Introduction to Financial Engineering and Algorithms

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Time Value of Money

Time Value of Money

- Consider a do-it-yourself fund based on regular savings invested in a bank account attracting interest at 5% per annum.
- When you retire 40 years after, you want to receive a pension equal to 50% of your final salary and payable for 20 years.
- Your earnings are assumed to grow at 2% annually, and you want the pension payments to grow at the same rate.

- What proportion of your salary must be given to achieve this?

Time Value of Money

- $100 to be received after 1 year is worth less than the same amount today.
  - Money due in the future or locked in a fixed-term account cannot be spend right away.
  - Prices may rise in the meantime (inflation).
  - There is risk that the money will not be received.
- The present value of future money is reduced to compensate the risks.

Time Value of Money

- What is the future value of an amount invested or borrowed today?
- What is the present value of an amount to be paid or received at a certain time in the future?

- The answer will depend on various factors!

Different Types of Interest Rates

- Why to study the interest rates?
    - The risk-free interest rate is a factor in the valuation of virtually all derivatives.
    - The interest rates can be underlying variables of derivatives.
- Three types of interest rates are introduced.
    - Treasury rates, LIBOR, and repo rates.
- Treasury rates.
    - The rates an investor earns on Treasury bills and Treasury bonds, which are government debts issued in its own currency.
    - Treasury rates are theoretically risk-free since the government is always able to pay the promised interest and principal payments.

Different Types of Interest Rates

- LIBOR and LIBID
  - The shorts for London Interbank Offered Rate and London Interbank Bid Rate (倫敦銀行間拆款利率).
  - A LIBOR (LIBID) quote, provided by and for AA credit rating banks, is the interest rate at which the bank is prepared to make (accept) a wholesale deposit with other banks.
  - Large banks quote LIBOR and LIBID for maturities up to 12 months in all major currencies.
  - Eurodollar futures and swaps can be used to imply the LIBOR rates beyond 12 months.

Different Types of Interest Rates

- LIBOR and LIBID trade in the Eurocurrency market, which is outside the control of any one government
  - Eurosterling, Euroyen, or Eurodollar
- Credit risk issue:
  - The credit risk of a AA-rated financial institution is small for short-term loans
  - Thus, LIBOR rates are close to risk-free

Different Types of Interest Rates

- Derivatives traders regard LIBOR rates as a better approximation of the “true” risk-free rate than Treasury rates.
- It is believed that Treasury rates is artificially low due to some tax advantage and regulatory issues for financial institutions.
- In the U.S., Treasury instruments are not taxed at the state level.
- Treasury instruments must to purchased by financial institutions to fulfill a variety of regulatory requirements.
- Minimal capital requirements for Treasury instruments is lower than those for other fixed-income securities.

- LIBOR rates reflect the true opportunity cost of fund for traders

LIBOR

- A example of employing simple interest is provided by the LIBOR rate.
  - This is the rate of interest valid for transactions between the largest London banks.
- LIBOR rates are quoted for various short periods of time up to one year.
- Used as reference values for a variety of transactions.
  - Typical commercial loan may be formulated as a particular LIBOR rate plus some additional margin.

LIBID

- The acronym LIBID stands for London Interbank Bid Rate.
- It is the bid rate that banks are willing to pay for eurocurrency deposits in the London interbank market.

Repo Rates

- Repo rates.
  - Repurchase agreement (repo) (附買回合約): a contract where a trader who owns securities agrees to sell them to a financial institutions now and buy them back at a slightly higher price.
    - Equivalent to raise funds with the securities as collaterals.
    - Thus, the repo loan involves very little credit risk.
  - Price margins reflect the interest earned by the financial institutions, which is known as the repo rate.
  - Overnight repos are most common, but there are also longer-term arrangements, known as term repos.

The Risk-Free Rate

- The short-term risk-free rate traditionally used by derivatives practitioners is LIBOR.
- The Treasury rate is considered to be artificially low for a number of reasons.
- As will be explained in later chapters:
  - Eurodollar futures and swaps are used to extend the LIBOR yield curve beyond one year.
  - The overnight indexed swap rate is increasingly being used instead of LIBOR as the risk-free rate.

Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.

Simple Interest

- The initial deposit amount is called the principal P.
- The future value of the deposit invested in a bank account with paid interest is $P +$ interest.
- Suppose the interest rate is $r > 0$, after one year the interest earned will be rP.
- Value of investment will become

$$V(1) = P + rP = (1 + r)P$$
$$V(2) = P + rP + rP = (1 + 2r)P$$

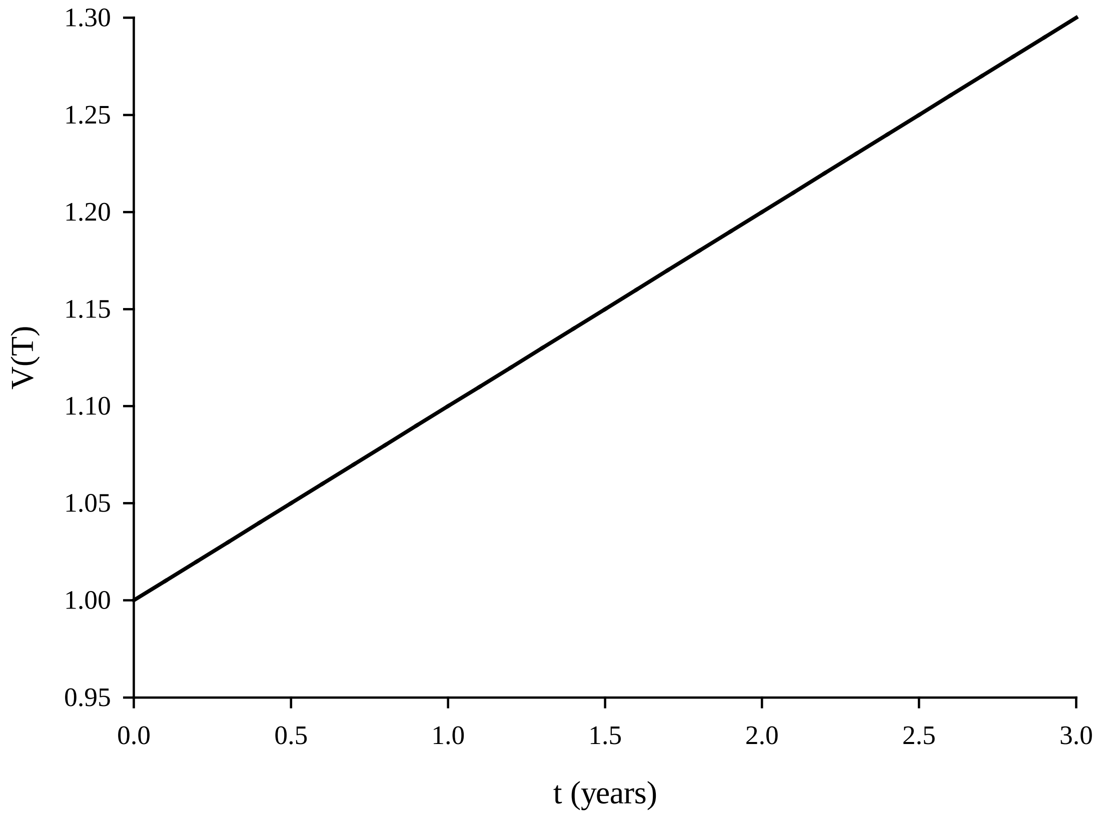
- Interest is typically calculated on a daily basis, the interest earned in one day will be

