Introduction to Financial Engineering and Algorithms

Lecturer: William W.Y. Hsu

Forwards Contracts

Forward Contract

- A forward contract is an OTC agreement between 2 parties to buy or sell an asset at a certain time in the future for a certain price.
- Most forward contracts lead to delivery of the asset physically or to final settlement in cash.
- There is no daily settlement process unless a collateralization agreement requires it.
- No money changes hands when first negotiated, so the initial value of the contract is zero.

Forward Contract

- The forward price for a contract is the delivery price that would be applicable to the contract if were negotiated today.
  - It is the delivery price that makes the contract worth exactly zero.
- The forward price may be different for contracts of different maturities.
- The forward value is how much the forward contract is worth, but the forward price is a term in the contract to specify the trading price of the underlying asset in the future

Forward Contract

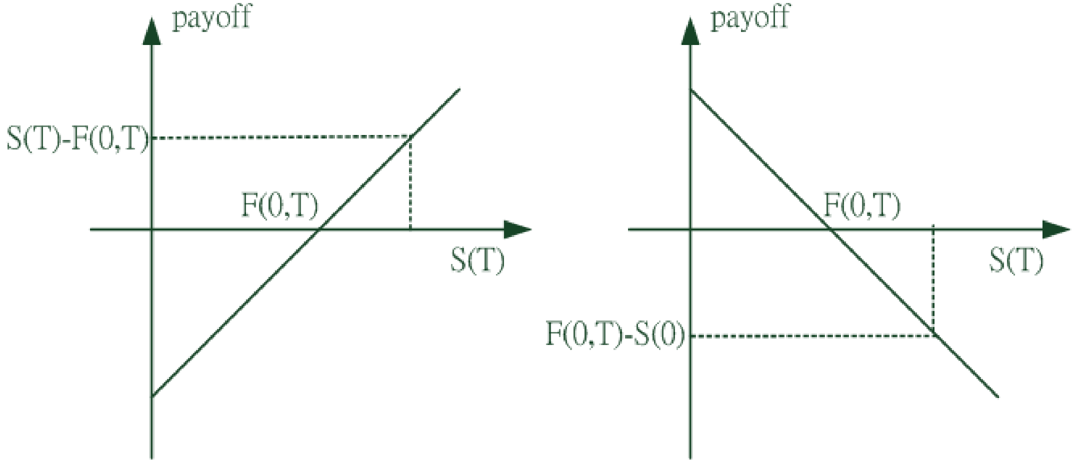
- We denote the market price of the underlying asset to be $S(t)$ at time $t \in [0, T]$.
  - $t = 0$ is the present time, $S(0)$ is the spot price of the underlying, known to everybody.
  - Mathematically, $S(t)$ can be represented as a positive random variable on a probability space Ω.
    $$S(t): \Omega \to (0, \infty)$$
  - The probability space Ω consists of all feasible price movement scenarios $w \in \Omega$.

Forward Contract

  - The probability space and the probability distribution of all $S(t)$ are the main ingredients of a market model.
- Holding a underlying asset may involve in some intermediate cash flow.
  - If $S(t)$ represents a stock, a dividend may be paid to the shareholder.
  - If $S(t)$ is a currency, interest may be paid if deposited in a bank.
  - If $S(t)$ is a commodity, storage fee may apply.
  - This is the cost of carry, the cash flow involved in maintaining a long or short position in the underlying asset.

Forward Contract

- Consider a forward contract exchanged at time 0 with delivery time T and forward price $F(0,T)$.
  - No payment is made by either party at time 0.
  - At delivery, the party with long forward positions will receive (or pay if negative) the amount $S(T) - F(0,T)$ by buying the underlying asset for $F(0,T)$ and selling it at market price for $S(T)$.
  - The payoff for short forward position is $F(0,T) - S(T)$.



Forward Price

- The no arbitrage principle makes it possible to obtain formulae for the forward prices of assets of various kinds.
  - With zero carry cost,

$$F(0,T) = S(0) \cdot \frac{A(T)}{A(0)}$$

- We assume that the future value of the money market account is know at time 0.
  - Constant interest rate with continuous compounding, $A(t) = A(0)e^{rt}$, then

$$F(0,T) = S(0)e^{rT}$$

Forward Price

- For an underlying asset with no cost of carry, if a forward contract is initiated at time $t \leq T$, then the forward price must be

$$F(t,T) = \frac{S(t)}{B(t,T)}$$

Forward Price

- Suppose that $F(t,T) > \frac{S(t)}{B(t,T)}$.
- At time t:
  - Sell $\frac{S(t)}{B(t,T)}$ zero-coupon bonds, receiving the amount $S(t)$.
  - Buy one share for $S(t)$.
  - Take a short forward position, agree to sell one share for $F(t,T)$ at time T.
- At time T:
  - Sell the stock for $F(t,T)$ using the forward contract.
  - Pay the face value \$1 for each unit bond sold.
- This brings risk-free profit of $F(t,T) - \frac{S(t)}{B(t,T)} > 0$.

Forward Price

- Suppose that $F(t,T) < \frac{S(t)}{B(t,T)}$.
- At time t:
  - Short sell one share for $S(t)$.
  - Buy $\frac{S(t)}{B(t,T)}$ zero-coupon bonds.
  - Enter a long forward contract with forward price $F(t,T)$.
- At time T:
  - Receive the face value \$1 for each unit bond held, collecting $\frac{S(t)}{B(t,T)}$.
  - Buy the stock for $F(t,T)$ using forward contract.
  - Close the short position by returning the stock to the owner.
- This brings risk-free profit of $\frac{S(t)}{B(t,T)} - F(t,T) > 0$.

Forward Price

- In a market with short sell restrictions, the inequality $F(t,T) < \frac{S(t)}{B(t,T)}$ does not necessarily lead to arbitrage opportunities.
- For example, short selling may require a security deposit.

Incomes in Forward Contracts

- Can be interest or dividends.
- Cost of carry is a kind of negative incomes, such as storage fee or insurance fee.
- Stocks may pay a dividend value d at an intermediate time t between initiating the forward contract and delivery.
  - At time t, the stock price will drop by the amount of dividend paid.
  - Forward price involves in the present stock price.
  - Formula for the forward price can be modified by subtracting the present value of the dividend.

Dividends in Forward Contract

- The forward price of a stock paying dividend d at time t, where $0 < t < T$ is

$$F(0,T) = [S(0) - B(0,t) \cdot d] \frac{1}{B(0,T)}$$

Dividends in Forward Contract

- Suppose that $F(0,T) > [S(0) - B(0,t) \cdot d]\frac{1}{B(0,T)}$
- At time 0:
  - Enter a short forward contract position with forward price $F(0,T)$ and delivery time T.
  - Borrow $S(0)$ by issuing $\frac{S(0)}{B(0,T)}$ zero coupon bonds.
  - Buy one share of $S(0)$.
  - Enter into $d \cdot \frac{B(0,t)}{B(0,T)}$ long forward contract position on a bond maturing at time T with delivery time t and forward price $\frac{B(0,T)}{B(0,t)}$.

