Introduction to Financial Engineering and Algorithms

Lecturer: William W.Y. Hsu

Binomial Trees (Lattices)

Introduction

- Introduce the binomial tree model in the one-period case.
- Discuss the risk neutral valuation relationship.
- Introduce the binomial tree model in the two-period case and the CRR binomial tree model.
- Consider the continuously compounding dividend yield in the binomial tree model.

One-Period Binomial Tree Model

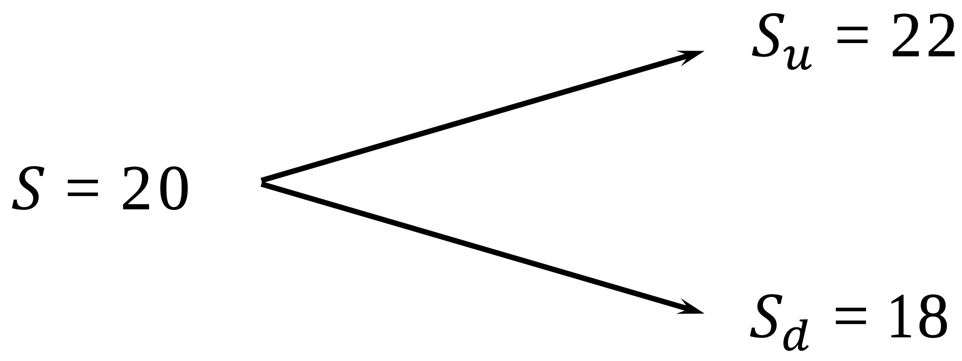
- The binomial tree model represents possible stock price at any time point based on a discrete-time and discrete-price framework.
- For a stock price at a time point, a binomial distribution models the stock price movement at the subsequent time point.
  - That is, there are two possible stock prices with assigned probabilities at the next time point.
- The binomial tree model is a general numerical method for pricing derivatives with various payoffs.
- The binomial tree model is particularly useful for valuing American options, which do not have analytic option pricing formulae.

Assumptions

- There are two (and only two) possible prices for the underlying asset on the next date. The underlying price will either:
  - Increase by a factor of u% (an uptick)
  - Decrease by a factor of d% (a downtick)
- The uncertainty is that we do not know which of the two prices will be realized.
- No dividends.
  The one-period interest rate, r, is constant over the life of the option (r% per period).
- Markets are perfect (no commissions, bid-ask spreads, taxes, price pressure, etc.)

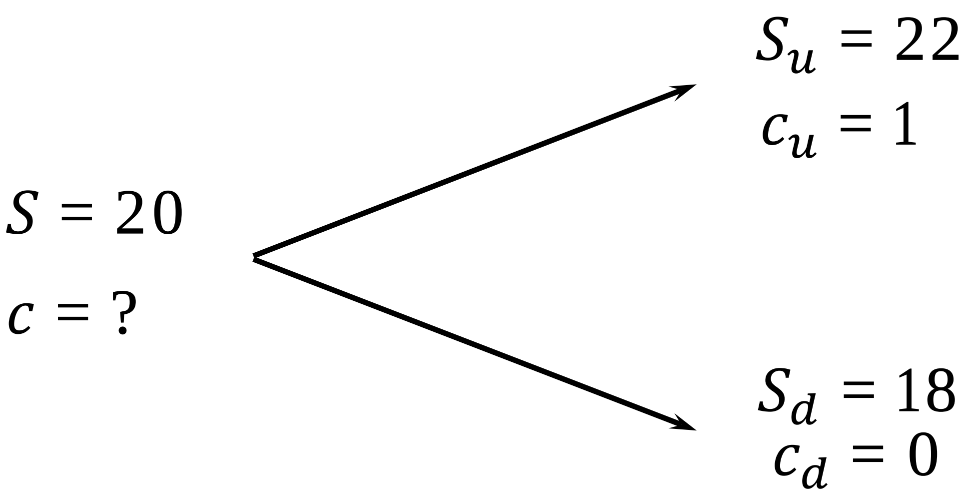
One-Period Binomial Tree Model

- One-period case for the binomial tree model.
  - The stock price S is currently \$20.
  - After three months, it will be either \$22 or \$18 for the upper and lower branches.



One-Period Binomial Tree Model

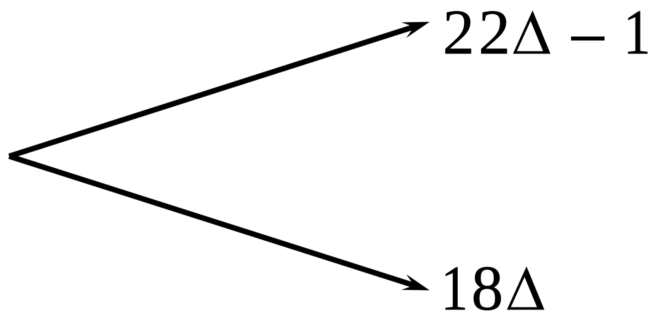
- Consider a 3-month call option on the stock with a strike price of 21.
  - Corresponding to the upper and lower movements in the stock price, the payoffs of this call option are c_u = \$1 and c_d = \$0.



- What is the theoretical value of this call today?

One-Period Binomial Tree Model

- Consider a portfolio P: long Δ shares, short 1 call option



- Portfolio P is riskless when 22Δ – 1 = 18Δ, which implies Δ = 0.25.
- The value of Portfolio P after 3 months is 22 x 0.25 – 1 = 18 * 0.25 = 4.50.

One-Period Binomial Tree Model

- Since Portfolio P is riskless, it should earn the risk-free interest rate according to the no-arbitrage argument.
  - If the return of Portfolio P is higher (lower) than the risk-free interest rate $\Rightarrow$ Portfolio P is more (less) attractive than other riskless assets $\Rightarrow$ Buy (Short) Portfolio P and short (buy) the riskless asset can arbitrage $\Rightarrow$ Purchasing (selling) pressure bid up (drive down) the price of Portfolio P, which causes the decline (rise) of the return of Portfolio P
- The value of the portfolio today is $4.5e^{-12\%\cdot 0.25} = 4.367$, where 12% is the risk-free interest rate today.
  - The amount of 4.367 should be the cost (or the initial investment) to construct Portfolio P.

One-Period Binomial Tree Model

- The riskless Portfolio P consists of long 0.25 shares and short 1 call option.
  - The cost to construct Portfolio P equals 0.25 x 20 – *c*.
- Solve for the theoretical value of this call today to be *c* = 0.633 by equalizing 0.25 * 20 – *c* with 4.367.

Generalization of One-Period Binomial Tree Model

- Consider any derivative f lasting for time T and its payoff is dependent on a stock.

$$
\begin{tikzcd}
 & \begin{array}{l} S_u = S \cdot u \\ f_u \end{array} \\
\begin{array}{l} S \\ f \end{array} \arrow[ur] \arrow[dr] & \\
 & \begin{array}{l} S_d = S \cdot d \\ f_d \end{array}
\end{tikzcd}
$$

  - Assume that $S_u = Su$ and $S_d = Sd$, where u and d are constant multiplying factors for the upper and lower branches.
  - f_u and f_d are payoffs of the derivative f corresponding to the upper and lower branches.