$$\frac{1}{365} rP$$

Simple Interest

- The interest earned in n days will be
$$\frac{n}{365}rP$$
- Rule of simple interest:
$$V(t) = (1 + tr)P$$
  - t is time in years.
  - $1 + tr$ is called the growth factor.
  - Interest rate r is assumed to be constant.
- If the principal P is invested at time s, then the value at time $t > s > 0$ is
$$V(t) = (1 + (t - s)r)P$$

Simple Interest Rate at $r = 10\%$



Example

- Consider a deposit of \$150 held for 20 days and attracting simple interest at 8%.
- This gives $t = \frac{20}{365}$ and $r = 0.08$.
- After 20 days the deposit will grow to

$$V\left(\frac{20}{365}\right) = \left(1 + \frac{20}{365} \times 0.08\right) \times 150 \cong 150.66$$

Simple Interest

- The return on a investment commencing at time s and terminating at time t will be denoted as $K(s,t)$.

$$K(s,t) = \frac{V(t) - V(s)}{V(s)}$$

- For simple interests, $K(s,t) = (t-s)r$.
  - The interest rate is equal to the return over 1 year, $K(0,1) = r$.
- Interest rate will always refer to a period of one year.

Present Value (Discounted Value)

- What is the initial amount whose vale at time t is given?
$$V(0) = V(T)(1 + rt)^{-1}$$
- This is called the present value or discounted value.
- The discount factor is $(1 + rt)^{-1}$.

Periodic (Discrete) Compounding

- In contrast to simple interest, the interest earned in periodic compounding will be added in to the principal.
    - Can be annually, semi-annually, quarterly, monthly, weekly, or even daily.
    - Subsequent interest attracted will be from not only the principal but also from the interest previously earned.

Periodic (Discrete) Compounding

- In case of monthly compounding with interest rate r.
  - The first interest payment will be $\frac{r}{12}P$.
  - Principal will be increased to $\left(1+\frac{r}{12}\right)P$.
  - The second month interest payment will be $\frac{r}{12}\left(1+\frac{r}{12}\right)P$.
  - The capital will be increased to $\left(1+\frac{r}{12}\right)^2 P$.
  - After n months, the capital will be $\left(1+\frac{r}{12}\right)^n P$.

Periodic (Discrete) Compounding

- In general, in m payments are made per annum, the time between two consecutive payments measured in years will be $\frac{1}{m}$.
- Each payment will increase the principal by a factor of $1 + \frac{r}{m}$.
- If the interest rate r remains unchanged, after t years the future value of an initial principal P will become

$$V(t) = \left(1 + \frac{r}{m}\right)^{tm} P$$

- The growth factor is $\left(1 + \frac{r}{m}\right)^{tm}$.

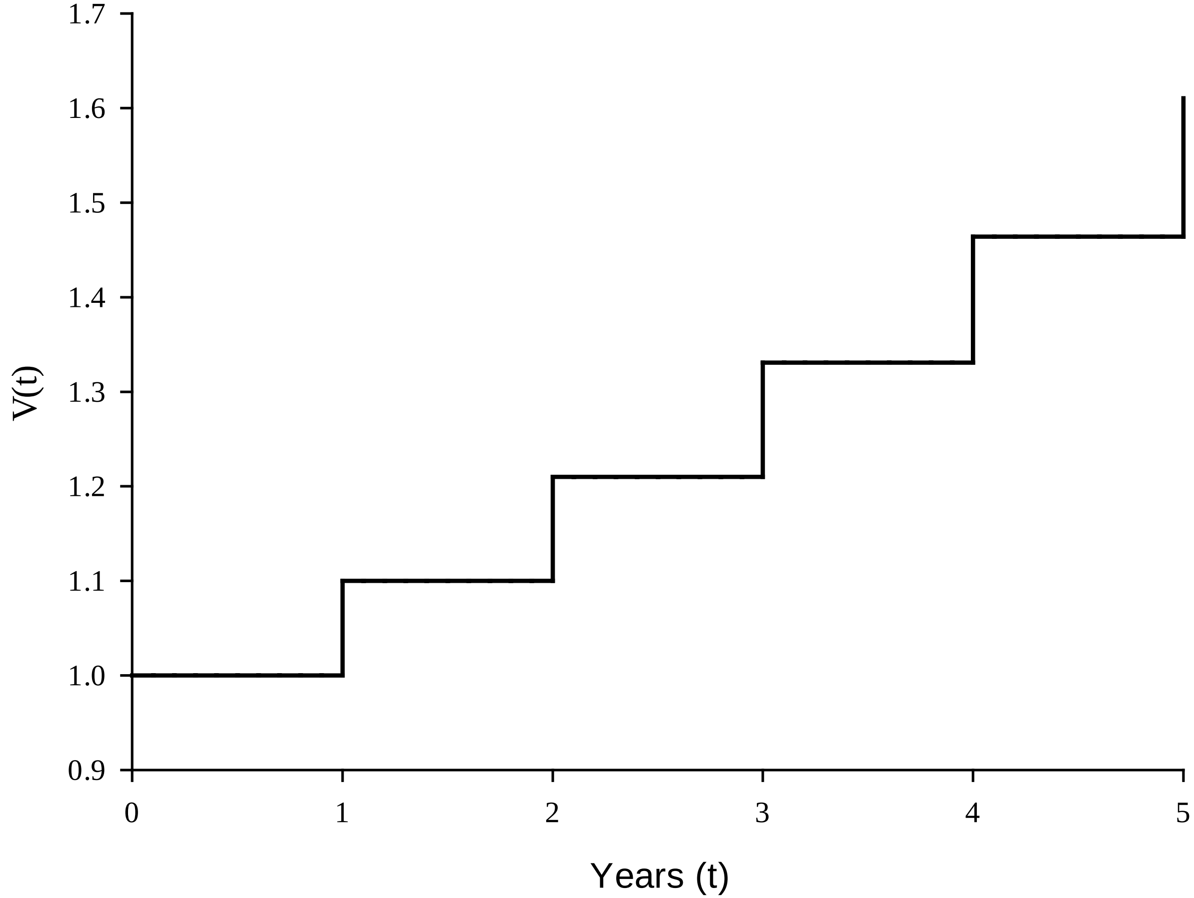
Periodic (Discrete) Compounding

- If $V(0) = P$, then $V\left(t + \frac{1}{m}\right) = V(t)\left(1 + \frac{r}{m}\right)$.

