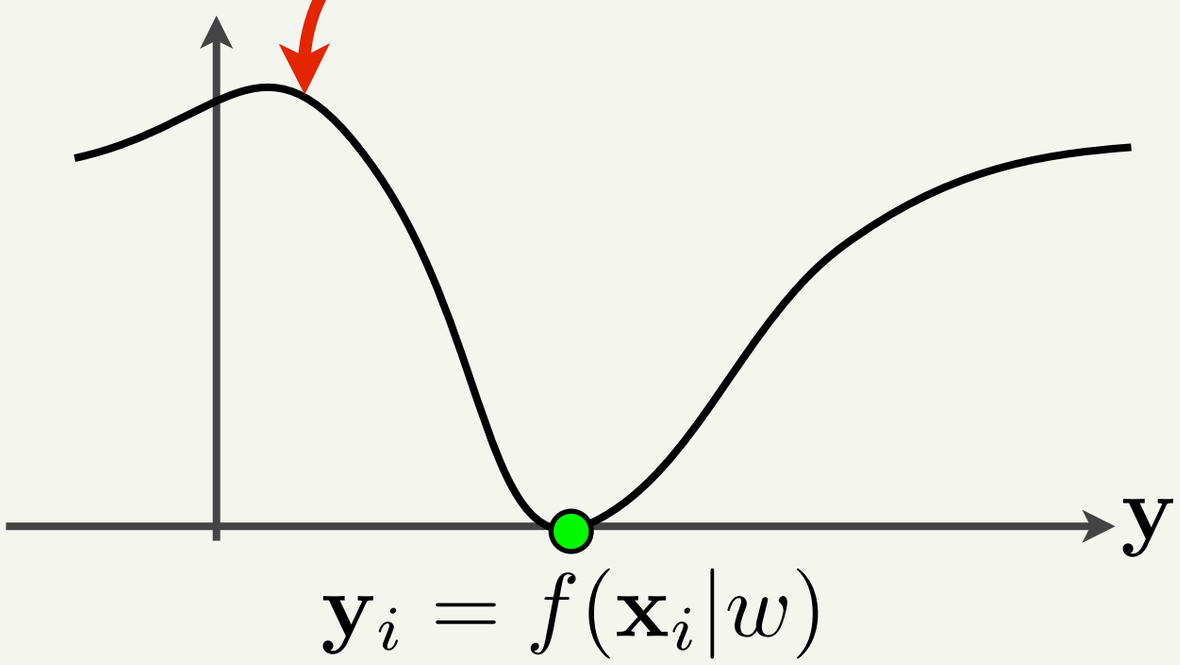

$$\forall i, \mathbf{y} \neq \mathbf{y}_i : \ \langle w, \delta\Psi_i(\mathbf{y}) \rangle > 0$$

Margin → max-margin

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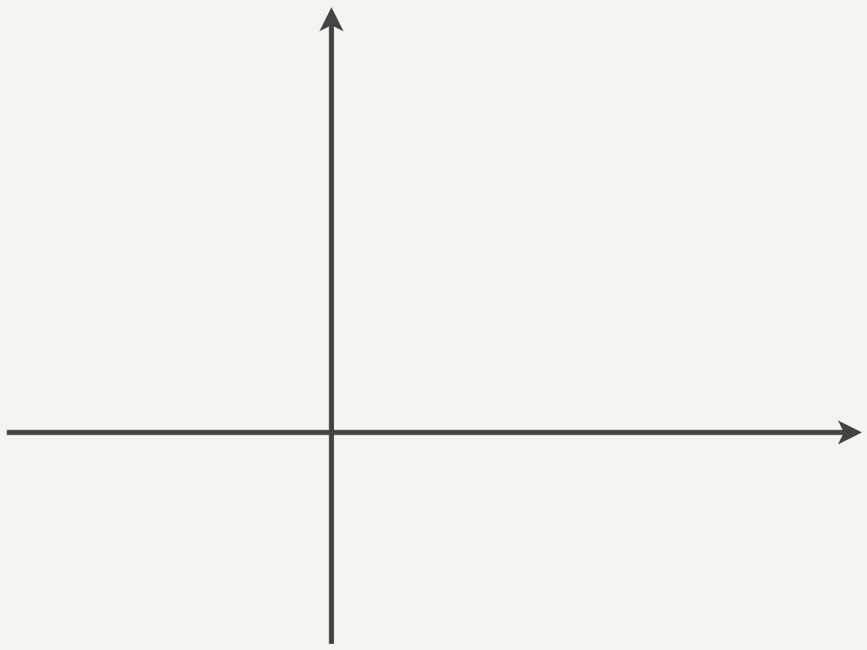

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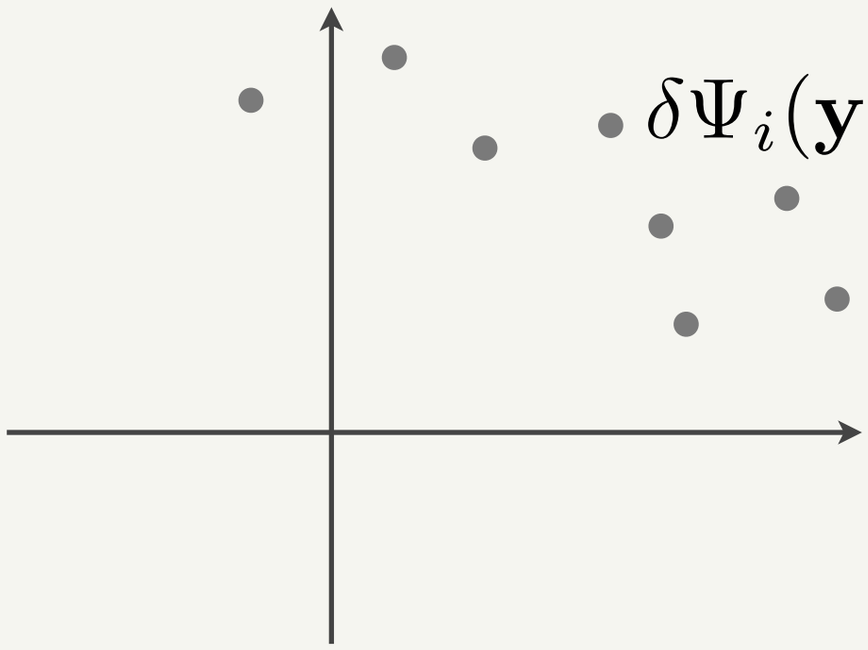

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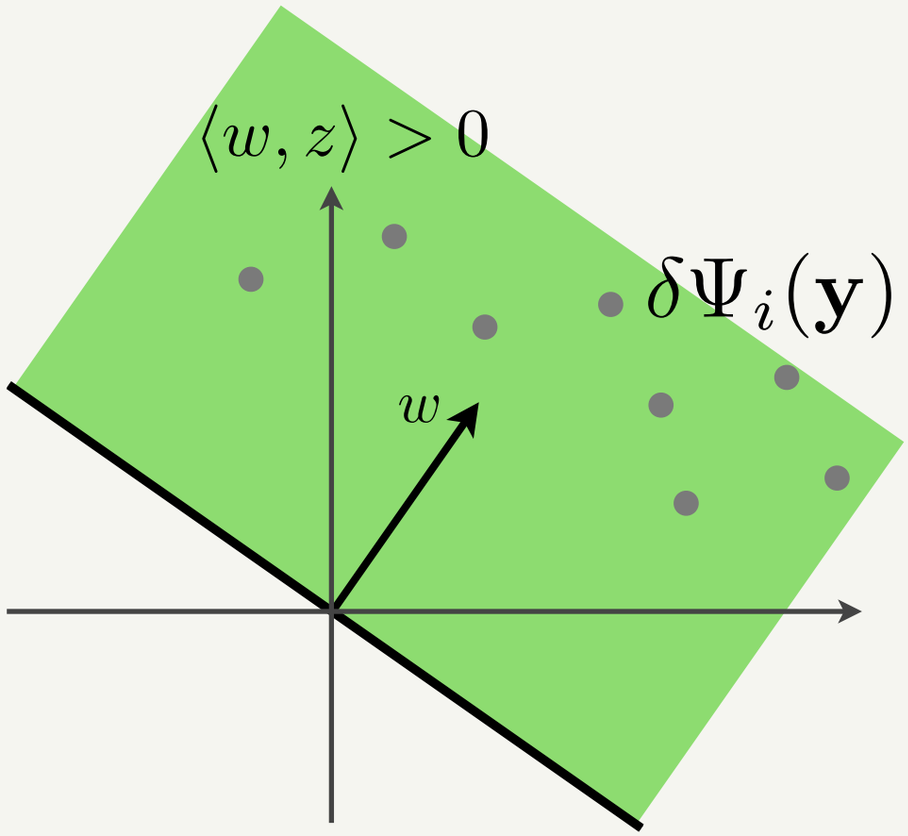

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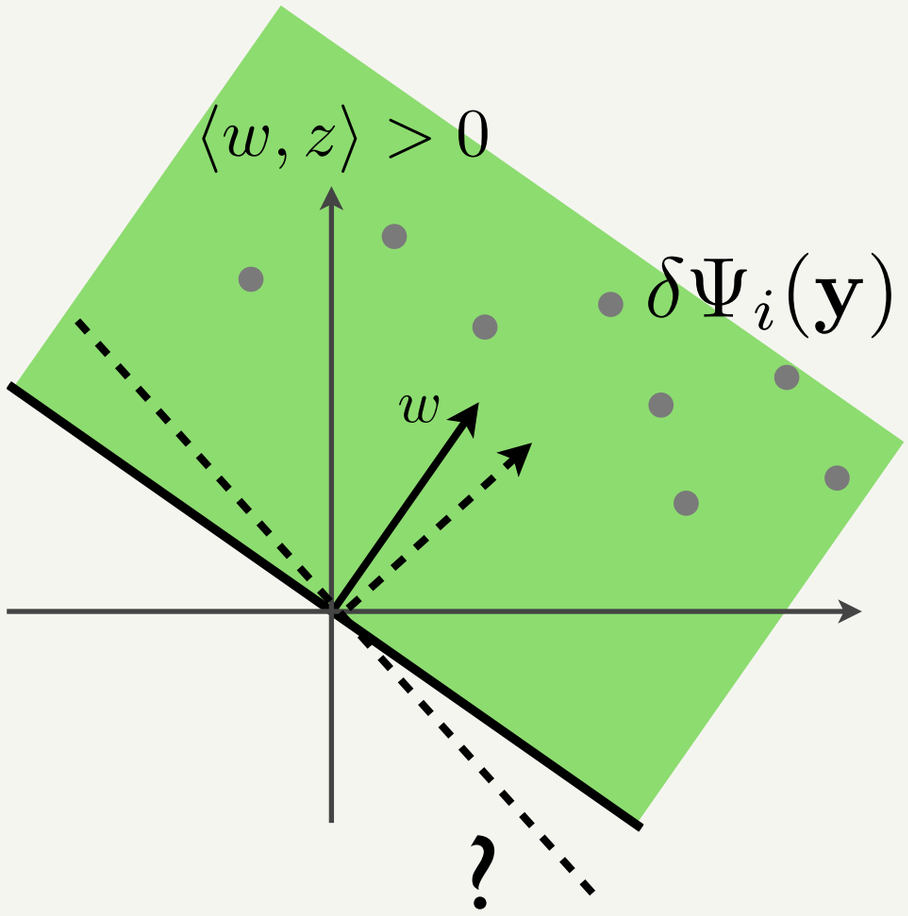

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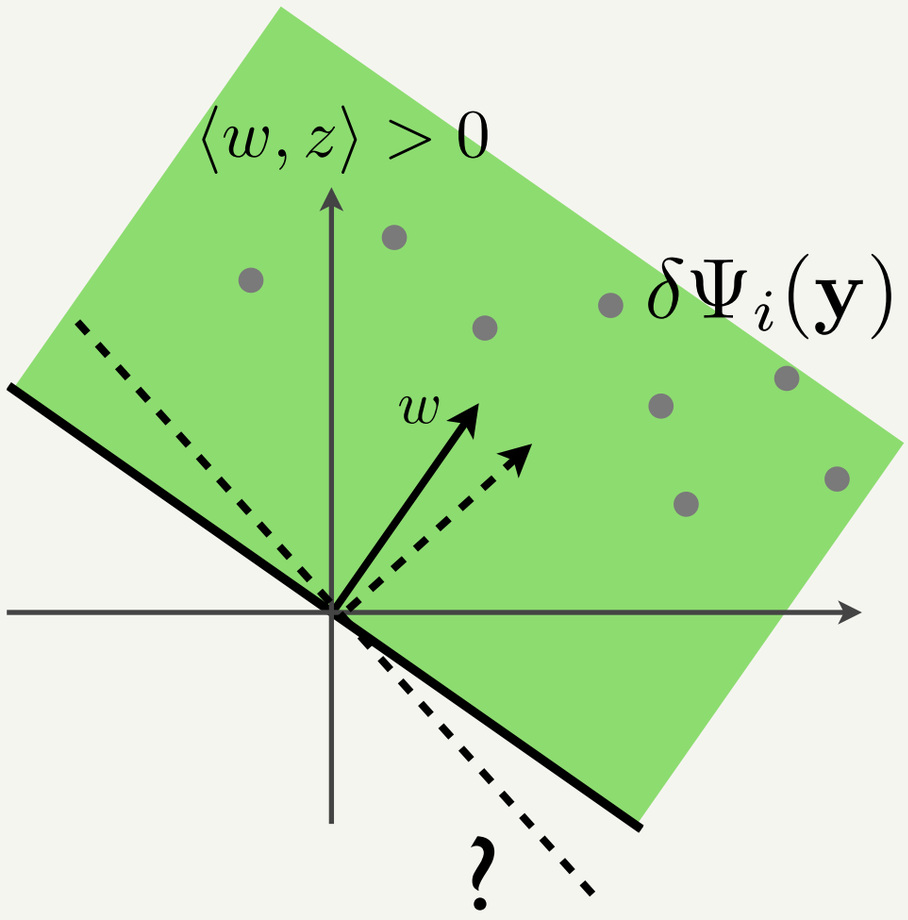

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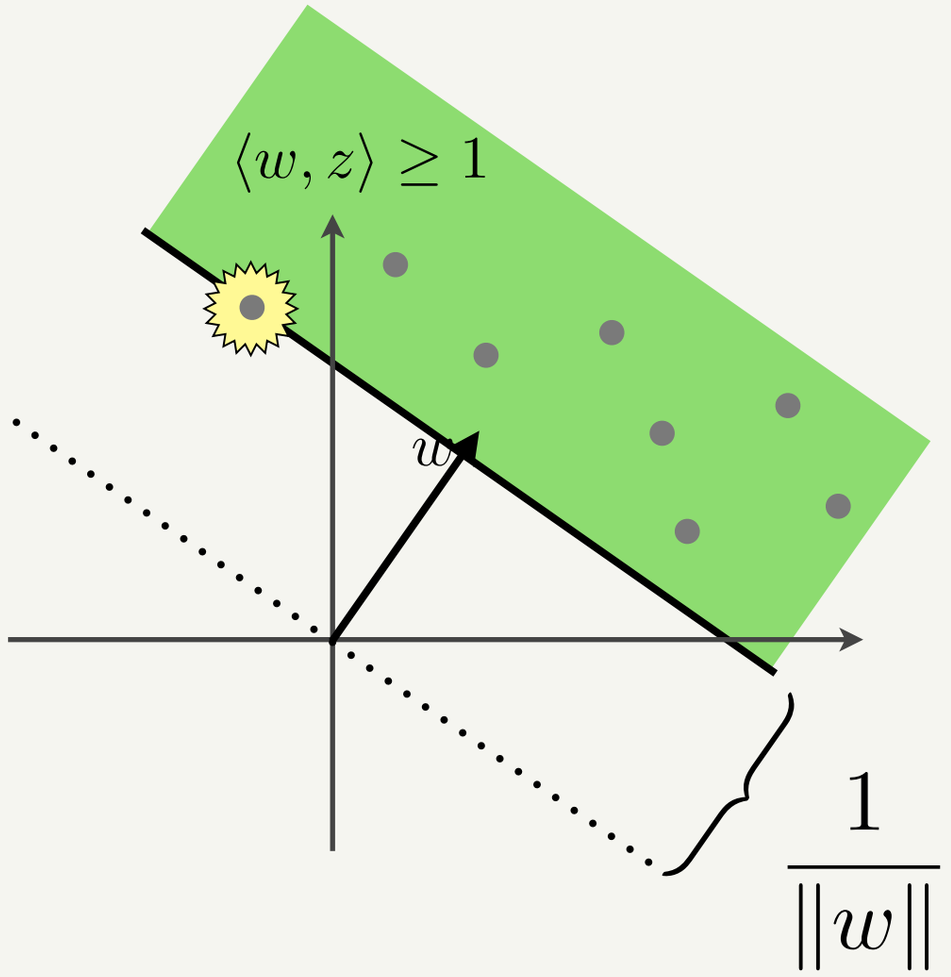

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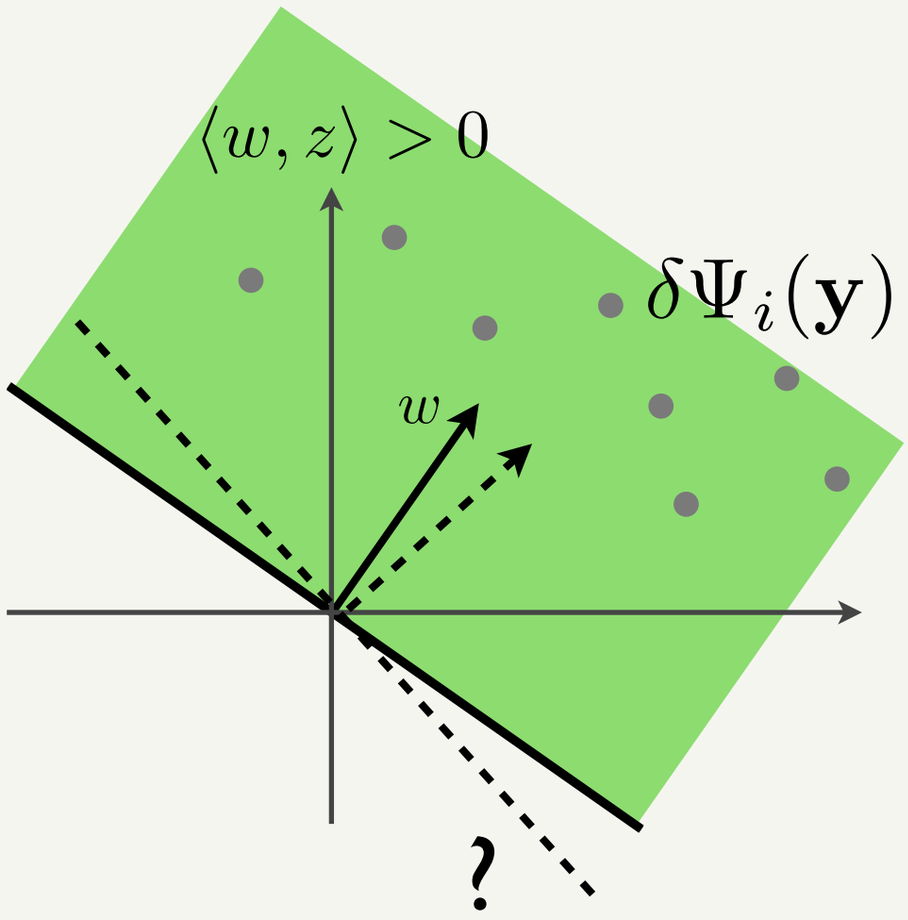

