Assignment - 2

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MA12B035

1) a) $x[k] = \phi_1 x[k-1] + e[k]$ ①

$$Corr(x[k], x[k-l]) = \frac{\sigma_{x[k], x[k-l]}}{\sqrt{\sigma_{x[k]x[k]}\,\sigma_{x[k-l]x[k-l]}}}$$

$$\sigma_{x[k]x[k-l]} = E[x[k]x[k-l]] - E[x[k]]E[x[k-l]]$$

$$x[k-l] = \phi_1 x[k-l-1] + e[k-l]$$

$$x[k] = \phi_1 x[k-1] + e[k]$$

$$x[k-1] = \phi_1 x[k-2] + e[k-2]$$

$$\therefore x[k] = (\phi_1)^l x[k-l] + \sum_{n=0}^{l-1} \phi^n e[k-n]$$

$$\therefore E(x[k]) = (\phi_1)^l E[x[k-l]]$$

$$E[x[k]\,x[k-l]] = (\phi_1)^l E[x^2[k-l]]$$

$$E[x[k]]\,E[x[k-l]] = (\phi_1)^l E^2[x[k-l]]$$

$$\therefore \sigma_{x[k]x[k-l]} = (\phi_1)^l \left[E[x^2[k-l]] - E^2[x[k-l]] \right]$$

$$= (\phi_1)^l \sigma^2_{x[k-l]} = (\phi_1)^l Var(x[k-l])$$

$$Corr(x[k], x[k-l]) = \frac{(\phi_1)^l Var(x[k-l])}{\left[Var(x[k])\,Var(x[k-l])\right]^{1/2}} = (\phi_1)^l \left[\frac{Var(x[k-l])}{Var(x[k])}\right]^{1/2}$$

$$\therefore \text{Corr}(x[k], x[k-\ell]) = \phi_1^{\ell} \left[\frac{\text{var}(x[k-\ell])}{\text{var}(x[k])} \right]^{1/2}$$ ②

Since $E[x[k]] \neq E[x[k-\ell]]$, $x[k]$ is not stationary.

b) $$x[k] = (\phi_1)^{\ell} x[k-\ell] + \sum_{n=0}^{\ell-1} \phi^n e[k-n]$$

as $\ell \to \infty$ $(\phi_1)^{\ell} \to 0$ $(|\phi_1| < 1)$

$$\therefore \lim_{\ell \to \infty} x[k] = \lim_{\ell \to \infty} (\phi_1)^{\ell} x[k-\ell] + \sum_{n=0}^{\ell-1} \phi^n e[k-n]$$

$$= 0 + \boxed{\sum_{n=0}^{\infty} \phi^n e[k-n]}$$

c) It is not initially stationary as the terms depend on $e[k]$ which we cannot predict.

d) The $x[0]$ should be constant so the process is stationary preferably be equal to 1 so it can be easily scaled.

2) a) MA(2) process

$$V[k] = e[k] + a_1 e[k-1] + a_2 e[k-2]$$

$$H(q^{-1}) = 1 + c_1 q^{-1} + c_2 q^{-2}$$

$$H(q) = 1 + c_1 q + c_2 q^2$$

$$g_\sigma(q^{-1}) = \sigma_{ee} H(q) H(q^{-1}) = \sum_{l=-\infty}^{\infty} \sigma_{vv}[l] q^{-l}$$

$$\therefore g_\sigma(z^{-1}) = \sigma_{ee}^2 (H(z))(H(z^{-1}))$$

$$\therefore g_\sigma(z^{-1}) = \sigma_{vv}[2] z^{-2} + \sigma_{vv}[1] z^{-1} + \sigma_{vv}[0] + \sigma_{vv}[-1] z + \sigma_{vv}[-2] z^2$$

$$\cancel{= \sigma_{vv}[0] + \sigma_{vv}[1][z^{-1}+z] + \sigma_{vv}[2][z^{-2}+z^2]}$$

$$= \sigma_{ee}^2 [1 + c_1 z + c_2 z^2][1 + c_1 z^{-1} + c_2 z^{-2}]$$

$$= \sigma_{ee}^2 [(1 + c_1^2 + c_2^2) + c_1(1+c_2) z^{-1} + c_2 z^{-2} + c_1(1+c_2) z + c_2 z^2]$$

$$\therefore \boxed{\begin{aligned} \sigma_{vv}[2] &= c_2 = \sigma_{vv}[-2] \\ \sigma_{vv}[1] &= c_1(1+c_2) = \sigma_{vv}[-1] \\ \sigma_{vv}[0] &= 1 + c_1^2 + c_2^2 \end{aligned}}$$

d) $\rho_{vv}[\ell] \leq 1$ (4)

$\rho_{vv}[0] = 1$

$\rho_{vv}[\ell] < 1 \quad \cancel{|\ell| >0}\ \ell \neq 0$

$\therefore \rho_{vv}[1] = \dfrac{c_1(1+c_2)}{1+c_1^2+c_2^2} < 1$

$\rho_{vv}[2] = \dfrac{c_2}{1+c_1^2+c_2^2} < 1$

$\therefore (c_1^2+c_2^2+1)\ \rho_{vv}[1] - c_1 - c_1c_2 = 0$

$[c_1^2+c_2^2+1]\ \rho_{vv}[2] - c_2 = 0$

$\therefore c_1^2 \rho_v^2[2] = c_2 - [c_2^2+1]\rho_v^2[2]$

$\Rightarrow c_1 = \sqrt{\dfrac{c_2}{\rho_{vv}[2]} - (c_2^2+1)}$

$\dfrac{c_2}{\rho_{vv}[2]} > 1 + c_2^2$

$\Rightarrow \dfrac{c_2}{\rho_{vv}[2]} - c_2^2 > 1$

$\Rightarrow \cancel{c_2^2 - \dfrac{c_2}{\rho_{vv}[2]}} < 4$

Causal holds only when $\boxed{|c_1| < 1 \ \& \ |c_2| < 2}$

c) $H(q^{-1}) = \dfrac{1}{1-1.3q^{-1}+0.4q^{-2}}$

$$v[k] = -d_1 v[k-1] - d_2 v[k-2] + e[k]$$

$$\boxed{-d_2 = 0.4}$$

$$\phi_{p+1,p+1} = \frac{\rho[p+1] - \sum_{j=1}^{p} \phi_{p,j}\,\rho[p+1-j]}{1 - \sum_{j=1}^{p} \phi_{p,j}\,\rho[j]}$$

