Introduction to Financial Engineering and Algorithms

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Interest Rate Futures

Introduce Two Interest Rate Futures

- Treasury bond futures
  - Quotation method
  - Delivery options (conversion factors and cheapest-to-deliver bonds)
  - Theoretical futures price on Treasury bonds
- Eurodollar futures
  - Quotation method
  - Futures vs. forward rates and convexity adjustment

Why "Count" Differently?

- In the capital markets, there are a number of ways that days between dates are computed for interest rate calculations
- These conventions developed in different markets in order to simplify calculations (before the days of calculators and computers), but they have continued to persist

Day Count Conventions in the U.S.

- Some fixed-income securities:
  - Treasury bonds (notes) are the coupon-bearing bonds issued by the U.S. government with the maturity longer (shorter) than 10 years.
  - Municipal bonds are coupon-bearing bonds issued by state or local governments, whose interest payments are not subject to federal and sometimes state and local tax.
  - Corporate bonds are the coupon-bearing bonds issued by business firms.
  - Money market instruments are short-term, highly liquid, and relatively low-risk debt instruments, and no coupon payments during the life of them.

Day Count Conventions in the U.S.

- The day count convention is usually expressed as *X* / *Y*
- When calculating the interest earned between two dates,
  - *X* defines the way in which the number of days between the two dates is calculated.
  - *Y* defines the way in which the total number of days in the reference period is measured.

$$\frac{\text{\# of days between dates}}{\text{\# of days in reference period}} \times \text{Interest earned in reference period}$$

  - Treasury bonds or notes: Actual/actual (in period).
  - Corporate and municipal bonds: 30/360.
  - Money market instruments: Actual/360.
- Note that conventions vary from country to country and instrument to instrument!

Five Basic Day Count Conventions

- Actual/360
- Actual/365
- Actual/Actual
- 30/360
- 30/360 European

Actual/360

- This calculates the actual number of days between two dates and assumes the year has 360 days.
- Many calculations for money market instruments with less than a year to maturity use this day count basis, including Treasury bills.

Actual/360 – Example

- What would the interest payment be for a $1 million six-month CD issued on 01/31/04 and maturing on 07/31/04, and that has a 6% coupon?
  - Actual number of days between 01/31/04 and 07/31/04 = 182 days
  - Interest = 0.06 × $1,000,000 × (182/360)
    = $30,333.33

Actual/365

- This calculates the actual number of days between two dates and assumes the year has 365 days.

Actual/365 – Example

- What would the interest payment be for the same $1 million six-month CD issued on 01/31/04 and maturing on 07/31/04, and that has a 6% coupon?
  - Actual number of days between 01/31/04 and 07/31/04 = 182 days
  - Interest = $0.06 \times \$1{,}000{,}000 \times (182/365)$

    $= \$29{,}917.81$

Actual/Actual

- This day count basis calculates the actual number of days between two dates and assumes the year has either 365 or 366 days depending on whether the year is a leap year.
  - More precisely, if the range of the date calculation includes February 29 (the leap day), the divisor is 366, otherwise it is 365
- This day count basis is used for Treasury bonds.

Actual/Actual – Example

- What would the interest payment be for the same $1 million six-month CD issued on 01/31/04 and maturing on 07/31/04, and that has a 6% coupon?
  - Actual number of days between 01/31/04 and 07/31/04 = 182 days
  - Interest = $0.06 \times \$1{,}000{,}000 \times (182/366)$
    $= \$29{,}836.07$

30/360

- This day count convention assumes that each month has 30 days and the total number of days in the year is 360.
  - There are adjustments for February and months with 31 days
  - Assume Date 1 is of the form M1/D1/Y1 and Date 2 is of the form M2/D2/Y2, with Date 2 being later than Date 1.
    - If D1 = 31, change D1 to 30
    - If D2 = 31 and D1 = 30, change D2 to 30
    - Days between dates = $(Y2 - Y1)\times 360 + (M2 - M1)\times 30 + (D2 - D1)$
- This day count basis is used for U.S. corporate and municipal bonds.

