Introduction to Financial Engineering and Algorithms

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Valuing Stock Options: The Black-Scholes Model

Introduction

- The assumption of the distribution of stock prices in Black-Scholes-Merton (BSM) model
- The risk neutral valuation relationship (RNVR) and BSM option pricing formulae
- Extension of RNVR on pricing forward contracts
- Implied volatility of options
- Effects of dividend payments on option prices

Black-Scholes-Merton Option Pricing Formulae

- In 1973, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in pricing European options
  - They developed a pricing formula for European options, which are known as the Black-Scholes or Black-Scholes-Merton model
  - Merton and Scholes won Nobel Prize in 1997 with this achievement
  - This chapter presents the Black-Scholes-Merton model for valuing European calls and puts on a non-dividend-paying stock first

Black-Scholes-Merton Option Pricing Formulae

- Based on the RNVR introduced, any derivative can be priced as the PV of its expected payoff in the risk neutral world
  - Consider a European call options, its value today is

$$c = e^{-rT} E[\max(S_T - K, 0) | \text{in the risk neutral world}]$$
$$= e^{-rT} \int_0^\infty \max(S_T - K, 0)\, f(S_T) dS_T,$$

  where $f(S_T)$ is the probability density function (PDF) of S_T in the risk neutral world

  - Therefore, to identify the distribution and thus the PDF of S_T in the risk-neutral world is the next step to derive the option pricing formula

Black-Scholes-Merton Option Pricing Formulae

- The distribution of S_T of the BSM model in the real world
  - The BSM model considers a non-dividend-paying stock and assumes the return on the stock in a short period of time is normally distributed, i.e.,

$$\frac{\Delta S}{S} \sim N(\mu \Delta t, \sigma^2 \Delta t),$$

where ΔS is the change in the stock price in a short period of time Δt, the mean of the return is $\mu \Delta t$, and the standard deviation of the return is $\sigma\sqrt{\Delta t}$

(Note that it is the variance of the return, not its standard deviation, that is proportional to Δt)

Black-Scholes-Merton Option Pricing Formulae

- The assumption of the normal distribution of the stock return implies that S_T is lognormally distributed as follows

$$\ln S_T \sim N(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma^2 T)$$

  - If $\ln(X) \sim N(\mu, \sigma^2)$, then $E[X] = e^{\mu+\sigma^2/2}$ and $\text{var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$
  - So, $E[S_T] = S_0 e^{\mu T}$ and $\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$
  - Rewrite the above distribution as follows

$$\ln S_T - \ln S_0 \sim N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$$

$$\ln \frac{S_T}{S_0} \sim N((\mu - \frac{\sigma^2}{2})T, \sigma^2 T)$$

Black-Scholes-Merton Option Pricing Formulae

- When $T = 1$, $\ln \frac{S_1}{S_0} \sim N(\mu - \frac{\sigma^2}{2}, \sigma^2)$
- Suppose R_1 is the annual continuously compounded return provided by the stock such that $S_1 = S_0 e^{R_1 \cdot 1}$
- We can derive $\ln \frac{S_1}{S_0} = R_1$ and thus $R_1 \sim N(\mu - \frac{\sigma^2}{2}, \sigma^2)$, so the mean and standard deviation of the continuously compounded return of the stock are $\mu - \frac{\sigma^2}{2}$ and σ, respectively

Black-Scholes-Merton Option Pricing Formulae

- Estimate $\mu - \frac{\sigma^2}{2}$ and σ given a series of historical stock prices
  - Suppose the length of the time interval of the observations of stock prices is τ, i.e., we have the series of
$$S_t, S_{t+\tau}, S_{t+2\tau}, \ldots, S_{t+n\tau}$$
  - Therefore, there are n observations of the log differences of the stock prices
$$u_1 = \ln\frac{S_{t+\tau}}{S_t}, u_2 = \ln\frac{S_{t+2\tau}}{S_{t+\tau}}, u_3 = \ln\frac{S_{t+3\tau}}{S_{t+2\tau}}, \ldots, u_n = \ln\frac{S_{t+n\tau}}{S_{t+(n-1)\tau}}$$
  - The lognormally distributed result on Slide 13.7 can be extended to any two time points t and $t+\tau$, i.e.,
$$\ln\frac{S_{t+\tau}}{S_t} \sim N((\mu - \frac{\sigma^2}{2})\tau, \sigma^2\tau)$$