Generalization of One-Period Binomial Tree Model

- Construct Portfolio P that longs Δ shares and shorts 1 derivative. The payoffs of Portfolio P are

$$
\begin{array}{l}
\nearrow \quad Su\Delta - f_u \\
\searrow \quad Sd\Delta - f_d
\end{array}
$$

- Portfolio P is riskless if $Su\Delta - f_u = Sd\Delta - f_d$ and thus

$$\Delta = \frac{f_u - f_d}{Su - Sd}$$

※Note that in the prior numerical example, $S = 20$, $u = 1.1$, $d = 0.9$, $f_u = 1$, and $f_d = 0$, so the solution of Δ for generating a riskless portfolio is 0.25

Generalization of One-Period Binomial Tree Model

- Value of Portfolio P at time T is $Su\Delta - f_u$ (or equivalently $Sd\Delta - f_d$)
- Value of Portfolio P today is thus $(Su\Delta - f_u)e^{-rT}$
- The initial investment (or the cost) for Portfolio P is $S\Delta - f$
- Hence $f = S\Delta - (Su\Delta - f_u)e^{-rT}$
- Substituting Δ for $\frac{f_u - f_d}{Su - Sd}$ in the above equation, we obtain

$$f = e^{-rT}[p \cdot f_u + (1-p) \cdot f_d],$$

where $p = \frac{e^{rT} - d}{u - d}$

Another Approach of Replicating Portfolio

The Stock Pricing 'Process'

- Time T is the expiration day of a call option. Time $T - 1$ is one period prior to expiration.

$$
\begin{tikzcd}[column sep=large]
 & S_{T,u} = (1+u)S_{T-1} \\
S_{T-1} \arrow[ur] \arrow[dr] & \\
 & S_{T,d} = (1+d)S_{T-1}
\end{tikzcd}
$$

- Suppose that S_{T-1} = 40, u = 25% and d = -10%. What are $S_{T,u}$ and $S_{T,d}$?

$$
\begin{tikzcd}[column sep=large]
 & S_{T,u} = \underline{\qquad\qquad} \\
40 \arrow[ur] \arrow[dr] & \\
 & S_{T,d} = \underline{\qquad\qquad}
\end{tikzcd}
$$

The Option Pricing Process

$C_{T,u} = \max(0, S_{T,u}-K) = \max(0,(1+u)S_{T-1}-K)$

C_{T-1}

$C_{T,d} = \max(0, S_{T,d}-K) = \max(0,(1+d)S_{T-1}-K)$

- Suppose that $K = 45$. What are $C_{T,u}$ and $C_{T,d}$?

$C_{T,u}$ = ______

C_{T-1}

$C_{T,d}$ = ______

The Equivalent Portfolio

- Buy Δ shares of stock and borrow $B.

$$\Delta S_{T-1}+B \nearrow \Delta(1+u)S_{T-1} + (1+r)B = \Delta S_{T,u} + (1+r)B$$

$$\Delta S_{T-1}+B \searrow \Delta(1+d)S_{T-1} + (1+r)B = \Delta S_{T,d} + (1+r)B$$

- Δ is not a "change" in S…. It defines the # of shares to buy. For a call, $0 < \Delta < 1$.
- Set the payoffs of the equivalent portfolio equal to $C_{T,u}$ and $C_{T,d}$, respectively.

$$\Delta(1+u)S_{T-1} + (1+r)B = C_{T,u}$$
$$\Delta(1+d)S_{T-1} + (1+r)B = C_{T,d}$$

These are two equations with two unknowns: Δ and B

A Key Point

- If two assets offer the same payoffs at time T, then they must be priced the same at time T-1.
- We have set the problem up so that the equivalent portfolio offers the same payoffs as the call.
- Hence the call's value at time T-1 must equal the $ amount invested in the equivalent portfolio.

$$C_{T-1} = \Delta S_{T-1} + B$$

Δ and B Define the "Equivalent Portfolio" of a Call

$$\Delta = \frac{C_{T,u} - C_{T,d}}{(u-d)S_{T-1}} = \frac{C_{T,u} - C_{T,d}}{S_{T,u} - S_{T,d}}; \quad 0 \le \Delta_c \le 1$$

$$B = \frac{(1+u)C_{T,d} - (1+d)C_{T,u}}{(u-d)(1+r)}; \quad B_c \le 0$$

$C_{T-1} = \Delta S_{T-1} + B$

A Shortcut

$$C_{T-1} = \frac{\frac{r-d}{u-d} C_{T,u} + \frac{u-r}{u-d} C_{T,d}}{(1+r)}$$

or,

$$C_{T-1} = \frac{pC_{T,u} + (1-p)C_{T,d}}{(1+r)}$$

where,

$$p = \frac{r-d}{u-d} \qquad \text{and} \qquad (1-p) = \frac{u-r}{u-d}$$

- In general:

$$C = \frac{pC_u + (1-p)C_d}{(1+r)}$$

Δ and B Define the "Equivalent Portfolio" of a Call

- Assume that the underlying asset can only rise by u% or decline by d% in the next period. Then in general, at any time:

$$\Delta = \frac{C_u - C_d}{(u-d)S} = \frac{C_u - C_d}{S_u - S_d}$$

$$B = \frac{(1+u)C_d - (1+d)C_u}{(u-d)(1+r)}$$

C = ΔS + B

Interpreting *p*

- p is the probability of an uptick in a **risk-neutral world**.
- In a **risk-neutral world**, all assets (including the stock and the option) will be priced to provide the same riskless rate of return, r.
- That is, the stock is priced to provide the same riskless rate of return as the call option.

$$p = \frac{e^{rT} - d}{u - d}$$

Interpreting Δ

- Delta, Δ, is the riskless hedge ratio; $0 < \Delta < 1$.
- Delta, Δ, is the number of shares needed to hedge one call.
    - If you are long Δ call, you can hedge your risk by selling 1 shares of stock.
- The number of calls to hedge one share is $1/\Delta$.
    - If you own 100 shares of stock, then sell $1/\Delta$ calls to hedge your position. Equivalently, buy Δ shares of stock and write one call.
- Delta is a slope.
    - An option's value is a function of the price of the underlying asset.
- In continuous time, $\Delta = \partial C/\partial S$ = the change in the value of a call caused by a (small) change in the price of the underlying asset.

Risk-Neutral Valuation Relationship

- Risk averse, risk neutral, and risk loving behaviors
  - A game of flipping a coin
    - For risk averse investors, they accept a risky game if its expected payoff is higher than the payoff of the riskless game and able to compensate investors for the risk they take
    - That is, risk averse investors requires compensation for risk
- For different investors, they have different tolerance for risk, i.e., they require different expected returns to accept the same risky game

Risk-Neutral Valuation Relationship

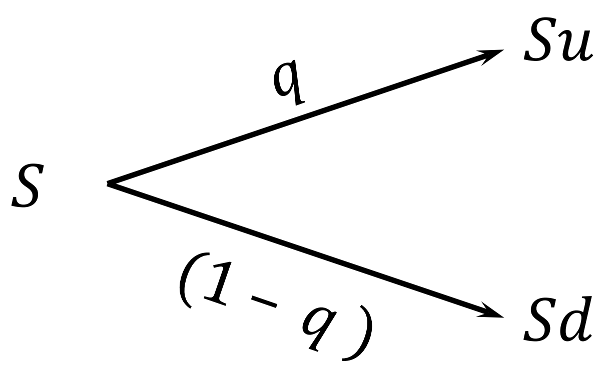
- For different investors, they have different tolerance for risk, i.e., they require different expected returns to accept the same risky game
- For risk neutral investors, they accept a risky game even if its expected payoff equals the payoff of the riskless game
  - That is, they require no compensation for risk
- For risk loving investors, they accept a risky game to enjoy the feeling of gamble even if its expected payoff is lower than the payoff of the riskless game
  - That is, they would like to sacrifice some benefit for entering a risky game

Risk-Neutral Valuation Relationship

- In a risk-averse financial market, securities with higher degree of risk need to offer higher expected returns
  - Since the above situation is consistent with the fundamental financial principle, the real world is a risk averse world
- In a risk-neutral financial market, the expected returns of all securities equal the risk free rate regardless of their degrees of risk
  - That is, even for derivatives, their expected returns equal the risk free rate in the risk neutral world
- In a risk-loving financial market, securities with higher degree of risk offer lower expected returns