30/360 – Example

- What would the interest payment be for the same $1 million six-month CD issued on 01/31/04 and maturing on 07/31/04, and that has a 6% coupon?
  - Number of days between 01/31/04 and 07/31/04 = 0 + (6 × 30) + 0 = 180 days
  - Interest = 0.06 × $1,000,000 × (180/360)
    = $30,000.00

30/360 European

- The 30/360 day count basis is different outside the United States, where the calculation was further simplified.
  - Assume Date 1 is of the form M1/D1/Y1 and Date 2 is of the form M2/D2/Y2, with Date 2 being later than Date 1
    - If D1 = 31, change D1 to 30
    - If D2 = 31, change D2 to 30
    - Days between dates = $(Y2 - Y1)\times 360 + (M2 - M1)\times 30 + (D2 - D1)$

30/360 European – Example

- What would the interest payment be for the same $1 million six-month CD issued on 01/31/04 and maturing on 07/31/04, and that has a 6% coupon?
  - Number of days between 01/31/04 and 07/31/04 = 0 + (6 × 30) + 0 = 180 days
  - Interest = 0.06 × $1,000,000 × (180/360)
    = $30,000.00

Date Calculation

- Calculate the number of days between 02/02/06 and 09/30/06, using:
  - An Actual day count: 240 days.
  - A 360 day count: 238 days.

Calculation of Interests for a Period

- Suppose we wish to calculate the interest earned between Mar. 1 and July 3 for coupon-bearing bonds.
- The reference period is from Mar. 1 (the last coupon payment day) to Sept. 1 (the next coupon payment day).
- The interest of \$4 is earned during the reference period.
- For Treasury bonds or notes, based on the actual/actual convention, the accrual interest is

$$\frac{124}{184} \times \$4 = \$2.6957$$

  - 124 (184) is the actual number of days between Mar. 1 and July 3 (Mar. 1 and Sept. 1).

Calculation of Interests for a Period

- For corporate or municipal bonds, based on the 30/360 convention, the accrual interest is

$$\frac{122}{180} \times \$4 = \$2.7111$$

  - 122 (=(4×30)+2) is the number of days between Mar. 1 and July 3, and 180 (=(6×30)) is the number of days between Mar. 1 and Sept. 1.

Calculation of Interests for a Period

- For a 124-day Treasury bill (from Mar. 1 to July 3) with the face value to be $100, if the rate of interest earned is known to be 8% (per annum) of the face value, based on the actual/360 convention, the interest income over the 124-day life is

$$\$100 \times 8\% \times \frac{124}{360} = \$2.7556$$

U.S. Treasury Securities

- Treasury bills (or T-Bills) mature in one year or less.
  - Like zero-coupon bonds, they do not pay interest prior to maturity.
  - Sold at a discount of the par value to create a positive yield to maturity.
  - Many regard Treasury bills as the least risky investment available to U.S. investors.
- Treasury notes (or T-Notes) mature in two to ten years, have a coupon payment every six months, and have denominations of $1,000.
  - In the basic transaction, one buys a "$1,000" T-Note for say, $950, collects interest over 10 years of say, 3% per year, which comes to $30 yearly, and at the end of the 10 years cashes it in for $1000.
  - So, $950 over the course of 10 years becomes $1300.

U.S. Treasury Securities

- Treasury bonds (T-Bonds, or the long bond) have the longest maturity, from twenty years to thirty years.
  - They have a coupon payment every six months like T-Notes, and are commonly issued with maturity of thirty years.
  - The secondary market is highly liquid, so the yield on the most recent T-Bond offering was commonly used as a proxy for long-term interest rates in general.
  - This role has largely been taken over by the 10-year note, as the size and frequency of long-term bond issues declined significantly in the 1990s and early 2000s.

Quotations for Treasury Bonds in the U.S.

- Quotations of T-bonds and T-notes.
- They are quoted in dollars and thirty-seconds of a dollar and at a percentage of pay value.
  - A quote of 95:07 indicates that the bond price is 95 + 7/32 = 95.2188 for a $100 bond.
- The quoted price is the clean price, which is not the same as the cash price at which the bond is traded.

Quotations for Treasury Bonds in the U.S.

- Cash price (dirty price) = Quoted price (clean price) + Accrued Interest since the last coupon date.
  - If the quoted price is 95:07 on July 3, and the interest of $4 is for the period between Mar. 1 and Sept. 1.

    Cash price (dirty price) = $95.2188 + $2.6957 = $97.9145
  - Note that the theoretical value for the dirty price is the sum of the PVs of all future cash flows.
  - The reason to distinguish clean and dirty prices.
    - Clean prices are more stable over time than dirty prices–when clean prices change, it usually reflects an economic reason, for instance a change in interest rates or in the bond issuer's credit quality.
    - Dirty prices, in contrast, change day to day depending on where the current date is in relation to the coupon payment dates, in addition to any economic reasons.

Quotations for Treasury Bonds in the U.S.

- The quotes of T-bills are annual discount rates of the face value with the bank-discount method.
  - For a Treasury bill that has n days to maturity,

    $$\text{Quoted Price (\%)} = \frac{360}{n}(100 - \text{Cash Price})$$

  - If the current market price is 99.5% of the face value for a 30-day T-bill, then.

    $$\text{Quoted Price (\%)} = \frac{360}{30}(100 - 99.5) = 6$$

  - This method provides a benchmark to compare with the performance of other fixed income securities in terms of rates of return (ROR).
    - Higher quoted prices indicate higher RORs and thus better investment targets if all other covenants are identical.