- The exact value of the investment may sometimes need to be known at time instants between interest payments.
  - Account may be closed on a day when no interest payment is due.
  - What is the value after 10 days of deposit of \$100 subjected to monthly compounding at 12%?

- The next plot shows annual compounding of $P = 1, r = 10\%$, and $m = 1$.

Periodic (Discrete) Compounding



- $P = 1$ and $r = 10\%$.

Periodic (Discrete) Compounding

- The future value of $V(t)$ increases if one of the parameters $m, t, r,$ or P increases, the other remains unchanged.
  - Proof of t, r, P is trivial.
- To prove m, we let $m < k$.
  - $\left(1+\frac{r}{m}\right)^{tm} < \left(1+\frac{r}{k}\right)^{tk}$.
  - This can be reduced to $\left(1+\frac{r}{m}\right)^{m} < \left(1+\frac{r}{k}\right)^{k}$.
  - Using the binomial formula $(a+b)^m = \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} a^i b^{m-i}$.

Periodic (Discrete) Compounding

- $$\left(1+\frac{r}{m}\right)^m = 1 + r + \frac{1-\frac{1}{m}}{2!}r^2 + \cdots + \frac{\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{m-1}{m}\right)}{m!}r^m$$
$$\leq 1 + \mathrm{r} + \frac{1-\frac{1}{k}}{2!}r^2 + \cdots + \frac{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdots\left(1-\frac{m-1}{k}\right)}{m!}r^m$$
$$< 1 + r + \frac{1-\frac{1}{k}}{2!}r^2 + \cdots + \frac{\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdots\left(1-\frac{m-1}{k}\right)}{m!}r^k$$
$$= \left(1+\frac{r}{k}\right)^k$$

Periodic (Discrete) Compounding

- The present value (discounted value) under periodic compounding is
$$V(0) = V(t)\left(1+\frac{r}{m}\right)^{-tm}$$
- The discount factor is $\left(1+\frac{r}{m}\right)^{-tm}$.
- Suppose $0 < t < T$, the value of an investment $V(t)$ is
$$V(t) = \left(1+\frac{r}{m}\right)^{-(T-t)m} V(T)$$
- The return is computed by
$$K(s,t) = \frac{V(t)-V(s)}{V(s)} = \left(1+\frac{r}{m}\right)^{(t-s)m} - 1$$

Compound Frequency

- For $A = \$1$, $n = 1$ year, and $R = 10\%$, analyze the effect of different compounding frequencies.

Compounding frequency	Terminal value of $1 at the end of 1 year
$m = 1$	1.10000000
$m = 2$	1.10250000
$m = 4$	1.10381289
$m = 12$	1.10471307
$m = 52$	1.10506479
$m = 365$	1.10515578
$m = \infty$	1.10517092

Continuous Compounding

- The future value of a principal P attracting interest at a rate $r > 0$ compounded m times a year is

$$V(t) = \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{tr} P$$

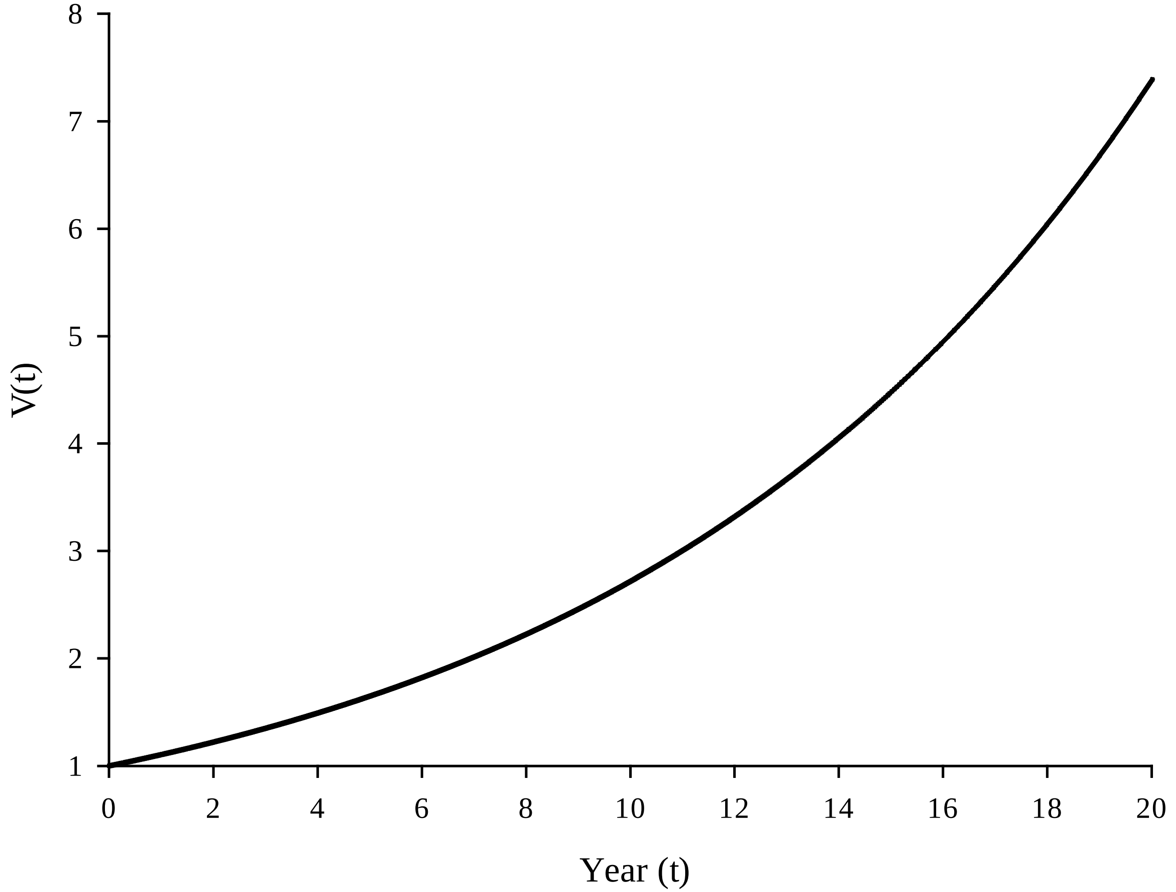
- In the limit $m \to \infty$,

$$V(t) = e^{rt} P$$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x}$$

- This is known as continuous compounding.
- The growth factor is e^{tr}.