Risk-Neutral Valuation Relationship

- Interpret p in $f = e^{-rT}[pf_u + (1-p)f_d]$ as a probability in the risk-neutral world
  - If the expected return of the stock price is μ in the real world, the expected stock price at the end of the period is $E(S_T) = Se^{\mu T}$



$$qSu + (1-q)Sd = Se^{\mu T} \Rightarrow q = \frac{e^{\mu T} - d}{u - d}$$

Risk-Neutral Valuation Relationship

- Comparing with $p = \frac{e^{rT} - d}{u - d}$, it is natural to interpret p and $1 - p$ as probabilities of upward and downward movements in the risk neutral world
  - This is because that the expected return of any security in the risk neutral world is the risk free rate

Risk-Neutral Valuation Relationship

- The formula $f = e^{-rT}[pf_u + (1-p)f_d]$ is consistent with the general rule to price derivatives
  - Note that in the risk neutral world, $[pf_u + (1-p)f_d]$ is the expected payoff of a derivative and e^{-rT} is the correct discount factor to derive the PV today
  - The complete version of the general derivative pricing method is that any derivative can be priced as the PV of its expected payoff *in the risk neutral world*

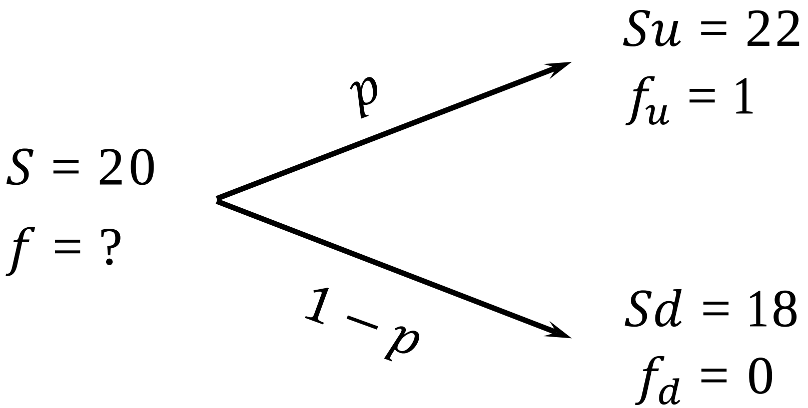
Risk-Neutral Valuation Relationship

- Risk-neutral valuation relationship (RNVR)
  - It states that any derivative can be priced with the general derivative pricing rule as if it and its underlying asset were in the risk neutral world
  - Since the expected returns of both the derivative and its underlying asset are the risk free rate
    - The probability of the upward movement in the prices of the underlying asset is $p = \frac{e^{rT} - d}{u - d}$
    - The discount rate for the expected payoff of the derivative is also r

※When we are valuing an option on a stock, the expected return on the underlying stock is irrelevant

Risk-Neutral Valuation Relationship

- Revisit the original numerical example in the risk-neutral world



  - Calculate $p = \frac{e^{rT}-d}{u-d} = \frac{e^{12\%\cdot 0.25}-0.9}{1.1-0.9} = 0.6523$
  - Calculate the option value according to the RNVR

$$e^{-12\%\cdot 0.25}[0.6523 \cdot 1 + (1-0.6523)\cdot 0] = 0.633$$

Risk-Neutral Probabilities

- Notice that $d < e^{rt} < u$.
- If not, there would have been an arbitrage strategy involving just the stock and cash.
- This also means that $0 < p < 1$.

Risk-Neutral Probabilities

- Notice that in our pricing formula, the option price is the present value of a weighted average of terminal payoffs.
- The weights were **p** and **(1-p)**
- Notice that the weights have the properties of a probability distribution
  - They are non-negative
  - They sum to one

Risk-Neutral Valuation

- These weights are called risk-neutral probabilities.
- Notice that our formula holds for general payoff functions (It is not specific to a call option).

Call: c(u) = max(0,Su-K)   c(d) = max(0,Sd-K)

Put: c(u) = max(0,K-Su)   c(d) = max(0,K-Sd)

Many others are possible

Risk-Neutral Valuation

- It is common for option pricing models to give the option price as a weighted average of terminal payoffs, where the weights are "risk neutral probabilities."
- Different models have different weights.
  (different risk neutral distributions).

An Amazing Observation

- Never this whole time have we said anything about the probability that the stock will go up or down.
- The option price is independent of the underlying asset’s expected return.
- The option price does not depend on whether you think the stock is more likely to go up or down.
- Two people can disagree about whether the stock will go up or down, but agree on the option price.

One Way to Think of It

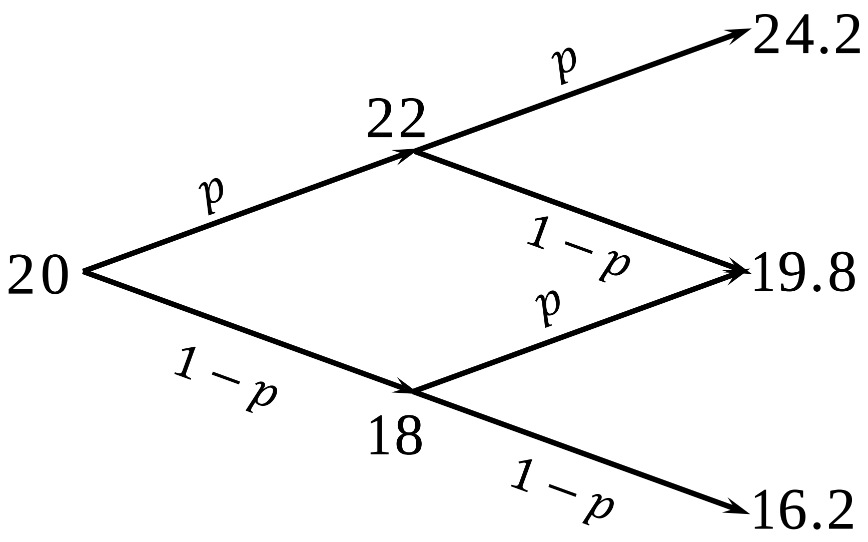
- You can believe that a stock is mispriced relative to fundamentals, and still believe that the option is correctly priced relative to the stock.
- Analogy: A Stock trading in two markets must have the same price in both markets.

Multi-Period Binomial Tree Model

- The Multi-Period Binomial Option Pricing Model is extremely flexible, hence valuable; it can value American options (which can be exercised early), and most, if not all, exotic options.