Treasury Bond Futures

- Treasury bond futures traded on CBOT is one the most popular long-term interest rate futures contracts.
  - Treasury bond futures prices are quoted in the same way as the Treasury bond prices, i.e., the quote of 115:245 indicates the futures price to be 115+24.5/32 = $115.7656.
  - Contract size: one contract involves the delivery of $100,000 face value of the bond.
  - Delivery can take place at any time during the delivery month.

Treasury Bond Futures

- For each trading day in the delivery month, the short side can deliver any government bond that has more than 15 years to maturity on the first day of the delivery month and is not callable within 15 years from that day.
  - T-note futures: deliver any government bond matured between 6.5 and 10 years.
  - 5-year T-note futures: deliver any government bond matured between four years and two months and five years and three months.

Treasury Bond Futures

- Cash received by party with short position = Most recent quoted futures price × Conversion factor + Accrued interest (an example is shown as follows).
  - Quoted price of bond futures = \$90.00
  - Conversion factor = 1.3800
  - Accrued interest on bond = \$3.00
  - Price received by the short side who delivers this bond is \$90.00×1.3800 + \$3.00 = \$127.20 (per \$100 of principal).

Treasury Bond Futures

- The conversion factor for a bond is approximately equal to the value of the bond on the assumption that the yield curve is flat at 6% with semiannual compounding.
  - To derive the conversion factor is through pricing bonds with 6% discount rate (or 3% discount rate for half a year).
- The two examples for calculating the conversion factor are shown as follows.

Conversion Factor

- Example 1
  - Consider a 10% coupon bond (paid semiannually) with 20 years and 2 months to maturity.
  - The bond is assumed to have exactly 20 years to maturity, and the first coupon payment is assumed to be made after six months.
    - The bond maturity and the dates to the coupon payments are rounded down to the multiples of three months for calculating the conversion factor.
  - The PV of this bond with the face value assumed to be \$1 is defined as the conversion factor:

$$\sum_{i=1}^{40} \frac{1 \cdot 5\%}{(1+3\%)^i} + \frac{1}{(1+3\%)^{40}} = 1.4623$$

Conversion Factor

- Example 2
  - Consider a 8% coupon bond (paid semiannually) with 18 years and 4 months to maturity.
  - For calculating the conversion factor, the bond is assumed to have exactly 18 years and 3 months to maturity, and the first coupon payment is assumed to be made after 3 months.
  - The value of the bond after 3 months (with the face value assumed to be $1) is:

$$1 \cdot 4\% + \sum_{i=1}^{36} \frac{1 \cdot 4\%}{(1+3\%)^{i}} + \frac{1}{(1+3\%)^{36}} = 1.2583$$

Conversion Factor

- The discount interest rate for the first three-month period is

$$\sqrt{(1+3\%)} - 1 = 1.4889\%$$

  - It is equivalent to find an interest rate to generate the identical cumulative return for six months, i.e.,

$$(1+r)^2 = 1 + 3\%$$

- The present value of this bond (with the face value assume to be \$1) is the conversion factor and can be derived as follows.

$$1.2583/(1+1.4889\%) = 1.2199$$

Cheapest-to-Deliver (CTD) Bond

- The trader with the short position can choose which of the available bonds is "Cheapest" to deliver, i.e., the CTD bond.
  - Find the CTD bond by minimizing the cost of purchasing a bond (at Quoted bond price + Accrued interest) minus the sales proceeds received from the T-bond futures (at Quoted futures price × Conversion factor + Accrued interest).
  - It is equivalent to deliver a bond for which Quoted bond price – (Quoted futures price × Conversion factor) is lowest.
  - Note that the CTD bond is defined as the bond most favorable to the short side for delivery rather than the bond with the cheapest value.

Cheapest-to-deliver (CTD) Bond

- Suppose there are three bonds which is deliverable and the most recent settlement futures price is 93.25.

Bond	Quoted bond price ($)	Conversion factor
1	99.50	1.0382
2	143.50	1.5188
3	119.75	1.2615

- The cost of delivering each of the bonds is as follows:
  - Bond 1: $99.50 – ($93.25 × 1.0382) = $2.69
  - Bond 2: $143.50 – ($93.25 × 1.5188) = $1.87
  - Bond 3: $119.75 – ($93.25 × 1.2615) = $2.12
- The CTD bond is Bond 2.

