

# ON DE CONCINI-KAC FORMS OF QUANTUM GROUPS

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## 1. SETTING

In this section, we establish notations and recall facts about the Drinfeld-Jimbo quantum groups as well as their Lusztig and De Concini-Kac forms.

Let  $\mathfrak{g}$  be a semisimple Lie algebra with simple positive roots  $\alpha_1, \dots, \alpha_r$  and fundamental weights  $\omega_1, \dots, \omega_r$ . Let  $P := \bigoplus_{i=1}^r \mathbb{Z}\omega_i$  be the weight lattice and  $Q := \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$  be the root lattice. Let  $P_+ := \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\omega_i$  be the set of all dominant weights in  $P$ , and  $Q_+ := \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0}\alpha_i$ . We fix a non-degenerate invariant bilinear form  $(\cdot, \cdot)$  on the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$ , and identify  $\mathfrak{h}^*$  with  $\mathfrak{h}$  using  $(\cdot, \cdot)$ . We set  $\mathsf{d}_i := \frac{(\alpha_i, \alpha_i)}{2}$ . The choice of  $(\cdot, \cdot)$  is such that  $\mathsf{d}_i = 1$  for short roots  $\alpha_i$ , in particular,  $\mathsf{d}_i \in \{1, 2, 3\}$  for any  $i$ . Define  $\omega_i^\vee := \frac{\omega_i}{\mathsf{d}_i}$  and  $\alpha_i^\vee := \frac{\alpha_i}{\mathsf{d}_i}$ , so that  $(\alpha_i, \omega_j^\vee) = (\omega_i, \alpha_j^\vee) = \delta_{i,j}$ . Thus, we shall identify the coweight and the coroot lattices of  $\mathfrak{g}$  with  $P^\vee := \bigoplus_{i=1}^r \mathbb{Z}\omega_i^\vee$  and  $Q^\vee := \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^\vee$ , respectively. Let  $(a_{ij})_{i,j=1}^r$  be the Cartan matrix of  $\mathfrak{g}$ :

$$a_{ij} = (\alpha_i^\vee, \alpha_j) = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i).$$

We also consider the symmetrized Cartan matrix  $B = (b_{ij})_{i,j=1}^r$  defined via:

$$(1.1) \quad b_{ij} := \mathsf{d}_i a_{ij} = (\alpha_i, \alpha_j).$$

Let  $v$  be a formal variable and let us consider  $\mathbb{Z}[v, v^{-1}]$  localized at  $\{v^{2k} - 1\}_{1 \leq k \leq \max\{\mathsf{d}_i\}_{i=1}^r}$ :

$$\mathcal{A} = \mathbb{Z}[v, v^{-1}] \left[ \left\{ \frac{1}{v^{2k}-1} \right\}_{1 \leq k \leq \max\{\mathsf{d}_i\}} \right], \quad ^1$$

---

<sup>1</sup>We note that [L3, L5] rather use  $\mathcal{A}$  for  $\mathbb{Z}[v, v^{-1}]$ , while [DCK, DCKP] use  $\mathcal{A}$  for  $\mathbb{C}[v, v^{-1}]$ .

with the quotient field  $\mathbb{Q}(v)$ . Given  $m \in \mathbb{Z}$ ,  $s \in \mathbb{N}$ , define  $[s]_v, [s]_v!, \begin{bmatrix} m \\ s \end{bmatrix}_v \in \mathbb{Z}[v, v^{-1}] \subset \mathcal{A}$  via:

$$[s]_v := \frac{v^s - v^{-s}}{v - v^{-1}}, \quad [s]_v! := \prod_{c=1}^s [c]_v = [1]_v \cdots [s]_v, \quad \begin{bmatrix} m \\ s \end{bmatrix}_v := \prod_{c=1}^s \frac{v^{m-c+1} - v^{-m+c-1}}{v^c - v^{-c}}.$$

Note that  $[0]_v! = 1$ . If  $m \geq s \geq 0$ , then the  $v$ -binomial coefficient equals  $\begin{bmatrix} m \\ s \end{bmatrix}_v = \frac{[m]_v!}{[s]_v! \cdot [m-s]_v!}$ .

For  $1 \leq i \leq r$ , we set  $v_i := v^{\mathbf{d}_i}$ , and define  $[s]_{v_i}, [s]_{v_i}!, \begin{bmatrix} m \\ s \end{bmatrix}_{v_i}$  accordingly.

Let  $q$  be an invertible element of a Noetherian ring  $R$  such that the elements  $q^{2k} - 1 \in R$  are invertible for all  $1 \leq k \leq \max\{\mathbf{d}_i\}_{i=1}^r$ . Define an algebra homomorphism:

$$(1.2) \quad \sigma: \mathcal{A} \longrightarrow R \quad \text{via} \quad v \mapsto q.$$

For  $m \in \mathbb{Z}, s \in \mathbb{N}$ , we define  $[s]_q, [s]_q!, \begin{bmatrix} m \\ s \end{bmatrix}_q \in R$  as the images of  $[s]_v, [s]_v!, \begin{bmatrix} m \\ s \end{bmatrix}_v \in \mathcal{A}$  under (1.2).

### 1.1. The Drinfeld-Jimbo quantum group $\mathbf{U}(\mathfrak{g})$ .

Let  $\mathbf{U}(\mathfrak{g})$  denote the Drinfeld-Jimbo quantum group of  $\mathfrak{g}$  over  $\mathbb{Q}(v)$ , that is, the associative unital  $\mathbb{Q}(v)$ -algebra generated by  $\{E_i, F_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in Q}$  with the following defining relations:

$$(1.3) \quad K^\mu K^{\mu'} = K^{\mu+\mu'}, \quad K^0 = 1,$$

$$(1.4) \quad K^\mu E_i K^{-\mu} = v^{(\alpha_i, \mu)} E_i, \quad K^\mu F_i K^{-\mu} = v^{-(\alpha_i, \mu)} F_i,$$

$$(1.5) \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}},$$

$$(1.6) \quad \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} E_i^{1-a_{ij}-m} E_j E_i^m = 0 \quad (i \neq j),$$

$$(1.7) \quad \sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} F_i^{1-a_{ij}-m} F_j F_i^m = 0 \quad (i \neq j),$$

with the standard notation  $K_i := K^{\alpha_i}$ .

Let  $\mathbf{U}^<, \mathbf{U}^>, \mathbf{U}^0$  denote the  $\mathbb{Q}(v)$ -subalgebras of  $\mathbf{U}(\mathfrak{g})$  generated by  $\{F_i\}_{i=1}^r, \{E_i\}_{i=1}^r$ , and  $\{K^\mu\}_{\mu \in Q}$ , respectively. The following is standard (see e.g. [J1, Theorem 4.21]):

**Lemma 1.1.** (a) (Triangular decomposition of  $\mathbf{U}(\mathfrak{g})$ ) The multiplication map

$$(1.8) \quad m: \mathbf{U}^< \otimes_{\mathbb{Q}(v)} \mathbf{U}^0 \otimes_{\mathbb{Q}(v)} \mathbf{U}^> \longrightarrow \mathbf{U}(\mathfrak{g})$$

is an isomorphism of  $\mathbb{Q}(v)$ -vector spaces.

(b) The subalgebras  $\mathbf{U}^<, \mathbf{U}^>, \mathbf{U}^0$  are isomorphic to the algebras generated by  $\{F_i\}_{i=1}^r, \{E_i\}_{i=1}^r$ , and  $\{K^\mu\}_{\mu \in Q}$ , with the defining relations (1.7), (1.6), and (1.3), respectively.

The algebra  $\mathbf{U}(\mathfrak{g})$  is  $Q$ -graded via:

$$(1.9) \quad \deg(E_i) = \alpha_i, \quad \deg(F_i) = -\alpha_i, \quad \deg(K^\mu) = 0.$$

Following [J1, Proposition 4.11], we also endow  $\mathbf{U}(\mathfrak{g})$  with the standard Hopf algebra structure:

$$(1.10) \quad \begin{aligned} \text{coproduct} \quad & \Delta: E_i \mapsto E_i \otimes 1 + K_i \otimes E_i, \quad F_i \mapsto F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad K^\mu \mapsto K^\mu \otimes K^\mu, \\ \text{antipode} \quad & S: E_i \mapsto -K_i^{-1}E_i, \quad F_i \mapsto -F_iK_i, \quad K^\mu \mapsto K^{-\mu}, \\ \text{counit} \quad & \varepsilon: E_i \mapsto 0, \quad F_i \mapsto 0, \quad K^\mu \mapsto 1. \end{aligned}$$

**Remark 1.2.** Sometimes (e.g. in Section 2) we need to consider a more general version of  $\mathbf{U}(\mathfrak{g})$ : we enlarge the Cartan part  $\mathbf{U}^0$  of  $\mathbf{U}(\mathfrak{g})$  getting an algebra over  $\mathbb{Q}(v^{1/N})$  for a suitable positive integer  $N$ . Let  $X$  be a  $\mathbb{Z}$ -lattice in  $\mathbb{Z}Q \otimes_{\mathbb{Z}} \mathbb{Q}$  containing the root lattice  $Q$ . Let  $N$  be a positive integer such that  $(X, Q) \in \frac{1}{N}\mathbb{Z}$ ; then for any  $\mu \in X$  and  $\lambda \in Q$ , we define  $v^{(\mu, \lambda)} := (v^{1/N})^{N(\mu, \lambda)}$ . Let  $\mathbf{U}(\mathfrak{g}, X)$  be an associative  $\mathbb{Q}(v^{1/N})$ -algebra generated by  $\{E_i, F_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in X}$  with the defining relations similar to the defining relations (1.3)–(1.7). We also endow  $\mathbf{U}(\mathfrak{g}, X)$  with the Hopf algebra structure as in (1.10) and the  $Q$ -grading as in (1.9).

Let us recall the  $\mathbb{Q}$ -algebra anti-involution  $\tau: \mathbf{U}(\mathfrak{g}) \rightarrow \mathbf{U}(\mathfrak{g})$  defined by

$$(1.11) \quad \tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K^\lambda) = K^{-\lambda}, \quad \tau(v) = v^{-1} \quad \forall 1 \leq i \leq r, \lambda \in Q.$$

There is Lusztig's braid group action on  $\mathbf{U}(\mathfrak{g})$  defined as follows, see [L3, Theorem 3.1], [L5, Part VI] (cf. [J1, §8.14–8.15]):

$$(1.12) \quad \begin{aligned} T_i(K^\mu) &= K^{s_{\alpha_i}\mu}, \quad T_i(E_i) = -F_iK^{\alpha_i}, \quad T_i(F_i) = -K^{-\alpha_i}E_i, \\ T_i(E_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k \frac{v_i^{-k}}{[-a_{ij} - k]_{v_i}! [k]_{v_i}!} E_i^{-a_{ij}-k} E_j E_i^k, \\ T_i(F_j) &= \sum_{k=0}^{-a_{ij}} (-1)^k \frac{v_i^k}{[-a_{ij} - k]_{v_i}! [k]_{v_i}!} F_i^k F_j F_i^{-a_{ij}-k}. \end{aligned}$$

The operators  $T_i$  are easily seen to commute with the map  $\tau$  of (1.11). With this braid group action, one can define the PBW basis of  $\mathbf{U}(\mathfrak{g})$ , cf. [J1, §8]. To do so, pick a reduced decomposition of the longest element  $w_0 = s_{i_1}s_{i_2} \cdots s_{i_N}$  of the Weyl group of  $\mathfrak{g}$  (where  $N$  is the cardinality of the positive root system  $\Delta_+$ ). Then the set of roots  $\beta_k = s_{i_1} \cdots s_{i_{k-1}} \alpha_{i_k}$  ( $1 \leq k \leq N$ ) provides a labeling of all positive roots  $\Delta_+$  of  $\mathfrak{g}$ , and using Lusztig's braid group action we define root vectors  $\{E_\beta, F_\beta\}_{\beta \in \Delta_+}$  in a standard way via:

$$(1.13) \quad E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} E_{i_k}, \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}} F_{i_k} = \tau(E_{\beta_k}) \quad \forall 1 \leq k \leq N.$$

For  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$ , consider the ordered monomials:

$$F^{\vec{k}} := F_{\beta_1}^{k_1} \cdots F_{\beta_N}^{k_N}, \quad E^{\vec{k}} := E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}, \quad F^{\vec{k}} := F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}, \quad E^{\vec{k}} := E_{\beta_N}^{k_N} \cdots E_{\beta_1}^{k_1}.$$

The following result follows from [J1, §8.24] and the triangular decomposition of Lemma 1.1:

- Lemma 1.3.** (a) The sets  $\{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v)$ -bases of  $\mathbf{U}^>$ .  
(b) The sets  $\{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v)$ -bases of  $\mathbf{U}^<$ .  
(c) The set  $\{K^\mu\}_{\mu \in Q}$  is a  $\mathbb{Q}(v)$ -basis of  $\mathbf{U}^0$ .  
(d) The sets  $\{F^{\vec{k}}K^\mu E^{\vec{r}}\}, \{F^{\vec{k}}K^\mu E^{\vec{r}}\}, \{F^{\vec{k}}K^\mu E^{\vec{r}}\}, \{F^{\vec{k}}K^\mu E^{\vec{r}}\}$ , with  $\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N$  and  $\mu \in Q$ , form  $\mathbb{Q}(v)$ -bases of  $\mathbf{U}(\mathfrak{g})$ .

### 1.2. The Lusztig form.

For  $1 \leq i \leq r$  and  $s \in \mathbb{N}$ , define the **divided powers**  $E_i^{[s]}, F_i^{[s]} \in \mathbf{U}(\mathfrak{g})$ <sup>2</sup> via:

$$E_i^{[s]} := \frac{E_i^s}{[s]_{v_i}!}, \quad F_i^{[s]} := \frac{F_i^s}{[s]_{v_i}!}.$$

For  $1 \leq i \leq r$ ,  $a \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , define the  $v$ -binomial coefficients  $\begin{bmatrix} K_i; a \\ n \end{bmatrix} \in \mathbf{U}(\mathfrak{g})$  via:

$$\begin{bmatrix} K_i; a \\ n \end{bmatrix} := \prod_{c=1}^n \frac{K_i v_i^{a-c+1} - K_i^{-1} v_i^{-a+c-1}}{v_i^c - v_i^{-c}}.$$

Following [L3, §1.3], define the **Lusztig form**  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g})$  generated by  $\{E_i^{[s]}, F_i^{[s]}, K^\mu\}_{1 \leq i \leq r}^{s \in \mathbb{N}, \mu \in Q}$ . Note that all elements  $\begin{bmatrix} K_i; a \\ n \end{bmatrix}$  lie in  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$ , due to a recursion

$$\begin{bmatrix} K_i; a \\ n \end{bmatrix} - v_i^{-n} \begin{bmatrix} K_i; a+1 \\ n \end{bmatrix} = -v_i^{-a-1} K_i^{-1} \begin{bmatrix} K_i; a \\ n-1 \end{bmatrix},$$

combined with the following important equality (see [L1, §4.3]):

$$(1.14) \quad E_i^{[p]} F_i^{[s]} = \sum_{c=0}^{\min(p,s)} F_i^{[s-c]} \begin{bmatrix} K_i; 2c-p-s \\ c \end{bmatrix} E_i^{[p-c]}.$$

We also note that  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  is clearly invariant under the map  $\tau$  of (1.11).

Evoking the standard Hopf algebra structure (1.10) on  $\mathbf{U}(\mathfrak{g})$ , we find (see [J1, §4.9]):

$$\begin{aligned} \Delta(E_i^{[s]}) &= \sum_{c=0}^s v_i^{c(s-c)} E_i^{[s-c]} K_i^c \otimes E_i^{[c]}, & S(E_i^{[s]}) &= (-1)^s v_i^{s(s-1)} K_i^{-s} E_i^{[s]}, \\ \Delta(F_i^{[s]}) &= \sum_{c=0}^s v_i^{-c(s-c)} F_i^{[c]} \otimes K_i^{-c} F_i^{[s-c]}, & S(F_i^{[s]}) &= (-1)^s v_i^{-s(s-1)} F_i^{[s]} K_i^s, \end{aligned}$$

so that  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  is actually a Hopf  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g})$ .

Let  $\dot{\mathcal{U}}_{\mathcal{A}}^<, \dot{\mathcal{U}}_{\mathcal{A}}^>, \dot{\mathcal{U}}_{\mathcal{A}}^0$  denote the  $\mathcal{A}$ -subalgebras of  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  generated by  $\{F_i^{[s]}\}$ ,  $\{E_i^{[s]}\}$ ,  $\left\{ \begin{bmatrix} K_i; m \\ s \end{bmatrix}, K^\mu \right\}$ , respectively. Evoking the construction of root generators (1.13), we define

$$(1.15) \quad F_{\beta_k}^{[s]} := \frac{F_{\beta_k}^s}{[s]_{v_{i_k}}!}, \quad E_{\beta_k}^{[s]} := \frac{E_{\beta_k}^s}{[s]_{v_{i_k}}!} \quad \forall 1 \leq k \leq N, s \in \mathbb{N}.$$

According to [L3, Theorem 6.7], we have:

**Lemma 1.4.** (a) Both subalgebras  $\dot{\mathcal{U}}_{\mathcal{A}}^>$  and  $\dot{\mathcal{U}}_{\mathcal{A}}^<$  are  $Q$ -graded via (1.9), and each of their degree components is a free  $\mathcal{A}$ -module of finite rank.

(b) (Triangular decomposition of  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$ ) The multiplication map

$$(1.16) \quad m: \dot{\mathcal{U}}_{\mathcal{A}}^< \otimes_{\mathcal{A}} \dot{\mathcal{U}}_{\mathcal{A}}^0 \otimes_{\mathcal{A}} \dot{\mathcal{U}}_{\mathcal{A}}^> \longrightarrow \dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$$

is an isomorphism of free  $\mathcal{A}$ -modules.

(c) The elements

$$\left\{ \prod_{i=1}^r \left( K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} \right) \mid t_i \geq 0, \delta_i \in \{0, 1\} \right\}$$

---

<sup>2</sup>Our superscript  $[s]$  differs from  $(s)$  in [L3, L5], as we shall use the latter for a modified version in Section 2.4.

form an  $\mathcal{A}$ -basis of  $\dot{\mathcal{U}}_{\mathcal{A}}^0$ .

- (d) The elements  $E^{[\vec{k}]} := E_{\beta_1}^{[k_1]} \dots E_{\beta_N}^{[k_N]}$  with all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$  form an  $\mathcal{A}$ -basis of  $\dot{\mathcal{U}}_{\mathcal{A}}^>$ . Similarly, the elements  $E^{[\vec{k}]} := E_{\beta_N}^{[k_N]} \dots E_{\beta_1}^{[k_1]}$  with all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$  form an  $\mathcal{A}$ -basis of  $\dot{\mathcal{U}}_{\mathcal{A}}^<$ .
- (e) The elements  $F^{[\vec{k}]} := F_{\beta_1}^{[k_1]} \dots F_{\beta_N}^{[k_N]}$  with all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$  form an  $\mathcal{A}$ -basis of  $\dot{\mathcal{U}}_{\mathcal{A}}^<$ . Similarly, the elements  $F^{[\vec{k}]} := F_{\beta_N}^{[k_N]} \dots F_{\beta_1}^{[k_1]}$  with all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$  form an  $\mathcal{A}$ -basis of  $\dot{\mathcal{U}}_{\mathcal{A}}^>$ .

**Remark 1.5.** We note that while [L3] treats only the first basis in (d) and the second basis in (e), we obtain the other two bases from (e) and (d) by applying the  $\mathbb{Q}$ -algebra anti-involution  $\tau$  of (1.11).

Given  $q \in R$  as in the paragraph preceding (1.2), we define the **Lusztig form**  $\dot{\mathcal{U}}_q(\mathfrak{g})$  as the base change of  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  with respect to  $\sigma: \mathcal{A} \rightarrow R$  of (1.2):

$$(1.17) \quad \dot{\mathcal{U}}_q(\mathfrak{g}) := \dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} R.$$

### 1.3. The De Concini-Kac form.

Following [DCK, §1.5], define the **De Concini-Kac form**  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g})$  generated by  $\{E_i, F_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in Q}$ . We note that  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  is clearly a Hopf  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g})$ . It is also clear that  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  is invariant under the map  $\tau$  of (1.11).

Let  $\mathcal{U}_{\mathcal{A}}^<, \mathcal{U}_{\mathcal{A}}^>, \mathcal{U}_{\mathcal{A}}^0$  denote the  $\mathcal{A}$ -subalgebras of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  generated by  $\{F_i\}_{i=1}^r, \{E_i\}_{i=1}^r$ , and  $\{K^\mu\}_{\mu \in Q}$ , respectively. Since  $\mathcal{A}$  contains  $(v^{2k} - 1)^{-1}$  for any  $1 \leq k \leq \mathsf{d}_i$  and any  $i$ , the braid group action preserves  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ , see (1.12). Hence the elements  $\{E_{\beta_k}, F_{\beta_k}\}_{k=1}^N$  of (1.13) are contained in  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ . The following result is a standard corollary of Lemma 1.1:

**Lemma 1.6.** (a) Both subalgebras  $\mathcal{U}_{\mathcal{A}}^>$  and  $\mathcal{U}_{\mathcal{A}}^<$  are  $Q$ -graded via (1.9), and each of their degree components is a free  $\mathcal{A}$ -module of finite rank.  
(b) (Triangular decomposition of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ ) The multiplication map

$$(1.18) \quad m: \mathcal{U}_{\mathcal{A}}^< \otimes_{\mathcal{A}} \mathcal{U}_{\mathcal{A}}^0 \otimes_{\mathcal{A}} \mathcal{U}_{\mathcal{A}}^> \longrightarrow \mathcal{U}_{\mathcal{A}}(\mathfrak{g})$$

is an isomorphism of free  $\mathcal{A}$ -modules.

- (c) The elements  $K^\mu$  with  $\mu \in Q$  form an  $\mathcal{A}$ -basis of  $\mathcal{U}_{\mathcal{A}}^0$ .
- (d) The sets  $\{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  form  $\mathcal{A}$ -bases of  $\mathcal{U}_{\mathcal{A}}^>$ .
- (e) The sets  $\{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  form  $\mathcal{A}$ -bases of  $\mathcal{U}_{\mathcal{A}}^<$ .

*Proof.* Let us show that the second set in part (d) is an  $\mathcal{A}$ -basis of  $\mathcal{U}_{\mathcal{A}}^>$ . It suffices to show that the inclusion  $\mathcal{U}_{\mathcal{A}}^> \supseteq \bigoplus_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} \mathcal{A} \cdot E^{\vec{k}}$  is actually an equality. To this end, pick any  $x \in \mathcal{U}_{\mathcal{A}}^>$ .

Evoking the Hopf pairing  $(\cdot, \cdot)$  of [J1, §6.12], cf. Proposition 3.2 below, we note that  $(\dot{\mathcal{U}}_{\mathcal{A}}^<, x) \in \mathcal{A}$ , with  $\dot{\mathcal{U}}_{\mathcal{A}}^<$  defined in Section 1.2 above. On the other hand, writing down  $x$  in the Lusztig PBW basis as  $x = \sum_{\vec{k}} c_{\vec{k}} E^{\vec{k}}$  and using the “duality” of the Lusztig basis with respect to the pairing  $(\cdot, \cdot)$ , see [J1, §§8.29–8.30] and (3.21) below, we conclude that all  $c_{\vec{k}} \in \mathcal{A}$  as  $\frac{1}{v_i - v_i^{-1}} \in \mathcal{A}$  for all  $i$ .

The proof that the second set in part (e) is an  $\mathcal{A}$ -basis of  $\mathcal{U}_{\mathcal{A}}^<$  is analogous.