Dividends in Forward Contract

- At time t:
  - Cash the dividend d and invest it in bonds $B(t,T)$, buying $d \cdot \frac{B(0,t)}{B(0,T)}$ bonds at the forward price $\frac{B(0,T)}{B(0,t)}$.
- At time T:
  - Sell the shares for $F(0,T)$.
  - Pay the bond holders $\frac{S(0)}{B(0,T)}$.
  - Receive $d \cdot \frac{B(0,t)}{B(0,T)}$ from the bonds purchased at time t.
  - The final balance will be

$$F(0,T) - \frac{S(0)}{B(0,T)} + d \cdot \frac{B(0,t)}{B(0,T)} > 0$$

  - Contradiction to the no arbitrage principle.

Continuous Dividend Yield

- Dividends are often assumed to be paid continuously at a specified rate, rather than at discrete time instant.
  - Highly diversify portfolio, discrete payments may be scattered all around.
- Suppose that a stock pays dividends continuously at a rate $r_{div} > 0$.
  - Called continuous dividend yield.
  - If we reinvest the dividend into the stock, then an investment in one share held a time 0 will increase to $e^{r_{div}T}$ shares at time T.
  - To have one share at time T, we should begin with $e^{-r_{div}T}$ shares at time 0.

Continuous Dividend Yield

- Forward price for stocks paying continuous dividend is

$$F(0,T) = e^{-r_{div}T} \cdot \frac{1}{B(0,T)}$$

- Suppose $F(0,T) > e^{-r_{div}T} \cdot \frac{1}{B(0,T)}$
- At time 0:
  - Enter a short forward contract position.
  - Borrow $S(0)e^{-r_{div}T}$ to buy $e^{-r_{div}T}$ shares of stock.
- Between time 0 and T
  - Collect the continuous dividend yield and reinvest them into the stock.
  - At time T, you will have 1 share of stock.

Continuous Dividend Yield

- At time T:
  - Sell the share for $F(0,T)$, closing the short forward position.
  - Pay $\frac{S(0)e^{-r_{div}T}}{B(0,T)}$, clearing the loan with interest.
  - The final payoff is $F(0,T) - \frac{S(0)e^{-r_{div}T}}{B(0,T)} > 0$.
  - Violating the no arbitrage principle.

- In general, if the contract is initiated at time $t < T$,

$$F(t,T) = e^{-r_{div}(T-t)} \cdot \frac{1}{B(t,T)}$$

Value of Forward Contract

- Every forward contract has zero value when initiated.
  - The underlying value will change as time goes by.
  - The value of the forward price will change correspondingly.
  - Long forward contract should have the value $S(T) - F(0, T)$ at delivery.
- Suppose a forward contract initiated at time t, $0 < t < T$, has a price $F(t, T)$.
  - If $F(t, T) > F(0, T)$, it is good new for the investor.
  - Enter a new long forward position with the same delivery date.
    - Sell $F(0, T)$ and buy $F(t, T)$.
    - The value at time 0 of $F(t, T)$ can be computed by discounting the value.

Value of Forward Contract

- The value of a long forward contract with forward price $F(0,T)$ at time $0 \le t \le T$ is
$$V(t) = [F(t,T) - F(0,T)]B(t,T)$$
- At time t, the holder of the long forward position may simultaneously enter the short position at no cost.
  - Done with the forward price $F(t,T)$.
- The combined payoff is
$$S(T) - F(0,T) - [S(T) - F(t,T)] = F(t,T) - F(0,T)$$
  - Final value is discounted by $B(t,T)$.

Value of Forward Contract

- For a stock paying no dividend

$$V(t) = \left[\frac{S(t)}{B(t,T)} - \frac{S(0)}{B(0,T)}\right] B(t,T) = S(t) - S(0) \cdot \frac{B(t,T)}{B(0,T)}$$

- If the stock price grows at the same rate as a risk-free investment, then the value of the forward contract is zero.
- If the stock price grows faster than the risk-free investment, the holder of the long forward position will gain money.

Contracts with Different Delivery Price

- If a forward contract has a delivery price X instead of $F(0,T)$, the value of this contract at time t is
$$V_X(t) = [F(t,T) - X)]B(t,T)$$
- Such a contract may have none-zero value initially.
- For stocks paying no dividends, the formula is
$$V_X(0) = [F(0,T) - X]B(0,T) = S(0) - XB(0,T)$$

Futures Contracts

Futures

- One of the two parties to a forward contract will be losing money.
  - There is always some risk of default by the party suffering a loss.
  - Futures contracts are design to eliminate such risk.
- Futures contract involves an underlying asset $S(t)$ and a specified time of delivery T.
  - Market dictates a futures price $f(t, T)$, $t \leq T$.
  - These prices are unknown except for $t = 0$.

Futures

- It cost nothing to initiate a futures contract.
  - Just like initiating a forwards contract.
  - The difference lies in the cash flow during the lifetime of the contract.
- A long forward contract involve in only a single payment $S(T) - F(0, T)$ at delivery.
- A futures contract involves a random cash flow, known as marketing to market.
  - Assume prescribe time instants $t_1 < t_2 < \cdots < t_N = T$.
  - Can be call trading days.
  - Holder of a long futures position will receive at time $t_n, n = 1, \ldots, N$
    $$f(t_n, T) - f(t_{n-1}, T)$$
  - Positive -> receive money, Negative -> pay money.

Futures

- Available on a wide range of underlying assets.
- Traded on exchanges.
- Standardized contracts defined by exchanges.
  - What can be delivered (quantity and quality of the underlying asset or some alternatives).
  - Choice of the contract size.
    - Too large $\Rightarrow$ investors wish to hedge or speculate relatively small position will be unable to use this contract.
    - Too small $\Rightarrow$ trading may be expensive since there is a transaction cost associated with each contract traded.
    - Decide the appropriate contract size to maximize the trading volume.

Futures

- Quality of the underlying asset.
  - For commodity underlying assets, usually a range of different grades of commodities can be delivered, but the price received by the short position party is according to the grade chosen by the short position party
  - For the corn futures on CBOT, the standard grade is “#2 Yellow,” but “#1 Yellow” (“# 3 Yellow”) is also deliverable for 1.5 cents per bushel higher (lower) than “#2 Yellow”.
  - For financial underlying assets, it is generally unambiguous about their quality.
  - The delivery alternatives occur in the futures on Treasury bonds and notes.

Futures

- When it can be delivered
  - For example, corn futures on CBOT have delivery months of March, May, July, September, and December.
  - Last trading date: the business day prior to the 15th calendar day of the contract month.
  - Last delivery date: second business day following the last trading day of the delivery month.
- Where it can be delivered
  - Several places are designated to deliver the underlying asset.
  - These places are usually the warehouses or largest trading markets of the underlying asset.

