Introduction to Financial Engineering and Algorithms

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Options

Case Study

- European call and put options with strike price €98 on a German company G with current stock price €99 cost €4.145 and €2.738 respectively.
- European call and put options with strike price £65 on a British company B with current stock price £66 cost £2.55 and £1.34 respectively.
- European call and put option on the £ with exercise price equal to the current rate, €1.5 to £1, cost €14.40 and €13.16, respectively, for 1000 units.
- All options have a maturity of 1 month.
- Assume ideal world with no overheads, is there any mispricing going on?

European Options

- European call option gives the holder the right to buy an asset S for a fixed price X in advance at a specific future time T.
  - S is called the underlying.
  - X is call the strike price or exercise price.
  - T is called the exercise time or expiry time.
- European put option gives the holder the right to sell an asset S for a fixed price X in advance at a specific future time T.

American Options

- American call option gives the holder the right to buy an asset S for a fixed price X in advance between now and a future time T.
- American put option gives the holder the right to sell an asset S for a fixed price X in advance between now and a future time T.

Options

- Options vs. Forward and Futures.
  1. An option gives the holder the right to do something, but the holder does not have to exercise this right.
     - In forwards or futures, the two parties have committed themselves to do some action in the future.
  2. Purchase of an option requires an up-front payment.
     - Forwards or futures costs a trader nothing (except for the margin requirements) when they are initiated.
- Special assumption in this chapter.
  - The time value of money is not considered to calculate the option profit, which equals the payoff (received at the end of the option life) minus the cost of the option (paid at the beginning of the option life).

European Options

- The payoff for an European call option is

$$\text{Payoff} = \begin{cases} S(T) - X, & \text{if } S(T) > X \\ 0, & \text{otherwise} \end{cases}$$

- The payoff for an European put option is

$$\text{Payoff} = \begin{cases} X - S(T), & \text{if } S(T) < X \\ 0, & \text{otherwise} \end{cases}$$

- We use $\text{Payoff} = (S(T) - \mathrm{X})^+$ to denote $\max(S(T) - X, 0)$.

Underlying Assets

- Underlying asset can denote almost anything.
  - Common forms can be stocks, commodities, foreign currencies, stock indexes, interest rates.
  - Other forms include weather, temperature, or even the snow/rain level.
  - For assets that are impossible to buy or sell, the option is cleared in cash.
    - Like settling a bet.
- Consider a holder of a European call option on the S&P500.
  - The strike price is 800.
  - On the exercise date, the price is 815.
  - The holder will gain $(815-800)*some amount of money.
  - If the price is lower than 800 on the exercise date, the holder will gain nothing.

Premiums

- Payoffs of an option are always non-negative with positive probabilities.
  - A premium has to be paid.
  - Without the premium, the holder will always gain money.
  - The premium is the market price of the option.

Example

- European calls on Cadbury Schweppes PLC stock on 20 Oct. 2007:
    - Strike price 640 pence.
    - Exercise date 21 Dec. 2007.
    - Market price (premium) is 22.5 pence.
    - Traded at Euronex London International Finance Futures Exchange (LIFFE).
    - Assume the loan is 5.23% continuously compounded.
- Borrowing 22.5 pence on Oct. and returning it on Dec. will cost $22.5e^{0.0523*\frac{2}{12}} \cong 22.7$.
- Investment will bring profit if the stock price goes above $640 + 22.7 = 662.7$ pence on the exercise date.

Payoffs with Premium

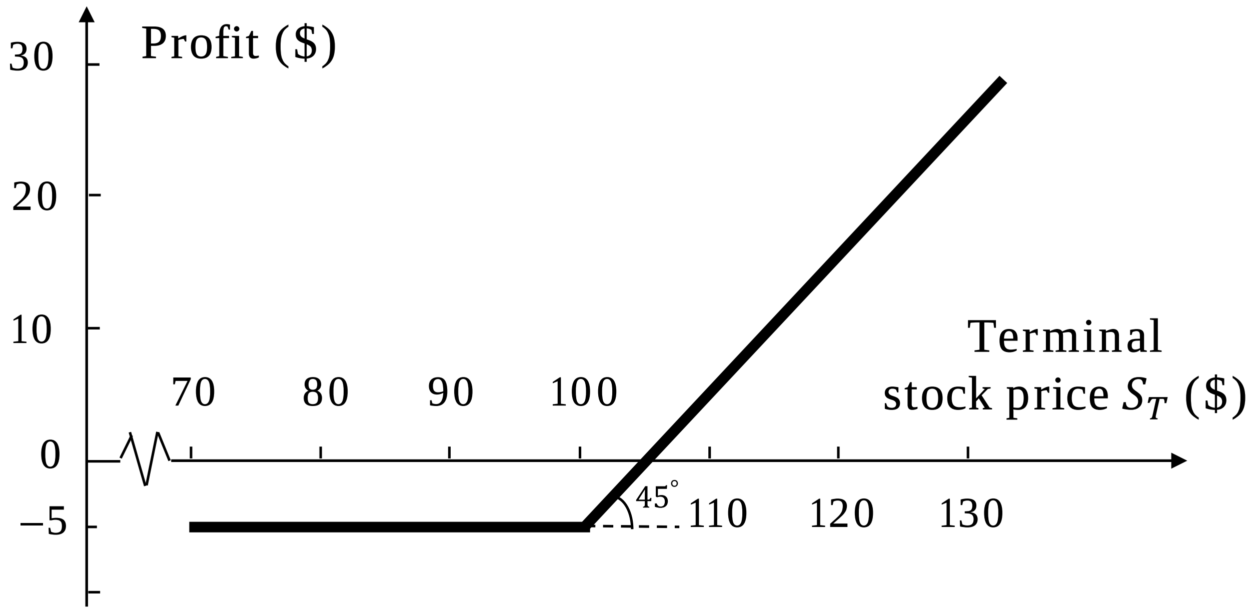
- The option holder will gain
  - European calls: $(S(T) - X)^{+} - C_E e^{rT}$.
  - European puts: $\left(X - S(T)\right)^{+} - P_E e^{rT}$.
- The option writer will gain
  - European calls: $C_E e^{rT} - (S(T) - X)^{+}$.
  - European puts: $P_E e^{rT} - \left(X - S(T)\right)^{+}$.
- The loss of a holder is limited to the premium paid.
- The loss for a writer is greater.
  - Writers of a call option has unbounded loss.

Option Positions

- Four positions for options:
  - Long call
  - Long put
  - Short call
  - Short put
- The term of “long” means to buy options, and the term of “short” means to sell or issue (or write) (發行) options
  - The writer of an option receives cash up front, but has potential liabilities later.
  - The holder of an option pay the up front cost to acquire the option and the right to do something in the future.

Profit of Longing a Call

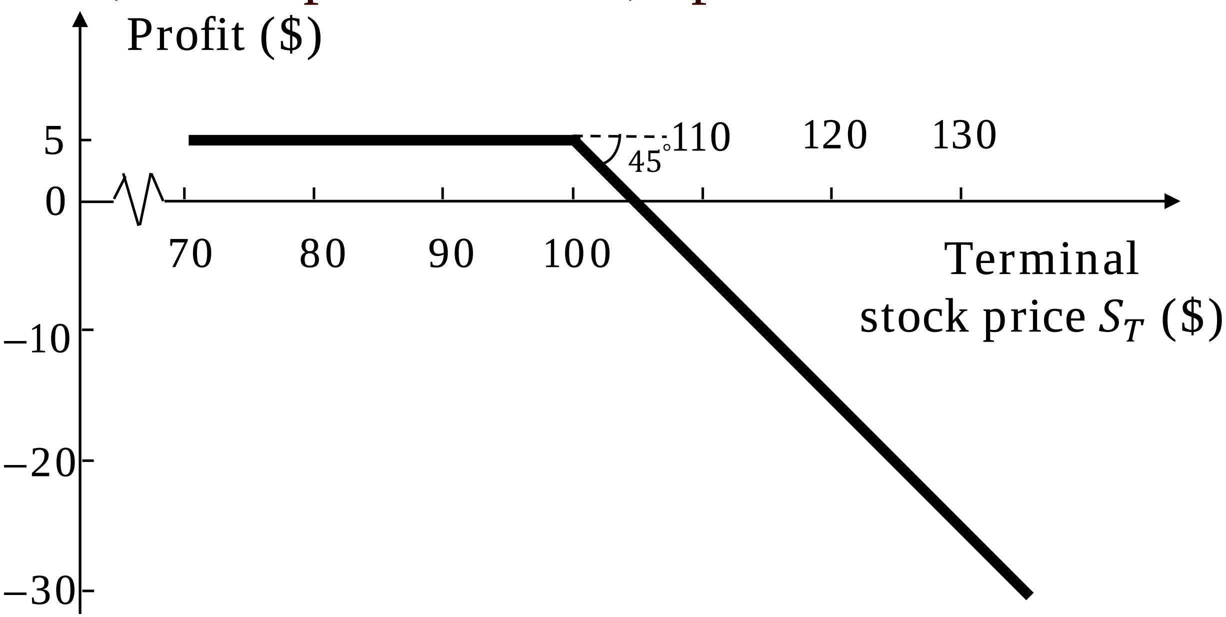
- Profit at maturity for buying one European call option: option price = \$5, strike price = \$100, option life = 4 months.



- Note that as long as S_T is higher (lower) than the strike price, the call holder should (should not) exercise this option.
- Since the cost to acquire the call option is \$5, the call holder earn a positive profit when S_T is higher than (strike price + \$5).