The anti-involution  $\tau$  of (1.11) induces an anti-isomorphism  $\mathcal{U}_{\mathcal{A}}^> \rightarrow \mathcal{U}_{\mathcal{A}}^<$ . Moreover,  $\tau(E^{\vec{k}}) = F^{\vec{k}}, \tau(F^{\vec{k}}) = E^{\vec{k}}$ . Therefore, the first sets in part (d) and (e) are  $\tau$ -images of the second sets in part (e) and (d), respectively, hence form  $\mathcal{A}$ -bases. Part (c) follows from Lemma 1.1(b). The rest of Lemma 1.6 immediately follows from the above combined with Lemma 1.1.  $\square$

Given  $q \in R$  as in the paragraph preceding (1.2), we define the **De Concini-Kac form**  $\mathcal{U}_q(\mathfrak{g})$  as the base change of  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$  with respect to  $\sigma: \mathcal{A} \rightarrow R$  of (1.2):

$$(1.19) \quad \mathcal{U}_q(\mathfrak{g}) := \mathcal{U}_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} R.$$

## 2. TWISTED COPRODUCT AND INTEGRAL FORMS

For a Hopf algebra  $(A, \Delta, S, \varepsilon)$ , the **left adjoint action**  $\text{ad}: A \curvearrowright A$  of  $A$  on itself is given by:

$$(2.1) \quad (\text{ad } a)(b) := a_{(1)} \cdot b \cdot S(a_{(2)}) \quad \forall a, b \in A.$$

Here we use the Sweedler's notation for the coproduct (suppressing the summation symbol):

$$(2.2) \quad \Delta(a) = a_{(1)} \otimes a_{(2)} \quad \forall a \in A.$$

Let us record the following basic property of the adjoint action (see e.g. [JL1, Lemma 2.2(ii)]):

$$(2.3) \quad (\text{ad } a)(b \cdot c) = (\text{ad } a_{(1)})(b) \cdot (\text{ad } a_{(2)})(c) \quad \forall a, b, c \in A.$$

Thus we have the adjoint action  $\mathbf{U}(\mathfrak{g}) \curvearrowright \mathbf{U}(\mathfrak{g})$ . However, this action does not restrict to an action of the Lusztig form  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  on the De Concini-Kac form  $\mathcal{U}_{\mathcal{A}}(\mathfrak{g})$ . To remedy this and for other purposes, we will modify the coproduct of  $\mathbf{U}(\mathfrak{g})$  via the twist construction. Then we introduce the **(twisted) Lusztig form**  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g})$  and the **even part** subalgebra  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ . The latter is a suitable alternative (based on a certain “Cartan twist” of generators) to the De Concini-Kac form so that we have an adjoint action  $\dot{\mathcal{U}}_{\mathcal{A}}(\mathfrak{g}) \curvearrowright U_{\mathcal{A}}^{ev}(\mathfrak{g})$ , which is of crucial importance for the rest of this paper.

### 2.1. Twisting.

To achieve the above goal, let us recall the standard twist construction (cf. [R, Theorem 1]):

**Proposition 2.1.** (a) For a (topological) Hopf algebra  $(A, m, \Delta, S, \varepsilon)$  and  $F \in A \otimes A$  (or in an appropriate completion  $A \widehat{\otimes} A$ ) satisfying

$$(2.4) \quad (\Delta \otimes \text{Id})(F) = F_{13}F_{23}, \quad (\text{Id} \otimes \Delta)(F) = F_{13}F_{12}, \quad F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}, \quad F_{12}F_{21} = 1,$$

the formulas

$$(2.5) \quad \Delta^{(F)}(a) = F\Delta(a)F^{-1}, \quad S^{(F)}(a) = uS(a)u^{-1}, \quad \varepsilon^{(F)}(a) = \varepsilon(a)$$

with  $u := m(\text{Id} \otimes S)(F)$ , endow  $A$  with a new Hopf algebra structure:  $(A, m, \Delta^{(F)}, S^{(F)}, \varepsilon^{(F)})$ .

(b) If  $(A, m, \Delta, S, \varepsilon)$  is a quasitriangular Hopf algebra with a universal  $R$ -matrix  $R \in A \otimes A$  (or in an appropriate completion  $A \widehat{\otimes} A$ ), then  $(A, m, \Delta^{(F)}, S^{(F)}, \varepsilon^{(F)})$  is also a quasitriangular Hopf algebra with a universal  $R$ -matrix

$$(2.6) \quad R^{(F)} = F^{-1}RF^{-1}.$$

The Hopf algebra  $(A, m, \Delta^{(F)}, S^{(F)}, \varepsilon^{(F)})$  is called the **twist** of  $(A, m, \Delta, S, \varepsilon)$  by  $F$ .

**Remark 2.2.** (a) The inverse of  $u$  is explicitly given by  $u^{-1} = m(S \otimes \text{Id})(F_{21})$ .

(b) For Proposition 2.1(a) alone, conditions (2.4) may be relaxed (cf. [CP, Proposition 4.2.13]):

$$F_{12}(\Delta \otimes \text{Id})(F) = F_{23}(\text{Id} \otimes \Delta)(F), \quad (\varepsilon \otimes \text{Id})(F) = 1 = (\text{Id} \otimes \varepsilon)(F), \quad F \text{ is invertible}.$$

## 2.2. Twisted Hopf algebra structure on $\mathbf{U}(\mathfrak{g})$ .

Following [R, §2], let us apply Proposition 2.1 to  $A = \mathbf{U}(\mathfrak{g})$  endowed with the standard Hopf algebra structure  $(A, \Delta, S, \varepsilon)$  of (1.10) and a Cartan type element  $F$ :

$$(2.7) \quad F = v^{\sum_{i,j=1}^r \phi_{ij} \omega_i^\vee \otimes \omega_j^\vee}.$$

Such  $F$  satisfies the conditions (2.4) iff the matrix  $\Phi = (\phi_{ij})_{i,j=1}^r \in \text{Mat}_{r \times r}(\mathbb{Q})$  is skew-symmetric.

To simplify our notations, we shall henceforth denote the corresponding twisted coproduct  $\Delta^{(F)}$ , antipode  $S^{(F)}$ , and counit  $\varepsilon^{(F)}$  of (2.5) by  $\Delta'$ ,  $S'$ , and  $\varepsilon'$ , respectively.

**Remark 2.3.** We note that  $\sum_{j=1}^r \phi_{ij} \omega_j^\vee$  is not contained in  $Q$ . Thus, in fact, we will perform the twist on a Hopf  $\mathbb{Q}(v^{1/N})$ -algebra  $\mathbf{U}(\mathfrak{g}, X)$  for a  $\mathbb{Z}$ -lattice  $X$  of  $\mathbb{Z}Q \otimes_{\mathbb{Z}} \mathbb{Q}$  containing all  $\{\sum_{j=1}^r \phi_{ij} \omega_j^\vee\}_{1 \leq i \leq r}$ , see Remark 1.2. Explicitly, the twisted Hopf structure is as follows:

$$(2.8) \quad \begin{aligned} \Delta'(K^\mu) &= K^\mu \otimes K^\mu, \\ \Delta'(E_i) &= E_i \otimes K^{\sum_{j=1}^r \phi_{ij} \omega_j^\vee} + K^{\alpha_i - \sum_{j=1}^r \phi_{ij} \omega_j^\vee} \otimes E_i, \\ \Delta'(F_i) &= F_i \otimes K^{-\alpha_i - \sum_{j=1}^r \phi_{ij} \omega_j^\vee} + K^{\sum_{j=1}^r \phi_{ij} \omega_j^\vee} \otimes F_i, \\ S'(K^\mu) &= K^{-\mu}, \quad S'(E_i) = -K^{-\alpha_i} E_i, \quad S'(F_i) = -F_i K^{\alpha_i}, \\ \varepsilon'(K^\mu) &= 1, \quad \varepsilon'(E_i) = \varepsilon'(F_i) = 0, \end{aligned}$$

for  $\mu \in X$ ,  $1 \leq i \leq r$ .

## 2.3. Special choice of $\Phi$ .

In this section, we spell out a special choice of  $\Phi$  which will be used in various constructions in the rest of this paper. Let  $\text{Dyn}(\mathfrak{g})$  denote the graph obtained from the Dynkin diagram of  $\mathfrak{g}$  by replacing all multiple edges by simple ones, e.g.  $\text{Dyn}(\mathfrak{sp}_{2r}) = \text{Dyn}(\mathfrak{so}_{2r+1}) = \text{Dyn}(\mathfrak{sl}_{r+1}) = A_r$ . Given an orientation  $\text{Or}$  of the graph  $\text{Dyn}(\mathfrak{g})$ , define the *associated matrix*  $(\epsilon_{ij})_{i,j=1}^r$  via:

$$(2.9) \quad \epsilon_{ij} := \begin{cases} 0 & \text{if } a_{ij} \geq 0 \\ 1 & \text{if } a_{ij} < 0 \text{ and } \text{Or} \text{ contains an oriented edge } i \rightarrow j \\ -1 & \text{if } a_{ij} < 0 \text{ and } \text{Or} \text{ contains an oriented edge } i \leftarrow j \end{cases}.$$

Then, we consider  $\Phi = (\phi_{ij})_{i,j=1}^r$  with

$$(2.10) \quad \phi_{ij} = \epsilon_{ij} \frac{b_{ij}}{2},$$

where  $b_{ij} = (\alpha_i, \alpha_j)$  as in (1.1). With this choice, we have  $\sum_{j=1}^r \phi_{ij} \omega_j^\vee \in P/2$  for all  $1 \leq i \leq r$ . Therefore, following Remark 2.3, we will consider the Hopf algebra  $\mathbf{U}(\mathfrak{g}, P/2)$  over  $\mathbb{Q}(v^{1/2})$  with the Hopf structure as in (2.8). The new Lusztig form  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  and the even part algebra  $U_{\mathcal{A}}^{ev}$  will be Hopf  $\mathcal{A}$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$ . To introduce those, we first define the **modified** elements  $\{\tilde{E}_i, \tilde{F}_i\}_{i=1}^r$  of  $\mathbf{U}(\mathfrak{g}, P/2)$ .

For any  $1 \leq i \leq r$ , let

$$(2.11) \quad \nu_i^> := -\alpha_i + \sum_{j=1}^r \phi_{ij} \omega_j^\vee, \quad \nu_i^< := \sum_{j=1}^r \phi_{ij} \omega_j^\vee = \alpha_i + \nu_i^>,$$

and consider

$$(2.12) \quad \tilde{E}_i := E_i K^{\nu_i^>} , \quad \tilde{F}_i := K^{-\nu_i^<} F_i .$$

We will also need the following elements of  $\mathfrak{h}^*$ :

$$(2.13) \quad \zeta_i^> := \alpha_i - 2 \sum_{j=1}^r \phi_{ij} \omega_j^\vee = -\nu_i^> - \nu_i^<, \quad \zeta_i^< := -\alpha_i - 2 \sum_{j=1}^r \phi_{ij} \omega_j^\vee.$$

One can show that the  $\mathbb{Q}(v^{1/2})$ -algebra  $\mathbf{U}(\mathfrak{g}, P/2)$  is generated by  $\{\tilde{E}_i, \tilde{F}_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in P/2}$  subject to the following relations:

$$(2.14) \quad \begin{aligned} K^\mu K^{\mu'} &= K^{\mu+\mu'}, \quad K^0 = 1, \\ K^\mu \tilde{E}_i K^{-\mu} &= v^{(\alpha_i, \mu)} \tilde{E}_i, \quad K^\mu \tilde{F}_i K^{-\mu} = v^{-(\alpha_i, \mu)} \tilde{F}_i, \\ \tilde{E}_i \tilde{F}_j &= v^{(\alpha_i, -\zeta_j^<)} \tilde{F}_j \tilde{E}_i \quad (i \neq j), \quad \tilde{E}_i \tilde{F}_i - v_i^2 \tilde{F}_i \tilde{E}_i = v_i \frac{1 - K_i^{-2}}{1 - v_i^{-2}}, \\ \sum_{m=0}^{1-a_{ij}} (-1)^m v^{m\epsilon_{ij}b_{ij}} \left[ \frac{1-a_{ij}}{m} \right]_{v_i} \tilde{E}_i^{1-a_{ij}-m} \tilde{E}_j \tilde{E}_i^m &= 0 \quad (i \neq j), \\ \sum_{m=0}^{1-a_{ij}} (-1)^m v^{m\epsilon_{ij}b_{ij}} \left[ \frac{1-a_{ij}}{m} \right]_{v_i} \tilde{F}_i^{1-a_{ij}-m} \tilde{F}_j \tilde{F}_i^m &= 0 \quad (i \neq j), \end{aligned}$$

with the standard notation  $K_i := K^{\alpha_i}$ . The algebra  $\mathbf{U}(\mathfrak{g}, P/2)$  is  $Q$ -graded via (cf. (1.9)):

$$(2.15) \quad \deg(\tilde{E}_i) = \alpha_i, \quad \deg(\tilde{F}_i) = -\alpha_i, \quad \deg(K^\mu) = 0.$$

Moreover, we have

$$(2.16) \quad \begin{aligned} \Delta'(K^\mu) &= K^\mu \otimes K^\mu, \quad \Delta'(\tilde{E}_i) = 1 \otimes \tilde{E}_i + \tilde{E}_i \otimes K^{-\zeta_i^>}, \quad \Delta'(\tilde{F}_i) = 1 \otimes \tilde{F}_i + \tilde{F}_i \otimes K^{\zeta_i^<} , \\ S'(K^\mu) &= K^{-\mu}, \quad S'(\tilde{E}_i) = -\tilde{E}_i K^{\zeta_i^>}, \quad S'(\tilde{F}_i) = -\tilde{F}_i K^{-\zeta_i^<}. \end{aligned}$$

Let

$$(2.17) \quad 2P := \{\mu \in \mathfrak{h}^* \mid \mu/2 \in P\} = \left\{ \mu \in P^\vee \mid \frac{(\alpha_i, \mu)}{2d_i} \in \mathbb{Z} \text{ for all } 1 \leq i \leq r \right\}.$$

**Lemma 2.4.** *The elements  $\{\zeta_i^>\}_{i=1}^r$  and  $\{\zeta_i^<\}_{i=1}^r$  are contained in  $2P$ .*

*Proof.* We have  $(\alpha_j, \zeta_i^>) = (\alpha_j, \alpha_i - 2 \sum_{k=1}^r \phi_{ik} \omega_k^\vee) = (\alpha_j, \alpha_i) - 2\phi_{ij} = (1 - \epsilon_{ij})(\alpha_i, \alpha_j)$ . Hence  $\frac{(\alpha_j, \zeta_i^>)}{2d_j} = \frac{1 - \epsilon_{ij}}{2} a_{ji}$ . If  $i = j$  then  $\frac{(\alpha_j, \zeta_i^>)}{2d_j} = 1$ . If  $i \neq j$ , then  $\frac{(\alpha_j, \zeta_i^>)}{2d_j} \in \{0, a_{ji}\}$ . This implies that  $\zeta_i^> \in 2P$ . The result for  $\zeta_i^<$  follows from  $\zeta_i^< = \zeta_i^> - 2\alpha_i$ .  $\square$

**Remark 2.5.** The elements  $\tilde{E}_i, \tilde{F}_i$  appeared in [S1], where the motivation was as follows. Let  $\mathbf{U}_\Phi^<, \mathbf{U}_\Phi^>$  be the  $\mathbb{Q}(v^{1/2})$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\{\tilde{F}_i\}, \{\tilde{E}_i\}$ , respectively. Then, the assignment  $\tilde{E}_i \mapsto 1, \tilde{F}_i \mapsto 1$  gives rise to algebra homomorphisms:

$$\chi^< : \mathbf{U}_\Phi^< \longrightarrow \mathbb{Q}(v^{1/2}), \quad \chi^> : \mathbf{U}_\Phi^> \longrightarrow \mathbb{Q}(v^{1/2}),$$

cf. [S1, Proposition 2, Theorem 4]. We are not going to use this fact in what follows.

## 2.4. The (twisted) Lusztig form $\dot{U}_q(\mathfrak{g})$ .

Let

$$(n)_v = \frac{1 - v^{-2n}}{1 - v^{-2}}, \quad (n)_v! = (1)_v \dots (n)_v, \quad \binom{m}{s}_v := \prod_{c=1}^s \frac{1 - v^{2(-m+c-1)}}{1 - v^{-2c}}.$$

We note that  $[n]_v = v^{n-1}(n)_v$ . Let us consider the following elements:

$$\binom{K_i; a}{n} = \frac{\prod_{s=1}^n (1 - K_i^{-2} v_i^{2(s-a-1)})}{\prod_{s=1}^n (1 - v_i^{-2s})}, \quad \tilde{E}_i^{(n)} = \frac{\tilde{E}_i^n}{(n)_v!}, \quad \tilde{F}_i^{(n)} = \frac{\tilde{F}_i^n}{(n)_v!}.$$

We define the **(twisted) Lusztig form**  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by the elements  $\{\tilde{E}_i^{(s)}, \tilde{F}_i^{(s)}, K^\mu\}_{1 \leq i \leq r}^{s \in \mathbb{N}, \mu \in 2P}$ . Later on we shall often refer to  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  as the Lusztig form, when there is no ambiguity. By (1.14), we have

$$(2.18) \quad \begin{aligned} \tilde{E}_i^{(p)} \tilde{F}_j^{(s)} &= v^{ps(\alpha_i, -\zeta_j^<)} \tilde{F}_j^{(s)} \tilde{E}_i^{(p)} \quad \text{for } i \neq j, \\ \tilde{E}_i^{(p)} \tilde{F}_i^{(s)} &= \sum_{c=0}^{\min(p,s)} v_i^{2ps-c^2} \tilde{F}_i^{(s-c)} \binom{K_i; 2c-p-s}{c} \tilde{E}_i^{(p-c)}. \end{aligned}$$

The last two relations in (2.14) can be rewritten as follows:

$$(2.19) \quad \begin{aligned} \sum_{m=0}^{1-a_{ij}} (-1)^m v_i^{ma_{ij}(\epsilon_{ij}-1)-m(m-1)} \tilde{E}_i^{(1-a_{ij}-m)} \tilde{E}_j \tilde{E}_i^{(m)} &= 0 \quad (i \neq j), \\ \sum_{m=0}^{1-a_{ij}} (-1)^m v_i^{ma_{ij}(\epsilon_{ij}-1)-m(m-1)} \tilde{F}_i^{(1-a_{ij}-m)} \tilde{F}_j \tilde{F}_i^{(m)} &= 0 \quad (i \neq j). \end{aligned}$$

Evoking the twisted Hopf algebra structure from (2.8) and (2.16), we have

$$(2.20) \quad \begin{aligned} \Delta'(\tilde{E}_i^{(s)}) &= \sum_{c=0}^s \tilde{E}_i^{(s-c)} \otimes \tilde{E}_i^{(c)} K^{-(s-c)\zeta_i^>} , \quad S'(\tilde{E}_i^{(s)}) = (-1)^s v_i^{s(s-1)} \tilde{E}_i^{(s)} K^{s\zeta_i^>} , \\ \Delta'(\tilde{F}_i^{(s)}) &= \sum_{c=0}^s v_i^{2c(s-c)} \tilde{F}_i^{(c)} \otimes \tilde{F}_i^{(s-c)} K^{c\zeta_i^<} , \quad S'(\tilde{F}_i^{(s)}) = (-1)^s v_i^{-s(s-1)} \tilde{F}_i^{(s)} K^{-s\zeta_i^<}. \end{aligned}$$

Combining these formulas with Lemma 2.4, we deduce that  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  is a Hopf  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  with respect to the twisted Hopf structure of (2.8) and (2.16).

Given  $q \in R$  as in the paragraph preceding (1.2), we define the **(twisted) Lusztig form**  $\dot{U}_q(\mathfrak{g})$  as the base change of  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  with respect to  $\sigma: \mathcal{A} \rightarrow R$  of (1.2):

$$(2.21) \quad \dot{U}_q(\mathfrak{g}) := \dot{U}_{\mathcal{A}}(\mathfrak{g}) \otimes_{\mathcal{A}} R.$$

We denote the images of  $\tilde{E}_i^{(s)}, \tilde{F}_i^{(s)}, K^\mu, \binom{K_i; a}{n}$  in  $\dot{U}_q(\mathfrak{g})$  by the same symbols.

We shall often be interested in idempotent versions of the Lusztig forms, see Section 4.1.1. In this context, the ‘‘Cartan twist’’ disappears and we get the same algebra as [L5] but with a different coproduct.

## 2.5. The even part algebra $U_q^{ev}(\mathfrak{g})$ .

We define the **even part** algebra  $U_{\mathcal{A}}^{ev}$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by the elements  $\{\tilde{E}_i, \tilde{F}_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in 2P}$ . By (2.16) and Lemma 2.4, it follows that  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  is a Hopf  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  with respect to the twisted Hopf structure of (2.8) and (2.16).

Given  $q \in R$  as in the paragraph preceding (1.2), we define the **even part** algebra  $U_q^{ev}(\mathfrak{g})$  as the base change of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  with respect to  $\sigma: \mathcal{A} \rightarrow R$  of (1.2):

$$(2.22) \quad U_q^{ev}(\mathfrak{g}) := U_{\mathcal{A}}^{ev}(\mathfrak{g}) \otimes_{\mathcal{A}} R.$$

We denote the images of  $\tilde{E}_i, \tilde{F}_i, K^\mu$  in  $U_q^{ev}(\mathfrak{g})$  by the same symbols.

## 2.6. Triangular decompositions and PBW bases for $\mathbf{U}(\mathfrak{g}, P/2), \dot{U}_{\mathcal{A}}(\mathfrak{g}), U_{\mathcal{A}}^{ev}(\mathfrak{g})$ .

Let  $\mathbf{U}_{P/2}^>, \mathbf{U}_{P/2}^<, \mathbf{U}_{P/2}^0$  be the  $\mathbb{Q}(v^{1/2})$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\{E_i\}_{i=1}^r, \{F_i\}_{i=1}^r, \{K^\mu\}_{\mu \in P/2}$ , respectively. Let  $\mathbf{U}_\Phi^>, \mathbf{U}_\Phi^<, \mathbf{U}_\Phi^0$  be the  $\mathbb{Q}(v^{1/2})$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated

by  $\{\tilde{E}_i\}_{i=1}^r, \{\tilde{F}_i\}_{i=1}^r, \{K^\mu\}_{\mu \in P/2}$ , respectively. We note that  $\mathbf{U}_{P/2}^0 = \mathbf{U}_\Phi^0$ . Likewise, the subalgebras generated by  $\mathbf{U}_{P/2}^>, \mathbf{U}_{P/2}^0$  and  $\mathbf{U}_\Phi^>, \mathbf{U}_\Phi^0$  (resp.  $\mathbf{U}_{P/2}^<, \mathbf{U}_{P/2}^0$  and  $\mathbf{U}_\Phi^<, \mathbf{U}_\Phi^0$ ) coincide, and will be denoted  $\mathbf{U}^>$  (resp.  $\mathbf{U}^<$ ). We also note that all these subalgebras are  $Q$ -graded via (2.15).

**Lemma 2.6.** *The following multiplication maps are isomorphisms of  $\mathbb{Q}(v^{1/2})$ -vector spaces:*

$$(2.23) \quad m: \mathbf{U}_{P/2}^< \otimes_{\mathbb{Q}(v^{1/2})} \mathbf{U}_{P/2}^0 \otimes_{\mathbb{Q}(v^{1/2})} \mathbf{U}_{P/2}^> \longrightarrow \mathbf{U}(\mathfrak{g}, P/2),$$

$$(2.24) \quad m: \mathbf{U}_\Phi^< \otimes_{\mathbb{Q}(v^{1/2})} \mathbf{U}_\Phi^0 \otimes_{\mathbb{Q}(v^{1/2})} \mathbf{U}_\Phi^> \longrightarrow \mathbf{U}(\mathfrak{g}, P/2).$$

*Proof.* The proof of (2.23) is the same as that of Lemma 1.1, while (2.24) follows from (2.23).  $\square$

The  $\mathbb{Q}$ -algebra anti-involution  $\tau$  of (1.11) can be extended to a  $\mathbb{Q}$ -algebra anti-involution  $\tau$  of  $\mathbf{U}(\mathfrak{g}, P/2)$  via:

$$(2.25) \quad \tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K^\mu) = K^{-\mu}, \quad \tau(v^{1/2}) = v^{-1/2},$$

for  $1 \leq i \leq r, \mu \in P/2$ .