Futures

- Price and position limits.
  - The daily price limits prevent unreasonably large price movements due to excess speculation.
    - For corn futures on CBOT, the price limit is $0.40 per bushel expandable to $0.60 when the market closes at limit up or limit down.
  - Position limits on the maximum number of contracts that a speculator can hold: to prevent the market from undue influence because of the trading behavior of the speculator.

Futures

- The futures price at deliver is $f(T,T) = S(T)$.
- For each time $t_n \leq T$, $n = 1, \ldots, N$ the futures price is $f(t_n, T)$.
    - It costs nothing to enter a futures contract at any time.

Futures and Forwards

Forward	Futures
Private contract between two parties (traded in OTC markets)	Traded on an exchange
Not standardized	Standardized
Usually one specified delivery date	Range of delivery dates, e.g., the week following the maturity date
Settled at the end of contract	Settled daily
Delivery or final settlement usual	Usually closed out prior to maturity
Some credit risk	Virtually no credit risk

Pricing Futures Contracts

- If the interest rate is constant, then $f(0,T) = F(0,T)$.

- Assume only one intermediate time instant t, $0 < t < T$. And $f(0,T) > F(0,T)$.
- At time 0:
  - Enter a long forward position. (Free!)
  - Open a fraction $e^{-r(T-t)}$ of a short forward position. (Free!)
- At time t:
  - Pay (receive) the amount $e^{-r(T-t)}[f(t,T) - f(0,T)]$ as a result of marketing to market.
  - Borrow (invest) $e^{-r(T-t)}[f(t,T) - f(0,T)]$ risk-free.
  - Increase the short futures position to 1 contract. (Free!)

Pricing Futures Contracts

- At time T:
  - Close the risk-free investment by paying $f(t, T) - f(0, T)$.
  - Close the short futures position by paying $S(T) - f(t, T)$. ($S(T) = f(T, T)$).
  - Close the long forward position by receiving $S(T) - F(0, T)$.
- The final balance is $f(0, T) - F(0, T) > 0$.
- Contradiction to the no arbitrage principle.
- The argument also applies to $f(t, T) = F(t, T)$.
- If the interest rate changes unpredictably, the above formula may not be true.

Pricing Futures Contracts

- The argument also applies to $f(t,T) = F(t,T)$.
- If the interest rate changes unpredictably, the above formula may not be true.
  - If interest rate movement is known, the formula can be modified and remain valid.
- With constant interest rate r, the simple structure of futures price is

$$f(t,T) = S(t)e^{r(T-t)}$$

  - Underlying $S(t)$ does not pay dividends.
  - Futures price is random due to randomness of $S(t)$.
  - If futures price depart from this formula, it reflects the markets view of future interest rate changes.

Example

- Consider 3 scenarios with the same stock price at maturity.
- Let $T = \frac{2}{365}$ be 2 days from now with marketing-to-market days $t_1 = \frac{1}{365}$ and $t_2 = T$.

- Scenario 1: Stock price goes up at a rate r.
  - $S(t) = S(0)e^{rt}$.
  - $f(t, T) = S(0)e^{rt}e^{r(T-t)} = S(0)e^{rT}$, for all t.
  - Marketing-to-market payments are 0.

Example

- Scenario 2: The price goes up more than scenario 1, $S(t_1) > S(0)e^{rt_1}$.
  - $f(t_1, T) = S(t_1)e^{r(T-t_1)} > f(0, T) = S(0)e^{rT}$.
    - Positive cash flow to the holder of long futures position.
    - $f(t_1, T) - f(0,0)$.
  - $S(T) = S(0)e^{rT} = f(T, T)$.
    - Negative cash flow on day two.
  - $f(T, T) - f(t1, T) = S(0)e^{rT} - S(t_1)e^{r(T-t_1)} = -\big(f(t_1, T) - f(0,0)\big)$.
    - Offsetting the payment received on day 1.
- Scenario 3: The price goes down below $S(0)e^{rt_1}$.
  - Negative payment on t_1 and positive payment on T.

Example

- Timing of the balancing payments is crucial.
  - The final cash flow is the same.
  - However, time value of money in scenario 2 is better than scenario 3.
  - The cash can be used to reinvest between time t_1 and T.

- An investor predicts stock price increase over the risk-free rate.
  - Enters a long futures position for advantage.
- An investor predicts stock price decrease over the risk-free rate.
  - Enters a short futures position for advantage.

Another Example

- Suppose S_0 = \$900, an income of \$40 occurs at 4 months, and 4-month and 9-month rates are 3% and 4% per annum. If the 9-month futures price is \$910 (or \$870), is there any arbitrage opportunity?
  - The PV of the income at 4 months is $\$40e^{-0.03\cdot(4/12)} = \$39.6$
  - The theoretical futures price is
$$F_0 = (\$900 - \$39.6)e^{0.04\cdot(9/12)} = \$886.6$$

- As long as the futures price deviates from this theoretical price, there is arbitrage opportunity.

Another Example

- For $F_0 = \$910$, which is overvalued than its theoretical value
  - At $t = 0$
    - Borrow \$900: \$39.6 for 4 months and \$860.4 for 9 months.
    - Buy one unit of asset at S_0 = \$900.
    - Enter into a short position of the 9-month futures ($F_0 = \$910$).
  - At $t = 4$ months
    - Receive $\$39.6e^{0.03\cdot(4/12)}$ =\$40 of income from the asset.
    - Use this \$40 to repay the first loan.
  - At $t = 9$ months
    - Sell the asset through the futures for \$910.
    - Use $\$860.4e^{0.04\cdot(9/12)}$ =\$886.6 to repay the second loan.
    - Profit realized = \$910 – \$886.6 = \$23.4.

Another Example

- For $F_0 = \$870$, which is undervalued than its theoretical value
  - At $t = 0$
    - Short sell one unit of asset at S_0 = \$900.
    - Invest \$39.6 for 4 months and \$860.4 for 9 months.
    - Enter into a long position of the 9-month futures ($F_0 = \$870$).
  - At $t = 4$ months
    - Receive $\$39.6e^{0.03\cdot(4/12)}$ =\$40 from the 4-month investment.
    - Use this \$40 to pay the lender of the asset.
  - At $t = 9$ months
    - Receive $\$860.4e^{0.04\cdot(9/12)}$ =\$886.6 from 9-month investment.
    - Buy the asset through the futures for \$870.
    - Return the asset to the lender.
    - Profit realized = \$886.6 – \$870 = \$16.6.

Margins

- Ensures that the obligations involved in a futures position are fulfilled, practical regulations are enforced.
  - Each investor entering in to a futures contract has to pay a deposit, called the initial margin.
  - Kept by the clearing house as a collateral.
  - For a long futures position:
    - $f(t_n, T) - f(t_{n-1}, T)$ is added (subtracted) to the deposit if positive (negative).
  - For a short futures position:
    - $f(t_n, T) - f(t_{n-1}, T)$ is subtracted (added) to the deposit if negative (positive).