Profit of Shorting a Call

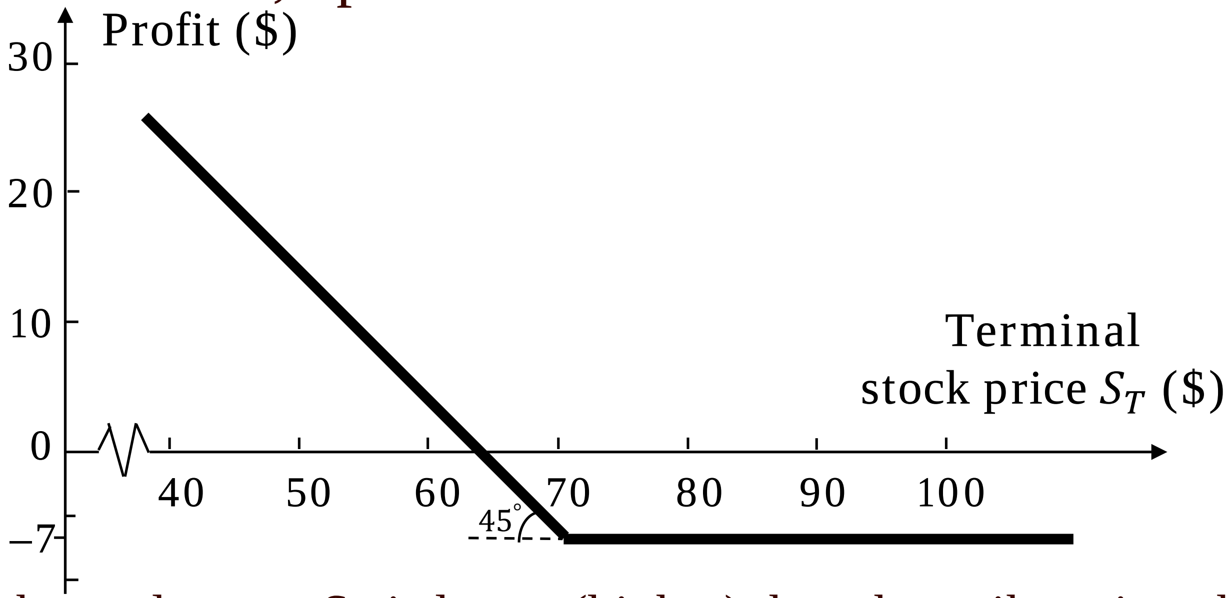
- Profit from writing the same European call option: option price = \$5, strike price = \$100, option life = 4 months.



  - When S_T is lower than the strike price, the call holder gives up his right and thus the option writer can earn the whole \$5 of option price.
  - If this call is exercised, the maximum losses of the option writer are unlimited.
  - The call writer's profit or loss is the reverse of that for the call holder.

Profit of Longing a Put

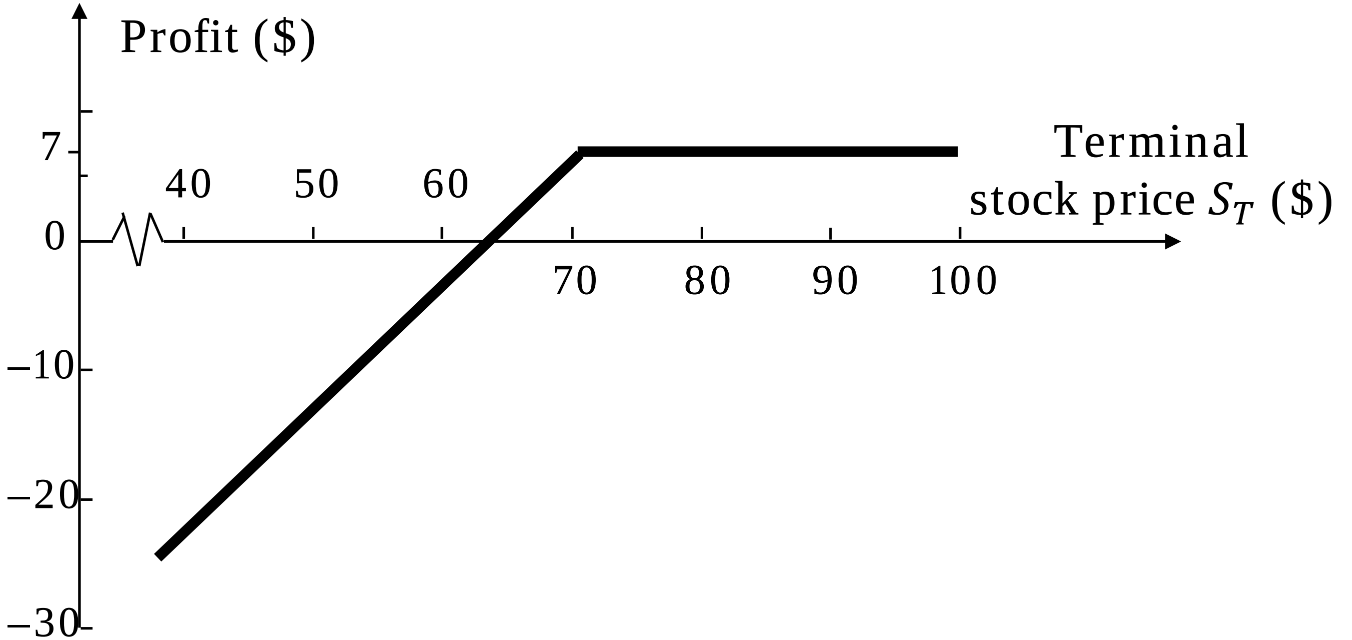
- Profit from buying a European put option: option price = $7, strike price = $70, option life = 3 months



  - Note that as long as S_T is lower (higher) than the strike price, the put holder should (should not) exercise this option.
  - Since the cost to acquire the put option is $7, the put holder earn a positive profit when S_T is lower than (strike price – $7).

Profit of Shorting a Put

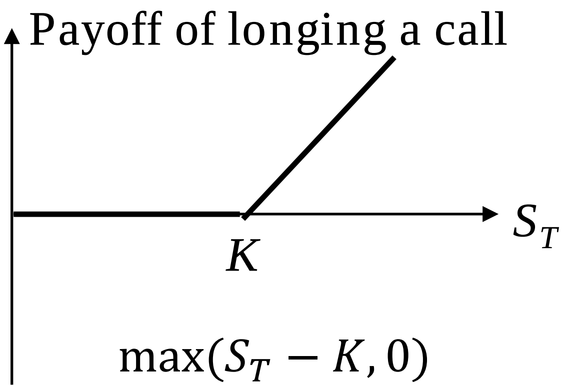
- Profit from writing the same European put option: option price = \$7, strike price = \$70, option life = 3 months.

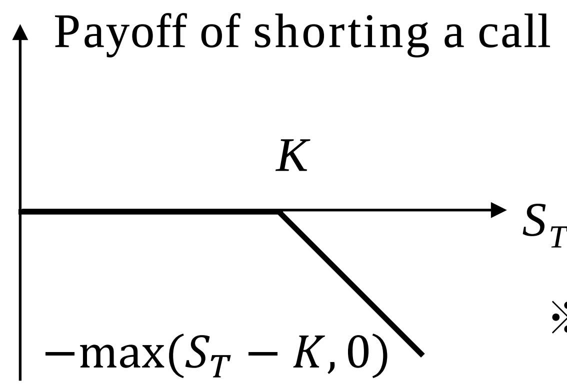


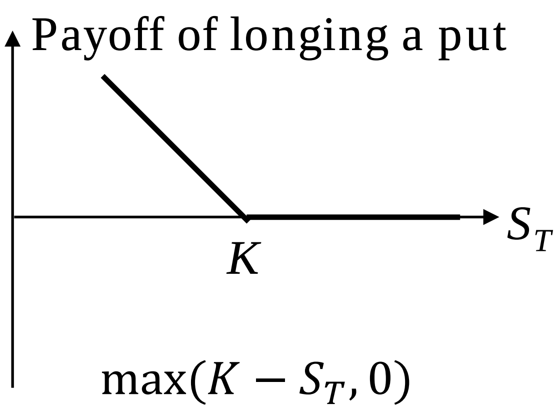
  - The put writer's profit or loss is the reverse of that for the put holder.
  - The put writer's profit declines when the stock price falls below the strike price (\$70), and the breakeven point for the put writer is \$63 (=\$70 – \$7) .

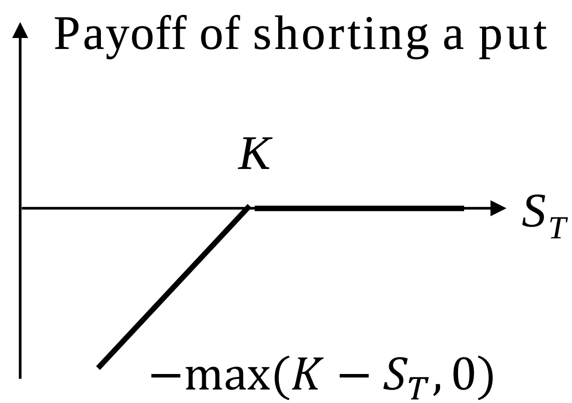
Payoffs from Positions in European Options

- Payoff is the final payment at maturity. The deduction of the option price is not necessary.









※ Note the negative relationship between the payoff functions of the option holder and writer.

