Assignment -6

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MM12B035

1a.R

Toshiba

Thu Nov 27 20:04:55 2014

```
library(itsmr)

## Warning: package 'itsmr' was built under R version 3.1.2

ma_2=arima.sim(1000,model=list(ma=c(1,0.21)))
arma_12=arima.sim(1000,model=list(ar=-0.4,ma=c(0.7,0.12)))

ma_2_sol=arma(ma_2)
arma_12_sol=arma(arma_12)

ma_sol1=hannan(ma_2,0,2)
arma_sol1=hannan(arma_12,1,2)
```

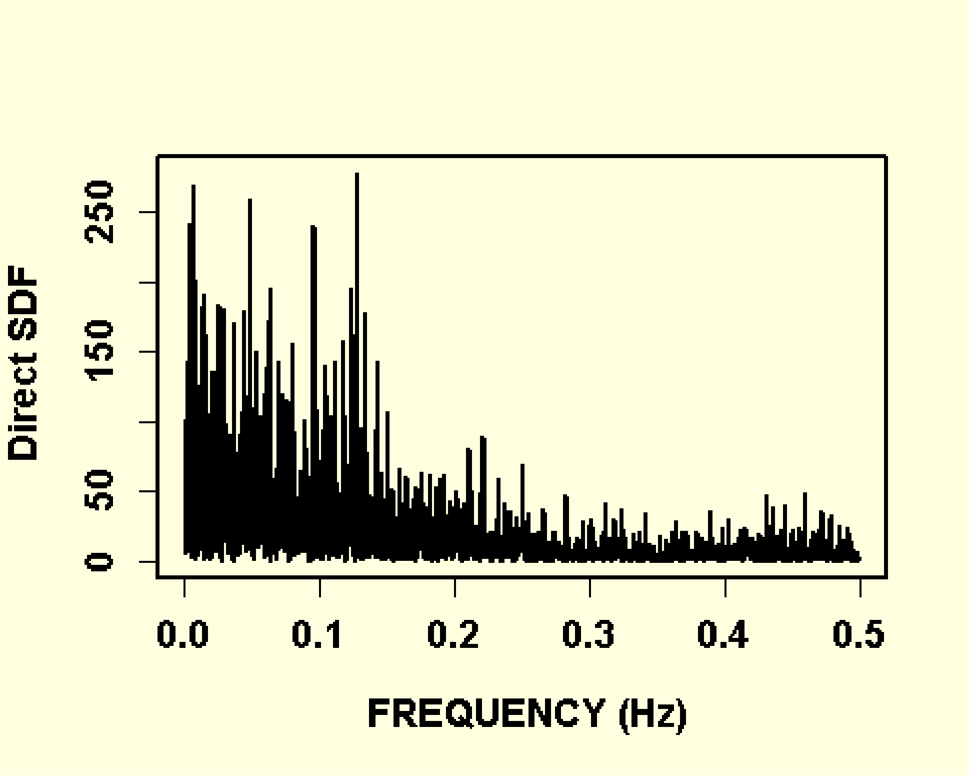
4a.R

Toshiba

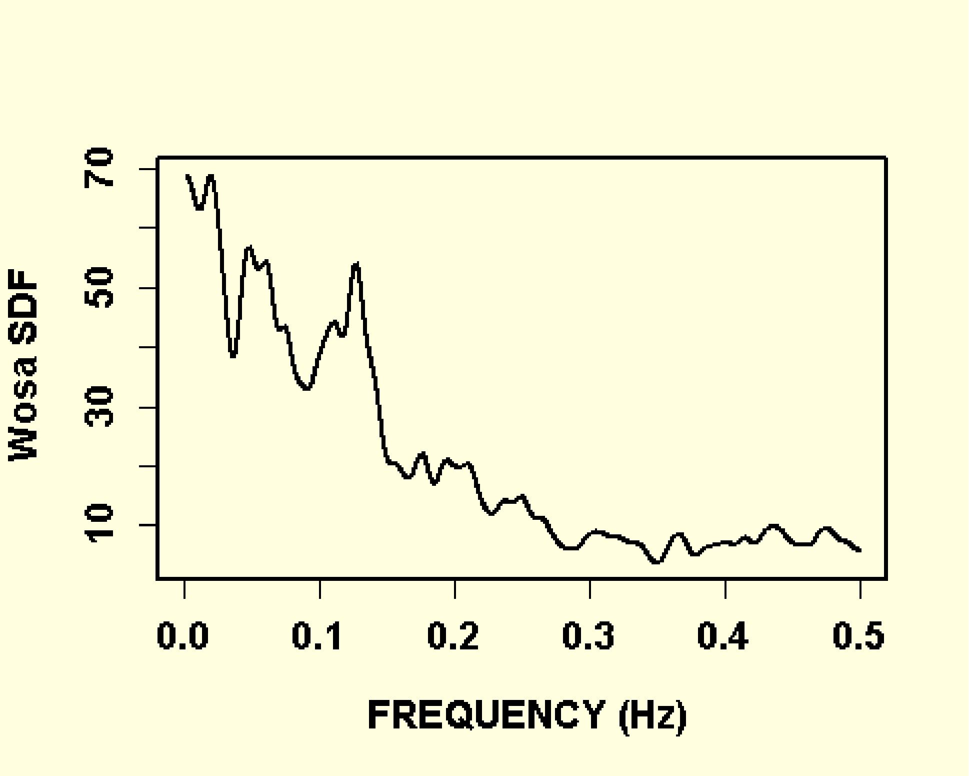
Thu Nov 27 18:51:56 2014

```
library(sapa)