Margins

- Futures contracts are settled daily
  - Coupled with margin mechanism (保證金機制) to reduce the default probability.
- A margin is cash or marketable securities deposited by an investor with his broker.
- The balance in the margin account is adjusted to reflect the investor's gains or losses.
  - This practice is referred to as daily settlement (每日結算) or marking to market (按市價計值).
  - This mechanism minimizes the possibility of a loss through a default on a contract.

Margins

- Excess build above the initial margin can be withdrawn by the investor.
    - Can be used for reinvestment.
- If the deposit drops below the maintenance margin:
    - Clearing house will issue a margin call.
    - The investor to make a payment and restore the deposit to the level of the initial margin.
- Futures position can be closed anytime.
    - Deposit will be returned to the investor.
- Futures position is closed immediately if the investor fails to respond to a margin call.
- Using margins eliminates the risk of default.

Margins

- An investor takes a long position in two December gold futures contracts on June 5.
  - Contract size is 100 oz.
  - Futures price is $900/oz.
  - Initial margin (初始保證金) is $2,000/contract .
    - To satisfy the initial margin requirements (but not the subsequent margin calls), an investor can sometimes deposit securities with the broker
    - Treasury bills (stock shares) is accepted as the equivalent cash deposit amount at 90% of their face value (50% of their market value).
  - Maintenance margin (維持保證金) is $1,500/contract.
  - Note that margin requirements are the same on short positions as they are on long positions.

Example of Margin Mechanism

Day	Futures Price (US$)	Daily Gain (Loss) (US$)	Cumulative Gain (Loss) (US$)	Margin Account Balance (US$)		Margin Call (US$)		
	900.00			4,000				
5-Jun	897.00	(600)	(600)	3,400		0		
⋮	⋮	⋮	⋮	⋮		⋮		
13-Jun	893.30	(420)	(1,340)	2,660	+	1,340	=	4,000
⋮	⋮	⋮	⋮	⋮		⋮	<	3,000
19-Jun	887.00	(1,140)	(2,600)	2,740	+	1,260	=	4,000
⋮	⋮	⋮	⋮	⋮		⋮		
26-Jun	892.30	260	(1,540)	5,060		0		

- A decline (rise) in futures price implies a loss (profit) for traders with a long position of futures.
- The decrease amount, e.g., (600) on 5-Jun, is transferred to the margin account of an investor with a short position.
- If the balance falls below the maintenance margin, the investor receives a margin call and is expected to top up the margin account to the initial margin level the next day. The extra funds deposited are known as variation margin.

Example

t-days	$f(t,T)$	Cash flow	Margin 1	Payment	Margin 2
0	140	Opening	0	-14	14
1	138	-2	12	0	12
2	130	-8	4	-9	13
3	140	+10	23	+9	14
4	150	+10	24	+9	15
		Closing	15	+15	0
			Total	10	

- Initial margin is 10% of $f(t,T)$.
- Maintenance margin is 5% of $f(t,T)$.

Margins

- A important feature of the futures market is liquidity.
- Made possible by the clearing house.
  - Standardized delivery arrangements for commodities such as gold and timber.
  - Act as an intermediary, matching the total of a large number of short and long futures position of various size.
  - Maintains the margin deposit for each investor to eliminate default risk.

- Forward contracts: negotiated directly between two parties.

Margins

- Closing out (平倉) a futures position involves entering into an offsetting trade.
  - Most futures contracts are closed out before maturity.
  - If an investor does not provide the variation margin, his position is forced to be closed out by the broker.
    - For an investor who does not meet margin calls in the scenario of 13-Jun in the previous example, two short-position gold futures with the futures price \$893.3 are purchased by the broker on behalf of the investor.
    - Two long-position futures (@ \$900/oz.) and two short- position futures (@ \$893.3/oz.) $\Rightarrow -\$6.7\times100\times2 = -\$1{,}340$.
    - The deduction in margin accounts already reflects this loss amount, so these two positions are cancelled automatically at maturity.

Margins

- Clearinghouse (結算所) and clearing margin
  - A clearinghouse acts as an intermediary in futures transactions, with the main task to calculate the net position of each of its members.
  - The broker is required to maintain margin with a clearinghouse member and the clearinghouse member is required to maintain a margin account with the clearinghouse.
  - The latter is known as a clearing margin.
    - There is an original margin but no maintenance margin.
    - Every day the account balance for each contract must be maintained at an amount equal to the original margin times the number of outstanding contracts.

Physical Delivery vs. Cash Settlement

- If a futures contract is not closed out before maturity, it is usually settled by delivering the assets underlying the contract.
  - When there are alternatives about what is delivered, when it is delivered, and where it is delivered, the party with the short position can choose among these alternatives.
- A few contracts (for example, those on stock indices and Eurodollars) are settled in cash.
  - That is, to pay directly the profit or loss of the futures contract, which is the difference between the spot and the delivery price at maturity.

Types of Orders

- Market order:
  - A request that a trade will be executed immediately at the best price available in the market.
- Limit order:
  - Executed only at the specified price or at the prices more favorable to the investor.
- Stop order:
  - Executed at the best available price once the specified price level is triggered.
    - Specifically, a stop-loss (stop-buy) order becomes a market sell (buy) order as soon as the specified price has been hit from above (from below).

Types of Orders

- Stop-limit order:
  - A combination of a stop and a limit orders. If the specified price in the stop order is triggered, it becomes a limit order with another specified price.
- Market-if-touched order:
  - Becomes the market sell order if the specified price is triggered from below (to realized the sufficiently large gains).
- Discretionary order (market-not-held order):
  - Traded as a market order except that execution may be delayed at the broker's discretion in an attempt to get a better price.

Investing Futures

- An investor believing the stock price going up can:
  - Invest the stock directly.
  - Enter a long futures position of the stock.
- Investing the futures only require a margin, which is a percentage of the stock price.
  - Less than buying the stock itself.
  - If prediction is correct, since the money engages is smaller, the return is more impressive!

Example

- Let $S(0) = \$100$, $r = 6\%$, $T = 0.5$ and marketing-to-market be restricted to one instant $t_1 = 0.25$.
- Let initial deposit be 20% of the price. $(20\% \times S(0) = \$20)$

- The investor can open 5 long futures position with \$100.
  - If $S(t_1) = 108$ then $f(t_1, T) \cong 106.39$.
  - $f(0, T) \cong 103.05$.
  - Market-to-marketing gain is $f(t_1, T) - f(0, T) = 3.34$.
  - 5 positions = $5 \times 3.34 = \$16.73$.
  - More than double gain from a direct investment in stock.

Hedging with Futures

- Entering a forward contract can reduce exposure to stock price variation.
  - But forward contract may not be readily available.
  - The default risk in forward contract!
- Use futures market.
  - Well-developed.
  - Liquid.
  - Protected from risk of default.