Example

- Find the expected gain for a holder of an European call option.
  - Strike price is $90.
  - Exercise date is 6 month later.
  - The stock prices on the exercise date is $87, $92, or $97 with probability $\frac{1}{3}$ each.
  - The premium is $8 and loan is 9% compounded continuously.
- $E[(S(T)-X)^+]-8e^{9\%\cdot\frac{1}{2}} = \frac{1}{3}(87-90)^+ + \frac{1}{3}(92-90)^+ + \frac{1}{3}(97-90)^+ - 8e^{0.09*0.5} \cong -5.368$

Types of Options

- Several exchange-traded options.
  - Stock options
    - Most stock options are traded on exchanges.
    - In the U.S., one stock options gives the holder the right to buy or sell 100 shares of stock at the specified strike price.
    - In the U.S., exchange-traded stock options are American-style options.
  - Foreign currency options
    - Most currency options trading is in the OTC market, but there is some exchange trading.
    - In the U.S., NASDAQ OMX offers European-style option contracts on a variety of different currencies.
    - One option contract is with the right to trade10,000 units of a foreign currency (1,000,000 units for Japanese yen).

Types of Options

- Index options
  - The most popular exchange-traded option contracts in the U.S. are the index options on S&P 500 index, S&P100 index , NASDAQ-100 index, and Dow Jones Industrial Average Index.
  - All are traded on Chicago Board Options Exchange.
  - Most index options are European.
    - An exception is the index option on S&P 100 index.
  - One index option contract is to buy or sell 100 times the index at the specified strike price.
    - Consider a call option with a strike price of 980, which is exercised when the index value is 992. The option writer pays the option holder $(992 - 980) \times \$100 = \$1,200$.

Types of Options

- Futures options
  - Notations: the strike price is K, the maturity date for the futures option is T, the maturity date for the underlying futures is T_1 $(> T)$, and the futures price at T is F_T.
  - Call futures option: when $F_T > K$, the holder exercises his right and receive a long position in the underlying futures contract plus a cash amount equal to $(F_T - K)$.
  - Put futures option: when $F_T < K$, the holder exercises his right and receive a short position in the underlying futures contract plus a cash amount equal to $(K - F_T)$.
  - Note that the futures contract received by the option holder at T is worth zero because the zero-value futures is always the case when it is initiated.
- If option holders close out the futures position immediately by entering into an offsetting position, they can finish the transaction completely and earn the received cash amount.

Types of Options

- When an exchange trades a futures contract, it often also trades options on that futures contract, e.g., futures and futures option on corn offered by CME Group.
- The reasons to trade options on futures rather than options on the underlying asset.
  1. A futures contract is more liquid than the underlying asset.
  2. The futures option is settled in cash (plus a futures position) rather than settled by physical delivering.

     (Note that most underlying futures contracts are closed out prior to delivery, and thus it is not necessary to concern the delivery options and delivery costs for physical delivering).
  3. Future options entail lower transaction costs than spot options in many circumstances.

Trading Options on Exchanges

- In the rest of this chapter, we will focus on stock options.
- Items in option contracts.
  - Expiration date (or maturity date) (到期日)
    - The last trading day is the third Friday of the expiration month.
    - The expiration date is the Saturday immediately following the last trading day.
    - The option holder has until 4:30 p.m. Central Time on the last trading day to instruct a broker to exercise the option.
    - The broker has until 10:59 p.m. the next day to complete the paperwork to notify the exchange about the exercise.

Trading Options on Exchanges

- Stock options are issued on a January, February, or March cycle
  - The January cycle consists of the months of Jan., Apr., July, and Oct. (similar for February and March cycles).
  - If the expiration date in the current month has not yet been reached, options trade for the current month (until the expiration date), the following month, and the next two months in the cycle.
  - If the expiration date in the current month has passed, options trade for the next month, the next-but-one month, and the next two months in the cycle.
  - When one option expires, trading in another month is started such that the above rules can be satisfied.

Trading Options on Exchanges

- Strike price
  - For each maturity, there is a series of strike prices spaced $2.5 (for stock price between $5 and $25), $5 (for stock price between $25 and $200), or $10 (for stock price above $200) apart.
  - When options with a new expiration month is introduced, the two or three strike prices closest to the current stock price are selected as the strike prices for the option contracts by the exchange.
    - If the stock price moves outside the range, a new strike price is introduced to extend the range to cover the stock price.
    - Suppose the stock price is $82, the initial strike prices may be $80, $85, $90. If the stock price rises above $90 (declines below $80), a strike price of $95 ($75) is introduced.

Trading Options on Exchanges

- European or American
  - In Taiwan, index option and stock option are European-style options, but the warrant (introduced later) in Taiwan is a American-style option.
- Some terminologies
  - Option class
    - All options of the same type (calls or puts) are referred to as an option class.
  - Option series
    - Consist of all the options of a given class with the same expiration date and strike price.
    - In other words, an option series refers to a particular contract that is traded.

Trading Options on Exchanges

- Intrinsic value (內含價值) vs. Time value (時間價值)
  - The intrinsic value of an option is defined as the maximum of zero and the payoff of the option if it were exercised immediately.
    - For calls, the intrinsic value is $\max(S(T)-X,0)$.
    - For puts, the intrinsic value is $\max(X-S(T),0)$.
    - An American-style option is worth at least as much as its intrinsic value because the holder always can realize the intrinsic value by exercising the option immediately.
  - Option value = Time value + Intrinsic value.

Trading Options on Exchanges

- Moneyness (價值狀況).
  - In the money (ITM): Options are referred to as in the money if they have positive intrinsic values, i.e., $S(T) > X$ for calls and $S(T) < X$ for puts.
  - Out of the money (OTM): Options are referred to as out of the money if they have zero intrinsic value, i.e., i.e.,$S(T) < X$for calls and $S(T) > X$ for puts.
  - At the money (ATM): $S(T) = X$ for both calls and puts.

※An ITM option will always be exercised on the expiration date, i.e., if $S(T) > X$for calls or $S(T) < X$ for puts, those options will always be exercised.

Trading Options on Exchanges

- Adjustments for dividends and stock splits.
- Suppose you own *N* options with a strike price of *K*:
  - Adjustment for cash dividends.
  - Some OTC options are cash dividend protected.
    - If a company declares a cash dividend, the strike price for options on the company's stock is reduced on the ex-dividend day (付息日) by the amount of the cash dividend.
  - Exchange-traded options are usually not adjusted for cash dividends.
  - Adjustment for *n*-for-*m* stock split or stock dividends.
    - An *n*-for-*m* stock split is to use *n* newly-issued stock shares to exchange for *m* outstanding stock shares.
    - A 20% stock dividend is equivalent to a 6-for-5 stock split because shareholders receive 0.2 additional shares for each 1 share owned.

Trading Options on Exchanges

- Since the effect of stock split and paying stock dividends is to issue more shares to replace existing shares, there is no impact on the asset value or the earning ability of a company.
- An n-for-m stock split or stock dividends should cause the stock price to go down to m/n of its previous value.
- The terms of option contracts are adjusted to reflect expected changes in a stock price arising from a stock split or a payment of stock dividends.
  - The strike price is reduced to m/n of its previous value.
  - The number of shares covered by one options is increased to n/m of its previous value.

Trading Options on Exchanges

- In Taiwan, warrants and stock options are with provisions for cash- and stock-dividend protection.
- Index options in Taiwan are stock-split and stock-dividend protected.
  - Even there is no clauses of adjustments for stock splits and stock dividends in index option contracts in Taiwan.
  - This is because the index level remains the same before and after stock splits and payments of stock dividends in Taiwan.

Example

- Consider a call option to buy 100 shares for the strike price to be \$20/share.
- How should terms be adjusted:
  - For a 2-for-1 stock split?
    - The strike price is reduced to \$10 (=\$20 $\times$ (1/2)).
    - The number of shares covered by one options is 200 (=100 $\times$ 2).
  - For a 25% stock dividend?
    - It is equivalent to a 5-for-4 stock split.
    - The strike price is reduced to \$16 (=\$20 $\times$ (4/5)).
    - The number of shares covered by one options is 125 (=100 $\times$ (5/4)).

Trading Options on Exchanges

- Position and exercise limits
  - A position (exercise) limit defines the maximum number of option contracts that an investor can hold (exercise) on one side of the market on CBOE.

  (Prevent the market from being unduly influenced by the activities of an individual investor).
    - Long calls and short puts are regarded to be on the same side of the market (bull's view).
    - Short calls and long puts are regarded to be on the same side of the market (bear's view).
    - The exercise limit usually equals the position limit.
    - For the largest and most frequently traded stocks, the position limit is 250,000 contracts.

Trading Options on Exchanges

- Commissions: charged by a broker and calculated as a fix cost plus a proportion of the dollar amount of the trade of options.
  - The commission is charged both when an option position is initiated and when it is closed out.
  - For retail investors, a typical commission schedule is

Dollar amount of trade	< $2,500	$2,500 to $10,000	> $10,000
Commission	$20 + 0.02 of dollar amount	$45 + 0.01 of dollar amount	$120 + 0.0025 of dollar amount

Trading Options on Exchanges

- In addition, maximum (minimum) commission is $30 per contracts for the first 5 contracts plus $20 ($2) for each additional contract.
- Note that if the option is exercised, the option holders need to pay the commission for trading the underlying shares, which is a proportion, e.g., 1.5%, of the trade on the underlying shares
- The commissions vary significantly from broker to broker

Trading Options on Exchanges

- Market Makers
  - Most exchanges use market makers to facilitate options trading.
    - The market makers are always ready to trade options with other traders and thus enhance liquidity to the market.
  - A market maker quotes both bid and offer prices when requested.
    - The bid (offer) is the price at which the market maker is prepared to buy (sell) the options.
    - The offer is always higher than the bid, and the bid-offer spread is the source of the profit for market makers.
    - Thus, the bid-offer spread is a hidden cost for option traders.