## Warning: package 'sapa' was built under R version 3.1.2

## Loading required package: ifultools

## Warning: package 'ifultools' was built under R version 3.1.2

## Loading required package: splus2R

## Warning: package 'splus2R' was built under R version 3.1.2

## Loading required package: MASS
```

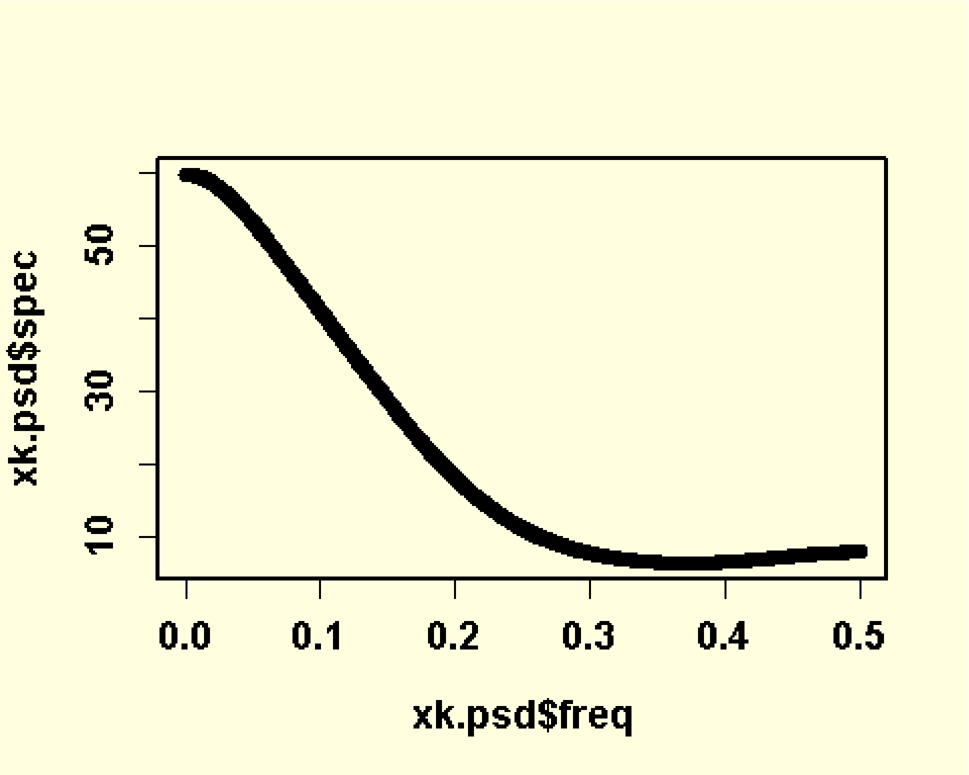
```r
xk=arima.sim(2000,model=list(order(0,0,2),ma=c(0.5,0.25)))
xk=4*xk

xk.psd <- SDF(xk,method='direct')

# Periodogram (could use spec.pgram)
par(bg='lightyellow',font.axis=2,font.lab=2,cex.axis=1.2,cex.lab=1.2,lwd=2
)
plot(xk.psd,yscale='linear')
```