$p=1$

$$\Rightarrow \phi_{2,2} = \frac{\rho[2] - \phi_{1,1}\rho[1]}{1-\phi_{1,1}\rho[1]} = 0.4$$

$$\phi_{p+1,j} = \phi_{p,j} - \phi_{p+1,p+1}\phi_{p,p-j+1}$$

$$\Rightarrow \phi_{2,1} = \phi_{1,1}[1-\phi_{2,2}] = \boxed{\rho[1](0.6)}$$

$$\sigma_{vv}[0] = -d_1\sigma_{vv}[1] - d_2\sigma_{vv}[2] + \sigma_{ev}[0]$$

$$\Rightarrow \sigma_{vv}[0] + d_1\sigma_{vv}[1] + d_2\sigma_{vv}[2] = \sigma_e^2$$

Similarly,

$$\sigma_{vv}[1] + d_1\sigma_{vv}[2] + d_2\sigma_{vv}[1] = 0$$

$$\sigma_{vv}[2] + d_1\sigma_{vv}[1] + d_2\sigma_{vv}[0] = 0$$

$$\begin{bmatrix} 1 & d_1 & d_2 \\ 0 & 1+d_2 & d_1 \\ d_2 & d_1 & 1 \end{bmatrix} \begin{bmatrix} \sigma[0] \\ \sigma[1] \\ \sigma[2] \end{bmatrix} = \begin{bmatrix} \sigma_z^2 \\ 0 \\ 0 \end{bmatrix}$$

$\downarrow$

D

$$|D| = (1+d_2-d_1^2) - d_1[-d_1 d_2] + d_2[-d_2[1+d_2]]$$

$$= 1+d_2-d_1^2 + d_1^2 d_2 - d_2^2 - d_2^3$$

$$d_2 = -0.4$$

$$|D| = 0.6 + d_1^2[-1.4] - 0.16 - 0.064$$

$$= -1.4d_1^2 + 0.6 - 0.224$$

$$= -1.4d_1^2 + 0.376$$

$$D^{-1} = \frac{1}{|D|}\begin{bmatrix} 1+d_2-d_1^2 & d_1 d_2 - d_1 & d_1^2-1-d_2 \\ d_1 d_2 & 1-d_2^2 & -d_1 \\ -d_2-d_2^2 & d_1 d_2 - d_1 & 1+d_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} \sigma[0] \\ \sigma[1] \\ \sigma[2] \end{bmatrix} = \frac{1}{|D|}\begin{bmatrix} (1+d_2-d_1^2)\sigma_z^2 \\ d_1 d_2 \sigma_z^2 \\ (-d_2-d_2^2)\sigma_z^2 \end{bmatrix}$$

$$\therefore \rho[2] = \frac{-[d_2+d_2^2]}{1+d_2-d_1^2} = -d_2$$

$$\Rightarrow d_1 = 0 \qquad \therefore \boxed{v[k] = 0.4\, v[k-2] + z[k]}$$

2) acf_v <- ARMAacf(ar=c(0,0.4),lag.max=30)

3) a) $y[k] = A\sin(2\pi f_0 k) + e[k]$ ⑦

$$E[y[k]] = A\,E[\sin(2\pi f_0 k)] + 0$$

= not constant

$\therefore$ not stationary

b) $$R_{yy}[l] = \frac{1}{N}\sum_{k=l+1}^{N}(y[k]-\bar{y})(y[k-l]-\bar{y})$$

$$= E[y[k]\,y[k-l]]$$ since $\bar{y}=0$ for large N

$$= E[A^2\sin(2\pi f_0 k)\sin(2\pi f_0(k-l))]$$

$$= A^2\,E[\sin(2\pi f_0 k)\sin(2\pi f_0(k-l))]$$

$$\frac{\cos(A-B)-\cos(A+B)}{2} = \sin A\sin B$$

$$\therefore = \frac{A^2}{2}E[\cos(2\pi f_0 l) - \cos(2\pi f_0(2k-l))]$$

$$= \frac{A^2}{2}\left[E[\cos 2\pi f_0 l] - E[\cos 2\pi f_0(2k-l)]\right]$$

$\therefore$ frequency of f_0

c) Based on ACF, we can tell if it is stationary or not and also if it is an MA or AR model

3d) k=c(1:1000)

t=sin(2*pi*0.2*k)

sigt=var(t)

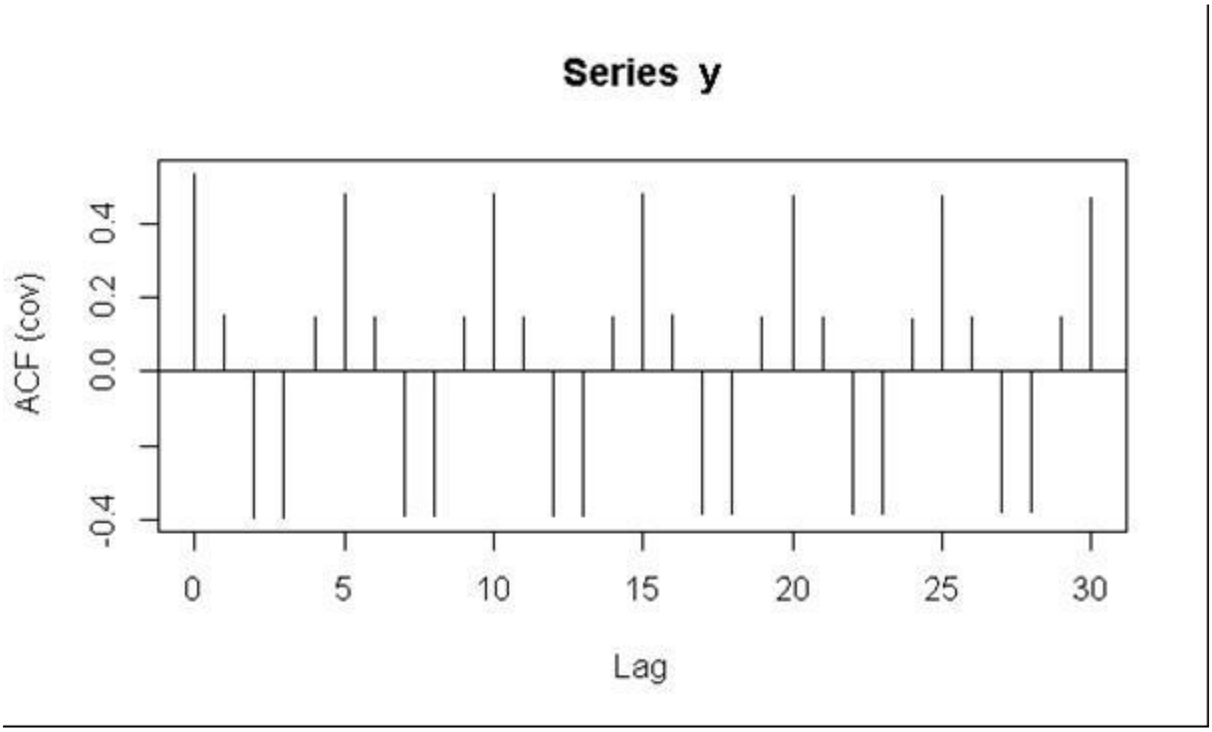
sige=sigt/10

e=rnorm(1000,0,sqrt(sige))

y=sin(2*pi*0.2*k)+e

acf(y,lag.max =NULL,type="covariance",plot=TRUE)

in the plot we can see that the graph repeats at lags of 5 which implies the frequency is 0.2