Trading Options on Exchanges

- Margins
  - In the U.S., when shares are purchased, an investor can either pay cash or borrow using a margin account (this is known as buying on margin).
  - For options, buying on margin is not always allowed.
    - When options with maturities less than 9 months, the option price must be paid in full.
    - For options with maturities greater than 9 months, investors can buy on margin, borrowing up to 25% of the option price.
    - The limit of buying on margin is because options already contain substantial leverage and it is inappropriate to further raise this leverage with buying on margin.

Trading Options on Exchanges

- For option writers, it is required to maintain funds in a margin account to minimize their default risk.
- The rule to determine the amount of the margin required for a naked option is the greater of the following two quantities:
  1. A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount (if any) by which the option is out of the money.
  2. A total of 100% of the proceeds of the sale plus 10% of the underlying share price.
- A naked option is an option that is not combined with an offsetting position in the underlying asset initially (an example of a non-naked option is shown on the next slide).
- The margin required by brokers can be higher than but no less than the amount calculated in the above process.

Trading Options on Exchanges

- The above calculation are performed everyday (but with the current prices of options and the underlying asset instead) to determine daily required margins.
- For index options, the 20% is replaced by 15% because a stock index is usually less volatile than the price of an individual stock.
- For other trading strategies, such as covered calls, protective puts, spreads, combinations, straddles, and strangles, the CBOE has special rules for determining the margin requirements.
  - For example, a covered call is a written call option when the shares that might have to be delivered are already owned by the option writer.
  - Therefore, the worst scenario for the writer of a covered call is to sell the shares he owns at the strike price when the call is exercised by the option holder.
  - Since the default risk of the writer of covered calls is minor, there is no margin requirements for writing covered calls.

Trading Options on Exchanges

- The Options Clearing Corporation (OCC)
  - The OCC performs the same function for option markets as the clearing house does for futures market.
  - The flow of orders: investors → broker → member of OCC → OCC.
  - The hierarchy of the required margin for writing options:
    - Option writers maintains a margin account with a broker.
    - The broker maintains a margin account with the OCC member.
    - The OCC member maintains a margin account with the OCC.

Trading Options on Exchanges

- Process of exercising options:
  - When an investor notifies a broker to exercise an option, the broker in turn notifies the OCC member which will place an exercise order with the OCC.
  - The OCC randomly selects a member with an outstanding short position in the same option, and that member selects a particular investor who has written the option according to some pre-specified rules.
- In the U.S., the Commodity Futures Trading Commission is responsible for regulating markets for options or futures.

Trading Options on Exchanges

- Taxation for options:
  1. When the option position is closed out or expires unexercised.
     - The gains and losses from the trading of stock options are taxed as capital gains or losses.
  2. When the option is exercised:
     - The gain or loss from the option is rolled into the position taken in the stock and recognized when the stock position is closed out.
     - For the holder who exercises a call (For the call writer), he can acquire a long (short) position in the underlying stock at the strike price plus the call price.
     - For the holder who exercises a put (For the put writer), he can acquire a short (long) position in the underlying stock and the net income (cost) equals the strike price less the put price.

Trading Options on Exchanges

- Trading options in OTC markets.
  - The OTC market for options has become increasingly important since the early 1980s and is now larger than the exchange-traded market.
  - OTC options on foreign exchange and interest rates are particularly popular.
  - The advantage of OTC options is that they can be custom-made to meet the precise needs of investors.
  - The disadvantage of OTC options is that the option writer may default.
    - To overcome this disadvantage, market participants usually require counterparties to post collateral.

Warrants, ESOs, and CBs

- Warrants, employee stock options (ESOs), and convertible bonds (CBs) are options or option-embedded securities issued by financial institutions or nonfinancial corporations.

- Warrant (權證):
  - Warrants are options issued by a financial institution or a nonfinancial corporation.
  - For example, a financial institution might issue put warrants (認售權證) on gold and then to proceed to create a market for trading these warrants.
  - The warrants issued by financial institutions can be traded on an exchange or in an OTC market.
  - If the warrant is traded in an OTC market, the issuing financial institution acts as the market maker for this warrant.

Warrants, ESOs, and CBs

- A common use of warrants by a nonfinancial corporation is to issue call warrants on its own stock and attaches them to the bond issue.
  - To make the corporate bond more attractive to investors.
  - To reduce the coupon rate of the corporate bond and thus save the funding cost.
- The issuer settles up with the holder when a warrant is exercised.

- Employee stock option (ESO) (員工股票選擇權):
  - ESOs are call options issued to employees by their company to align the interests between the employees and the shareholders.
  - Usually are issued to be at the money at the time of issue.
    - If the performance of the company is satisfied and thus the share price rises, the ESO becomes in the money and brings profit for employees.

Warrants, ESOs, and CBs

- ESOs cannot be exercised within a period of time (usually 1 to 4 years).
- ESOs cannot be sold to others.
- ESOs can last for as long as 10 or 15 years.
- Today, they are recorded as expenses at the fair market value of the ESO in the company’s income statement in most nations (rather than recorded as the distribution of the stock dividends in the past).
  - This accounting principle makes ESOs a less attractive form of compensation than they used to be.

Warrants, ESOs, and CBs

- Convertible bond (CB) (可轉換公司債)
  - CBs are bonds issued by a company that can be converted into equity at certain times using a predetermined conversion ratio.
  - Note that CBs are different from bonds with an embedded call option on the company's stock.
    - This is because you cannot separate the conversion right clearly from the CB.
  - Very often a convertible is callable.
  - The call provision is a way in which the issuer can force the conversion at a time earlier than the holder might otherwise choose.

Warrants, ESOs, and CBs

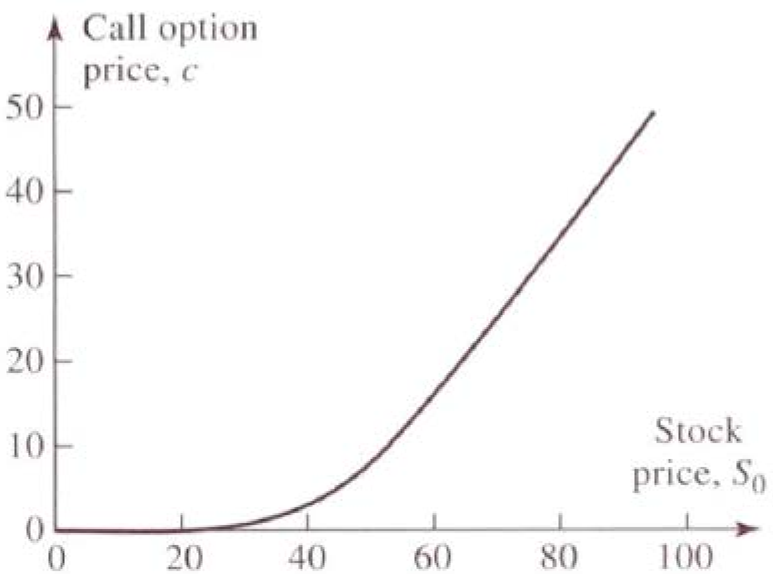
- Different from the options traded on exchanges, the numbers of outstanding warrants, ESOs, and CBs are predetermined on the issue day.
  - The numbers of outstanding warrants, ESOs, and CBs are determined by the size of the original issue and changes only when they are exercised or when they expire.
  - In contrast, as more people trade a particular option series on an exchange, the number of outstanding exchanged-traded options increases.

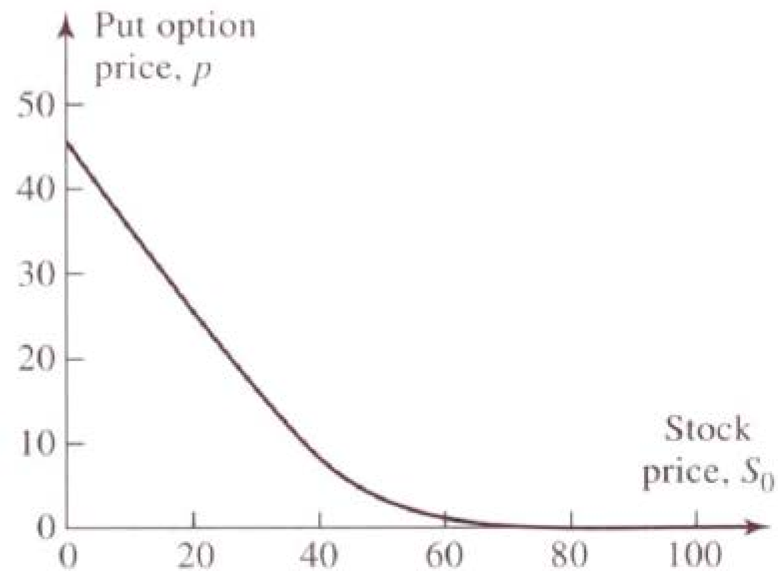
Warrants, ESOs, and CBs

- When these three instruments are exercised, the company issues more shares of its own stock and sells the stock shares to the option holder for the strike (or conversion) price.
  - Thus, the exercise of these three instruments leads to an increase in the number of outstanding shares of the issuing company.
  - Note that the above statement is not true for warrants issued by financial institutions.

