Chapter 3

Binomial Model

The proper valuation of options has been a focus of economists and mathematicians. Between the two types of derivatives, futures and options, it is relatively easy to establish the price of a futures contract. When the cost of a stock rises, the price of the futures contract on the stock increases by the same relative amount, which makes the relationship linear. For options, there is no such simple link with the underlying asset.

It was Black and Scholes who first created the option-pricing model using stochastic processes.[1] Before the Black-Scholes model came along, options were bought and sold for the value that individual traders thought they ought have. The strike price of an option was usually the *forward price* which is just the current price of the stock or currency adjusted for interest-rate effects.

Before the Black-Scholes model we will study the binomial model suggested by Cox, Ross and Rubinstein[2]. They assume that stock price movements are composed of a large number of small binomial movements.

[1] F. Black & M. Scholes, "*The pricing of options and corporate liabilities*," Journal of Political Economy **81**, 637-659 (1973)

[2] J. Cox, S. Ross & M. Rubinstein, "*Option pricing: A simplified approach*," Journal of Financial Economics **7**, 229-264 (1979)

3.1 Boundary conditions for option prices

Let us denote the prices of European and American call options of the strike price X and the spot price of the asset S at time to mature $\tau = T - t$, by $c(S, \tau, X)$ and $C(S, \tau, X)$, respectively and the prices of European and American put options by $p(S, \tau, X)$ and $P(S, \tau, X)$. Because an option has only limited liability,

$$c(S, \tau, X), C(S, \tau, X), p(S, \tau, X) \text{ and } P(S, \tau, X) \geq 0. \tag{3.1}$$

At the expiry $t = T$, an option is worth its intrinsic value;

$$\begin{aligned} c(S, \tau = 0, X) &= C(S, \tau = 0, X) = \max(S_T - X, 0) \\ p(S, \tau = 0, X) &= P(S, \tau = 0, X) = \max(X - S_T, 0). \end{aligned} \tag{3.2}$$

An American option must be worth at least their corresponding intrinsic values at anytime of its life;

$$\begin{aligned} C(S, \tau, X) &\geq \max(S - X, 0) \\ P(S, \tau, X) &\geq \max(X - S, 0). \end{aligned} \tag{3.3}$$

An American option is worth at least its European counterpart because of the privilege of early exercise;

$$C(S, \tau, X) \geq c(S, \tau, X) \quad \text{and} \quad P(S, \tau, X) \geq p(S, \tau, X). \tag{3.4}$$

If all other conditions are the same, the option with a longer lifetime is worth more; for $\tau_1 < \tau_2$

$$\begin{aligned} c(S, \tau_1, X) < c(S, \tau_2, X) \quad &, \quad C(S, \tau_1, X) < C(S, \tau_2, X) \\ p(S, \tau_1, X) < p(S, \tau_2, X) \quad &, \quad P(S, \tau_1, X) < P(S, \tau_2, X). \end{aligned} \tag{3.5}$$

If all other conditions are the same, the call (put) option with a lower (higher) strike price is worth more; for $X_1 < X_2$

$$\begin{aligned} c(S, t, X_1) > c(S, t, X_2) \quad &, \quad C(S, t, X_1) > C(S, t, X_2) \\ p(S, t, X_1) < p(S, t, X_2) \quad &, \quad P(S, t, X_1) < P(S, t, X_2). \end{aligned} \tag{3.6}$$

Call (Put) option price functions are decreasing (increasing) functions of their strike prices. Note also that call (put) option price functions are increasing (decreasing) functions of the *asset price*.