Since the lattice  $P/2$  is stable under the action of the Weyl group  $W$ , the formulas (1.12) still define a braid group action on  $\mathbf{U}(\mathfrak{g}, P/2)$ . Therefore, we can still define the root vectors  $E_{\beta_k}, F_{\beta_k}$  as in (1.13). However, this braid group action as well as the map  $\tau$  of (2.25) do not preserve  $\dot{U}_{\mathcal{A}}(\mathfrak{g}), U_{\mathcal{A}}^{ev}(\mathfrak{g})$ . We also note that the elements  $E_{\beta_k}, F_{\beta_k}$  are not contained in  $\mathbf{U}_\Phi^>, \mathbf{U}_\Phi^<$ , respectively. To construct the bases of  $\mathbf{U}_\Phi^>, \mathbf{U}_\Phi^<$  we shall first introduce **modified** elements  $\tilde{E}_{\beta_k}, \tilde{F}_{\beta_k}$ . To this end, for any  $\beta = \sum_{i=1}^r a_i \alpha_i \in Q_+$ , define (cf. (2.11)):

$$(2.26) \quad \nu_\beta^> = \sum_{i=1}^r a_i \nu_i^>, \quad \nu_\beta^< = \sum_{i=1}^r a_i \nu_i^<.$$

**Lemma 2.7.** *For any  $\alpha, \beta \in Q_+$ , we have that  $(\nu_\alpha^>, \beta) \pm (\alpha, \nu_\beta^<) \in \mathbb{Z}$ .*

*Proof.* We have  $(\nu_i^>, \alpha_j) - (\alpha_i, \nu_j^<) = (\epsilon_{ij} - 1)(\alpha_i, \alpha_j) \in \mathbb{Z}$ ,  $(\nu_i^>, \alpha_j) + (\alpha_i, \nu_j^<) = -(\alpha_i, \alpha_j) \in \mathbb{Z}$ . The general case of  $\alpha, \beta \in Q_+$  follows immediately now.  $\square$

For the positive root  $\beta_k = \sum_{i=1}^r a_i \alpha_i$  of  $\mathfrak{g}$ , set

$$(2.27) \quad b_{\beta_k}^> = \sum_{i < j} a_i a_j (\nu_i^>, \alpha_j), \quad b_{\beta_k}^< = - \sum_{i < j} a_i a_j (\nu_j^<, \alpha_i),$$

and define

$$(2.28) \quad \tilde{E}_{\beta_k} := v^{b_{\beta_k}^>} E_{\beta_k} K^{\nu_{\beta_k}^>}, \quad \tilde{F}_{\beta_k} := v^{b_{\beta_k}^< } K^{-\nu_{\beta_k}^<} F_{\beta_k}, \quad \tilde{E}_{\beta_k}^{(s)} := \frac{\tilde{E}_{\beta_k}^s}{(s)_{v_{i_k}}!}, \quad \tilde{F}_{\beta_k}^{(s)} := \frac{\tilde{F}_{\beta_k}^s}{(s)_{v_{i_k}}!}.$$

For any  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$  (with  $N = |\Delta_+|$  as before), we define the ordered monomials:

$$\begin{aligned} \tilde{E}^{\vec{k}} &:= \tilde{E}_{\beta_1}^{k_1} \dots \tilde{E}_{\beta_N}^{k_N}, & \tilde{F}^{\vec{k}} &:= \tilde{F}_{\beta_1}^{k_1} \dots \tilde{F}_{\beta_N}^{k_N}, & \tilde{E}^{\vec{k}} &:= \tilde{E}_{\beta_N}^{k_N} \dots \tilde{E}_{\beta_1}^{k_1}, & \tilde{F}^{\vec{k}} &:= \tilde{F}_{\beta_N}^{k_N} \dots \tilde{F}_{\beta_1}^{k_1}, \\ \tilde{E}^{(\vec{k})} &:= \tilde{E}_{\beta_1}^{(k_1)} \dots \tilde{E}_{\beta_N}^{(k_N)}, & \tilde{F}^{(\vec{k})} &:= \tilde{F}_{\beta_1}^{(k_1)} \dots \tilde{F}_{\beta_N}^{(k_N)}, \\ \tilde{E}^{(\vec{k})} &:= \tilde{E}_{\beta_N}^{(k_N)} \dots \tilde{E}_{\beta_1}^{(k_1)}, & \tilde{F}^{(\vec{k})} &:= \tilde{F}_{\beta_N}^{(k_N)} \dots \tilde{F}_{\beta_1}^{(k_1)}. \end{aligned}$$

Let  $\dot{U}_{\mathcal{A}}^<, \dot{U}_{\mathcal{A}}^>, \dot{U}_{\mathcal{A}}^0$  denote the  $\mathcal{A}$ -subalgebras of  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  generated by  $\{\tilde{F}_i^{(s)}\}, \{\tilde{E}_i^{(s)}\}, \{{K_i; a \choose s}\}, K^\mu$ , respectively. Let  $U_{\mathcal{A}}^{ev<}, U_{\mathcal{A}}^{ev>}, U_{\mathcal{A}}^{ev0}$  denote the  $\mathcal{A}$ -subalgebras of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  generated by  $\{\tilde{F}_i\}, \{\tilde{E}_i\}$  and  $\{K^\mu\}$ , respectively.

**Lemma 2.8.** *For all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$ , we have:*

$$\tilde{E}^{\vec{k}}, \tilde{E}^{\vec{k}} \in U_{\mathcal{A}}^{ev>} ; \quad \tilde{F}^{\vec{k}}, \tilde{F}^{\vec{k}} \in U_{\mathcal{A}}^{ev<} ; \quad \tilde{E}^{(\vec{k})}, \tilde{E}^{(\vec{k})} \in \dot{U}_{\mathcal{A}}^>; \quad \tilde{F}^{(\vec{k})}, \tilde{F}^{(\vec{k})} \in \dot{U}_{\mathcal{A}}^<.$$

*Proof.* Show  $\tilde{E}^{\vec{k}} \in U_{\mathcal{A}}^{ev >} :$  Let  $\mathcal{I}_{\beta_k}$  denote the collection of all tuples  $I = (i_1, \dots, i_m)$  with  $1 \leq i_1, \dots, i_m \leq r$  and  $\sum_{p=1}^m \alpha_{i_p} = \beta_k$ . Any tuple  $I$  in  $\mathcal{I}_{\beta_k}$  is obtained from the following tuple  $I_0$  by permuting the indices:  $I_0 = (1, \dots, 1, 2, \dots, 2, \dots, r, \dots, r)$  with the index  $j$  repeated  $a_j$  times. For any  $I \in \mathcal{I}_{\beta_k}$ , let  $c_I = -\sum_{p=1}^{m-1} (\nu_{i_p}^>, \alpha_{i_{p+1}} + \dots + \alpha_{i_m})$ . In particular,  $b_{\beta_k}^> + c_{I_0} = \sum_{i=1}^r \mathsf{d}_i a_i (a_i - 1) \in \mathbb{Z}$ . We now prove that  $c_I - c_{I_0} \in \mathbb{Z}$  for any  $I \in \mathcal{I}_{\beta_k}$ . Since  $I$  can be obtained from  $I_0$  via permuting the indices, it is enough to show that  $c_{I_1} - c_{I_2} \in \mathbb{Z}$  for two tuples  $I_1, I_2 \in \mathcal{I}_{\beta_k}$  obtained from each other by permuting two consecutive indices. To this end, assume that  $I_2$  is obtained from  $I_1$  by permuting indices  $i_p$  and  $i_{p+1}$ . Then  $c_{I_1} - c_{I_2} = (\nu_{i_{p+1}}^>, \alpha_{i_p}) - (\nu_{i_p}^>, \alpha_{i_{p+1}}) = 2\phi_{i_{p+1}, i_p} \in \mathbb{Z}$ . We thus conclude that  $b_{\beta_k}^> + c_I \in \mathbb{Z}$  for any tuple  $I$  in  $\mathcal{I}_{\beta_k}$ .

For any  $I = (i_1, \dots, i_m) \in \mathcal{I}_{\beta_k}$ , let  $E_I = E_{i_1} \dots E_{i_m}$ ,  $\tilde{E}_I = \tilde{E}_{i_1} \dots \tilde{E}_{i_m}$ . Since  $E_{\beta_k}$  are contained in  $U_{\mathcal{A}}^>$  and have degree  $\beta_k$ , we have  $E_{\beta_k} = \sum_{I \in \mathcal{I}_{\beta_k}} p_I(v) E_I$  with  $p_I(v) \in \mathcal{A}$ , and so

$$\tilde{E}_{\beta_k} = v^{b_{\beta_k}^>} E_{\beta_k} K^{\nu_{\beta_k}^>} = \sum_{I \in \mathcal{I}_{\beta_k}} v^{b_{\beta_k}^> + c_I} p_I(v) \tilde{E}_I.$$

Since  $b_{\beta_k}^> + c_I \in \mathbb{Z}$ , it follows that  $\tilde{E}_{\beta_k} \in U_{\mathcal{A}}^{ev >}$ , hence  $\tilde{E}^{\vec{k}} \in U_{\mathcal{A}}^{ev >}$  for any  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$ .

Show  $\tilde{E}^{(\vec{k})} \in \dot{U}_{\mathcal{A}}^>$ : It is enough to show that  $\tilde{E}_{\beta_k}^{(n)} \in \dot{U}_{\mathcal{A}}^>$  for all  $n \in \mathbb{N}$ . We have

$$\tilde{E}_{\beta_k}^{(n)} = v^{(\nu_{\beta_k}^>, \beta_k)(n-1)n/2} v^{nb_{\beta_k}^>} E_{\beta_k}^{(n)} K^{n\nu_{\beta_k}^>}, \quad \text{in which } (\nu_{\beta_k}^>, \beta_k) \in \mathbb{Z}.$$

Now

$$E_{\beta_k}^{(n)} = \sum_{\mathbf{a}=(a_1, \dots, a_l)} p_{\mathbf{a}}(v) E_{i_1}^{(a_1)} \dots E_{i_l}^{(a_l)},$$

for some tuple  $\mathbf{a} = (a_1, \dots, a_l)$  such that  $\sum_{j=1}^l a_j \alpha_{i_j} = n\beta_k$  and  $p_{\mathbf{a}}(v) \in \mathcal{A}$ . Then arguing as in the case of  $\tilde{E}_{\beta_k}$ , one can show that  $v^{nb_{\beta_k}^>} E_{\beta_k}^{(n)} K^{n\nu_{\beta_k}^>} \in \dot{U}_{\mathcal{A}}$ , hence  $\tilde{E}_{\beta_k}^{(n)} \in \dot{U}_{\mathcal{A}}^>$ .

The proofs of the other statements are analogous. □

We are now ready to establish the PBW-bases for  $\mathbf{U}(\mathfrak{g}, P/2), \dot{U}_{\mathcal{A}}(\mathfrak{g}), U_{\mathcal{A}}^{ev}(\mathfrak{g})$ . We start with  $\mathbf{U}(\mathfrak{g}, P/2)$ . We also recall the ordered monomials  $F^{\vec{k}}, E^{\vec{k}}, F^{\vec{k}}, E^{\vec{k}}$  introduced before Lemma 1.3.

**Lemma 2.9.** (a1) The sets  $\{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{E^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}_{P/2}^>$ .

(a2) The sets  $\{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{F^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}_{P/2}^<$ .

(a3) The set  $\{K^\mu\}_{\mu \in P/2}$  is a  $\mathbb{Q}(v^{1/2})$ -basis of  $\mathbf{U}_{P/2}^0$ .

(a4) The sets  $\{F^{\vec{k}} K^\mu E^{\vec{r}}\}, \{F^{\vec{k}} K^\mu E^{\vec{r}}\}, \{F^{\vec{k}} K^\mu E^{\vec{r}}\}, \{F^{\vec{k}} K^\mu E^{\vec{r}}\}$ , with  $\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N$  and  $\mu \in P/2$ , form  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}(\mathfrak{g}, P/2)$ .

(b1) The sets  $\{\tilde{E}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{E}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}_{\Phi}^>$ .

(b2) The sets  $\{\tilde{F}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{F}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}_{\Phi}^<$ .

(b3) The set  $\{K^\mu\}_{\mu \in P/2}$  is a  $\mathbb{Q}(v^{1/2})$ -basis of  $\mathbf{U}_{\Phi}^0$ .

(b4) The sets  $\{\tilde{F}^{\vec{k}} K^\mu \tilde{E}^{\vec{r}}\}, \{\tilde{F}^{\vec{k}} K^\mu \tilde{E}^{\vec{r}}\}, \{\tilde{F}^{\vec{k}} K^\mu \tilde{E}^{\vec{r}}\}, \{\tilde{F}^{\vec{k}} K^\mu \tilde{E}^{\vec{r}}\}$ , with  $\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N$  and  $\mu \in P/2$ , form  $\mathbb{Q}(v^{1/2})$ -bases of  $\mathbf{U}(\mathfrak{g}, P/2)$ .

*Proof.* Parts (a1)–(a4) follow from Lemma 1.3, while parts (b1)–(b4) follow directly from (a1)–(a4). □

Before stating the next result, it will be useful to record some of the relations among Cartan elements (cf. [L4, §2.3]):

$$(2.29) \quad \binom{t+t'}{t}_i \binom{K_i; 0}{t+t'} = \binom{K_i; 0}{t} \sum_{k=0}^{t'} (-1)^k v_i^{2t'(t+k)-k(k+1)} \binom{t+k-1}{k}_i \binom{K_i; 0}{t'-k},$$

$$(2.30) \quad \binom{K_i; -c}{t} = \sum_{0 \leq k \leq t} (-1)^k v_i^{2t(c+k)-k(k+1)} \binom{c+k-1}{k}_i \binom{K_i; 0}{t-k} \quad (c \geq 1),$$

$$(2.31) \quad \binom{K_i; c}{t} = \sum_{0 \leq k \leq t} v_i^{2k(t-k)} \binom{c}{k}_i K_i^{-2k} \binom{K_i; 0}{t-k} \quad (c \geq 0),$$

for all  $t, t' \geq 0$ .

**Lemma 2.10.** (a1) Subalgebras  $\dot{U}_{\mathcal{A}}^<, \dot{U}_{\mathcal{A}}^>, U_{\mathcal{A}}^{ev <}, U_{\mathcal{A}}^{ev >}$  are  $Q$ -graded via (2.15), and each of their degree components is a free  $\mathcal{A}$ -module of finite rank.

(a2) The multiplication maps

$$\begin{aligned} m: \dot{U}_{\mathcal{A}}^< \otimes_{\mathcal{A}} \dot{U}_{\mathcal{A}}^0 \otimes_{\mathcal{A}} \dot{U}_{\mathcal{A}}^> &\longrightarrow \dot{U}_{\mathcal{A}}(\mathfrak{g}), \\ m: U_{\mathcal{A}}^{ev <} \otimes_{\mathcal{A}} U_{\mathcal{A}}^{ev 0} \otimes_{\mathcal{A}} U_{\mathcal{A}}^{ev >} &\longrightarrow U_{\mathcal{A}}^{ev}(\mathfrak{g}) \end{aligned}$$

are isomorphisms of free  $\mathcal{A}$ -modules.

(b1) The sets  $\{\tilde{E}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{E}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $U_{\mathcal{A}}^{ev >}$ .

(b2) The sets  $\{\tilde{F}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{F}^{\vec{k}}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $U_{\mathcal{A}}^{ev <}$ .

(b3) The set  $\{K^\mu\}_{\mu \in 2P}$  is an  $\mathcal{A}$ -basis of  $U_{\mathcal{A}}^{ev 0}$ .

(c1) The sets  $\{\tilde{E}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{E}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $\dot{U}_{\mathcal{A}}^>$ .

(c2) The sets  $\{\tilde{F}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{\tilde{F}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $\dot{U}_{\mathcal{A}}^<$ .

(c3) The subalgebra  $\dot{U}_{\mathcal{A}}^0$  has the following  $\mathcal{A}$ -basis:

$$(2.32) \quad \left\{ K^{2\varsigma_j} \cdot \prod_{i=1}^r \left( K_i^{2\lfloor \frac{t_i}{2} \rfloor} \binom{K_i; 0}{t_i} \right) \mid t_i \geq 0, 1 \leq j \leq k \right\},$$

where  $\varsigma_1, \dots, \varsigma_k \in P$  is a set of representatives of the left cosets  $P/Q$ .

*Proof.* We will present the proof of (b1), (b2), (c1), (c2) after Lemma 3.10. Part (b3) easily follows from part (b3) of Lemma 2.9 that does not depend on this lemma.

Let us sketch the proof of (c3). Let  $\widetilde{\dot{U}_{\mathcal{A}}^0}$  denote the  $\mathcal{A}$ -submodule of  $\dot{U}_{\mathcal{A}}^0$  linearly generated by the elements in (2.32). It is not hard to see that elements in (2.32) form a  $\mathbb{Q}(v^{1/2})$ -basis for the  $\mathbb{Q}(v^{1/2})$ -subalgebra  $\mathbf{U}^0(2P)$  of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\{K^\mu\}_{\mu \in 2P}$ . Hence  $\widetilde{\dot{U}_{\mathcal{A}}^0}$  is a free  $\mathcal{A}$ -module with the  $\mathcal{A}$ -basis (2.32).

*Step 1:* We will show that  $K_i^{2k} \binom{K_i; 0}{m}$  for  $m \in \mathbb{N}, k \in \mathbb{Z}$  are  $\mathcal{A}$ -linear combinations of elements  $K_i^{2\lfloor \frac{t_i}{2} \rfloor} \binom{K_i; 0}{t_i}$ . Hence  $K^\mu, \binom{K_i; a}{m}$  are contained in  $\widetilde{\dot{U}_{\mathcal{A}}^0}$  for all  $\mu \in 2P, a \in \mathbb{Z}, m \in \mathbb{N}$  by using (2.30)–(2.31).

We will prove the statement for the even number  $2m$ , and the proof for odd number  $2m+1$  is the same. The statement holds for  $K_i^{2m} \binom{K_i; 0}{2m}$ . Consider

$$(1 - v_i^{-4m}) K^{2m} \binom{K_i; 0}{2m+1} = K^{2m} \binom{K_i; 0}{m} - v_i^{2m} K_i^{2m-2} \binom{K_i; 0}{2m},$$

hence the statement holds for  $K_i^{2m-2} \binom{K_i;0}{2m}$ . Consider

$$\begin{aligned} \prod_{i=0}^1 (1 - v_i^{-4m-i}) K_i^{2m+2} \binom{K_i;0}{2m+2} &= K_i^{2m+2} \binom{K_i;0}{2m} - (v_i^{4m} + v_i^{4m+2}) K_i^{2m} \binom{K_i;0}{2m} \\ &\quad + v_i^{8m+2} K_i^{2m-2} \binom{K_i;0}{2m}, \end{aligned}$$

hence the statement holds for  $K_i^{2m+2} \binom{K_i;0}{2m}$ . Keep doing this procedure, we can show that the statement holds for  $K_i^{2k} \binom{K_i;0}{2m}$  for all  $k \in \mathbb{Z}$ .

*Step 2:* We will show that  $\tilde{\dot{U}}_{\mathcal{A}}^0$  is stable under the left multiplication by  $K^\mu, \binom{K_i;a}{m}$ . By (2.30)-(2.31), it is enough to show that  $\tilde{\dot{U}}_{\mathcal{A}}^0$  is stable under the left multiplication by  $K_i^{2k} \binom{K_i;0}{m}$ . On the other hand, (2.29) (by induction) implies that  $\binom{K_i;0}{t} \binom{K_i;0}{t'}$  is a  $\mathcal{A}$ -linear combination of  $\binom{K_i;0}{t''}$ . Therefore, after multiplying any element of  $\tilde{\dot{U}}_{\mathcal{A}}^0$  by  $K_i^{2k} \binom{K_i;0}{m}$ , we get a  $\mathcal{A}$ -linear combination of  $\prod_{i=1}^r K_i^{2k_i} \binom{K_i;0}{t_i}$  for some  $k_i \in \mathbb{Z}$ . Using Step 1 again, the latter  $\mathcal{A}$ -linear combination belongs to  $\tilde{\dot{U}}_{\mathcal{A}}^0$ .

Therefore, we have  $\tilde{\dot{U}}_{\mathcal{A}}^0 = \dot{U}_{\mathcal{A}}^0$  and part (c3) follows.

Part (a1) follows from (b1), (b2), (c1), and (c2). Let us prove the first isomorphism in part (a2). One can show that  $\dot{U}_{\mathcal{A}}^< \dot{U}_{\mathcal{A}}^0 \dot{U}_{\mathcal{A}}^> = \dot{U}_{\mathcal{A}}(\mathfrak{g})$ . Indeed,  $1 \in \dot{U}_{\mathcal{A}}^< \dot{U}_{\mathcal{A}}^0 \dot{U}_{\mathcal{A}}^>$  while  $\dot{U}_{\mathcal{A}}^< \dot{U}_{\mathcal{A}}^0 \dot{U}_{\mathcal{A}}^>$  is closed under the left multiplications by  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  as can be seen by using the commutation relations between  $\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\mu, \binom{K_i;0}{m}$ .

Let  $\mathbf{U}(\mathfrak{g}, 2P)$  denote the  $\mathbb{Q}(v^{1/2})$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\{\tilde{E}_i, \tilde{F}_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in 2P}$ . By Lemma 2.9 and the proof of (c3), the algebra  $\mathbf{U}(\mathfrak{g}, 2P)$  has the following  $\mathbb{Q}(v^{1/2})$ -basis:

$$(2.33) \quad \tilde{F}^{\vec{k}} K^{2\zeta_j} \prod_{i=1}^r \left( K_i^{2\lfloor \frac{t_i}{2} \rfloor} \binom{K_i;0}{t_i} \right) \tilde{E}^{\vec{r}},$$

with the suitable indices as above. Combining this with (c1)–(c3), we see that  $\dot{U}_{\mathcal{A}}^< \dot{U}_{\mathcal{A}}^0 \dot{U}_{\mathcal{A}}^>$  is a free  $\mathcal{A}$ -module with the  $\mathcal{A}$ -basis (2.33), hence the first isomorphism of part (a) follows. The proof of the second isomorphism is similar.  $\square$

## 2.7. The adjoint action of $\dot{U}_{\mathcal{A}}(\mathfrak{g})$ on $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ .

For the twisted Hopf algebra structure in (2.8), we have the left adjoint action  $\text{ad}'$  of  $\mathbf{U}(\mathfrak{g}, P/2)$  on itself. Explicitly, for any  $1 \leq i \leq r, \mu \in P/2, x \in \mathbf{U}(\mathfrak{g}, P/2)$ , we have:

$$(2.34) \quad \text{ad}'(K^\mu)(x) = K^\mu x K^{-\mu}, \quad \text{ad}'(\tilde{E}_i)(x) = [\tilde{E}_i, x] K^{\zeta_i^>} , \quad \text{ad}'(\tilde{F}_i)(x) = [\tilde{F}_i, x] K^{-\zeta_i^<} ,$$

due to (2.16). It turns out that this action restricts to an action  $\dot{U}_{\mathcal{A}}(\mathfrak{g}) \curvearrowright U_{\mathcal{A}}^{ev}(\mathfrak{g})$ :

**Proposition 2.11.** *The adjoint action  $\text{ad}'$  of  $\dot{U}_{\mathcal{A}}(\mathfrak{g}) \subset \mathbf{U}(\mathfrak{g}, P/2)$  preserves the even part  $U_{\mathcal{A}}^{ev}(\mathfrak{g}) \subset \mathbf{U}(\mathfrak{g}, P/2)$ , thus giving rise to*

$$(2.35) \quad \text{ad}' : \dot{U}_{\mathcal{A}}(\mathfrak{g}) \curvearrowright U_{\mathcal{A}}^{ev}(\mathfrak{g}).$$

Given  $q \in R$  as in the paragraph preceding (1.2), we obtain

$$(2.36) \quad (\text{left}) \text{ adjoint action } \text{ad}' : \dot{U}_q(\mathfrak{g}) \curvearrowright U_q^{ev}(\mathfrak{g})$$

after applying the base change  $\sigma : \mathcal{A} \rightarrow R$  of (1.2) to (2.35).

---

<sup>3</sup>In contrast, while  $\text{ad}(F_i)(x) = [F_i, x] K_i$ , there is no similar formula relating  $\text{ad}(E_i)(x)$  to  $[E_i, x]$ .

*Proof of Proposition 2.11.* Due to property (2.3) and the explicit coproduct formulas in (2.16) and (2.20), it suffices to verify the claim for the generators of  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  acting on the generators of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ . To this end, we first note that  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  is obviously stable under  $\{\text{ad}'(K^\beta) \mid \beta \in 2P\}$ .

Next, let us consider  $\text{ad}'(\tilde{E}_i^{(m)})$ . First, we apply  $\text{ad}'(\tilde{E}_i^{(m)})$  to  $\{K^\beta\}_{\beta \in 2P}$ . To this end, recall:

$$\text{ad}'(\tilde{E}_i)(x) = [\tilde{E}_i, x] K^{\zeta_i^>}$$

with  $\zeta_i^> \in 2P$  by Lemma 2.4. Combining this with  $[\tilde{E}_i, \tilde{E}_i^s K^\beta] = (1 - v^{(\beta, \alpha_i)}) \tilde{E}_i^{s+1} K^\beta$ , we obtain:  $\text{ad}'(\tilde{E}_i)(\tilde{E}_i^s K^\beta) = (1 - v^{(\beta, \alpha_i)}) \tilde{E}_i^{s+1} K^{\beta + \zeta_i^>}$ . As  $(\zeta_i^>, \alpha_i) = (\alpha_i, \alpha_i) = 2d_i$ , we finally get:

$$(2.37) \quad \text{ad}'(\tilde{E}_i^m)(K^\beta) = \prod_{s=0}^{m-1} \left(1 - v^{(\beta, \alpha_i) + 2s d_i}\right) \cdot \tilde{E}_i^m K^{\beta + m \zeta_i^>}.$$

Note that  $(\beta, \alpha_i) \in 2d_i \mathbb{Z}$  for  $\beta \in 2P$ , so that:

$$(2.38) \quad \frac{\prod_{s=0}^{m-1} (1 - v^{(\beta, \alpha_i) + 2s d_i})}{(m)_{v_i}!} \in \mathcal{A}.$$

Hence, we get the following equality in  $\mathbf{U}(\mathfrak{g})$ :

$$(2.39) \quad \text{ad}'(\tilde{E}_i^{(m)})(K^\beta) = \frac{\prod_{s=0}^{m-1} (1 - v^{(\beta, \alpha_i) + 2s d_i})}{(m)_{v_i}!} \cdot \tilde{E}_i^m K^{\beta + m \zeta_i^>} ,$$

the right-hand side of which lies in  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ , due to (2.38).