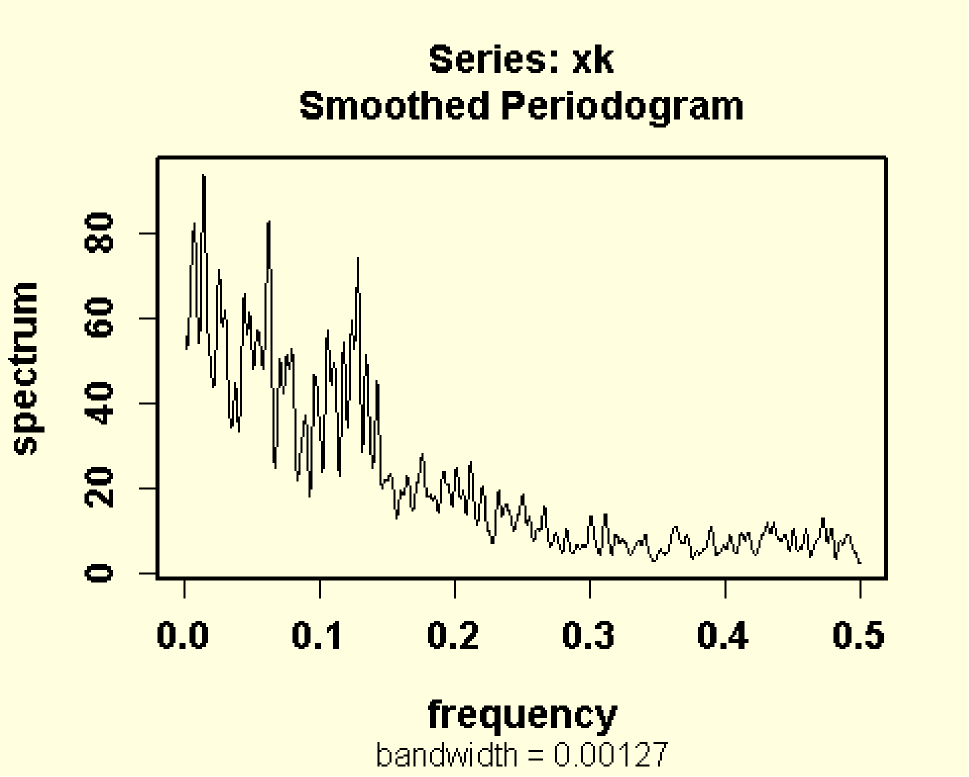

```r
# Welch's overlapping segment averaging method; 128 samples per segment
xk.psd <- SDF(xk,method='wosa',blocksize=128)
plot(xk.psd,yscale='linear')
```




```
xk.psd <- spec.ar(xk,plot=F)# Parametric method (could give misleading res
ults)
par(bg='lightyellow',font.axis=2,font.lab=2,cex.axis=1.2,cex.lab=1.2,lwd=2
)
plot(xk.psd$freq,xk.psd$spec)
```



```r
#Daniell Smoother
xk.psd <- spec.pgram(xk,span=c(7,7),taper=0,log='no')
```



3) $x[k] = e[k] - e[k-1]$ ①

$\hat{x}[k+1 \mid k, k-1, \dots 1] = -e[k]$.

$x[k] = e[k] - e[k-1]$

$x[k-1] = e[k-1] - e[k-2]$

$x[k-2] = e[k-2] - e[k-3]$

$\vdots \qquad \vdots \qquad \vdots$

$$\Rightarrow \sum_{i=0}^{k-1} x[k-i] = e[k]$$

$$\boxed{\hat{x}[k+1 \mid k \dots] = \sum_{i=1}^{k} \alpha_i \, x[i]}$$

$$\epsilon[k+1] = x[k+1] - \hat{x}[k+1 \mid k]$$

$$E(\epsilon[k+1] \cdot x[i]) = 0 \qquad i = 1, 2, \dots k.$$

$$\therefore E\left(\left(x[k+1] - \sum_{i=1}^{k} \alpha_i \, x[i]\right) x[j]\right) = 0$$