3.1.1 Upper boundary

For there to be no arbitrage opportunity, a call option is not worth more than its underlying stock:

$$c(S,\tau,X),\quad C(S,\tau,X) \le S, \tag{3.7}$$

where S is the spot price for the underlying stock. If $c(S,\tau,X) > S$, an arbitrageur can gain a risk-free profit by shorting a call option and buying a stock at the same time. Under the no arbitrage condition, a put option is not worth more than its strike price:

$$p(S,\tau,X),\quad P(S,\tau,X) \le X, \tag{3.8}$$

where X is the strike price. If an American put option is priced more than its strike price, shorting a put option an arbitrageur can get a risk-free profit. Furthermore, for a European put option, the option is not exercised till its maturity so its value cannot exceed the present value of its strike price:

$$p(S,\tau,X) \le X\mathrm{e}^{-r(T-t)}. \tag{3.9}$$

3.1.2 Lower boundary for a European option

Non-dividend-paying stock

Let us consider the following portfolios:

- Portfolio A: a long position in one European call option and cash $X\mathrm{e}^{-r(T-t)}$
- Portfolio B: a long position in one share.

At time T, the value of portfolio A is $\max(S_T, X)$ while that of portfolio B is only S_T so that portfolio A is worth more than portfolio B. At time t, thus,

$$c(S,\tau,X) + X\mathrm{e}^{-r(T-t)} \ge S \quad\Longrightarrow\quad c(S,\tau,X) \ge \max(S - X\mathrm{e}^{-r(T-t)}, 0). \tag{3.10}$$

To find the lower bound for European put option, let us consider the following portfolio

- Portfolio C: a long position in one European put option and a long position in one share
- Portfolio D: cash amount $X\mathrm{e}^{-r(T-t)}$

It is obvious that portfolio C is worth at least portfolio D so we have the following boundary condition

$$p(S,\tau,X) + S \geq X\mathrm{e}^{-r(T-t)} \implies p(S,\tau,X) \geq \max(X\mathrm{e}^{-r(T-t)} - S, 0). \quad (3.11)$$

$\boxed{\text{Dividend-paying stock}}$

If dividend is payed on the underlying stock during the lifetime of an option, the value of the stock has to be adjusted from S to $S - I$ where I is *the present value of the dividend.*

3.2 put-call parity

Consider a call option $c = c(S,\tau,X)$ on a stock S. If one has a portfolio composed of a call option and cash of the strike price X on the exercise day T, the portfolio values either the stock price or the strike price whichever is bigger. If the risk-free interest rate is r, $Xe^{-r(T-t)}$ at time $t < T$ is equivalent to X at T so that

$$c + Xe^{-r(T-t)} \text{ (at } t) \Rightarrow \max[S_T, X] \text{ (at } T). \quad (3.12)$$

Consider another portfolio composed of a put option $p = p(S,\tau,X)$ of the strike price X and a stock. This portfolio values the maximum of X and S_T at the exercise day:

$$p + S \text{ (at } t) \Rightarrow \max[S_T, X] \text{ (at } T). \quad (3.13)$$

Thus using Eqs. (3.12, 3.13) the relation between the put and call options on the same strike price is obtained as

$$c + Xe^{-r(T-t)} = p + S. \quad (3.14)$$

This is the so-called *put-call parity.*

If the underlying stock pays out a dividend, the put-call parity is somewhat modified to

$$c + I + Xe^{-r(T-t)} = p + S \tag{3.15}$$

where I is the present value of the dividend.

3.3 Price boundary and put-call parity for American options

An American call on a non-dividend paying asset carries the same value as its European counterpart. When an American call option is exercised, its value immediately becomes $\max(S - X, 0)$ which is smaller than the lower limit of European call option value shown in Eq.(3.10);

$$c(S, \tau, X) = C(S, \tau, X). \tag{3.16}$$

An American put option on a non-dividend paying asset is worth at least its European counter part. When an American put option is exercised, it realises its intrinsic value $\max(X - S, 0)$, which is larger than the lower limit of the put option $\max(X\mathrm{e}^{-r(T-t)} - S, 0)$ in Eq.(3.11);

$$P(S, \tau, X) \geq p(S, \tau, X). \tag{3.17}$$

What does the put-call parity look like for American options?
Using the put-call parity (3.14) for European options and Eqs.(3.16) and (3.17), we find

$$P > C + X\mathrm{e}^{-r(T-t)} - S \Longrightarrow C - P < S - X\mathrm{e}^{-r(T-t)}, \tag{3.18}$$

where the argument (S, τ, X) has been omitted. To find the lower boundary of $C - P$ we consider the two portfolios:

- Portfolio A: a long position in one European call option and an amount of cash equal to X in a risk-free account

- Portfolio B: a long position in one American put option and one share.