Continuous Compounding



- $P = 1$ and $r = 10\%$.

Continuous Compounding

- The growth rate under continuous compounding is proportional to the current wealth.
$$V'(t) = re^{tr}P = rV(t)$$
- We can use continuous compounding to estimate periodic compounding when m is large.

Continuous Compounding

- Continuous compounding produces higher future value than periodic compounding with any frequency m, given the same principal P and interest rate r.

$$e^{tr} > \left(1 + \frac{r}{m}\right)^{tm} = \left[\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}\right]^{rt}$$

- The sequence $\left(1 + \frac{r}{m}\right)^{\frac{m}{r}}$ is increasing and converges to e as $m \to \infty$.

Conversion Formulas

$$R_C = m\ln(1+\frac{R_m}{m})$$
$$R_m = m(e^{\frac{R_c}{m}}-1)$$

- R_c: Continuously compounded rate.
- R_m: Same rate with compounding *m* times per year.

Examples

- 10% with semiannual compounding is equivalent to $2 * \ln(1.05) = 9.758\%$ with continuous compounding.
- 8% with continuous compounding is equivalent to $4 * \left(e^{\frac{0.08}{4}} - 1\right) = 8.08\%$ with quarterly compounding.
- Rates used in option pricing are nearly always expressed with continuous compounding.

Doubling Time

- The time required to double the principal under continuous compounding is

$$T = \frac{\ln 2}{r}$$

Example

- In 1626 Peter Minuit, governor of the colony of New Netherland, bought the island of Manhattan from Indians paying with goods worth \$24.
- Find the value of this sum in year 2009 at 5% compounded continuously.

$$\$24e^{0.05\cdot(2009-1626)} \cong \$4{,}976{,}804{,}376$$

Streams of Payment

- Annuity is a sequence of finitely many payments of a fixed amount due at equal time intervals.
- Suppose that payments of amount C are to be made once a year for n years.
  - Assume the interest rate is r and annual compounding is used.
  - The present value of this stream of payments is

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^n}$$

Streams of Payment

- We can denote this as the present value factor for an annuity.

$$\mathrm{PA}(r,n) = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \cdots + \frac{1}{(1+r)^n}$$
$$= \frac{1-(1+r)^{-n}}{r}$$

- The present value of an annuity in concise form is

$$C \times \mathrm{PA}(r,n)$$

Example

- Consider a loan of \$1000 to be paid back in 5 equal installments due at yearly intervals.
- The installments include both the interest payable each year calculated at 15% of the current outstanding balance and a repayment of a fraction of the loan.
- This is an amortized loan.
- The amount of each installment can be computed as

$$\frac{1000}{\mathrm{PA}(15\%, 5)} \cong 298.32$$

- The loan is equivalent to an annuity from the point of view of the lender.

Example

Principal	Interest	Loan	Payment
$1,000.00	$150.00	$1,150.00	$298.32
$851.68	$127.75	$949.43	$298.32
$681.11	$102.17	$783.28	$298.32
$484.96	$72.74	$557.70	$298.32
$259.38	$38.91	$298.29	$298.32
-$0.03			

Streams of Payment

- A perpetuity is an infinite sequence of payments of a fixed amount C occurring at the end of each year.
- The formula for the present value of a perpetuity is

$$\lim_{n\to\infty} \mathrm{PA}(r,n) \times C = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots = \frac{C}{r}$$

  - Note that the installments pay only covers the interest.

Example

- An individual who plans to retire in 20 years has decided to put an amount *A* in the bank at the beginning of each of the next 240 months.
- After which she will withdraw $1,000 at the beginning of each of the following 360 months.
- Assuming a nominal yearly interest rate of 6% compounded monthly, how large does *A* need to be?

Solution

- The present value of all her deposits is (assume r is a monthly rate)
$$A+\frac{A}{1+r}+\cdots+\frac{A}{(1+r)^{239}}=A(1+r)\cdot\mathrm{PA}(r,240)$$
- Let W be the amount withdrawn in the following 360 months. The present value of this cash flow is
$$\frac{W}{(1+r)^{240}}+\frac{W}{(1+r)^{241}}+\cdots+\frac{W}{(1+r)^{599}}$$
$$=\frac{W}{(1+r)^{239}}\cdot\mathrm{PA}(r,360)$$

Solution

- For her to fund all withdraws with no money left

$$A(1+r)\cdot \mathrm{PA}(r,240)=\frac{W}{(1+r)^{239}}\cdot \mathrm{PA}(r,360)$$

- Solving for A

$$A=\frac{W}{(1+r)^{240}}\cdot\frac{\mathrm{PA}(r,360)}{\mathrm{PA}(r,240)}$$

- Using $\mathrm{PA}(r,n)=\frac{1-(1+r)^{-n}}{r}$

$$A=\frac{W}{(1+r)^{240}}\cdot\frac{\frac{1-(1+r)^{-360}}{r}}{\frac{1-(1+r)^{-240}}{r}}$$

Solution

- Simplify the formula

$$A = \frac{W}{(1+r)^{240}} \cdot \frac{1-(1+r)^{-360}}{1-(1+r)^{-240}}$$

- We assume r to be monthly rate, thus $r = \frac{6\%}{12} = 0.005$. The money withdrawn in the later phase is $W = \$1000$.
- Solving for A

$$A = \frac{1000}{(1+0.005)^{240}} \cdot \frac{1-(1+0.005)^{-360}}{1-(1+0.005)^{-240}} \cong 360.99$$

Logarithmic Returns

- The returns $K(s,t)$ on an investment is not additive.
- The logarithmic return is additive, defined as

$$k(s,t) = \ln\frac{V(t)}{V(s)}$$

$$k(s,t) + k(t,u) = \ln\frac{V(t)}{V(s)} + \ln\frac{V(u)}{V(t)}$$

$$= \ln\frac{V(t) \cdot V(u)}{V(s) \cdot V(t)}$$

$$= \ln\frac{V(u)}{V(s)}$$

$$= k(s,u)$$

Logarithmic Returns

- If $V(t) = e^{rt}P$, then $k(s,t) = r(t-s)$.
- The interest rate can be recovered as

$$r = \frac{k(s,t)}{t-s}$$

- Suppose that the logarithmic return over 2 months on an investment subject to continuous compounding is 3%. What is the interest rate?

Comparing Compounding Methods

- When the interest rate and principal is the same, frequent compounding will produce higher future value than less frequent compounding.
- We want to know given the same principal, which compounding method is better.