Treasury Bond Futures

- The delivery options in the Treasury bond futures contract are summarized as follows:
- Timing option: delivery can be made any time during the delivery month.
- Delivery alternatives: any of a range of eligible bonds can be delivered (CTD bond will be delivered).
- The wild card play.
  - Futures market is closed at 2:00 p.m.
  - Spot market is closed at 4:00 p.m.
  - Short-side traders have the time until 8:00 p.m. to issue to the clearinghouse a notice of intention to deliver.
  - The invoice price is calculated on the closing price at 2:00 p.m. (If the spot price declines after 2:00 p.m., there is a benefit for the short-side traders).

Treasury Bond Futures

- The theoretical futures price for the Treasury bond futures contracts is difficult to determine.
  - Due to the delivery options mentioned above.
  - It is inappropriate to consider a constant risk-free interest rate since the change of prices of Treasury bonds implies a stochastic risk-free interest rates.
  - Suppose the CTD bond and the delivery date are known and the risk-free interest rate applicable to a time to maturity T is constant.
    - Since the bond is a security providing known dollar incomes, then the futures price on T-bonds is:

$$F_0 = (S_0 - I_0)e^{rT},$$

where S_0 is the spot price of the CTD bond, and I_0 is the PV of all coupons during the life of the futures contracts.

10-Year Treasury Bond Futures on Taiwan Futures Exchange (TAIFEX)

Item	Description
Name	10-year Government Bond Futures
Ticker Symbol	GBF
Underlying Asset	10-year government bonds with a face value of NTD5M and 3% coupon rate
Deliverable Bonds	10-year ROC government bonds that pay interests once a year and mature in not less than 8 years and 6 months and not more than 10 years from the expiration of the futures contract
Delivery Months	The three successive months of March, June, September, and December cycle (introduced in Ch. 9)
Price Quotation	Quoted by per NTD100 face value
Minimum Price Movement	NTD0.005 per NTU100 (NTD250 per contract)
Daily Settlement Price	The volume-weighted average trading price based on the transactions within the last one minute
Daily Price Limit	+/– NTD3 based on the settlement price on the previous trading day
Last Trading Day	The second Wednesday of the delivery month
Delivery	Physical delivery
Delivery Day	The second business day following the last trading day
Final Settlement Price	Determined by the volume-weighted average of trading prices in the last 15 minutes before closing on the last trading day; if there are less than 20 transactions in the last 15-minute interval, determined by the volume-weighted average of the last 20 transaction prices of the last trading day, excluding the two highest and two lowest transaction prices; if there are less than 20 transactions on that day, determined by the volume-weighted average of actual trading prices of the last trading day
	If there are no transactions on the last trading day or if the aforesaid price is apparently unreasonable, TAIFEX will determine the final settlement price
Position Limit	Any investor's same-side positions shall not exceed 1000 contracts for any single contracts and 2000 contracts for all GBFs.
	Institutional investors may apply for an exemption from the above limit for hedging purpose
Margin	The initial and maintenance margin levels shall not be less that those regulated by TAIFEX

Eurodollar Futures

- A Eurodollar is a dollar deposited in a bank outside the United States.
  - The Eurodollar interest rate is the rate of interest earned on Eurodollars deposited by one AA-rated bank with another AA-rated bank, i.e., the LIBOR.
- Eurodollar futures are interest rate futures which can lock the 3-month Eurodollar forward deposit rate at the maturity date of the futures.
  - One contract is on the rate earned on $1 million.

Eurodollar Futures

- Quotes and values of Eurodollar futures.
  - If Z is the quoted price of a three-month Eurodollar futures contract, the value of this contract is $10{,}000 \times [100 - 0.25 \times (100 - Z)]$.
  - $(100 - Z)$ represents the LIBOR rate (per annum); $0.25 \times (100 - Z)$ represents the LIBOR rate for 3 months; $[100 - 0.25 \times (100 - Z)]$ can be understood as the market price of a virtual three-month zero coupon bond (with the face value of \$100) corresponding to the LIBOR = $100 - Z$.
  - For example, if $Z = 95.53$ (95.54), the value of the three-month Eurodollar futures is \$988,825 (\$988,850).
- A change of 0.01 in a Eurodollar futures quote corresponds to a contract value change of \$25.
  - For the long position, $Z\uparrow$ (LIBOR$\downarrow$) $\Rightarrow$ gain
  - For the short position, $Z\downarrow$ (LIBOR$\uparrow$) $\Rightarrow$ gains

Eurodollar Futures

- A Eurodollar futures contract is settled in cash.
- When it expires (on the third Wednesday of the delivery month), *Z* is set equal to 100 minus the actual 3-month Eurodollar deposit rate on that day and all contracts are closed out at this value.
  - Suppose you take a long position of a Eurodollar futures contract on November 1.
  - The contract expires on December 21.
  - The series of futures prices are as shown on the next page.
  - How much do you gain or lose a) on the first day, b) on the second day, c) over the whole period until expiration?

