Assignment - 5

MM12B035
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5) a)

$$\min_{g(x)} E\left((Y-g(x))^2\right)$$

$$\frac{\partial E}{\partial(g(x))} = E\left(-2(Y-g(x))\right) = 0$$

$$\Rightarrow E(Y) = E(g(x))$$

$$\Rightarrow \boxed{E(Y/X) = g(x)}$$

c)

$$y = x^2 + z \qquad x \sim N(0,1).$$

$$\hat{y} = x^2 \quad (\text{wrt } x) \qquad z \sim N(0,1)$$

$$MSE = E\left((y-\hat{y})^2\right)$$

$$= \boxed{E(z^2) = 1}$$

$$MSE = 1$$

d) $\hat{y} = \alpha x + c$

$$E\left((y-\hat{y})^2\right) = E\left((x^2+z-\alpha x-c)^2\right)$$

$$= E\left((x^2+z-\alpha x-c)(x^2+z-\alpha x-c)\right) = E\big[x^4 + x^2 z - \alpha x^3 - cx^2 + zx^2 + z^2 - \alpha z x$$
$$- cz - \alpha x^3 - \alpha z x + \alpha^2 x^2 + \alpha c x$$
$$- cx^2 - cz + c\alpha x + c^2\big]$$

$$= E\left[x^4 + x^3(-2\alpha) + x^2[2z - 2c + \alpha^2] + x[2\alpha c - 2\alpha z] + z^2 - 2cz + c^2\right]$$

$$= E(\cancel{x^4})^0 - 2\alpha E(\cancel{x^3})^0 + E\left(x^2(2z - 2c + \alpha^2)\right) - 2\alpha E\left(x(c+z)\right)$$
$$+ E\left(z^2 - 2cz + c^2\right)$$

$$= 2E(\cancel{x^2 z})^0 - 2cE(x^2) + \alpha^2 E(x^2) - 2\alpha \cancel{c E(x)}^0 - 2\alpha \cancel{E(xz)}^0$$
$$+ E(z^2) - 2\cancel{c E(z)}^0 + E(c^2)$$

$$= -2c + \alpha^2 + 1 + c^2 = \boxed{1 + \alpha^2 + c^2 - 2c}$$

1) a) $y[k] = x^T[k]\,\theta + e[k]$

$$\hat{\sigma}_\epsilon^2 = \frac{\sum_{k=0}^{N-1} \left(\epsilon[k]\right)^2}{N-p}$$

$$\epsilon = y - \hat{y} = y - Py = P^\perp y$$

$$\hat{\sigma}_\epsilon^2 = \frac{\epsilon \epsilon^T}{N-p} = (P^\perp y)(P^\perp y)^T = \frac{(P^\perp)^2 y^T y}{N-p} \qquad P^\perp y = P^\perp e$$

$$= \frac{(P^\perp)^2 e^2}{N-p} = \frac{\sum_{k=0}^{N-1} \left(P_k\, e[k]\right)^2}{N-p} = \frac{\sum_{k=0}^{N-p} e^2[k]}{N-p}$$

$$E(\hat{\sigma}_\epsilon^2) = E\left(\frac{\sum_{k=0}^{N-p} e^2[k]}{N-p}\right) = \frac{\sum_{k=0}^{N-p} E(e^2[k])}{N-p}$$

$$= \frac{N-p}{N-p}\sigma_e^2 = \boxed{\sigma_e^2}$$

a) $y[k] = -d_1 y[k-1] \dots - d_p y[k-p] + e[k]$

L-S estimator:-

$$\hat{\theta} = \left(\Sigma_{yy}\right)^{-1} \sigma_e^2$$

Y-W estimator:-

$$\sigma_{yy}[l] = -d_1 \sigma_{yy}[l-1] - d_2 \sigma_{yy}[l-2) \dots - d_p \sigma_{yy}[l-p] + \sigma_{ee}[l]$$

$$\Rightarrow \sigma_{yy}[l] + d_1 \sigma_{yy}[l-1] + \dots d_p \sigma_{yy}[l-p] = \sigma_{ee}[l]$$