Black-Scholes-Merton Option Pricing Formulae

- The arithmetic average and variance of the series of $u_i = \ln\frac{S_{t+i\tau}}{S_{t+(i-1)\tau}}$ can estimate $(\mu - \frac{\sigma^2}{2})\tau$ and $\sigma^2\tau$, respectively

$$(\mu - \frac{\sigma^2}{2})\tau = \bar{u} = \frac{1}{n}(\ln\frac{S_{t+\tau}}{S_t} + \ln\frac{S_{t+2\tau}}{S_{t+\tau}} + \cdots + \ln\frac{S_{t+n\tau}}{S_{t+(n-1)\tau}})$$

$$= \frac{1}{n}(\ln R_1 + \ln R_2 + \cdots + \ln R_n)$$

$$= \ln((R_1 R_2 \ldots R_n)^{\frac{1}{n}}) = \ln(\sqrt[n]{R_1 R_2 \ldots R_n})$$

$$\Rightarrow \mu - \frac{\sigma^2}{2} = \frac{\bar{u}}{\tau}$$

Black-Scholes-Merton Option Pricing Formulae

(Note that $(\mu - \frac{\sigma^2}{2})\tau$ is in essence the geometric average of the stock returns)

- $\sigma^2\tau = s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2$

$$\Rightarrow \sigma^2 = \frac{s^2}{\tau} \Rightarrow \sigma = \frac{s}{\sqrt{\tau}}$$

Black-Scholes-Merton Option Pricing Formulae

- Estimate μ given a series of historical stock prices
  - Extend $E[S_T] = S_0 e^{\mu T}$ to consider any time points t and $t+\tau$ as follows

$$E[S_{t+\tau}] = S_t e^{\mu\tau} \Rightarrow E\left(\frac{S_{t+\tau}}{S_t}\right) = e^{\mu\tau} \Rightarrow \ln\left(E\left(\frac{S_{t+\tau}}{S_t}\right)\right) = \mu\tau$$

  - The natural logarithm of the arithmetic average of the series of the gross returns $R_i = \frac{S_{t+i\tau}}{S_{t+(i-1)\tau}}$ can estimate $\mu\tau$:

$$\mu\tau = \ln\left(\frac{1}{n}\left(\frac{S_{t+\tau}}{S_t} + \frac{S_{t+2\tau}}{S_{t+\tau}} + \cdots + \frac{S_{t+n\tau}}{S_{(n-1)t}}\right)\right)$$

$$= \ln\left(\frac{1}{n}(R_1 + R_2 + \cdots + R_n)\right) = \ln(\bar{R})$$

$\Rightarrow \mu = \frac{\ln(\bar{R})}{\tau}$ (which is in essence the arithmetic average of the stock returns)

Black-Scholes-Merton Option Pricing Formulae

- Suppose that returns in successive 5 years are 15%, 20%, 30%, –20% and 25%
  - The arithmetic average of the returns is 14% (= μ in annual compounding)
    - Based on the formula, $\mu = \frac{\ln(\bar{R})}{\tau} = 13.10\%$ in continuously compounding
  - The geometric average of the returns is 12.40% (= $\mu - \frac{\sigma^2}{2}$ in annual compounding)
    - Based on the formula, $\mu - \frac{\sigma^2}{2} = 11.69\%$ in continuously compounding

※The same results can be derived from the transformation $R_c = m \ln\left(1 + \frac{R_m}{m}\right)$

Black-Scholes-Merton Option Pricing Formulae

- Trading days vs. Calendar days
  - Usually the daily observations of stock prices are used to estimate $\mu - \frac{\sigma^2}{2}$ and σ
  - Since the variation (or the volatility) of stock prices occurs only on trading days, so only trading days are considered to estimate $\mu - \frac{\sigma^2}{2}$ and σ
    - When the market is closed, the variation of stock prices is zero
  - As a result, τ is set to $\frac{1}{\text{\# of trading days per year}}$ and the number of trading days in a year is assumed to be 252 in the U.S.
  - Finally, $\mu - \frac{\sigma^2}{2} = \frac{\bar{u}}{\tau} = \bar{u} \cdot 252$ and $\sigma = \frac{s}{\sqrt{\tau}} = s \cdot \sqrt{252}$