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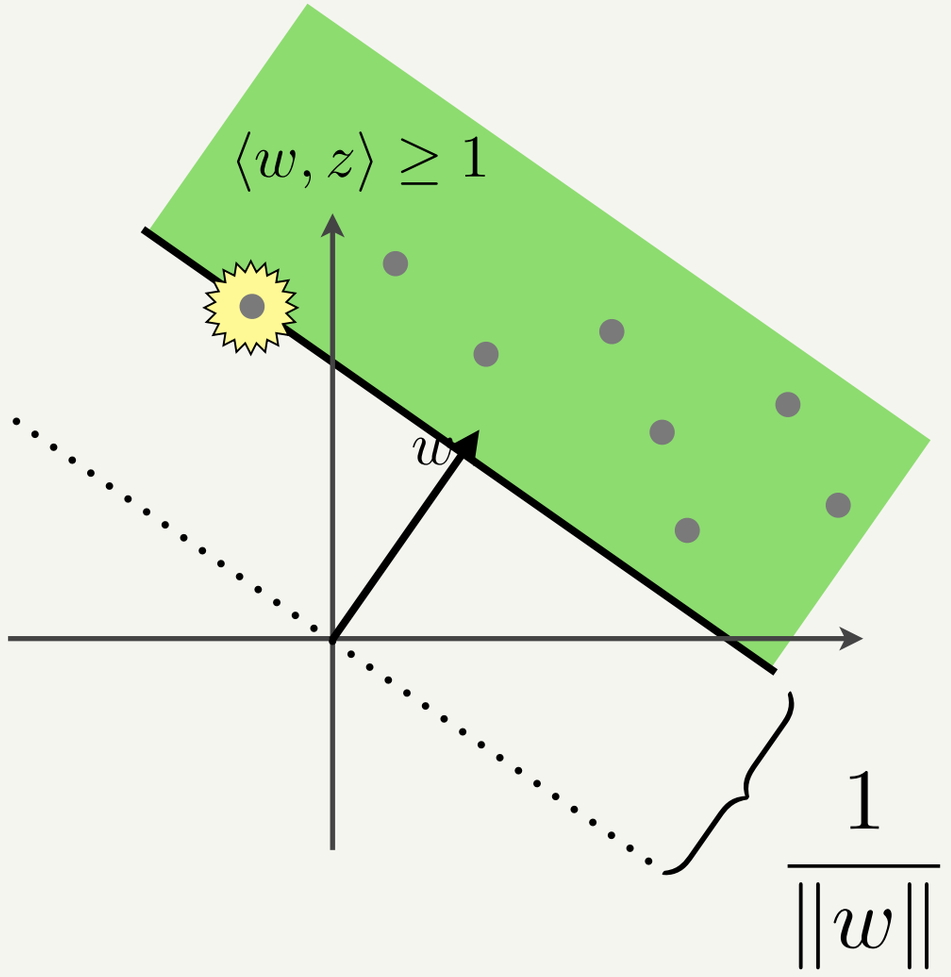
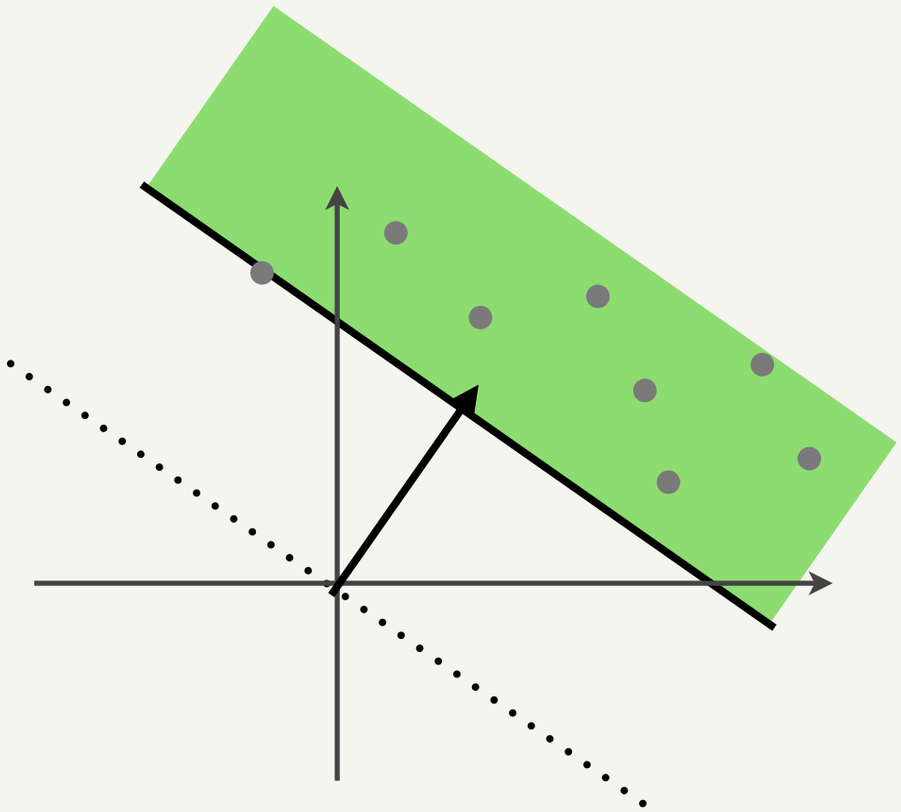


$$\min \tfrac{1}{2}\|w\|^2,$$
$$\langle w, \delta\Psi_i(\mathbf{y}) \rangle \geq 1 \quad \forall i, \mathbf{y} \neq \mathbf{y}_i$$

Non-separable case



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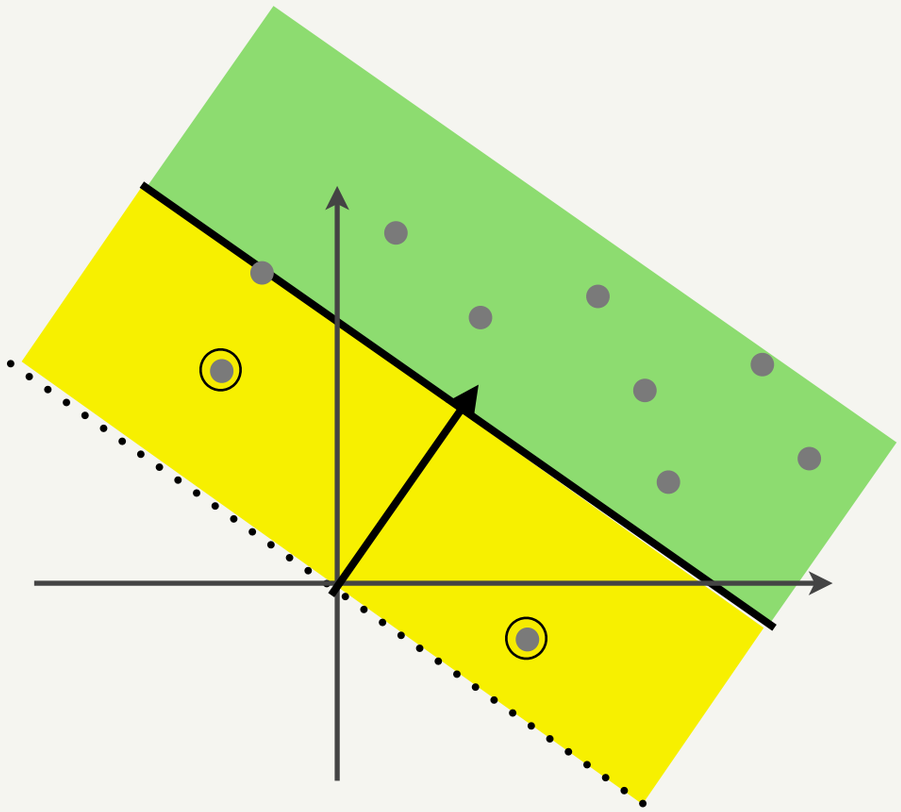

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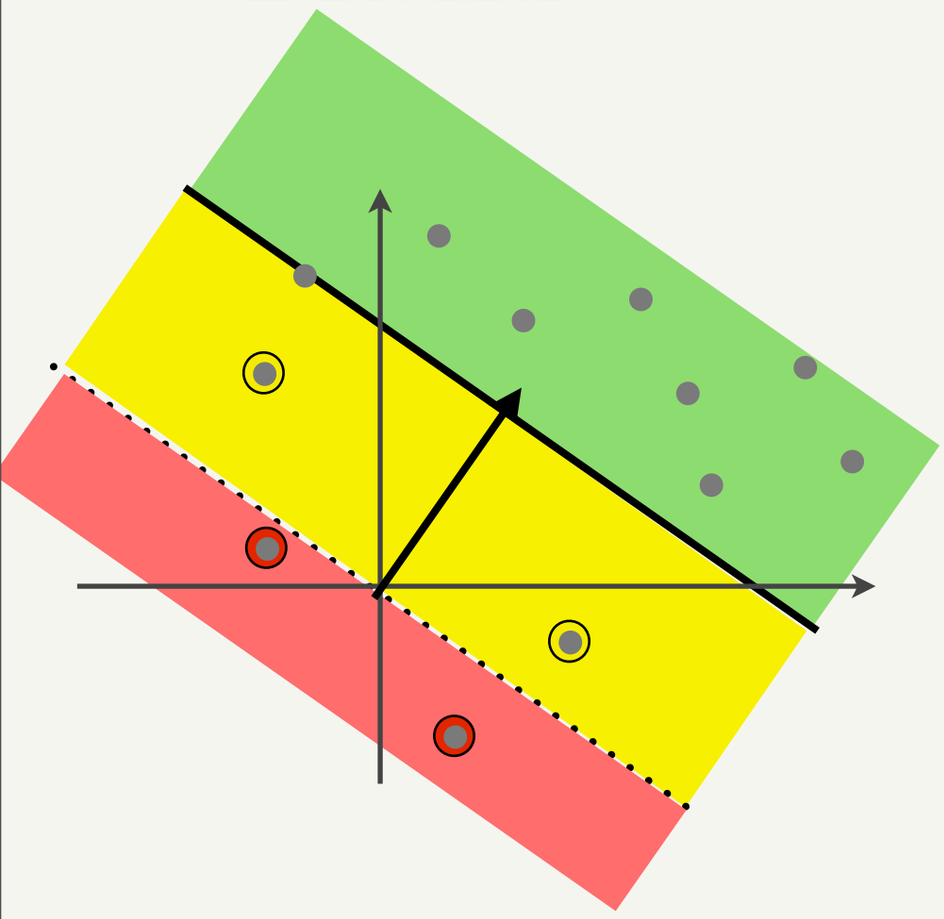

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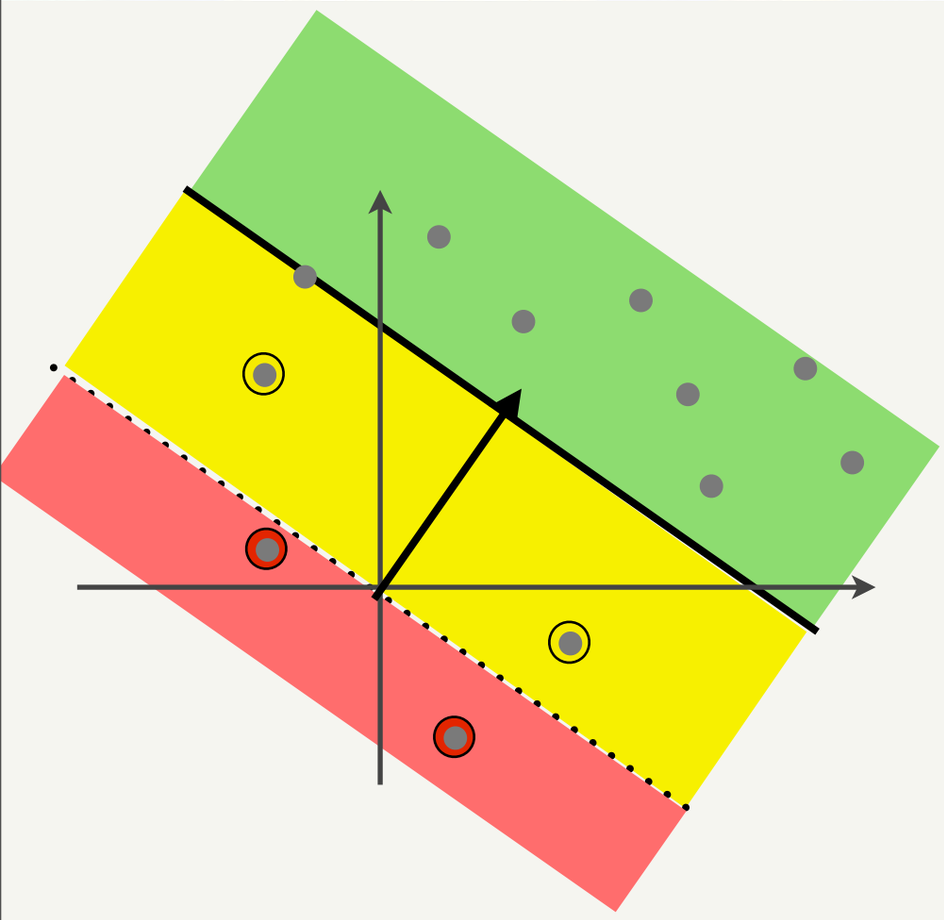

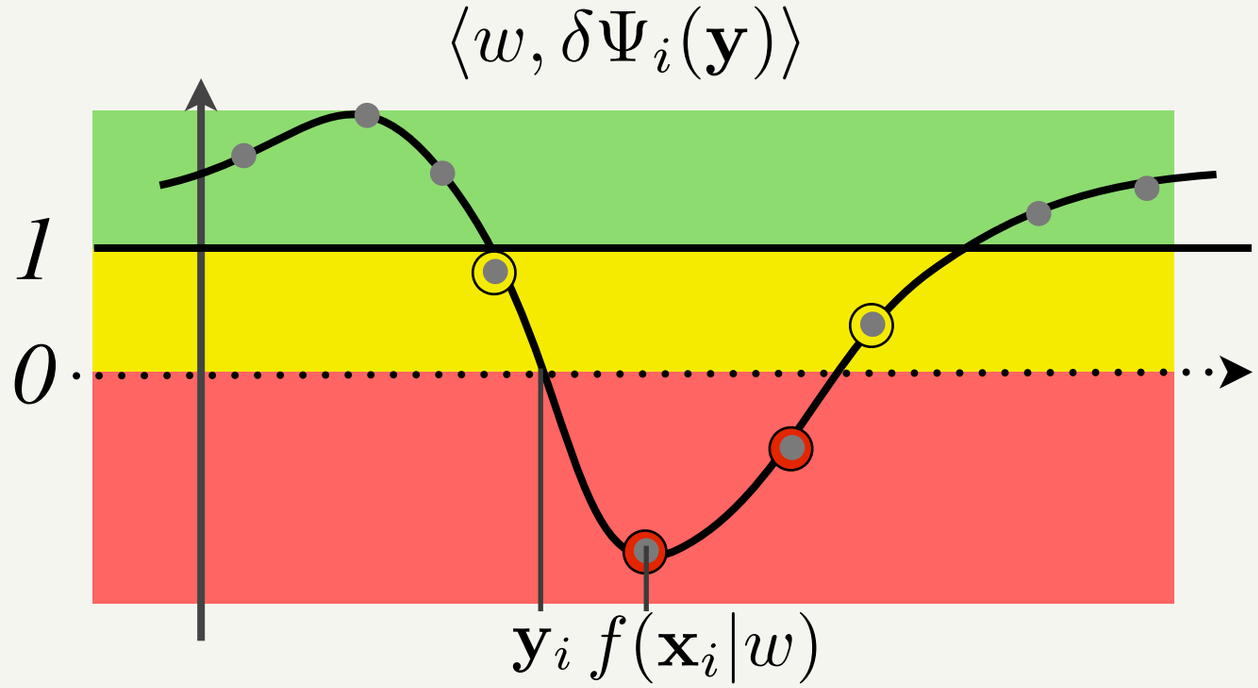

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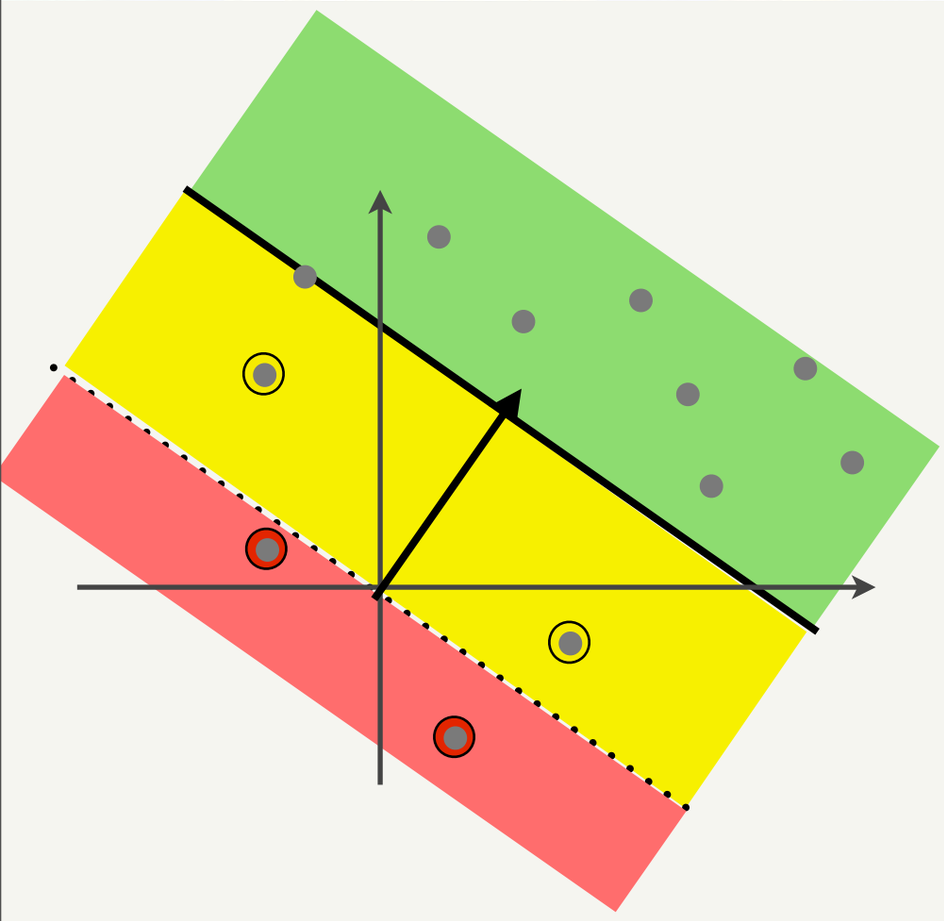