4 a) (i) $x_1[k] = 0.7x[k-1] - 0.12x[k-2] + e[k]$ ②

(ii) $x_2[k] = e[k] + 0.4e[k-1]$

(i) $ACVF(\ell) = E[x_1[k]\, x_1[k-\ell]]$

$$= E\left[\left(0.7x_1[k-1] - 0.12x[k-2] + e[k]\right) x[k-\ell]\right]$$

$$= 0.7\, E\left[x[k-1]\, x[k-\ell]\right] - 0.12\, E\left[x[k-2]x[k-\ell]\right] + E\left[e[k]\, x[k-\ell]\right]$$

$$ACVF[0] = 0.7\sigma_x^2[1] - 0.12\sigma_x^2[2] + \sigma_e^2$$

$$\sigma_x^2[0] = 0.7\sigma_x^2[1] - 0.12\sigma_x^2[2] + \sigma_e^2$$

$$\sigma_x^2[1] = 0.7\sigma_x^2[0] - 0.12\sigma_x^2[1]$$

$$\Rightarrow \sigma_x^2[1] = \frac{0.7}{1.12}\sigma_x^2[0]$$

$$\sigma_x^2[2] = 0.7\sigma_x^2[1] - 0.12\sigma_x^2[0]$$

$$= \left[\frac{0.49}{1.12} - 0.12\right]\sigma_x^2[0]$$

$$\therefore \rho_{xx}[1] = \frac{70}{112} = 0.625$$

$$\rho_{xx}[2] = \left[\frac{0.49 - 0.12(1.12)}{1.12}\right] = 0.3175$$

II) $x_2[K] = e[K] + 0.4\,e[K-1]$ (9)

$\sigma_{xx}[l] = E[x[K]\,x[K-l]]$

$= \sigma_{ex}[l] + 0.4\sigma_{ex}[l-1]$

$= E\left[e[K]\left(e[K-l] + 0.4\,e[K-l-1]\right)\right]$

$+ 0.4\,E\left[e[K-1]\left[e[K-l] + 0.4\,e[K-l-1]\right]\right]$

$= \sigma_e^2[l] + 0.4\,\sigma_e^2[l+1] + 0.4\,\sigma_{ee}[l-1] + 0.16\,\sigma_e^2[l]$

$\sigma_x^2[0] = 1.16\,\sigma_e^2$

$\sigma_{xx}[1] = 0.4\,\sigma_e^2$

$\sigma_{xx}[2] = 0$

$\therefore \rho_{xx}[1] = \frac{0.4}{1.16} = \frac{10}{29}$

$\rho_{xx}[2] = 0$

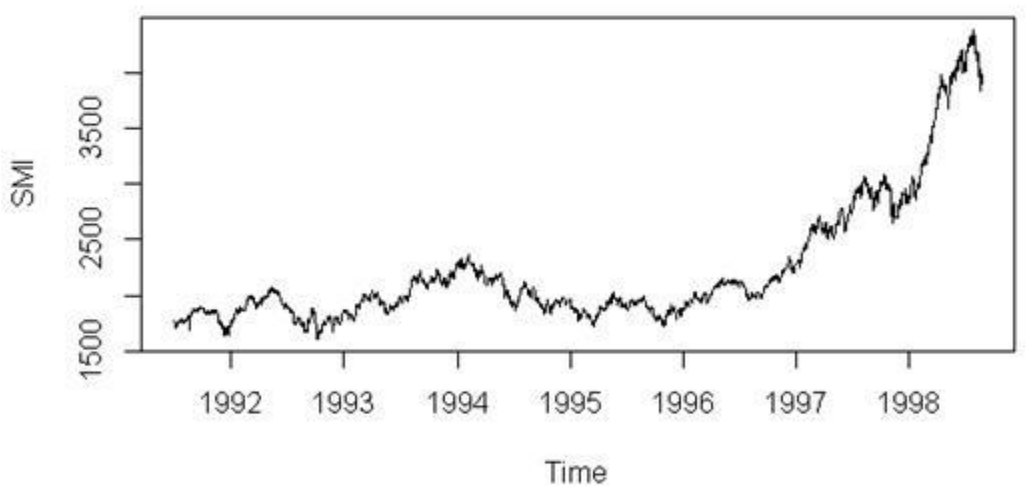
$\therefore \boxed{\phi_{xx}[1] = \frac{10}{29}}$

4a)

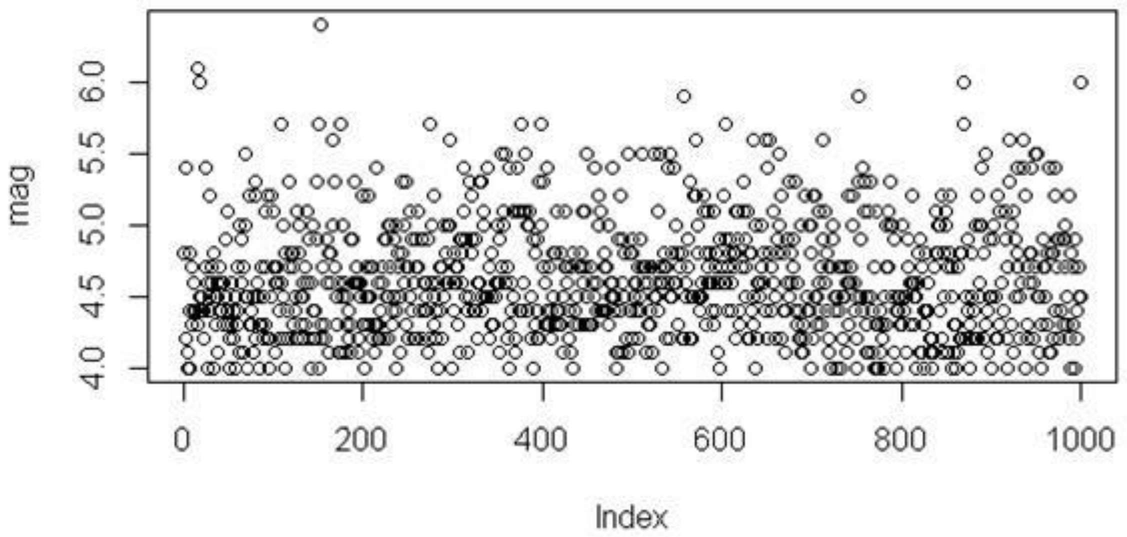
```
acf_x1 <- ARMAacf(ar=c(0.7,-0.12),lag.max=4,pacf= TRUE)

acf_x2 <- ARMAacf(ma=c(0.4),lag.max=4,pacf = TRUE)

pacf_x1_1=acf_x1[1]# pacf at lag 1 for x1

pacf_x1_2=acf_x1[2]# pacf at lag 2 for x1

pacf_x2_1=acf_x2[1]# pacf at lag 1 for x2

pacf_x2_2=acf_x2[2]# pacf at lag 2 for x2
```

