2. Getting started

Hsu, Lih-Hsing

Computer Theory Lab.

2.1 Insertion sort

- **Example:** Sorting problem
  - Input: A sequence of *n* numbers $< a_1, a_2, \ldots, a_n >$
  - Output: A permutation $< a_1', a_2', \ldots, a_n' >$ of the input sequence such that $a_1' \le a_2' \le \ldots \le a_n'$.

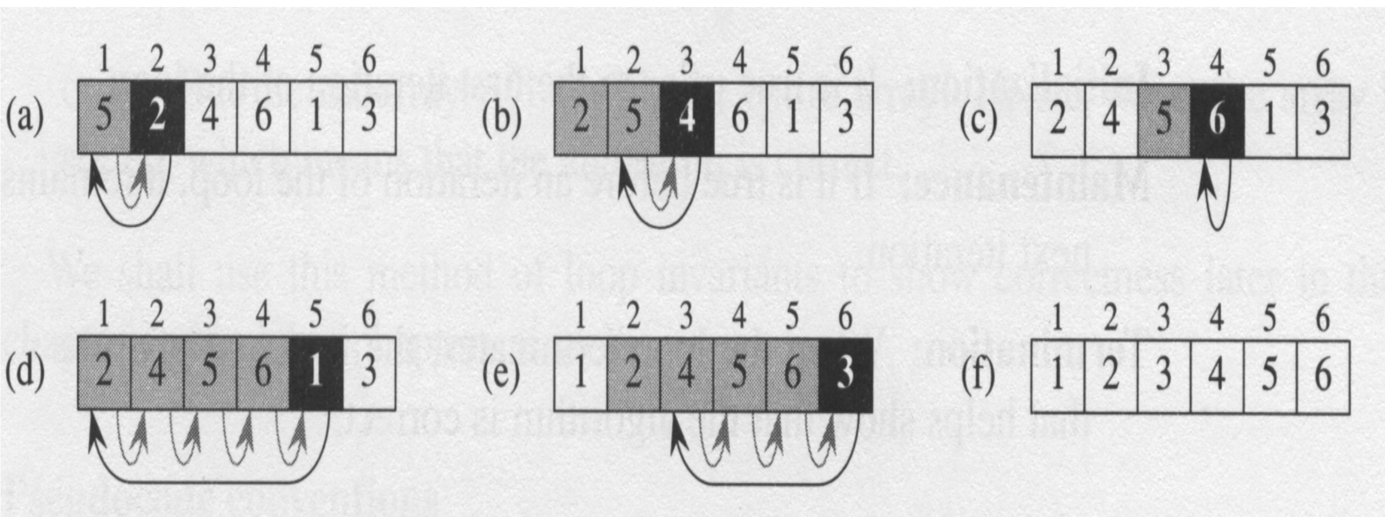
The number that we wish to sort are known as the *keys*.

Pseudocode

Insertion sort

```
Insertion-sort(A)
for j ← 2 to length[A]
  do key ← A[j]
     ∗Insert A[j] into the sorted sequence A[1..j-1]
     i ← j - 1
     while i>0 and A[i]>key
         do A[i+1] ← A[i]
            i ← i - 1
     A[i +1] ← key
```


The operation of Insertion-Sort



- *Sorted in place* :
  - The numbers are rearranged within the array A, with at most a constant number of them sorted outside the array at any time.
- *Loop invariant* :
  - At the start of each iteration of the for loop of line 1-8, the subarray A[1..j-1] consists of the elements originally in A[1..j-1] but in sorted order.

2.2 Analyzing algorithms

Analyzing an algorithm has come to mean predicting the resources that the algorithm requires.

- ***Resources:*** memory, communication, bandwidth, logic gate, **time**.
- **Assumption:** one processor, ***RAM***
- (We shall have occasion to investigate models for parallel computers and digital hardware.)