Next, we apply  $\text{ad}'(\tilde{E}_i)$  to  $\tilde{E}_j K^\beta$  with a suitably chosen  $\beta \in 2P$ :

- For  $j = i$ , we have  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_i) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$ , due to:

$$\text{ad}'(\tilde{E}_i)(\tilde{E}_i) = [\tilde{E}_i, \tilde{E}_i] K^{\zeta_i^>} = 0 \implies \text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_i) = 0 \text{ for } m \geq 1 .$$

- For  $j \neq i$ , set  $\beta = \zeta_j^> \in 2P$ . Evoking (2.20), we thus obtain:

$$\text{ad}'(\tilde{E}_i^{1-a_{ij}})(\tilde{E}_j K^\beta) = \left( \sum_{m=0}^{1-a_{ij}} (-1)^m v^{m \epsilon_{ij} b_{ij}} \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_{v_i} \tilde{E}_i^{1-a_{ij}-m} \tilde{E}_j \tilde{E}_i^m \right) K^{\beta + (1-a_{ij}) \zeta_i^>} = 0 ,$$

with the last equality due to (2.14). Therefore, we have:

$$\text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_j K^\beta) = 0 \in U_{\mathcal{A}}^{ev}(\mathfrak{g}) \quad \text{for } m > -a_{ij} .$$

We also have:

$$\text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_j K^\beta) = \frac{1}{(m)_{v_i}!} \text{ad}'(\tilde{E}_i^m)(\tilde{E}_j K^\beta) \in U_{\mathcal{A}}^{ev}(\mathfrak{g}) \quad \text{for } 1 \leq m \leq -a_{ij} ,$$

as  $1/(m)_{v_i}! \in \mathcal{A}$  (here, we use that if  $a_{ij} < 0$ , then either  $a_{ij} = -1$  or  $a_{ji} = -1$ ).

Combining the inclusions  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_j K^\beta) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$  established above for all  $m$ , formulas (2.3, 2.20), and the inclusion  $\text{ad}'(\tilde{E}_i^{(m)})(K^{-\beta}) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$  (also verified above), we conclude that  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{E}_j)$  is indeed an element of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  for any  $m \geq 1$ .

Finally, we apply  $\text{ad}'(\tilde{E}_i^{(m)})$  to  $\tilde{F}_j K^\beta$  with  $\beta := -\zeta_j^< \in 2P$ :

- For  $j \neq i$ , we have

$$\text{ad}'(\tilde{E}_i)(\tilde{F}_j K^\beta) = \left( \tilde{E}_i \tilde{F}_j - v^{(\alpha_i, -\zeta_j^<)} \tilde{F}_j \tilde{E}_i \right) K^{\beta + \zeta_i^>} = 0 ,$$

with the last equality due to the third line of (2.14). Thus, we get  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{F}_j K^\beta) = 0$  for any  $m \geq 1$ . Combining this with the inclusion  $\text{ad}'(\tilde{E}_i^{(m)})(K^{-\beta}) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$  verified above and formulas (2.3, 2.20), we obtain the desired inclusion  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{F}_j) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$  for any  $m \geq 1$ .

- For  $j = i$ , we have

$$\begin{aligned} \text{ad}'(\tilde{E}_i)(\tilde{F}_i K^\beta) &= [\tilde{E}_i, \tilde{F}_i K^{-\zeta_i^<}] K^{\zeta_i^>} = (\tilde{E}_i \tilde{F}_i - v_i^2 \tilde{F}_i \tilde{E}_i) K^{\zeta_i^> - \zeta_i^<} \\ &= v_i \frac{1 - K^{-2\alpha_i}}{1 - v_i^2} K^{2\alpha_i} = \frac{v_i}{1 - v_i^{-2}} (K^{2\alpha_i} - 1) \in U_{\mathcal{A}}^{ev}(\mathfrak{g}), \end{aligned}$$

due to the third line of (2.14). Combining this with (2.37), we obtain for  $m \geq 2$ :

$$\text{ad}'(\tilde{E}_i^m)(\tilde{F}_i K^\beta) = \frac{v_i}{1 - v_i^{-2}} \prod_{s=0}^{m-2} (1 - v^{4d_i + 2sd_i}) \tilde{E}_i^{m-1} K^{2\alpha_i + (m-1)\zeta_i^>}.$$

Therefore, we have:

$$(2.40) \quad \text{ad}'(\tilde{E}_i^{(m)})(\tilde{F}_i K^\beta) \in U_{\mathcal{A}}^{ev}(\mathfrak{g}) \quad \text{for } m \geq 1,$$

due to the obvious inclusion

$$\frac{v_i \prod_{s=0}^{m-2} (1 - v^{4d_i + 2sd_i})}{(1 - v_i^{-2})(m)_{v_i}!} \in \mathcal{A} \quad \text{for } m \geq 2.$$

Combining (2.40), formulas (2.3, 2.20), and the inclusion  $\text{ad}'(\tilde{E}_i^{(m)})(K^{-\beta}) \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$  verified above, we conclude that  $\text{ad}'(\tilde{E}_i^{(m)})(\tilde{F}_i)$  is indeed an element of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  for any  $m \geq 1$ .

The case of  $\text{ad}'(\tilde{F}_i^{(m)})$  is treated analogously; we leave details to the interested reader.  $\square$

### 3. HOPF AND INVARIANT PAIRINGS

Let  $\mathbf{U}^>, \mathbf{U}^<$  denote the  $\mathbb{Q}(v^{1/2})$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\{E_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in P/2}$  and  $\{F_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in P/2}$ , respectively. We note that both  $\mathbf{U}^>$  and  $\mathbf{U}^<$  are Hopf  $\mathbb{Q}(v^{1/2})$ -subalgebras of  $\mathbf{U}(\mathfrak{g}, P/2)$  endowed with either the standard Hopf algebra structure  $(\Delta, S, \varepsilon)$  of (1.10) or the twisted one  $(\Delta', S', \varepsilon')$  of (2.8). Let  $U_{\mathcal{A}}^{ev>} \text{ and } U_{\mathcal{A}}^{ev<}$  (respectively,  $\dot{U}_{\mathcal{A}}^>$  and  $\dot{U}_{\mathcal{A}}^<$ ) denote the  $\mathcal{A}$ -subalgebras of  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  (respectively, of  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$ ) generated by  $U_{\mathcal{A}}^{ev>}, U_{\mathcal{A}}^{ev0}$  and  $U_{\mathcal{A}}^{ev<}, U_{\mathcal{A}}^{ev0}$  (respectively,  $\dot{U}_{\mathcal{A}}^>, \dot{U}_{\mathcal{A}}^0$  and  $\dot{U}_{\mathcal{A}}^<, \dot{U}_{\mathcal{A}}^0$ ). Likewise, define the  $R$ -algebras  $U_q^{ev>}, U_q^{ev0}$  and  $U_q^{ev<}, U_q^>$ .

The goal of this section is to generalize the Hopf pairing of [J1, §6.12] and the invariant pairing of [J1, §6.20]<sup>4</sup> to our twisted setup and then construct similar pairings (now involving both the modified Lusztig form and the even part algebra):

$$\begin{aligned} (\ , \ )' : U_q^{ev<} \times U_q^{ev>} &\longrightarrow R, \quad (\ , \ )' : U_q^{ev<} \times \dot{U}_q^> \longrightarrow R, \\ \langle \ , \ \rangle' : U_q^{ev}(\mathfrak{g}) \times \dot{U}_q(\mathfrak{g}) &\longrightarrow R. \end{aligned}$$

#### 3.1. Two endomorphisms of $\mathfrak{h}^*$ .

Let us recall the skew-symmetric matrix  $\Phi = (\phi_{ij})_{i,j=1}^r$  from Section 2.3. It gives rise to two endomorphisms  $\kappa, \gamma \in \text{End}(\mathfrak{h}^*)$  defined in the basis  $\{\alpha_i\}_{i=1}^r$  via:

$$(3.1) \quad \kappa(\alpha_i) := \alpha_i + \sum_{j=1}^r 2\phi_{ij}\omega_j^\vee = -\zeta_i^< \quad \text{and} \quad \gamma(\alpha_i) := \alpha_i - \sum_{j=1}^r 2\phi_{ij}\omega_j^\vee = \zeta_i^>,$$

where we use the notations (2.13). These endomorphisms are adjoint to each other:

$$(3.2) \quad (\kappa(\alpha), \beta) = (\alpha, \gamma(\beta)) \quad \text{for any } \alpha, \beta \in \mathfrak{h}^*,$$

with respect to the pairing  $( \ , \ )$  on  $\mathfrak{h}^*$  that is naturally induced from the one on  $\mathfrak{h}$ .

**Lemma 3.1.** *We have  $\kappa(Q) = \gamma(Q) = 2P$ . In particular,  $\kappa$  and  $\gamma$  are invertible.*

<sup>4</sup>As noted in [J1, §6.18], this pairing slightly differs from the one in [L5, §3.1.7-8].

*Proof.* Since the Dynkin diagram of  $\mathfrak{g}$  has no cycles, we can reorder  $\{\alpha_i\}_{i=1}^r$  (respectively  $\{\omega_i\}_{i=1}^r$ ) so that  $\epsilon_{ij} = -1$  for  $i > j$  with  $(\alpha_i, \alpha_j) \neq 0$ . Then, evoking the symmetrized Cartan matrix  $B$  of (1.1), we get:

$$\begin{pmatrix} \kappa(\alpha_1) \\ \vdots \\ \kappa(\alpha_r) \end{pmatrix} = (B + 2\Phi) \cdot \begin{pmatrix} \omega_1^\vee \\ \vdots \\ \omega_r^\vee \end{pmatrix} = K \cdot \text{diag}(2\mathbf{d}_1, \dots, 2\mathbf{d}_r) \cdot \begin{pmatrix} \omega_1^\vee \\ \vdots \\ \omega_r^\vee \end{pmatrix} = K \cdot \begin{pmatrix} 2\omega_1 \\ \vdots \\ 2\omega_r \end{pmatrix}.$$

Here,  $K = (k_{ij})_{i,j=1}^r$  is an upper-triangular matrix with  $k_{ii} = 1$  and  $k_{ij} = a_{ij}$ ,  $i < j$ , which thus maps the lattice  $2P = \bigoplus_{i=1}^r \mathbb{Z} \cdot 2\omega_i$  bijectively to itself. This completes the proof of  $\kappa(Q) = 2P$ .

The proof of  $\gamma(Q) = 2P$  is completely analogous.  $\square$

### 3.2. Twisted Hopf pairing.

Let us pick an even  $\mathsf{N} \in \mathbb{N}$  such that  $\frac{1}{2}(P/2, P/2) \in \frac{1}{\mathsf{N}}\mathbb{Z}$  and  $(\kappa^{-1}(P/2), P/2) \in \frac{1}{\mathsf{N}}\mathbb{Z}$ . The following result is just an extension of [J1, §6.12]:

**Proposition 3.2.** *There exists a unique  $\mathbb{Q}(v^{1/2})$ -bilinear pairing*

$$(3.3) \quad (\ , \ ): \mathbf{U}^\leq \times \mathbf{U}^\geq \longrightarrow \mathbb{Q}(v^{1/\mathsf{N}})$$

such that

$$(3.4) \quad (y, xx') = (\Delta(y), x' \otimes x), \quad (yy', x) = (y \otimes y', \Delta(x)),$$

$$(3.5) \quad (F_i, E_j) = -\frac{\delta_{i,j}}{v_i - v_i^{-1}}, \quad (K^\mu, K^{\mu'}) = v^{-(\mu, \mu')}, \quad (F_i, K^{\mu'}) = (K^\mu, E_i) = 0,$$

for any  $1 \leq i, j \leq r$ ,  $\mu, \mu' \in P/2$ ,  $x, x' \in \mathbf{U}^\geq$ , and  $y, y' \in \mathbf{U}^\leq$ .

The pairing (3.3) involves the standard Hopf structure on  $\mathbf{U}(\mathfrak{g}, P/2)$ . The following result provides an appropriate generalization of the above proposition to our twisted setup:

**Proposition 3.3.** *There exists a unique  $\mathbb{Q}(v^{1/2})$ -bilinear pairing*

$$(3.6) \quad (\ , \ )': \mathbf{U}^\leq \times \mathbf{U}^\geq \longrightarrow \mathbb{Q}(v^{1/\mathsf{N}})$$

such that

$$(3.7) \quad (y, xx')' = (\Delta'(y), x' \otimes x)', \quad (yy', x)' = (y \otimes y', \Delta'(x))',$$

$$(3.8) \quad (\tilde{F}_i, \tilde{E}_j)' = -\frac{\delta_{i,j}}{v_i - v_i^{-1}}, \quad (K^\mu, K^{\mu'})' = v^{-(\kappa^{-1}(\mu), \mu')}, \quad (\tilde{F}_i, K^{\mu'})' = (K^\mu, \tilde{E}_i)' = 0,$$

for any  $1 \leq i, j \leq r$ ,  $\mu, \mu' \in P/2$ ,  $x, x' \in \mathbf{U}^\geq$ , and  $y, y' \in \mathbf{U}^\leq$ .

*Proof.* Follows by applying a line-to-line reasoning as in [J1, §6.8–6.12].  $\square$

**Remark 3.4.** We have the following equalities:

$$(3.9) \quad (\mathbf{U}_{-\nu}^\leq, \mathbf{U}_\mu^\geq) = 0, \quad (\mathbf{U}_{-\nu}^\leq, \mathbf{U}_\mu^\geq)' = 0 \quad \text{for } \nu \neq \mu,$$

$$(3.10) \quad (yK^\lambda, xK^{\lambda'}) = (y, x)v^{-(\lambda, \lambda')} \quad \text{for } y \in \mathbf{U}_{P/2}^<, x \in \mathbf{U}_{P/2}^>, \lambda, \lambda' \in P/2,$$

$$(3.11) \quad (\tilde{y}K^\lambda, \tilde{x}K^{\lambda'})' = (\tilde{y}, \tilde{x})'v^{(\lambda, \deg(\tilde{x})) - (\kappa^{-1}(\lambda), \lambda')} \quad \text{for } \tilde{y} \in \mathbf{U}_\Phi^\leq, \tilde{x} \in \mathbf{U}_\Phi^>, \lambda, \lambda' \in P/2,$$

where we write  $\deg(\tilde{x})$  for the weight of  $\tilde{x}$ .

**Remark 3.5.** (a) Given two Hopf algebras  $(A, \Delta_A, S_A, \varepsilon_A)$  and  $(B, \Delta_B, S_B, \varepsilon_B)$  over a commutative ring  $R$ , an  $R$ -bilinear pairing  $(\cdot, \cdot): A \times B \rightarrow R$  is called a **Hopf pairing** if it satisfies:

$$(3.12) \quad \begin{aligned} (y, xx') &= (\Delta_A(y), x' \otimes x) = (y_{(1)}, x')(y_{(2)}, x), \\ (yy', x) &= (y \otimes y', \Delta_B(x)) = (y, x_{(1)})(y', x_{(2)}), \end{aligned}$$

$$(3.13) \quad (S_A(y), S_B(x)) = (y, x),$$

$$(3.14) \quad (y, 1_B) = \varepsilon_A(y), \quad (1_A, x) = \varepsilon_B(x),$$

for any  $y, y' \in A$  and  $x, x' \in B$ , where we use Sweedler notation (2.2) for the coproducts.

(b) The pairing  $(\cdot, \cdot)'$  of (3.6) is actually a Hopf pairing, that is, it satisfies not only (3.12), but also (3.13) and (3.14). While the latter is obvious (as it suffices to verify it on the generators, due to (3.12)), to see the former, we note that the pairing  $(\cdot, \cdot)^{\text{op}}: \mathbf{U}^{\leqslant} \times \mathbf{U}^{\geqslant} \rightarrow \mathbb{Q}(v^{1/N})$  defined via  $(y, x)^{\text{op}} := (S'(y), S'(x))'$  satisfies both properties (3.7, 3.8), and thus coincides with  $(\cdot, \cdot)'$ .

**Remark 3.6.** Proposition 3.3 can be similarly stated for a more general skew-symmetric matrix  $\Phi$  and Hopf algebra  $\mathbf{U}(\mathfrak{g}, X)$  as long as the linear maps  $\kappa, \gamma$  are invertible. To simplify the exposition, we decided not to present that generality.

**Corollary 3.7.** For any  $1 \leq i \leq r$  and  $n \in \mathbb{N}$ , we have:

$$(\tilde{F}_i^n, \tilde{E}_i^n)' = (-1)^n v_i^{-n} \frac{(n)_{v_i}!}{(1 - v_i^{-2})^n}.$$

*Proof.* Due to (3.7) and (2.20), we have:

$$\begin{aligned} (\tilde{F}_i^n, \tilde{E}_i^n)' &= (\tilde{F}_i^{n-1} \otimes \tilde{F}_i, \Delta'(\tilde{E}_i^n))' = (n)_{v_i} \cdot (\tilde{F}_i^{n-1}, \tilde{E}_i^{n-1})' (\tilde{F}_i, \tilde{E}_i K^{-(n-1)\zeta_i^>})' = \\ &\quad \frac{-(n)_{v_i}}{v_i - v_i^{-1}} (\tilde{F}_i^{n-1}, \tilde{E}_i^{n-1})'. \end{aligned}$$

The result now follows by induction on  $n$ .  $\square$

The next proposition relates the pairings (3.3) and (3.6). Consider the linear endomorphisms  $\varpi^{\leqslant}$  of  $\mathbf{U}^{\leqslant}$  and  $\varpi^{\geqslant}$  of  $\mathbf{U}^{\geqslant}$  defined via:

$$(3.15) \quad \varpi^{\leqslant}: F_{i_1} \cdots F_{i_n} K^\mu \mapsto K^{-\nu_{i_1}^< - \cdots - \nu_{i_n}^<} F_{i_1} \cdots F_{i_n} K^{\kappa(\mu)},$$

$$(3.16) \quad \varpi^{\geqslant}: E_{j_n} \cdots E_{j_1} K^\nu \mapsto E_{j_n} \cdots E_{j_1} K^{\nu - \nu_{j_n}^> - \cdots - \nu_{j_1}^>},$$

for any  $n \geq 0$ ,  $1 \leq i_1, \dots, i_n, j_1, \dots, j_n \leq r$ ,  $\mu, \nu \in P/2$ , with  $\nu_j^< = \sum_{p=1}^r \phi_{jp} \omega_p^\vee$  of (2.11),  $\kappa \in \text{End}(\mathfrak{h}^*)$  of (3.1). These maps are well-defined as they preserve the  $v$ -Serre relations (1.6, 1.7).

**Proposition 3.8.** For any  $y \in \mathbf{U}^{\leqslant}$  and  $x \in \mathbf{U}^{\geqslant}$ , we have:

$$(y, x) = (\varpi^{\leqslant}(y), \varpi^{\geqslant}(x))'.$$

*Proof.* Let us first evaluate the pairing  $(y, x)$  for any  $y = F_{i_1} \cdots F_{i_n} \in \mathbf{U}_{P/2}^<$  and  $x = E_{j_n} \cdots E_{j_1} \in \mathbf{U}_{P/2}^>$ . Applying  $n-1$  times the first equality of (3.4), we obtain:

$$(y, x) = (F_{i_1} \cdots F_{i_n}, E_{j_n} \cdots E_{j_1}) = (\Delta^{(n-1)}(F_{i_1}) \cdots \Delta^{(n-1)}(F_{i_n}), E_{j_1} \otimes \cdots \otimes E_{j_n}).$$

Here, the  $(n-1)$ -fold coproduct  $\Delta^{(n-1)}(F_i) \in \mathbf{U}^{\leqslant \otimes n}$  is explicitly given by:

$$\Delta^{(n-1)}(F_i) = \sum_{a=1}^n \underbrace{1 \otimes \cdots \otimes 1}_{a-1 \text{ times}} \otimes F_i \otimes \underbrace{K^{-\alpha_i} \otimes \cdots \otimes K^{-\alpha_i}}_{n-a \text{ times}}.$$

Therefore, we have:

$$\begin{aligned}\Delta^{(n-1)}(F_{i_1}) \cdots \Delta^{(n-1)}(F_{i_n}) &= \sum_{\sigma \in S_n} \overrightarrow{\bigotimes}_{1 \leq l \leq n} \prod_{m < l}^{\sigma(m) < \sigma(l)} K^{-\alpha_{i_{\sigma(m)}}} \cdot F_{i_{\sigma(l)}} \cdot \prod_{m < l}^{\sigma(m) > \sigma(l)} K^{-\alpha_{i_{\sigma(m)}}} + (\dots) \\ &= \sum_{\sigma \in S_n} \overrightarrow{\bigotimes}_{1 \leq l \leq n} v^{\sum_{1 \leq m < l}^{\sigma(m) < \sigma(l)} (\alpha_{i_{\sigma(m)}}, \alpha_{i_{\sigma(l)}})} \cdot F_{i_{\sigma(l)}} \cdot \prod_{m < l} K^{-\alpha_{i_{\sigma(m)}}} + (\dots).\end{aligned}$$

Here,  $\overrightarrow{\bigotimes}_{1 \leq l \leq n}$  denotes the ordered tensor product (that is,  $\overrightarrow{\bigotimes}_{1 \leq l \leq n} x_i = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ ), while  $(\dots)$  denotes all other terms with at least one tensorand being of degree zero, which thus have a trivial  $(\ , \ )$ -pairing with  $E_{j_1} \otimes \cdots \otimes E_{j_n}$  by (3.9). As  $(F_i K^\mu, E_j) = (F_i, E_j)$ , we thus get:

$$(3.17) \quad (F_{i_1} \cdots F_{i_n}, E_{j_n} \cdots E_{j_1}) = \sum_{\sigma \in S_n} v^{\sum_{1 \leq m < l \leq n}^{\sigma(m) < \sigma(l)} (\alpha_{i_{\sigma(m)}}, \alpha_{i_{\sigma(l)}})} \cdot (F_{i_{\sigma(1)}}, E_{j_1}) \cdots (F_{i_{\sigma(n)}}, E_{j_n}).$$

Let us now similarly compute the twisted Hopf pairing  $(\varpi^{\leq}(y), \varpi^{\geq}(x))'$  for  $x$  and  $y$  as above. Rewriting (3.15) as  $\varpi^{\leq}(F_{i_1} \cdots F_{i_n}) = v^{\sum_{1 \leq m < l \leq n} \phi_{i_l i_m}} \tilde{F}_{i_1} \cdots \tilde{F}_{i_n}$ , we obtain:

$$\begin{aligned}(\varpi^{\leq}(y), \varpi^{\geq}(x))' &= \\ &v^{\sum_{1 \leq m < l \leq n} \phi_{i_l i_m}} (\Delta'^{(n)}(\tilde{F}_{i_1}) \cdots \Delta'^{(n)}(\tilde{F}_{i_n}), K^{-\nu_{j_1}^{\leq} - \cdots - \nu_{j_n}^{\leq}} \otimes E_{j_1} \otimes \cdots \otimes E_{j_n})'.\end{aligned}$$

Here, the  $n$ -fold coproduct  $\Delta'^{(n)}(\tilde{F}_i) \in \mathbf{U}^{\leq \otimes (n+1)}$  is explicitly given by:

$$\begin{aligned}\Delta'^{(n)}(\tilde{F}_i) &= \sum_{a=1}^{n+1} \underbrace{1 \otimes \cdots \otimes 1}_{a-1 \text{ times}} \otimes \tilde{F}_i \otimes \underbrace{K^{\zeta_i^{\leq}} \otimes \cdots \otimes K^{\zeta_i^{\leq}}}_{n+1-a \text{ times}} \\ &= \sum_{a=1}^{n+1} \underbrace{1 \otimes \cdots \otimes 1}_{a-1 \text{ times}} \otimes \tilde{F}_i \otimes \underbrace{K^{-\kappa(\alpha_i)} \otimes \cdots \otimes K^{-\kappa(\alpha_i)}}_{n+1-a \text{ times}}.\end{aligned}$$