Multi-Period Binomial Tree Model

- Values of the parameters of the binomial tree
  - $S = 20$, $r = 12\%$, $u = 1.1$, $d = 0.9$, $T = 0.5$, the number of time steps is $n = 2$, and thus the length of each time step is $\Delta t = T/n = 0.25$
  - Hence, the risk-neutral probability $p = \frac{e^{r\Delta t}-d}{u-d} = \frac{e^{12\% \cdot 0.25}-0.9}{1.1-0.9} = 0.6523$

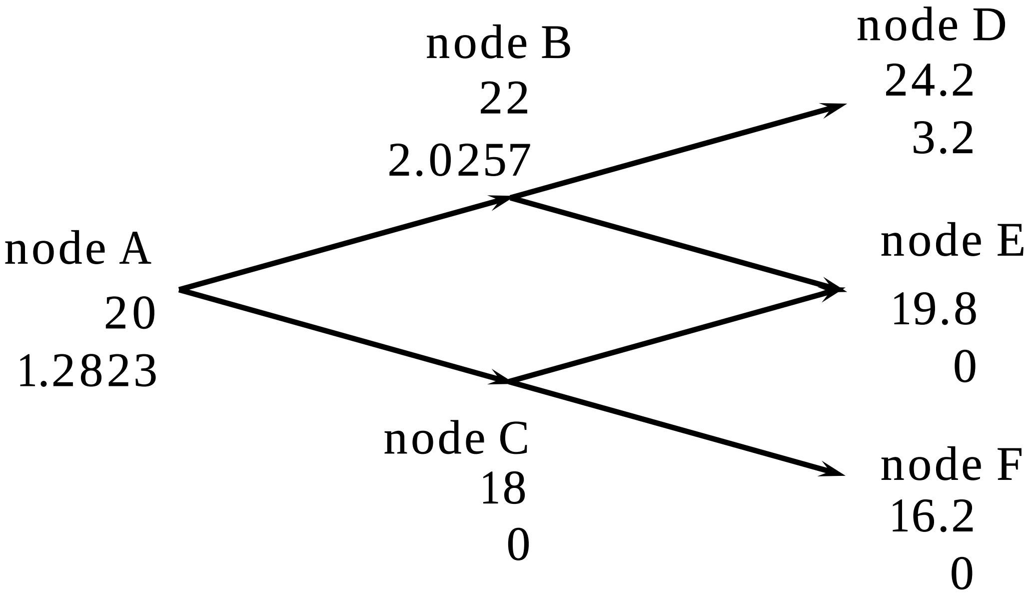


※Note the recombined feature can limit the growth of the number of nodes on the binomial tree in a acceptable manner

Multi-Period Binomial Tree Model

- For a European call option with the strike price to be 21, perform the backward induction method (逆向歸納法) recursively on the binomial tree
  - Option value at node B: $e^{-12\% \cdot 0.25}(0.6523 \cdot 3.2 + 0.3477 \cdot 0) = 2.0257$
  - Option value at node C: $e^{-12\% \cdot 0.25}(0.6523 \cdot 0 + 0.3477 \cdot 0) = 0$
  - Option value at node A (the initial or root node): $e^{-12\% \cdot 0.25}(0.6523 \cdot$

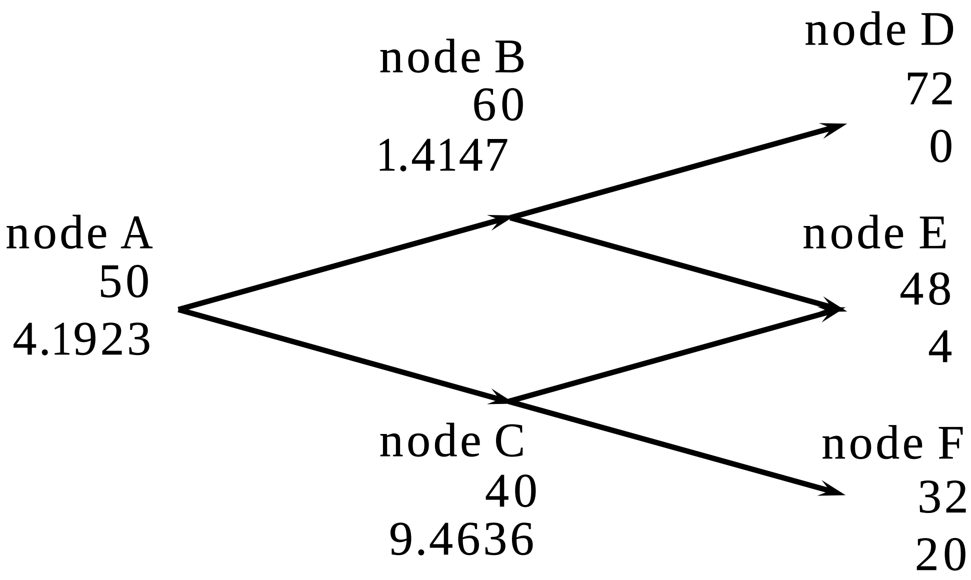
Multi-Period Binomial Tree Model



Multi-Period Binomial Tree Model

- For a European put with $K = 52$ and $T = 2$
  - $S = 50, r = 5\%, u = 1.2, d = 0.8, n = 2, \Delta t = 1$, and $p = 0.6282$
  - Option value at node B: $e^{-5\% \cdot 1}(0.6282 \cdot 0 + 0.3718 \cdot 4) = 1.4147$
  - Option value at node C: $e^{-5\% \cdot 1}(0.6282 \cdot 4 + 0.3718 \cdot 20) = 9.4636$
  - Option value at node A: $e^{-5\% \cdot 1}(0.6282 \cdot 1.4147 + 0.3718 \cdot$

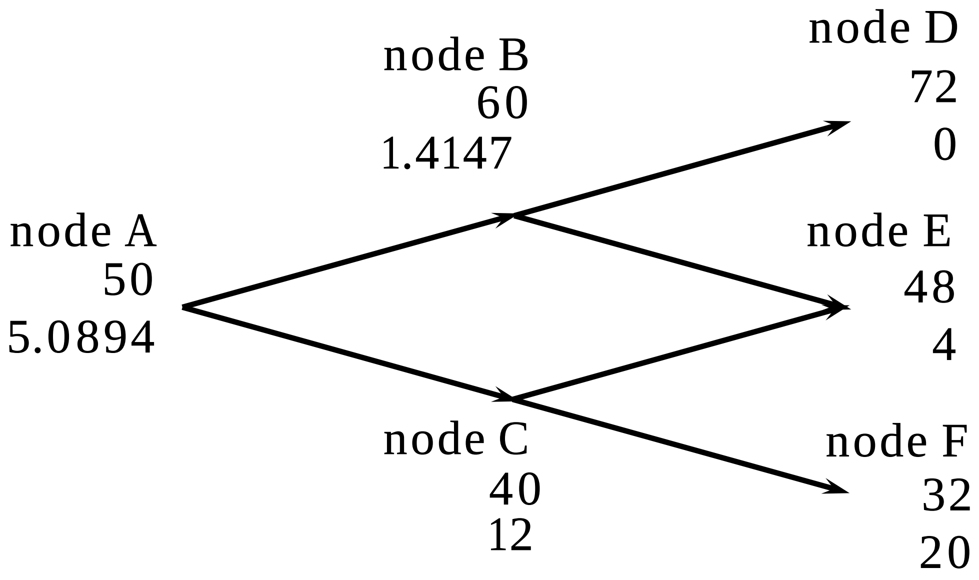
Multi-Period Binomial Tree Model



Multi-Period Binomial Tree Model

- For an American put with $K = 52$ and $T = 2$
  - Option value at node B: $e^{-5\%\cdot 1}(0.6282 \cdot 0 + 0.3718 \cdot 4) = 1.4147$
  - Option value at node C: $e^{-5\%\cdot 1}(0.6282 \cdot 4 + 0.3718 \cdot 20) = 9.4636$, which is smaller than the exercise value $\max(K -$