Eurodollar Futures

1. On Nov. 1, you plan to lend \$1 million for three months on Dec 21, the Eurodollar futures can lock the lending rate to be (100 – 97.12)% = 2.88% and thus the interest income should be \$1,000,000×0.25×2.88% = \$7,200.
2. On Nov. 2, you have a gain of (97.23 – 97.12)×100×\$25 = \$275 (Note that the increase of 0.01 in *Z* generates a gain of \$25).

Date	Quote
Nov 1	97.12
Nov 2	97.23
Nov 3	96.98
⋮	⋮
Dec 21	97.42

Eurodollar Futures

3. On Nov. 3, you have a loss of

$$(96.98 – 97.23)\times 100\times \$25 = -\$625.$$

4. On Dec. 21, you earn (100 – 97.42)% = 2.58% on lending $1 million for three months (= $1,000,000×0.25×2.58% = $6,450) and make a gain on the futures contract to be

$$(97.42 – 97.12)\times 100\times \$25 =\$750.$$

- (Note that in the above example, the actual 3-month LIBOR rate declines to become 2.58%).
- The total payoff is $7,200 (=$6,450 + $750), which is the same as the payoff if the 3-month lending rate on Dec. 12 was fixed at 2.88%.

Forward Rates and Eurodollar Futures

- Eurodollar futures contracts can last as long as 10 years.
- For Eurodollar futures lasting beyond two years, we cannot assume that the forward rate equals the futures rate.
- For forward rates, they are implied from the current term structure of interest rates; for futures rates, they are implied from (100 – Z)%, where Z is the quote of Eurodollar futures.
  - $(100 - Z_0)$% implies the futures rate applicable to the following three-month period at T.
  - For example, if $Z_0 = 97.12$ and T is two years, then (100 – 97.12)% = 2.88% is the futures rate for the period of three months after two years.

Forward Rates and Eurodollar Futures

- Two reasons to explain the difference between the forward rate and futures rate.
  1. Futures is settled daily, whereas forward is settled once.
     - The futures price of Eurodollar futures contracts, Z, is highly negatively correlated with other interest rates since $(100 – Z)\%$ represents a 3-month interest rate applicable to a future time point.
     - The futures price Z should be lower than the counterpart forward price.
     - So, the futures rate $(=(100 – Z)\%)$ should be higher than the counterpart forward rate.

Forward Rates and Eurodollar Futures

2. A (Eurodollar) futures is settled at the beginning of the reference three-month period (i.e., at T); A forward is settled at the end of the reference three-month period (i.e., at T + 0.25).
   - If all others are the same, a futures price Z should be the 3-month-ahead PV of the counterpart forward price, i.e., the future price is lower than the counterpart forward price
   - The futures rate (=(100 – Z)%) should be higher than the counterpart forward rate

Forward Rates and Eurodollar Futures

- A "convexity adjustment" often made is

  Forward rate = Futures rate – $\frac{1}{2}\sigma^2 T_1 T_2$,

  where T_1 is the time to maturity of the futures contract, T_2 is the time to maturity of the rate underlying the futures contract (90 days later than T_1) and σ is the standard deviation of the short-term interest rate changes per year (typically σ is about 1.2%).
- Note that the above formula is for rates with continuous compounding.

Forward Rates and Eurodollar Futures

- Suppose we wish to calculate the forward rate when the 8-year Eurodollar futures price quote is 94.
  - The convexity adjustment is $\frac{1}{2} \times 0.012^2 \times 8 \times 8.25 = 0.00475$
  - The futures rate is 6% per annum on an actual/360 basis with quarterly compounding $\Rightarrow$ 6.083% (=6%×365/360) per annum on an actual/365 basis with quarterly compounding $\Rightarrow$ 6.038% with continuously compounding.
  - The forward rate is therefore 6.038% – 0.00475 = 5.563% per annum with continuous compounding.

Forward Rates and Eurodollar Futures

- Convexity adjustment for different maturities when σ is about 1.2%.

Maturity of Eurodollar Futures (years)	Convexity Adjustment
2	0.032%
4	0.122%
6	0.270%
8	0.475%
10	0.738%

- The above table shows that for a longer time to maturity, the difference between the futures and forward rates is more pronounced.