Therefore, we have:

$$\begin{aligned}\Delta'^{(n)}(\tilde{F}_{i_1}) \cdots \Delta'^{(n)}(\tilde{F}_{i_n}) &= 1 \otimes \sum_{\sigma \in S_n} \overrightarrow{\bigotimes}_{1 \leq l \leq n} \prod_{m < l}^{\sigma(m) < \sigma(l)} K^{-\kappa(\alpha_{i_{\sigma(m)}})} \cdot \tilde{F}_{i_{\sigma(l)}} \cdot \prod_{m < l}^{\sigma(m) > \sigma(l)} K^{-\kappa(\alpha_{i_{\sigma(m)}})} + (\dots) \\ &= 1 \otimes \sum_{\sigma \in S_n} \overrightarrow{\bigotimes}_{1 \leq l \leq n} v^{\sum_{m < l}^{\sigma(m) < \sigma(l)} (\kappa(\alpha_{i_{\sigma(m)}}, \alpha_{i_{\sigma(l)}})} \cdot \tilde{F}_{i_{\sigma(l)}} \cdot \prod_{m < l} K^{-\kappa(\alpha_{i_{\sigma(m)}})} + (\dots),\end{aligned}$$

where  $(\dots)$  denotes all other terms which, for degree reasons (3.9), have a zero pairing with  $K^{-\nu_{j_1}^{\leq} - \cdots - \nu_{j_n}^{\leq}} \otimes E_{j_1} \otimes \cdots \otimes E_{j_n}$  via  $(\ , \ )'$ . Due to the second equality of (3.7), we also have:

$$(\tilde{F}_i K^{-\kappa(\mu)}, E_j)' = (\tilde{F}_i \otimes K^{-\kappa(\mu)}, E_j \otimes K^{\sum_{p=1}^r \phi_{j_p} \omega_p^{\vee}})' = v^{\sum_{p=1}^r \phi_{j_p}(\mu, \omega_p^{\vee})} (\tilde{F}_i, E_j)'.$$

Combining all the above, we thus obtain:

$$\begin{aligned} (\varpi^{\leq}(F_{i_1} \cdots F_{i_n}), \varpi^{\geq}(E_{j_n} \cdots E_{j_1}))' = \\ \sum_{\sigma \in S_n} \left( v^{\sum_{1 \leq m < l \leq n}^{\sigma(m) < \sigma(l)} (\alpha_{i_{\sigma(m)}}, \alpha_{i_{\sigma(l)}})} \cdot (\tilde{F}_{i_{\sigma(1)}}, E_{j_1})' \cdots (\tilde{F}_{i_{\sigma(n)}}, E_{j_n})' \times \right. \\ \left. v^{\sum_{m < l} \phi_{i_l i_m} + \sum_{m < l} \phi_{i_{\sigma(l)} i_{\sigma(m)}} + \sum_{m < l}^{\sigma(m) < \sigma(l)} 2\phi_{i_{\sigma(m)} i_{\sigma(l)}}} \right). \end{aligned}$$

But the exponent of  $v$  in the last line above vanishes due to  $\phi_{ab} = -\phi_{ba}$ , so that:

$$\begin{aligned} (\varpi^{\leq}(F_{i_1} \cdots F_{i_n}), \varpi^{\geq}(E_{j_n} \cdots E_{j_1}))' = \\ (3.18) \quad \sum_{\sigma \in S_n} v^{\sum_{1 \leq m < l \leq n}^{\sigma(m) < \sigma(l)} (\alpha_{i_{\sigma(m)}}, \alpha_{i_{\sigma(l)}})} \cdot (\tilde{F}_{i_{\sigma(1)}}, E_{j_1})' \cdots (\tilde{F}_{i_{\sigma(n)}}, E_{j_n})'. \end{aligned}$$

Comparing (3.17, 3.18), evoking (3.9) and  $(F_i, E_j) = -\frac{\delta_{i,j}}{v_i - v_i^{-1}} = (\tilde{F}_i, E_j)'$ , we obtain:

$$(y, x) = (\varpi^{\leq}(y), \varpi^{\geq}(x))' \quad \text{for any } y \in \mathbf{U}_{P/2}^<, x \in \mathbf{U}_{P/2}^>.$$

It remains only to incorporate the Cartan part. For any homogeneous  $y \in \mathbf{U}_{P/2}^<, x \in \mathbf{U}_{P/2}^>$ :

$$(y K^\mu, x K^\nu) = v^{-(\mu, \nu)} \cdot (y, x),$$

in accordance with (3.10). We also note that  $\hat{y} := \varpi^{\leq}(y) = \varpi^{\leq}(y K^\mu) K^{-\kappa(\mu)}$  lies in  $\mathbf{U}_\Phi^<$ , while  $\hat{x} := \varpi^{\geq}(x) K^{-\gamma(\deg(x))} = \varpi^{\geq}(x K^\nu) K^{-\gamma(\deg(x)) - \nu}$  lies in  $\mathbf{U}_\Phi^>$ , so that by (3.11) we get:

$$(\varpi^{\leq}(y K^\mu), \varpi^{\geq}(x K^\nu))' = v^{-(\mu, \nu)} \cdot (\hat{y}, \hat{x})' = v^{-(\mu, \nu)} \cdot (\hat{y}, \hat{x} K^{\gamma(\deg(x))})' = v^{-(\mu, \nu)} \cdot (\varpi^{\leq}(y), \varpi^{\geq}(x))'.$$

Combining the above two equalities with the earlier part of the proof, we obtain

$$(\varpi^{\leq}(y K^\mu), \varpi^{\geq}(x K^\nu))' = v^{-(\mu, \nu)} \cdot (\varpi^{\leq}(y), \varpi^{\geq}(x))' = v^{-(\mu, \nu)} \cdot (y, x) = (y K^\mu, x K^\nu).$$

This completes our proof of Proposition 3.8.  $\square$

Recall the elements  $\{\tilde{E}_{\beta_k}, \tilde{F}_{\beta_k}\}_{k=1}^N$  of (2.28) and the constants  $\{b_{\beta_k}^>, b_{\beta_k}^<\}_{k=1}^N$  of (2.27).

**Corollary 3.9.** *For any  $\vec{k} = (k_1, \dots, k_N), \vec{r} = (r_1, \dots, r_N) \in \mathbb{Z}_{\geq 0}^N$ , we have*

$$(\tilde{F}^{\vec{k}}, \tilde{E}^{\vec{r}})' = \delta_{\vec{k}, \vec{r}} v^{A_{\vec{k}}} \prod_{p=1}^N v_{i_p}^{\frac{k_p(k_p-1)}{2}} \frac{[k_p]_{v_{i_p}}!}{(v_{i_p}^{-1} - v_{i_p})^{k_p}},$$

with  $A_{\vec{k}}$  explicitly given by

$$A_{\vec{k}} = \sum_{i=1}^N k_i (b_{\beta_i}^> + b_{\beta_i}^<) + \sum_{i=1}^N \frac{k_i(k_i-1)}{2} ((\nu_{\beta_i}^>, \beta_i) - (\nu_{\beta_i}^<, \beta_i)) + \sum_{i < j} k_i k_j ((\nu_{\beta_j}^>, \beta_i) - (\nu_{\beta_i}^<, \beta_j)).$$

*Proof.* First, we note that  $A_{\vec{k}} \in \mathbb{Z}$ , due to Lemma 2.7. Recall that  $\tilde{F}_{\beta_k} = v^{b_{\beta_k}^<} K^{-\nu_{\beta_k}^<} F_{\beta_k}$  and  $\tilde{E}_{\beta_k} = v^{b_{\beta_k}^>} E_{\beta_k} K^{\nu_{\beta_k}^>}$ . Hence

$$\begin{aligned} (3.19) \quad \tilde{F}^{\vec{k}} &= \prod_{1 \leq i \leq N}^{\leftarrow} (v^{b_{\beta_i}^<} K^{-\nu_{\beta_i}^<} F_{\beta_i})^{k_i} = \prod_{1 \leq i \leq N}^{\leftarrow} \left( v^{k_i b_{\beta_i}^<} v^{-(\nu_{\beta_i}^<, \beta_i) \frac{k_i(k_i-1)}{2}} K^{-k_i \nu_{\beta_i}^<} F_{\beta_i}^{k_i} \right) \\ &= v^{\sum_i k_i b_{\beta_i}^<} v^{-\sum_i (\nu_{\beta_i}^<, \beta_i) \frac{k_i(k_i-1)}{2}} v^{-\sum_{i < j} k_i k_j (\nu_{\beta_i}^<, \beta_j)} \cdot K^{-\sum_i k_i \nu_{\beta_i}^<} F^{\vec{k}} \\ &= v^{\sum_i k_i b_{\beta_i}^<} v^{-\sum_i (\nu_{\beta_i}^<, \beta_i) \frac{k_i(k_i-1)}{2}} v^{-\sum_{i < j} k_i k_j (\nu_{\beta_i}^<, \beta_j)} \cdot \varpi^{\leq}(F^{\vec{k}}) \end{aligned}$$

and

$$\begin{aligned}
(3.20) \quad \tilde{E}^{\tilde{r}} &= \prod_{1 \leq i \leq N}^{\leftarrow} (v^{b_{\beta_i}^>} E_{\beta_i} K^{\nu_{\beta_i}^>})^{r_i} = \prod_{1 \leq i \leq N}^{\leftarrow} \left( v^{r_i b_{\beta_i}^>} v^{(\nu_{\beta_i}^>, \beta_i) \frac{r_i(r_i-1)}{2}} E_{\beta_i}^{r_i} K^{r_i \nu_{\beta_i}^>} \right) \\
&= v^{\sum_i r_i b_{\beta_i}^>} v^{\sum_i (\nu_{\beta_i}^>, \beta_i) \frac{r_i(r_i-1)}{2}} v^{\sum_{i < j} r_i r_j (\nu_{\beta_j}^>, \beta_i)} \cdot E^{\tilde{r}} K^{\sum_i r_i \nu_{\beta_i}^>} \\
&= v^{\sum_i r_i b_{\beta_i}^>} v^{\sum_i (\nu_{\beta_i}^>, \beta_i) \frac{r_i(r_i-1)}{2}} v^{\sum_{i < j} r_i r_j (\nu_{\beta_j}^>, \beta_i)} \cdot \varpi^{\geqslant}(E^{\tilde{r}}) K^{\sum_i r_i (\nu_{\beta_i}^> + \nu_{\beta_i}^<)}.
\end{aligned}$$

Since  $\varpi^{\leqslant}(F^{\tilde{k}}) \in \mathbf{U}_{\Phi}^{\leqslant}$  by (3.19), we get

$$(\tilde{F}^{\tilde{k}}, \tilde{E}^{\tilde{r}})' = \delta_{\tilde{k}, \tilde{r}} v^{A_{\tilde{k}}} \left( \varpi^{\leqslant}(F^{\tilde{k}}), \varpi^{\geqslant}(E^{\tilde{k}}) K^{\sum_i k_i (\nu_{\beta_i}^> + \nu_{\beta_i}^<)} \right)' = \delta_{\tilde{k}, \tilde{r}} v^{A_{\tilde{k}}} \left( \varpi^{\leqslant}(F^{\tilde{k}}), \varpi^{\geqslant}(E^{\tilde{k}}) \right)',$$

due to (3.19, 3.20) and (3.11). By Proposition 3.8, we thus obtain:

$$(\tilde{F}^{\tilde{k}}, \tilde{E}^{\tilde{r}})' = \delta_{\tilde{k}, \tilde{r}} v^{A_{\tilde{k}}} (F^{\tilde{k}}, E^{\tilde{k}}).$$

But according to [J1, §8.29–8.30], we have

$$(3.21) \quad (F^{\tilde{k}}, E^{\tilde{k}}) = \prod_{p=1}^N v_{i_p}^{\frac{k_p(k_p-1)}{2}} \frac{[k_p]_{v_{i_p}}!}{(v_{i_p}^{-1} - v_{i_p})^{k_p}}.$$

This implies the equality of the corollary.  $\square$

Let us now investigate the behavior of the pairing (3.6) with respect to  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  and  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ :

**Lemma 3.10.** *The restriction of (3.6) gives rise to  $\mathcal{A}$ -valued pairings:*

$$(3.22) \quad (\ , \ )': \dot{U}_{\mathcal{A}}^{\leqslant} \times U_{\mathcal{A}}^{ev \geqslant} \longrightarrow \mathcal{A},$$

$$(3.23) \quad (\ , \ )': U_{\mathcal{A}}^{ev \leqslant} \times \dot{U}_{\mathcal{A}}^{\geqslant} \longrightarrow \mathcal{A}.$$

Both (3.22, 3.23) are uniquely determined by the properties (3.7, 3.8).

This result also admits a non-twisted version.

*Proof.* To show that  $(y, x)' \in \mathcal{A}$  for any  $y \in \dot{U}_{\mathcal{A}}^{\leqslant}$  and  $x \in U_{\mathcal{A}}^{ev \geqslant}$ , we first apply the formulas (3.7) to reduce to the case  $x \in \{\tilde{E}_i, K^{\nu}\}_{1 \leq i \leq r}^{\nu \in 2P}$  and  $y \in \{\tilde{F}_j^{(s)}, K^{\mu}, \binom{K_j; a}{n}\}_{1 \leq j \leq r, \mu \in 2P}^{a \in \mathbb{Z}, n \in \mathbb{N}}$ . Hence, it remains to apply the following explicit formulas (easily derived from (3.7) and (3.8)):

$$\begin{aligned}
(3.24) \quad &(\tilde{F}_j^{(s)}, \tilde{E}_i)' = -\frac{\delta_{i,j} \delta_{s,1}}{v_i - v_i^{-1}}, \\
&(\tilde{F}_j^{(s)}, K^{\nu})' = 0, \quad (K^{\mu}, \tilde{E}_i)' = \left( \binom{K_j; a}{n}, \tilde{E}_i \right)' = 0, \\
&(K^{\mu}, K^{\nu})' = v^{-(\kappa^{-1}(\mu), \nu)}, \quad \left( \binom{K_j; a}{n}, K^{\nu} \right)' = \left( a - (\kappa^{-1}(\alpha_j^{\vee}), \nu), n \right)_{v_j}.
\end{aligned}$$

We note that  $(\kappa^{-1}(\mu), \nu) \in \mathbb{Z}$  and  $(\kappa^{-1}(\alpha_j^{\vee}), \nu) = (\alpha_j^{\vee}, \gamma^{-1}(\nu)) \in \mathbb{Z}$  for any  $\mu, \nu \in 2P$  by (3.2) and Lemma 3.1. So, indeed  $(y, x)' \in \mathcal{A}$ .

The verification of  $(y, x)' \in \mathcal{A}$  for any  $y \in U^{ev} \leqslant$  and  $x \in \dot{U}_{\mathcal{A}}^{\geqslant}$  is likewise based on:

$$(3.25) \quad \begin{aligned} (\tilde{F}_j, \tilde{E}_i^{(s)})' &= -\frac{\delta_{i,j}\delta_{s,1}}{v_i - v_i^{-1}}, \\ (\tilde{F}_j, K^\mu)' &= \left(\tilde{F}_j, \binom{K_i; a}{n}\right)' = 0, \quad (K^\nu, \tilde{E}_i^{(s)})' = 0, \\ (K^\nu, K^\mu)' &= v^{-(\kappa^{-1}(\nu), \mu)}, \quad \left(K^\nu, \binom{K_i; a}{n}\right)' = \left(a - (\kappa^{-1}(\nu), \alpha_i^\vee)\right)_{v_i}. \end{aligned}$$

Finally, the uniqueness part of Lemma 3.10 is obvious.  $\square$

We can now complete the proof of Lemma 2.10:

*Proof of (b1), (b2), (c1), (c2) in Lemma 2.10.* First, let us show that the second set in (b1) forms an  $\mathcal{A}$ -basis of  $U_{\mathcal{A}}^{ev} \geqslant$ . According to Lemmas 2.8–2.9, we have an inclusion  $U_{\mathcal{A}}^{ev} \geqslant \supseteq \bigoplus_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} \mathcal{A} \cdot \tilde{E}^{\vec{k}}$ . For any  $x \in U_{\mathcal{A}}^{ev} \geqslant$ , we have  $(\tilde{F}^{(\vec{k})}, x)' \in \mathcal{A}$  by Lemma 3.10. On the other hand, writing  $x = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} c_{\vec{k}} \tilde{E}^{\vec{k}}$  with  $c_{\vec{k}} \in \mathbb{Q}(v^{1/2})$  via Lemma 2.9 and using Corollary 3.9, we get  $(\tilde{F}^{(\vec{k})}, x)' = f_{\vec{k}}(v)c_{\vec{k}}$ . Here,  $f_{\vec{k}}(v)$  is an invertible element of  $\mathcal{A}$ , and therefore  $c_{\vec{k}} \in \mathcal{A}$ . We thus conclude that  $U_{\mathcal{A}}^{ev} \geqslant = \bigoplus_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} \mathcal{A} \cdot \tilde{E}^{\vec{k}}$ . The proofs that the second sets in (b2), (c1), (c2) form  $\mathcal{A}$ -bases for the corresponding algebras are completely analogous.

Let us now prove that the first set in (b1) is an  $\mathcal{A}$ -basis of  $U_{\mathcal{A}}^{ev} \geqslant$ . Recall the map  $\tau$  of (2.25). Since  $\tau(F_{\beta_k}) = E_{\beta_k}$  and  $\tau(E_{\beta_k}) = F_{\beta_k}$ , we have:

$$\tau(\tilde{F}_i) = \tilde{E}_i K^{\alpha_i}, \quad \tau(\tilde{F}_{\beta_k}) = v^{-b_{\beta_k}^< - b_{\beta_k}^>} \tilde{E}_{\beta_k} K^{\beta_k} \quad \forall 1 \leq i \leq r, 1 \leq k \leq N,$$

in which  $-b_{\beta_k}^< - b_{\beta_k}^> \in \mathbb{Z}$  by definition (2.27) and Lemma 2.7, and we used  $\nu_{\beta_k}^< - \nu_{\beta_k}^> = \beta_k$ . Let  $\mathcal{U}_{\mathcal{A}}^{ev} \geqslant$  be the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}(\mathfrak{g}, P/2)$  generated by  $\tilde{E}_i K^{\alpha_i}$  for  $1 \leq i \leq r$ . Then the map  $\tau: U_{\mathcal{A}}^{ev} \leqslant \rightarrow \mathcal{U}_{\mathcal{A}}^{ev} \geqslant$  is a  $\mathbb{Z}$ -algebra anti-isomorphism. Therefore,  $\mathcal{U}_{\mathcal{A}}^{ev} \geqslant$  has an  $\mathcal{A}$ -basis consisting of elements  $\prod_{1 \leq j \leq N} (\tilde{E}_{\beta_j} K^{\beta_j})^{k_j}$ . On the other hand, we have  $Q_+$ -gradings:

$$U_{\mathcal{A}}^{ev} \geqslant = \bigoplus_{\mu \in Q_+} U_{\mu}^{ev} \geqslant, \quad \mathcal{U}_{\mathcal{A}}^{ev} \geqslant = \bigoplus_{\mu \in Q_+} \mathcal{U}_{\mu}^{ev} \geqslant,$$

so that  $U_{\mu}^{ev} \geqslant = \mathcal{U}_{\mu}^{ev} \geqslant K^{-\mu}$  for all  $\mu \in Q_+$ . Therefore, the elements  $\tilde{E}^{\vec{k}}$  form an  $\mathcal{A}$ -basis of  $U_{\mathcal{A}}^{ev} \geqslant$ . The proofs that the first sets in (b2), (c1), (c2) form  $\mathcal{A}$ -bases of the corresponding algebras are analogous.  $\square$

After the base change with respect to  $\sigma: \mathcal{A} \rightarrow R$  of (1.2), we obtain:

**Corollary 3.11.** *There exist unique  $R$ -valued Hopf pairings*

$$(3.26) \quad (\ , \ )': \dot{U}_q^{\leqslant} \times \dot{U}_q^{\geqslant} \longrightarrow R,$$

$$(3.27) \quad (\ , \ )': U_q^{ev} \leqslant \times \dot{U}_q^{\geqslant} \longrightarrow R,$$

satisfying (3.24) and (3.25), respectively, where  $v = q$ .

For  $\lambda \in P$ , let us define the character  $\hat{\chi}_\lambda: \dot{U}_q^0 \rightarrow R$  as follows:

$$(3.28) \quad \hat{\chi}_\lambda: \quad K^\mu \mapsto q^{(\mu, \lambda)}, \quad \binom{K_j; a}{n} \mapsto \binom{a + (\alpha_j^\vee, \lambda)}{n}_{q_j} \quad \forall \mu \in 2P, n \in \mathbb{N}.$$

**Lemma 3.12.** (a) Characters  $\{\hat{\chi}_\lambda\}_{\lambda \in P}$  are pair-wise distinct.

(b) Suppose  $\sum a_\lambda \hat{\chi}_\lambda = 0$  for finitely many  $a_\lambda \neq 0 \in R$  then  $a_\lambda = 0$  for all  $\lambda$ .

We need the following lemma:

**Lemma 3.13.** Let  $q \in R$  be invertible. If  $m \in \mathbb{Z}$  is such that  $q^{2m} = 1$  and  $\binom{m}{n}_q = 0$  for all  $n \in \mathbb{N}$ , then  $m = 0$ .

*Proof.* If  $q$  is not a root of unity then  $q^{2m} = 1$  implies that  $m = 0$ . So we can assume  $q$  is a root of unity in  $R$ . Let  $\mathfrak{m}$  be a maximal ideal of  $R$  and  $\mathbb{F} = R/\mathfrak{m}$ . Let  $\bar{q}$  be the image of  $q$  in  $\mathbb{F}^\times$ . Let  $\ell$  be the order of  $\bar{q}^2$  in  $\mathbb{F}$ . In the field  $\mathbb{F}$ , we have  $\bar{q}^{2m} = 1$  and  $\binom{m}{n}_{\bar{q}} = 0$  for all  $n \in \mathbb{N}$ . Since  $\bar{q}^{2m} = 1$ , we have  $m = \ell a$  for some  $a \in \mathbb{Z}$ . We consider two cases:

- if  $\text{char}(\mathbb{F}) = 0$ , then  $a = \binom{m}{\ell}_q = 0$  in  $\mathbb{F}$  by Lemma 4.1(a), which implies that  $a = 0$  and so  $m = 0$ ;
- if  $\text{char}(\mathbb{F}) = p$ , then  $\binom{a}{p^n} = \binom{m}{p^n \ell}_q = 0$  in  $\mathbb{F}$  for all  $n \in \mathbb{N}$ , see Lemma 4.1(a) below.

Since  $p \nmid \binom{a}{p^n}$  when either  $p^n \leq a < p^{n+1}$  or  $p^n - p^{n+1} \leq a < 0$  it follows that  $a = 0$  and so  $m = 0$  in this case.

This completes the proof.  $\square$

*Proof of Lemma 3.12.* (a) Now assume that  $\hat{\chi}_\lambda = \hat{\chi}_\mu$  for  $\lambda, \mu \in P$ . We want to show that  $\lambda = \mu$ . By  $\hat{\chi}_\lambda(K^{2\alpha_i}) = \hat{\chi}_\mu(K^{2\alpha_i})$ , we have  $q_i^{2(\lambda-\mu, \alpha_i^\vee)} = 1$ . By  $\hat{\chi}_\lambda(x) = \hat{\chi}_\mu(x)$  with  $x = \binom{K_i:(\mu, \alpha_i^\vee)}{n}$ , we have  $\binom{(\lambda-\mu, \alpha_i^\vee)}{n}_{q_i} = 0$  for all  $n \in \mathbb{N}$ . Hence  $(\lambda - \mu, \alpha_i^\vee) = 0$  for all  $1 \leq i \leq r$ , due to Lemma 3.13. Therefore,  $\lambda - \mu = 0$ .

(b) Assume we have  $\sum_{i=1}^k a_i \hat{\chi}_i = 0$  with all  $a_i \neq 0$  and  $k > 0$ . We have

$$0 = \sum_{i=1}^k a_i \hat{\chi}_i(y_1 y) - \hat{\chi}_1(y_1) \sum_{i=1}^k a_i \hat{\chi}_i(y) = \sum_{i=2}^k a_i (\hat{\chi}_i(y_1) - \hat{\chi}_1(y_1)) \hat{\chi}_i(y)$$

for all  $y, y_1 \in \dot{U}_q^0$ . Repeating this process, we get  $a_k \prod_{i=1}^{k-1} (\hat{\chi}_k(y_i) - \hat{\chi}_i(y_i)) = 0$  for all  $y_1, \dots, y_{k-1} \in \dot{U}_q^0$ . Let  $I$  be the annihilator of  $a_k$  in  $R$ , then  $\prod_{i=1}^{k-1} (\hat{\chi}_k(y_i) - \hat{\chi}_i(y_i)) \in I$  for all  $y_1, \dots, y_{k-1} \in R$ .

Since  $a_k \neq 0$ , there is a maximal ideal  $\mathfrak{m}$  of  $R$  containing  $I$ . Let  $\mathbb{F} = R/\mathfrak{m}$ . Consider the induced character  $\underline{\chi}_i : \dot{U}_{\mathbb{F}}^0 \rightarrow \mathbb{F}$  of  $\hat{\chi}_i$ . Since the characters  $\{\underline{\chi}_i\}_{i=1}^{k-1}$  are pairwise distinct by part (a), it follows that there are  $\bar{y}_1, \dots, \bar{y}_{k-1} \in \dot{U}_{\mathbb{F}}^0$  such that  $\prod_{i=1}^{k-1} (\underline{\chi}_k(\bar{y}_i) - \underline{\chi}_i(\bar{y}_i)) \neq 0$ . Thus, there are  $y_1, \dots, y_{k-1} \in \dot{U}_q^0$  such that  $\prod_{i=1}^{k-1} (\hat{\chi}_k(y_i) - \hat{\chi}_i(y_i)) \notin \mathfrak{m}$ , hence  $\prod_{i=1}^{k-1} (\hat{\chi}_k(y_i) - \hat{\chi}_i(y_i)) \notin I$ , a contradiction.  $\square$

We conclude this subsection with the following key observation:

**Theorem 3.14.** (a) The pairing  $(\cdot, \cdot)'$  of (3.26) has the zero kernel in the second argument.

(b) The pairing  $(\cdot, \cdot)'$  of (3.27) has the zero kernel in the first argument.