Multi-Period Binomial Tree Model

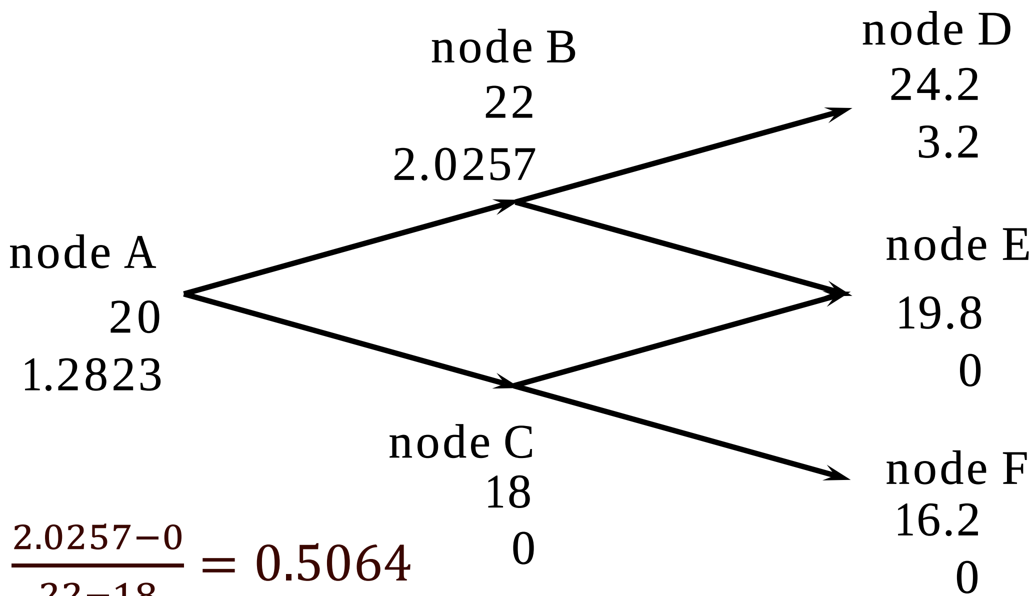


Delta

- Delta (Δ)
  - The formula to calculate Δ in the binomial tree model is $\frac{f_u - f_d}{Su - Sd}$.
  - In the binomial tree model, Δ is the number of shares of the stock we should hold for each option shorted in order to create a riskless portfolio
  - For the one-period example, the delta of the call option is $\frac{1-0}{22-18} = 0.25$

Delta

- The value of Δ varies from node to node.



  - Δ at node A: $\frac{2.0257-0}{22-18} = 0.5064$
  - Δ at node B: $\frac{3.2-0}{24.2-19.8} = 0.7273$
  - Δ at node C: $\frac{0-0}{19.8-16.2} = 0$

Delta

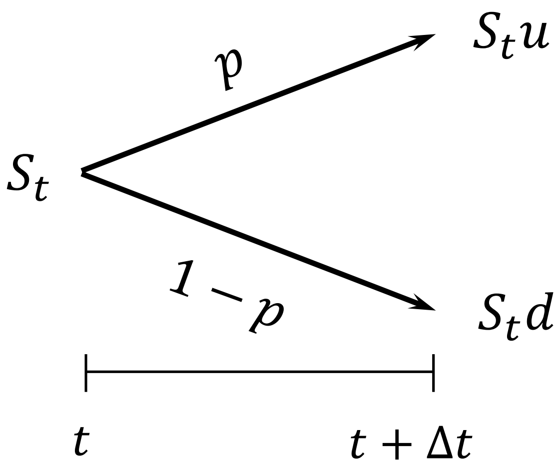
- The delta hedging strategy is a procedure to eliminate the price risk and construct a riskless portfolio for a period of time
- The method to decide the value of the delta in the binomial tree model is in effect to perform the delta hedging strategy
  - Since the value of Δ changes over time, the delta hedging strategy needs rebalances over time
  - For node A, Δ is decided to be 0.5024 such that the portfolio is riskless during the first period of time
  - If the stock price rises (falls) to reach node B (C), Δ changes to 0.7273 (0), which means that we need to increase (reduce) the number of shares held to make the portfolio risk free in the second period

Delta

- Theoretically speaking, Δ is defined as the ratio of the change in the price of a stock option with respect to the change in the price of the underlying stock, i.e., $\Delta \equiv \frac{\partial f}{\partial S}$
  - Furthermore, the delta hedging strategy can generate a riskless portfolio for a very short period of time

CRR Binomial Tree Model

- How to determine u and d
  - In practice, given any stock price at the time point t, u and d are determined to match the variance of the stock price at the next time point $t + \Delta t$



CRR Binomial Tree Model

$$\because \mathrm{var}(S_{t+\Delta t}) = E\left[S_{t+\Delta t}^2\right] - E[S_{t+\Delta t}]^2$$

$$\therefore S_t^2\sigma^2\Delta t = pS_t^2u^2 + (1-p)S_t^2d^2 - (S_te^{r\Delta t})^2$$

$$\Rightarrow \sigma^2\Delta t = pu^2 + (1-p)d^2 - e^{2r\Delta t}$$

where σ is the volatility and Δ*t* is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

$$e^{r\Delta t} = pu + (1-p)d$$

$$\sigma^2\Delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$

CRR Binomial Tree Model

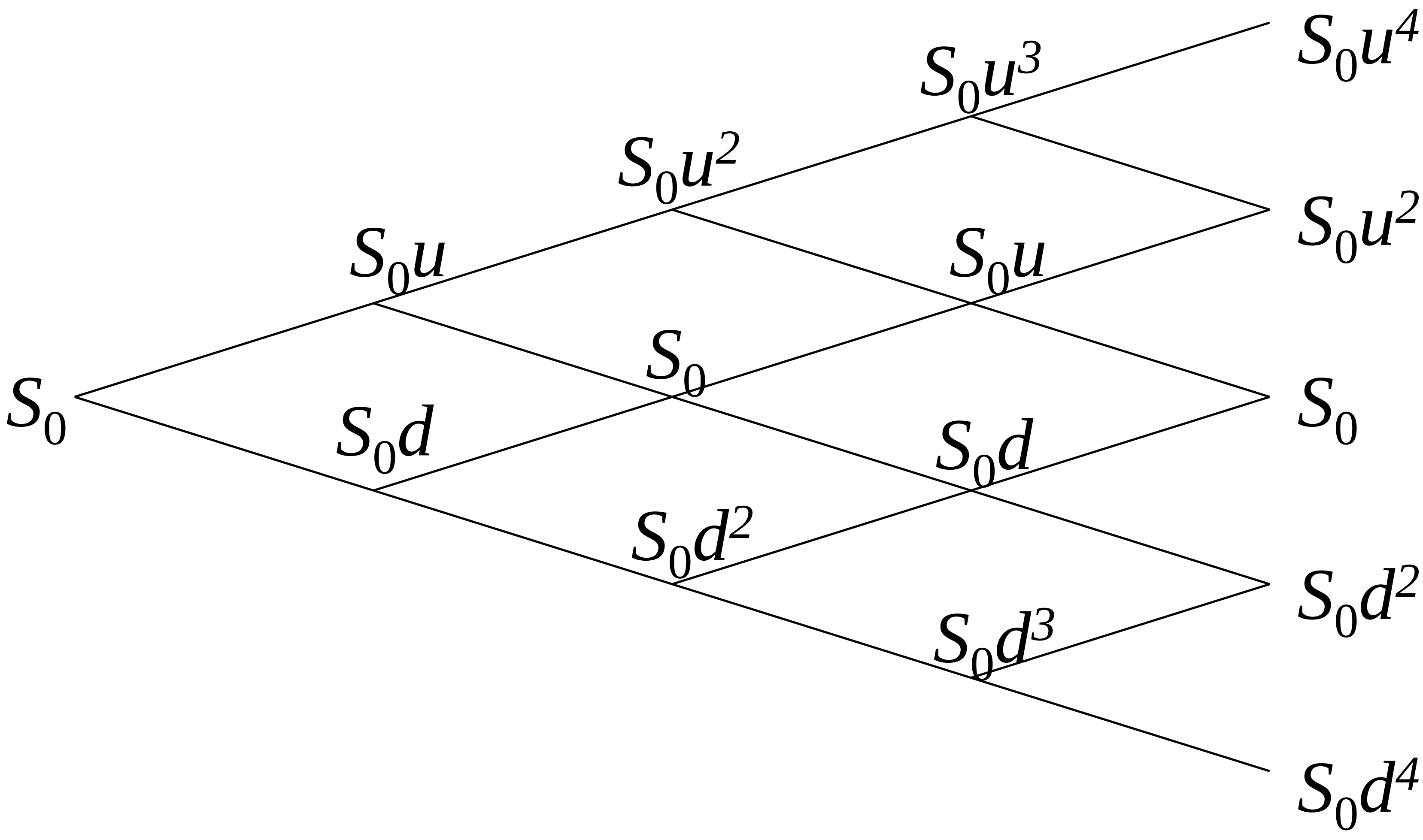
- With $p = \frac{e^{r\Delta t} - d}{u - d}$ and the assumption of $ud = 1$

  $\Rightarrow u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$

CRR Binomial Tree Model

- The validity of the CRR binomial tree model depends on the risk-neutral probability p being in [0,1]
- In practice, the life of the option is typically partitioned into hundreds time steps
  - First, ensure the validity of the risk-neutral probability, p, which approaches 0.5 if Δt approaches 0
  - Second, ensure the convergence to the Black-Scholes model.