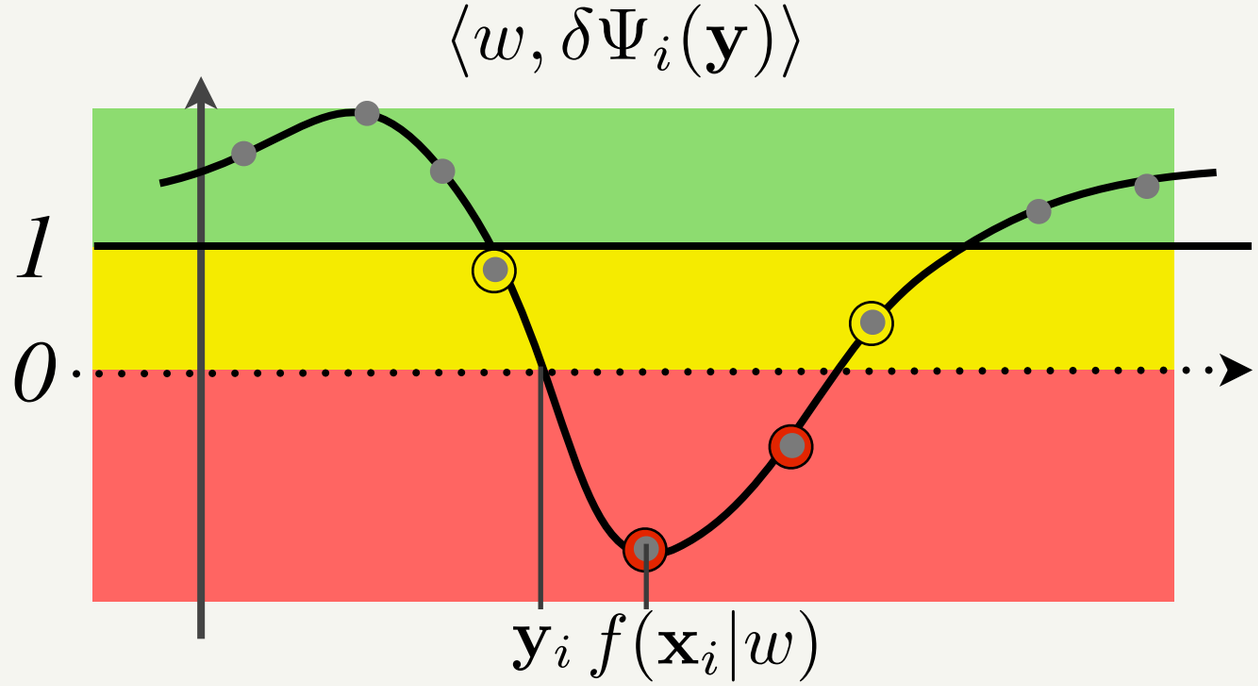

$$\langle w, \Psi_i(\mathbf{y}) \rangle \geq 1 - \xi_i / \Delta(\mathbf{y}_i, \mathbf{y})$$

$$\xi_i \geq 0$$

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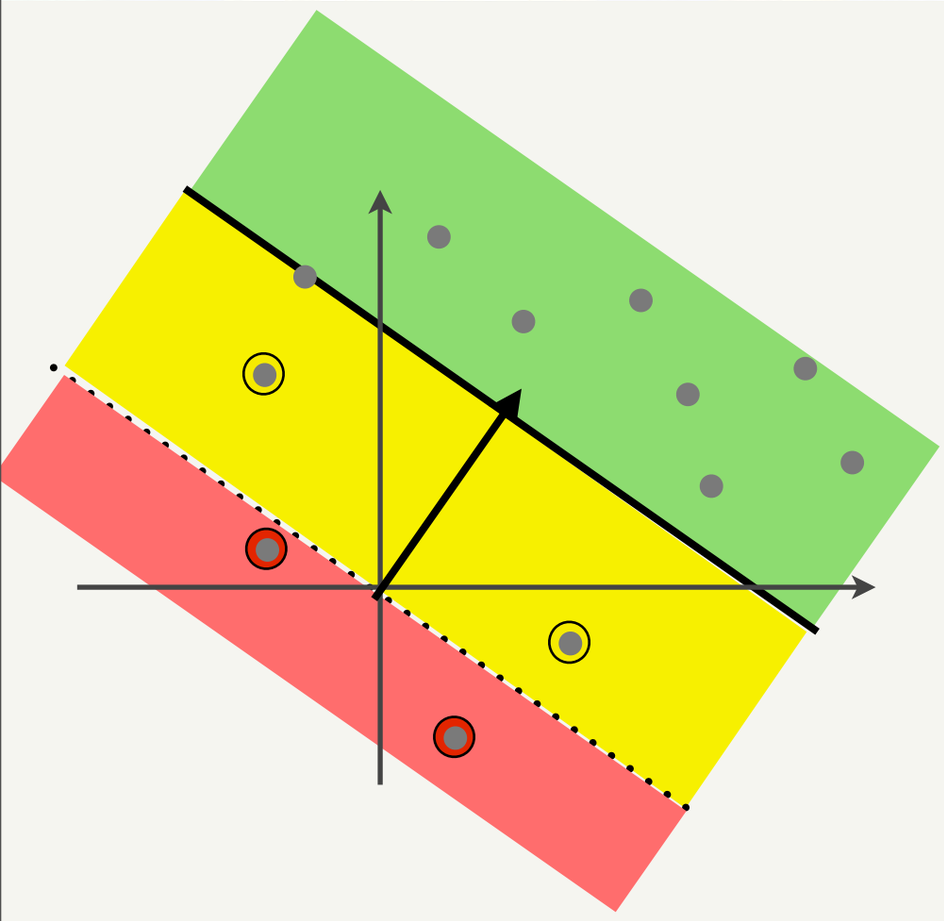

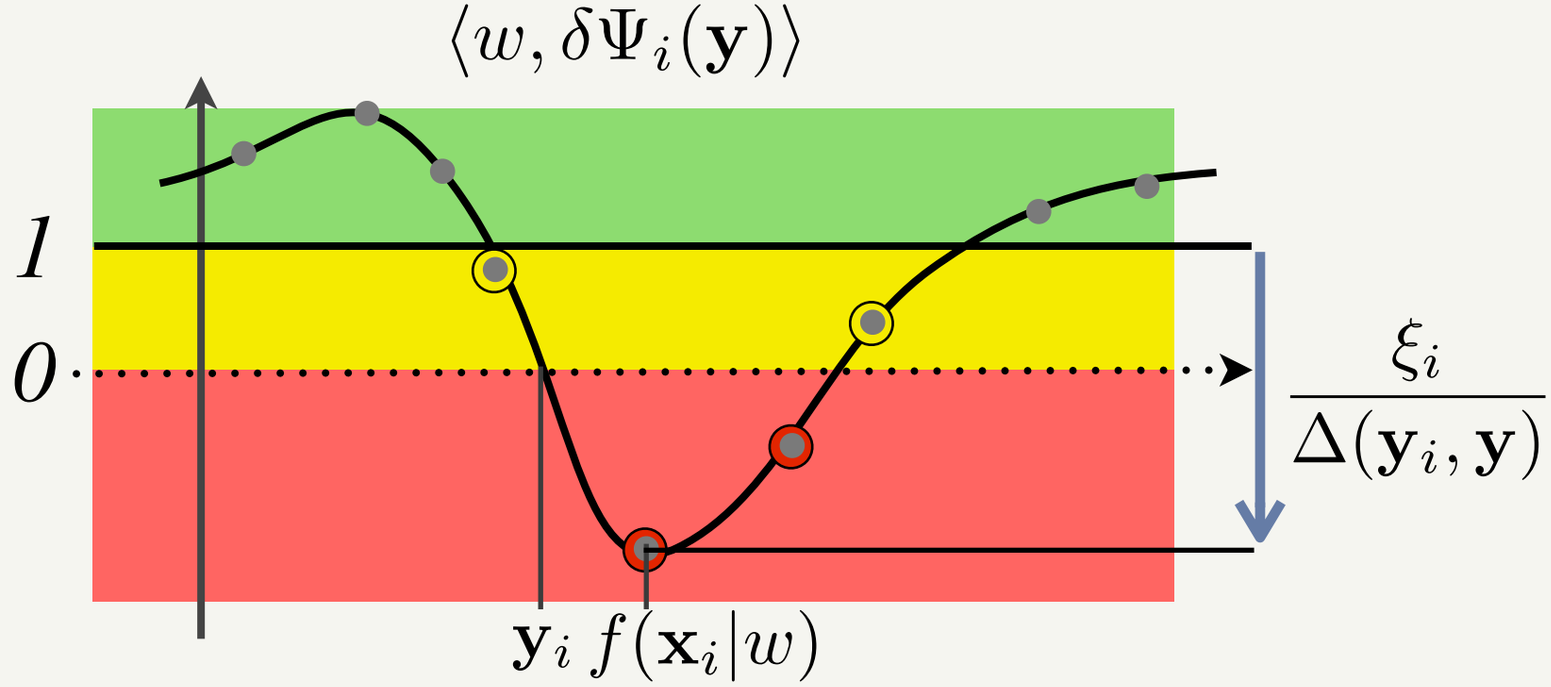

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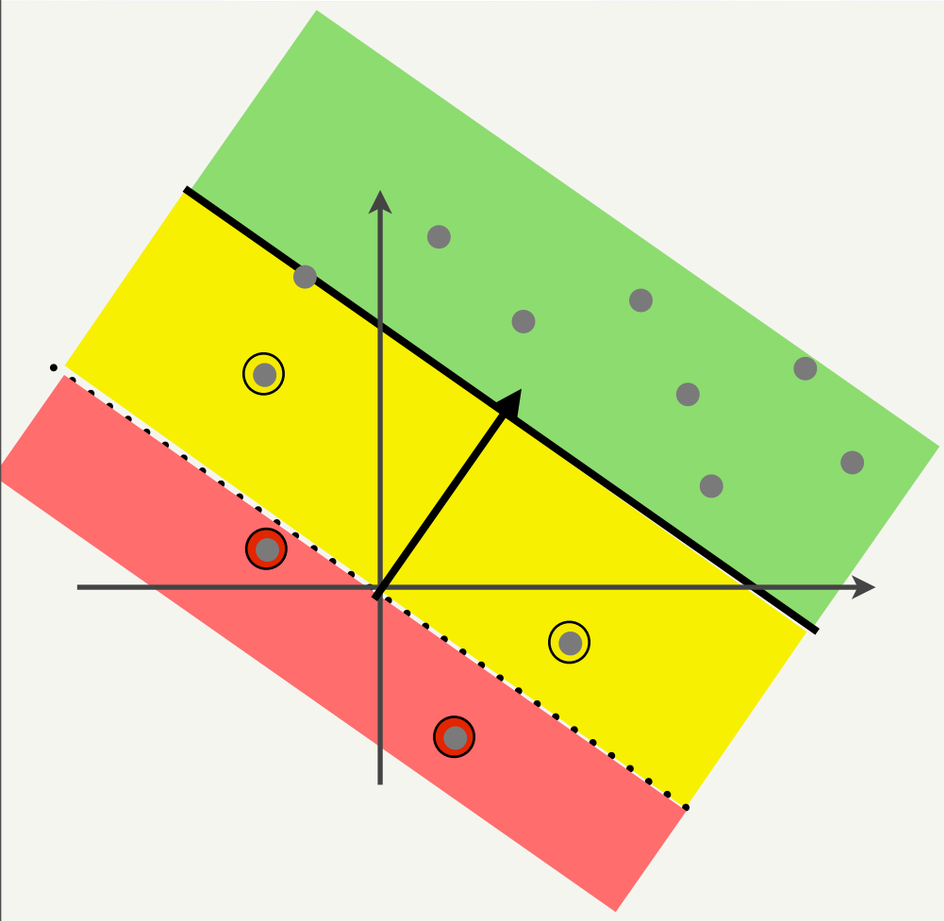

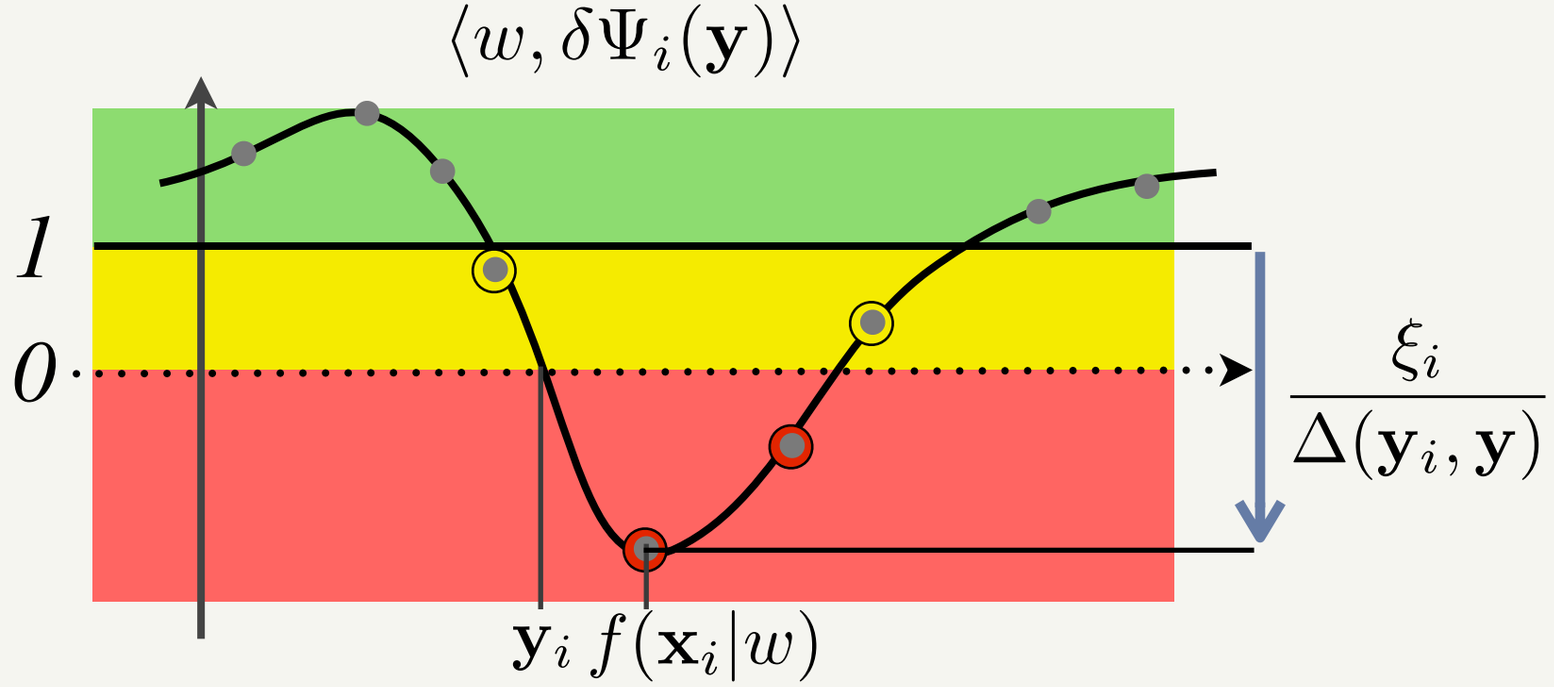

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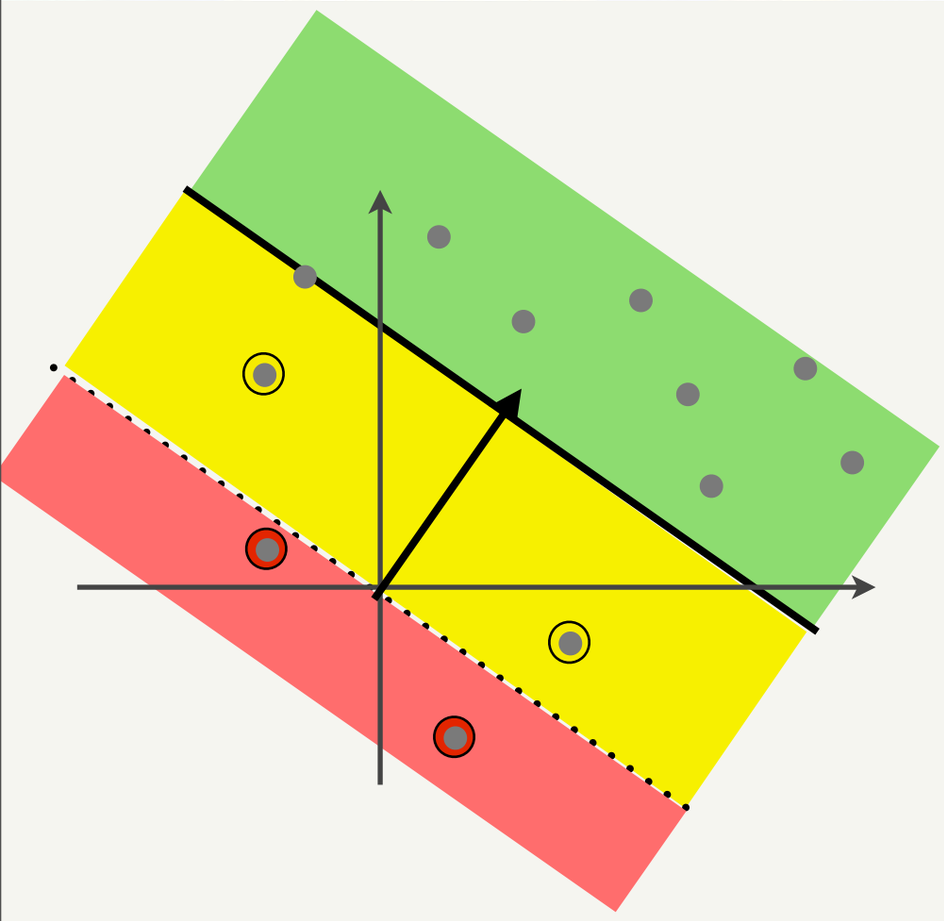