Duration

- The change in the value of a fixed income security that will result from a 1% change in interest rates.
- Duration is stated in years.
    - For example, a 5 year duration means the bond will decrease in value by 5% if interest rates rise 1% and increase in value by 5% if interest rates fall 1%.
- Duration is a weighted measure of the length of time the bond will pay out.
- Unlike maturity, duration takes into account interest payments that occur throughout the course of holding the bond. Basically, duration is a weighted average of the maturity of all the income streams from a bond or portfolio of bonds.

Duration

- For a two-year bond with 4 coupon payments every six months of $50 and a $1000 face value, duration (in years) is 0.5(50/1200) + 1(50/1200)+ 1.5(50/1200)+ 2(50/1200) + 2(1000/1200) = 1.875 years.
- Notice that the duration on any bond that pays coupons will be less than the maturity because there is some amount of the payments that are going to come before the maturity date.

Duration

- Investors use duration to measure the volatility of the bond. Generally, the higher the duration (the longer an investor needs to wait for the bulk of the payments), the more its price will drop as interest rates go up.
- Added risk comes greater expected returns.
- If an investor expects interest rates to fall during the course of the time the bond is held, a bond with a long duration would be appealing because the bond's price would increase more than comparable bonds with shorter durations.

Duration

- Bond duration: defined as the weighted average of the time points for each payment of a bond.
  - The weight of each payment is the ratio of the present value of that payment over the bond price.
  - For continuous compounding:
    - Duration of a bond with cash payment c_i at t_i is

$$D \equiv \sum_{t_i} t_i \frac{c_i e^{-y t_i}}{B},$$

    - $B = \sum_{t_i} c_i e^{-y t_i}$ is the bond price and y is its yield (Note that to calculate the duration, all discounting is done at the bond yield rate y and thus B equals the market value of the bond).
- The above definition leads to $D = -\frac{dB/B}{dy}$.

Duration

- Prove that $D = -\frac{dB/B}{dy}$, where $D \equiv \sum_{t_i} t_i \frac{c_i e^{-yt_i}}{B}$

$$B = \sum_{t_i} c_i e^{-yt_i} \quad \therefore -\frac{(dB/B)}{dy} = \frac{1}{B}\left[-\sum_{t_i} c_i e^{-yt_i}(-t_i)\right]$$

$$= \frac{1}{B}\left(\sum_{t_i} c_i e^{-yt_i} t_i\right)$$

$$= \sum_{t_i} t_i \frac{c_i e^{-yt_i}}{B} = D$$

Duration

- $D = -\frac{dB/B}{dy}$ can be approximated by $D \approx -\frac{\Delta B/B}{\Delta y}$, which indicates that the duration can measure the interest rate risk– the percentage change in bond price due to the change in yield.
  - The negative sign implies the inverse relationship between the changes in bond prices and yields.
  - The market convention is to say the absolute change rather than the percentage change in interest rates.

Duration

- Rewrite $D \approx -\frac{\Delta B/B}{\Delta y}$ as $\Delta B \approx -BD\Delta y$, based upon which we can estimate the change in the bond price if the duration is known and the change in its yield is estimated.

Time (years)	Cash flow ($)	Present Value (y=0.12)	Weight	Time × Weight
0.5	5	4.709	0.050	0.025
1.0	5	4.435	0.047	0.047
1.5	5	4.176	0.044	0.066
2.0	5	3.933	0.042	0.083
2.5	5	3.704	0.039	0.098
3.0	105	73.256	0.778	2.333
Total		94.213	1	2.653

- If the yield rises by 0.1%, the estimated change in the bond price is –94.213×2.653×0.1% = –0.250

Duration

- Examination the accuracy of $\Delta B \approx -BD\Delta y$: when the bond yield increases by 0.1% to become 12.1%, the actual bond price is
  $$5e^{-0.121\cdot 0.5} + 5e^{-0.121\cdot 1.0} + 5e^{-0.121\cdot 1.5} + 5e^{-0.121\cdot 2.0} + 5e^{-0.121\cdot 2.5} + 105e^{-0.121\cdot 3.0} = 93.963$$
  - Lower than the original bond price (94.213) by 0.250 (to three decimal places)
- Note that the estimate of $\Delta B \approx -BD\Delta y$ is accurate only when the absolute value of Δy is small.

Duration

- If Δy=2%, $\Delta B \approx -BD\Delta y = -94.213 \times 2.653 \times 2\% = -4.999$.
- Applying the discount yield as 14%, the actual bond price is
$$5e^{-0.14\cdot 0.5} + 5e^{-0.14\cdot 1.0} + 5e^{-0.14\cdot 1.5} + 5e^{-0.14\cdot 2.0} + 5e^{-0.14\cdot 2.5} + 105e^{-0.14\cdot 3.0} = 89.354$$
  - Lower than the original bond price (94.213) by 4.86.