$$y[k] - \bar{y} = -d_1 (y[k-1] - \bar{y}) \dots - d_p (y[k-p] - \bar{y}) + e[k]$$

$$\Rightarrow \sigma_{yy}[l] - \bar{y}\mu = -d_1 [\sigma_{yy}[l-1) - \mu\bar{y}] \dots - d_p [\sigma_{yy}[l-p] - \mu\bar{y}] + \sigma_{ee}[l]$$

$$\Rightarrow (\sigma_{yy}[l] - \mu\bar{y}) + d_1 (\sigma_{yy}[l-1] - \mu\bar{y}) + \dots + d_p (\sigma_{yy}[l-p] - \mu\bar{y}) = \sigma_{ee}[l]$$

$$\Rightarrow \begin{bmatrix} \sigma_{yy}[0]-\mu\bar{y} & \sigma_{yy}[1]-\mu\bar{y} & \dots & \sigma_{yy}[p]-\mu\bar{y} \\ \vdots & & & \\ \vdots & & & \\ \sigma_{yy}[p]-\mu\bar{y} & \dots & \dots & \sigma_{yy}[0]-\mu\bar{y} \end{bmatrix} \begin{bmatrix} 1 \\ d_1 \\ \vdots \\ d_p \end{bmatrix} = \begin{bmatrix} \sigma_{ee}[0] \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \left(\Sigma_{yy} - \mu\bar{y} I_{(p+1)\times(p+1)}\right) \hat{\theta} = \sigma_e^2$$

$$\hat{\theta} = \left(\Sigma_{yy} - \mu \bar{y}\, I_{(p+1)\times(p+1)} \right)^{-1} \sigma_e^2 \quad (\text{Y-W estimator})$$

as $N \to \infty$, $\bar{y} = 0$

$$\Rightarrow \hat{\theta} = \left(\Sigma_{yy} \right)^{-1} \sigma_e^2 \quad (\text{Y-W estimator})$$

as $N \to \infty$ $\boxed{\hat{\theta}_{Y-W} = \hat{\theta}_{L-S}}$

$$y[k] = -d_1\, y[k-1] + e[k]$$

$$E[y[k]] = E[-d_1 y[k-1]] + E[e[k]]$$

$$\Rightarrow (1+d_1)\mu = 0 \Rightarrow \boxed{\mu = 0}$$

$$\cancel{y[k] y[k-l] = -d_1 y[k-1] y[k-l] + e[k] y[k-l]}$$

$$\Rightarrow \cancel{\sigma_{yy}[l] = -d_1 \sigma_{yy}[l-1] + \sigma_{ey}[l]}$$

$$y[k]\, y[k] = -d_1 y[k-1]\left[-d_1 y[k-1] + e[k]\right] + e[k]\, y[k]$$

$$\Rightarrow \sigma_{yy}[0] = d_1^2 \sigma_{yy}[0] + \sigma_e^2$$

$$\Rightarrow \boxed{\sigma_y^2 = \frac{\sigma_e^2}{1-d_1^2}}$$

$$\cancel{\hat{y} = -d} \quad \hat{y}[k] = -d_1 y[k-1]$$

$$\epsilon[k] = y[k] - \hat{y}[k] = y[k] + d_1 y[k-1]$$

$$f(y,\theta) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(\frac{-1\, y^2[0]}{2\sigma_y^2}\right) * \frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left(\frac{-1}{2\sigma_e^2}\left[y[1] + d_1 y[0]\right)^2\right)$$

$$= \frac{\sqrt{1-d_1^2}}{2\pi\sigma_e^2} \exp\left[\frac{-1}{2\sigma_e^2}\left[(1-d_1^2)\, y^2[0] + \left(y[1] + d_1 y[0]\right)^2\right]\right]$$

$$= \frac{\sqrt{1-d_1^2}}{2\pi\sigma_e^2} \exp\left[\frac{-1}{2\sigma_e^2}\left[y^2[0] + y^2[1] + 2\, y[1]\, y[0]\, d_1\right]\right]$$