$$\Rightarrow E(x[k+1] \, x[j]) = \sum_{i=1}^{k} \alpha_i \, E[x[i] \, x[j]]$$

$$\Rightarrow E[(e[k+1] - e[k])(e[j] - e[j-1])] = \sum_{i=1}^{k} \alpha_i \, E((e[j] - e[j-1])(e[i] - e[i-1]))$$

$$\Rightarrow E[e[k+1][j] - e[k+1] e[j-1] - e[k] e[j] + e[k] e[j-1]]$$
$$= \sum_{i=1}^{k} \alpha_i \, E[e[i] e[j] - e[j] e[i-1] - e[j-1] e[i] + e[j-1] e[i-1]]$$

$$\Rightarrow \sigma_e^2[k+1-j] - \sigma_e^2[k+2-j] - \sigma_e^2[k-j] + \sigma_e^2[k+1-j]$$

$$= \sum_{i=1}^{k} \alpha_i \left[2\sigma_{ee}[i-j] - \sigma_{ee}[j+1-i] - \sigma_{ee}[j-1-i]\right]$$

$j = 1 \cdots k.$

$$\Rightarrow \boxed{\sigma_e^2[k-j] = \sum_{i=1}^{k} \alpha_i \left[\sigma_{ee}[j+1-i] + \sigma_{ee}[j-1-i] - 2\sigma_{ee}[j-i]\right]}$$

$$\Rightarrow \sigma_e^2[k-j] = \sigma_{ee}[j]\,\alpha_1 + \sigma_{ee}[j-1][\alpha_2 - 2\alpha_1] + \cancel{\sigma_{ee}}[j$$

$$+ \sigma_{ee}[j-2][\alpha_1 - 2\alpha_2 + \alpha_3] + \cdots$$

$$+ \sigma_{ee}[j-k][\alpha_{k-1} - 2\alpha_k]$$

$$\sigma_e^2[k-1] = \sigma_e^2[\alpha_2 - 2\alpha_1]$$

$$\sigma_e^2[k-2] = \sigma_e^2[\alpha_1 - 2\alpha_2 + \alpha_3]$$

$$\sigma_e^2[k-3] = \sigma_e^2[\alpha_2 - 2\alpha_3 + \alpha_4]$$

$$\vdots$$

$$\sigma_e^2[1] = \sigma_e^2[\alpha_{k-2} - 2\alpha_{k-1} + \alpha_k]$$

$$+\sigma_e^2[0] = \sigma_e^2[\alpha_{k-1} - 2\alpha_k]$$

$$\therefore -2\alpha_1 + \alpha_2 = 0$$

$$\alpha_1 - 2\alpha_2 + \alpha_3 = 0$$

$$\alpha_2 - 2\alpha_3 + \alpha_4 = 0$$

$$\vdots$$

$$\alpha_{k-2} - 2\alpha_{k-1} + \alpha_k = 0$$

$$\alpha_{k-1} - 2\alpha_k = 1$$

On adding all equations

$$\boxed{\alpha_1 + \alpha_k = -1}$$

on re substituting.

$$\alpha_2 = 2\alpha_1 \qquad \alpha_3 = 3\alpha_1 \qquad \alpha_4 = 4\alpha_1 \Rightarrow \alpha_k = k\alpha_1$$

$$\Rightarrow \alpha_i = \left(\frac{-i}{k+1}\right)$$

$$\therefore \alpha_i = \frac{-i}{K+1}$$

$$\therefore \boxed{\hat{x}[K+1|K..] = \frac{-1}{K+1} \sum_{i=1}^{K} i\, x[i]} \longrightarrow BLP$$

$$E\left(\left(x[K+1] - \sum_{i=1}^{K} \alpha_i\, x[i]\right)^2\right)$$

$$= E\left(\left(\sum_{i=1}^{K} \frac{i\, x[i]}{1+K}\right)^2\right)$$

$$= \frac{1}{(1+K)^2} E\left[\left(\sum_{i=1}^{K} i\, x[i]\right)^2\right]$$

$$= \frac{1}{(1+K)^2} E\left[\left(x[1] + 2x[2] + \cdots K x[K]\right)^2\right]$$