At the expiry, $t = T$, the portfolio A is worth

$$\max(S_T, X) + X\mathrm{e}^{r(T-t)} - X \tag{3.19}$$

and the portfolio B is

$$\max(S_T, X). \tag{3.20}$$

We can clearly see that portfolio A is worth more than portfolio B. This argument can easily be extended to the same inequality at anytime. Thus

$$c + X > P + S \Longrightarrow C + X > P + S \Longrightarrow C - P > S - X. \tag{3.21}$$

Eqs.(3.18) and(3.21) give us

$$S - X < C - P < S - X\mathrm{e}^{-r(T-t)}, \tag{3.22}$$

3.4 Binomial trees

We consider a useful technique to price an option. The technique is based on an assumption that the underlying asset price movements consist of a large number of small binomial movements.

Assume, at time $t = 0$, the asset price is S and, at a later time δt, the asset price could go up to uS with the probability p or down to dS with the probability $q = 1 - p$ where $u > 1$ and $0 < d < 1$.

3.4.1 Risk-neutral assumption

Consider the following portfolio

- Portfolio: a long position in Δ number of shares and a short position in one call option.

The movement of the share price shows binomial behaviour. It can go up or down. We want to calculate Δ which gives the same value of the portfolio regardless of

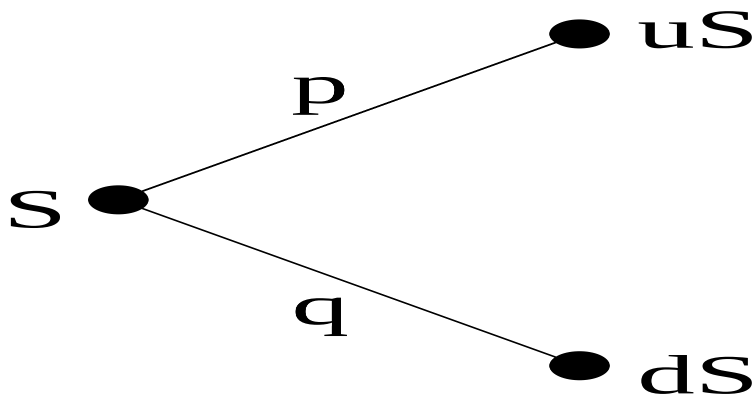

Figure 3.1: Binomial tree

the share price movement. As an example, assume that the share price is £30 at $t = 0$ and we know that, at $t = \delta t$ the price can go up to £35 and it can go down to £29. The strike price of the call option is £32. Then the portfolio is worth either £35Δ − £3 (up) or £29Δ (down). So Δ we want to find is obtained from the following equation

$$\pounds 35\Delta - \pounds 3 = \pounds 29\Delta. \tag{3.23}$$

We need to have $\Delta = 0.5$ to be in the *risk-neutral position*.

More generally, we denote the value of the call options at time δt by

$$\begin{aligned} c_u &= \max(uS - X, 0) \quad : \quad \text{after upward movement} \\ c_d &= \max(dS - X, 0) \quad : \quad \text{after downward movement.} \end{aligned} \tag{3.24}$$

To calculate Δ to be in the risk-neutral position,

$$\Delta uS - c_u = \Delta dS - c_d. \tag{3.25}$$

Solving the equations, we obtain

$$\Delta = \frac{c_u - c_d}{(u-d)S}. \tag{3.26}$$

Taking the risk-neutral position, the value of the portfolio at the future time δt is 100% secured: Denoting the secured value of the portfolio by A

$$A = \frac{dc_u - uc_d}{u - d} \tag{3.27}$$

Now, under the arbitrage-free condition, the initial investment, $\Delta S - c$ has to be the same as the present value of A so we write

$$
\begin{aligned}
& \Delta S - c = A\mathrm{e}^{-r\delta t} \\
\Rightarrow \quad & c = \Delta S - A\mathrm{e}^{-r\delta t} \\
\Rightarrow \quad & c = \left(\frac{\mathrm{e}^{r\delta t} - d}{u - d} c_u + \frac{u - \mathrm{e}^{r\delta t}}{u - d} c_d \right) \mathrm{e}^{-r\delta t}. \qquad (3.28)
\end{aligned}
$$