Duration

- When the yield y is expressed with compounding m times per year.
  - Bond price formula $B = \sum_{t_i} \frac{c_i}{(1+y/m)^{t_i m}}$
  - Following the definition of duration $D \equiv \sum_{t_i} t_i \frac{c_i/(1+y/m)^{t_i m}}{B}$
  - The above duration formula leads to

$$-\frac{dB/B}{dy} = \frac{1}{1+y/m} D = D^*,$$

    where D^* is referred to as the "modified duration".
  - Thus, the duration relationship to estimate the change in the bond price is $\Delta B \approx -BD^*\Delta y$.

Duration

- Prove that $-\frac{dB/B}{dy} = \frac{1}{1+y/m} D$, where $D \equiv \sum_{t_i} t_i \frac{c_i/(1+y/m)^{t_i m}}{B}$

$$\because B = \sum_{t_i} \frac{c_i}{(1+y/m)^{t_i m}}$$

$$\therefore -\frac{dB/B}{dy} = \frac{1}{B}\left[-\sum_{t_i} t_i \frac{c_i}{(1+y/m)^{t_i m+1}} (-m t_i)\left(\frac{1}{m}\right)\right]$$

$$= \frac{1}{B}\sum_{t_i} \frac{c_i}{(1+y/m)^{t_i m+1}} =$$

$$\frac{1}{B}\sum_{t_i} \frac{c_i}{(1+y/m)^{t_i m}} \frac{1}{(1+y/m)}$$

$$= \frac{1}{(1+y/m)} \sum_{t_i} \frac{c_i/(1+y/m)^{t_i m}}{B}$$

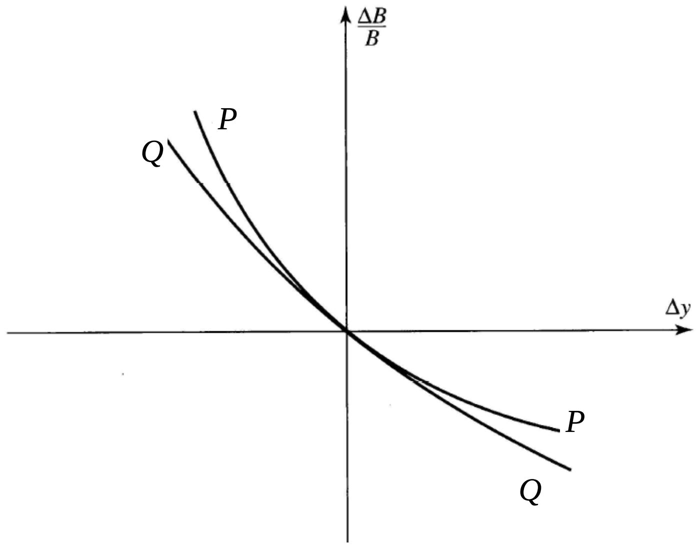
$$= \frac{1}{1+y/m} D = D^*$$

Duration

- $\frac{1}{1+y/m}D$ is known as the modified duration D^*, which can measure the IR risk more precisely than the duration D.
- The difference between D^* and D decreases with the compounding frequency, and when continuous compounding is considered, the difference disappears.

Duration

- Two bonds with the same duration have the identical gradients at the origin



- The above figure implies that for one small change in yield, Δy, the percentage change in the bond prices of P and Q are close, which inspires the duration-based hedging.

Duration Matching Hedge

- To hedge the IR risk of a bond P with nonzero dP/dy, consider to include an proper position of a bond Q such that $\frac{dP}{dy} + \frac{dQ}{dy} = 0$ ($\frac{\Delta P}{\Delta y} + \frac{\Delta Q}{\Delta y} = 0$ approximately).
  - With a change in yield, Δy, the changes of the prices of the positions P and Q offset with each other.
  - Duration relationship: $\frac{\Delta P}{\Delta y} = -PD_P$ and $\frac{\Delta Q}{\Delta y} = -QD_Q$.
  - If we can find a bond Q such that $PD_P = QD_Q$, we can SHORT the bond Q to generate a position so that $\frac{\Delta Q}{\Delta y} = -\frac{\Delta P}{\Delta y}$ and the portfolio of (P + Q) is free of the interest rate risk.
  - Note that the above idea is true for all risk factors, not only for the risk factor of interest rate.