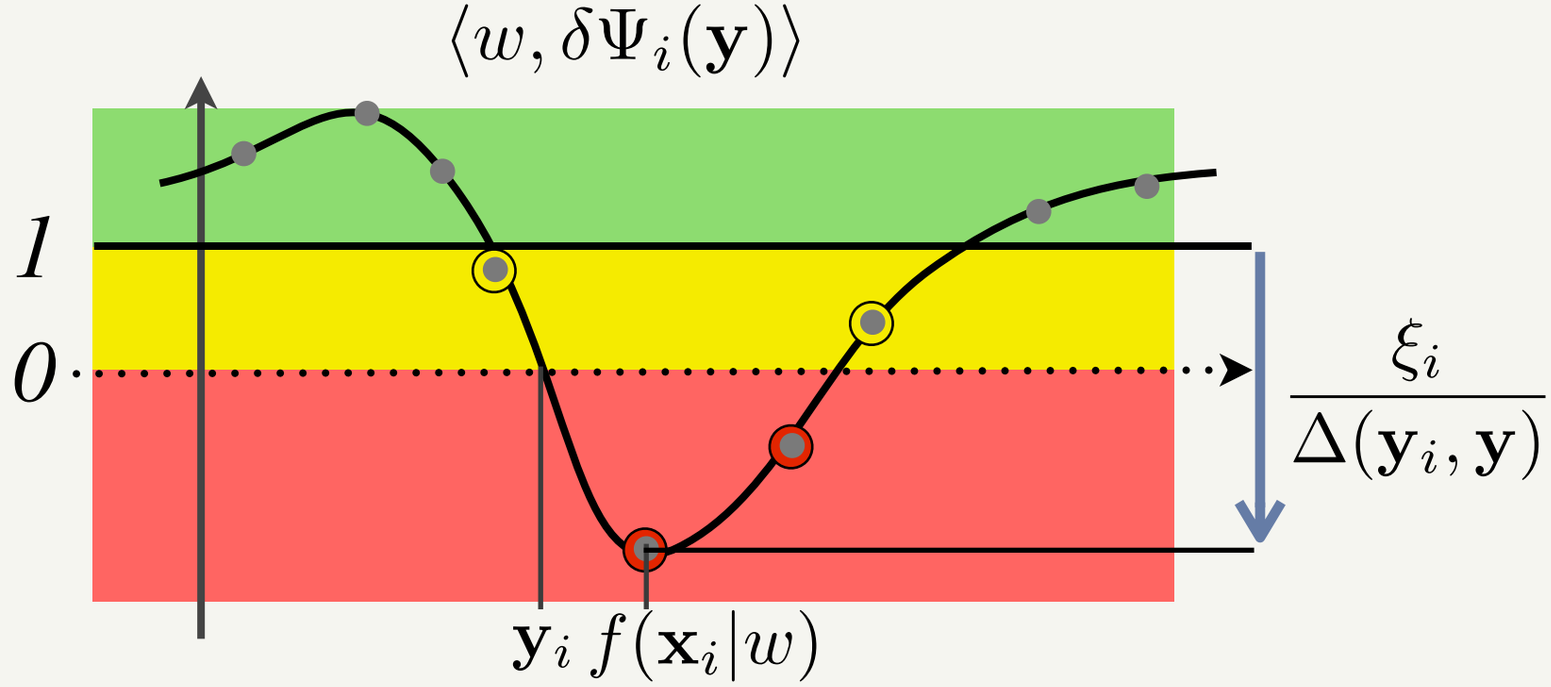

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Recap

SVM*struct* regression

$$\min \tfrac{1}{2}\|w\|^2 + \tfrac{C}{N}\sum_{i=1}^{N}\xi_i,$$
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- Max-margin: regularized (and sparse) solution
- Soft-margin: non-separable case
- One constraint for each data point $\boldsymbol{x}_i$ and output value $\mathbf{y}$
- One slack variable for each data point $\boldsymbol{x}_i$
- Minimize upper bound on empirical error
- (Generalization results)

Kernelization

- Compute a kernel $k((\mathbf{x},\mathbf{y}),(\mathbf{x}',\mathbf{y}'))$ instead of $\psi(\mathbf{x},\mathbf{y})$

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$$w = \sum_{i\mathbf{y}} c_{i\mathbf{y}} \Psi(\mathbf{x}_i, \mathbf{y})$$

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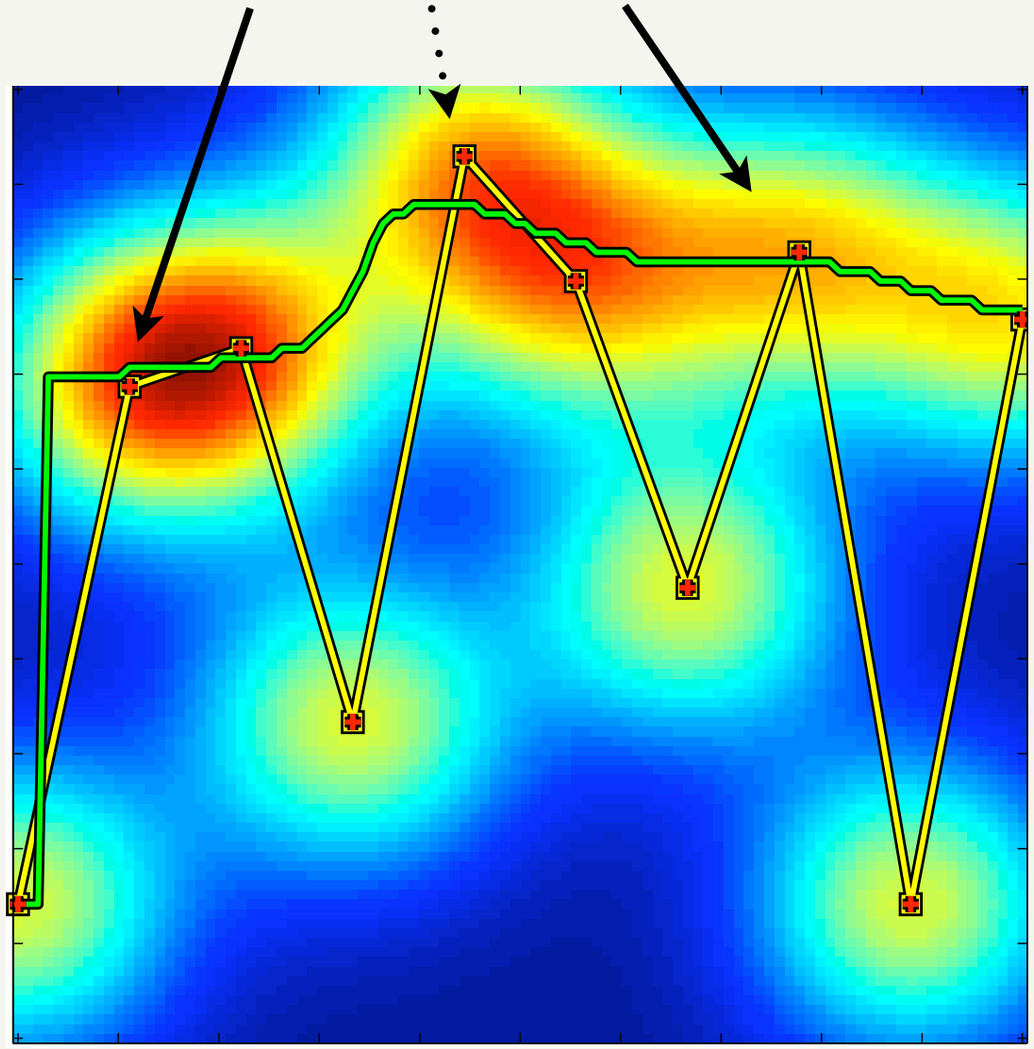
$$\sum_{j\mathbf{y}'i\mathbf{y}} c_{j\mathbf{y}'} c_{i\mathbf{y}} k(j\mathbf{y}', i\mathbf{y})$$

$$\sum_{j\mathbf{y}'} c_{j\mathbf{y}'} (k(j\mathbf{y}', i\mathbf{y}_i) - k(j\mathbf{y}', i\mathbf{y}))$$

Example: Real function

- Fit the real function to data $f(x|w) : \mathbb{R} \to \mathbb{R}$

data $(x_1, y_1), \ldots, (x_N, y_N)$



$$F(x, y|c) = \sum_{jy'} c_{jy'} k((x_j, y'), (x, y))$$

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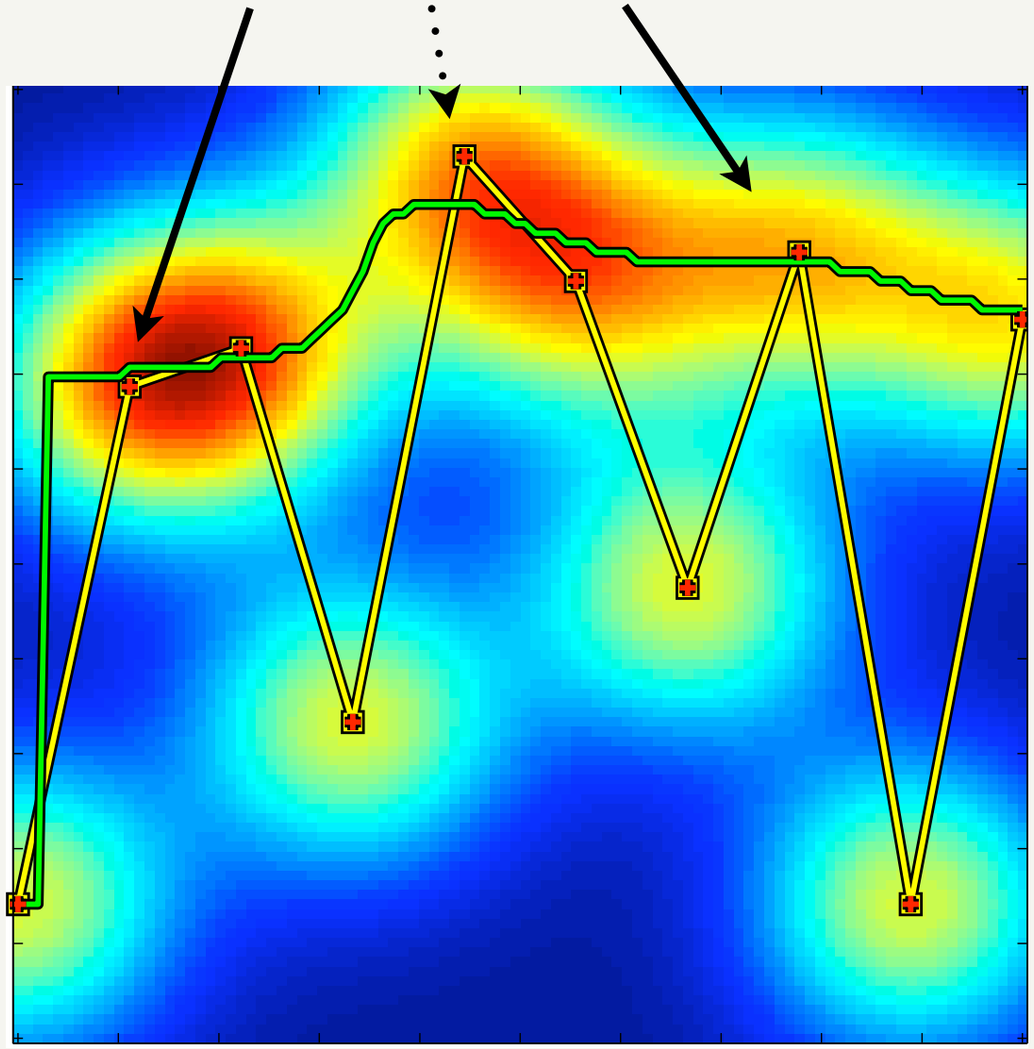

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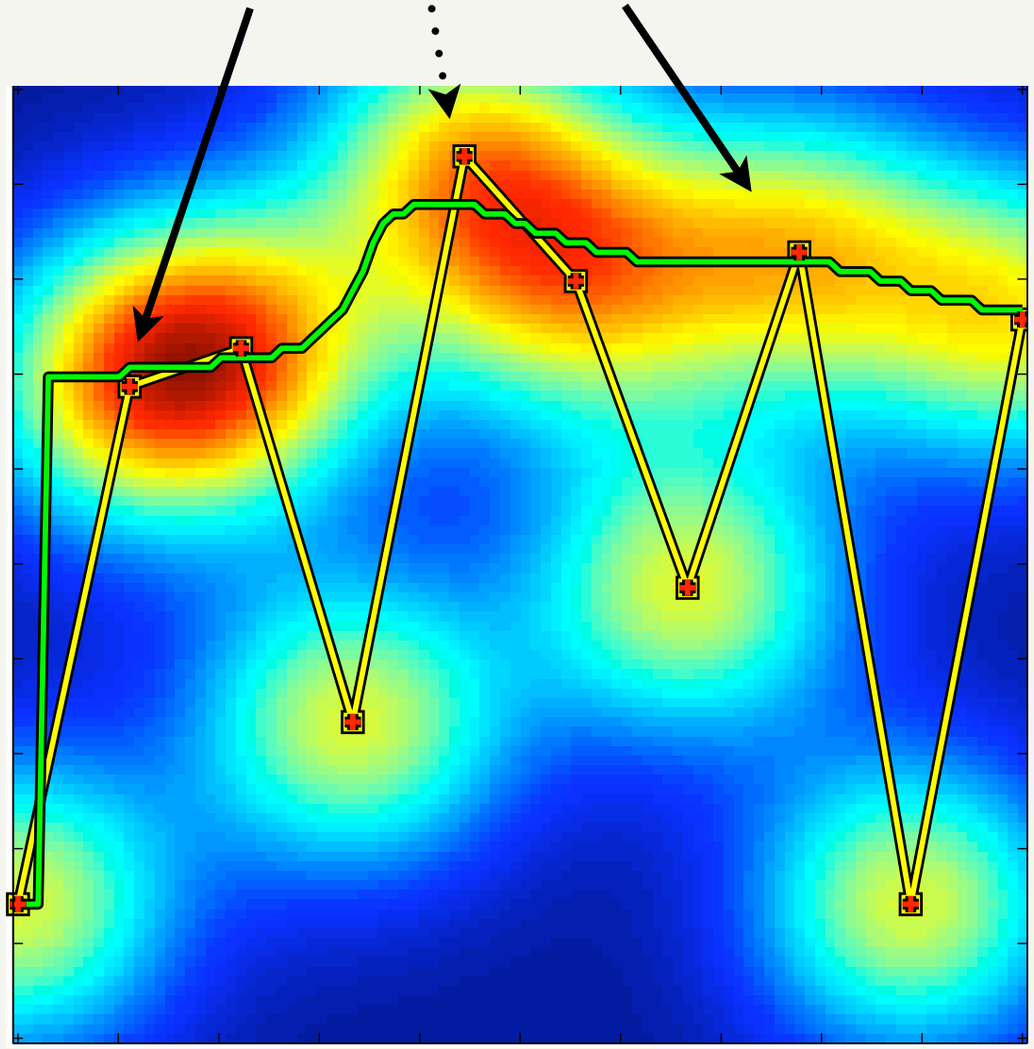

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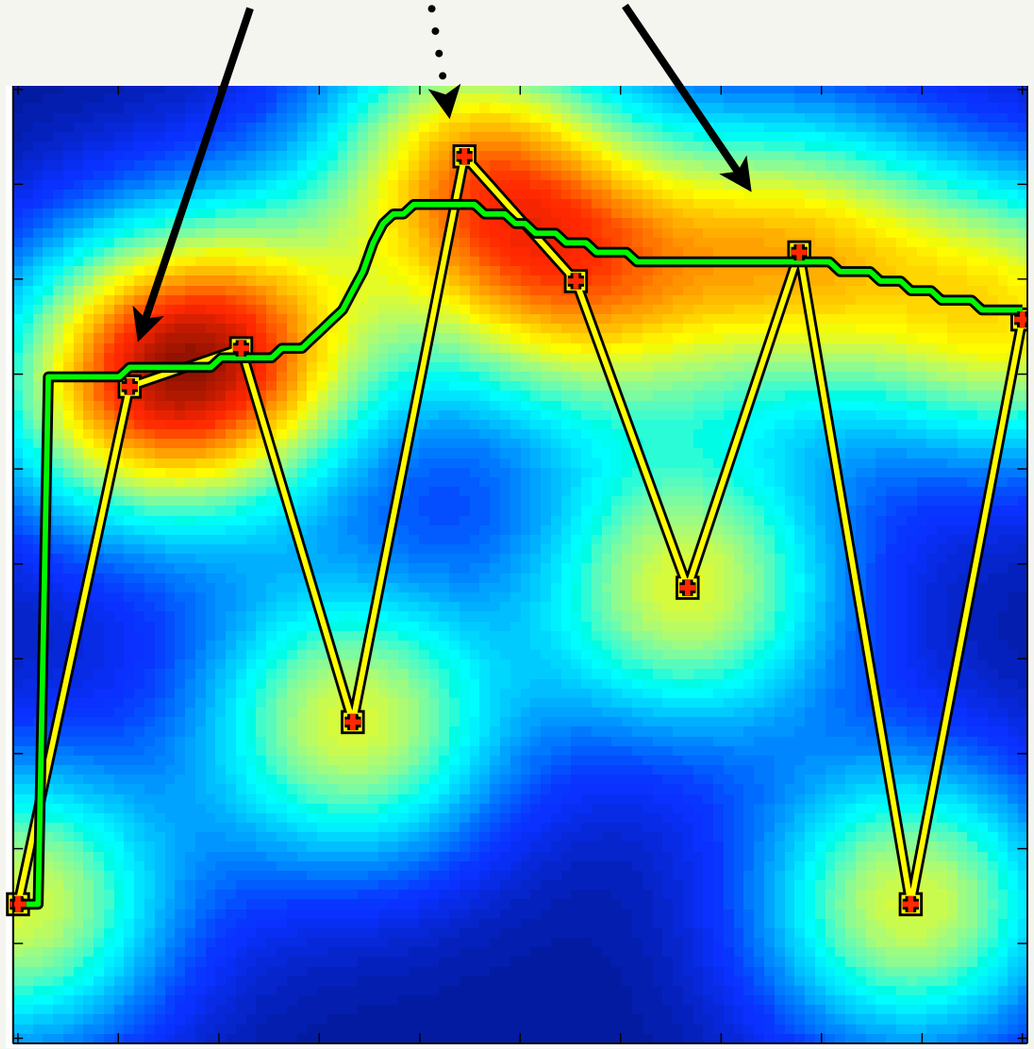