※ Note that if the number of calendar days is used, $\mu - \frac{\sigma^2}{2} = \bar{u} \cdot 365$ and $\sigma = s \cdot \sqrt{365}$ will be overestimated

Risk Neutral Valuation Relationship and Black-Scholes-Merton Option Pricing Formulae

RNVR and BSM Model

- Assumptions underlying the BSM model
  - Stock price behavior follows the lognormal model.
    - There are no transaction costs or taxes
  - All securities are perfectly divisible
    - It is possible to buy 1/100th or 1/3 of a share
  - There are no dividends on the stock in $[0,T]$
  - There are no riskless arbitrage opportunities
  - Security trading is continuous
  - The risk-free interest rate, r, is constant in $[0,T]$
  - Investors can borrow or lend at r unlimitedly

RNVR and BSM Model

- RNVR states that any derivative can be priced with the general derivative pricing rule as if it and its underlying asset were in the risk neutral world
  - Consequently, the return of the stock price is r, i.e.,

$$\frac{\Delta S}{S} \sim N(r\Delta t, \sigma^2 \Delta t),$$

(Note that $\frac{\Delta S}{S}$ measures the total return on stock, including the capital gains and dividend yield)

  - The lognormal distribution of S_T becomes

$$\ln S_T \sim N(\ln S_0 + (r - \frac{\sigma^2}{2})T, \sigma^2 T)$$

RNVR and BSM Model

- The probability density function for the lognormally distributed S_T is

$$f(S_T) = \frac{1}{S_T\sqrt{2\pi}\sigma\sqrt{T}} e^{-\frac{1}{2}\left(\frac{\ln S_T - (\ln S_0 + \left(r - \frac{\sigma^2}{2}\right)T)}{\sigma\sqrt{T}}\right)^2}$$

- Based on RNVR, the risk-free interest rate should be employed to discount the expected payoff of any derivative
- As a result, European calls can be valued as

$$c = e^{-rT} E[\max(S_T - K, 0)|\text{in the risk neutral world}]$$

$$= e^{-rT} \int_0^\infty \max(S_T - K, 0)\, f(S_T) dS_T,$$

$$= e^{-rT} \int_K^\infty (S_T - K) f(S_T) dS_T$$

RNVR and BSM Model

- European puts can be valued as

$$p = e^{-rT} \int_0^K (K - S_T) f(S_T) dS_T$$

- The BSM option price formulae can be derived by evaluating the above two integrals

$$c = S_0 N(d_1) - K e^{-rT} N(d_2),$$

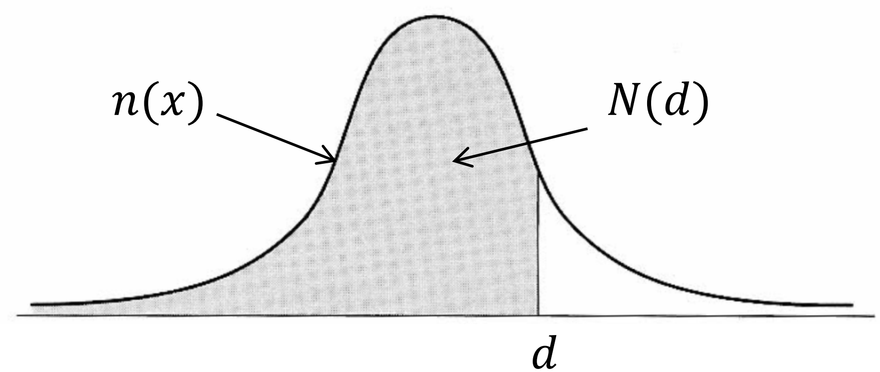
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1),$$

where $d_1 = \frac{\ln(S_0/K) + \left(r + \sigma^2/2\right)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0/K) + \left(r - \sigma^2/2\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

RNVR and BSM Model

- $N(d)$ is the cumulative distribution function for a standardized normal variable
  - It returns the probability that a variable with a standard normal distribution is less than the constant number d
  - By definition, $N(d) = \int_{-\infty}^{d} n(x)dx = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$



  - There is no closed-form solution for the integral to compute $N(d)$, so it needs to be solved with numerical techniques
  - In Excel 2010, "NORM.S.DIST(z,1)" returns $N(z)$ and "NORM.S.DIST(z,0)" returns $n(z)$