Duration Matching Hedge

- Notations
  - V_F: Contract value for one interest rate futures contract
  - D_F: Duration of the asset underlying futures at the maturity of the futures contract
  - P: Value of the portfolio being hedged at the maturity of the hedge
  - D_P: Duration of the portfolio at the maturity of the hedge
- Duration-based hedge ratio ($\frac{\Delta P}{\Delta y} + \frac{\Delta(NV_F)}{\Delta y} = 0$)

$$N^* = \frac{PD_P}{V_F D_F} \text{ interest rate futures should be SHORTED}$$

$$\text{(due to } \frac{\Delta P}{\Delta y} = -PD_P \text{ and } \frac{\Delta V_F}{\Delta y} = -V_F D_F\text{)}$$

Example for Hedging a Bond Portfolio

- In August, a fund manager has \$10 million (P) invested in a portfolio of government bonds with a duration of 6.8 years (D_P) after three months and wants to hedge against interest rate moves in futures three months.
- The manager decides to use December T-bond futures. The futures price is 93:02 or 93.0625 (per \$100 face value) and the duration of the cheapest to deliver bond is 9.2 years (D_F) in December.
- The number of contracts that should be shorted is

$$\frac{\$10{,}000{,}000 \times 6.8}{\$93.0625 \times 1{,}000 \times 9.2} = 79.42$$

Example for Hedging a Floating-Rate Loan

- On the last trading day in April, a company considers to borrow \$15 million for the following three months
  - The IRs for each of the following three months will be the one-month LIBOR rate plus 1%
  - For May, the one-month LIBOR rate is known to be 8% per annum and thus the interest payments for this month is $\$15{,}000{,}000 \times (8\% + 1\%) \times (1/12) = \$112{,}500$ certainly
- For June
  - The IR applied to this month is determined by the one-month LIBOR rate on the last trading day before June

Example for Hedging a Floating-Rate Loan

- The IR risk can be hedged by taking a short position in the June Eurodollar futures contracts today.
  - The company requires a short position in Eurodollar futures since it will lose money if the IR rises and gain if the IR falls.
  - Note that for the short-side traders of Eurodollar futures, they can obtain gains when the IR rises.
- The duration of the asset underlying the Eurodollar futures at maturity is three months (0.25 years), and the duration of the liability being hedged is one month (0.0833 years).
  - The underlying asset of the Eurodollar futures is the 3-month LIBOR rate on the delivery date.
    - Suppose the lending principal is $1, the underlying asset of the Eurodollar futures can be viewed as the interest payment (=$1× the 3-month LIBOR rate) at the end of the 3-month lending period.
  - For lending or borrowing money, the only cash flow occurs at the end of the reference period, so the duration equals the length of the reference period.

Example for Hedging a Floating-Rate Loan

- Suppose the quoted price for June Eurodollar futures is 91.88. Thus the contract value is

$$10{,}000 \times [100 - 0.25 \times (100 - 91.88)] = \$979{,}700$$

- The number of the short position of the June Eurodollar futures is

$$\frac{\$15{,}000{,}000 \times 0.08333}{\$979{,}700 \times 0.25} = 5.10 \approx 5$$

- For July
  - The Sept. Eurodollar futures contract is used, and its quoted price is 91.44. Thus, the value of this futures is

$$10{,}000 \times [100 - 0.25 \times (100 - 91.44)] = \$978{,}600$$

  - The number of the short position of the July Eurodollar futures is

$$\frac{\$15{,}000{,}000 \times 0.08333}{\$978{,}600 \times 0.25} = 5.11 \approx 5$$

The Effectiveness of the Hedges

- For June
  - On the last trading day before June, the one-month LIBOR rate proves to be 8.88% and the June Eurodollar futures price proves to be 91.12 and its value is $977,800.
  - By closing out the futures, the company gains 5×($979,700 – $977,800) = $9,500.
    - For the short position of Eurodollar futures, the payoff equals (initial futures value – final futures value).
    - For the long position of Eurodollar futures, the payoff equals (final futures value – initial futures value).
  - The interest payment for June is $15,000,000× (8.88%+1%)×(1/12) = $123,500, which is higher than the interest payment in May (i.e., $112,500) by $11,000.

The Effectiveness of the Hedges

- For July
  - On the last trading day before July, the one-month LIBOR rate proves to be 9.84% and the Sept. Eurodollar futures price proves to be 90.16 and its value is $975,400.
  - By closing out the futures, the company gains 5×($978,600 – $975,400) = $16,000.
  - The interest payment for July is $15,000,000 ×(9.84%+1%)×(1/12) = $135,500, indicating extra interest costs of $23,000 (= $135,500 – $112,500) higher than that in May.

※The deterioration of the hedging effect is due to the large change in LIBOR rate. Note that the LIBOR rate for July (9.84%) is significantly higher than the LIBOR rate for May (8%).

Limitations of Duration-Based Hedging

- Assume that only parallel shifts in yield curve take place.
  - That is, assume that all yields with different time maturities move by Δy simultaneously.
- Assumes that the change in yield, Δy, is small.