$$= \frac{\sqrt{1-d_1^2}}{2\pi\sigma_e^2} \exp\left[\frac{-1}{2\sigma_e^2}\left[x^2[1] + x^2[2] + 2 d_1 x[1] x[2]\right]\right]$$

$$L(y,\theta) = \frac{1}{2}\ln(1-d_1^2) - \ln(2\pi\sigma_e^2) - \frac{1}{2\sigma_e^2}\left[x^2[1] + x^2[2] + 2 d_1 x[1] x[2]\right]$$

$$\frac{\partial L}{\partial \sigma_e^2} = \cancel{-\frac{1}{\sigma_e^2} + \frac{d_1 x[1] x[2]}{(\sigma_e^2)^2}} = 0$$

$$\Rightarrow \cancel{\sigma_e^2 = d_1 x[1] x[2]}$$

$$\frac{\partial L}{\partial \sigma_e^2} = \frac{-1}{\sigma_e^2} + \frac{1}{2\sigma_e^{4}}\left[x^2[1] + x^2[2] + 2 d_1 x[1] x[2]\right] = 0$$

$$\Rightarrow \sigma_e^2 = \frac{1}{2}\left[x^2[1] + x^2[2] + 2 d_1 x[1] x[2]\right]$$

$$\frac{\partial L}{\partial d_1} = \frac{1(-2d_1)}{2(1-d_1^2)} - \frac{1}{2\sigma_e^2}\left[2x[1]x[2]\right] = 0$$

$$\Rightarrow \frac{d_1}{1-d_1^2} + \frac{x[1]x[2]}{\sigma_e^2} = 0$$

$$\Rightarrow (1-d_1^2)\,x[1]x[2] = -d_1\sigma_e^2$$

$$\Rightarrow x[1]x[2]\,d_1^2 - d_1\sigma_e^2 - x[1]x[2] = 0$$

$$\Rightarrow d_1^2 - \frac{d_1\sigma_e^2}{x[1]x[2]} - 1 = 0$$

$$d_1 = \frac{\frac{\sigma_e^2}{x[1]x[2]} \pm \sqrt{\left(\frac{\sigma_e^2}{x[1]x[2]}\right)^2 + 4}}{2}$$

$$= \boxed{\frac{\sigma_e^2}{2x[1]x[2]} \pm \sqrt{\left(\frac{\sigma_e^2}{2x[1]x[2]}\right)^2 + 1}}$$

b) $$\hat{\theta} = (\phi^T\phi)^{-1}\phi^T y$$

$$= (\Sigma_{yy})^{-1}\sigma_e^2$$

$$\Sigma_{yy} = \begin{bmatrix} x^2[0] & x[0]x[1] \\ x[0]x[1] & x^2[1] \end{bmatrix}$$