RNVR and BSM Model

- Properties of the BSM formulae
  - As S_0 becomes very large, calls (puts) are extremely ITM (OTM)
    - $d_1 \to \infty \Rightarrow N(d_1) \to 1 \Rightarrow N(-d_1) \to 0$
    - $d_2 \to \infty \Rightarrow N(d_2) \to 1 \Rightarrow N(-d_2) \to 0$
    - The call value $c = S_0 N(d_1) - Ke^{-rT} N(d_2) \to S_0 - Ke^{-rT}$
    - The put value $p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \to 0$
  - As S_0 becomes very small, calls (puts) are extremely OTM (ITM)
    - $d_1 \to -\infty \Rightarrow N(d_1) \to 0 \Rightarrow N(-d_1) \to 1$
    - $d_2 \to -\infty \Rightarrow N(d_2) \to 0 \Rightarrow N(-d_2) \to 1$
    - The call value $c = S_0 N(d_1) - Ke^{-rT} N(d_2) \to 0$
    - The put value $p = Ke^{-rT} N(-d_2) - S_0 N(-d_1) \to Ke^{-rT} - S_0$

RNVR and BSM Model

- A full proof of the BSM formulae is beyond the scope of this course!
  - In fact, Black and Scholes derived the option pricing formulae in another way which is similar to the binomial tree model
    - The basic idea is to construct a riskless portfolio by determining proper weights for the option and its underlying asset
    - This is because the option price and the underlying asset price share the same source of uncertainty
    - The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
    - Thus, a partial differential equation (偏微分方程式) is derived, and the solution of it is the BSM option pricing formula

RNVR and BSM Model

- The procedure of the BSM model also leads to the RNVR
  - The no arbitrage argument implies the return of the riskless portfolio should be the risk free interest rate
  - We can obtain that the expected growth rates for both the underlying asset price and the option price are the same to be the risk-free interest rate
  - As a result, an option can be priced as if it and its underlying asset were in the risk neutral world
    - In fact, the general rule to price an option as the PV of its expected payoff in the risk neutral world, i.e.,

$$e^{-rT}E[\text{payoff}(S_T)|\text{in the risk neutral world}],$$

is the unique solution of the partial differential equation considered in the BSM model

Apply Risk Neutral Valuation Relationship to Pricing Forward Contracts

RNVR for Forward Contracts

- Apply the RNVR to deriving the theoretical value of the forward contracts today
  - Consider a long forward contract that matures at time T with the delivery price K
    - The payoff of the contract at maturity is $S_T - K$

Steps 1 and 2: The expected payoff in the risk-neutral world is

$$E[S_T - K|\text{in the risk–neutral world}]$$
$$= E[S_T|\text{in the risk–neutral world}] - K = S_0 e^{rT} - K$$

Step 3: The theoretical value of the forward contract today is

$$f = e^{-rT}(S_0 e^{rT} - K) = S_0 - K e^{-rT}$$

RNVR for Forward Contracts

- We know the value of a long forward contract at t is

$$f_t = (F_t - K)e^{-r(T-t)}$$

- Considering $t = 0$, we have the value of a long forward contract today as follows

$$f_0 = (F_0 - K)e^{-rT},$$

where F_0 is the forward price today

- The theoretical forward price today is $F_0 = S_0 e^{rT}$ and derived with the no arbitrage argument
- Substitute F_0 for $S_0 e^{rT}$, and the theoretical value of the long forward contract today is

$$f_0 = (F_0 - K)e^{-rT} = (S_0 e^{rT} - K)e^{-rT} = S_0 - Ke^{-rT}$$

Implied Volatility

- The only parameter in the BSM pricing formulae that cannot be observed directly is the volatility of the stock price
  - Note that the parameters in the BSM model include S_0, K, r, σ, and T
  - The implied volatility of an option is the volatility for which the BSM option price equals the market price
  - The is a one-to-one correspondence between prices and implied volatilities
  - The bisection method (二分逼近法) to find the implied volatility given $S_0 = 21$, $K = 20$, $r = 0.1$, $T = 0.25$, and $c = 1.9 \Rightarrow$ The implied $\sigma = 24.2\%$
- The implied volatility σ_{imp} is the value that makes the BSM price = market price.