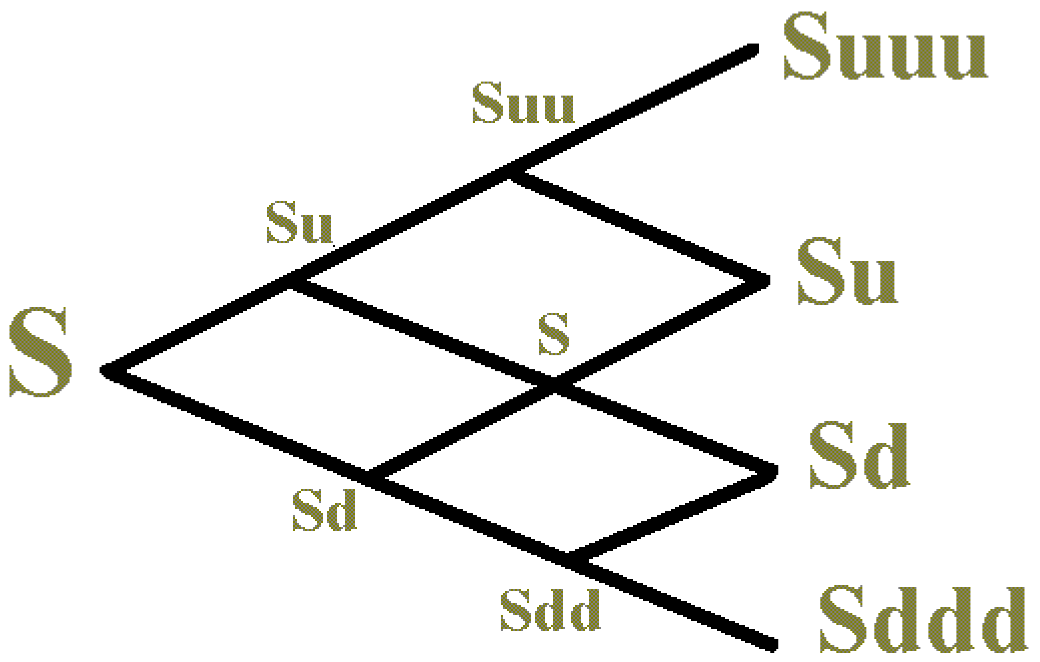
A Complete Tree



Dividend Yield in the Binomial Tree Model

- In the risk-neutral world, the total return from dividends and capital gains is r
- If the dividend yield is q, the return of capital gains in the stock price should be $r - q$
- Hence, $pS_t u + (1-p)S_t d = S_t e^{(r-q)\Delta t} \Rightarrow p = \frac{e^{(r-q)T} - d}{u-d}$
- The dividend yield does NOT affect the volatility of the stock price and thus does NOT affect the multiplying factors u and d
- So, $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ in the CRR model still can be used

Example: Three Step Binomial Tree



Example: Three Step Binomial Tree

- Suppose the parameters are:

  $u = 1.1375$, $d = 0.8791$

  $T = 1$ year 34

  $N = 3$ steps

  $S = 100$

  $X = 110$

  Length of one time step:

  $$\Delta t = T/N = 1/3$$

  Discount factor for one time step:

  $$e^{-r\Delta t} = e^{-.05(1/3)} = .98347$$

  $$1/D = e^{r\Delta t} = 1.0168$$

Example: Three Step Binomial Tree

The stock prices are:

			147.18
		129.39	
	113.75		113.75
100		100	
	87.91		87.91
		77.29	
			67.94

Solving for the Option Price

- You can find the option price by working back through the tree.
  - First, write down the option prices at the terminal nodes (these come from the payoff function).
  - Calculate values at second-to-last nodes just like we did for the one step tree.
  - Keep working back to time zero.

Terminal Nodes

The 4 terminals:

$$\max(0, 147.18 - 110) = 37.18$$
$$\max(0, 113.75 - 110) = 3.75$$
$$\max(0, 87.91 - 110) = 0$$
$$\max(0, 67.94 - 110) = 0$$

Option Price Tree

			37.18
		?	
	?		3.75
?		?	
	?		0
		?	
			0

Calculate Risk-Neutral Probability of an Up Move

- u = 1.1375, d = 0.8791
- p = $\frac{e^{r\Delta t}-d}{u-d}$ = (1.0168 - 0.8791) / (1.1375 - 0.8791)
  = .5329
- 1-p = .4671

Work Back Through the Tree

37.18

21.21

? 3.75

? 1.97

? 0

0

0

Work Back Through the Tree

37.18

21.21

12.02

3.75

6.77

1.97

1.03

0

0

0

Example: Put Option

$S_0 = 50$; $X = 50$; $r = 10\%$; $\sigma = 40\%$;

$T = 5$ months $= 0.4167$;