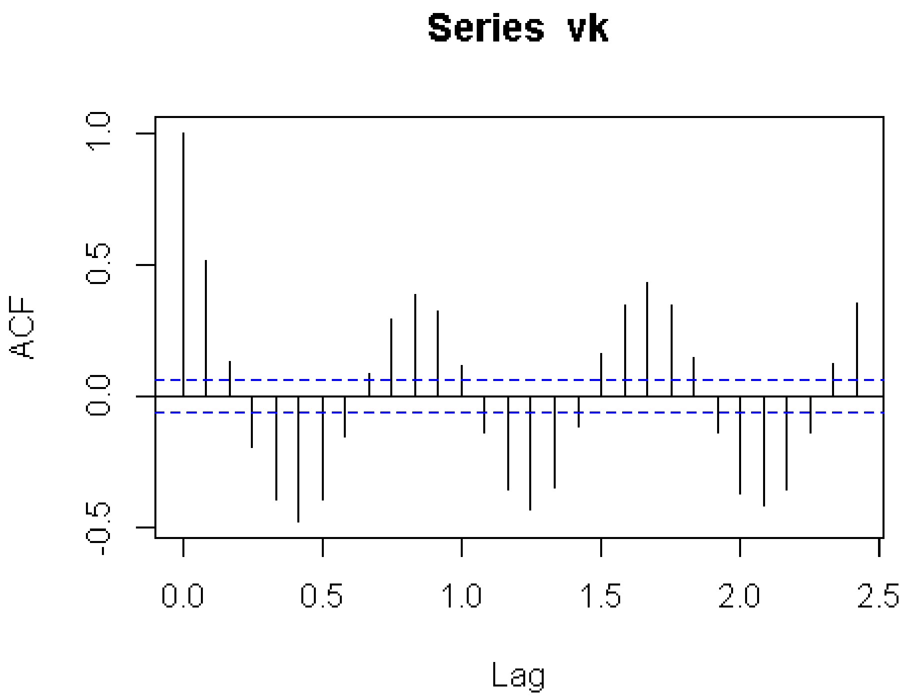
$$\sigma_e^2 = \sigma_y^2[1-d_1^2]$$

4b.R

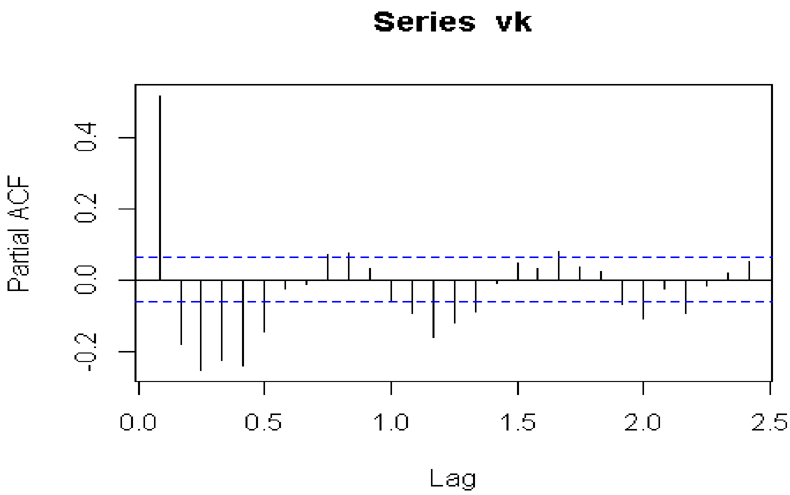
Vishal Subbiah

Sun Nov 09 15:07:30 2014

```
load('sarima_data.Rdata')
#plot(yk)
#acf(yk,xlab="Lag",ylab='ACF')
#pacf(yk)
vk=diff(yk)
acf(vk)
```




```
pacf(vk)
```