Example

- Suppose that certificate promising to pay \$120 after one year can be purchase or sold now, or at anytime during this year for \$100.
  - This is consistent with a constant interest rate of 20% under annual compounding.
- If an investor decided to sell such a certificate half a year after the purchase, what price should the investor ask for?
  - Is \$110 a fair price? (half the profit of one year \$20)

Example

- No! The price is too high, giving arbitrage chances.
    - Borrow $1000 to buy 10 certificates for $100 each.
    - After six months sell the 10 certificates for $110 each, gaining $1100.
    - Use the $1100 to buy 11 certificates for $100 each.
    - After another six month, sell the 11 certificates for $110 each, gaining $1210.
    - Clear the loan $1200 ($1000+20% interest).
    - You earn $10!

- What will happen if the certificate after six months is $109?

Example

- What should the correct price be?
- The price of the certificate after six month is related to the interest rate under semi-annual compounding.
  - If this interest rate is r, then the price will be $(1+\frac{r}{2})$.
  - Arbitrage will disappear if the growth factor $\left(1+\frac{r}{2}\right)^2$ over 1 year is equal to the growth factor 1.2 under annual compounding.

$$\left(1+\frac{r}{2}\right)^2 = 1.2$$

  - Solving the equation we get $r \cong 0.1908902300207$.
  - The certificate price after 6 month should be $100\left(1+\frac{r}{2}\right) \cong 109.545$.

Comparing Compounding Methods

- We say that two compounding methods are equivalent if the corresponding growth factor over a period of one year are the same.
- If one of the growth factor exceeds the other, then the corresponding compounding method is said to be preferable.
- We can switch from one compounding method to another by recalculating the interest rates.

Effective Rates

- For a compounding method with interest rate r the effective rate r_e is one that gives the same growth factor over one year period under annual compounding.
- Periodic compounding with frequency m and interest rate r

$$\left(1+\frac{r}{m}\right)^m = 1 + r_e$$

- For continuous compounding with interest rate r

$$e^r = 1 + r_e$$

Example

- Which is better? Daily compounding with interest rate of 15% or semi-annual compounding with interest rate 15.5%?

Example

- Using the effective rate formula,

$$1 + r_e = \left(1 + \frac{0.15}{365}\right)^{365} \cong 1.1618$$

$$1 + r_e' = \left(1 + \frac{0.155}{2}\right)^{2} \cong 1.1610$$

- Effective rate for daily compounding at 15% is 16.18%
- Effective rate for semi-annual compounding at 15% is 16.10%

Example

- Suppose you have just spoken to a bank about borrowing $100,000 to purchase a house, and the loan officer has told you that a $100,000 loan, to be repaid in monthly installments over 15 years with an interest rate of .6% per month.
- If the bank charges a loan initiation fee of $600, a house inspection fee of $400, and 1 "point," what is the effective annual interest rate of the loan being offered?

Solution

- Let A be the monthly mortgage payment and r be the monthly interest rate.

$$A \cdot \mathrm{PA}(r, 180) = \$100{,}000$$

- Solving for A

$$A = \frac{\$100000}{PA(r, 180)} = \frac{\$100000 \cdot r}{1 - (1 + r)^{-180}}$$

- Using $r = 0.006$,

$$A = \frac{\$100000 \times 0.006}{1 - (1 + 0.006)^{-180}} \cong \$910.05$$

Solution

- Your monthly payment will be \$910.05, however, after the bank charging fees \$600+\$400+1 “point”.
  - 1 point = \$100,000*1%=\$1000.
  - You only received \$100,000-(\$600+\$400+\$1000)=\$98,000!
- The effective monthly interest r under this case is

$$A \cdot \mathrm{PA}(r, 180) = \$98{,}000$$

$$\mathrm{PA}(r, 180) = \frac{\$98{,}000}{A} = \frac{\$98{,}000}{\$910.05} \cong 107.6867767$$

$$\mathrm{PA}(r, 180) = \frac{1-(1+r)^{-180}}{r} \cong 107.6867767$$

$$r \cong 0.62736\%$$

Solution

- The effective annual rate is
  $$(1+r)^{12}-1 \cong 1.0062736^{12}-1 \cong 0.0779 = 7.79\%$$

- Without the fees, the effect annual rate is
  $$(1+r)^{12}-1 \cong 1.006^{12}-1 \cong 0.0744241 \cong 7.44\%$$

Money Markets

- The money market consists of risk-free (default-free) securities.
  - Bonds, a financial security promising the holder a sequence of guaranteed future payments.
  - Risk-free means that the payments will be delivered with certainty.
  - Treasury bills, treasury notes, treasury, mortgage, debenture bonds, commercial papers, are types of bond.

Zero Coupon Bonds

- Zero coupon bonds (零息債卷) involves just a single payment.
  - The issuing institution promises to exchange the bond for a certain amount of money F (face value) on a given day T (maturity date).
  - Typical life span is 1 year.
  - Sold at a price P lower than F.
  - Buying the zero coupon bond equals to lending money.

Zero Coupon Bonds

- Given the interest rate, the present value of the bond can be computed.
- A bond with $F = 100$, $r = 12\%$, and $T = 1$. The present value should be
$$V(0) = F(1 + r)^{-1} \cong 89.29$$
- However, the bonds are freely traded in the market, and its price determines the implied interest rate.
- Suppose a 1 year bond with $F = 100$ is traded at \$91. The implied interest rate is
$$F(1 + r)^{-1} = 91$$
$$r \cong 9.89\%$$

Zero Rates

- A zero rate (or spot rate) for maturity T is the rate of interest earned on an investment that provides a payoff only at time T.
- An example of zero rates with different maturities.

Maturity (years)	Zero rate (continuous compounding)	Current value of the corresponding zero coupon bond (零息債券) (face value = \$1 paid at maturity)
0.5	5.0%	$\$1 \cdot e^{-5.0\% \cdot 0.5} = \$0.9753$
1.0	5.8%	$\$1 \cdot e^{-5.8\% \cdot 1.0} = \$0.9436$
1.5	6.4%	$\$1 \cdot e^{-6.4\% \cdot 1.5} = \$0.9085$
2.0	6.8%	$\$1 \cdot e^{-6.8\% \cdot 2.0} = \$0.8728$

Unit Bonds

- We consider unit bonds with face value equal to one unit of home currency, $F = 1$.

Trading Zero Coupon Bonds

- A bond can be sold anytime prior to maturity at the market price.
- We denote the price at time t by $B(t,T)$.
  - $B(0,T)$ is the current, time 0 bond price.
  - $B(T,T) = 1$ equals to the face value of the bond.
- Prices of the bond is also affected by the interest rate.
  - $B(t,T) = \left(1 + \frac{r}{m}\right)^{-m(T-t)}$.
  - For continuous compounding, $B(t,T) = e^{-r(T-t)}$

Equivalency of Earning

- One year deposit at interest rate 8%.
- One year $100 bond selling at $92.59.