Effect of Factors on Option Pricing

- Stock price $S(0)$ ↑
  - For both European and American calls, prob. of being ITM ↑ and thus call values ↑.
  - For both European and American puts, prob. of being ITM ↓ and thus put values ↓.

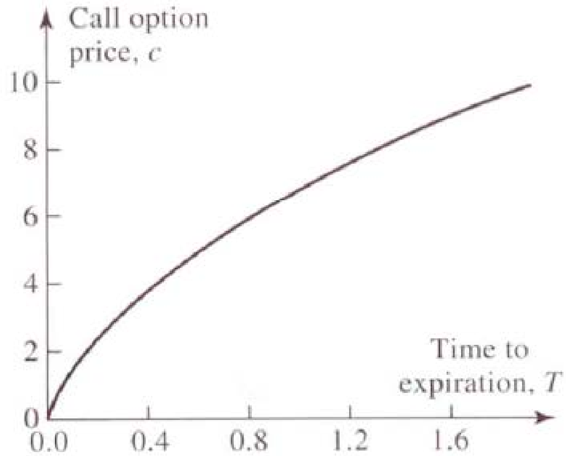


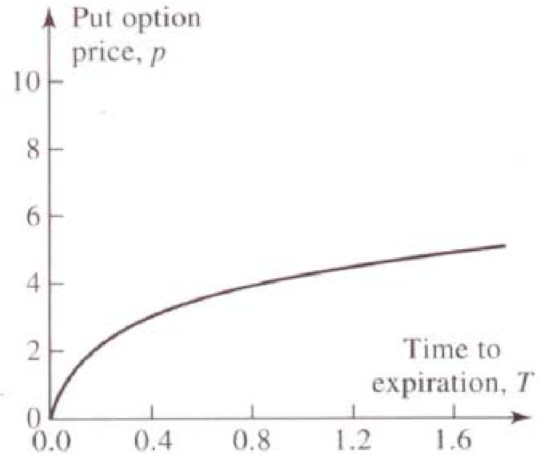


*$K = 50, r = 5\%, \sigma = 30\%, D = 0,$ and $T = 1$

Effect of Factors on Option Pricing

- Time to maturity T ↑
  - For American options, the owner of the long-life option has all the exercise opportunities open to the owner of the short-life option–and more $\Rightarrow$ The long-life American option must be worth as least as the short-life American option.
  - European calls and puts generally (not always) become more valuable as the time to expiration increases.

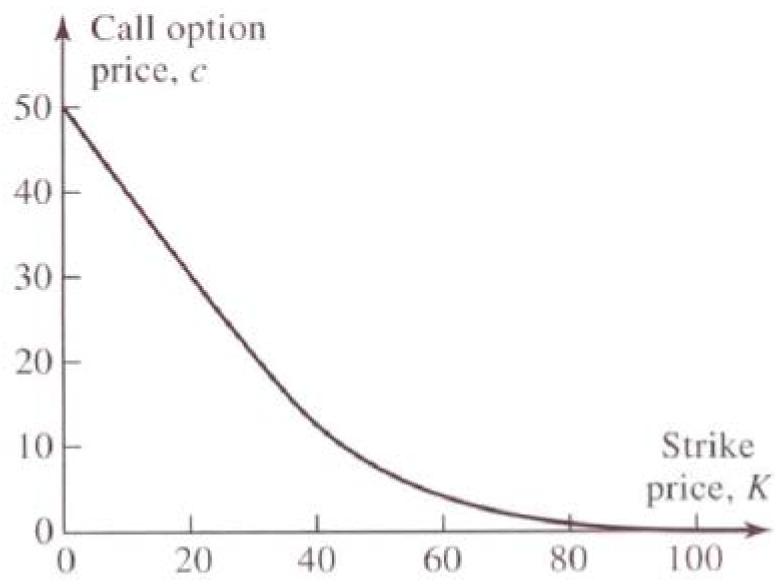


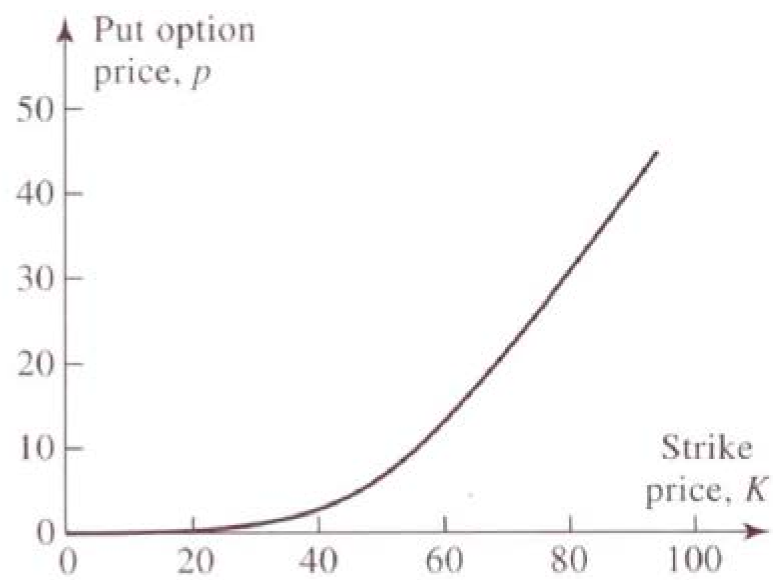


$S(0) = 50, K = 50, r = 5\%, \sigma = 30\%,$ and $D = 0$

Effect of Factors on Option Pricing

- Strike price $K \uparrow$
  - For both European and American calls, prob. of being ITM $\downarrow$ and thus call values $\downarrow$.
  - For both European and American puts, prob. of being ITM $\uparrow$ and thus put values $\uparrow$.





$S(0) = 50, r = 5\%, \sigma = 30\%, D = 0,$ and $T = 1$

Effect of Factors on Option Pricing

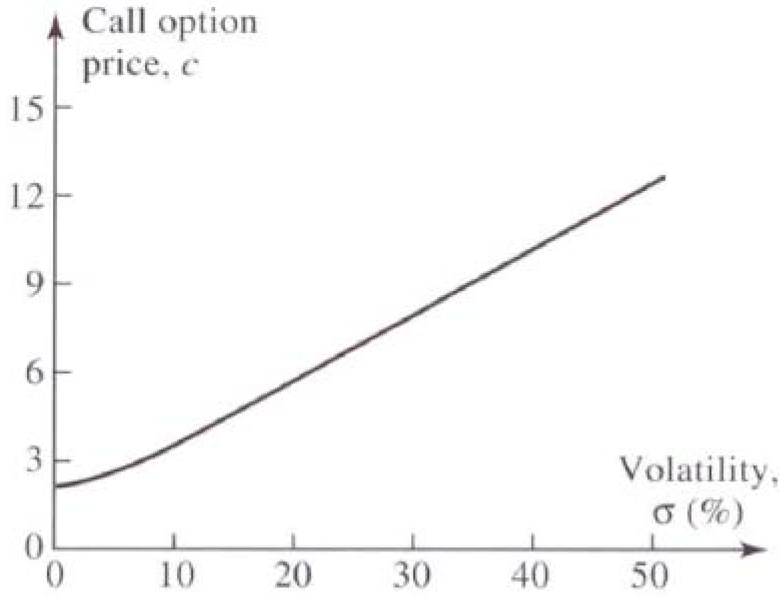
- For European calls,
  - Suppose two European call options on a stock with different time maturity T_1 and T_2 $(> T_1)$.
  - If there is a very large cash dividends paid in $[T_1, T_2]$, the stock price declines so that the short-life call could be worth more than the long-life call.

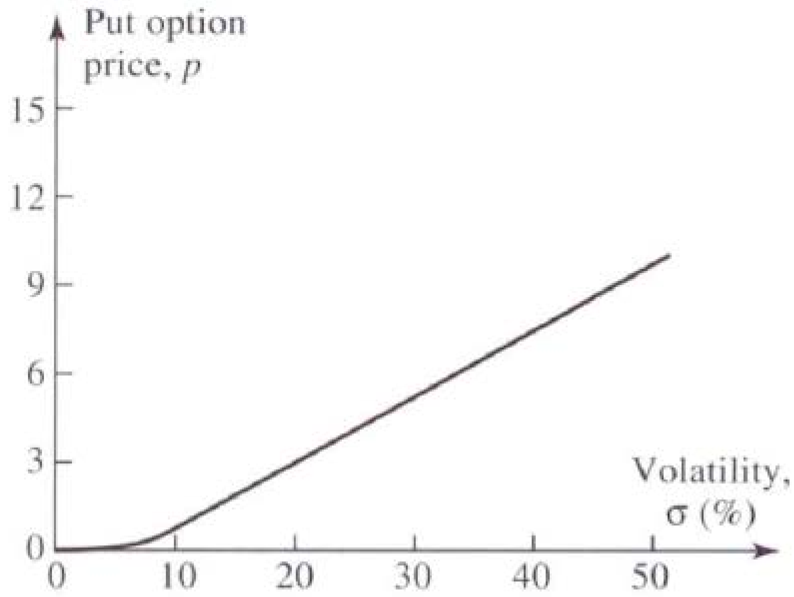
Effect of Factors on Option Pricing

- For deeply ITM European put options, short-life put (with T_1 time to maturity) could be worth more than the long-life put (with T_2 time to maturity).
  - Note that the put value can be derived as $e^{-rT}E[\max(K - S_T, 0)]$.
  - Consider an extreme case in which the stock price is close to 0 so that S_T can be almost ignored when calculating payoffs of puts.
  - The option values of the above two put options are $e^{-rT_1}E[K -$

Effect of Factors on Option Pricing

- Volatility $\sigma \uparrow$
  - The chance that the stock will do well or poor increases.
  - For calls (puts) which have limited downside (upside) risk, call (put) values benefits from the higher prob. of price increases (decreases) $\Rightarrow$ option value $\uparrow$ when $\sigma \uparrow$.