*Proof.* (a) By Lemma 2.10 and Corollary 3.9, we see that the pairing  $(\cdot, \cdot)' : \dot{U}_q^< \times U_q^{ev>} \rightarrow R$  is non-degenerate in each argument. Both algebras  $\dot{U}_q^<$  and  $U_q^{ev>}$  are  $Q$ -graded via (2.15), with each graded component being a free  $R$ -module of finite rank, and the pairing  $(\cdot, \cdot)'$  is of degree zero (3.9). So it is enough to prove the restriction  $(\cdot, \cdot)' : \dot{U}_q^0 \times U_q^{ev0} \rightarrow R$  is non-degenerated in the second argument.

Since  $\{K^\nu\}_{\nu \in 2P} \in U_q^{ev0}$  are group-like, we obtain characters  $\chi_\nu^+ : \dot{U}_q^0 \rightarrow R$  for  $\nu \in 2P$  defined via  $\chi_\nu^+(x) = (x, K^\nu)'$ . It is easy to see that  $\chi_\nu^+ = \hat{\chi}_{-\gamma^{-1}(\nu)}$ , here  $\gamma^{-1}(\nu) \in Q$  since  $\nu \in 2P$ . The

non-degeneracy of the second argument in the pairing  $(\ , \ )': \dot{U}_q^0 \times U_q^{ev0} \rightarrow 0$  is equivalent to the statement that if  $\sum_\nu a_\nu \chi_\nu^+ = 0$  for finitely many  $a_\nu \neq 0 \in R$  then  $a_\nu = 0$  for all  $\nu$ . But the latter statement follows by Lemma 3.12.b).

(b) The proof is identical.  $\square$

### 3.3. Twisted invariant pairing.

Consider a function  $G$  on  $(\mathfrak{h}^*)^4 = \mathfrak{h}^* \times \mathfrak{h}^* \times \mathfrak{h}^* \times \mathfrak{h}^*$  defined via:

$$G(\lambda_1, \lambda_2, \mu_1, \mu_2) := \frac{(\lambda_1, \kappa(\mu_1) + \gamma(\mu_2)) + (\lambda_2, \kappa(\mu_2) + \gamma(\mu_1)) + (\gamma(\mu_1 - \mu_2), \gamma(\mu_1 - \mu_2)) - (\lambda_1, \lambda_2)}{2} + (2\rho, \mu_2),$$

with the endomorphisms  $\kappa, \gamma \in \text{End}(\mathfrak{h}^*)$  defined in (3.1).

Evoking the Hopf pairing  $(\ , \ )'$  of (3.6), the triangular decomposition (2.24) of  $\mathbf{U}(\mathfrak{g}, P/2)$ , and the  $Q$ -grading (2.15) of  $\mathbf{U}_\Phi^>$  and  $\mathbf{U}_\Phi^<$  (cf. paragraph preceding Lemma 2.6), we define a  $\mathbb{Q}(v^{1/2})$ -bilinear pairing:

$$(3.29) \quad \langle \ , \ \rangle': \mathbf{U}(\mathfrak{g}, P/2) \times \mathbf{U}(\mathfrak{g}, P/2) \longrightarrow \mathbb{Q}(v^{1/\mathsf{N}})$$

via

$$(3.30) \quad \langle y_1 K^{\lambda_1} x_1, y_2 K^{\lambda_2} x_2 \rangle' := (y_1, x_2)' \cdot (y_2, x_1)' \cdot v^{G(\lambda_1, \lambda_2, \mu_1, \mu_2)},$$

for all  $x_1 \in \mathbf{U}_{\Phi, \mu_1}^>, x_2 \in \mathbf{U}_{\Phi, \mu_2}^>, y_1 \in \mathbf{U}_{\Phi, -\nu_1}^<, y_2 \in \mathbf{U}_{\Phi, -\nu_2}^<$  with  $\mu_1, \mu_2, \nu_1, \nu_2 \in Q_+, \lambda_1, \lambda_2 \in P/2$ .

**Remark 3.15.** To simplify the exponent of  $v$  in (3.30), we can rewrite the pairing (3.29) as:

$$(3.31) \quad \left\langle (y_1 K^{\kappa(\nu_1)}) K^{\lambda_1} (x_1 K^{\gamma(\mu_1)}), (y_2 K^{\kappa(\nu_2)}) K^{\lambda_2} (x_2 K^{\gamma(\mu_2)}) \right\rangle' = (y_1, x_2)' \cdot (y_2, x_1)' \cdot v^{-\frac{(\lambda_1, \lambda_2)}{2} + (2\rho, \nu_1)}$$

for all  $x_1 \in \mathbf{U}_{\Phi, \mu_1}^>, x_2 \in \mathbf{U}_{\Phi, \mu_2}^>, y_1 \in \mathbf{U}_{\Phi, -\nu_1}^<, y_2 \in \mathbf{U}_{\Phi, -\nu_2}^<$  with  $\mu_1, \mu_2, \nu_1, \nu_2 \in Q^+, \lambda_1, \lambda_2 \in P/2$ . Here, we use  $\gamma + \kappa = 2\text{Id}_{\mathfrak{h}^*}$ , so that  $\gamma\kappa = \kappa\gamma$  and  $(\gamma(\mu), \gamma(\mu')) = (\kappa(\mu), \kappa(\mu'))$  due to (3.2). Formula (3.31) also clarifies our first condition in the choice of  $\mathsf{N}$  in the beginning of Section 3.2.

We note that:

$$(3.32) \quad \langle \mathbf{U}_{\Phi, -\nu_1}^< \mathbf{U}_\Phi^0 \mathbf{U}_{\Phi, \mu_1}^>, \mathbf{U}_{\Phi, -\nu_2}^< \mathbf{U}_\Phi^0 \mathbf{U}_{\Phi, \mu_2}^> \rangle' = 0 \quad \text{unless } \nu_1 = \mu_2, \nu_2 = \mu_1.$$

The following result is completely analogous to [J1, Proposition 6.20]:

**Proposition 3.16.** *The above pairing  $\langle \ , \ \rangle'$  of (3.29) on  $\mathbf{U}(\mathfrak{g}, P/2)$  satisfies:*

$$(3.33) \quad \langle \text{ad}'(x)y, z \rangle' = \langle y, \text{ad}'(S'(x))z \rangle' \quad \text{for any } x, y, z \in \mathbf{U}(\mathfrak{g}, P/2).$$

*Proof.* Follows by applying a line-to-line reasoning as in [J1, §6.14–6.20].  $\square$

**Remark 3.17.** (a) Given a Hopf algebra  $(A, S, \Delta, \varepsilon)$  over a commutative ring  $R$ , a pairing  $\langle \ , \ \rangle: A \times A \rightarrow R$  is called an **invariant pairing** if it satisfies the condition:

$$\langle \text{ad}(x)y, z \rangle = \langle y, \text{ad}(S(x))z \rangle \quad \text{for any } x, y, z \in A.$$

(b) Equivalently, a pairing  $\langle \ , \ \rangle: A \times A \rightarrow R$  on a Hopf  $R$ -algebra  $A$  is invariant iff the corresponding  $R$ -linear map  $A \rightarrow A^*$ , given by  $v \mapsto \varphi_v$  with  $\varphi_v(w) := \langle v, w \rangle$ , is an  $A$ -module homomorphism, where  $A \curvearrowright A^*$  denotes the left dual of the left adjoint  $A$ -action of (2.1).

(c) Equivalently, a pairing  $\langle \ , \ \rangle: A \times A \rightarrow R$  on a Hopf  $R$ -algebra  $A$  is invariant iff it intertwines the  $A$ -actions (the right-hand side being endowed with the trivial  $A$ -action):

$$(3.34) \quad \langle \text{ad}(x_{(1)})y, \text{ad}(x_{(2)})z \rangle = \varepsilon(x)\langle y, z \rangle \quad \text{for any } x, y, z \in A,$$

where we use the Sweedler's notation (2.2) for the coproduct.

Let us now investigate the behavior of the pairing (3.29) with respect to  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  and  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ .

**Proposition 3.18.** *The restriction of (3.29) gives rise to:*

$$(3.35) \quad \langle , \rangle': U_{\mathcal{A}}^{ev}(\mathfrak{g}) \times \dot{U}_{\mathcal{A}}(\mathfrak{g}) \longrightarrow \mathcal{A}[v^{\pm 1/\mathbb{N}}].$$

Evoking the adjoint action  $\text{ad}' : \dot{U}_{\mathcal{A}}(\mathfrak{g}) \curvearrowright U_{\mathcal{A}}^{ev}(\mathfrak{g})$  of Proposition 2.11, the pairing (3.35) satisfies:

$$(3.36) \quad \langle \text{ad}'(x)y, z \rangle' = \langle y, \text{ad}'(S'(x))z \rangle' \quad \text{for any } x, z \in \dot{U}_{\mathcal{A}}(\mathfrak{g}), y \in U_{\mathcal{A}}^{ev}(\mathfrak{g}).$$

*Proof.* The restriction of the pairing (3.29) to  $U_{\mathcal{A}}^{ev}(\mathfrak{g}) \times \dot{U}_{\mathcal{A}}(\mathfrak{g})$  satisfies the condition (3.36) due to (3.33), and takes values in  $\mathcal{A}[v^{\pm 1/\mathbb{N}}]$  due to (3.31) and Lemma 3.10.  $\square$

Evoking the pairings  $(, )'$  of Lemma 3.10 (uniquely determined by (3.7, 3.24) and (3.7, 3.25)), the triangular decompositions of  $\dot{U}_{\mathcal{A}}(\mathfrak{g})$  and  $U_{\mathcal{A}}^{ev}(\mathfrak{g})$  of Lemma 2.10, the  $Q$ -grading (2.15) of both algebras, Lemma 3.1, and formula (3.31), we note that the pairing (3.35) is given by:

$$(3.37) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), (\dot{y} K^{\kappa(\dot{\nu})}) K^{\dot{\lambda}} \prod_{i=1}^r \binom{K_i; 0}{s_i} (\dot{x} K^{\gamma(\dot{\mu})}) \right\rangle' = \\ (\dot{y}, x)' \cdot (y, \dot{x})' \cdot v^{-\frac{(\lambda, \dot{\lambda})}{2} + (2\rho, \nu)} \cdot \prod_{i=1}^r \left( -\frac{(\alpha_i^\vee, \lambda)}{s_i^2} \right)_{v_i}$$

for any  $\dot{x} \in \dot{U}_{\mathcal{A}, \dot{\mu}}^>, x \in U_{\mathcal{A}, \mu}^{ev>}, \dot{y} \in \dot{U}_{\mathcal{A}, -\dot{\nu}}^<, y \in U_{\mathcal{A}, -\nu}^{ev<}, \dot{\mu}, \mu, \dot{\nu}, \nu \in Q_+, \dot{\lambda}, \lambda \in 2P$ , and  $s_i \geq 0$ . In particular, we note that  $-(\alpha_i^\vee, \lambda)/2 \in \mathbb{Z}$ , and so  $\left( -\frac{(\alpha_i^\vee, \lambda)}{s_i^2} \right)_{v_i}$  is well-defined.

Given  $q \in R$  as in the paragraph preceding (1.2), suppose  $q$  has an  $\mathbb{N}$ -th root in  $R$ , and fix such a root  $q^{1/\mathbb{N}}$ . Lift  $\sigma: \mathcal{A} \rightarrow R$  of (1.2) to  $\mathcal{A}[v^{1/\mathbb{N}}] \rightarrow R$  via  $v^{1/\mathbb{N}} \mapsto q^{1/\mathbb{N}}$ . We thus obtain:

**Proposition 3.19.** *Define an  $R$ -bilinear pairing*

$$(3.38) \quad \langle , \rangle': U_q^{ev}(\mathfrak{g}) \times \dot{U}_q(\mathfrak{g}) \longrightarrow R$$

via (3.37), where  $v^{1/\mathbb{N}} = q^{1/\mathbb{N}}$  and the pairings  $(\dot{y}, x)', (y, \dot{x})'$  refer to (3.26, 3.27). Then:

(a) *The pairing (3.38) is  $\text{ad}'$ -invariant:*

$$\langle \text{ad}'(x)y, z \rangle' = \langle y, \text{ad}'(S'(x))z \rangle' \quad \text{for any } x, z \in \dot{U}_q(\mathfrak{g}), y \in U_q^{ev}(\mathfrak{g}).$$

(b) *The pairing (3.38) has the zero kernel in the first argument.*

*Proof.* Part (a) is obvious. Let us prove part (b). Let  $x \in U_q^{ev}(\mathfrak{g})$  then, by using the PBW-bases of Lemma 2.10,

$$x = \sum_{\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N} (\tilde{F}^{\vec{k}} K^{-\kappa(\text{wt}(\tilde{F}^{\vec{k}}))}) a_{\vec{k}, \vec{r}} (\tilde{E}^{\vec{r}} K^{\gamma(\text{wt}(\tilde{E}^{\vec{r}}))}),$$

for finitely many nonzero  $a_{\vec{k}, \vec{r}} \in U_q^{ev0}$ . Suppose  $\langle x, u \rangle' = 0$  for all  $u \in \dot{U}_q(\mathfrak{g})$ . Apply this to  $u = (\tilde{F}^{\vec{r}} K^{-\kappa(\text{wt}(\tilde{F}^{\vec{r}}))}) u_0 (\tilde{E}^{\vec{k}} K^{\gamma(\text{wt}(\tilde{E}^{\vec{k}}))})$  for any  $u_0 \in \dot{U}_q^0$  and using the computations in Corollary 3.9, we obtain  $\langle a_{\vec{k}, \vec{r}}, u_0 \rangle' = 0$  for all  $u_0 \in \dot{U}_q^0$ . Note that the map  $\dot{U}_q^0 \rightarrow R$  defined by  $u_0 \mapsto \langle K^{2\lambda}, u_0 \rangle'$  is equal to  $\hat{\chi}_{-\lambda}$  for all  $\lambda \in P$ . Therefore, by Lemma 3.12,  $a_{\vec{k}, \vec{r}} = 0$ , hence  $x = 0$ .  $\square$

#### 4. TWISTED QUANTUM FROBENIUS HOMOMORPHISM

We impose restrictions on  $\ell$  as in Section 4.5 below. Let  $\mathcal{A}'$  be the quotient of  $\mathcal{A}$  by the ideal generated by  $\ell$ -cyclotomic polynomial  $f_\ell \in \mathcal{A}$ . In this section, we assume that the given ring homomorphism  $\sigma: \mathcal{A} \rightarrow R$  factors through a ring homomorphism  $\mathcal{A}' \rightarrow R$ . To define the coproduct and the braiding for certain Hopf algebras below, we shall further assume that there is  $\epsilon^{1/N} \in R$ , where  $N$  satisfies  $\frac{1}{2}(P/2, P/2) \subset \frac{1}{N}\mathbb{Z}$ , and we fix such  $N, \epsilon^{1/N}$ . We set  $\epsilon_i = \epsilon^{\mathbf{d}_i}$ , where  $\mathbf{d}_i = (\alpha_i, \alpha_i)/2 \in \{1, 2, 3\}$ . Define a positive integer  $\ell_i$  by

$$(4.1) \quad \ell_i := \ell / \text{GCD}(2d_i, \ell).$$

More generally, for any  $\alpha \in \Delta_+$ , let

$$(4.2) \quad \ell_\alpha := \ell / \text{GCD}((\alpha, \alpha), \ell).$$

In this section, we recall the quantum Frobenius homomorphism of [L5] with small modifications.

Henceforth, we shall often use the following result:

- Lemma 4.1.** (a) For any  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ , we have  $\binom{a}{b}_{\epsilon_i} = \binom{a_0}{b_0}_{\epsilon_i} \cdot \binom{a_1}{b_1}_{\epsilon_i}$ , where  $a = \ell_i a_1 + a_0$  and  $b = \ell_i b_1 + b_0$  with  $a_1 \in \mathbb{Z}$ ,  $b_1 \in \mathbb{N}$ , and  $0 \leq a_0, b_0 \leq \ell_i - 1$ .  
(b) The  $v = \epsilon$  specialization of the  $v$ -multibinomial coefficient  $\frac{(n\ell_i)_{v_i}!}{((\ell_i)_{v_i}!)^n} \in \mathbb{Z}[v, v^{-1}]$  equals  $n!$ .  
(c)  $(-1)^{\ell_i} \epsilon_i^{\ell_i(\ell_i+1)} = -1$ .

*Proof.* (a) Follows from [L5, Lemma 34.1.2(c)] upon using  $\binom{a}{b}_{\epsilon_i} = \epsilon_i^{b(a-b)} \binom{a}{b}_{\epsilon_i}$  and  $\epsilon_i^{2\ell_i} = 1$ .

(b) Follows by induction on  $n$  by applying part (a) to  $a = n\ell_i, b = \ell_i$ .

(c) We have  $(-1)^{\ell_i} \epsilon_i^{\ell_i(\ell_i+1)} = (-1)^{\ell_i} \epsilon^{\mathbf{d}_i \ell_i(\ell_i+1)}$ . If  $\ell_i$  is odd, then  $(-1)^{\ell_i} = -1, \epsilon^{\mathbf{d}_i \ell_i(\ell_i+1)} = 1$ . If  $\ell_i$  is even, then we note that  $\epsilon^{\mathbf{d}_i \ell_i} = -1$  and also  $(-1)^{\ell_i} = 1$ . The result follows.  $\square$

##### 4.1. The domain of the quantum Frobenius homomorphism.

Let us recall the Lusztig form  $\dot{U}_\epsilon(\mathfrak{g}) = \dot{U}_\mathcal{A}(\mathfrak{g}) \otimes_{\mathcal{A}} R$  from (2.21), and let  $\dot{U}_\epsilon^>, \dot{U}_\epsilon^<$  be its  $R$ -subalgebras generated by  $\{\tilde{E}_i^{(n)}\}_{1 \leq i \leq r}^{n \geq 1}, \{\tilde{F}_i^{(n)}\}_{1 \leq i \leq r}^{n \geq 1}$ . We recall the  $q$ -Serre relations, cf. (2.19):

$$\begin{aligned} & \sum_{m=0}^{1-a_{ij}} (-1)^m \epsilon_i^{ma_{ij}(\epsilon_{ij}-1)-m(m-1)} \tilde{E}_i^{(1-a_{ij}-m)} \tilde{E}_j \tilde{E}_i^{(m)} = 0 \quad (i \neq j), \\ & \sum_{m=0}^{1-a_{ij}} (-1)^m \epsilon_i^{ma_{ij}(\epsilon_{ij}-1)-m(m-1)} \tilde{F}_i^{(1-a_{ij}-m)} \tilde{F}_j \tilde{F}_i^{(m)} = 0 \quad (i \neq j). \end{aligned}$$

The generators  $\tilde{E}_i^{(n)}$  and  $\tilde{F}_j^{(n)}$  also satisfy the following relations, cf. (2.18):

$$(4.3) \quad \begin{aligned} \tilde{E}_i^{(p)} \tilde{F}_j^{(s)} &= \epsilon^{ps(\alpha_i, \kappa(\alpha_j))} \tilde{F}_j^{(s)} \tilde{E}_i^{(p)} \quad \text{for } i \neq j, \\ \tilde{E}_i^{(p)} \tilde{F}_i^{(s)} &= \sum_{c=0}^{\min(p,s)} \epsilon_i^{2ps-c^2} \tilde{F}_i^{(s-c)} \binom{K_i; 2c-p-s}{c} \tilde{E}_i^{(p-c)}. \end{aligned}$$

##### 4.1.1. The idempotent Lusztig form $\widehat{U}_\epsilon(\mathfrak{g}, X)$ .

For any lattice  $X$  with  $Q \subseteq X \subseteq P$ , let  $\widehat{U}_\epsilon(\mathfrak{g}, X)$  be the **idempotent Lusztig form** defined similarly to [L5, Chapter 23] with generators

$$\left\{ \tilde{E}_i^{(n)} 1_\lambda, \tilde{F}_i^{(n)} 1_\lambda \mid 1 \leq i \leq r, n \geq 0, \lambda \in X \right\}.$$

We record the topological coproduct in  $\widehat{U}_\epsilon(\mathfrak{g}, X)$ :

$$(4.4) \quad \begin{aligned} \Delta(\tilde{E}_i^{(r)} 1_\lambda) &= \sum_{c=0}^r \prod_{\lambda' + \lambda'' = \lambda} \epsilon^{-(r-c)(\zeta_i^>, \lambda'')} \tilde{E}_i^{(r-c)} 1_{\lambda'} \otimes \tilde{E}_i^{(c)} 1_{\lambda''}, \\ \Delta(\tilde{F}_i^{(r)} 1_\lambda) &= \sum_{c=0}^r \prod_{\lambda' + \lambda'' = \lambda} \epsilon_i^{2c(r-c)} \epsilon^{c(\zeta_i^<, \lambda'')} \tilde{F}_i^{(c)} 1_{\lambda'} \otimes \tilde{F}_i^{(r-c)} 1_{\lambda''}, \end{aligned}$$

cf. (2.20), as well as some relations in  $\widehat{U}_\epsilon(\mathfrak{g}, X)$ :

$$(4.5) \quad \begin{aligned} \tilde{E}_i^{(p)} 1_\lambda \tilde{F}_j^{(s)} &= \epsilon^{ps(\alpha_i, \kappa(\alpha_j))} \tilde{F}_j^{(s)} 1_{\lambda+s\alpha_j+p\alpha_i} \tilde{E}_i^{(p)} \quad \text{for } i \neq j, \\ \tilde{E}_i^{(p)} 1_\lambda \tilde{F}_i^{(s)} &= \sum_{c \geq 0} \epsilon_i^{2ps-c^2} \binom{(\lambda, \alpha_i^\vee) + s + p}{c}_{\epsilon_i} \tilde{F}_i^{(s-c)} 1_{\lambda+(p+s-c)\alpha_i} \tilde{E}_i^{(p-c)}, \end{aligned}$$

cf. (4.3), with  $\lambda, \lambda', \lambda'' \in X$  in the formulas above. We note that in (4.4), the element  $\epsilon^{1/N}$  is used in the definition of non-integer powers of  $\epsilon$ , e.g.  $\epsilon^{c(\zeta_i^<, \lambda'')} := (\epsilon^{1/N})^{Nc(\zeta_i^<, \lambda'')}$ .

## 4.2. The codomain of the quantum Frobenius homomorphism.

Let us consider the following data:

- The lattices  $P^* = \bigoplus_{i=1}^r \mathbb{Z}\omega_i^*$  and  $Q^* = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^*$  with  $\omega_i^* := \ell_i \omega_i$  and  $\alpha_i^* := \ell_i \alpha_i$ . We also set  $\omega_i^{*\vee} := \omega_i^\vee / \ell_i$  and  $\alpha_i^{*\vee} := \alpha_i^\vee / \ell_i$ .
- The new Cartan matrix with  $(i, j)$ -entry

$$(4.6) \quad a_{ij}^* = 2(\alpha_i^*, \alpha_j^*) / (\alpha_i^*, \alpha_i^*) = 2\ell_j(\alpha_i, \alpha_j) / \ell_i(\alpha_i, \alpha_i).$$

- The bilinear form on  $P^*$  induced from the bilinear form on  $P$  via the inclusion  $P^* \subset P$ .

The fact that  $(a_{ij}^*)_{i,j=1}^r$  is a Cartan matrix of a semisimple Lie algebra follows from [L5, §2.2.4]. In the case when  $\ell$  is divisible by  $2d_i$  for all  $i$ , we have  $a_{ij}^* = a_{ji}$  for all  $i, j$ , so that  $(a_{ij}^*)$  is the Cartan matrix of the Langlands dual  $\mathfrak{g}^\vee$  of the Lie algebra  $\mathfrak{g}$ .