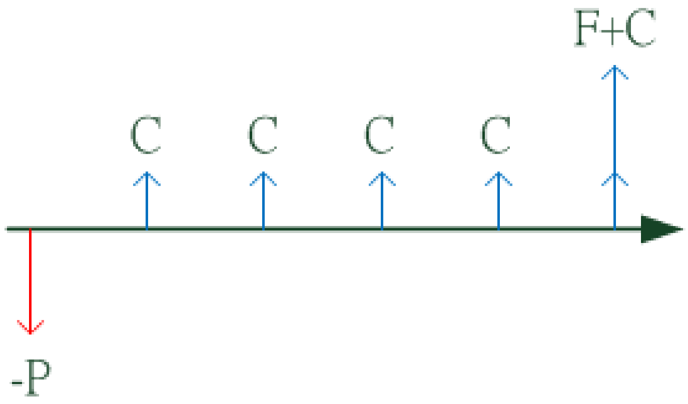
- Which statement is more clear?

Coupon Bonds

- Bonds the promise a sequence of payments are called coupon bonds.
- These payments include the face value at maturity, and coupons paid regularly.
  - Can be annual, semi-annual, quarterly, monthly, …

Example

- Consider a bond with face value $F = 100$, maturity $T = 5$ years, coupon of $C = 10$ dollars paid annually, and continuous compounding rate $r = 12\%$.
- The streams of payment is 10,10,10,10,10+100.



- Price of the bond is

$$V(0) = 10e^{-r} + 10e^{-2r} + 10e^{-3r} + 10e^{-4r} + (10 + 100)e^{-5r}$$
$$\cong 90.27$$

Example

- After 1 year, once the first coupon is cashed, the bond becomes a four year bond $T = 4$ worth
  $$V(1) = 10e^{-r} + 10e^{-2r} + 10e^{-3r} + (10 + 100)e^{-4r} \cong 91.78$$
- Total wealth at time 1 is $V(1) + C = V(0)e^{r}$.
- After 1.5 years, the value of the bond becomes

$$V(1.5) = 10e^{-0.5r} + 10e^{-1.5r} + 10e^{-2.5r} + (10 + 100)e^{-3.5r} \cong 97.45$$

- After four years, the bond becomes a zero coupon bond with face value \$110, worth
  $$V(4) = 110e^{-r} \cong 97.56$$

Bond Price with Different Interest Rates

- Each cash payment is discounted at the appropriate zero rate.
  - More specifically, to discount a cash payment matured at t, a zero rate with the time to maturity t should be considered
  - Based on the zero rates on silde page 59, the theoretical price of a two-year bond providing a 6% coupon semiannually is

$$3e^{-0.05\cdot 0.5} + 3e^{-0.058\cdot 1.0} + 3e^{-0.064\cdot 1.5} + 103e^{-0.068\cdot 2.0}$$
$$= 98.39$$

Bond Yield (or Yield to Maturity)

- The bond yield (or yield to maturity) is a constant discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond
  - Suppose that the market price of the bond in our example equals its theoretical price, 98.39.
  - The bond yield is given by solving

$$3e^{-y\cdot 0.5} + 3e^{-y\cdot 1.0} + 3e^{-y\cdot 1.5} + 103e^{-y\cdot 2.0} = 98.39$$

to get $y = 0.0676$ or 6.76%

Clean Price and Dirty Price

- Pricing bonds between coupon payments may be somewhat complicated.
- The dirty price is the present value of future payments.
    - The price of the bond sold between coupon payments.
- Accrued interest accumulated since the last coupon payment is evaluated by applying the simple interest rule.
- The clean price is then quoted by subtracting accrued interest from the dirty price.

Coupon Rates

- The coupon can be expressed as a fraction of the face value.
- If the coupon is paid annually, then $C = iF$, where i is called the coupon rate.

- Whenever coupons are paid annually, the coupon rate is equal to the interest rate for annual compounding if and only if the price of the bond is equal to its face value.
  - The bond is trading at par.

Proof

- Suppose $F = 100,\ T = 3$, and annual compounding $r = i$.

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{F+C}{(1+r)^3} = \frac{rF}{1+r} + \frac{rF}{(1+r)^2} + \frac{F+rF}{(1+r)^3}$$

$$= \frac{\mathrm{rF}}{1+\mathrm{r}} + \frac{\mathrm{rF}}{(1+\mathrm{r})^2} + \frac{\mathrm{F}}{(1+\mathrm{r})^2} = \frac{\mathrm{rF}}{1+\mathrm{r}} + \frac{\mathrm{F}(1+\mathrm{r})}{(1+\mathrm{r})^2} = \mathrm{F}$$

- Conversely,

$$F = \frac{C}{1+i} + \frac{C}{(1+i)^2} + \frac{F+C}{(1+i)^3} = \frac{F + 3C + 3Cr + Cr^2}{(1+r)^3}$$

$$(1+r)^3 F = F + 3C + 3Cr + Cr^2$$

$$F + 3rF + 3r^2F + r^3F = F + 3C + 3Cr + Cr^2$$

$$C = F\frac{(3r + 3r^2 + r^3)}{(3 + 3r + r^2)} = rF$$

Coupon Rates

- If a bond is selling below the face value, it means that the implied interest rate is higher than the coupon rate.
  - Bond price decreases when the interest rate goes up.
- If the bond price is higher than the face value, it means that the interest rate is lower than the coupon rate.
- This information is valuable in open markets.
  - Give indication of interest rates.

Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
  - For the same example, we solve

$$\frac{c}{2}e^{-0.05\cdot 0.5}+\frac{c}{2}e^{-0.058\cdot 1.0}+\frac{c}{2}e^{-0.064\cdot 1.5}+\left(100+\frac{c}{2}\right)e^{-0.068\cdot 2.0}=100$$

to get $c = 6.87$.

The Bootstrap Method

- Bootstrap method: to determine treasury zero rates sequentially from the shortest maturity to the longest maturity based on market prices of Treasury bills and bonds.
  - The sequence must be followed because the information of zero rates with shorter maturities is needed to solve the zero rate with a longer maturity (shown in the following numerical example).
  - The name of “bootstrap”: To unfasten the shoelace and next take off your shoes, make sure you first loosen the upper part of the shoelace and then loosen the lower part.