Denoting

$$
p = \frac{\mathrm{e}^{r\delta t} - d}{u - d}. \qquad (3.29)
$$

The risk-neutral call option price at the present time is

$$
c = [pc_u + (1 - p)c_d]\mathrm{e}^{-r\delta t} \qquad (3.30)
$$

If we take p as the risk-neutral probability for the share price to go up, the call option price is the present value of the value of call option which will be realised in the future time δt.

3.4.2 Risk-neutral probability

Let us consider the risk-neutral probability defined in Eq.(3.29). If it is really the probability of the share price to go up, we may write the expectation value of the share price,

$$
E[S] = puS + (1 - p)dS \qquad (3.31)
$$

and using Eq.(3.29) we find $E[S] = S\mathrm{e}^{r\delta t}$ which makes very good sense.

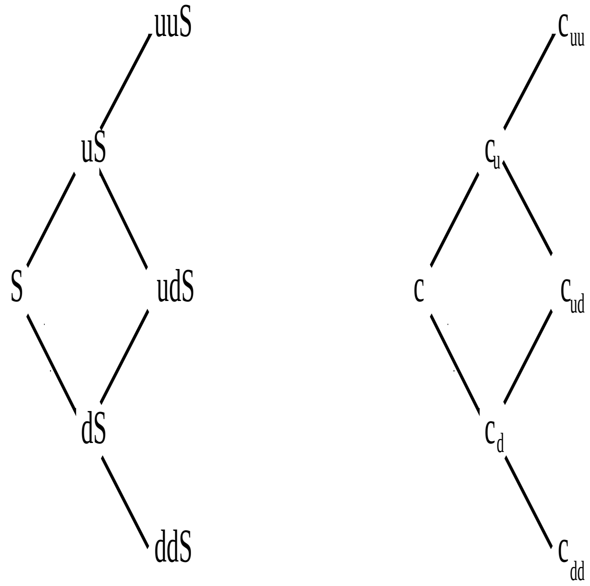

Figure 3.2: Binomial tree for asset price and call price in a two-period model. $c_{uu} = \max(u^2S - X, 0)$, $c_{ud} = \max(udS - X, 0)$ and $c_{dd} = \max(d^2S - X, 0)$

We recall the earlier example and assume that the risk-free interest rate is 5% per annum and δt is 0.3 year. The risk-neutral probability (3.29) can be written as

$$p = \frac{S\mathrm{e}^{r\delta t} - Sd}{Su - Sd}. \tag{3.32}$$

Substituting relevant values, we find the risk-neutral probability $p = 0.24$ The value of a call option at $t = \delta t$ is $0.24 \times £3 + (1 - 0.24) \times £0 = £0.72$. The present value is the value of the call option;

$$c = \mathrm{e}^{-0.015} \times £0.72 = £0.71. \tag{3.33}$$

The extension of the binomial model with two periods is straightforward. Let c_{uu} denote the call value at $2\delta t$ with two consecutive upward moves of the asset price and similar notational interpretation for c_{ud} and c_{dd}:

$$c_{uu} = \max(u^2S - X, 0)\ ,\ c_{ud} = \max(udS - X, 0)\ ,\ c_{dd} = \max(d^2S - X, 0). \quad (3.34)$$

Assuming p the risk-neutral probability, the values of call options at δt are

$$\begin{aligned} c_u &= \mathrm{e}^{-r\delta t}[pc_{uu} + (1-p)c_{ud}] \\ c_d &= \mathrm{e}^{-r\delta t}[pc_{ud} + (1-p)c_{dd}]. \end{aligned} \quad (3.35)$$