4b)

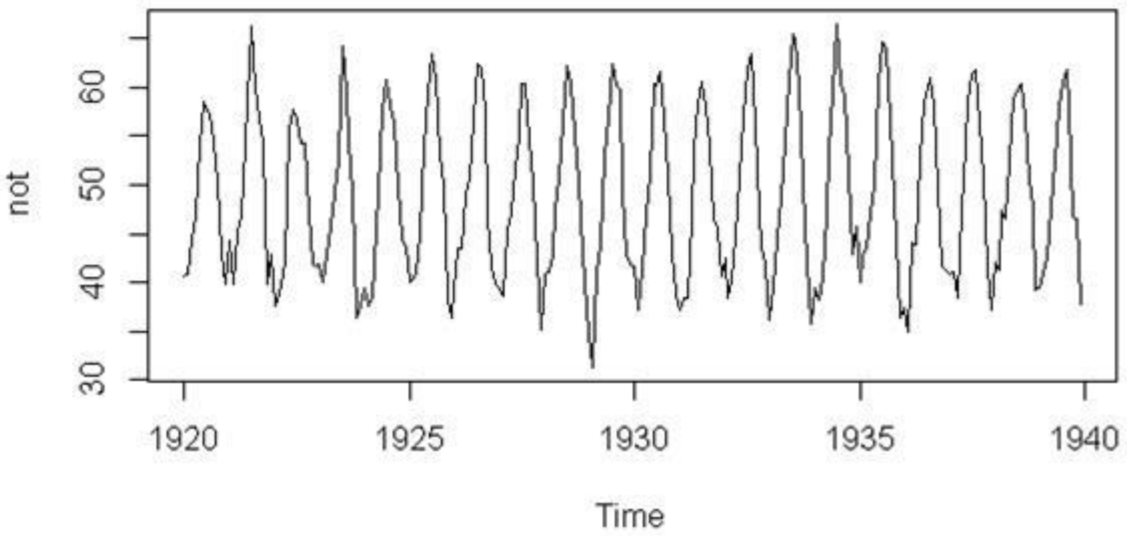
```
SMI=EuStockMarkets[,3]

mag=quakes[,4]

not=nottem

k=c(1:1000)

e=rnorm(1000,0,1)

x=0.01*k + e

#time series

plot(SMI)
```



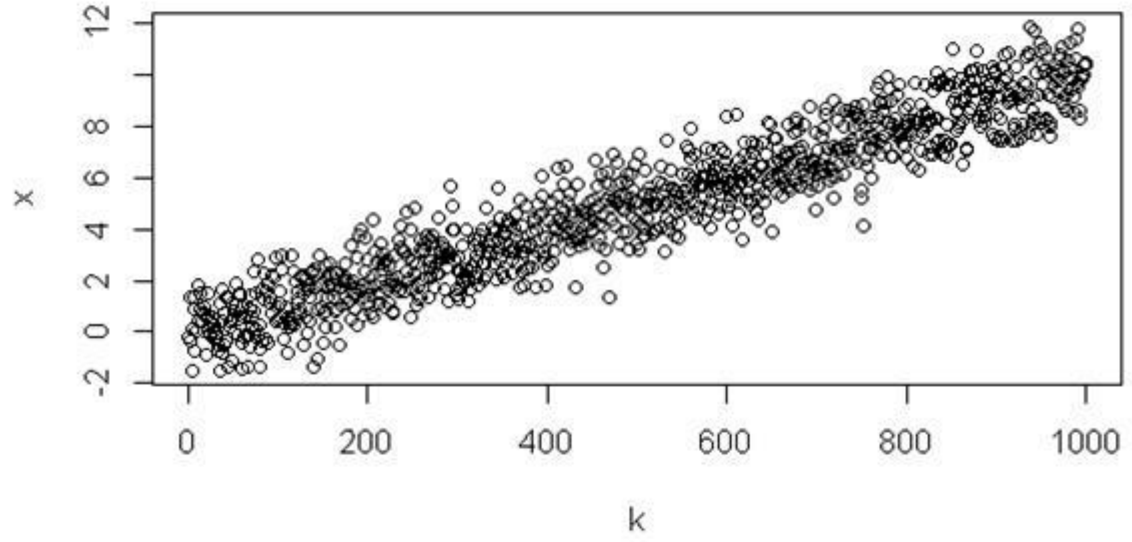
plot(mag)



plot(not)