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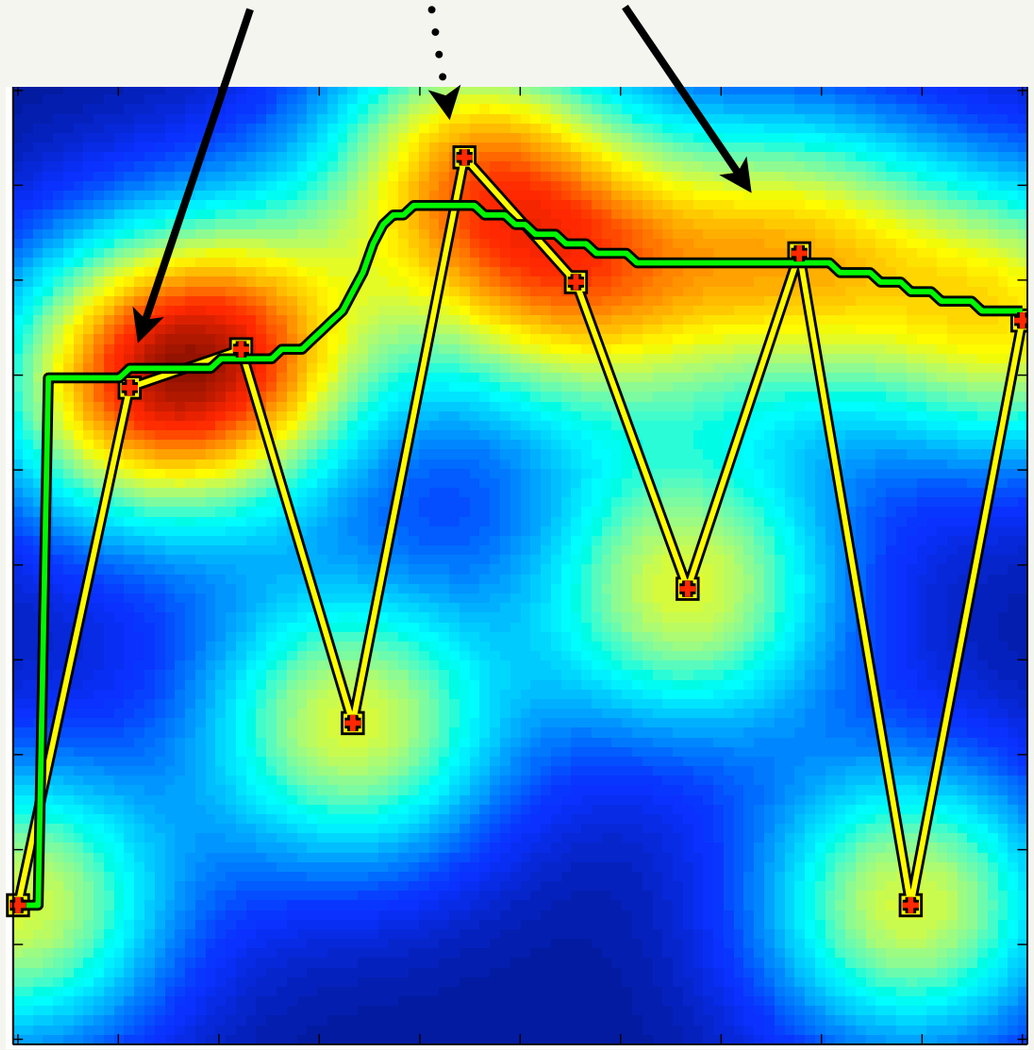

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Dual

Primal in matrix notation

$$\min \tfrac{1}{2}\|w\|^2 + \tfrac{C}{N}\sum_{i=1}^{N}\xi_i,$$
$$\xi_i \geq 0 \quad \forall i,$$
$$\langle w, \delta\Psi_i(\mathbf{y})\rangle \geq 1 - \xi_i/\Delta(\mathbf{y}_i, \mathbf{y}) \quad \forall i, \mathbf{y} \neq \mathbf{y}_i$$

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- Optimize primal variables w, ξ
  - partial = 0 yields $\mathbf{c} = E\alpha$
  - bounded minimum yields $A\alpha \leq \frac{C}{N}\mathbf{1}$

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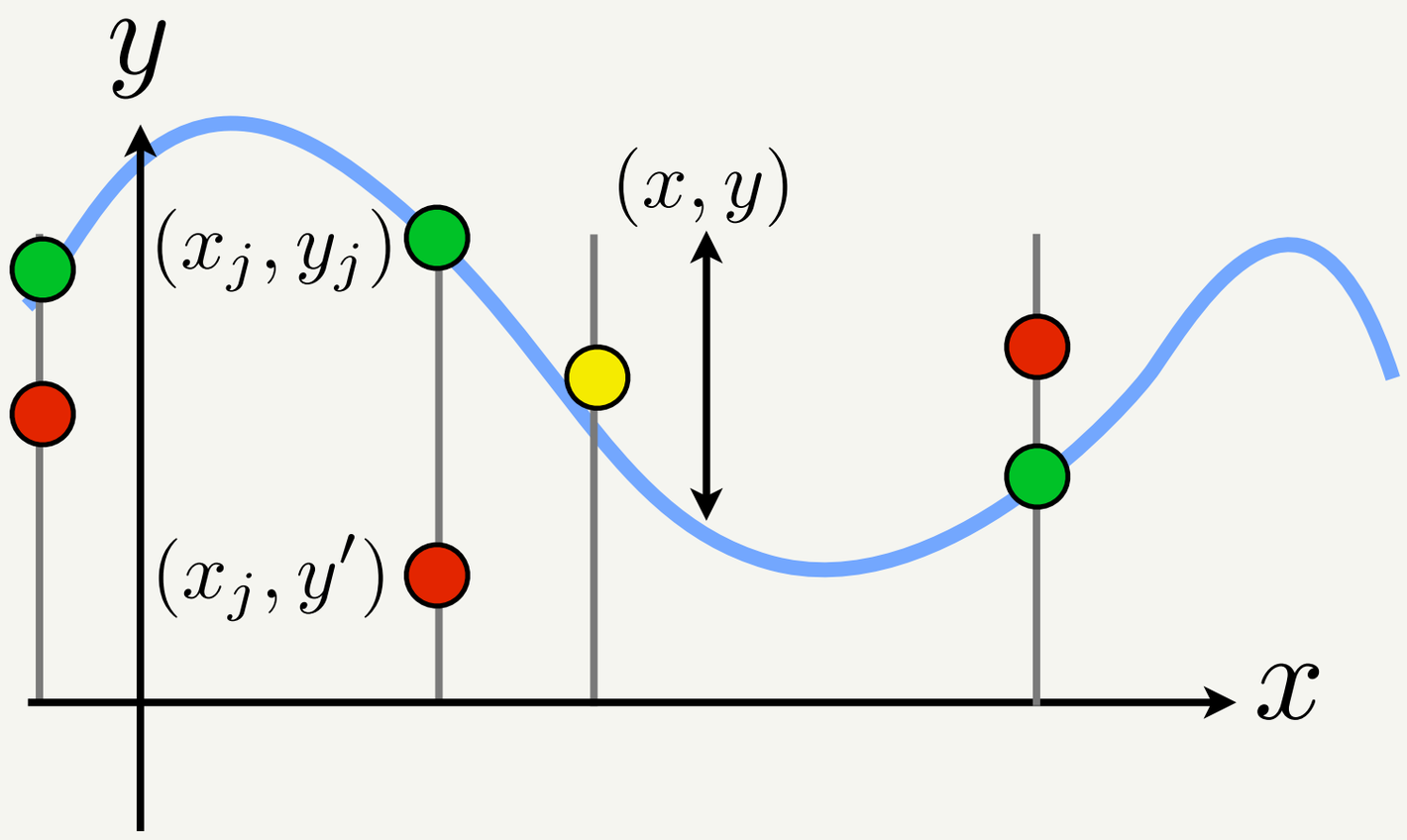

Optimization

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- Nice facts:
  - stops with an ε - optimal solution ...
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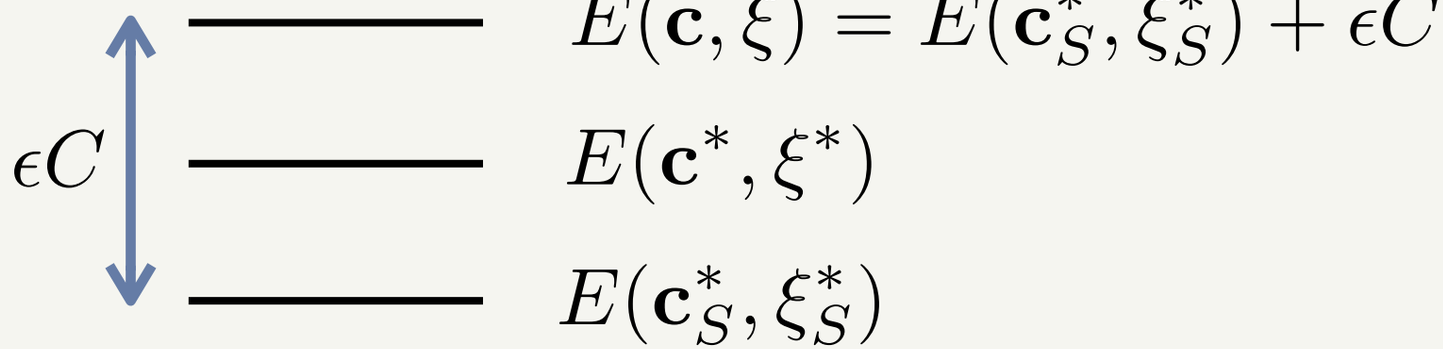

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- Stop if $\varepsilon' < \varepsilon$

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  - ***Theorem*** (convergence). Let

$$\bar{R} = \max_{i,\mathbf{y}} \|\delta\Psi_i(\mathbf{y})\| \qquad \bar{\Delta} = \max_{i,\mathbf{y}} \Delta(\mathbf{y}_i, \mathbf{y})$$

Then the algorithm stops within

$$\frac{C\bar{\Delta}^2\bar{R}^2 + N\bar{\Delta}}{\epsilon^2}$$

iterations.

Demo: Fitting a real function

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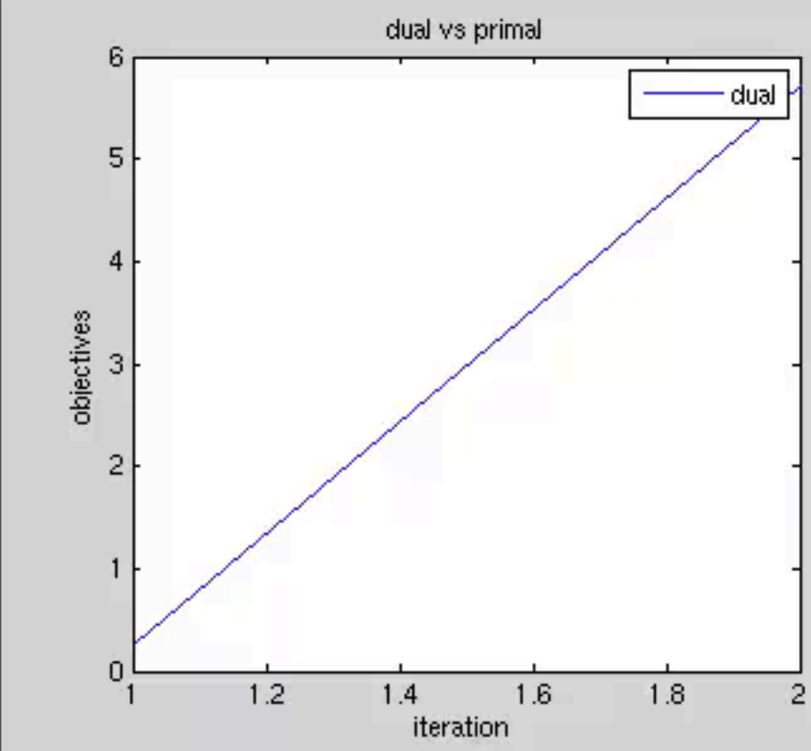
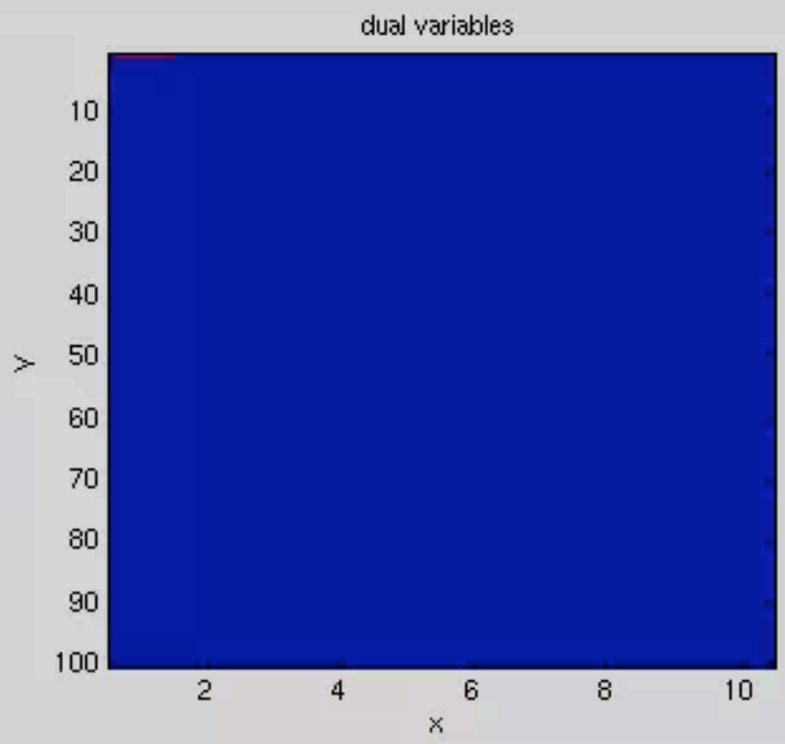
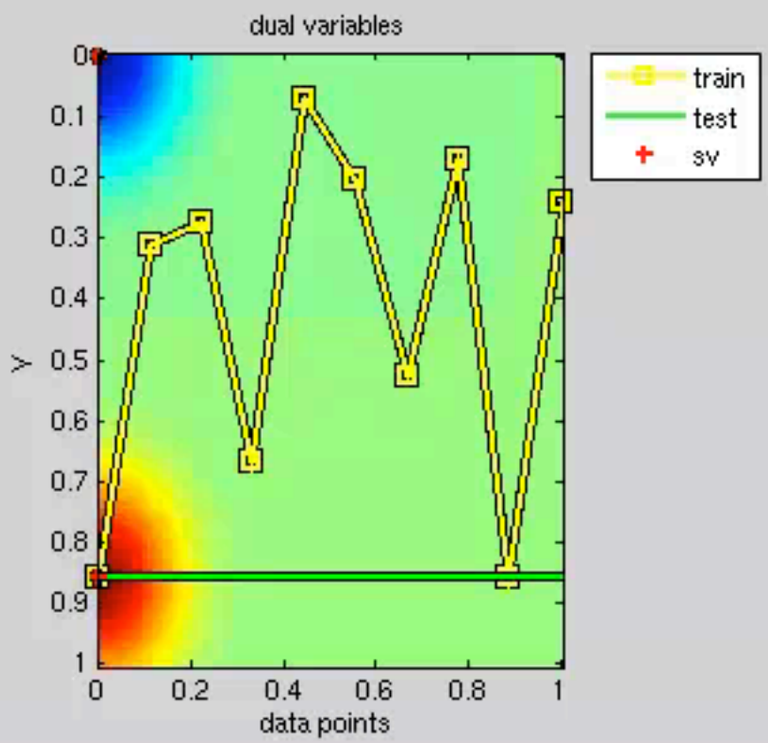
Demo: Fitting a real function

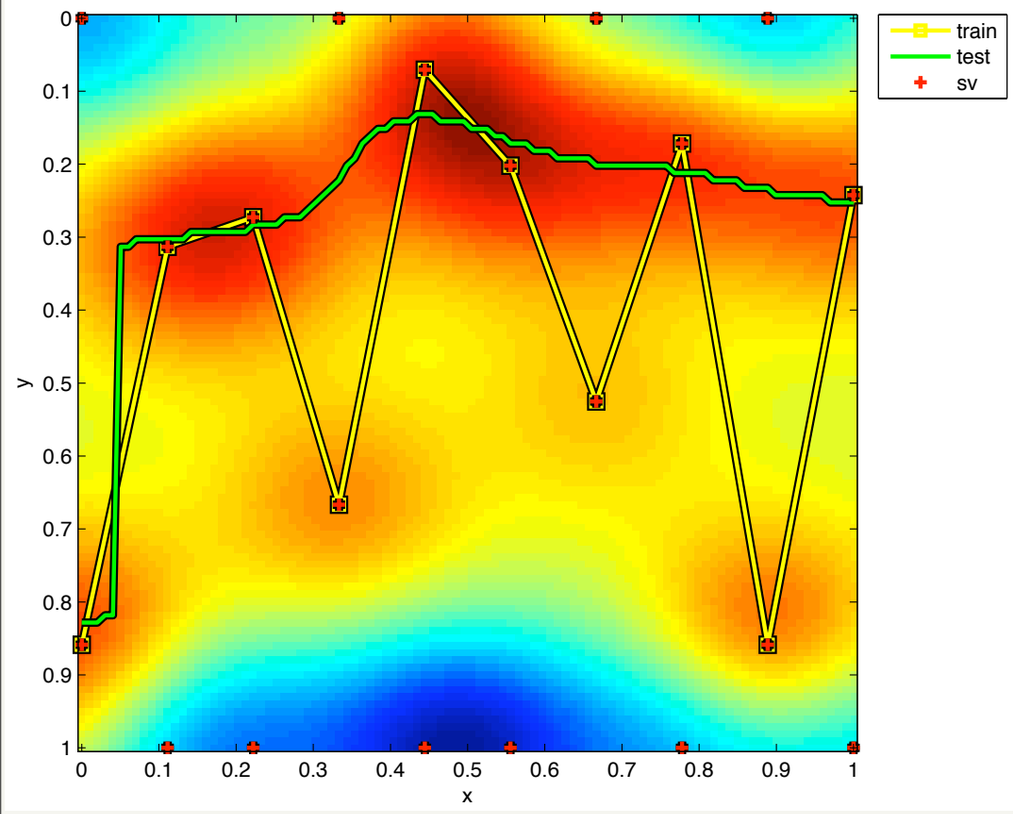
- Hunting constraints to fit a real function