We substitute c_u and c_d in Eq. (3.35) into Eq. (3.30) and obtain the current call value:

$$c = \mathrm{e}^{-2r\delta t}[p^2c_{uu} + 2p(1-p)c_{ud} + (1-p)^2c_{dd}]. \quad (3.36)$$

We can generalise the result to value an option which expires at $t = n\delta t$. With use of Eq.(3.34) in Eq.(3.36), the option value is obtained as

$$c = \mathrm{e}^{-nr\delta t}\sum_{j=0}^{n} {}_nC_j p^j(1-p)^{n-j}\max(u^jd^{n-j}S - X, 0), \quad (3.37)$$

where $c_{u^jd^{n-j}} = \max(u^jd^{n-j}S - X, 0)$ and ${}_nC_j = n!/[j!(n-j)!]$ is the combination formula.

Let us assume that k is the smallest integer which makes $u^kd^{n-k}S \geq X$, then Eq.(3.37) is written as

$$c = S\mathrm{e}^{-nr\delta t}\sum_{j=k}^{n} {}_nC_j p^j(1-p)^{n-j}u^jd^{n-j} - X\mathrm{e}^{-nr\delta t}\sum_{j=k}^{n} {}_nC_j p^j(1-p)^{n-j}. \quad (3.38)$$

This is the present value of the call option. ${}_nC_j p^j(1-p)^{n-j}$ is the risk-neutral probability of jth event to occur and $u^jd^{n-j}S$ is the value of the asset in this case. The term $\mathrm{e}^{-nr\delta t}$ is there to consider the present value. The second term is the present value of the cost to exercise the option. We can make the equation even tidier by using the complimentary binomial distribution function defined as

$$\Phi(n, k, p) = \sum_{j=k}^{n} {}_nC_j p^j(1-p)^{(n-j)} \quad (3.39)$$

and by denoting $p' = up\mathrm{e}^{-r\delta t}$. The call value (3.38) becomes

$$c = S\Phi(n, k, p') - X\mathrm{e}^{-nr\delta t}\Phi(n, k, p). \quad (3.40)$$

The value of a put option can be obtained using the put-call parity.

3.5 Delta

The delta is the ratio of the change in the price of an option to the change in the price of the underlying stock. It can be understood as the number of the underlying stock we should hold for each option shorted to create a risk-neutral portfolio. The delta is Δ in Eq.(3.26) at $t = 0$;

$$\Delta_0 = \frac{c_u - c_d}{(u-d)S} \tag{3.41}$$

This is the same as *alpha* in Eq.(3.26). At $t = \delta t$, there are two delta values;

$$\begin{aligned} \Delta_1 &= \frac{c_{uu} - c_{ud}}{(uu - ud)S} \\ \Delta_2 &= \frac{c_{ud} - c_{dd}}{(ud - dd)S}. \end{aligned} \tag{3.42}$$

The delta of a call option is positive while the delta of a put option is negative. It is easily shown that if the call is deep out-of-money, the delta is zero while if it is deep in-the-money, the delta is unity. The delta value changes from node to node which means that we have to adust the number of options to hold to be in the risk-neutral position. This is called the *delta hedging*. This is seen in the example leading to Eq. (3.23).

3.6 How to evaluate u and d

We still do not know how to determine u and d. Eq.(3.29) can be written as

$$pu + (1-p)d = e^{r\delta t}. \tag{3.43}$$

The term in the right-hand side is the expectation value, $E[S_{\delta t}/S]$, at $t = \delta t$. The diffusion of the process of the asset price is determined by the variance, $\mathrm{var}[S_{\delta t}/S]$,

$$\mathrm{var}[S_{\delta t}/S] = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2 = pu^2 + (1-p)d^2 - \mathrm{e}^{2r\delta t} \tag{3.44}$$

Assuming the Black-Scholes formula is right,[3] we take the variance given by the Black-Scholes model:

$$\mathrm{var}[S_{\delta t}/S] = \mathrm{e}^{2r\delta t}(\mathrm{e}^{\sigma^2\delta t} - 1) \tag{3.45}$$

where σ is the volatility. We need another equation to get the values for two variables u and d.