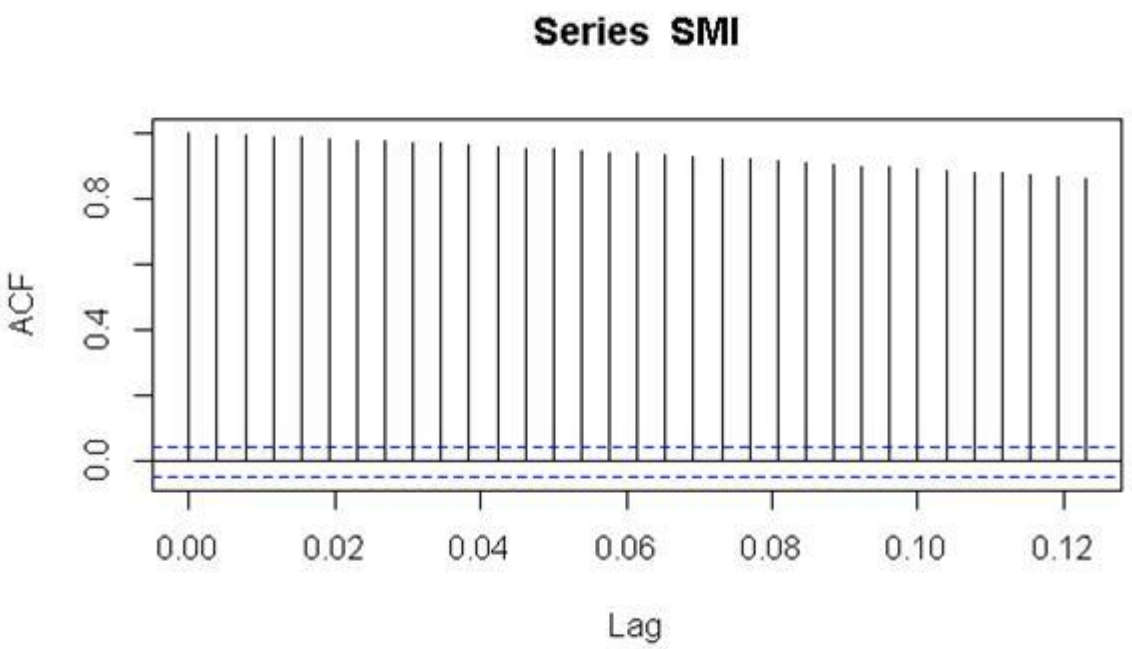


plot(k,x)

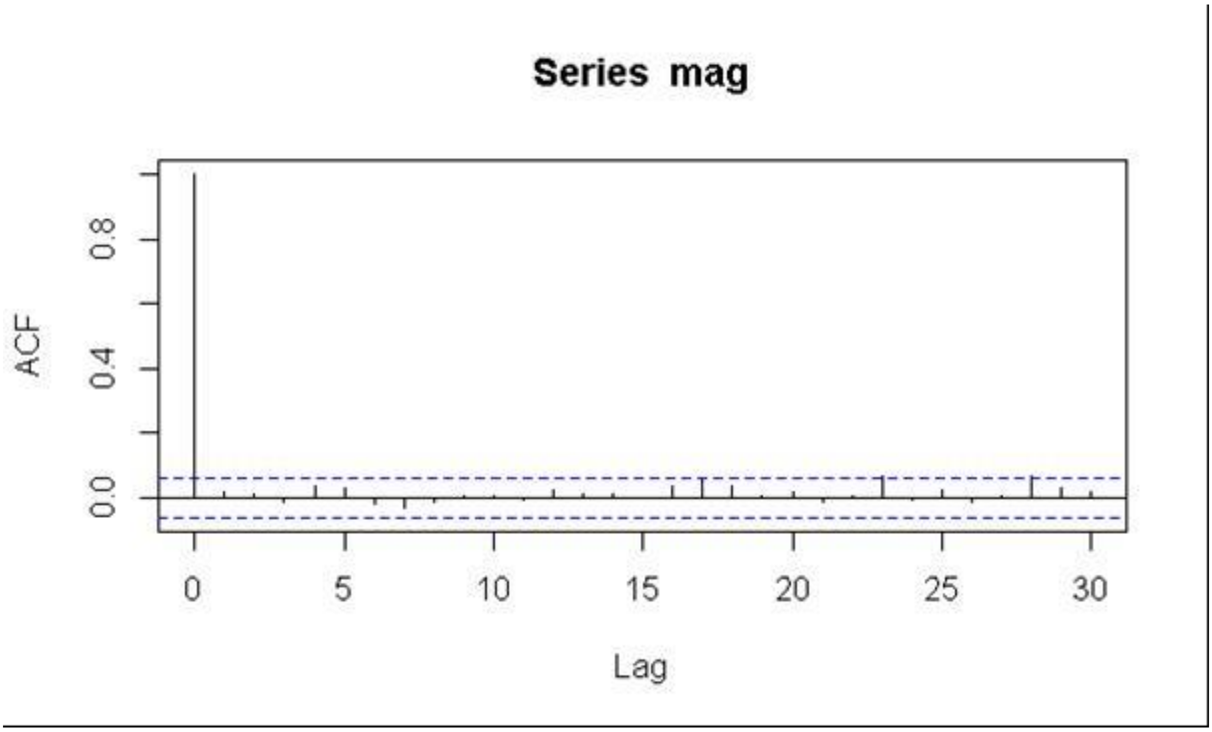


#ACF

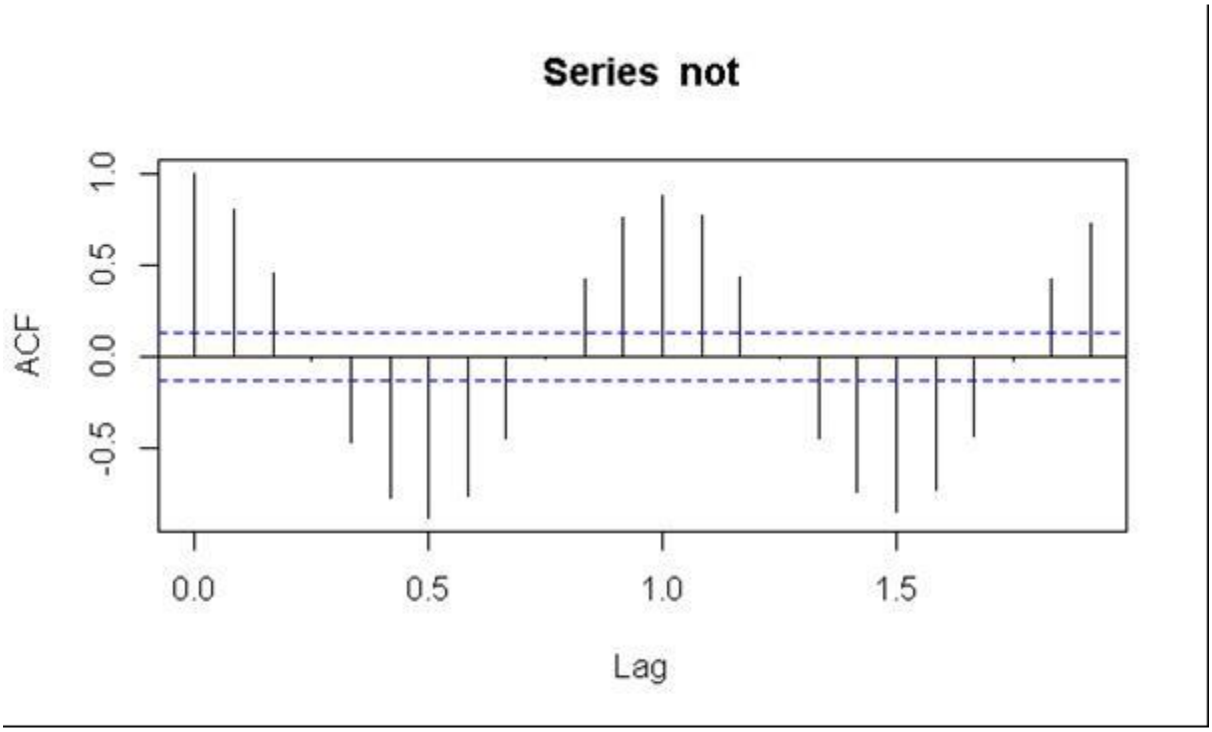
acf(SMI,lag.max =NULL,type="correlation",plot=TRUE)