The Bootstrap Method

- Hypothetic data for Treasury bills and bonds.
- Find the zero rates corresponding to the time to maturities of 0.25, 0.5, 1, 1.5, and 2 years. (Semi-annual coupon)

Bond Principal ($)	Time to Maturity (years)	Annual Coupon ($)	Bond Price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

The Bootstrap Method

- Step 1 (for $R(0.25)$):
  - For this zero coupon bond, an amount of \$2.5 can be earned on the investment of \$97.5 in 3 months.
  - The 3-month rate is 4 times \$2.5/\$97.5 or 10.2564% with quarterly compounding.
  - This is equivalent 10.1271% with continuous compounding.
    - Method 1: Exploit the conversion formula to solve R_c with m and R_m to be 4 and 10.2564%.
    - Method 2: Solve $R(0.25)$ from $\$97.5e^{R(0.25)\cdot 0.25} = \$100$.
- Step 2 (for $R(0.5)$ and $R(1)$):
  - Similarly, the 6-month and 1-year continuous compounding zero rates are 10.4693% and 10.5361%.

The Bootstrap Method

- Step 3 (for $R(1.5)$):
  - Solve the following equation for $R(1.5)$

$$4e^{-0.104693\cdot 0.5} + 4e^{-0.105361\cdot 1} + 104e^{-R(1.5)\cdot 1.5} = 96$$

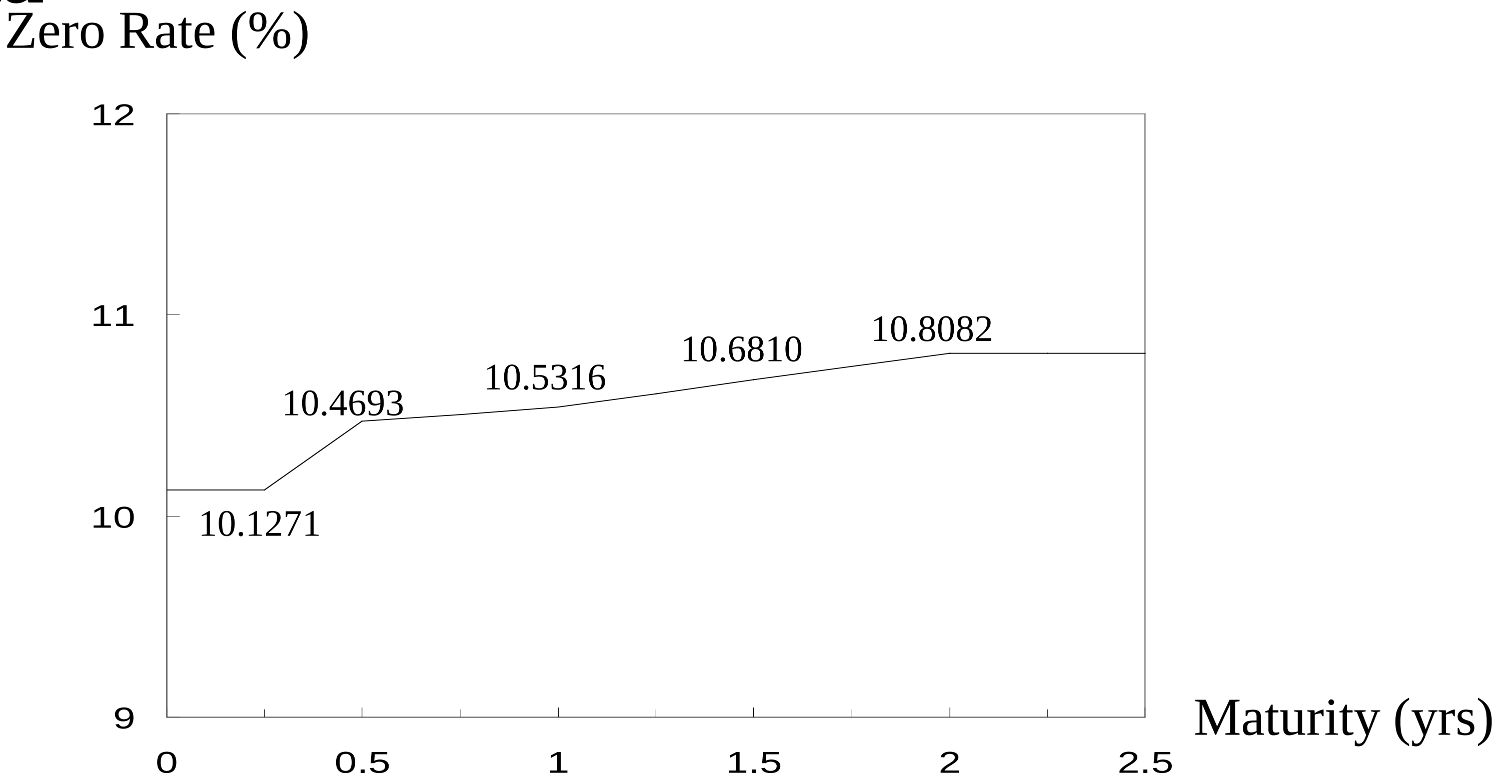
$$\Rightarrow R(1.5) = 10.6810\%$$

- Step 4 (for $R(2)$):
  - Solve the following equation for $R(2)$

$$6e^{-0.104693\cdot 0.5} + 6e^{-0.105361\cdot 1} + 6e^{-0.106810\cdot 1.5} + 106e^{-R(2)\cdot 2} = 101.6$$

$$\Rightarrow R(2) = 10.8082\%$$

Zero Curve Calculated From the Hypothetic Data



- The zero curve is also known as the term structure (期間結構) of interest rates or yield curve.
- Bond prices are determined with the demand and supply ⇒ Bond prices are stochastic ⇒ interest rates are stochastic.

Forward Rates

- The forward rate is the future zero rate implied by today's term structure of interest rates.
- Formula to calculate forward rates:
  - Suppose that the zero rates for time periods T_1 and T_2 are R_1 and R_2 with both rates continuously compounded.
  - Formula for the forward rate between T_1 and T_2 is

    $$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1},$$

    which is the future zero rate at T_1 with the time to maturity $(T_2 - T_1)$ implied from the current term structure.

Forward Rates

- The intuition for the formula is the equality of
  1. Cumulative return compounding at R_2 until T_2
  2. Cumulative return compounding at R_1 until T_1 and next compounding at R_F between T_1 and T_2

$$e^{R_2 T_2} = e^{R_1 T_1} e^{R_F (T_2 - T_1)}$$

- An example of calculation of forward rates:

Years (T)	Zero rate for an T-year investment	Forward rate for T-th year
1	3.0%	
2	4.0%	5.0%
3	4.6%	5.8%
4	5.0%	6.2%
5	5.3%	6.5%

Upward vs. Downward Sloping Yield Curve

- Rewrite the formula for the forward rates as

$$R_F = R_2 + (R_2 - R_1)\frac{T_1}{T_2 - T_1}$$

  - For an upward sloping yield curve, i.e., $R_2 > R_1$:

    forward rate R_F (applicable for the interval $[T_1, T_2]$) > zero rate R_2 (matured at T_2).