$$i\, x[i] = v[i]$$

$$= \frac{1}{(1+K)^2} \left[\sum_{i=1}^{K} E\left(\left(i\, x[i]\right)^2\right) + 2 \sum_{j=1} \sum_{\substack{i=1 \\ j \neq i}} E\left[ij\, x[i]\, x[j]\right]\right]$$

$$= \frac{1}{(1+K)^2} \left(\sum_{i=1}^{K} i^2 E\left[\left(x[i]\right)^2\right] + 2 \sum_{j=1} \sum_{\substack{i=1 \\ i \neq j}} ij\, E\left[x[i]\, x[j]\right]\right)$$

$$x^2[K] = e^2[K] + e^2[K-1] - 2e[K]\, e[K-1]$$

$$\sigma_x^2[0] = 2\sigma_e^2$$

$$\sigma_{xx}[i-j] = -\left[\sigma_{ee}[i-j+1] + \sigma_{ee}[i-j-1]\right]$$

$\therefore \sum_{i=1}^{k} i^2 E(x[i]^2)$

$\Rightarrow 2\sum_{i=1}^{k} i^2 \sigma_e^2$

$\Rightarrow \boxed{2\sigma_e^2 \left(\frac{k(k+1)(2k+1)}{6} \right)}$

$\Rightarrow 2\sigma_e^2 (k+1) \left[\frac{k(2k+1)}{6} \right]$

$-2 \sum_{\substack{i=1 \\ i \neq j}} \sum_{j=1} ij \left[\sigma_{ee}[i-j+1] + \sigma_{ee}[i-j-1] \right]$

$= 2(k+1)\sigma_e^2 \left[\frac{-\sum_{\substack{i=1 \\ i \neq j}} \sum_{j=1} ij \left[\sigma_{ee}(i-j+1) + \sigma_{ee}[i-j-1] \right]}{2(k+1)\sigma_e^2} \right]$

$= 2(k+1)\sigma_e^2 \left[k^2 \cancel{k} \frac{3+k-k^2}{3} \right]$

$\therefore 2\sigma_e^2 (k+1) \left[\frac{k(2\cancel{k}+1)}{6} + \frac{\cancel{6}+2k-2k^2}{\cancel{3}} \right]$

$\Rightarrow \cancel{2}\sigma_e^2 (k+1) \left[\frac{\cancel{6}+1k}{\cancel{6}\cancel{3}} \right]$

$\Rightarrow \sigma_e^2 [k+1](2+k).$

$\therefore E\left(\sum_{i=1}^{k+1} (i x[i])^2 \right) = \boxed{\sigma_e^2 [k+1](2+k)}$

$$\therefore E\left[\left(x[k+1] - \hat{x}[k+1|k]\right)^2\right] = \frac{\sigma_e^2 (k+1)(2+k)}{(k+1)^2}$$

$$= \left(\frac{k+2}{k+1}\right) \sigma_e^2$$

$$\therefore \boxed{E\left[\left(x[k+1] - \hat{x}[k+1|k]\right)^2\right] = \frac{k+2}{k+1} \sigma_e^2}$$

4 a). $x[k] = e[k-2] + 2\cancel{5}e[k-1] + 4e[k]$

$$x[k-l] = e[k-2-l] + 2e[k-1-l] + 4e[k-l]$$

$$x[k]\,x[k-l] = \left(e[k-2] + 2e[k-1] + 4e[k]\right)\left(e[k-2-l] + 2e[k-1-l] + 4e[k-l]\right)$$

$$\Rightarrow \sigma_{xx}[l] = 21\sigma_{ee}[l] + 10\left[\sigma_{ee}[l-1] + \sigma_{ee}[l+1]\right] + 4\left[\sigma_{ee}[l-2] + \sigma_{ee}[l+2]\right]$$