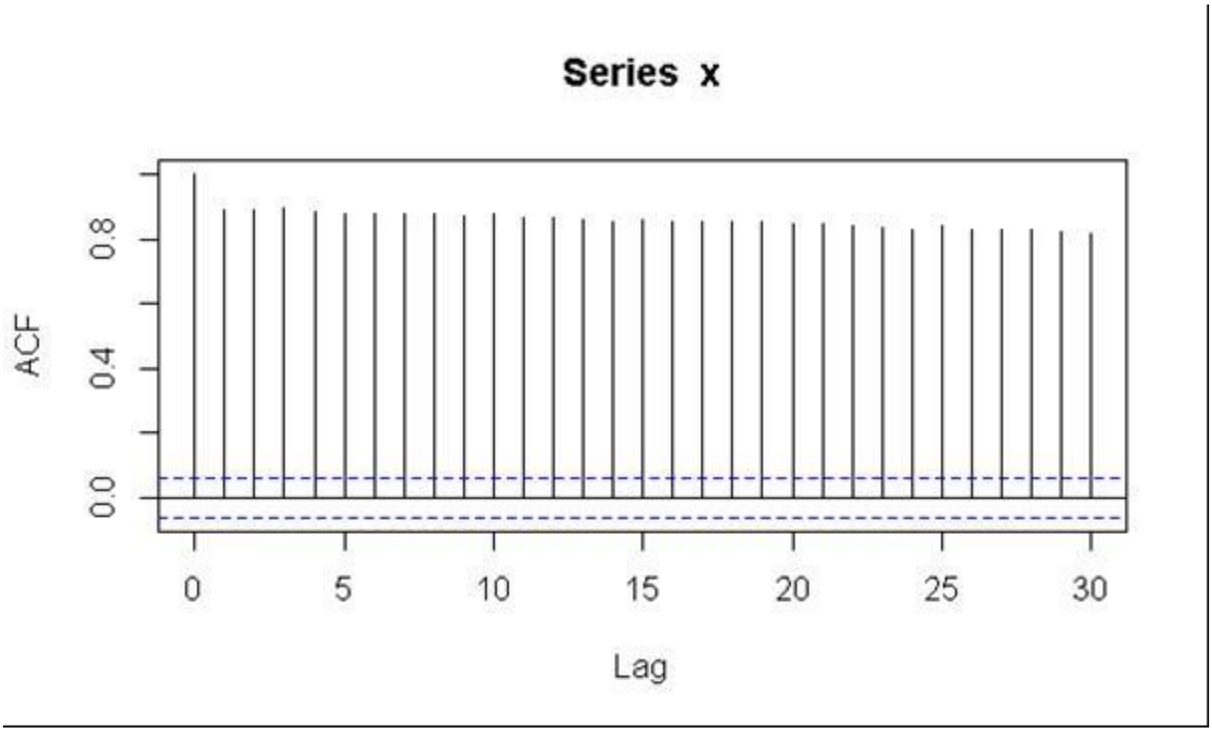
acf(mag,lag.max =NULL,type="correlation",plot=TRUE)



acf(not,lag.max =NULL,type="correlation",plot=TRUE)

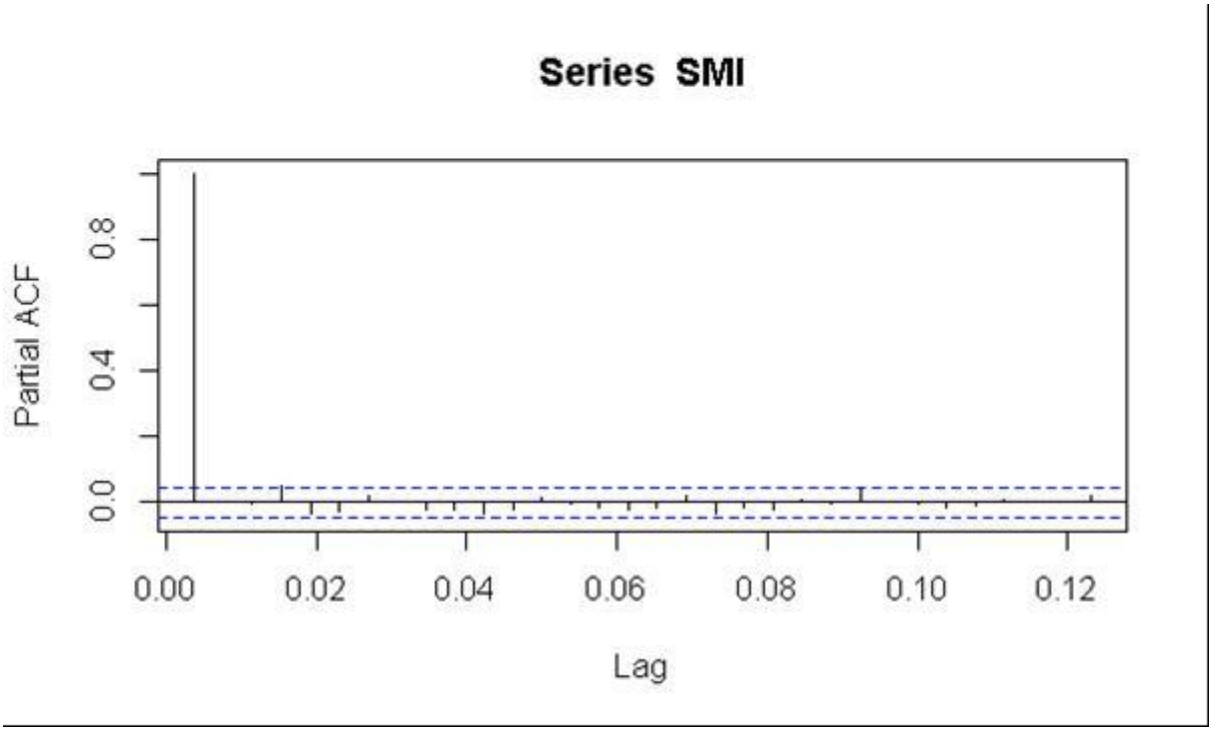


acf(x,lag.max =NULL,type="correlation",plot=TRUE)