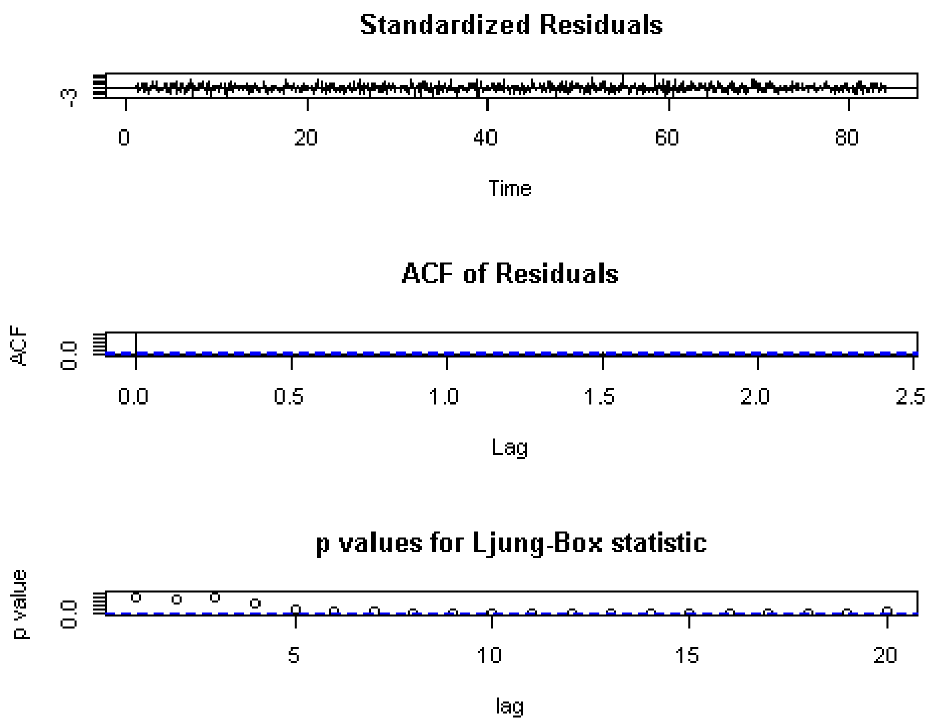
```r
mod_sarima =
arima(vk,order=c(3,0,5),seasonal=list(order=c(0,0,1),period=12),include.me
an=F)
print(mod_sarima)
```

```
##
## Call:
## arima(x = vk, order = c(3, 0, 5), seasonal = list(order = c(0, 0, 1),
period = 12),
##     include.mean = F)
##
## Coefficients:
```

```
## Warning in sqrt(diag(x$var.coef)): NaNs produced
```

```
##          ar1      ar2     ar3      ma1    ma2     ma3     ma4      ma5
##       1.7802  -1.2625  0.1622  -1.4609  0.693  0.1477  0.0978  -0.0958
## s.e.  0.0265   0.0429  0.0265      NaN    NaN  0.0096  0.0059   0.0012
##          sma1
##       -0.0432
## s.e.   0.0341
##
## sigma^2 estimated as 1.073:  log likelihood = -1456.68,  aic = 2933.37
```

```r
tsdiag(mod_sarima,gof.lag=20)
```



1c.R

```
vk1=arima.sim(n=1000,model=list(ar=c(1.1,-0.28)))
ols_1=ar.ols(vk1)
yw_1=ar.yw(vk1)
vk2=arima.sim(n=10000,model=list(ar=c(1.1,-0.28)))
ols_2=ar.ols(vk2)
yw_2=ar.yw(vk2)
```

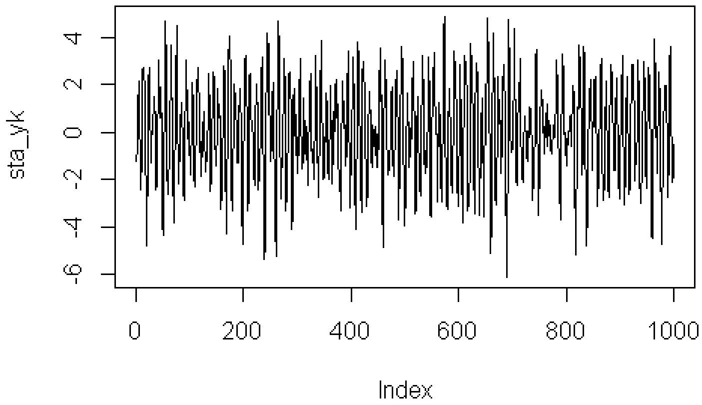
3.R

```
data(cmort)

vko=cmort[3:508]

vk1=cmort[2:507]

vk2=cmort[1:506]

mod_ar2_lr=lm(vko ~ I(vk1)+ I(vk2))

vk_yw=ar.yw(cmort)

vk_ols=ar.ols(cmort)
```