$\sigma_{xx}[-2] = 4\sigma_e^2$

$\sigma_{xx}[-1] = 10\sigma_e^2$

$\sigma_{xx}[0] = 21\sigma_e^2$

$\sigma_{xx}[1] = 10\sigma_e^2$

$\sigma_{xx}[2] = 4\sigma_e^2$

$$\gamma_x(\omega) = \frac{1}{2\pi} \sum_{l=-\infty}^{\infty} \sigma_{xx}[l]\, e^{-j\omega l}$$

$$= \frac{\sigma_e^2}{2\pi}\left[4\left[e^{2j\omega} + e^{-2j\omega}\right] + 10\left[e^{j\omega} + e^{-j\omega}\right] + 21\right]$$

$$\therefore \boxed{\gamma_{xx}(\omega) = \frac{\sigma_e^2}{2\pi}\left[21 + 20\cos\omega + 8\cos 2\omega\right]}$$

2) $v[k] = \phi_{p,1} v[k-1] + \phi_{p,2} v[k-2] + \cdots \phi_{p,p} v[k-p] + e[k]$ (6)

$$\theta^{(p)} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_p \end{bmatrix} ; \quad \theta^{(p-1)} = \begin{bmatrix} d_1 \\ \vdots \\ d_{p-1} \end{bmatrix} ; \quad \bar{\theta}^{(p-1)} = \begin{bmatrix} d_{p-1} \\ \vdots \\ d_1 \end{bmatrix}$$

$$\theta^{(p-1)} + k_p \bar{\theta}^{(p-1)}$$

$$= \begin{bmatrix} d_1 \\ \vdots \\ d_{p-1} \end{bmatrix} + k_p \begin{bmatrix} d_{p-1} \\ \vdots \\ d_1 \end{bmatrix}$$

$$= \begin{bmatrix} d_1 + k_p d_{p-1} \\ \vdots \\ d_{p-1} + k_p d_1 \end{bmatrix} = \begin{bmatrix} -\phi_{p,i} + \phi_{pp} \phi_{p,p-i} \\ {}_{p-1} \end{bmatrix}$$

$$-\underset{p-1}{\phi_{p,i}} + \underset{p-1}{\phi_{p,p-1}} \underset{p-1}{\phi_{p,p-i}} \quad i \neq p$$

$$\phi_{p,i} = \phi_{p-1,i} - \phi_{p,p} \phi_{p-1,p-i}$$

$$\phi_{pp} = \frac{\rho[p] - \sum_{j=1}^{p-1} \phi_{p,j} \rho[p-j]}{1 - \sum_{j=1}^{p-1} \phi_{p-1,j} \rho[j]}$$

$\phi_{p,i} = \phi_{p-1,i} - \phi_{p,p}\,\phi_{p-1,p-i}$

$d_1 = -\phi_{p,1}$.

$d_i = -\phi_{p-1,i}$; $k_{pp} = -\phi_{pp}$.

$\phi_{p,i} = \phi_{p-1,i} + k_p\,\phi_{p-1,p-i}$

$\Rightarrow \theta^{(p)} = \theta^{(p-1)} + k_p\,\bar{\theta}^{(p-1)}$ for 1 to $p-1$.

and $\phi_{pp} = k_{pp}$.

$$\therefore \boxed{\theta^{(p)} = \begin{bmatrix} \theta^{(p-1)} + k_p\,\bar{\theta}^{(p-1)} \\ k_p \end{bmatrix}}$$

i) $\epsilon_F^{(p)}[k] = \epsilon_F^{(p-1)}[k] + k_p\,\epsilon_B^{(p-1)}[k-p]$

$\epsilon_F^{(p-1)}[k] = \left[v[k] \cdots v[k-(p-1)]\right] \begin{bmatrix} 1 \\ \theta^{(p-1)} \end{bmatrix}$ $\qquad V^T = v[k] \cdots v[k-(p-1)]$

$\epsilon_B^{(p-1)}[k-p] = \left[v[k] \cdots v[k-(p-1)]\right] \begin{bmatrix} \bar{\theta}^{(p-1)} \\ 1 \end{bmatrix}$

$\epsilon_F^{(p-1)}[k] + k_p\,\epsilon_B^{(p-1)}[k-p]$

$= v[k] + \left[\sum_{i=1}^{p-1} d_i\, v[k-i]\right] + k_{pp}\left[v[k-(p-1)]\right] + \sum_{i=1}^{p-1} d_i\, v[k-(p-1)+i]$