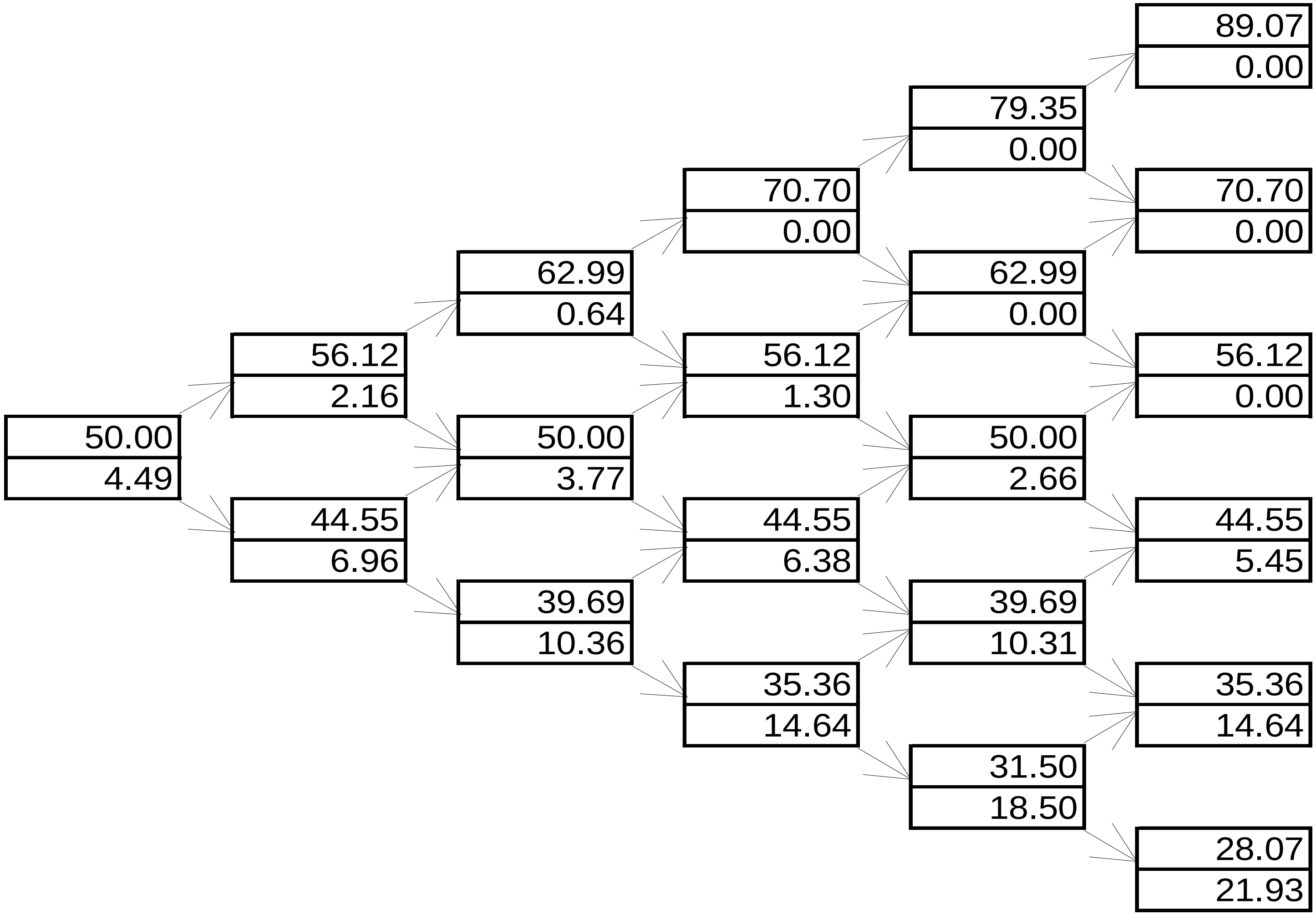
$\Delta t = 1$ month $= 0.0833$

The parameters imply

$u = 1.1224$; $d = 0.8909$;

$a = 1.0084$; $p = 0.5076$

Example



89.07
0.00
79.35
0.00
70.70
0.00
70.70
0.00
62.99
0.64
62.99
0.00
56.12
2.16
56.12
1.30
56.12
0.00
50.00
4.49
50.00
3.77
50.00
2.66
44.55
6.96
44.55
6.38
44.55
5.45
39.69
10.36
39.69
10.31
35.36
14.64
35.36
14.64
31.50
18.50
28.07
21.93

How to Estimate u and d

- First, estimate the standard deviation (σ) of stock returns.
- Then, pick: $u = e^{\sigma\sqrt{\Delta t}}$, $d = \frac{1}{u}$

Alternative Binomial Tree

- Jarrow and Rudd (1982):
- Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}}$$

$$d = e^{\left(r - \frac{\sigma^2}{2}\right)\Delta t - \sigma\sqrt{\Delta t}}$$

For European Options

- It is not really necessary to work step by step through the tree.
- Just find risk-neutral probabilities for the terminal nodes
- Then multiply terminal payoffs by the risk-neutral probabilities.

For American Options

- You generally have to work back through the tree, checking for early exercise at each node.
- Unless you already know that early exercise in never optimal (as is the case for American calls when there are no dividends)

For American Options

- The value of the option if it is left "alive" (i.e., unexercised) is given by the value of holding it for another period, equation.
- The value of the option if it is exercised is given by $max\ (0, S - K)$ if it is a call and $max\ (0, K - S)$ if it is a put.
- For an American call, the value of the option at a node is given by

$$C(S, K, t) = max\ (S - K, e^{-rh}\,[C(uS, K, t + h)\,p^* + C(dS, K, t + h)\,(1 - p^*)])$$

For American Options

- The valuation of American options proceeds as follows:
  - At each node, we check for early exercise.
  - If the value of the option is greater when exercised, we assign that value to the node. Otherwise, we assign the value of the option unexercised.
  - We work backward through the three as usual.

Options on Other Assets

- The model developed thus far can be modified easily to price options on underlying assets other than nondividend-paying stocks.
- The difference for different underlying assets is the construction of the binomial tree and the risk-neutral probability.
- We examine options on
  - stock indexes,
  - currencies,
  - futures contracts,
  - commodities,
  - bonds.

Options on a Stock Index

- Suppose a stock index pays continuous dividends at the rate Δ.
- The procedure for pricing this option is equivalent to that of the first example, which was used for our derivation. Specifically,
  - the up and down index moves.
  - the replicating portfolio.
  - the option pricing equations
  - the risk-neutral probability.

Options on Currency

- With a currency with spot price x_0, the forward price is

$$F_{0,t} = x_0 e^{(r-r_f)t},$$

where r_f is the foreign interest rate.

- Thus, we construct the binomial tree using

$$ux = xe^{(r-r_f)h+\delta\sqrt{h}}$$

$$dx = xe^{(r-r_f)h-\delta\sqrt{h}}$$

Options on Currency

- Investing in a "currency" means investing in a money-market fund or fixed income obligation denominated in that currency.
- Taking into account interest on the foreign-currency denominated obligation, the two equations are

$$\Delta \times uxe^{r_f h} + e^{rh} \times B = C_u$$
$$\Delta \times dxe^{r_f h} + e^{rh} \times B = C_d$$