2.2 Analyzing algorithms

- The best notion for **input size** depends on the problem being studied..
- The **running time** of an algorithm on a particular input is the number of primitive operations or “steps” executed. It is convenient to define the notion of step so that it is as machine-independent as possible

Insertion-sort(A)	cost	times
1 **for** $j \leftarrow 2$ **to** length[A]	c_1	n
2 **do** key $\leftarrow$ A[j]	c_2	$n-1$
3 ∗Insert A[j] into the sorted sequence A[1..j -1]	0	
4 $i \leftarrow j$ -1	c_4	$n-1$
5 **while** i>0 and A[i]>key	c_5	$\sum_{j=2}^{n} t_j$
6 **do** A[i +1] $\leftarrow A$[i]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
7 $i \leftarrow i$ -1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
8 A[i +1] $\leftarrow$ key	c_8	$n-1$

t_j : the number of times the while loop test in line 5 is executed for the value of j.

the running time

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j +$$

$$c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

- $t_j = 1$ for j = 2,3,…,n : Linear function on n

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1)$$
$$= (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$$

the running time

- $t_j = j$ for j = 2,3,…,n : quadratic function on n.

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5\left(\frac{n(n+1)}{2} - 1\right) +$$

$$c_6\left(\frac{n(n-1)}{2}\right) + c_7\left(\frac{n(n-1)}{2}\right) + c_8(n-1)$$

$$= \left(\frac{c_5 + c_6 + c_7}{2}\right)n^2 - \left(c_1 + c_2 + c_4 + \frac{c_5 - c_6 - c_7}{2} + c_8\right)n$$
$$-(c_2 + c_4 + c_5 + c_8)$$

Worst-case and average-case analysis

- Usually, we concentrate on finding only on the ***worst-case running time***
- Reason:
  - It is an upper bound on the running time
  - The worst case occurs fair often
  - The average case is often as bad as the worst case. For example, the insertion sort. Again, quadratic function.

Order of growth

- In some particular cases, we shall be interested in ***average-case***, or ***expect*** running time of an algorithm.

- It is the ***rate of growth***, or ***order of growth***, of the running time that really interests us.

2.3 Designing algorithms

- There are many ways to design algorithms:

- **Incremental approach**: insertion sort
- **Divide-and-conquer**: merge sort
  - recursive:
    - *divide*
    - *conquer*
    - *combine*

Merge(A,p,q,r)

$n_1 \leftarrow q - p + 1$

$n_2 \leftarrow r - q$

create array L[1 .. $n_1 + 1$] and R[1 ..$n_2 + 1$]

for $i \leftarrow 1$ ***to*** n_1

    do L[i] ←A[$p + i - 1$]

for $j \leftarrow 1$ ***to*** n_2

    do R[i] ←A[$q + j$]