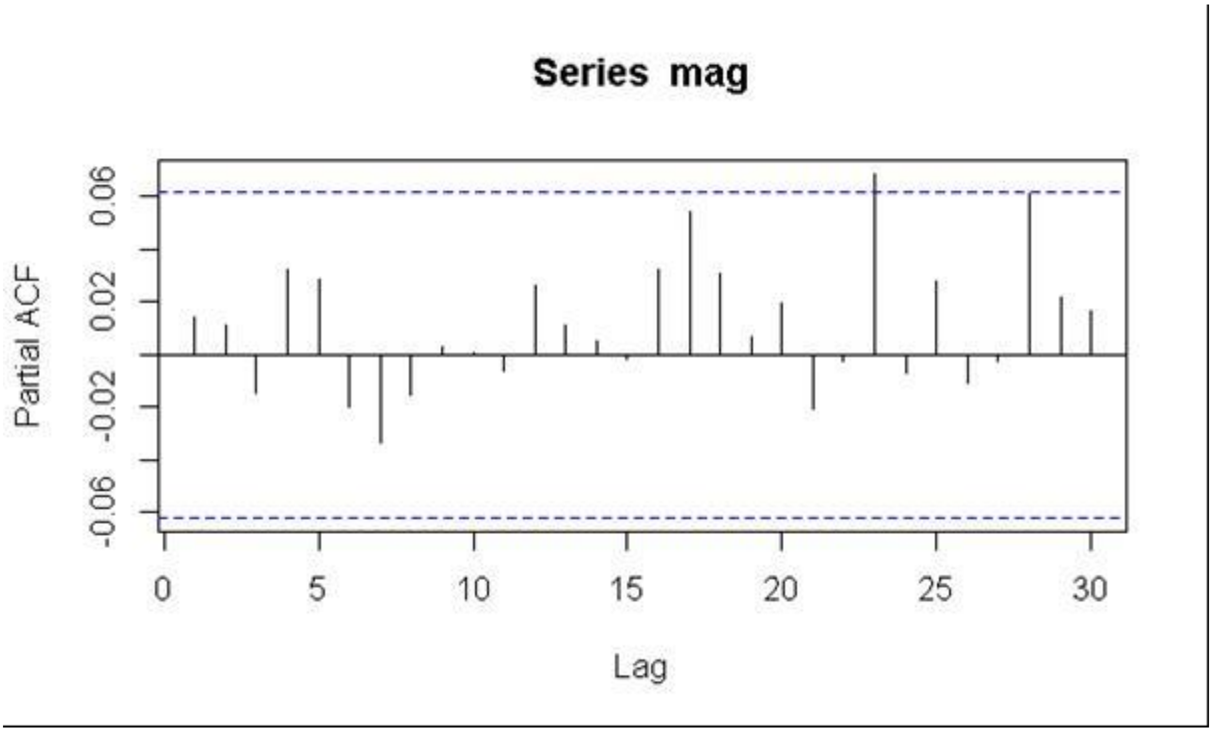


#PACF

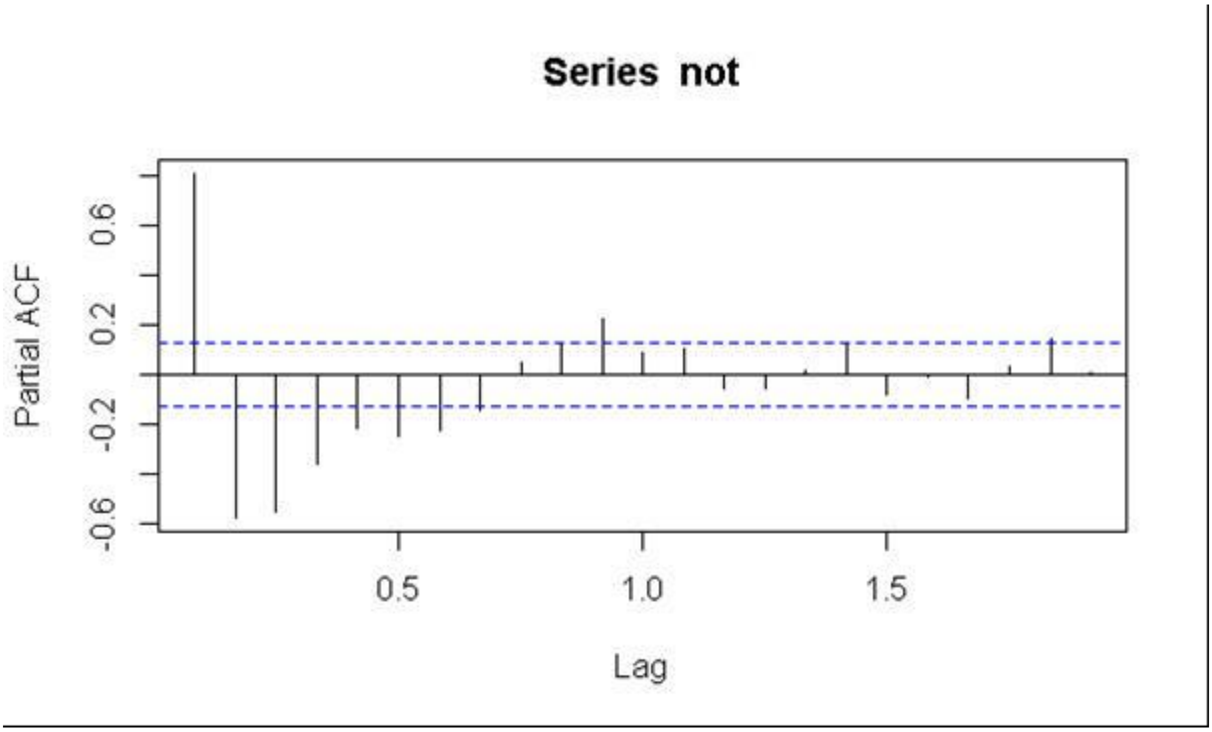
pacf(SMI,lag.max =NULL,plot=TRUE)