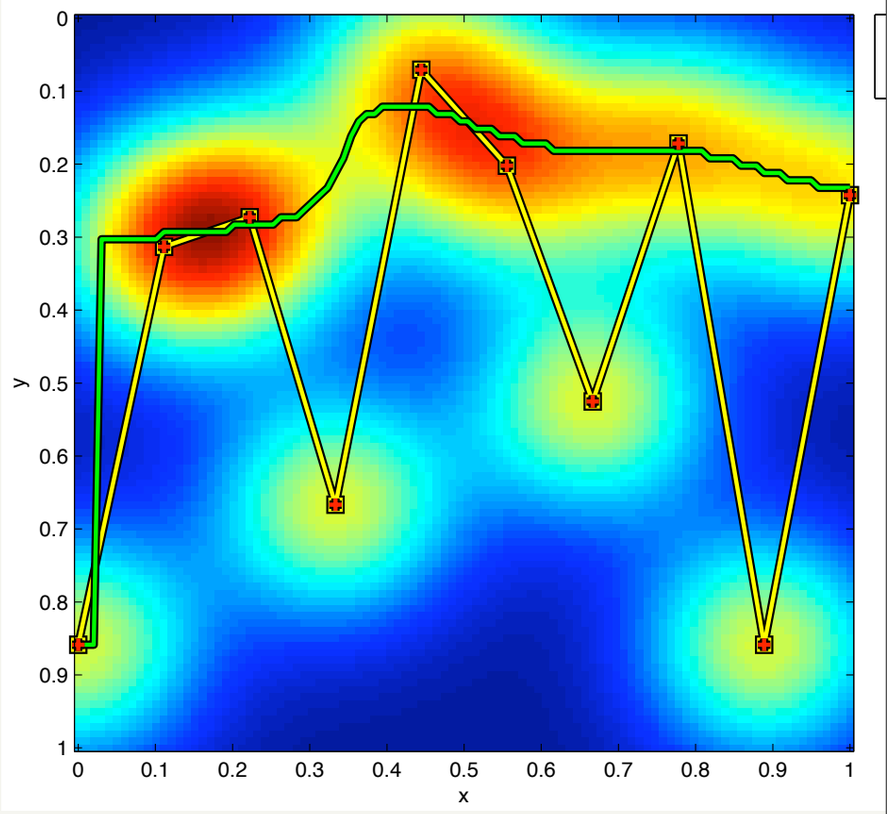


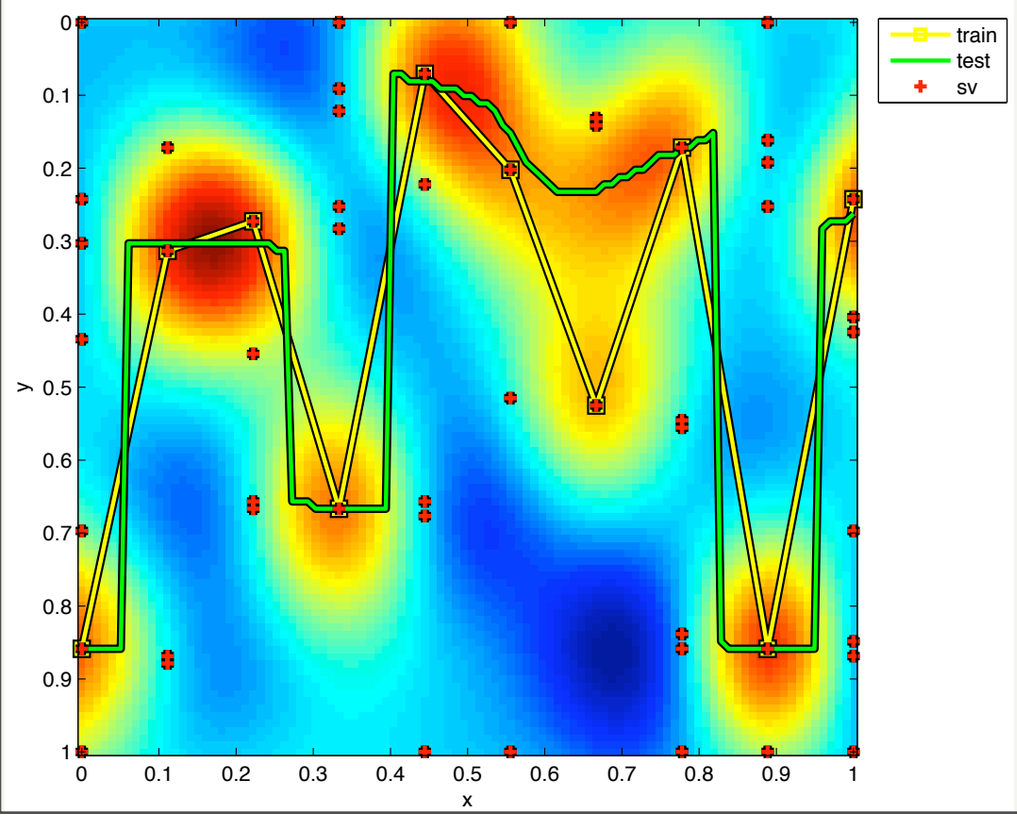
Demo: Effect of C



C=1

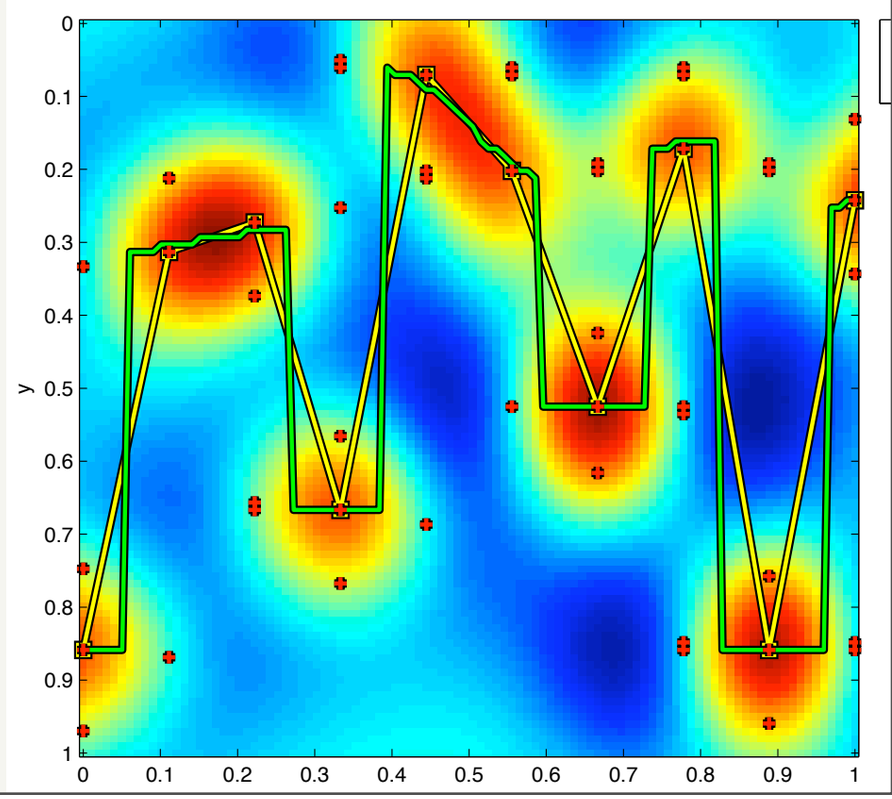
C=10





C=100

C=1000



An Application: Localization

Blaschko Lampert 08

The problem

- **Goal.** Learn a function from an image to a bounding box and label.
  - ***input*** x = image
  - ***output*** y = (label, bounding box)

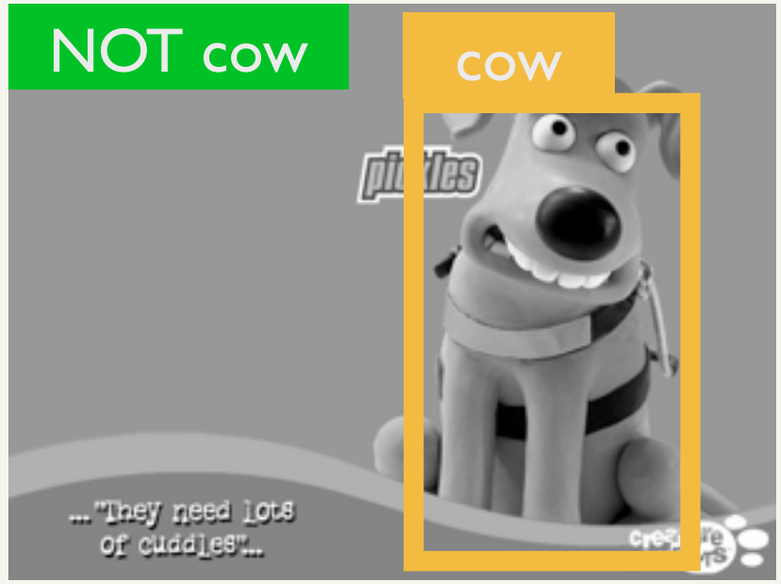
Training data

- Pairs (x_1, y_1), ..., (x_N, y_N) :

  (image, label + bounding box)

Loss

$\Delta(\mathbf{y}, \hat{\mathbf{y}})$ = overlap err.

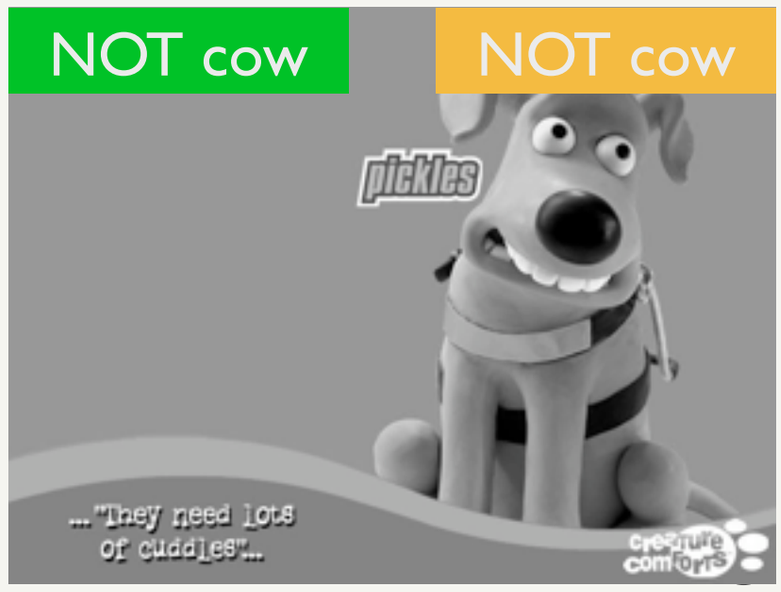


$\Delta(\mathbf{y}, \hat{\mathbf{y}}) = 1$

$$\Delta(\mathbf{y}, \hat{\mathbf{y}})$$

$\Delta(\mathbf{y}, \hat{\mathbf{y}}) = 1$

$\Delta(\mathbf{y}, \hat{\mathbf{y}}) = 0$



Kernel

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- Intuition:
  $x \approx x'$ and $y \approx y'$ then $k((x,y), (x', y'))$ large

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$$k((x,y),(x',y')) = \begin{cases} 0 & \text{if } \mathrm{lab}(y) = \mathrm{NOT} \\ & \text{or } \mathrm{lab}(y') = \mathrm{NOT} \end{cases}$$

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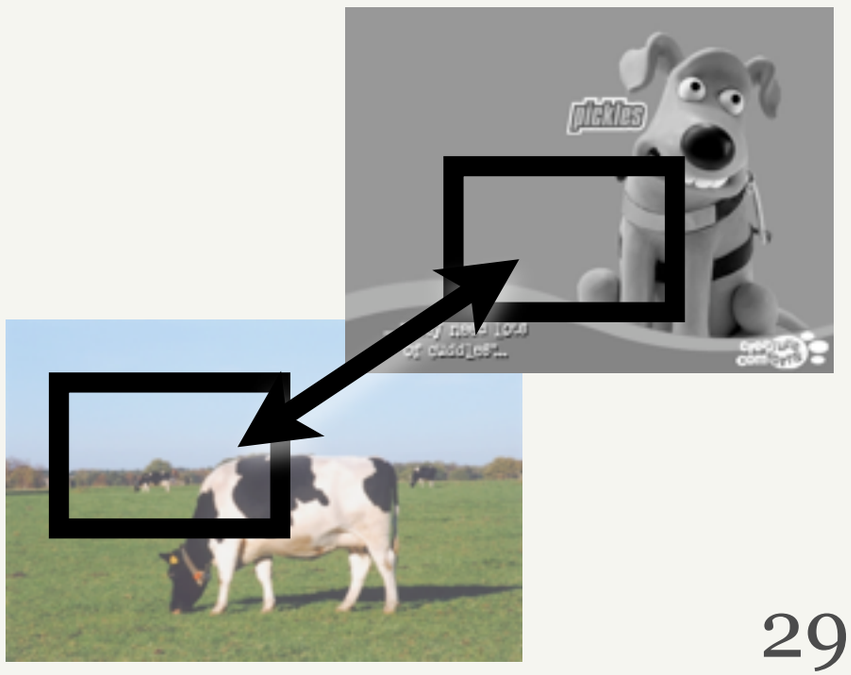
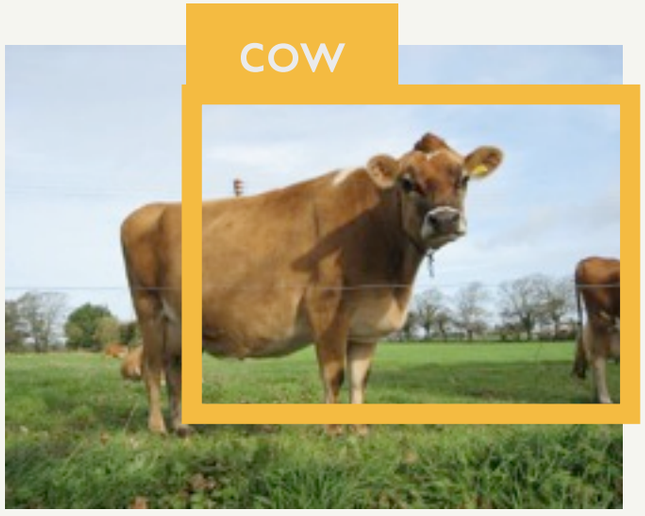
$$k((x,y), (x', y')) = \begin{cases} 0 & \text{if } \mathrm{lab}(y) = \mathrm{NOT} \text{ or } \mathrm{lab}(y') = \mathrm{NOT} \\ k(crop(x|y), crop(x'|y')) & \text{if } \mathrm{lab}(y) = \mathrm{lab}(y') = \mathrm{COW} \end{cases}$$

"image patch similarity"



SVM evaluation

Hypothesis



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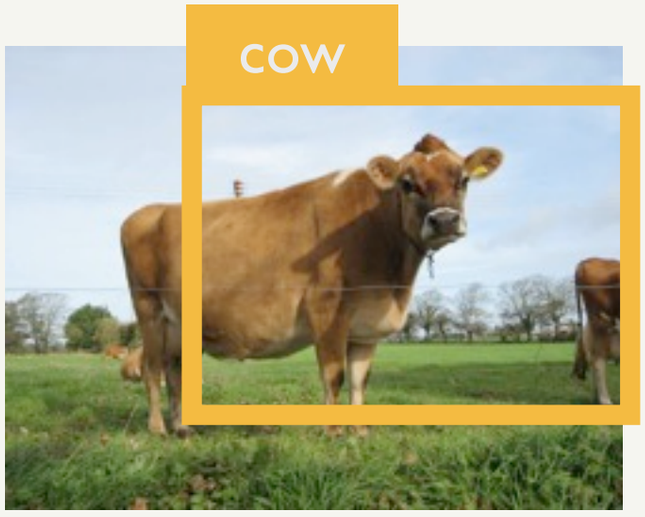
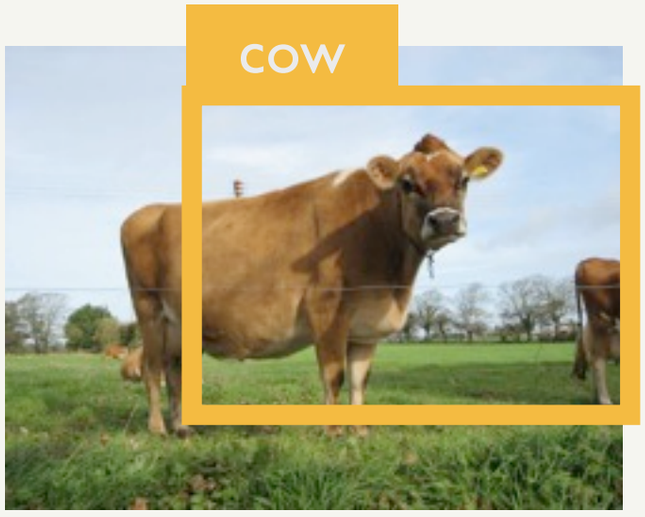