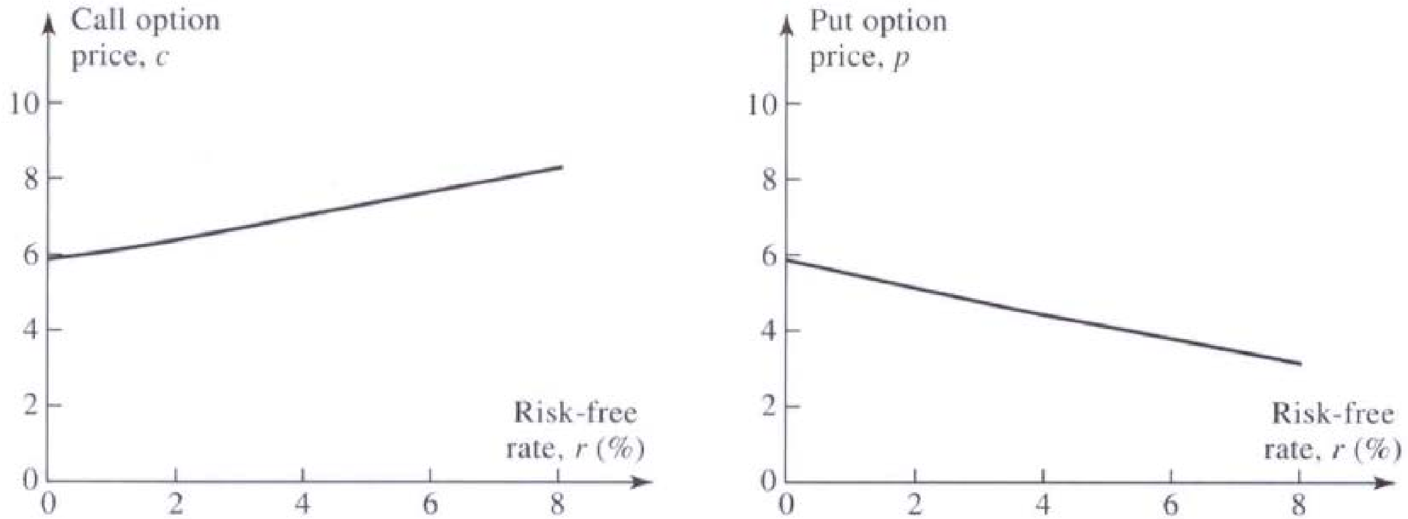




$S(0) = 50, K = 50, r = 5\%, D = 0, \text{and } T = 1$

Effect of Factors on Option Pricing

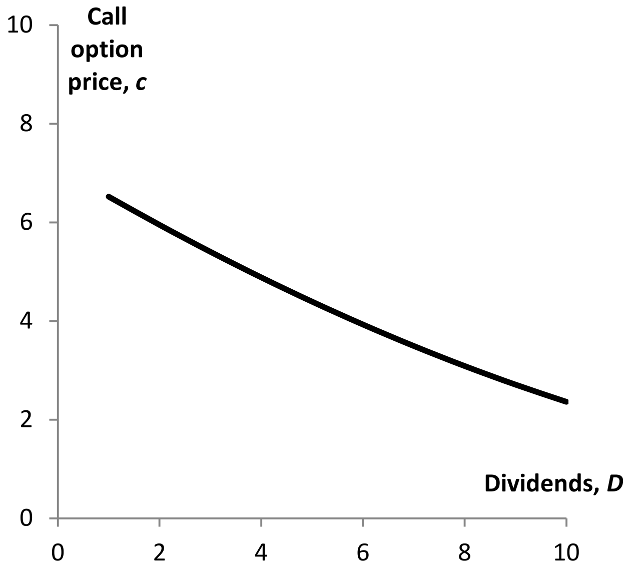
- Risk-free rate r ↑
  - For calls, option value ↑ because the higher expected S_T and the higher prob. to be ITM dominate the effect of lower PVs.
  - For puts, option value ↓ due to the higher expected S_T, the lower prob. to be ITM, and the effect of lower PVs.

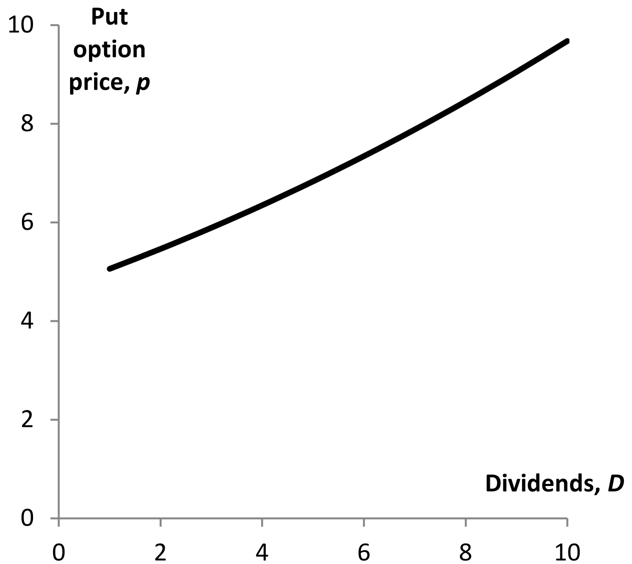


$S(0) = 50, K = 50, \sigma = 30\%, D = 0,$ and $T = 1$

Effect of Factors on Option Pricing

- Dividend payment ↑
  - Dividends have the effect of reducing the stock price on the ex-dividend date (除息日).
  - For calls, prob. of being ITM ↓ and thus call values ↓ .
  - For puts, prob. of being ITM ↑ and thus put values ↑.



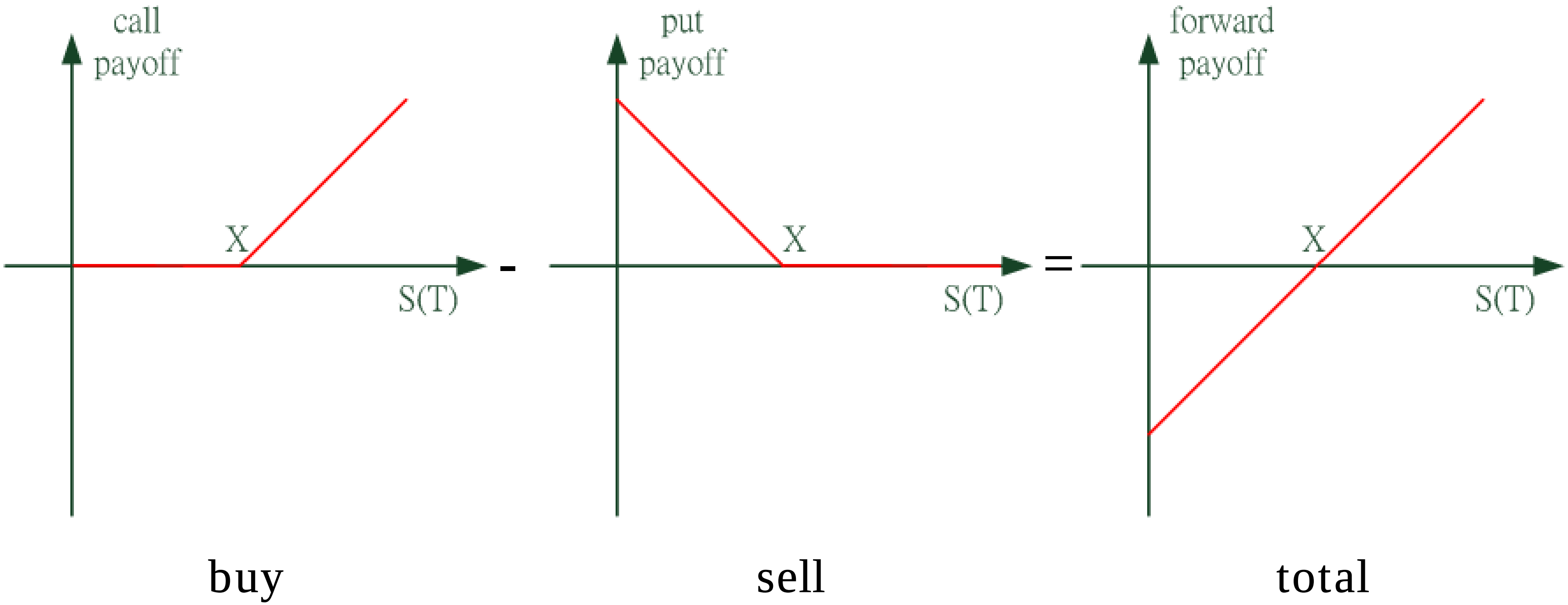


$S(0) = 50, K = 50, r = 5\%, \sigma = 30\%, \text{and } T = 1$

Put-Call Parity

- Consider a portfolio constructed by writing and selling 1 put and buying 1 call.
  - Both have the same strike price X and maturity T.
  - Adding the long position in the call and the short position in the put, we obtain the payoff of a long forward contract with forward price X and delivery time T.
  - If $S(T) \geq X$, the call will pay $S(T) - X$ and put will be worthless.
  - If $S(T) \leq X$, the put will pay $X - S(T)$ and call will be worthless.
  - The total value of the portfolio will be $S(T) - X$ at expiry.
    - This is the price of a long forward position.
  - The portfolio of the option should be worth $S(0) - Xe^{-rT}$, value of a forward contract. (Assuming r is constant)