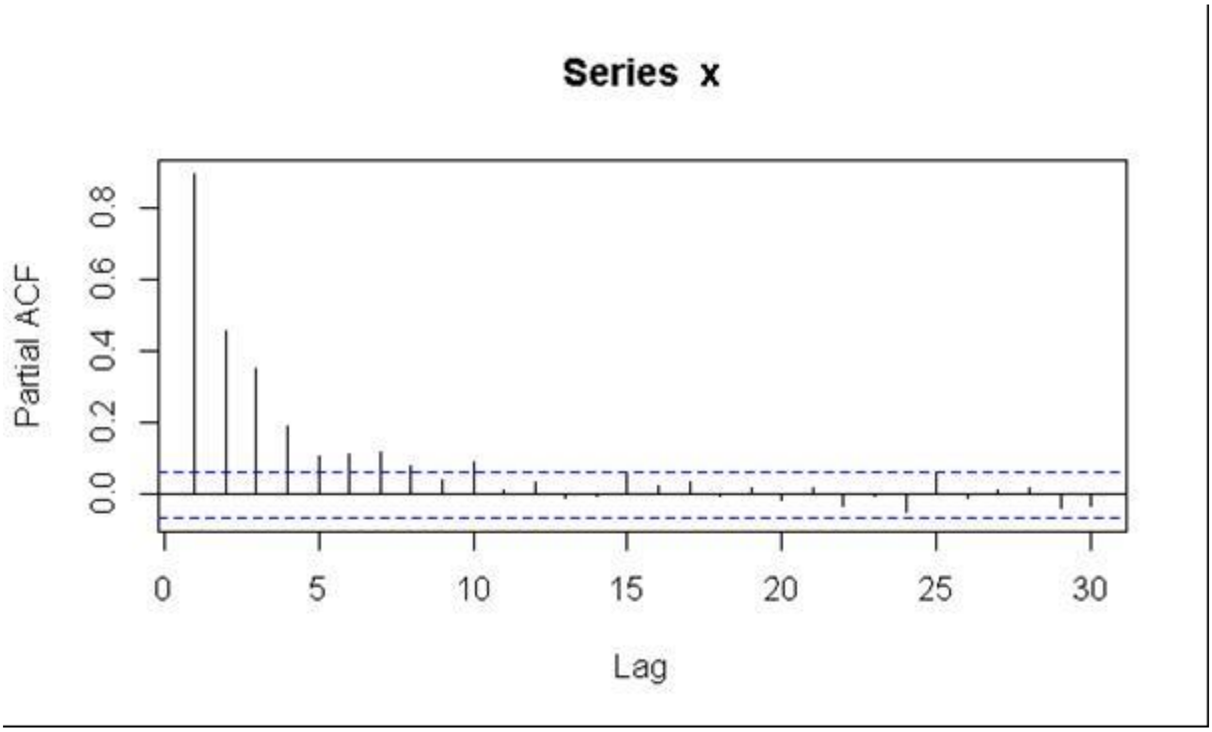
pacf(mag,lag.max =NULL,plot=TRUE)



pacf(not,lag.max =NULL,plot=TRUE)



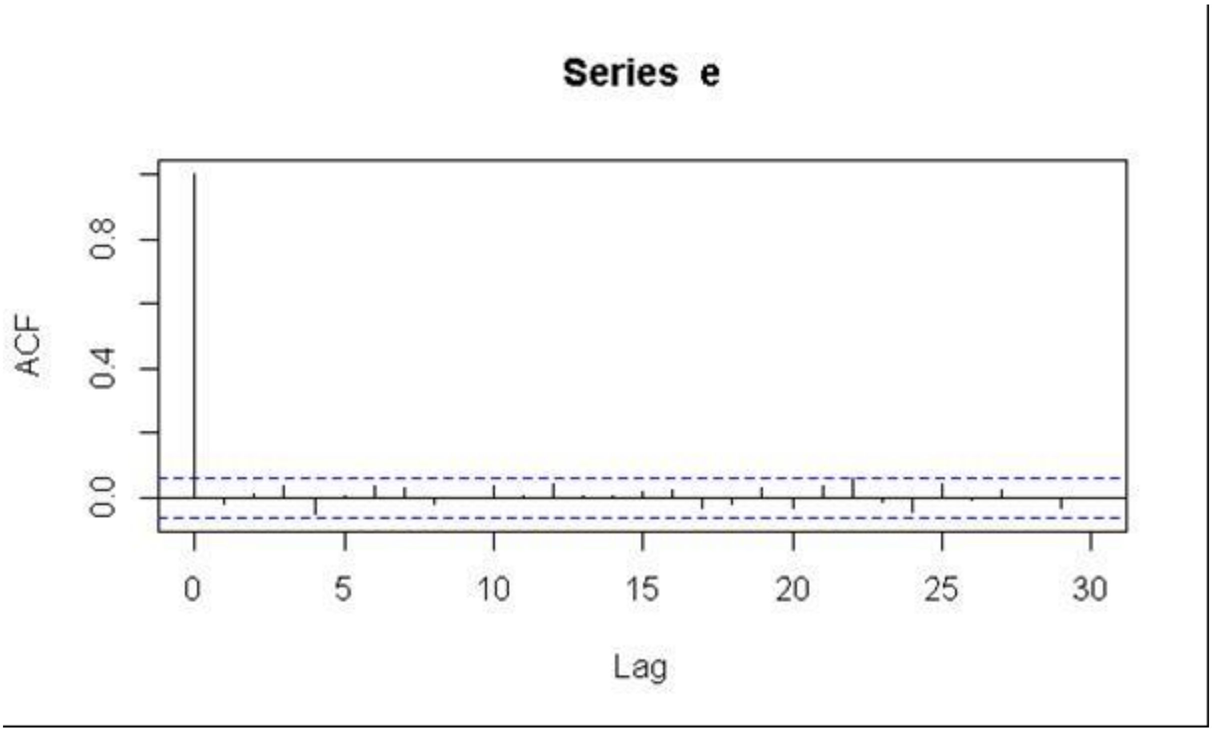
pacf(x,lag.max =NULL,plot=TRUE)



	Stationary	Periodicity
SMI	Yes (AR model)	-
Quakes(mag)	Yes(MA model)	-
nottem	No	Yes(every 5 lags)
x	Yes(AR Model)	-

5)

```
e=rnorm(1000,0,1)

t=acf(e,lag.max =NULL,type="correlation",plot=TRUE)
```




```
y=t$acf

z=acf(y,lag.max =20,type="correlation",plot=TRUE)
```