Example

- Let $S(0) = \$100$ and let the risk-free arte be constant at $r = 8\%$.
- Marketing-to-market takes place once a month.
- Suppose we want to sell the stock after 3 month.

- Lets enter 1 short futures contract on the stock with same maturity.

Example

$t - months$	$S(t)$	$f\left(t, \frac{1}{4}\right)$	Marketing to Market	Interest
0	100	102.02		
1	102	103.37	-1.35	-0.02
2	101	101.67	+1.69	+0.01
3	105	105.00	-3.32	0.00
		Total	**-2.98**	**-0.01**

- In this scenario, we sell the stock for $105.
- Marketing to market brings losses, reducing the sum to $105-2.98-0.01=$102.01.
- If marketing to market payments were not invested at risk-free rate, the realized sum will be $102.02.
  - Equal to the futures price $f(0, \frac{1}{4})$.

Example

$t - months$	$S(t)$	$f\left(t, \frac{1}{4}\right)$	Marketing to Market	Interest
0	100	102.02		
1	98	99.31	+2.70	+0.02
2	97	97.65	+1.67	+0.01
3	92	92.00	+5.65	0.00
		Total	**+10.02**	**+0.03**

- In this scenario, we sell the stock for $92.
- Marketing to market bring gain, increasing the sum to $92+10.02+0.03=$102.05.
- If marketing to market payments were not invested at risk-free rate, the realized sum will be $102.02.
  - Equal to the futures price $f(0, \frac{1}{4})$.

Discussion

- We have ignore the presence of initial margin.
- Limitations and standardization of futures contracts.
  - Matching the terms of the contract to our need.
  - Exercise dates for futures are typically certain fixed days.
    - Third Friday in March, June, September, and December.
  - If we want to close our investment at the end of April, we need to hedge with futures contracts with delivery date beyond the end of April.

Example

- Suppose we wish to sell stock after 2 month.
- Hedge using futures with delivery in 3 month.
  - Assume 2 month delivery is not available.

$t - months$	$S(t)$	$f\left(t, \frac{1}{4}\right)$	Marketing to Market	Interest
0	100	102.02		
1	102	103.37	-1.35	-0.02
2	101	101.67	+1.69	+0.01
		Total	**+0.34**	**-0.01**

- We sell the stock for \$101, together with marketing to market, we gain \$101+\$0.34-\$0.01 = \$101.33.

Example

$t - months$	$S(t)$	$f\left(t, \frac{1}{4}\right)$	Marketing to Market	Interest
0	100	102.02		
1	98	99.31	+2.70	+0.02
2	97	97.65	+1.67	+0.00
		Total	**+4.37**	**+0.02**

- We sell the stock for \$97, together with marketing to market, we gain \$97+\$4.37+\$0.02 = \$101.39.
- The futures price $f\left(0, \frac{2}{12}\right) \cong 101.34$. (Compounded at risk-free rate).
  - We almost hit our target!

Basis

- The difference between the spot and futures prices is call the basis.

$$b(t,T) = S(t) - f(t,T)$$

- The basis converges to 0 as $t \to T$.
  - $f(T,T) = S(T)$.
- In market with constant interest rates it is given explicitly by

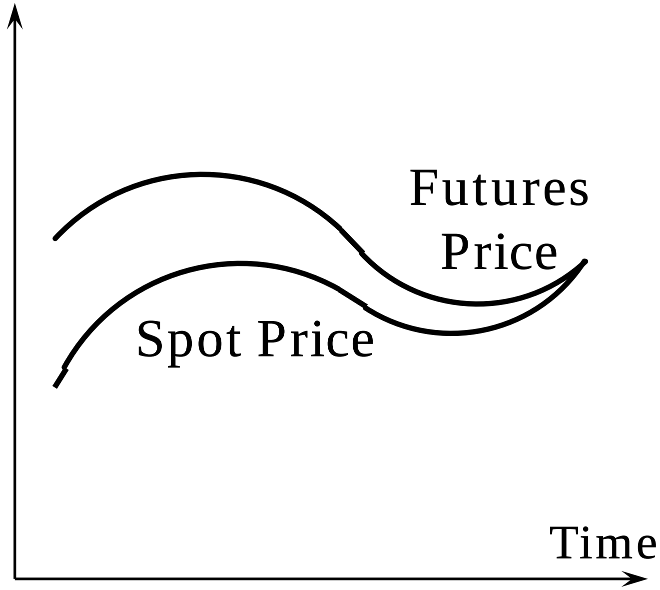
$$b(t,T) = S(t)(1 - e^{r(T-t)})$$

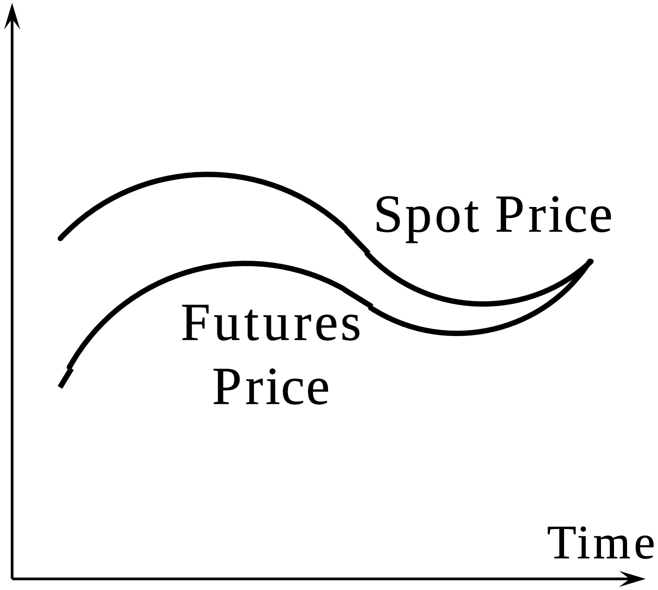
- If the asset pays dividends at a rate $r_{div} > r$

$$b(t,T) = S(t)(1 - e^{(r-r_{div})(T-t)})$$

Convergence of Futures to Spot

- Hedge initiated at time t_1 and closed out at time t_2.
- As long as the basis is not zero, no matter positive or negative, there is a basis risk.





Hedging with Futures

- A long futures hedge is appropriate when you know you need to purchase an asset in the future and want to lock in the price.
  - A manufacturer needs to buy 100,000 pounds of copper after one month ⇒ take a long position of 4 contracts (each can deliver 25,000 pounds of copper after one month) on NYMEX.