⑧

$$= V[K] + V[K-1]\,[\underbrace{d_{p-1,1} + K_{pp}\, d_{p-1,p-2}}_{\to\, d_{p,1}}]$$
$$+ V[K-2]\,[\underbrace{d_{p-1,2} + K_{pp}\, d_{p-1,p-3}}_{\to\, d_{p,2}}]$$
$$+ \vdots$$
$$V[K-p]\,[\underbrace{K_{pp}}_{\to\, d_{p,p}}]$$
$$\vdots$$

$$= V[K] + d_{p,1} V[K-1] + d_{p,2} V[K-2] + \cdots d_{p,p} V[K-p]$$

$$= \left[V[K] \cdots V[K-p]\right] \begin{bmatrix} 1 \\ a(p) \end{bmatrix}$$

$$= \boxed{\epsilon_F^{(p)}[K]}$$

$$\therefore\ \epsilon_F^{(p)}[K] = \epsilon_F^{(p-1)}[K] + K_p\, \epsilon_B^{(p-1)}[K-p]$$

(ii) $$\epsilon_B^{(p)}[K-p] = \epsilon_B^{(p-1)}[K-p] + K_p\, \epsilon_F^{(p-1)}[K]$$

$$\epsilon_B^{(p-1)}[K-p] + K_p\, \epsilon_F^{(p-1)}[K]$$

$$= V[K-p] + \sum_{i=1}^{p-1} d_{p-1,i}\, V[K-(p-1)+i] + K_p \left[V[K] + \sum_{i=1}^{p-1} d_{p-1,i}\, V[K-i] \right]$$

$$= V[K-p] + K_{pp} V[K] + (\underbrace{d_{p-1,p-1} + K_{pp}\, d_{p-1,1}}_{\to\, d_{p,1}}) V[K-1]$$
$$+ \vdots (\underbrace{d_{p-1,p-2} + K_{pp}\, d_{p-1,2}}_{\to\, d_{p,2}}) V[K-2] + \cdots$$

$$= \boxed{V[K-p] + \sum_{i=1}^{p} d_{p,i}\, V[(K-p)+i]}$$

$$= \boxed{\epsilon_B^{(p)}[K-p]}$$

(9)

$$\min_{K_p} \sum_{K=p}^{N-1} \left(\epsilon_F^2[K] + \epsilon_B^2[K-p] \right)$$

$$\sum_{K=p}^{N-1} \left(\epsilon_F^{2\,(p)}[K] + \epsilon_B^{2\,(p)}[K-p] \right]$$

$$L = \sum_{K=p}^{N-1} \left(\left(\epsilon_F^{(p-1)}[K] + K_p\, \epsilon_B^{(p-1)}[K-p] \right)^2 + \left(\epsilon_B^{(p-1)}[K-p] + K_p\, \epsilon_F^{(p-1)}[K] \right)^2 \right]$$

$$\frac{\partial L}{\partial K_p} = 2 \sum_{K=p}^{N-1} \left[\left(\epsilon_F^{(p-1)}[K] + K_p\, \epsilon_B^{(p-1)}[K-p] \right) \epsilon_B^{(p-1)}[K-p] + \left(\epsilon_B^{(p-1)}[K-p] + K_p\, \epsilon_F^{(p-1)}[K] \right) \epsilon_F^{(p-1)}[K] \right] = 0$$

$$\Rightarrow 2\epsilon_F^{(p-1)}[K]\, \epsilon_B^{(p-1)}[K-p] + K_p^* \left[\left(\epsilon_F^{(p-1)}[K] \right)^2 + \left(\epsilon_B^{(p-1)}[K] \right)^2 \right] = 0$$

$$\Rightarrow \boxed{\hat{K}_p^{*\,(i)} = \frac{-2\, \epsilon_F^{(p-1)}[K]\, \epsilon_B^{(p-1)}[K-p]}{\left(\epsilon_F^{(p-1)}[K] \right)^2 + \left(\epsilon_B^{(p-1)}[K] \right)^2}}$$

$$\therefore \boxed{\hat{K}_p^{*} = \frac{-2 \sum_{K=p}^{N-1} \epsilon_F^{(p-1)}[K]\, \epsilon_B^{(p-1)}[K-p]}{\sum_{K=p}^{N-1} \left(\left(\epsilon_F^{(p-1)}[K] \right)^2 + \left(\epsilon_B^{(p-1)}[K] \right)^2 \right)}}$$