- The risk-neutral probability of an up move is

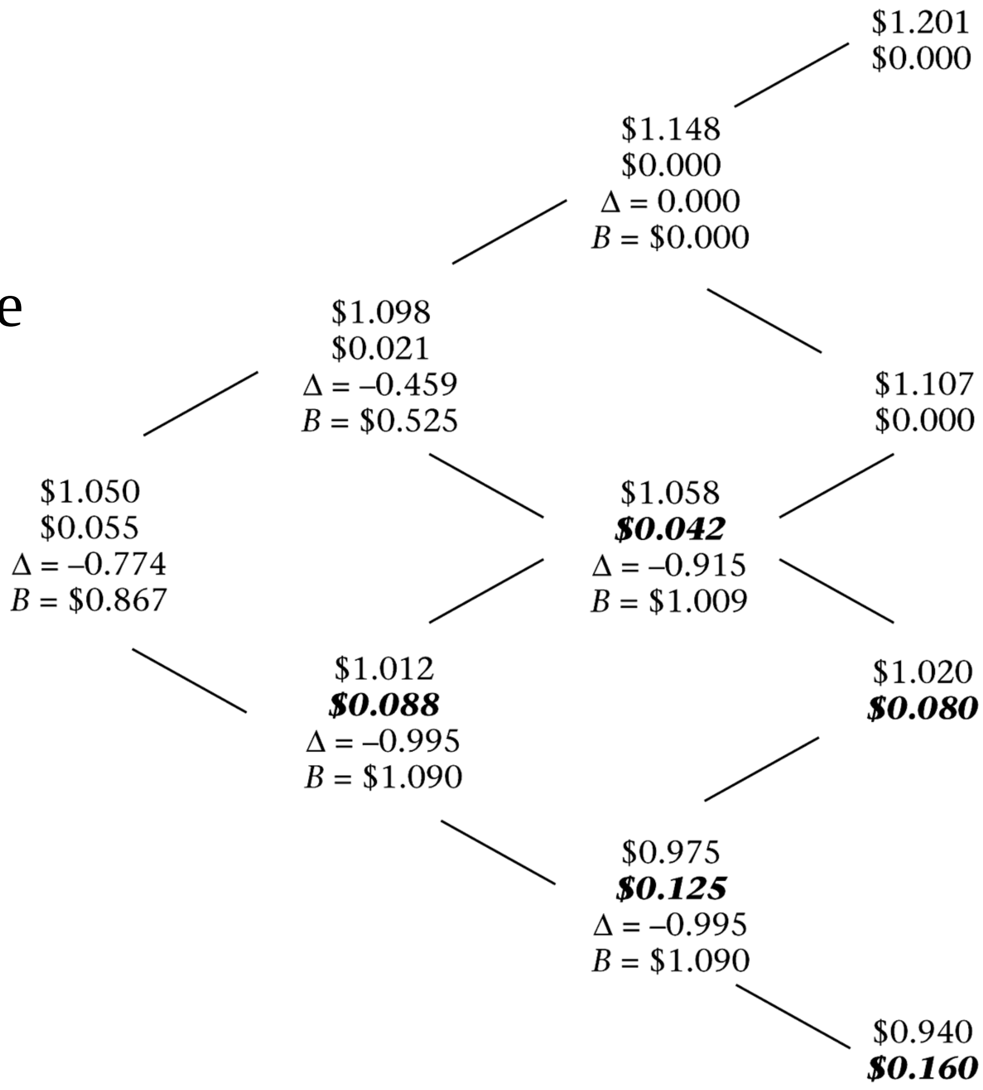
$$p^* = \frac{e^{(r-r_f)h} - d}{u-d}$$

Options on Currency

- Consider a dollar-denominated American put option on the euro, where
  - the current exchange rate is $1.05/€,
  - the strike is $1.10/€,
  - the euro-denominated interest rate is 3.1%,
  - the dollar-denominated rate is 5.5%.

Options on Currency

- The binomial tree for the American put option on the euro:



Options on Futures Contracts

- Assume the forward price is the same as the futures price.
- The nodes are constructed as

$$u = e^{\sigma\sqrt{h}}$$
$$d = e^{-\sigma\sqrt{h}}$$

- We need to find the number of futures contracts, Δ, and the lending, *B*, that replicates the option.
  - A replicating portfolio must satisfy

$$\Delta \times (uF - F) + e^{rh} \times B = C_u$$
$$\Delta \times (dF - F) + e^{rh} \times B = C_d$$

Options on Futures Contracts

- Solving for Δ and B gives

$$\Delta = \frac{C_u - C_d}{F(u-d)}$$

$$B = e^{-rh}\left(C_u \frac{1-d}{u-d} + C_d \frac{u-1}{u-d} \right)$$

Δ tells us how many futures contracts to hold to hedge the option, and B is simply the value of the option.

- We can again price the option.
- The risk-neutral probability of an up move is given by

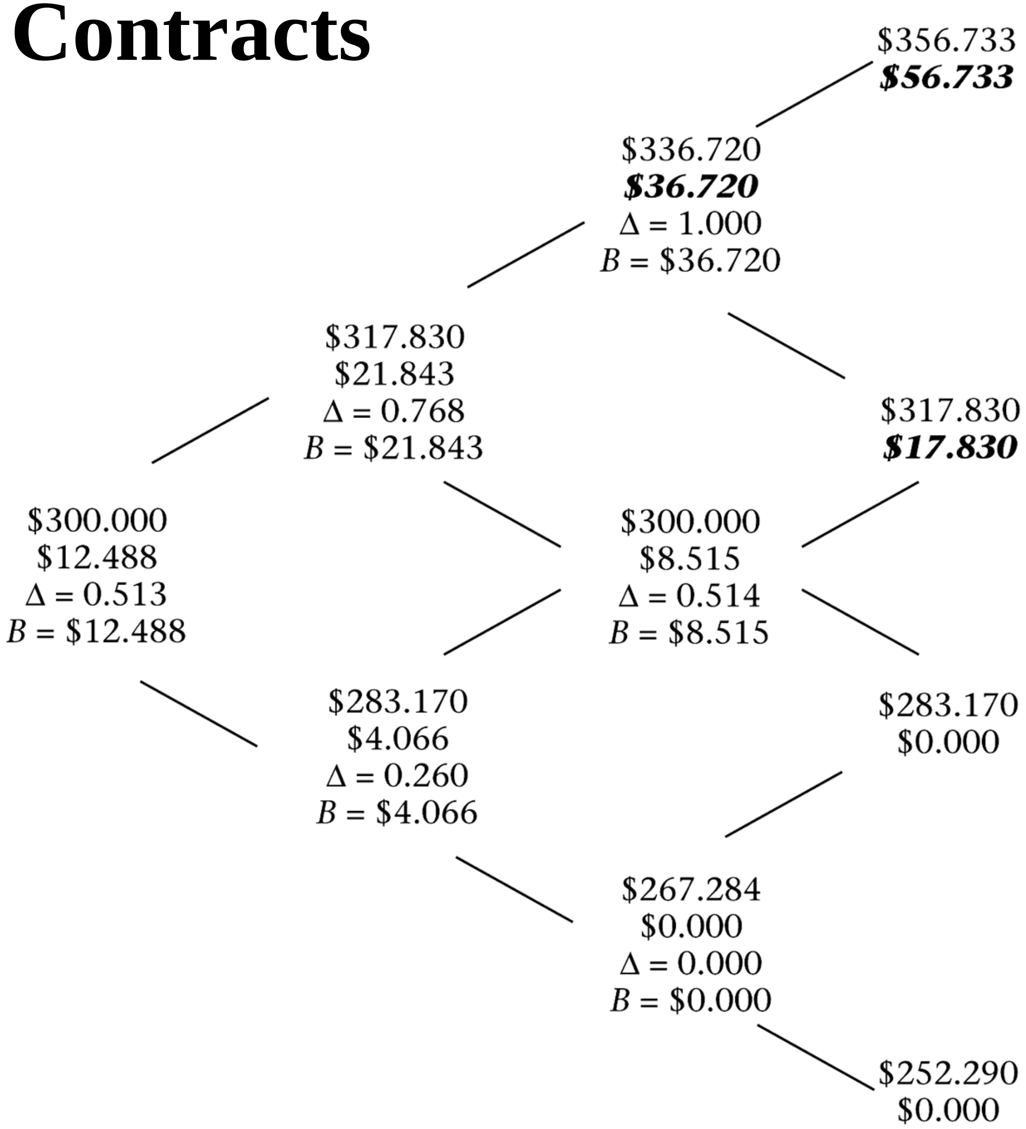
$$p^* = \frac{1-d}{u-d}$$

Options on Futures Contracts

- The motive for early-exercise of an option on a futures contract is the ability to earn interest on the mark-to-market proceeds.
  - When an option is exercised, the option holder pays nothing, is entered into a futures contract, and receives mark-to-market proceeds of the difference between the strike price and the futures price.

Options on Futures Contracts

- A tree for an American call option on a gold futures contract:



Options on Commodities

- It is possible to have options on a physical commodity.
- If there is a market for lending and borrowing the commodity, then pricing such an option is straightforward.
  - In practice, however, transactions in physical commodities often have greater transaction costs than for financial assets, and short-selling a commodity may not be possible.
- From the perspective of someone synthetically creating the option, the commodity is like a stock index, with the lease rate equal to the dividend yield.

Options on Bonds

- Bonds are like stocks that pay a discrete dividend (a coupon).
- Bonds differ from the other assets in two respects:
    1. The volatility of a bond decreases over time as the bond approaches maturity.
    2. The assumptions that interest rates are the same for all maturities and do not change over time are logically inconsistent for pricing options on bonds.

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year.
- The standard deviation of the return in time Δt is $\sigma\sqrt{\Delta t}$.
- If a stock price is $50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data

- Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years.
- Calculate the continuously compounded return in each interval as $u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$.
- Calculate the standard deviation s of the u_i's.
- The historical volatility estimate is $\hat{\sigma} = s/\sqrt{\tau}$.

Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed.
- For this reason time is usually measured in "trading days" not calendar days when options are valued.