Put-Call Parity



Put-Call Parity

- For a stock that pays no dividends the following relation holds between the price of European call and put options, both with strike price X and exercise time T.

$$C_E - P_E = S(0) - Xe^{-rT}$$

Put-Call Parity (Proof)

- Suppose that $C_E - P_E > S(0) - Xe^{-rT}$.
- At time 0:
  - Buy 1 share of stock for $S(0)$.
  - Buy 1 put option for P_E.
  - Write and sell 1 call for C_E.
  - Invest $-S(0) - P_E + C_E$ in the money market at the interest rate r.
    - If the value is negative then borrow the money at interest rate r.
- At time T:
  - Close the money market position, collecting $(-S(0) - P_E + C_E)e^{rT}$.
    - Pay the amount if borrowing money.
  - Sell the stock for X:
    - Exercise the put if $S(T) \leq X$.
    - Settling the short call position if $S(T) > X$.
- Total balance will be $(-S(0) - P_E + C_E)e^{rT} + X > 0$.

Put-Call Parity (Proof)

- Suppose that $C_E - P_E < S(0) - Xe^{-rT}$.
- At time 0:
  - Short sell 1 share of stock for $S(0)$.
  - Write and sell 1 put option for P_E.
  - Buy 1 call for C_E.
  - Invest $S(0) + P_E - C_E$ in the money market at the interest rate r.
    - If the value is negative then borrow the money at interest rate r.
- At time T:
  - Close the money market position, collecting $(S(0) + P_E - C_E)e^{rT}$.
    - Pay the amount if borrowing money.
  - Buy 1 share of stock for X and close the short position:
    - Exercise the call if $S(T) > X$.
    - Settling the short put position if $S(T) \leq X$.
- Total balance will be $(S(0) + P_E - C_E)e^{rT} - X > 0$.

Put-Call Parity with Dividends

- Suppose a stock pays a dividend between the times 0 and T.

$$C_E - P_E = S(0) - div_0 - Xe^{-rT}$$

- If the dividend is paid continuously at a rate r_{div},

$$C_E - P_E = S(0)e^{-r_{div}T} - Xe^{-rT}$$

Example (1999 FRM Exam Q.35)

- 根據Put-call parity，賣一個「賣權」等同:

  A. 買一個「買權」，買股票，借出錢；
  B. 賣一個「買權」，買股票，借入錢；
  C. 賣一個「買權」，買股票，借出錢；
  D. 賣一個「買權」，賣股票，借入錢；

Example (2002 FRM Exam Q.47)

- 兩年期的歐式買權價值$50，其執行價格為$140，現貨價$100，每年支付股利2%，年利率為5%；則執行價格為$140的兩年期的歐式賣權價值為:
  A. $77
  B. $10
  C. $90
  D. $81

American Put-Call Parity Estimates

- For American options put-call parity gives only some estimates.
- The prices of American call and put options with the same strike price X and expiry time T on a stock that pays no dividends satisfies

$$S(0) - Xe^{-rT} \geq \mathrm{C_A} - \mathrm{P_A} \geq S(0) - X$$

American Put-Call Parity Estimates

- If $C_A - P_A > S(0) - Xe^{-rT}$
  - We can write and sell a call.
  - Buy a put and a share of stock.
  - Finance the transaction in the money market.
- Exercising the call at time $t \leq T$.
  - We receive X for the share and settle the money market position.
  - We end up with a put and a positive amount.

$$X + \left(C_A - P_A - S(0)\right)e^{rt} = \left(Xe^{-rt} + C_A - P_A - S(0)\right)e^{rt}$$
$$\geq \left(Xe^{-rT} + C_A - P_A - S(0)\right)e^{rt} > 0$$

American Put-Call Parity Estimates

- If the call is not exercised.
  - We can exercise the put at time T.
  - Close the money market position.
  - Ending up with a positive amount.

$$X + \left(C_A - P_A - S(0)\right)e^{rT} > 0$$

Put-Call Parity

- Consider Portfolios A and C:
  - Portfolio A: 1 European call option plus a zero-coupon bond that provides a payoff of X at time T
  - Portfolio C: 1 European put plus 1 share of the stock

Portfolio A	$S_T > X$	$S_T < X$
Call option	$S_T - X$	0
Zero-coupon bond	X	X
Total	S_T	X

Portfolio C	$S_T > X$	$S_T < X$
Put option	0	$X - S_T$
1 share of stock	S_T	S_T
Total	S_T	X

Put-Call Parity

- Due to the law of one price, Portfolios A and C must therefore be worth the same today.

$$C_E + Xe^{-rT} = P_E + S(0)$$

- Is there any arbitrage opportunity if $P_E = 1$ or $P_E = 2.25$ given $C_E = 3$, $S(0) = 31$, $\mathrm{X} = 30$, $r = 10\%$, $D = 0$, and $T = 0.25$?
  - The theoretical price of the put option is 1.26 by solving $3 + 30e^{-0.1 \cdot 0.25} = P_E + 31$
  - The arbitrage strategies for $P_E = 2.25$ and $P_E = 1$ are shown in the following table.
- Rewrite the put-call parity: $C_E + Xe^{-rT} = P_E + S(0) \Rightarrow C_E + Xe^{-rT} - S(0) = P_E$, based on which it is simpler to identify the arbitrage opportunity.

Put-Call Parity

Three-month put price = $2.25 (Long $C_E + Xe^{-rT} - S(0)$ and short P_E)	**Three-month put price = $1 (Short $C_E + Xe^{-rT} - S(0)$ and long P_E)**
Buy the call at $3, short the stock to realize $31, and short the put to realize $2.25 $\Rightarrow$ Deposit the net cash flow $30.25 at 10% for 3 months	Short the call to realize $3, buy the stock at $31, buy put at $1, and borrow $29 at 10% for 3 months $\Rightarrow$ The net cash flow is 0
If $S_T > 30$ after 3 months: Receive $31.02 from the deposit, exercise the call to buy the stock at $30 $\Rightarrow$ Net profit = $1.02	If $S_T > 30$ after 3 months: The call is exercised and thus need to sell the stock for $30, and use $29.73 to repay loan $\Rightarrow$ Net profit = $0.27
If $S_T < 30$ after 3 months: Receive $31.02 from the deposit, the put is exercised and thus need to buy the stock at $30 $\Rightarrow$ Net profit = $1.02	If $S_T < 30$ after 3 months: Exercise the put to sell the stock for $30, and use $29.73 to repay loan $\Rightarrow$ Net profit = $0.27

Put-Call Parity

- Extension of the put-call parity for the American call and put

$$S_0 - K \le C - P \le S_0 - Ke^{-rT}$$

- Identify the upper and lower bounds of P_E given $C_E = 1.5$, $S(0) = 19$, $X = 20$, $r = 10\%$, $D = 0$, and $T = 5/12$

$$19 - 20 \le 1.5 - P_E \le 19 - 20e^{-0.1 \cdot 5/12}$$

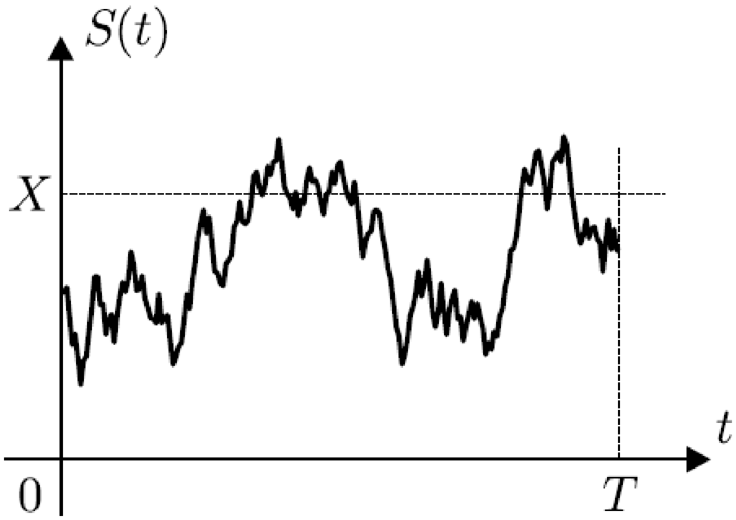
$$\Rightarrow 1.68 \le P_E \le 2.50$$

Bounds on Option Prices

- European and American options with the same strike price X and expiry time T:

$$C_E \leq C_A \text{ and } P_E \leq P_A$$

  - American option gives at least the same rights as the corresponding European option.
- American option has a probability of bring positive payoff whereas European option can not.