Case 1

Assume $u = 1/d$. Substituting it into Eqs.(3.43) and (3.44), and using Eq.(3.45), we find

$$u = \frac{R^2 + 1 + \sqrt{(R^2+1)^2 - 4\mathrm{e}^{2r\delta t}}}{2\mathrm{e}^{r\delta t}}, \tag{3.46}$$

where $R^2 = \mathrm{e}^{2r\delta t}\mathrm{e}^{\sigma^2\delta t}$. [4]

Case 2

Assume $p = 1/2$ and take Eqs.(3.43), (3.44) and (3.45), we then find

$$u = \mathrm{e}^{r\delta t}(1 + \sqrt{\mathrm{e}^{\sigma^2\delta t} - 1}), \quad d = \mathrm{e}^{r\delta t}(1 - \sqrt{\mathrm{e}^{\sigma^2\delta t} - 1}). \tag{3.47}$$

3.7 Numerical implementation

Using the binomial model, we can calculate the option value numerically. One of the important advantages of this numerical approach is that we can calculate the American option price as well as the European option price.

We use Case 1 ($u = 1/d$) to simplify the problem and to reduce the computation time. Then, in general, at time $t = i\delta t$, $i + 1$ stock prices are considered. These are

$$Su^j d^{i-j} \quad , \quad j = 0, 1, \cdots, i. \tag{3.48}$$

[3] We will fully explore the Black-Scholes model in Chap. 5. See also the variance for the geometric Brownian motion in Eq. (4.21).

[4] Taylor expand Eq.(3.46) in powers of $\sqrt{\delta t}$ and assume $O(\sqrt{\delta t}^3) \approx 0$ then we get

$$u = \frac{1}{d} = \mathrm{e}^{\sigma\sqrt{\delta t}}.$$

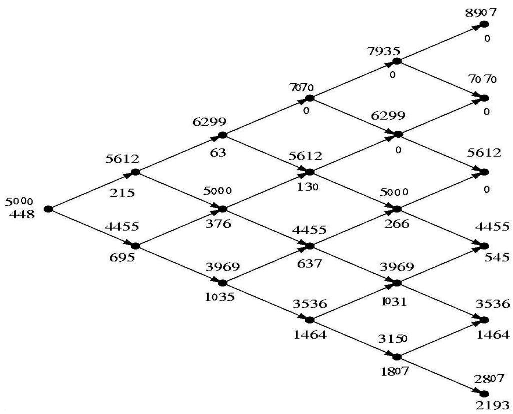

Figure 3.3: Binomial tree for American put option in cents.

To illustrate the approach, let us consider an example. Consider a five-month American put option on a non-dividend-paying stock when the stock price $S =$ \$50. The option is at the money, *i.e.*, $X =$ \$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. Suppose we divide the life of the option into five intervals of length one month ($\delta t = 1/12$ year). We then easily find that

$$u = \mathrm{e}^{\sigma\sqrt{\delta t}} = 1.1224 \qquad d = \mathrm{e}^{-\sigma\sqrt{\delta t}} = 0.8909$$
$$\mathrm{e}^{r\delta t} = 1.0084 \qquad p = \frac{\mathrm{e}^{r\delta t} - d}{u - d} = 0.5076.$$

With use of Eq.(3.48) we can easily calculate the value of the stock at the node as shown in the top figures in Fig. 3.3.

Options are evaluated by starting at the *end* of the tree and working backward. For example, a put option is worth $\max(X - S_T, 0)$. Because a risk-neutral world is being assumed, the value at each node at time $T - \delta t$ can be calculated as the expected value at time T discounted at rate r for a time period δt. Similarly, the value at each node at time $T - 2\delta t$ can be calculated as the expected value at time $T - \delta t$ discounted for a time period δt at rate r, and so on. In our example of the American option, it is necessary to check at each node to see whether early exercise is preferable. The bottom figure at each node in Fig. 3.3 is the option price for the example in hand.