  - For a downward sloping yield curve, i.e., $R_2 < R_1$:

    forward rate R_F (applicable for the interval $[T_1, T_2]$) < zero rate R_2 (matured at T_2).

Upward vs. Downward Sloping Yield Curve

- Expectations Theory:
  - Upward (downward) sloping yield curves indicate that the market is expecting higher (lower) forward rates.
  - Based on the cumulative returns to derive the forward rate, i.e., $e^{R_2 T_2} = e^{R_1 T_1} e^{R_F (T_2 - T_1)}$, we can infer that $R_2 > R_1$ ($R_2 < R_1$) if and only if $R_F > R_1$ ($R_F < R_1$)
- Liquidity Preference Theory:
  - Explain upward sloping yield curves according to the liquidity preference of lenders and borrowers.
  - Lenders prefer to preserve their liquidity and invest funds for short periods of time $\Rightarrow$ they demand lower (higher) rates for short- (long-) term loans.

Upward vs. Downward Sloping Yield Curve

- To avoid the re-borrowing risk, borrowers prefer to borrow at fixed rates for long periods of time $\Rightarrow$ they would like to pay lower (higher) rates for short- (long-) term loans.
- The above two forces lead to a convergent result which is an upward sloping yield curve.

Upward vs. Downward Sloping Yield Curve

- The Preferred Habitat Theory.
  - Another variation on the Pure Expectations Theory, states that in addition to interest rate expectations, investors have distinct investment horizons and require a meaningful premium to buy bonds with maturities outside their ‘preferred’ maturity.
  - Proponents of this theory believe that short-term investors are more prevalent in the fixed-interest market and therefore, longer-term rates tend to be higher than short-term rates.
- The mixture of the above theories can explain the occurrence of hump-shaped yield curves in markets.
  - The hump-shaped yield curve is first rising and then falling along the maturity dimension.

Forward Rate Agreement

- A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period.

Forward Rate Agreement: Key Results

- An FRA is equivalent to an agreement where interest at a predetermined rate, R_K, is exchanged for interest at the market rate.
- An FRA can be valued by assuming that the forward LIBOR interest rate, R_F, is certain to be realized.
- This means that the value of an FRA is the present value of the difference between the interest that would be paid at interest rate R_F and the interest that would be paid at rate R_K.

FRA Example

- A company has agreed that it will receive 4% on $100 million for 3 months starting in 3 years.
- The forward rate for the period between 3 and 3.25 years is 3%.
  - The difference is (4% - 3%)=1%.
  - The value of the contract to the company is +$250,000 discounted from time 3.25 years to time zero.

- Suppose rate proves to be 4.5% (with quarterly compounding).
- The payoff is –$125,000 at the 3.25 year point.
- This is equivalent to a payoff of –$123,609 at the 3-year point.

Money Market Account

- Investment in the money market can be realized by means of a financial intermediary.
  - Investment banks.
  - Buys and sells behalf of its customer. (Reduces transaction costs)
  - Risk-free position of an investor is given by the level of his or her account with the bank.
  - Long position, buying an asset thus investing money.
  - Short position, borrowing an asset thus borrowing money.

Money Market Account

- Given an initial amount $A(0)$.
  - We can purchase $\frac{A(0)}{B(0,T)}$ bonds.
  - The value of each bond at time t is $B(t,T) = e^{-(T-t)r} = e^{rt}e^{-rT} = e^{rt}B(0,T)$.
  - The investment's value at time t is $A(t) = \frac{A(0)}{B(0,T)} \cdot B(t,T) = A(0)e^{rt}$.
- The investment in a bond has a finite time horizon, which terminates at time T.
  - Position can be extended by reinvesting the amount $A(T)$ into a new bond.
  - If the bond is of the same kind, the investment can be prolonged for as long as required, with the formula $A(t) = A(0)e^{rt}$.

Money Market Account

- Coupon bonds can be a tool to manufacture an investment in the money market.
  - Suppose that the first coupon C is due at time t.
  - At time 0 we buy $\frac{A(0)}{V(0)}$ coupon bonds.
  - At time t we cash the coupon and sell the bond for $V(t)$.
  - We receive $C + V(t) = V(0)e^{rt}$.
  - Since the interest rate is constant, the sum of money is certain.
  - We created a zero-coupon bond with face value $V(0)e^{rt}$.

Case Discussion

- It is clear we have to save some money on a regular basis.
- Let us formulate the problem as: what fixed percentage of your salary should you be paying into this pension fund?
  - We assume annual salary and fixed interest rate.
  - Assume payments are made annually at the end of each year.
  - Let S be the initial salary, x be the percentage of salary to be invested, g be the salary growth rate, and r be the interest rate.
  - The present value of the saving after 40 years will be

$$V(0) = \sum_{n=1}^{40} \frac{xS(1+g)^n}{(1+r)^n}$$

  - Set $g = 2\%$ and $r = 5\%$.

Case Discussion

- Define the growing annuity factor GAF to be
$$\mathrm{GAF}(r,g,N)=\frac{1+g}{r-g}\left(1-\frac{(1+g)^N}{(1+r)^N}\right)$$
- $V(0)=xS(0)\cdot\mathrm{GAF}(r,g,N)$
- $\mathrm{GAF}(5\%,2\%,40)\cong 23.34$
- After 40 years, you will accumulate $V(40)=V(0)(1+r)^{40}$.
- The final salary is $S(1+g)^{40}$ after 40 years.
  - We want the initial pension to be 50% of that, and growing at rate g in the following years.

Case Discussion

- The pension will also be a growing annuity with present value (at 40 years)

$$50\% \cdot S(1+g)^{40}\mathrm{GAF}(r,g,20)$$

- This value should be equal to the amount accrued.

$$xS(1+r)^{40}\mathrm{GAF}(r,g,40) = 50\% \cdot S(1+g)^{40}\mathrm{GAF}(r,g,20)$$

- Solving this equation we get $x = 10.05\%$.

Case Discussion

- Over the course of 60 years, if the rates r and g fluctuate, the values of x will change.
- r and g tend to increase or decrease together with the rate of inflation.
  - $r - g$ tends to remain stable over the years.

	$\boldsymbol{r = 4\%}$	$\boldsymbol{r = 5\%}$	$\boldsymbol{r = 6\%}$
$g = 1\%$	9.96%	7.24%	5.24%
$g = 2\%$	13.70%	10.05%	7.34%
$g = 3\%$	18.62%	13.78%	10.14%