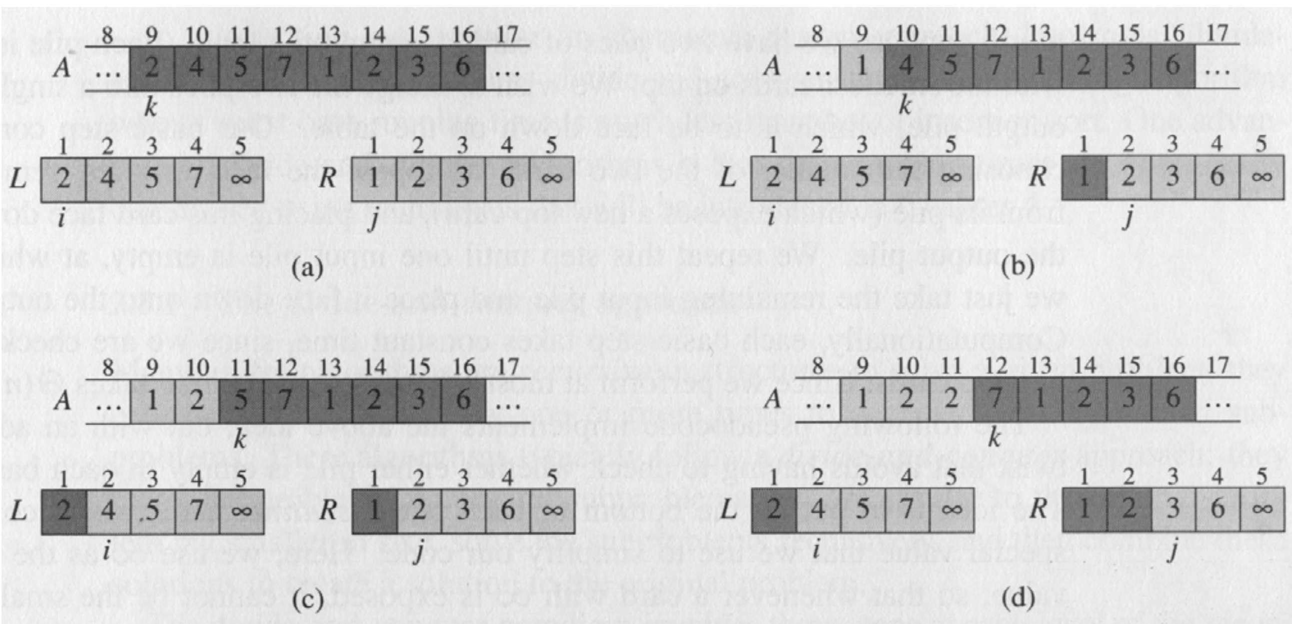
L[$n_1 + 1$] ← ∞

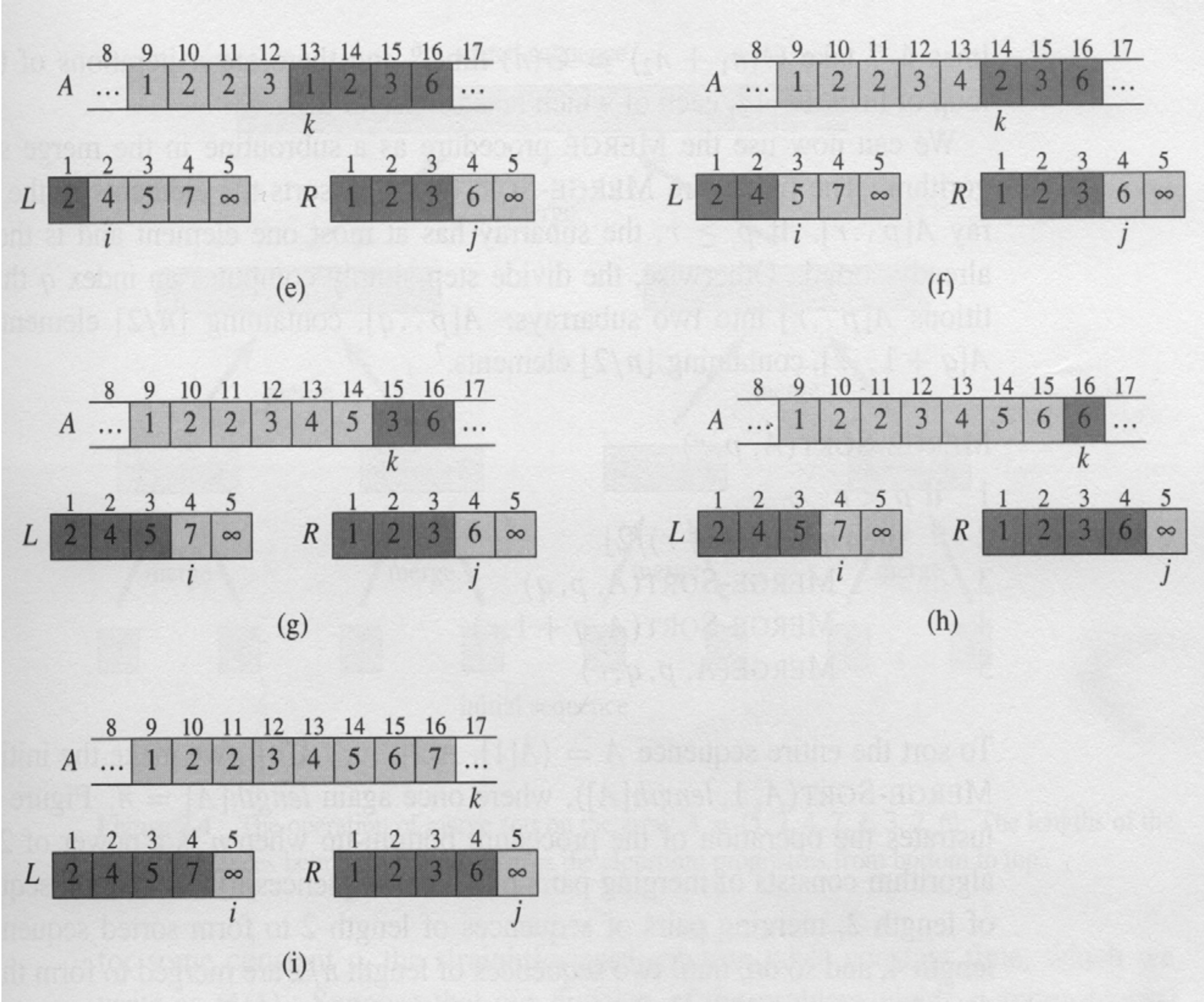
R[$n_2 + 1$] ← ∞

Merge(A,p,q,r)

```
i ← 1
j ← 1
for k ← p to r
   do if L[ i ] ≤ R[ j ]
          then A[ k ] ← L[ i ]
               i ← i + 1
      else A[ k ] ← R[ j ]
               j ← j + 1
```

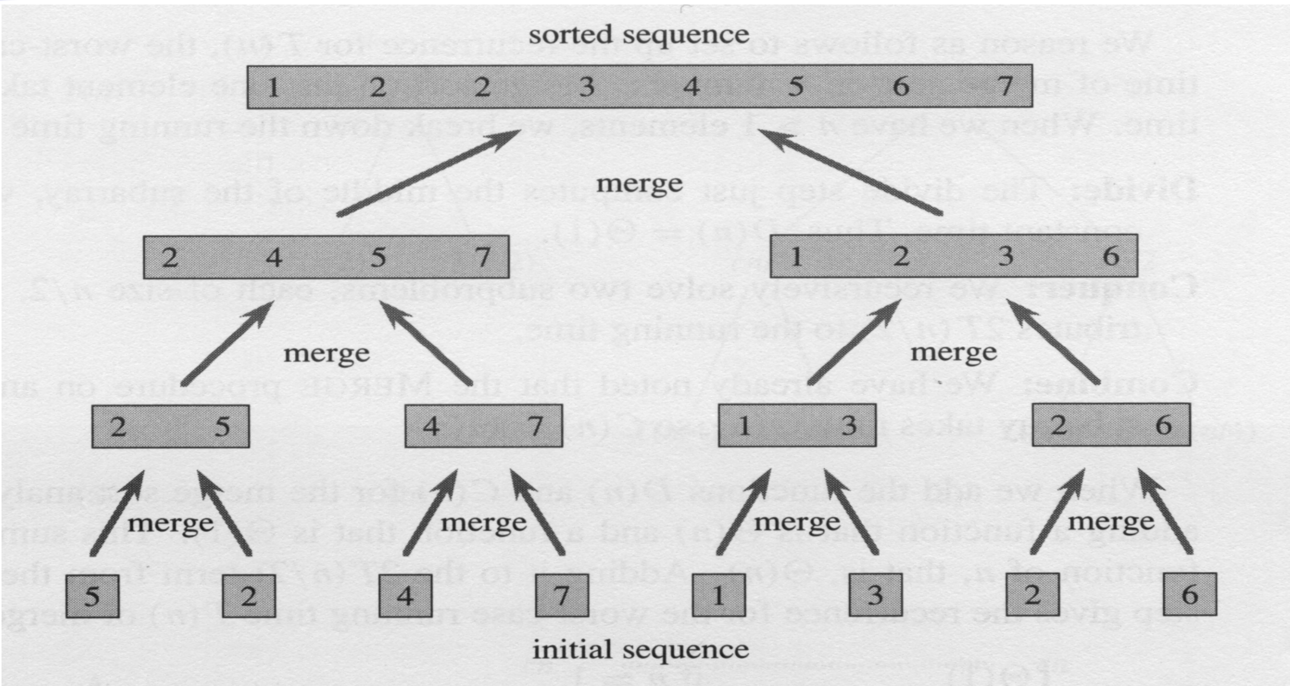

Example of Merge Sort





MERGE-SORT(A,p,r)

```
if p < r
   then q←⌊(p+r)/2⌋
        MERGE-SORT(A,p,q)
        MERGE-SORT(A,q+1,r)
        MERGE(A,p,q,r)
```



Analysis of merge sort

- Analyzing divide-and-conquer algorithms

$$T(n) = \begin{cases} \Theta(1) & \text{if} \quad n \le c \\ aT(n/b) + D(n) + c(n) & \text{otherwise} \end{cases}$$