**Remark 4.2.** We note that  $\kappa(Q^*) = \gamma(Q^*) = 2P^*$  similarly to Lemma 3.1.

### 4.2.1. The $\mathbb{Q}(v^{1/2})$ -Hopf algebra $\mathbf{U}^*(\mathfrak{g}, P^*/2)$ .

All constructions in Sections 1–3 can be carried out with the above datum. Let  $d_i^* = \frac{(\alpha_i^*, \alpha_i^*)}{2} = d_i \ell_i^2$ ,  $v_i^* = v^{d_i^*}$ , and respectively  $\epsilon_i^* = \epsilon^{d_i^*} \in \{\pm 1\}$ . Let  $\mathfrak{g}^d$  be the semisimple Lie algebra with the Cartan matrix  $(a_{ij}^*)$ , the weight lattice  $P^*$  and the root lattice  $Q^*$ . The graph  $\text{Dyn}(\mathfrak{g}^d)$ , see definition in Section 2.3, is the same as  $\text{Dyn}(\mathfrak{g})$ . Let us consider the associated matrix  $(\epsilon_{ij})$  as in (2.9) and then the following twist:

$$F^* := \prod_{\lambda, \mu \in P^*} v^{\sum_{i,j} \phi_{ij}^*(\omega_i^{*\vee}, \lambda)(\omega_j^{*\vee}, \mu)} 1_\lambda \otimes 1_\mu = \prod_{\lambda, \mu \in P^*} v^{\sum_{i,j} \phi_{ij}(\omega_i^\vee, \lambda)(\omega_j^\vee, \mu)} 1_\lambda \otimes 1_\mu,$$

where  $\phi_{ij}^* = \epsilon_{ij}(\alpha_i^*, \alpha_j^*)/2 = \ell_i \ell_j \phi_{ij}$ , cf. (2.7) and (2.10).

With the above data, we define the  $\mathbb{Q}(v^{1/2})$ -Hopf algebra  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  generated by the set  $\{\hat{e}_i, \hat{f}_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in P^*/2}$  as in Section 1.1. We also have the Lusztig's braid group action on

$\mathbf{U}^*(\mathfrak{g}, P^*/2)$  defined by:

$$(4.7) \quad \begin{aligned} T_i^*(K^\mu) &= K^{s_i^*\mu}, & T_i^*(\hat{e}_i) &= -\hat{f}_i K^{\alpha_i^*}, & T_i^*(\hat{f}_i) &= -K^{-\alpha_i^*} \hat{e}_i, \\ T_i^*(\hat{e}_j) &= \sum_{k=0}^{-a_{ij}^*} (-1)^k \frac{(v_i^*)^{-k}}{[-a_{ij}^* - k]_{v_i^*}! [k]_{v_i^*}!} \hat{e}_i^{-a_{ij}^*-k} \hat{e}_j \hat{e}_i^k & (i \neq j), \\ T_i^*(\hat{f}_j) &= \sum_{k=0}^{-a_{ij}^*} (-1)^k \frac{(v_i^*)^k}{[-a_{ij}^* - k]_{v_i^*}! [k]_{v_i^*}!} \hat{f}_i^k \hat{f}_j \hat{f}_i^{-a_{ij}^*-k} & (i \neq j), \end{aligned}$$

in which  $s_i^* = s_{\alpha_i^*}$ . The Weyl group of  $\mathfrak{g}^d$  is the same as the Weyl group of  $\mathfrak{g}$  via identifying  $s_i^*$  with  $s_i$ , so we also denote the Weyl group of  $\mathfrak{g}^d$  by  $W$ . Fix the same reduced decomposition of the longest element  $w_0$  in  $W$  as in Section 1.1. Then  $\beta_k^* = s_{i_1}^* \dots s_{i_{k-1}}^* \alpha_{i_k}^* = \ell_{i_k} \beta_k$  ( $1 \leq k \leq N$ ) provides a labeling of all positive roots  $\Delta_+^d$  of  $\mathfrak{g}^d$ . We then define root vectors  $\{\hat{e}_{\beta_k^*}, \hat{f}_{\beta_k^*}\}_{1 \leq k \leq N}$  in a standard way via:

$$\hat{e}_{\beta_k^*} = T_{i_1}^* \dots T_{i_{k-1}}^* \hat{e}_{i_k}, \quad \hat{f}_{\beta_k^*} = T_{i_1}^* \dots T_{i_{k-1}}^* \hat{f}_{i_k}.$$

Following (2.11, 2.12), for  $1 \leq i \leq r$ , let

$$\nu_i^{*>} := -\alpha_i^* + \sum_{1 \leq j \leq r} \phi_{ij}^* \omega_j^{*\vee} = \ell_i \nu_i^{>}, \quad \nu_i^{*<} := \sum_{1 \leq j \leq r} \phi_{ij}^* \omega_j^{*\vee} = \ell_i \nu_i^{<},$$

and set

$$\tilde{e}_i := \hat{e}_i K^{\nu_i^{*>}}, \quad \tilde{f}_i := K^{-\nu_i^{*<}} \hat{f}_i.$$

Then the algebra  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  is also generated over  $\mathbb{Q}(v^{1/2})$  by  $\{\tilde{e}_i, \tilde{f}_i, K^\mu\}_{1 \leq i \leq r}^{\mu \in P^*/2}$  subject to the following relations:

$$(4.8) \quad \begin{aligned} K^\mu K^{\mu'} &= K^{\mu+\mu'}, & K^0 &= 1, \\ K^\mu \tilde{e}_i K^{-\mu} &= v^{(\alpha_i^*, \mu)} \tilde{e}_i, & K^\mu \tilde{f}_i K^{-\mu} &= v^{-(\alpha_i^*, \mu)} \tilde{f}_i, \\ \tilde{e}_i \tilde{f}_j &= v^{(\alpha_i^*, -\zeta_j^{*<})} \tilde{f}_j \tilde{e}_i \quad (i \neq j), & \tilde{e}_i \tilde{f}_i - (v_i^*)^2 \tilde{f}_i \tilde{e}_i &= v_i^* \frac{1 - (K_i^*)^{-2}}{1 - (v_i^*)^{-2}}, \\ \sum_{m=0}^{1-a_{ij}^*} (-1)^m (v_i^*)^{ma_{ij}^*(\epsilon_{ij}-1)-m(m-1)} \tilde{e}_i^{(1-a_{ij}^*-m)} \tilde{e}_j \tilde{e}_i^{(m)} &= 0 \quad (i \neq j), \\ \sum_{m=0}^{1-a_{ij}^*} (-1)^m (v_i^*)^{ma_{ij}^*(\epsilon_{ij}-1)-m(m-1)} \tilde{f}_i^{(1-a_{ij}^*-m)} \tilde{f}_j \tilde{f}_i^{(m)} &= 0 \quad (i \neq j), \end{aligned}$$

where  $K_i^* = K^{\alpha_i^*}$ ,  $\tilde{e}_i^{(m)} = \tilde{e}^m / (m)_{v_i^*}$ ,  $\tilde{f}_i^{(m)} = \tilde{f}^m / (m)_{v_i^*}$ , and we set  $\zeta_j^{*<} = \ell_j \zeta_j^{<}$ ,  $\zeta_j^{*>} = \ell_j \zeta_j^{>}$ . Following the twisted construction in Section 2, we also have a twisted Hopf algebra structure on  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  as follows:

$$\begin{aligned} \Delta'(K^\mu) &= K^\mu \otimes K^\mu, & \Delta'(\tilde{e}_i) &= 1 \otimes \tilde{e}_i + \tilde{e}_i \otimes K^{-\zeta_i^{*>}}, & \Delta'(\tilde{f}_i) &= 1 \otimes \tilde{f}_i + \tilde{f}_i \otimes K^{\zeta_i^{*<}}, \\ S'(K^\mu) &= K^{-\mu}, & S'(\tilde{e}_i) &= -\tilde{e}_i K^{\zeta_i^{*>}}, & S'(\tilde{f}_i) &= -\tilde{f}_i K^{-\zeta_i^{*<}}. \end{aligned}$$

The algebra  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  is  $Q^*$ -graded via:

$$(4.9) \quad \deg(\tilde{e}_i) = \alpha_i^*, \quad \deg(\tilde{f}_i) = -\alpha_i^*, \quad \deg(K^\mu) = 0.$$

**Remark 4.3.** Similarly to Section 1, we define the (untwisted) Lusztig form  $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  generated by elements  $\{\hat{e}_i^{[n]}, \hat{f}_i^{[n]}, K^\mu\}_{1 \leq i \leq r, n \geq 1}^{\mu \in Q^*}$ , in which  $\hat{e}_i^{[n]} := \hat{e}_i^n / [n]_{v_i^*}!$ ,  $\hat{f}_i^{[n]} := \hat{f}_i^n / [n]_{v_i^*}!$ . Then we also consider the specialization  $\dot{\mathcal{U}}_\epsilon^*(\mathfrak{g}) := \dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g}) \otimes_{\mathcal{A}} R$ .

#### 4.2.2. The $\mathcal{A}$ -Hopf algebra $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$ and its specialization $\dot{\mathcal{U}}_\epsilon^*(\mathfrak{g})$ .

We define the (twisted) Lusztig form  $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$  as the  $\mathcal{A}$ -subalgebra of  $\mathbf{U}^*(\mathfrak{g}, P^*/2)$  generated by the elements  $\{\tilde{e}_i^{(n)}, \tilde{f}_i^{(n)}, K^\mu\}_{1 \leq i \leq r}^{\mu \in 2P^*}$  as in Section 2.4. It is a Hopf algebra over  $\mathcal{A}$ . We record the Hopf structure of  $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$ :

(4.10)

$$\begin{aligned} \Delta'(\tilde{e}_i^{(s)}) &= \sum_{c=0}^s \tilde{e}_i^{(s-c)} \otimes \tilde{e}_i^{(c)} K^{-(s-c)\zeta_i^*}, \quad S'(\tilde{e}_i^{(s)}) = (-1)^s (v_i^*)^{s(s-1)} \tilde{e}_i^{(s)} K^{s\zeta_i^*}, \\ \Delta'(\tilde{f}_i^{(s)}) &= \sum_{c=0}^s (v_i^*)^{2c(s-c)} \tilde{f}_i^{(c)} \otimes \tilde{f}_i^{(s-c)} K^{c\zeta_i^*}, \quad S'(\tilde{f}_i^{(s)}) = (-1)^s (v_i^*)^{-s(s-1)} \tilde{f}_i^{(s)} K^{-s\zeta_i^*}, \\ \Delta'(K^\mu) &= K^\mu \otimes K^\mu, \quad S'(K^\mu) = K^{-\mu}, \\ \varepsilon'(\tilde{e}_i^{(s)}) &= \varepsilon'(\tilde{f}_i^{(s)}) = 0, \quad \varepsilon'(K^\mu) = 1. \end{aligned}$$

The elements  $\binom{K_i^*; a}{n} := \frac{\prod_{s=1}^n (1 - (K_i^*)^{-2}(v_i^*)^{2(s-a-1)})}{\prod_{s=1}^n (1 - (v_i^*)^{-2s})}$  belong to  $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$ . Let  $\dot{\mathcal{U}}_{\mathcal{A}}^{*<} \subset \dot{\mathcal{U}}_{\mathcal{A}}^{*>} \subset \dot{\mathcal{U}}_{\mathcal{A}}^{*0}$

denote the  $\mathcal{A}$ -subalgebras of  $\dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$  generated by  $\{\tilde{e}_i^{(s)}\}$ ,  $\{\tilde{f}_i^{(s)}\}$ ,  $\left\{K^\mu, \binom{K_i^*; a}{n}\right\}$ , respectively.

For any  $\beta^* = \sum_{i=1}^r a_i \alpha_i^* \in Q_+^*$ , let

$$(4.11) \quad \begin{aligned} \nu_{\beta^*}^> &:= \sum_i a_i \nu_i^*, & \nu_{\beta^*}^< &:= \sum_i a_i \nu_i^*, \\ b_{\beta^*}^> &:= \sum_{i < j} a_i a_j (\nu_i^*, \alpha_j^*), & b_{\beta^*}^< &:= - \sum_{i < j} a_i a_j (\nu_j^*, \alpha_i^*), \end{aligned}$$

cf. (2.26)–(2.27). Similarly to (2.28) for any  $\beta_k^* \in \Delta_+^d$ , define:

$$(4.12) \quad \tilde{e}_{\beta_k^*} := v^{b_{\beta_k^*}^>} \hat{e}_{\beta_k^*} K^{\nu_{\beta_k^*}^>}, \quad \tilde{f}_{\beta_k^*} := v^{b_{\beta_k^*}^<} K^{-\nu_{\beta_k^*}^<} \hat{f}_{\beta_k^*}, \quad \tilde{e}_{\beta_k^*}^{(n)} := \frac{\tilde{e}_{\beta_k^*}^n}{(n)_{v_{i_k}^*}!}, \quad \tilde{f}_{\beta_k^*}^{(n)} := \frac{\tilde{f}_{\beta_k^*}^n}{(n)_{v_{i_k}^*}!}.$$

For any  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$ , we also define the ordered monomials:

$$\begin{aligned} \tilde{e}^{(\vec{k})} &:= \tilde{e}_{\beta_1^*}^{(k_1)} \dots \tilde{e}_{\beta_N^*}^{(k_N)}, & \tilde{f}^{(\vec{k})} &:= \tilde{f}_{\beta_1^*}^{(k_1)} \dots \tilde{f}_{\beta_N^*}^{(k_N)}, \\ \tilde{e}^{(\bar{k})} &:= \tilde{e}_{\beta_N^*}^{(k_N)} \dots \tilde{e}_{\beta_1^*}^{(k_1)}, & \tilde{f}^{(\bar{k})} &:= \tilde{f}_{\beta_N^*}^{(k_N)} \dots \tilde{f}_{\beta_1^*}^{(k_1)}. \end{aligned}$$

We have the following results analogous to Lemma 2.10:

**Lemma 4.4.** (a) The subalgebras  $\dot{\mathcal{U}}_{\mathcal{A}}^{*<} \subset \dot{\mathcal{U}}_{\mathcal{A}}^{*>} \subset \dot{\mathcal{U}}_{\mathcal{A}}^{*0}$  are  $Q^*$ -graded via (4.9), and each of their degree components is a free  $\mathcal{A}$ -module of finite rank.

(b) The multiplication map:

$$m: \dot{\mathcal{U}}_{\mathcal{A}}^{*<} \otimes_{\mathcal{A}} \dot{\mathcal{U}}_{\mathcal{A}}^{*0} \otimes_{\mathcal{A}} \dot{\mathcal{U}}_{\mathcal{A}}^{*>} \longrightarrow \dot{\mathcal{U}}_{\mathcal{A}}^*(\mathfrak{g})$$

is an isomorphism of free  $\mathcal{A}$ -modules.

(c) The sets  $\{\tilde{e}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$ ,  $\{\tilde{e}^{(\bar{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $\dot{\mathcal{U}}_{\mathcal{A}}^{*<}$ .

(d) The sets  $\{\tilde{f}^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$ ,  $\{\tilde{f}^{(\bar{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathcal{A}$ -bases of  $\dot{\mathcal{U}}_{\mathcal{A}}^{*>}$ .

(e) The subalgebra  $\dot{U}_{\mathcal{A}}^{*0}$  has the following  $\mathcal{A}$ -basis:

$$(4.13) \quad \left\{ K^{2\varsigma_j^*} \cdot \prod_{i=1}^r \left( (K_i^*)^2 \lfloor \frac{t_i}{2} \rfloor \binom{K_i^*; 0}{t_i} \right) \mid t_i \geq 0, 1 \leq j \leq k \right\},$$

where  $\varsigma_1^*, \dots, \varsigma_k^* \in P^*$  is a set of representatives of the left cosets  $P^*/Q^*$ .

Evoking the algebra homomorphism  $\sigma: \mathcal{A} \rightarrow R$  of (1.2), we define:

$$\dot{U}_\epsilon^*(\mathfrak{g}) := \dot{U}_{\mathcal{A}}^*(\mathfrak{g}) \otimes_{\mathcal{A}} R.$$

Let  $\tilde{e}_i^{(n)}, \tilde{f}_i^{(n)}, K^\mu, \binom{K_i^*; a}{n}$  denote the corresponding elements in  $\dot{U}_\epsilon^*(\mathfrak{g})$ , and let  $\dot{U}_\epsilon^{*>}, \dot{U}_\epsilon^{*<}, \dot{U}_\epsilon^{*0}$  be the corresponding  $R$ -subalgebras of  $\dot{U}_\epsilon^*(\mathfrak{g})$ .

We also note the following equalities in  $R$ :

$$\epsilon^{(\alpha_i^*, \kappa(\alpha_j^*))} = 1 = (\epsilon_i^*)^{ma_{ij}^*(\epsilon_{ij}-1)-m(m-1)}.$$

Therefore, in  $\dot{U}_\epsilon^*(\mathfrak{g})$ , the last two relations of (4.8) simplify as follows:

$$(4.14) \quad \begin{aligned} & \sum_{m=0}^{1-a_{ij}^*} (-1)^m \tilde{e}_i^{(1-a_{ij}^*-m)} \tilde{e}_j \tilde{e}_i^{(m)} = 0 \quad \text{for } i \neq j, \\ & \sum_{m=0}^{1-a_{ij}^*} (-1)^m \tilde{f}_i^{(1-a_{ij}^*-m)} \tilde{f}_j \tilde{f}_i^{(m)} = 0 \quad \text{for } i \neq j. \end{aligned}$$

Moreover, the generators  $\tilde{e}_i^{(n)}$  and  $\tilde{f}_j^{(n)}$  also satisfy the following relations, cf. (4.3):

$$(4.15) \quad \begin{aligned} \tilde{e}_i^{(p)} \tilde{f}_j^{(s)} &= \epsilon^{ps(\alpha_i^*, \kappa(\alpha_j^*))} \tilde{f}_j^{(s)} \tilde{e}_i^{(p)} = \tilde{f}_j^{(s)} \tilde{e}_i^{(p)} \quad \text{for } i \neq j, \\ \tilde{e}_i^{(p)} \tilde{f}_i^{(s)} &= \sum_{c=0}^{\min(p,s)} (\epsilon_i^*)^{2ps-c^2} \tilde{f}_i^{(s-c)} \binom{K_i^*; 2c-p-s}{c} \tilde{e}_i^{(p-c)}. \end{aligned}$$

#### 4.2.3. The idempotent Lusztig form $\widehat{U}_\epsilon^*(\mathfrak{g}, X^*)$ .

For any lattice  $X^*$  with  $Q^* \subseteq X^* \subseteq P^*$ , we also form the **idempotent Lusztig form**  $\widehat{U}_\epsilon^*(\mathfrak{g}, X^*)$  with the generators

$$\left\{ \tilde{e}_i^{(n)} 1_\lambda, \tilde{f}_i^{(n)} 1_\lambda \mid 1 \leq i \leq r, n \geq 0, \lambda \in X^* \right\}.$$

We record the topological coproduct in  $\widehat{U}_\epsilon^*(\mathfrak{g}, X^*)$ , cf. (4.4, 4.10):

$$(4.16) \quad \begin{aligned} \Delta(\tilde{e}_i^{(r)} 1_\lambda) &= \sum_{c=0}^r \prod_{\lambda' + \lambda'' = \lambda} \epsilon^{-(r-c)(\zeta_i^{*>} \lambda'', \lambda')} \tilde{e}_i^{(r-c)} 1_{\lambda'} \otimes \tilde{e}_i^{(c)} 1_{\lambda''}, \\ \Delta(\tilde{f}_i^{(r)} 1_\lambda) &= \sum_{c=0}^r \prod_{\lambda' + \lambda'' = \lambda} \epsilon^{c(\zeta_i^{*<} \lambda'', \lambda')} \tilde{f}_i^{(c)} 1_{\lambda'} \otimes \tilde{f}_i^{(r-c)} 1_{\lambda''}, \end{aligned}$$

as well as some relations in  $\widehat{U}_\epsilon^*(\mathfrak{g}, X^*)$ , cf. (4.5, 4.15):

$$(4.17) \quad \begin{aligned} \tilde{e}_i^{(p)} 1_\lambda \tilde{f}_j^{(s)} &= \tilde{f}_j^{(s)} 1_{\lambda+s\alpha_j^*+p\alpha_i^*} \tilde{e}_i^{(p)} \quad \text{for } i \neq j, \\ \tilde{e}_i^{(p)} 1_\lambda \tilde{f}_i^{(s)} &= \sum_{c \geq 0} (\epsilon_i^*)^{2ps-c^2} \binom{(\lambda, \alpha_i^{*\vee}) + s + p}{c} \tilde{f}_i^{(s-c)} 1_{\lambda+(p+s-c)\alpha_i^*} \tilde{e}_i^{(p-c)}, \end{aligned}$$

with  $\lambda, \lambda', \lambda'' \in X^*$  in the formulas above.

### 4.3. The Kostant form $\dot{U}_R(\mathfrak{g}^d)$ .

It turns out that the algebra  $\widehat{U}_\epsilon^*(\mathfrak{g}, X^*)$  is closely related to a classical object, the Kostant form of the universal enveloping algebra.

Let us recall the semisimple Lie algebra  $\mathfrak{g}^d$  with the Cartan matrix (4.6), the weight lattice  $P^*$ , and the root lattice  $Q^*$  from the previous subsections. Let  $\{e_i, f_i, h_i\}_{1 \leq i \leq r}$  be the Chevalley generators of  $\mathfrak{g}^d$ . Following [Ko], the **Kostant form**  $\dot{U}_\mathbb{Z}(\mathfrak{g}^d)$  is the  $\mathbb{Z}$ -subalgebra of the enveloping algebra  $U_\mathbb{Q}(\mathfrak{g}^d)$  over  $\mathbb{Q}$  generated by the following elements:

$$e_i^{(n)} := \frac{e_i^n}{n!}, \quad f_i^{(n)} := \frac{f_i^n}{n!}, \quad \binom{h_i; a}{n} := \frac{(h_i + a)(h_i + a - 1) \cdots (h_i + a - n + 1)}{n!}.$$

Let  $\dot{U}_\mathbb{Z}^>(\mathfrak{g}^d), \dot{U}_\mathbb{Z}^<(\mathfrak{g}^d), \dot{U}_\mathbb{Z}^0(\mathfrak{g}^d)$  be the  $\mathbb{Z}$ -subalgebras of  $U_\mathbb{Q}(\mathfrak{g}^d)$  generated by elements  $\{e_i^{(n)}\}, \{f_i^{(n)}\}, \{\binom{h_i; a}{n}\}$ , respectively. We have the standard braid group action on  $U_\mathbb{Q}(\mathfrak{g}^d)$  defined by

$$\begin{aligned} T_i^*(h_j) &= s_i^*(h_j), & T_i^*(e_i) &= -f_i, & T_i^*(f_i) &= -e_i, \\ T_i^*(e_j) &= \sum_{k=0}^{-a_{ij}^*} (-1)^k e_i^{(-a_{ij}^*-k)} e_j e_i^{(k)}, & T_i^*(f_j) &= \sum_{k=0}^{-a_{ij}^*} (-1)^k f_i^{(k)} f_j f_i^{(-a_{ij}^*-k)} & (i \neq j). \end{aligned}$$

Evoking the reduced decomposition of the longest element  $w_0 = s_{i_1}^* \dots s_{i_N}^*$  in the Weyl group of  $\mathfrak{g}^d$  and the resulting labeling  $\beta_k^*$  of all  $\Delta_+^d$ , one defines root vectors of  $\mathfrak{g}^d$  in a standard way:

$$e_{\beta_k^*} := T_{i_1}^* \dots T_{i_{k-1}}^* e_{i_k}, \quad f_{\beta_k^*} := T_{i_1}^* \dots T_{i_{k-1}}^* f_{i_k}.$$

For any  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$ , we define

$$e^{(\vec{k})} = e_{\beta_1^*}^{(k_1)} \dots e_{\beta_N^*}^{(k_N)}, \quad e^{(\vec{k})} := e_{\beta_N^*}^{(k_N)} \dots e_{\beta_1^*}^{(k_1)}, \quad f^{(\vec{k})} := f_{\beta_1^*}^{(k_1)} \dots f_{\beta_N^*}^{(k_N)}, \quad f^{(\vec{k})} := f_{\beta_N^*}^{(k_N)} \dots f_{\beta_1^*}^{(k_1)}.$$

This construction gives a Chevalley-type basis of  $\mathfrak{g}^d$  as used in [Ko] (we left the proof to interested readers). Then we can apply [Ko, Theorem 1] to get:

**Lemma 4.5.** (a) *The multiplication map:*

$$m: \dot{U}_\mathbb{Z}^<(\mathfrak{g}^d) \otimes_{\mathbb{Z}} \dot{U}_\mathbb{Z}^0(\mathfrak{g}^d) \otimes_{\mathbb{Z}} \dot{U}_\mathbb{Z}^>(\mathfrak{g}^d) \longrightarrow \dot{U}_\mathbb{Z}(\mathfrak{g}^d)$$

is an isomorphism of free  $\mathbb{Z}$ -modules.

(b) The sets  $\{e^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{e^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Z}$ -bases of  $\dot{U}_\mathbb{Z}^>(\mathfrak{g}^d)$ .

(c) The sets  $\{f^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}, \{f^{(\vec{k})}\}_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}$  are  $\mathbb{Z}$ -bases of  $\dot{U}_\mathbb{Z}^<(\mathfrak{g}^d)$ .

(d) The set  $\left\{ \prod_{i=1}^r \binom{h_i; 0}{t_i} \mid t_i \geq 0 \right\}$  is a  $\mathbb{Z}$ -basis of  $\dot{U}_\mathbb{Z}^0(\mathfrak{g}^d)$ .