Copper price ($/pound)	3	3.2	3.4	3.6	3.8	4
Cost for buying 100,000 pounds in the market	-300,000	-320,000	-340,000	-360,000	-380,000	-400,000
Payoff from futures (F = $3.75/pound)	-75,000	-55,000	-35,000	-15,000	5,000	25,000
Net cost	-375,000	-375,000	-375,000	-375,000	-375,000	-375,000

Hedging with Futures

- A short futures hedge is appropriate when you know you will sell an asset in the future and want to lock in the price.
  - An oil producer will sell 10,000 barrels of crude oil after two months $\Rightarrow$ take a short position of 10 contracts (each can deliver 1,000 barrels of crude oil after two months) on NYMEX.

Oil price ($/bbl.)	80	90	100	110	120
Income for selling 10,000 bbl. in the market	800,000	900,000	1,000,000	1,100,000	1,200,000
Payoff from futures (F = $100/bbl.)	200,000	100,000	0	-100,000	-200,000
Net income	1,000,000	1,000,000	1,000,000	1,000,000	1,000,000

Basis Risk

- Basis risk could be due to a mismatch between the expiration date of the futures and the actual trading date of the asset
  - Thus, the spot and futures prices do not converge on the actual trading date
  - Basis risk arises because of the uncertainty about the difference between the spot and futures prices when the hedge is closed out on the actual trading date
  - For this type of basis risk, the basis is defined as the difference between the prices of spot and futures.

Basis Risk

- Basis risk also arises if the asset to be hedged is different from the asset underlying futures, e.g., natural gas price vs. crude oil futures.
  - This type of basis risk is the uncertainty of the difference between the price change of natural gas and the price change of crude oil.

Basis Risk

- Two criteria for choosing contracts to minimize the basis risk
  - Choose a delivery date that is as close as possible to, but later than, the end of the life of the hedge.
    - The basis risk increases with the difference between the actual trading date and the delivery date.
    - If the delivery date is earlier than the hedging expiration date, the extreme price movement in the unhedged period could result in a large loss.
  - When there is no futures contract on the asset to be hedged, choose the contract whose futures price is most highly correlated with the asset price.

Rolling The Hedge Forward

- Sometimes the expiration date of the hedge is later than the delivery dates of all available futures contracts.
  - We can use a series of futures contracts to increase the life of a hedge.
  - Each time when we switch from 1 futures contract to another, a basis risk is incurred.
- This method is called rolling the hedge forward.

Rolling The Hedge Forward

- In April 2010 a company realizes that it will have 100,000 bbl. of oil to sell in June 2011.
  - Suppose that only the first six delivery months have sufficient liquidity to meet the company’s need.

1. Short 100 Oct. 2010 futures contracts now .
2. Roll the hedge into the Mar. 2011 futures contracts in Sept. 2010.
3. Roll the hedge into the Aug. 2011 futures contracts in Feb. 2011.

Rolling The Hedge Forward

Date	Apr. 2010	Sept. 2010	Feb. 2011	June 2011
Oct. 2010 futures price	68.20	67.40		
Mar. 2011 futures price		67.00	66.50	
July 2011 futures price			66.30	65.90
Spot price	69.00			66.00

- One possible scenario (for demonstration).
- The payoff from rolling the short positions of futures is
- (68.20 – 67.40) + (67.00 – 66.50) + (66.30 – 65.90) = 1.70.
- The selling price in June 2011 (66.00) plus the profit from futures (1.70) equals 67.70, which is lower than the original futures price expired in Oct. 2010 (68.20).

Rolling The Hedge Forward

- Basis risk:
  - In Sept. 2010, the futures price for Oct. 2010 (67.40) is different from the futures prices for Mar. 2011(67.00).
  - In Feb. 2011, the futures price for Mar. 2011 (66.50) is different from the futures prices for July 2011(66.30).
  - In June 2011, the futures price for July 2011 (65.90) is different from the spot price (66.00).
- The magnitudes of those differences are uncertain, so they are basis risks.
  - The total payoff from the basis risk in this scenario (67.00 – 67.40) + (66.30 – 66.50) + (66.00 – 65.90) = –0.5, which reflects the difference between the original futures price (68.20) and the final payoff when the rolling hedge strategy is considered (67.7).

Hedging with Futures

- Suppose that we wish to sell an asset at time $t < T$.
- To protect us against a decrease in the asset price, at time 0 we can short a futures contract with futures price $f(0, T)$.
  - At time t, we receive $S(t)$ from selling the asset.
  - We also gain the cash flow $f(0, T) - f(t, T)$.
    - Marketing to market in futures market.
    - Neglecting any other intermediate cash flow.
  - We gain $f(0, T) + S(t) - f(t, T) = f(0, T) + b(t, T)$.
- The price is known at time 0, $f(0, T)$.
  - The new risk lies in the unknown level of the basis.
  - This uncertainty is mainly concerned with unknown future interest rate.

Hedge Ratio

- Ultimate goal of a hedger is to minimize risk.
  - Hedging using the optimal hedge ratio N.
  - N not necessarily equal to the number of shares of the underlying asset held.
- Compute the risk by measuring the variance of the basis

$$b_N(t,T) = S(t) - N \cdot f(t,T)$$

$$\mathrm{Var}\big(b_N(t,T)\big) = \sigma_{S(t)}^2 + N^2\sigma_{f(t,T)}^2 - 2N\sigma_{f(t,T)}\sigma_{S(t)f(t,T)}$$

  - $\sigma_{S(T)f(t,T)}$ is the correlation between the spot and futures price.
  - $\sigma_{S(T)}$, $\sigma_{f(t,T)}$ are the standard deviations.
- Solving for minimum (quadratic function), exists at
- $N = \sigma_{S(t)f(t,T)}\dfrac{\sigma_{S(t)}}{\sigma_{f(t,T)}}$, the optimal hedge ratio.

Hedge Ratio

- Optimal number of futures contracts:

$$N_Q^* = N \cdot \frac{Q_A}{Q_F}$$

Q_A is the size of position being hedged (units of the asset to be hedged)

Q_F is the size of one futures contact (units of the asset underlying futures)

N_Q^* is the optimal hedge ratio for one unit of the asset to be hedged

Hedge Ratio

- Alternative way to determine the optimal number of futures contracts.
  - Instead of comparing the units of assets to be hedged and for hedging, the values of assets to be hedged and for hedging are used alternatively to calculate the optimal number of contracts

$$N_V^* = N \cdot \frac{V_A}{V_F},$$

where V_A and V_F are the dollar values of the position to be hedged and one futures contract.

Cross Hedge and Optimal Hedge Ratio

- Cross hedge example:
  - An airline that concerns about the future price of jet fuel.
  - Since the jet fuel futures are not actively traded, it might choose to use heating oil futures contracts to hedge its exposure.
- When the asset underlying the futures is the same as the asset to be hedged, it is natural to use a hedge ratio of 1.0.
- When the cross hedge is used, an optimal hedge ratio to minimize the variance of the hedged and hedging positions can be derived.