Implied Volatility

- The implied volatility reflect the consensus of traders in the option market on the volatility of the underlying asset price for a future period of time
  - The implied volatility is a forward-looking estimation, which is the expected volatility about a future period time over which the option will exist
  - The historical volatility works if the price behavior of the underlying asset in the immediate future is the same as that in the recent past
- The implied volatility of an option DOES depend on its strike price and time to maturity
- Traders and brokers often quote implied volatilities rather than dollar prices

Implied Volatility

- VIX index
  - The CBOE publishes indices of implied volatility
  - The most popular index, the S&P

Implied Volatility

- VIX index
  - The CBOE publishes indices of implied volatilities
  - The most popular index, the S&P VIX, is an index of the implied volatility of 30-day S&P 500 index options calculated from a wide range of calls and puts
  - The S&P VIX, with a normal range between 15% and 25%, can be interpreted as the expectation of the volatility of the S&P 500 index in the future one month
  - Trading in futures on the VIX started in 2004 and trading in options on the VIX started in 2006
    - Note that the underlying asset of those derivatives is the VIX index, which is not a tradable asset

Effects of Dividend Payments on Option Prices

- European options on dividend-paying stocks are valued by substituting S_0 for $S_0 - D_0$ in the BSM option pricing formulae
  - D_0 is the sum of the PV (discounted at r) of the dividend payments during the life of the option
  - Note that this technique is used to determine the forward price in Ch. 5 and to modify the lower bounds and the put-call parity of options in Ch. 10
  - Note that that on the ex-dividend dates, the stock prices are expected to reduce by the amounts of the dividend payments

Example

- European call option on a stock.
- There are ex-dividend dates in two months and five months.
- The dividend on each ex-dividend date is expected to be \$0.50.
- The current share price is \$40, the exercise price is \$40, the stock price volatility is 30% p.a, the risk-free rate of interest is 9% p.a, and the time to maturity is six months.
- The present value of the dividends is

$$D = 0.5e^{-0.09*\frac{2}{12}} + 0.5e^{-0.09*\frac{5}{12}} = 0.9741$$

Example

- Thus use the Black-Scholes formula with
- $\hat{S} = S_0 - D = 40 - 0.9741 = 39.0259$, $X = 40$, $r = 0.09$, $\sigma = 0.3$, and $T = 0.5$

$$d_1 = \frac{\ln\left(\frac{39.0259}{40}\right) + 0.135 * 0.5}{0.3\sqrt{0.5}} = 0.2017$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = -0.0104$$

- $N(d_1) = 0.5800$, $N(d_2) = 0.4959$
- Call price is $c = 39.0259 * 0.5800 - 40e^{-0.09*0.5} * 0.4959 = 3.67$

Effects of Dividend Payments on Option Prices

- Therefore, the distributions of S_T are identical under the following two situations
  1. The initial stock price is S_0 and there are dividend payments D_t's during the life of the option
  2. The initial stock price is $S_0 - D_0$ and there is no dividend payment during the life of the option

※Since the value of a European option is

$$e^{-rT} E[\text{payoff}(S_T) | \text{in the risk neutral world}]$$

$$= e^{-rT} \int_0^\infty \text{payoff}(S_T) f(S_T) dS_T,$$

it can be expected that the option value should be the same under the above two situations due to the identical probability density function $f(S_T)$

Effects of Dividend Payments on Option Prices

- For American options, it can be priced with only some numerical method like the binomial tree model
  - To take the dividend payments into account, the replacement of S_0 with $S_0 - D_0$ is no more valid
  - This is because in order to make the optimal early exercise decision, we need the correct probability distribution of the stock price at any time point t, which cannot be achieved by the replacement of S_0 with $S_0 - D_0$

Effects of Dividend Payments on Option Prices

- Fischer Black proposed an approximation for the value of an American call based on the BSM model if there are dividend payments during the life of the option
  - The well-known early exercise behavior of American calls
    - An American call on a non-dividend-paying stock should never be exercised early
    - An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date

Effects of Dividend Payments on Option Prices

- Approximate the American call equal to the maximum of two European option prices:
  1. The 1st European option price is for an option maturing at the same time as the American option
  2. The 2nd European option price is for an option maturing just before the final ex-dividend date

  (The strike price, initial stock price, risk-free interest rate, and the volatility are the same for the option under consideration)
- Note this method can generate only an approximation for an American call, but the binomial tree model can generate the exact option values for both American calls or puts