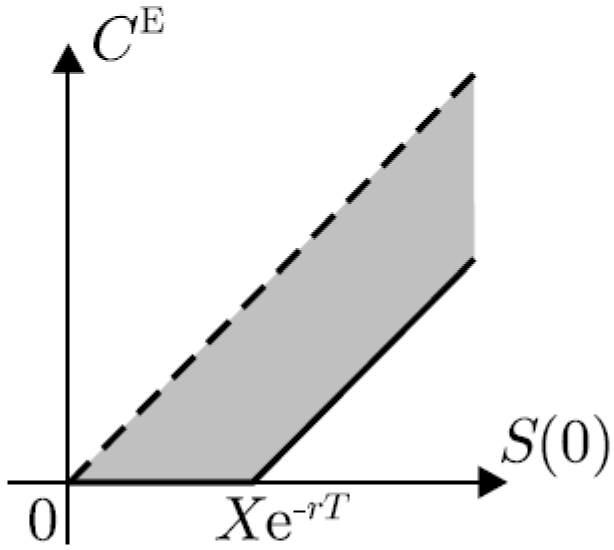
Bounds on Call Options

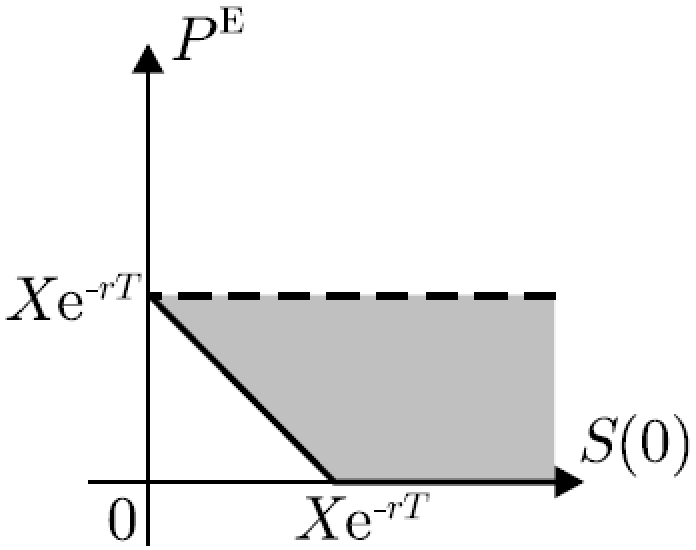
- Observe that, for European call options:

$$(S(0) - Xe^{-rT})^{+} < C_E < S(0)$$

- By the put-call parity:

$$(-S(0) + Xe^{-rT})^{+} \le P_E < Xe^{-rT}$$





Bounds on Call Options

- Considering the effect of dividends div_0:

$$(S(0) - div_0 - Xe^{-rT})^+ < C_E < S(0) - div_0$$

$$(-S(0) + div_0 + Xe^{-rT})^+ \leq P_E < Xe^{-rT}$$

Calls on Non-Dividend-Paying Stock

- The prices of European and American call options with the same strike price and expiry date on a non-dividend paying stock is equal.
  - $C_E = C_A$
  - It seems unreasonable at first, compared to previous slides.

Calls on Non-Dividend-Paying Stock

- From the previous slides, we know $C_A \geq C_E$. If $C_A > C_E$:
- At time 0:
  - Write and sell an American call at C_A.
  - Buy an European call at C_E.
  - Invest the extra money $(C_A - C_E)$ at interest rate r.
- At time $t \leq T$:
  - If the American call is exercised, borrow a share of stock and sell it at X.
    - Your obligation as a call writer.
  - Invest the X at interest rate r.
- At time T:
  - Exercise your European call and buy a share at X. Close the short position.
  - Total cash flow is $(C_A - C_E)e^{rT} + Xe^{r(T-t)} - X > 0$.

Calls on Non-Dividend-Paying Stock

- This is only true for European style options.
- It is not true for dividend paying stocks.
- In most cases, American options should be exercised prematurely.

Example – European Call

- Is there any an arbitrage opportunity if $C_E = 3$, $S(0) = 20$, $\mathrm{X} = 18$, $r = 10\%$, $D = 0$, and $T = 1$?
  - Since the call price violates the lower bound constraint ($\$20 - \$18e^{-0.1\cdot 1} = \$3.71$) , the following strategy can arbitrage from this distortion
    - Buy the underestimated call and short one share of stock ⇒ Generate a cash inflow of \$20 – \$3 = \$17.
    - Deposit \$17 at $r = 10\%$ for one year ⇒ Generate an income of $\$17e^{10\%\cdot 1} = \$18.79$ at the end of the year.
    - If $S_T > \$18$, exercise the call to purchase one share of stock at \$18 and close out the short position ⇒ The net income is \$18.79 – \$18 = \$0.79.
    - If $S_T < \$18$, give up the right of the call, purchase 1 share at S_T in the market, and close out the short position ⇒ The net income is \$18.79 – S_T, which must be higher than \$0.79.

Example – European Put

- Is there any arbitrage opportunity if $P_E = 1$, $S(0) = 37$, $X = 40$, $r = 5\%$, $D = 0$, and $T = 0.5$?
  - Since the put price violates the lower bound constraint ($\$40e^{-0.05\cdot 0.5} - \$37 = \$2.01$) , the following strategy can arbitrage from this distortion
    - Borrow \$38 at $r = 5\%$ for 6 months $\Rightarrow$ Need to pay off $\$38e^{5\%\cdot 0.5} = \$38.96$ after half a year
    - Use the borrowing fund to buy the underestimated put and one share of stock
    - If $S_T > \$40$, discard the put, sell the stock for S_T, and repay the loan $\Rightarrow$ The net income is S_T – \$38.96 > 0
    - If $S_T < \$40$, exercise the right of the put to sell the share of stock at \$40 and repay the loan $\Rightarrow$ The net income is \$40 – \$38.96 = \$1.04

Early Exercise

- Usually there is some chance that an American option will be exercised early.
- An exception is an American call on a non-dividend paying stock, which should never be exercised early

$$\because c \geq S_0 - Ke^{-rT} \text{ and } C_A \geq c$$

$$\therefore C_A \geq c \geq S_0 - Ke^{-rT} \geq S_0 - K = \text{exercise value}$$

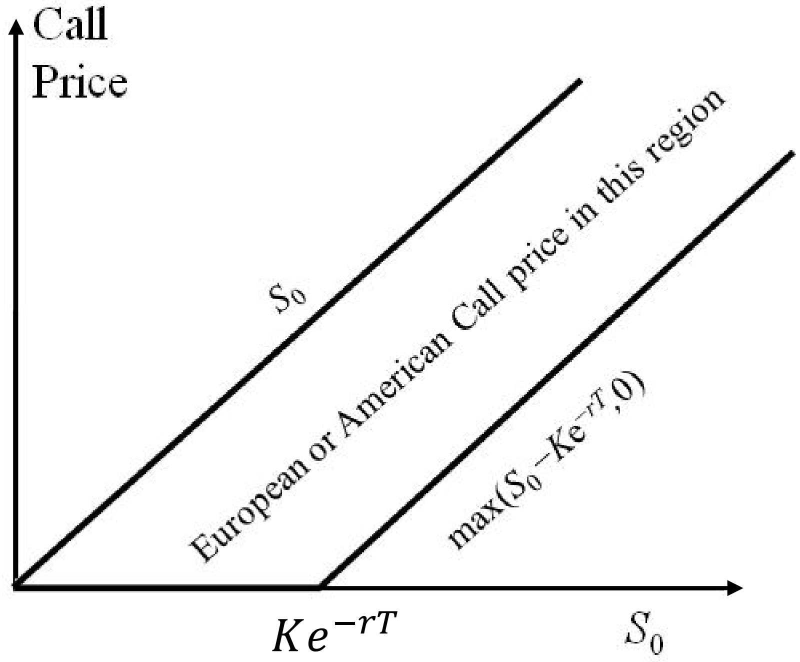
$\Rightarrow$ It is not optimal to exercise American call option if there is no dividend payments.

(Note the early exercise occurs when $C_A <$ exercise value)

Early Exercise

- So, American calls are equivalent to European calls if there is no dividend payment.

$$\max(S_0 - Ke^{-rT}, 0) \le c, C_A \le S_0$$



Early Exercise

- For a deeply ITM American call option: $C_A = 42$, $S_0 = 100$, $K = 60$, $T = 0.25$, and $D = 0$. Should you exercise the call immediately?
  - What should you do if

1. You want to hold the stock for the next 3 months?
   - No, it is better to delay paying the strike price for the stock 3 months later.
2. You still want to hold the stock but you do not feel that the stock is worth holding for the next 3 months?
   - No, it is possible to purchase the stock at a price lower than the strike price 3 months later.
3. You want to sell the stock share immediately after the exercise?
   - No, selling the American call for \$42 is better than undertaking this strategy, which is with the payoff of \$100 – \$60 = \$40.

Early Exercise

- Reasons for not exercising an American call early if there are no dividends
  - Due to no dividends, no income is sacrificed if you hold the American call instead of holding the underlying stock shares
  - Payment of the strike price can be delayed
  - Holding the call provides the possibility that the purchasing price could be lower than but never higher than the strike price
  - The payoff from exercising the American call is lower than the payoff from selling the American call directly

Early Exercise

- It can be optimal to exercise American put option on a non-dividend-paying stock early

  $\because p \geq Ke^{-rT} - S_0$ and $P_A \geq p$

  $\therefore P_A \geq p \geq Ke^{-rT} - S_0$,

  which is lower than the exercise price $K - S_0$

  $\Rightarrow$ The relationship between the American put price, P_A, and its exercise value, $K - S_0$, is uncertain

  $\Rightarrow$ For American puts, as long as their values are lower than $\max(K - S_0, 0)$, they are early exercised and the option value rises to become $\max(K - S_0, 0)$