Example

- An airline company uses the heating oil futures (F) to hedge the price risk of the jet fuel (S).
  - $\sigma_S = 0.0263,\ \sigma_F = 0.0313,\ \rho_{SF} = 0.928$

$$\Rightarrow N = \rho_{SF}\frac{\sigma_S}{\sigma_F} = 0.928 \times \frac{0.0263}{0.0313} = 0.778$$

  - Each heating oil contract traded on NYMEX is for 42,000 gallons of heating oil and the airline has an exposure to the price of 2 million gallons of jet fuel.

$$\Rightarrow N_Q^* = N \cdot \frac{Q_A}{Q_F} = 0.778 \times \frac{2{,}000{,}000}{42{,}000} = 37.03$$

※ About 37 heating oil futures is needed for hedging

Tailing the Hedge

- Tailing the hedge
  - The trader slightly adjusts the hedge ratio to offset the interest that can be earned from daily settlement profits or paid on daily settlement losses from his margin account
  - The alternative approach to decide N_V^* can reflect the tailing adjustment for futures

$$N_V^* = N \cdot \frac{V_A}{V_F} = N \cdot \frac{Q_A \times \text{spot price}}{Q_F \times \text{futures price}} = N_Q^* \frac{\text{spot price}}{\text{futures price}} = N_Q^* \frac{1}{(1+r)^T},$$

where $1/(1+r)^T$ is the tailing factor and smaller than 1, which reflects that the tailing adjustment involves a reduction in the futures position.

Tailing the Hedge

- Intuition for the reduction adjustment in the futures position
  - The essence of a hedge is to match a spot position with an offsetting position in futures.
  - Futures prices are more volatile than spot prices (due to futures price=spot price((1+r)^T)).
  - In the process of daily settlement, the excess movements in futures prices will generate the interest gains or losses in excess of the needed amounts to offset the spot position.
  - The optimal number of futures contracts should reduce if the daily settlement is considered.
- Note that for forwards, since there is no daily settlement, the tailing adjustment is not necessary.

Index Futures

- Stock exchange index is a weighted average of selection of stock prices with weights proportional to the market capitalization of stock.
  - Standard and Poor Index S&P500 – computed using 500 stocks.
    - Representing 80% of trade at the NYSE.
  - Consider the index as a portfolio.
    - Transaction cost would impede trading in this portfolio (practically).

Index Futures

- The underlying asset of stock index futures is a stock index level, which can be viewed as an investment asset paying a yield income.
  - A stock index reflects the performance of a virtual portfolio of stocks.
  - It is infeasible to take the PVs of all cash dividends of all stocks in this portfolio into account.
  - In practice, the continuous compounding dividend yield is estimated for this virtual portfolio.
  - The futures and spot prices of a stock index futures follows $F_0 = S_0 e^{(r-q)T}$, where q is the continuous compounding dividend yields on the stock index portfolio during the life of the futures contract.

Index Arbitrage

- When $F_0 > S_0 e^{(r-q)T}$, an arbitrageur buys the portfolio of stocks underlying the index and takes a short position of stock index futures.
- When $F_0 < S_0 e^{(r-q)T}$, an arbitrageur takes a long position of futures and (short) sells the portfolio of stocks underlying the index.

Index Arbitrage

- Index arbitrage involves simultaneous trades in futures and many different stocks
  - Very often a computer program is used to generate the trades, which is known as program trading.
  - During a financial crisis, simultaneous trades could be not possible and the no-arbitrage relationship between F_0 and S_0 does not hold.
    - On Oct. 19, 1987 (Black Monday), the S&P 500 index was 225.06 (down 57.88 on that day) and the futures price for the Dec. S&P 500 index futures was 201.5 (down 80.75 on that day).
    - The overloaded system on exchanges delays the execution of orders and thus the index arbitrage becomes infeasible.

Index Futures

- Let $f(t, T)$ represent the futures price, expressed in index points.
  - Marketing to market is given by the difference $f(t_n, T) - f(t_{n-1}, T)$ multiplied by a fixed amount.
    - \$500 for futures on S&P500.
    - TWSE
      - Stock index 200NTD / 1 point..
      - Electronics 200NTD/ 0.05 point.
      - Finance 200NTD/ 0.2 point.

Index Futures

- The futures contract on the Nikkei 225 Index in CME views 5 times the Nikkei 225 Index, which is measured in yen, as a dollar number.
  - Suppose you take a long position of the Nikki 225 index futures with F_0 to be 1000, and on the delivery date, the Nikki 225 index is 1100.
    - Your payoff is USD$5×(1100 – 1000) = USD$500.
  - Note that traders cannot trade the stock index portfolio underlying the Nikkei 225 Index in USD.
    - Therefore, the formula of $F_0 = S_0 e^{(r-q)T}$ cannot apply to Nikkei 225 index futures, which in fact is a "**quanto**" futures where the underlying asset is measured in one currency and the payoff is in another currency.

Foreign Exchange Quotes for Futures and Forward Contracts

- Futures exchange rates are always quoted as the number of USD per unit of the foreign currency.
- Forward exchange rates are quoted in the same way as spot exchange rates. This means that GBP, EUR, AUD, and NZD are USD per unit of foreign currency. Other currencies (e.g., CAD and JPY) are quoted as units of the foreign currency per USD.