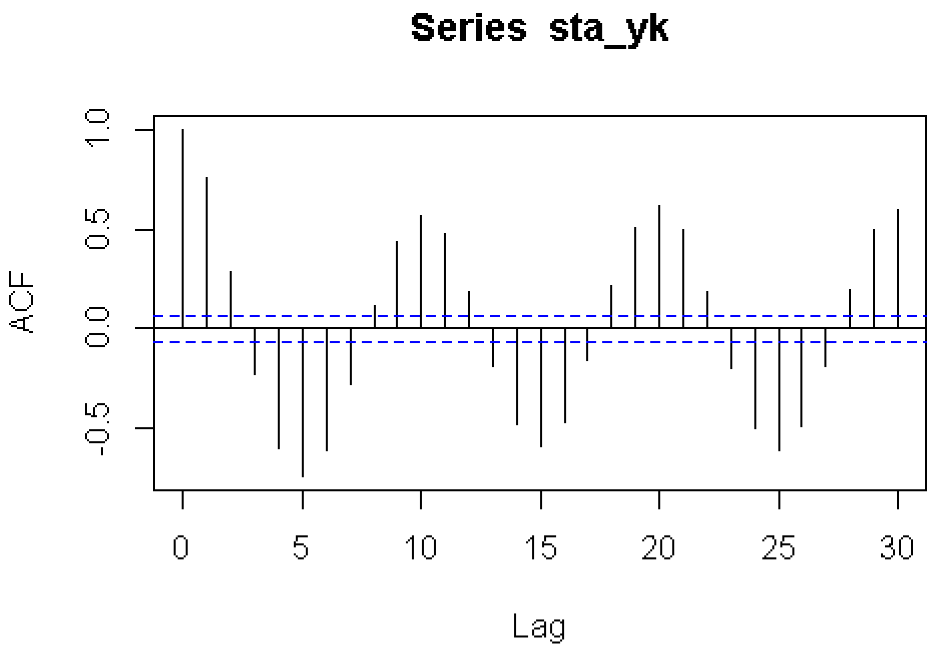
4a.R

```
load('sarima_data.Rdata')

yk_1=stl(yk,s.window="periodic")

#plot(yk,type='l')

sea_yk=yk_1$time.series[1:1000,1]# seasonal
```

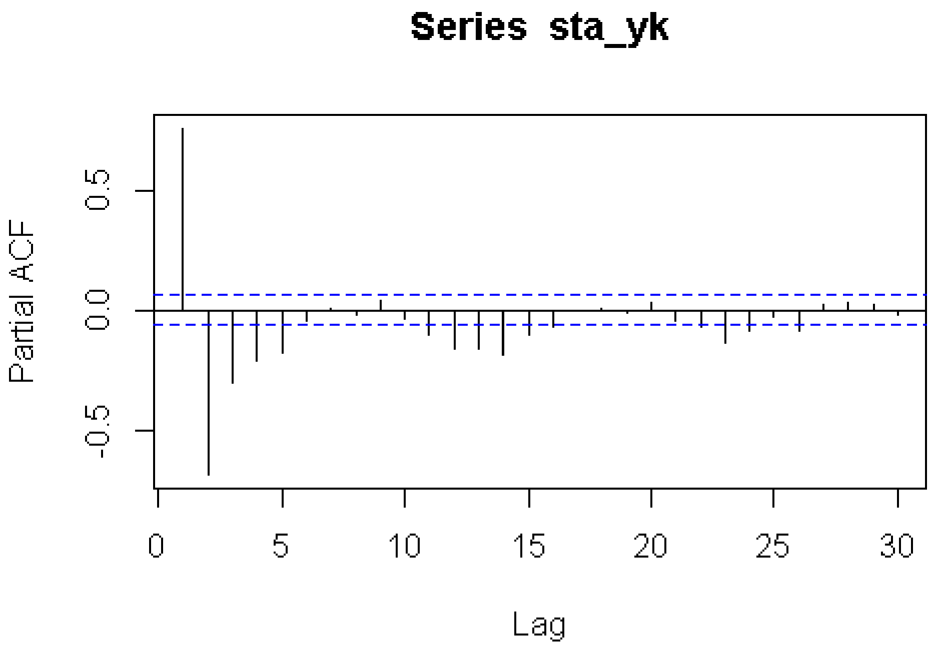
```
tre_yk=yk_1$time.series[1:1000,2]# trend

sta_yk=yk_1$time.series[1:1000,3]# stationary


#plot(sea_yk,type='l')

#plot(tre_yk,type='l')

plot(sta_yk,type='l')
```




```
acf(sta_yk)
```




```
pacf(sta_yk)
```



vk=arma(sta_yk,order=c(2,5))

acf(vk$residuals[10:1000], type="covariance") # residuals is white