  See Chapter 4

- Analysis of merge sort

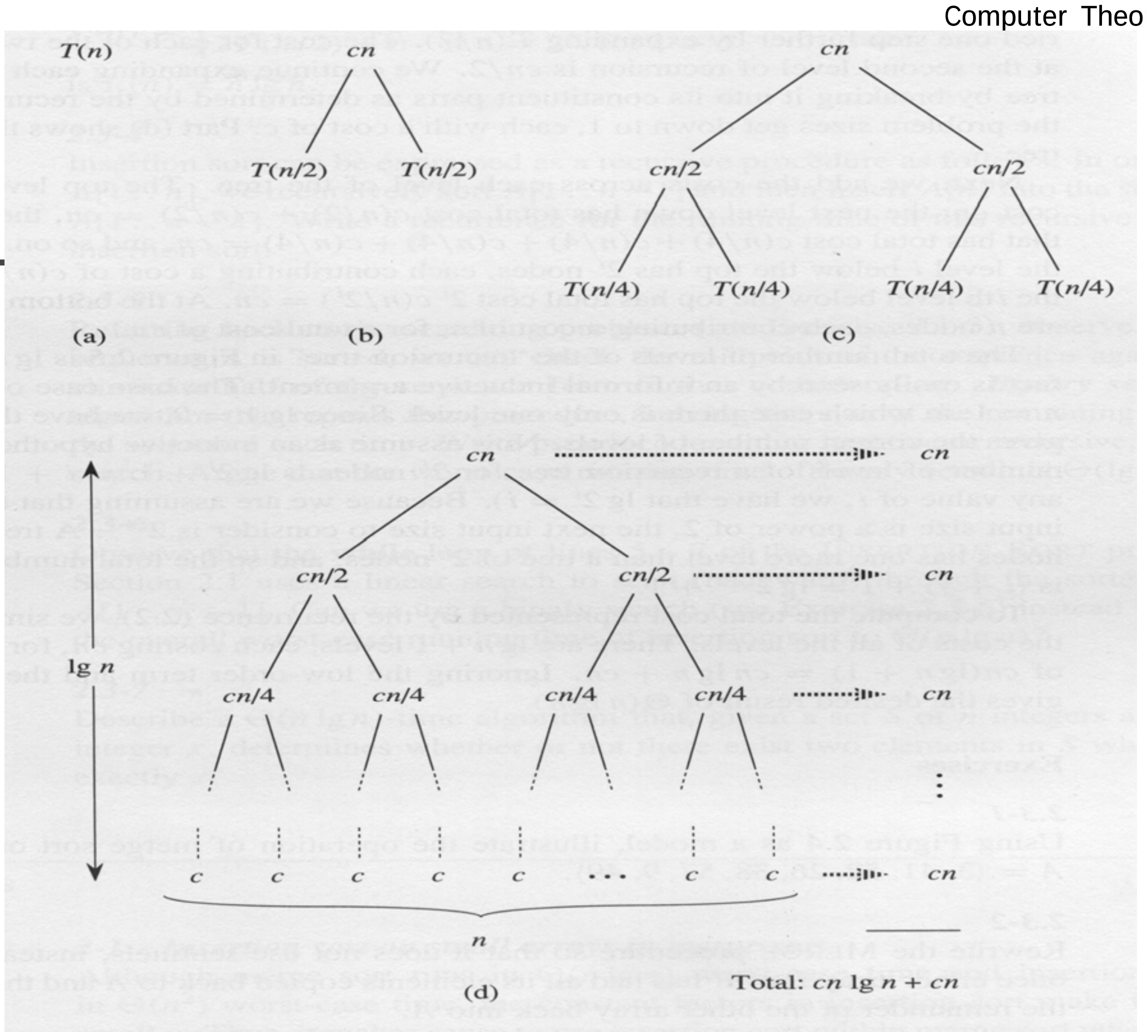
$$T(n) = \begin{cases} \Theta(1) & \text{if} \quad n = 1 \\ 2T(n/2) + \Theta(n) & \text{if} \quad n > 1 \end{cases}$$

$$T(n) = \Theta(n \log n)$$

Analysis of merge sort

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

where the constant *c* represents the time require to solve problems of size 1 as well as the time per array element of the divide and combine steps.



Outperforms insertion sort!