3.8 Variations of binomial models

3.8.1 Underlying stock paying a dividend

The value of a share reflects the value of the company. After a dividend is payed, the value of the company is reduced so the value of the share.

Let us assume a single dividend during the life of an option and the dividend, which is given as a proportion of the share price, is *known*. Assume the dividend is proportional to the share price at the ex-dividend day $i\delta t$, then the share prices at nodes are

$$\begin{aligned} &Su^j d^{k-j} \qquad && j = 0, 1, 2, \cdot\cdot, k \quad && k = 0, 1, 2, \cdot\cdot, i-1 \\ &S(1-\gamma)u^j d^{k-j} \qquad && j = 0, 1, 2 \cdot\cdot, k \quad && k = i, i+1, \cdot\cdot \end{aligned} \tag{3.49}$$

where γ is the dividend yield.

We have assumed that the dividend is not proportional to the share price but fixed per share. If the dividend D is payed on $i\delta t$,

$$\begin{aligned} &Su^j d^{k-j} \qquad && j = 0, 1, 2, \cdot\cdot, k \quad && k = 0, 1, 2, \cdot\cdot, i-1 \\ &Su^j d^{k-j} - D \qquad && j = 0, 1, 2, \cdot\cdot, k \quad && k = i. \end{aligned} \tag{3.50}$$

3.8.2 Underlying stock with continuous dividend yield

A stock index is composed of several hundred different shares. Each share gives dividend away at a different time so the stock index can be assumed to provide a dividend continuously. If we modify the risk-free interest rate from r to $r - q$

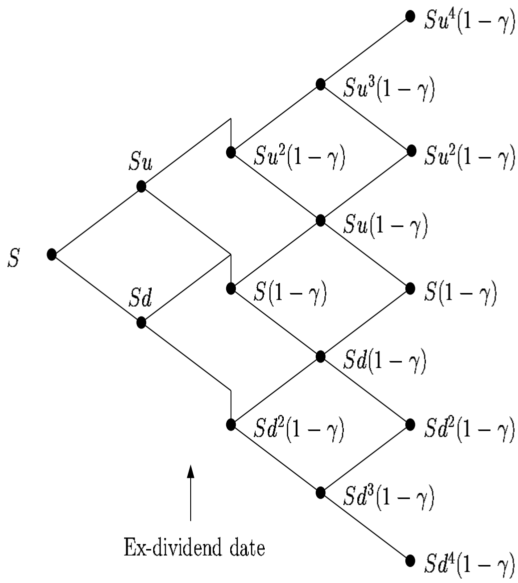

Figure 3.4: Binomial tree for stock paying a known dividend yield

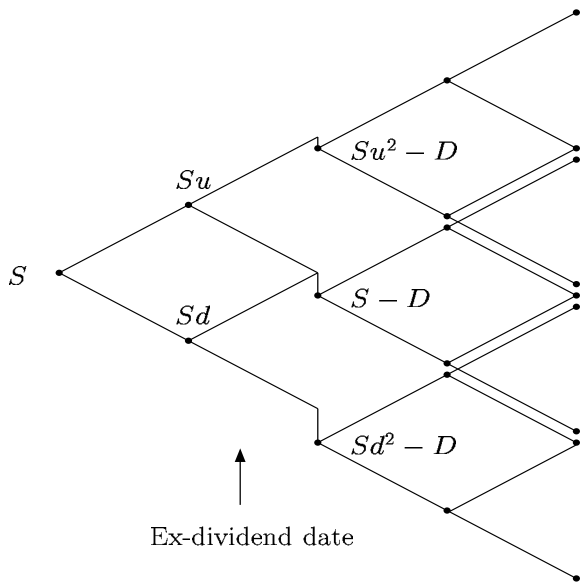

Figure 3.5: Binomial tree for stock paying a known dividend D per share at $\delta t < t < 2\delta t$

for the risk-neutral probability p in Eq.(3.29) and upward and downward return rates in Eq.(3.46), the model we have considered works for the options on the underlying stock with the dividend yield q.

When a foreign currency is the underlying asset, the foreign risk-free interest rate is the relevant dividend yield.