Scoring

SVM evaluation

Hypothesis



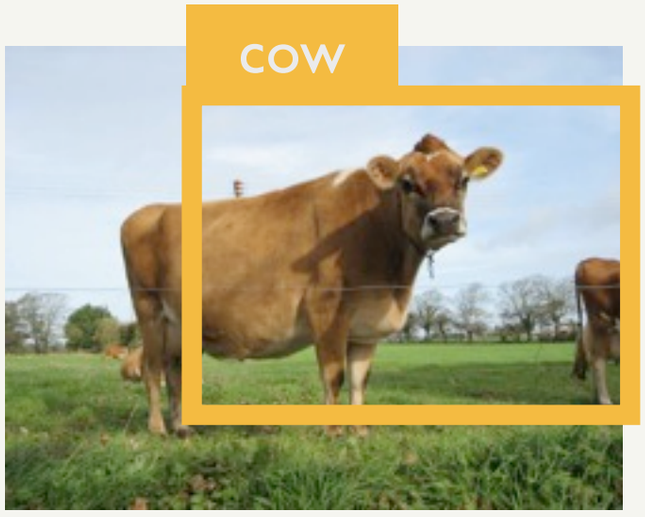
Scoring

$$F(\boldsymbol{x},\boldsymbol{y}|w) = 0$$

"do nothing"

SVM evaluation

Hypothesis



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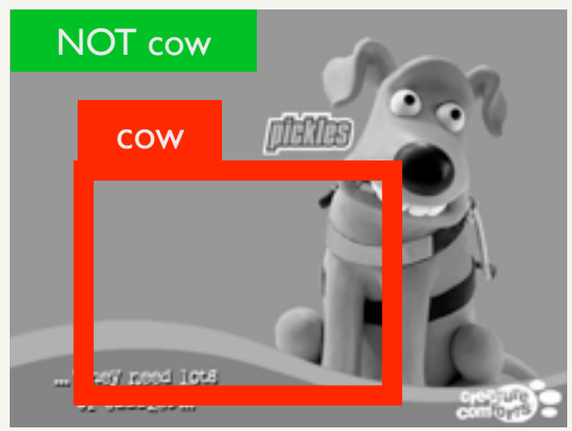

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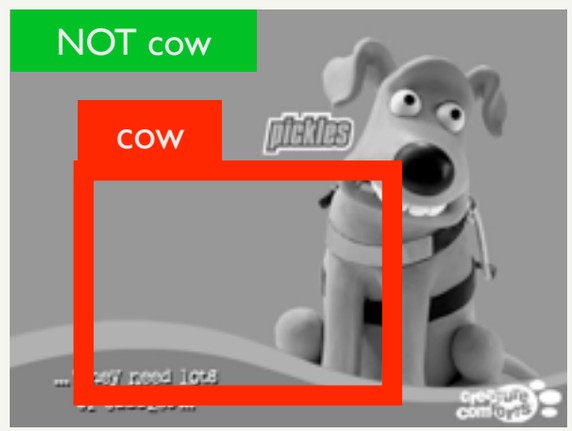

Return best hypothesis

Most violated constraints

$$max\{slack \times \Delta(y,y')\}$$

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new constraint

Most violated constraints

$$max\{slack \times \Delta(y,y')\}$$

new constraint

- Finding most violated constraints
  - similar to evaluating the SVM
  - scoring weighed by $\Delta(y,y')$
  - repeated many times
    - needs a fast search method (e.g. branch and bound)

M^3 networks

Taskar Guestrin Koller 03

M³ network

M³ network

- Structured SVM entails a large number of constraints
  - So far, handled by adding one constraint per time

M^3 network

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- ***M^3 network*** is a graphical model
  - Markov network (encodes indep. relations)
  - Inference in structured output framework
  - Reduction of exp to poly number of constraints

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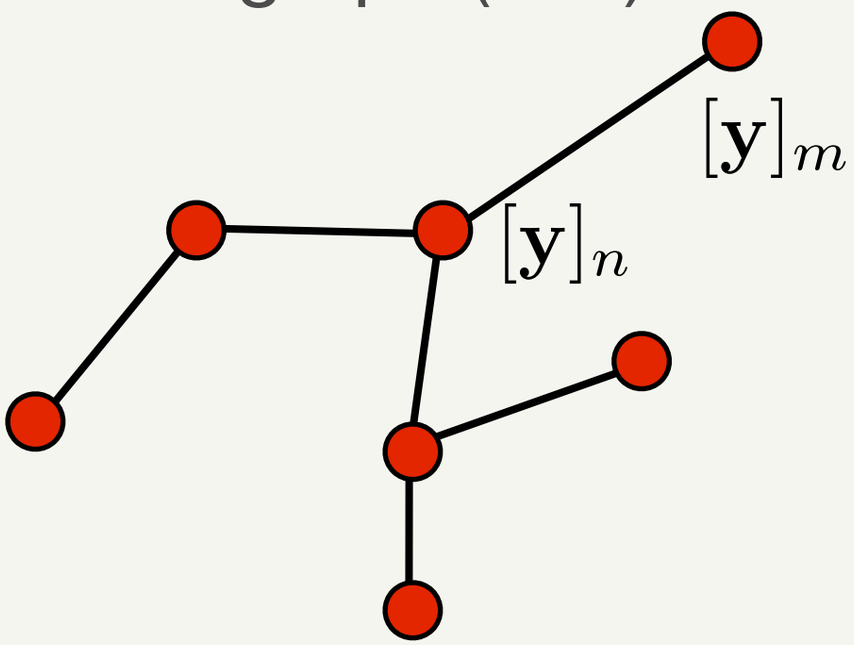
- ***M^3 network*** is a graphical model
  - Markov network (encodes indep. relations)
  - Inference in structured output framework
  - Reduction of exp to poly number of constraints
- *M^3 network* = Max-Margin Markov network

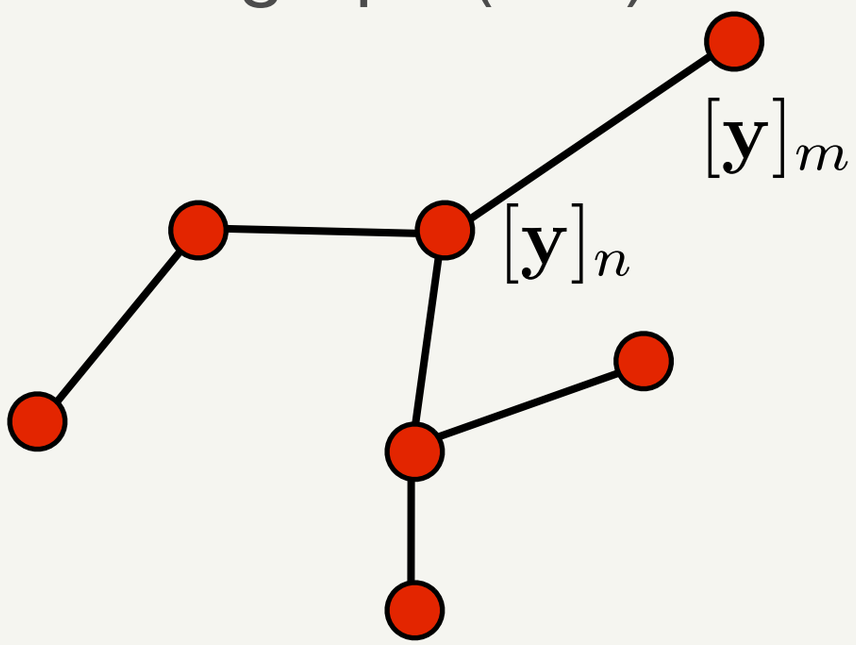
M^3 Model

graph (tree)



M^3 Model

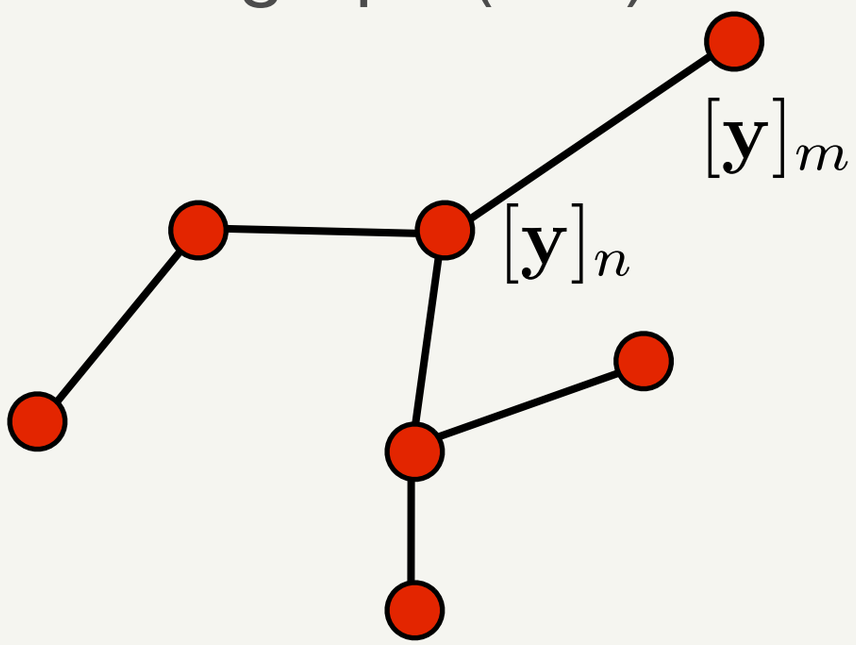
graph (tree)



State $\mathbf{y} = \begin{bmatrix} 1 & 1 & 0 & 1 & \dots & 0 \end{bmatrix}$

M^3 Model

graph (tree)



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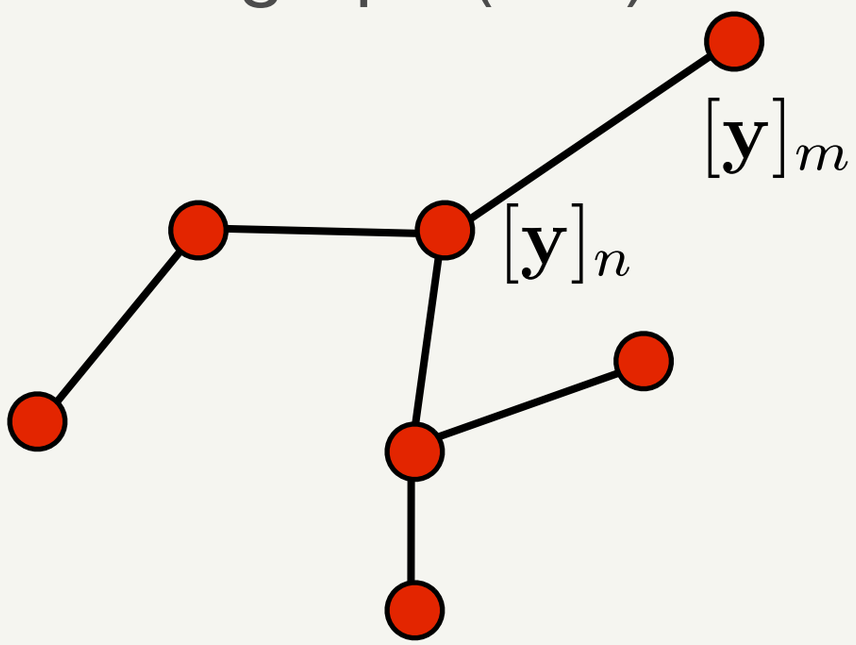

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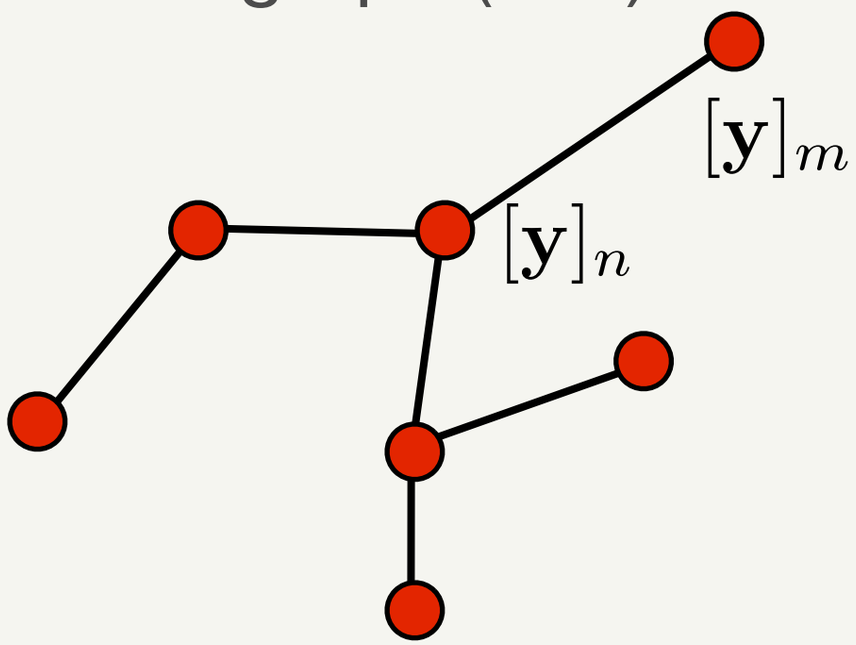

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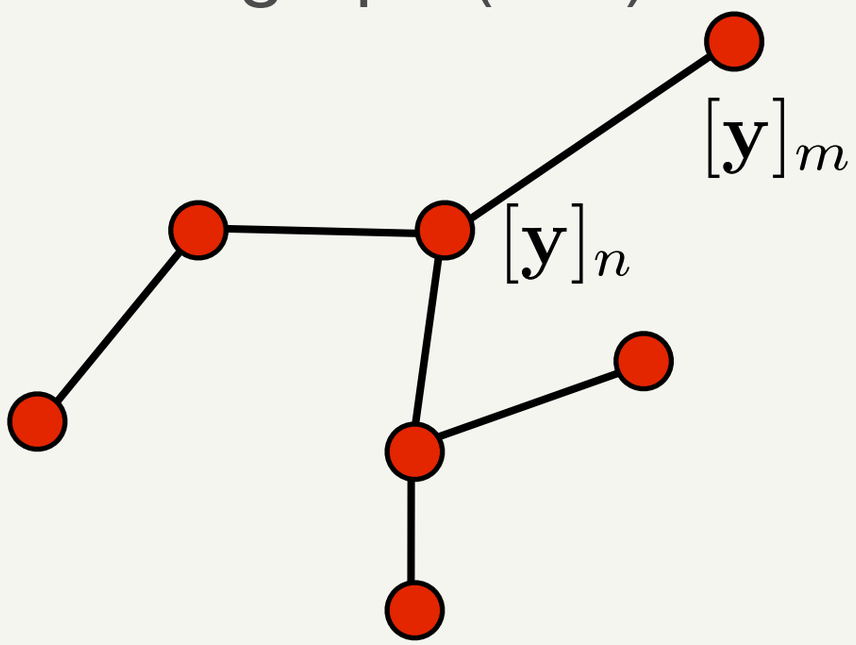

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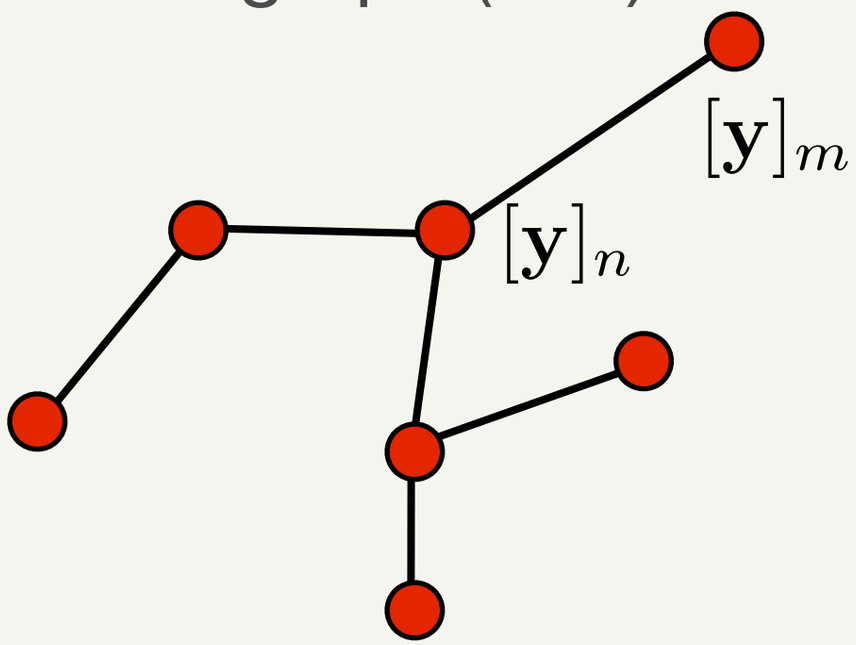

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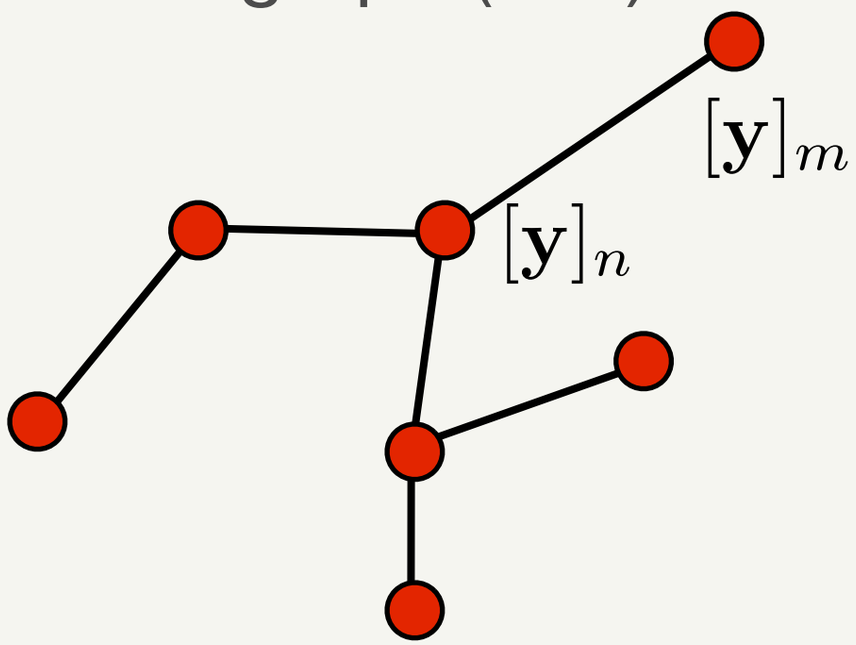