Futures and Forwards on Currencies

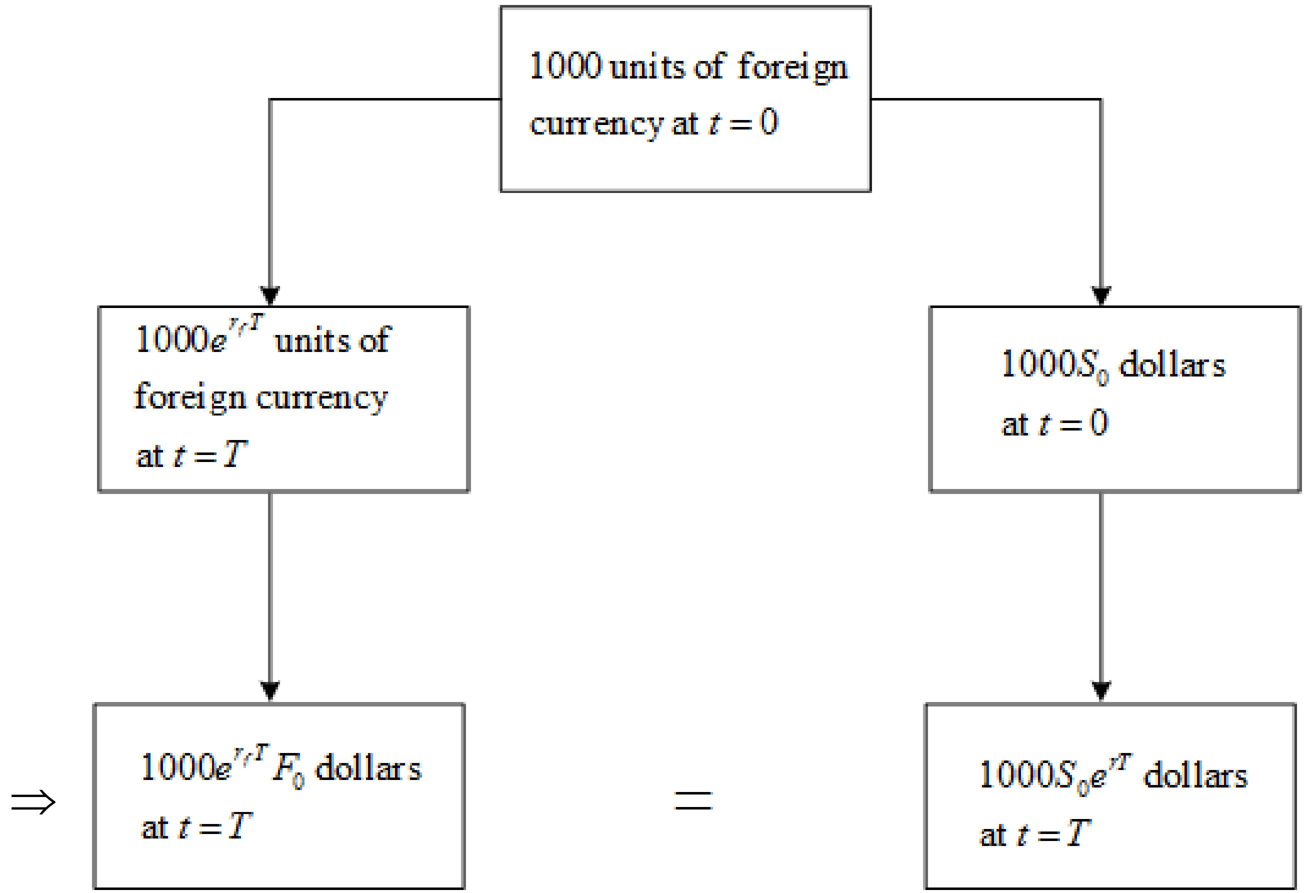
- A foreign currency is analogous to a security providing a yield income.
  - The foreign risk-free interest rate is the yield an investor can earn if he holds that currency.
  - It follows that if the dividend yield is replace with the foreign risk-free interest rate, we can derive the futures price as

$$F_0 = S_0 e^{(r - r_f)T}$$

    - S_0 (F_0) is the spot (futures) price of the foreign currency in terms of the domestic currency, i.e., the current exchange rate (the exchange rate applied on the delivery date).
    - r and r_f are the domestic and foreign risk-free interest rates, respectively.

Futures and Forwards on Currencies

Enter into a foreign currency futures to sell $1000e^{r_f T}$ units of foreign currency at F_0



- If $1000e^{r_f T}F_0 \neq 1000S_0e^{rT}$, an arbitrage opportunity occurs.
- To eliminate all arbitrage opportunities, we can derive $F_0 = S_0e^{(r-r_f)T}$.

Forward vs Futures Prices

- Forward and futures prices are usually assumed to be the same. When interest rates are uncertain, they are, in theory, slightly different.
- The difference between forward and futures prices can be significant if there exists a relationship between the interest rate and the underlying variable.

Forward vs Futures Prices

- A positive correlation between interest rates and the asset price.
  - With the increase of the asset price, the futures holder can earn doubly from the increase of the balance of the margin account and the higher interest rate.
  - With the decrease of the asset price, the futures holder losses, but the opportunity cost of fund for these losses is low due to the lower interest rate.
  - Thus, the futures contract is more attractive and demanded so that futures price > forward price .

Forward vs Futures Prices

- A negative correlation between interest rates and the asset price.
  - With the increase of the asset price, the benefit of the increase of the balance of the margin account will be offset by the lower interest rate.
  - With the decrease of the asset price, the balance of the margin account decreases such that the futures holder cannot fully enjoy the higher interest rate.
  - Thus, the futures contract is relatively not attractive so that futures price < forward price.

Futures on Consumption Assets

- Commodities that are consumption assets rather than investment assets usually provide no income, but are subject to significant storage costs.
  - The first way to model the storage cost:
  $$F_0 = S_0 e^{(r+u)T},$$
  where u is the storage cost per unit time as a percent of the asset value.
  - Alternative way (for the one-time payment of storage costs):
  $$F_0 = (S_0 + U_0)e^{rT},$$
  where U_0 is the present value of the storage costs.

Futures on Consumption Assets

- If $F_0 > (S_0 + U_0)e^{rT}$
  - Borrow $(S_0 + U_0)$ at the risk-free rate and use it to purchase one unit of the commodity and to pay storage costs.
  - Short a futures contract on one unit of the commodity.

  $\Rightarrow$Lead to a positive payoff of $F_0 - (S_0 + U_0)e^{rT} \Rightarrow F_0 > (S_0 + U_0)e^{rT}$ cannot hold.
- If $F_0 < (S_0 + U_0)e^{rT}$
  - Sell the commodity, save the storage costs, and invest the proceeds to earn the risk-free rate.
  - Take a long position in a futures contract.

  $\Rightarrow$Lead to a positive payoff of $(S_0 + U_0)e^{rT} - F_0$

Futures on Consumption Assets

- Due to the concern for the need of using or consuming commodities, the relationship between the futures and spot prices of a consumption commodity is

  $$F_0 \leq S_0 e^{(r+u)T},$$

  where u is the storage cost per unit time as a percent of the asset value, and is

  $$F_0 \leq (S_0 + U_0) e^{rT},$$

  where U_0 is the present value of the storage costs.

Convenience Yield and Cost of Carry

- The benefits from holding the physical asset are referred to as the convenience yield provided by the commodity. Denote the convenience yield as y, then we can derive

$$F_0 e^{yT} = (S_0 + U_0)e^{rT}$$

$$F_0 e^{yT} = S_0 e^{(r+u)T} (\Rightarrow F_0 = S_0 e^{(r+u-y)T})$$

  - The convenience yield reflects the concern of the future availability of the commodity.
    - The greater the possibility that shortages will occur, the higher the convenience yield.
  - Using $F_0 = S_0 e^{(r+u-y)T}$ to explain normal and inverted markets.
    - If $r + u > y \Rightarrow F_0$ increases with $T \Rightarrow$ normal market.
    - If $r + u < y \Rightarrow F_0$ decreases with $T \Rightarrow$ inverted market.

Convenience Yield and Cost of Carry

- The relationship between futures and spot prices can be summarized in terms of cost of carry (持有成本).
  - The cost of carry, c, is the interest costs plus the storage cost less the income earned, i.e., $c = r + u - q$.
  - For an investment asset, $F_0 = S_0 e^{cT}$.
  - For a consumption asset, $F_0 \leq S_0 e^{cT}$.
  - The convenience yield on the consumption asset, y, is defined so that $F_0 = S_0 e^{(c-y)T}$.
- Note that for investment assets, the convenience yield must be zero to eliminate arbitrage opportunities.