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Node n state $[\mathbf{y}]_n$

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$$\begin{aligned} &\max \mathbf{1}^\top \alpha - \tfrac{1}{2}\alpha^\top E^\top K E \alpha \\ &\alpha \geq 0, \\ &A\alpha \leq \tfrac{C}{N}\mathbf{1} \end{aligned}$$

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  - total: $N \times 2^M$

Marginalizing dual variables

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$$A = \begin{bmatrix} \dots & \Delta(\mathbf{y}_1\mathbf{y})^{-1} & \dots & & \dots 0 \dots \\ & & \ddots & & \\ & \dots 0 \dots & & \dots & \Delta(\mathbf{y}_N\mathbf{y})^{-1} & \dots \end{bmatrix}$$

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$$\begin{aligned}\mathbf{1}^\top\alpha &= \sum_i \sum_{\mathbf{y}} \Delta(\mathbf{y}_i,\mathbf{y})\beta_{i\mathbf{y}} \\ &= \sum_i \sum_{\mathbf{y}} \sum_n \Delta([\mathbf{y}_i]_n,[\mathbf{y}]_n)\beta_{i\mathbf{y}} \\ &= \sum_i \sum_n \sum_{\mathbf{y}} \Delta([\mathbf{y}_i]_n,[\mathbf{y}]_n)\beta_{i\mathbf{y}} \\ &= \sum_i \sum_n \sum_{[\mathbf{y}]_n} \Delta([\mathbf{y}_i]_n,[\mathbf{y}]_n)\mu_{i[\mathbf{y}]_n}\end{aligned}$$

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$$A = \begin{bmatrix} \dots & \Delta(\mathbf{y}_1\mathbf{y})^{-1} & \dots & \dots 0 \dots \\ & & \ddots & \\ \dots 0 \dots & & \dots & \Delta(\mathbf{y}_N\mathbf{y})^{-1} & \dots \end{bmatrix}$$

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$$\begin{aligned} \mathbf{1}^\top \alpha &= \sum_i \sum_{\mathbf{y}} \Delta(\mathbf{y}_i, \mathbf{y}) \beta_{i\mathbf{y}} \\ &= \sum_i \sum_{\mathbf{y}} \sum_n \Delta([\mathbf{y}_i]_n, [\mathbf{y}]_n) \beta_{i\mathbf{y}} \\ &= \sum_i \sum_n \sum_{\mathbf{y}} \Delta([\mathbf{y}_i]_n, [\mathbf{y}]_n) \beta_{i\mathbf{y}} \\ &= \sum_i \sum_n \sum_{[\mathbf{y}]_n} \Delta([\mathbf{y}_i]_n, [\mathbf{y}]_n) \mu_{i[\mathbf{y}]_n} \end{aligned}$$

$$\mu_{i[\mathbf{y}]_n} = \sum_{\mathbf{y} \sim [\mathbf{y}]_n} \beta_{i\mathbf{y}}$$

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  - compute $1\times$ and $2\times$ marginal variables

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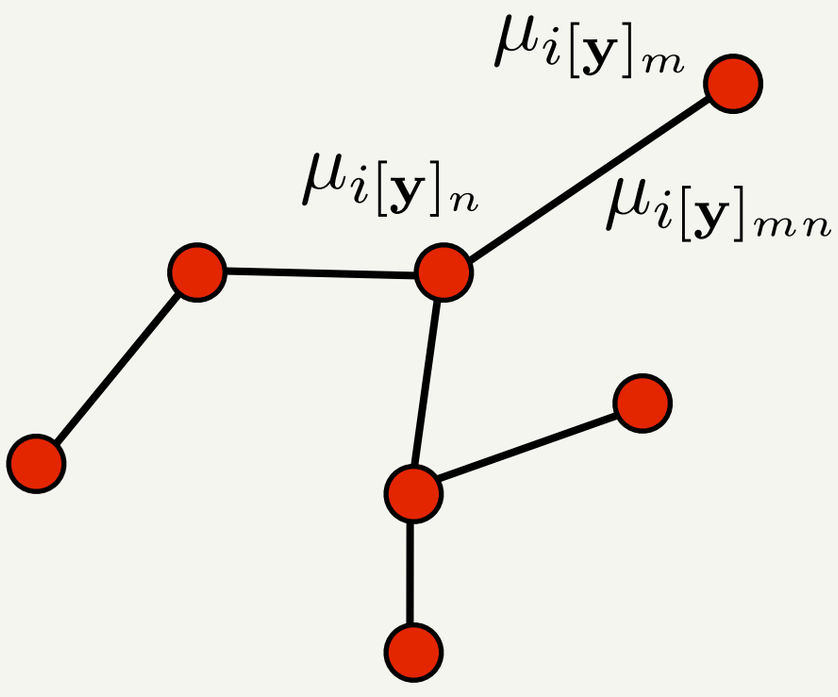

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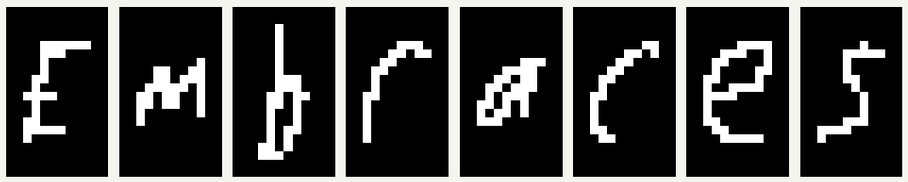
- If not a tree the simplified dual is a relaxation

Example application

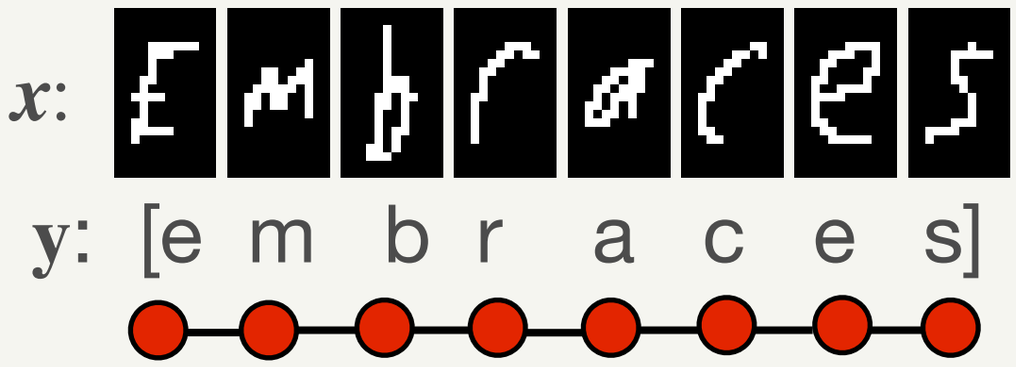
x: 

y: [e m b r a c e s]

$[x]_n$: pixel data for one characters

$[y]_n$: one of {a,b,c, ..., z}

Example application



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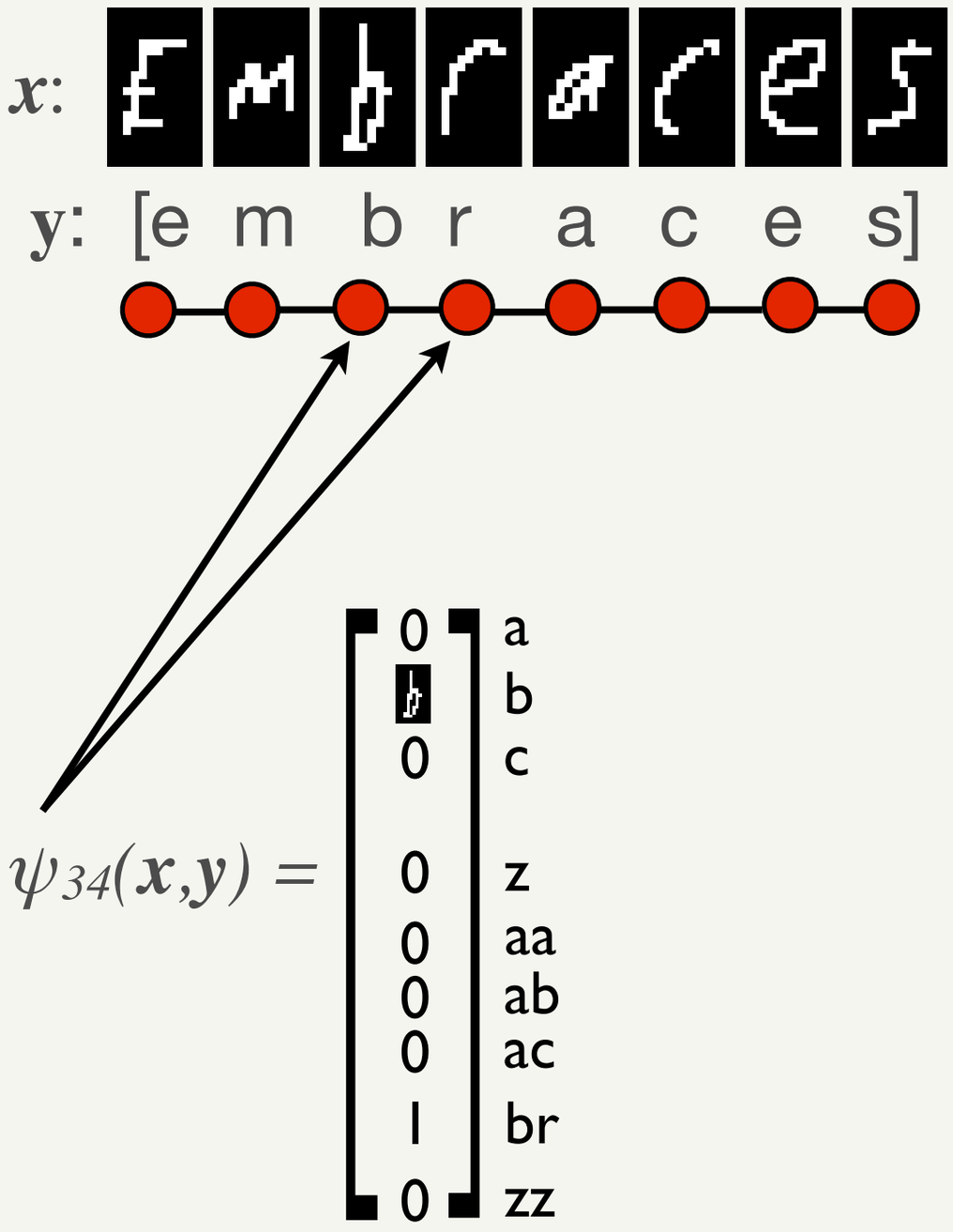

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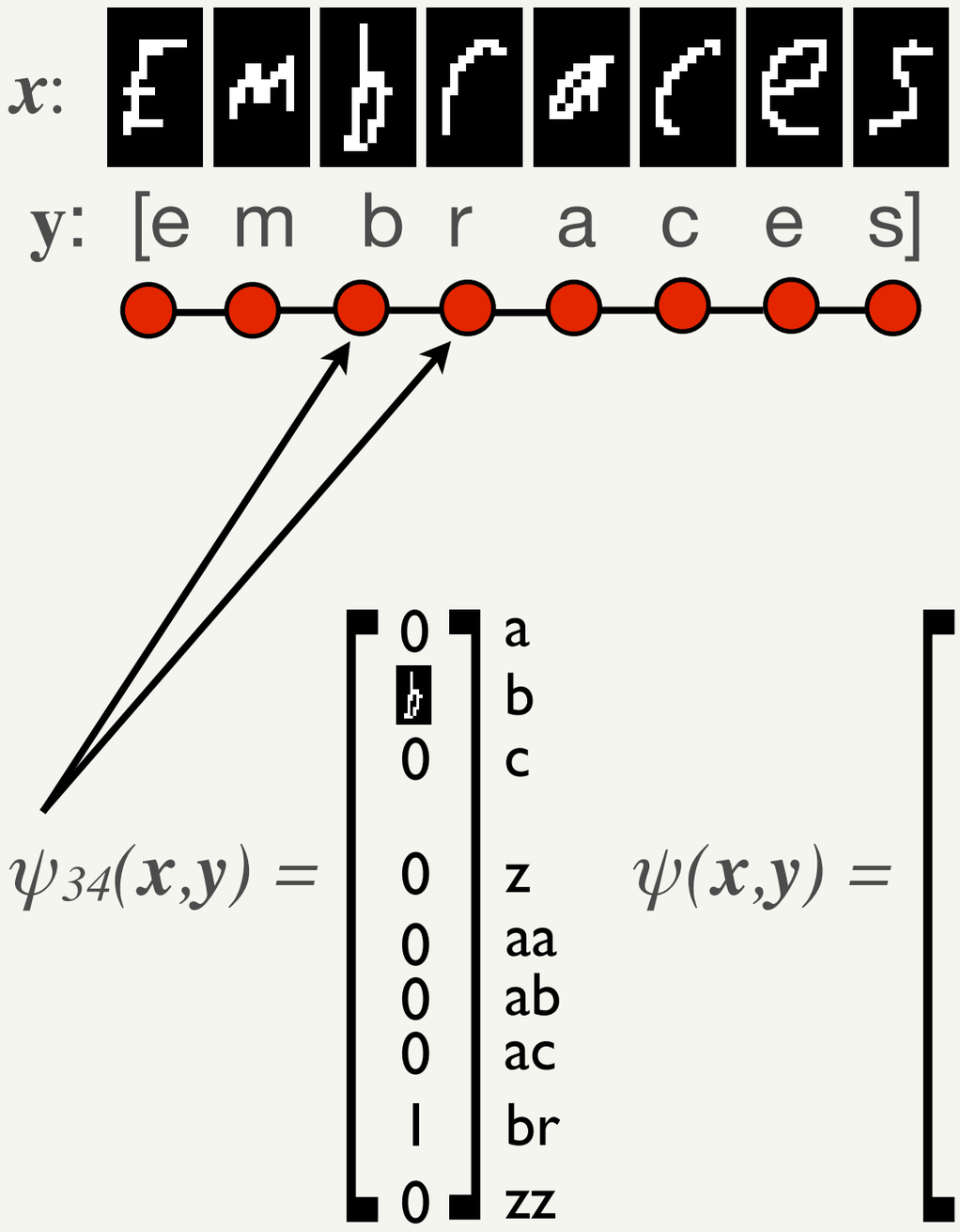

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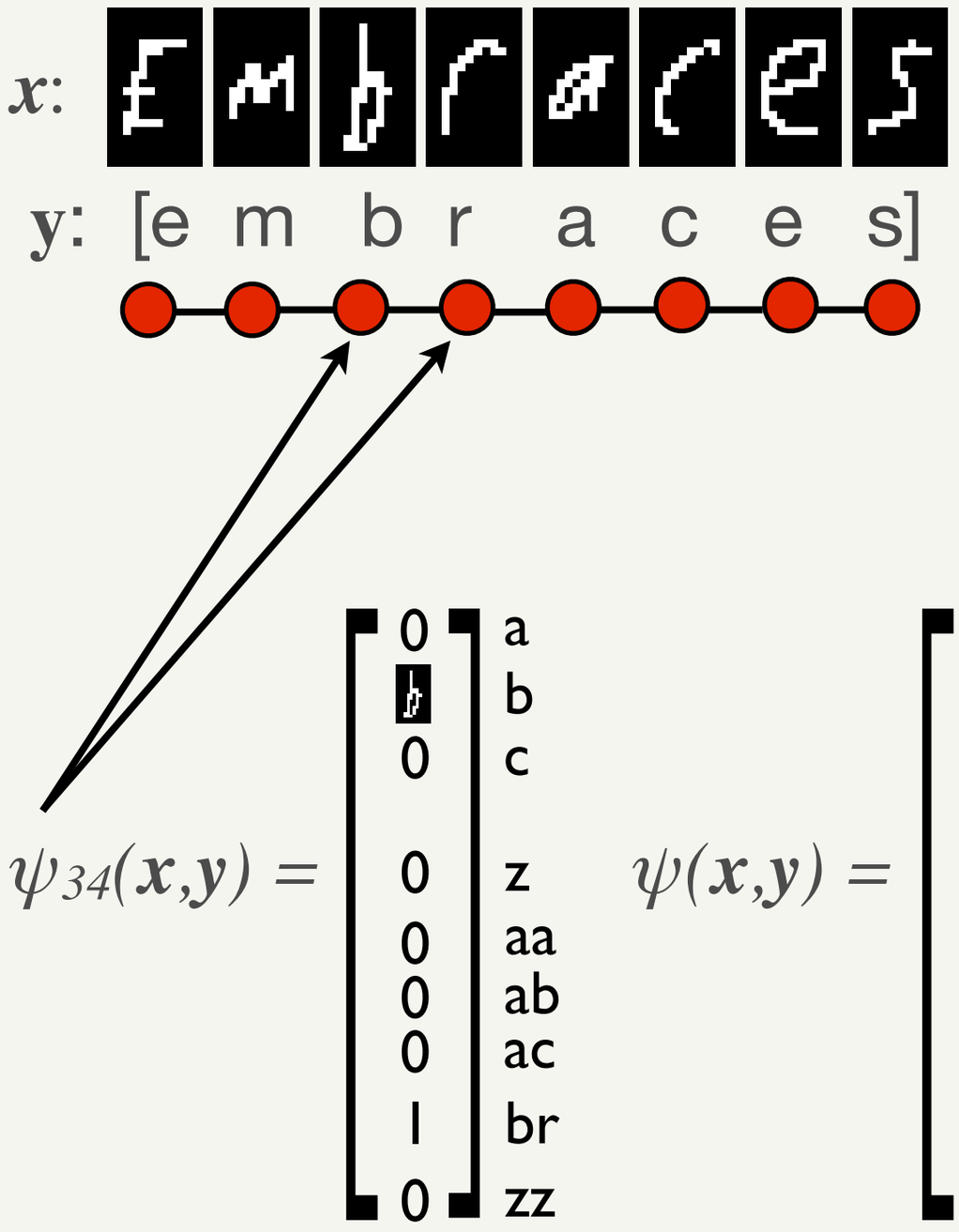

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Comparing words

$\langle \psi(x_1,y_1), \psi(x_2,y_2) \rangle$

correl. homogeneous chars +
correl. char co-occurence matrix

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  - very general

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  - very general
- Function evaluation entails a maximization (inference) step
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- Risk minimization is a large convex program.
  Handled by:
- Iteratively adding one constraint
  - also requires inference step
- M^3 marginalization