For any commutative ring  $R$ , we define the **Kostant form**  $\dot{U}_R(\mathfrak{g}^d)$  via:

$$\dot{U}_R(\mathfrak{g}^d) := \dot{U}_\mathbb{Z}(\mathfrak{g}^d) \otimes_{\mathbb{Z}} R.$$

Let  $\dot{U}_R^>(\mathfrak{g}^d), \dot{U}_R^0(\mathfrak{g}^d), \dot{U}_R^<(\mathfrak{g}^d)$  be the corresponding  $R$ -subalgebras of  $\dot{U}_R(\mathfrak{g}^d)$ . The next lemma relates the positive and negative subalgebras of  $\dot{U}_\epsilon^*(\mathfrak{g})$  and  $\dot{U}_R(\mathfrak{g}^d)$ , compare to [L5, §33.2].

**Lemma 4.6.** (a) *There is a unique isomorphism of  $R$ -algebras  $\mathcal{F}^>: \dot{U}_\epsilon^*> \rightarrow \dot{U}_R^>(\mathfrak{g}^d)$  such that  $\mathcal{F}^>(\tilde{e}_i^{(n)}) = e_i^{(n)}$  for all  $n \geq 1, 1 \leq i \leq r$ .*

(b) *There is a unique isomorphism of  $R$ -algebras  $\mathcal{F}^<: \dot{U}_\epsilon^*< \rightarrow \dot{U}_R^<(\mathfrak{g}^d)$  such that  $\mathcal{F}^<(\tilde{f}_i^{(n)}) = f_i^{(n)}$  for all  $n \geq 1, 1 \leq i \leq r$ .*

*Proof.* We write  $\dot{U}_\epsilon^{*>}$  instead of  $\dot{U}_R^{*>}$  to keep track of the base ring  $R$ .

(a) The uniqueness is obvious. To prove the existence, we note that the general case follows from the case  $R = \mathcal{A}'$  through the base change. Since  $\mathcal{A}'$  is an integral domain, it is embedded into its fraction field  $\mathcal{K}$ . The algebra  $\dot{U}_{\mathcal{A}'}^{*>}$  is thus embedded into  $\dot{U}_{\mathcal{K}}^{*>}$  as the  $\mathcal{A}'$ -subalgebra generated by  $\{\tilde{e}_i^{(n)}\}$  by Lemma 4.4. Likewise, the algebra  $\dot{U}_{\mathcal{A}'}^>(\mathfrak{g}^d)$  is embedded in  $\dot{U}_{\mathcal{K}}^>(\mathfrak{g}^d)$  as the  $\mathcal{A}'$ -subalgebra generated by  $\{e_i^{(n)}\}$  by Lemma 4.5. Therefore, the existence of the claimed isomorphism in the case  $R = \mathcal{A}'$  follows from its existence in the case  $R = \mathcal{K}$ .

Since  $\mathcal{K}$  is a field of characteristics 0, the algebra  $\dot{U}_{\mathcal{K}}^>(\mathfrak{g}^d)$  is generated by the elements  $e_i$  subject to the usual Serre relations. Therefore, due to (4.14), there is a  $\mathcal{K}$ -algebra homomorphism  $(\mathcal{F}^>)^{-1}: \dot{U}_{\mathcal{K}}^>(\mathfrak{g}^d) \rightarrow \dot{U}_\epsilon^{*>}$  defined by  $(\mathcal{F}^>)^{-1}(e_i) = \tilde{e}_i$ , so that  $(\mathcal{F}^>)^{-1}(e_i^{(n)}) = \tilde{e}_i^{(n)}$  for all  $n \geq 1$ ,  $1 \leq i \leq r$ . Therefore,  $(\mathcal{F}^>)^{-1}$  is surjective. Furthermore,  $(\mathcal{F}^>)^{-1}$  preserves the  $Q^*$ -grading on both algebras. By Lemmas 4.4 and 4.5, the finite dimensional  $\mathcal{K}$ -vector spaces  $\dot{U}_{\mathcal{K},\mu}^{*>}$  and  $\dot{U}_{\mathcal{K},\mu}^>(\mathfrak{g}^d)$  have the same dimension for any  $\mu \in Q_+^*$ . Therefore,  $(\mathcal{F}^>)^{-1}$  is an isomorphism of  $\mathcal{K}$ -algebras. The inverse is the claimed isomorphism  $\mathcal{F}^>$  for  $R = \mathcal{K}$ .

(b) The proof is the same.  $\square$

**Corollary 4.7.** *If  $R$  is a field of characteristics 0, then the algebra  $\dot{U}_\epsilon^{*>}$  is an algebra generated by  $\{\tilde{e}_i\}_{1 \leq i \leq r}$  subject to  $\sum_{m=0}^{1-a_{ij}^*} (-1)^m \tilde{e}_i^{(1-a_{ij}^*-m)} \tilde{e}_j \tilde{e}_i^{(m)} = 0$  for  $i \neq j$ , while the algebra  $\dot{U}_\epsilon^{*<}$  is an algebra generated by  $\{\tilde{f}_i\}_{1 \leq i \leq r}$  subject to  $\sum_{m=0}^{1-a_{ij}^*} (-1)^m \tilde{f}_i^{(1-a_{ij}^*-m)} \tilde{f}_j \tilde{f}_i^{(m)} = 0$  for  $i \neq j$ .*

#### 4.4. Completed forms $\tilde{U}_\epsilon(\mathfrak{g}, P)$ , $\tilde{U}_\epsilon^*(\mathfrak{g}, P^*)$ , $\tilde{U}_\epsilon^*(\mathfrak{g}, Q^*)$ .

Let us consider the following completion of  $\dot{U}_\epsilon(\mathfrak{g}, P)$

$$\tilde{U}_\epsilon(\mathfrak{g}, P) := \bigoplus_{\mu, \lambda \in Q_+} \dot{U}_{-\mu}^< \otimes_R \prod_{\nu \in P} R1_\nu \otimes_R \dot{U}_\lambda^>.$$

The topology of  $\tilde{U}_\epsilon(\mathfrak{g}, P)$  comes from the factor  $\prod_{\nu \in P} R1_\nu$  as follows: small neighborhoods of zero in  $\prod R1_\nu$  contain elements  $\prod a_\nu 1_\nu$  with  $a_\nu = 0$  for all  $\nu$  in a suitably large ball  $\mathcal{B}$  centered at the origin in the real vector space  $\mathfrak{h}_{\mathbb{R}}^*$  containing  $P$ . Note that the topological Hopf structure on  $\widehat{U}_\epsilon(\mathfrak{g}, P)$  extends to a topological Hopf structure on  $\tilde{U}_\epsilon(\mathfrak{g}, P)$ .

Similarly, we consider the topological Hopf algebras which are completions of  $\widehat{U}^*(\mathfrak{g}, P^*)$  and  $\widehat{U}^*(\mathfrak{g}, Q^*)$ , respectively:

$$\tilde{U}_\epsilon^*(\mathfrak{g}, P^*) := \bigoplus_{\mu, \lambda \in Q_+^*} \dot{U}_{-\mu}^< \otimes_R \prod_{\nu \in P^*} R1_\nu \otimes_R \dot{U}_\lambda^{*>}, \quad \tilde{U}_\epsilon^*(\mathfrak{g}, Q^*) := \bigoplus_{\mu, \lambda \in Q_+^*} \dot{U}_{-\mu}^< \otimes_R \prod_{\nu \in Q^*} R1_\nu \otimes_R \dot{U}_\lambda^{*>}.$$

There are natural algebra homomorphisms:

$$(4.18) \quad \psi_1: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P) \quad \text{and} \quad \psi_2: \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P).$$

The map  $\psi_2$  is a natural embedding. The map  $\psi$  is defined by  $\psi_1(yu_0x) = y(\prod_\nu \hat{\chi}_\nu(u_0)1_\nu)x$  for  $y \in \dot{U}_\epsilon^<$ ,  $x \in \dot{U}_\epsilon^>$  and  $u_0 \in \dot{U}_\epsilon^0$ .

**Lemma 4.8.** *Both  $\psi_1$  and  $\psi_2$  are Hopf algebra homomorphisms. The images of  $\psi_1$  and  $\psi_2$  from (4.18) are dense subalgebras in the codomains.*

*Proof.* The image of  $\psi_2$  is clearly dense in  $\tilde{U}_\epsilon(\mathfrak{g}, P)$ . To show that the image of  $\psi_1$  is dense in  $\tilde{U}_\epsilon(\mathfrak{g}, P)$ , it is enough to show that the image of the natural map  $\psi_1^0: \dot{U}_\epsilon^0(\mathfrak{g}) \rightarrow \prod_\nu R1_\nu$  is dense. To this end, it suffices to show that for any ball  $\mathcal{B}$  centered at the origin, the map  $\dot{U}_\epsilon^0(\mathfrak{g}) \rightarrow \prod_{\nu \in \mathcal{B} \cap P} R1_\nu$  is surjective. Since the cardinality of  $\mathcal{B} \cap P$  is finite, it is enough to show that for any finite set  $S \subset P$ , the map  $\psi_1^S: \dot{U}_\epsilon^0(\mathfrak{g}) \rightarrow \prod_{\nu \in S} R1_\nu$  is surjective. The last

statement follows from the case  $R = \mathcal{A}_N := \mathcal{A}[v^{\pm 1/N}]$  through the base change, which we shall deduce from the case when  $R$  is a field.

*Step 1:  $R = \mathbb{F}$  is a field.* Following (3.28), let us consider the algebra homomorphisms  $\hat{\chi}_\nu : \dot{U}_\epsilon^0(\mathfrak{g}) \rightarrow \mathbb{F}$  defined by

$$(4.19) \quad \hat{\chi}_\nu : \quad K^\lambda \mapsto \epsilon^{(\lambda, \nu)}, \quad \binom{K_i; 0}{t} \mapsto \binom{(\alpha_i^\vee, \nu)}{t}_{\epsilon_i}.$$

The characters  $\{\hat{\chi}_\nu\}_{\nu \in S}$  of characters of  $\dot{U}_\epsilon^0(\mathfrak{g})$  are linearly independent, cf. the proof of Proposition 3.19(b). The map  $\dot{U}_\epsilon^0(\mathfrak{g}) \rightarrow \prod_{\nu \in \mathcal{B} \cap P} R1_\nu$  sends an element  $a$  to  $(\hat{\chi}_\nu(a))_{\nu \in S}$  finishing the proof.

*Step 2:  $R = \mathcal{A}_N$ .* For a maximal ideal  $\mathfrak{p}$  of  $\mathcal{A}_N$ , let  $\mathcal{A}_{N\mathfrak{p}}$  and  $\mathbb{F}_\mathfrak{p}$  be the localization and the residue field of  $\mathcal{A}_N$ . The map  $\psi_1^S$  is surjective if the maps  $\psi_{1, \mathcal{A}_{N\mathfrak{p}}}^S : \dot{U}_{\mathcal{A}_{N\mathfrak{p}}}^0(\mathfrak{g}) \rightarrow \prod_{\nu \in S} \mathcal{A}_{N\mathfrak{p}} 1_\nu$  are surjective for all maximal ideals  $\mathfrak{p}$  of  $\mathcal{A}_N$  because the cokernel of  $\psi_1^S$  is a finitely generated module annihilated by the localization at any maximal ideal and such a module must be 0. Since  $\prod_{\nu \in S} \mathcal{A}_{N\mathfrak{p}} 1_\nu$  is a finitely generated  $\mathcal{A}_{N\mathfrak{p}}$ -module, due to Nakayama's lemma,  $\psi_{1, \mathcal{A}_{N\mathfrak{p}}}^S$  is surjective iff the map  $\psi_{1, \mathbb{F}_\mathfrak{p}}^S : \dot{U}_{\mathbb{F}_\mathfrak{p}}^0(\mathfrak{g}) \rightarrow \prod_{\nu \in S} \mathbb{F}_\mathfrak{p} 1_\nu$  is surjective. The latter follows by Step 1.  $\square$

**Remark 4.9.** Similarly, we have Hopf algebra homomorphisms

$$\psi_1^* : \dot{U}_\epsilon^*(\mathfrak{g}) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*), \quad \psi_2^* : \widehat{U}_\epsilon^*(\mathfrak{g}, P^*) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*),$$

with dense images in the codomains. Moreover,  $\psi_2^*$  is an inclusion.

**Lemma 4.10.** (a) We have the following algebra homomorphisms:

$$(4.20) \quad \phi_P : \dot{U}_R(\mathfrak{g}^d) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*), \quad \phi_Q : \dot{U}_R(\mathfrak{g}^d) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, Q^*),$$

defined by

$$\phi_X(e_i^{(n)}) = (\epsilon_i^*)^n \tilde{e}_i^{(n)} \prod_{\nu \in X^*} 1_\nu, \quad \phi_X(f_i^{(n)}) = \tilde{f}_i^{(n)} \prod_{\nu \in X^*} 1_\nu, \quad \phi_X \left( \binom{h_i; 0}{t_i} \right) = \prod_{\nu \in X^*} \binom{(\nu, \alpha_i^{*\vee})}{t_i} 1_\nu,$$

with  $1 \leq i \leq r$ ,  $t_i \in \mathbb{Z}_{\geq 0}$ , and  $X$  is either  $P$  or  $Q$ .

(b)  $\phi_P, \phi_Q$  are inclusions with dense images.

*Proof.* (a) The general case follows by the base change from the case when  $R = \mathcal{A}'$ . Since  $\dot{U}_R(\mathfrak{g}^d)$  and  $\tilde{U}_\epsilon^*(\mathfrak{g}, X^*)$  are torsion free over the domain  $R = \mathcal{A}'$ , the case when  $R = \mathcal{A}'$  follows by the case when  $R = \mathcal{K}$ , the fraction field of  $\mathcal{A}'$ .

When  $R = \mathcal{K}$ , the existence of algebra homomorphisms follows by using the Serre presentation of the enveloping algebra  $\dot{U}_\mathcal{K}(\mathfrak{g}^d)$  combining with (4.17) and Lemma 4.6.

(b) It is enough to show that  $\phi_X^0 : \dot{U}_R^0(\mathfrak{g}^d) \rightarrow \prod_{\nu \in X^*} R1_\nu$  is injective. Let  $T_X^d$  be the torus with the lattice  $X^*$ , i.e.,  $R[T_X^d] = \bigoplus_{\nu \in X^*} R\chi_\nu$ , hence,  $\prod_{\nu \in X^*} R1_\nu = \text{Hom}_R(R[T_X^d], R)$ . On the other hand  $\dot{U}_R^0(\mathfrak{g}^d)$  is  $\text{Dist}(T_X^d)$ , the distribution algebra of  $T_X^d$  at identity. Then one see that the map  $\phi_X^0$  can be obtained from the pairing

$$\text{Dist}(T_X^d) \times R[T_X^d] \rightarrow R.$$

This pairing is nondegenerate on the first argument, hence  $\phi_X^0$  is injective. The proof of dense images are the same as the one of Lemma 4.8.  $\square$

**Remark 4.11.** While  $\phi_Q$  is a Hopf algebra homomorphism,  $\phi_P$  is not, e.g., by looking at the coproduct

$$\tilde{\Delta}(\tilde{e}_i \prod_{\nu \in P^*} 1_\nu) = \tilde{e}_i \prod_{\nu \in P^*} 1_\nu \otimes \prod_{\nu \in P^*} \epsilon^{(-\zeta_i^*, \nu)} 1_\nu + \prod_{\nu \in P^*} 1_\nu \otimes \tilde{e}_i \prod_{\nu \in P^*} 1_\nu.$$

#### 4.5. Quantum Frobenius homomorphism.

Following [L5, Section 35.1.2], let us assume that for any  $i \neq j$  such that  $\ell_j \geq 2$ , we have  $\ell_i \geq 1 - a_{ij}$ . The excluded values of  $\ell$  for each type of simple Lie algebras are recorded in the following table:

A	B	C	D	E	F	G
$\emptyset$	4	4	$\emptyset$	$\emptyset$	4	3, 4, 6

We have the following analogue of [L5, Theorem 35.1.9]:

**Proposition 4.12.** *There is a unique  $R$ -algebra homomorphism (twisted quantum Frobenius)*

$$(4.21) \quad \tilde{\text{Fr}}: \widehat{U}_\epsilon(\mathfrak{g}, P) \longrightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$$

such that

- $\tilde{\text{Fr}}(1_\lambda)$  equals  $1_\lambda$  if  $\lambda \in P^*$  and is zero otherwise,
- $\tilde{\text{Fr}}(\tilde{E}_i^{(n)} 1_\lambda)$  equals  $\tilde{e}_i^{(n/\ell_i)} 1_\lambda$  if  $\lambda \in P^*$  and  $n$  is divisible by  $\ell_i$ , and is zero otherwise;
- $\tilde{\text{Fr}}(\tilde{F}_i^{(n)} 1_\lambda)$  equals  $\tilde{f}_i^{(n/\ell_i)} 1_\lambda$  if  $\lambda \in P^*$  and  $n$  is divisible by  $\ell_i$ , and is zero otherwise.

Furthermore, this homomorphism is surjective and compatible with the comultiplications.

We note that the assumption that there is  $\epsilon^{1/\mathbb{N}}$  in  $R$  is only needed when we talk about the comultiplications. The proof of Proposition 4.12 is similar to that of [L5, Theorem 35.1.9]. We start with the following result:

**Proposition 4.13.** (a) *There is a unique  $R$ -algebra homomorphism  $\widehat{\text{Fr}}^<: \dot{U}_\epsilon^*< \rightarrow \dot{U}_\epsilon^<$  such that  $\widehat{\text{Fr}}^<(\tilde{f}_i^{(n)}) = \tilde{F}_i^{(n\ell_i)}$  for all  $i$  and  $n \geq 1$ . Similarly, there is a unique  $R$ -algebra homomorphism  $\widehat{\text{Fr}}^>: \dot{U}_\epsilon^*> \rightarrow \dot{U}_\epsilon^>$  such that  $\widehat{\text{Fr}}^>(\tilde{e}_i^{(n)}) = \tilde{E}_i^{(n\ell_i)}$  for all  $i$  and  $n \geq 1$ .*

(b) *There is a unique  $R$ -algebra homomorphism:*

$$(4.22) \quad \tilde{\text{Fr}}^>: \dot{U}_\epsilon^> \longrightarrow \dot{U}_\epsilon^*>$$

such that  $\tilde{\text{Fr}}^>(\tilde{E}_i^{(n)})$  equals  $\tilde{e}_i^{(n/\ell_i)}$  if  $n$  is divisible by  $\ell_i$ , and equals zero otherwise. Similarly, there is a unique  $R$ -algebra homomorphism

$$(4.23) \quad \tilde{\text{Fr}}^<: \dot{U}_\epsilon^< \longrightarrow \dot{U}_\epsilon^*<$$

such that  $\tilde{\text{Fr}}^<(\tilde{F}_i^{(n)})$  equals  $\tilde{f}_i^{(n/\ell_i)}$  if  $n$  is divisible by  $\ell_i$ , and equals zero otherwise.

*Proof.* The proof is the same as in [L5]. One reduces to the case when  $R = \mathcal{K}$ , the fraction field of  $\mathcal{A}'$ .

(a) The uniqueness is clear. The existence follows by using the generators and relations of  $\dot{U}_\epsilon^*<$  in Corollary 4.7 and the following identity

$$\sum_{n+m=1-a_{ij}^*} (-1)^m \tilde{F}_i^{(\ell_i n)} \tilde{F}_j^{(\ell_j)} \tilde{F}_i^{(\ell_i m)} = 0.$$

This identity is deduced from the formula [L5, §35.2.3(b)].

(b) Let us prove (4.23) only. We use the following lemma which is a version of Theorem 35.4.2(b) in [L5] incorporating a Cartan twist

**Lemma 4.14.** *Let  $\tilde{\mathfrak{f}}$  be the  $\mathcal{K}$ -subalgebra of  $\dot{U}_\epsilon^<$  generated by  $\tilde{F}_i$  with  $\ell_i \geq 2$ . There is an isomorphism of  $\mathcal{K}$ -vector spaces  $\tilde{\mathfrak{f}}: \dot{U}_\epsilon^{*<} \otimes \tilde{\mathfrak{f}} \rightarrow \dot{U}_\epsilon^<$  defined by  $x \otimes y \mapsto \widehat{\text{Fr}}^<(x)y$ .*

With this lemma, one proceeds exactly the same as the proof of [L5, Theorem 35.1.7] to finish the proof.  $\square$

**Remark 4.15.** First, we note that the compositions  $\tilde{\text{Fr}}^> \circ \widehat{\text{Fr}}^>$  and  $\widehat{\text{Fr}}^< \circ \tilde{\text{Fr}}^<$  are identities. Second, we claim that the kernel of the homomorphism  $\tilde{\text{Fr}}^>: \dot{U}_\epsilon^> \rightarrow \dot{U}_\epsilon^{*>}$  from (4.22) is a two-sided ideal generated by  $\{\tilde{E}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}$ , and likewise the kernel of the homomorphism  $\tilde{\text{Fr}}^<: \dot{U}_\epsilon^< \rightarrow \dot{U}_\epsilon^{*<}$  from (4.23) is a two-sided ideal generated by  $\{\tilde{F}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}$ . Let us establish the kernel of the map  $\tilde{\text{Fr}}^>$ . It is enough to show that  $\tilde{\text{Fr}}^>(x) = 0$  implies  $x = 0$  for the elements of the following form:  $x = \sum a_{n_1, \dots, n_m} \tilde{E}_{i_1}^{(n_1)} \dots \tilde{E}_{i_m}^{(n_m)}$  with  $\ell_{i_j} \mid n_j$  and  $a_{n_1, \dots, n_m} \in R$ . For such  $x$ , consider  $\tilde{x} = \sum a_{n_1, \dots, n_m} \tilde{e}_{i_1}^{(n_1/\ell_{i_1})} \dots \tilde{e}_{i_m}^{(n_m/\ell_{i_m})}$ , so that  $x = \widehat{\text{Fr}}^>(\tilde{x})$  and  $\tilde{x} = \tilde{\text{Fr}}^>(x)$ . Therefore, if  $\tilde{\text{Fr}}(x) = 0$  then  $x = 0$ .

*Proof of Proposition 4.12.* In [L5], the proof of Theorem 35.1.9 is based on the presentation of the idempotent version by generators and relations, established in Section 31.1.3. The similar presentation also holds in our Cartan-twisted setup since the Cartan elements are all invertible. The only non-trivial relations we need to check are the ones between  $\tilde{E}_i^{(n)}$  and  $\tilde{F}_j^{(m)}$ , but those directly follow by comparing the formulas (4.5) and (4.17). It is straightforward to check that the homomorphism  $\tilde{\text{Fr}}$  is compatible with the coproducts in (4.4) and (4.16).  $\square$

**Remark 4.16.** We have a natural adjoint action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $\widehat{U}_\epsilon(\mathfrak{g}, P)$  via  $\text{ad}'_l$ . Via the Hopf morphism  $\tilde{\text{Fr}}: \dot{U}_\epsilon(\mathfrak{g}, P) \rightarrow \dot{U}_\epsilon^*(\mathfrak{g}, P^*)$ , the Hopf algebra  $\dot{U}_\epsilon(\mathfrak{g})$  naturally acts on  $\dot{U}_\epsilon^*(\mathfrak{g}, P^*)$  via  $\text{ad}'_l$  as follows:  $\text{ad}'_l(x)b = \tilde{\text{Fr}}(\text{ad}'_l(x)b')$  for  $b \in \dot{U}_\epsilon^*(\mathfrak{g}, P^*)$  and any  $b' \in \widehat{U}_\epsilon(\mathfrak{g}, P)$  such that  $\tilde{\text{Fr}}(b') = b$ .

Let us recall the Kostant form  $\dot{U}_R(\mathfrak{g}^d)$  from Subsection 4.3. Then, we have the following twisted Frobenius homomorphism from the original Lusztig forms:

**Proposition 4.17.** <sup>5</sup> *There is a unique  $R$ -Hopf algebra homomorphism (twisted quantum Frobenius)*

$$\check{\text{Fr}}: \dot{U}_\epsilon(\mathfrak{g}) \longrightarrow \dot{U}_R(\mathfrak{g}^d)$$

such that

$$\tilde{E}_i^{(n)} \mapsto (\epsilon_i^*)^{-n/\ell_i} e_i^{(n/\ell_i)}, \quad \tilde{F}_i^{(n)} \mapsto f_i^{(n/\ell_i)}, \quad K^\lambda \mapsto 1,$$

where  $\lambda \in 2P$  and we set  $e_i^{(n/\ell_i)} = f_i^{(n/\ell_i)} = 0$  if  $\ell_i$  does not divide  $n$ .

*Proof.* Let us recall the topological Hopf algebras  $\tilde{U}_\epsilon(\mathfrak{g}, P)$  and  $\tilde{U}_\epsilon^*(\mathfrak{g}, Q^*)$  in Section 4.4. The Hopf algebra homomorphism in Proposition 4.12 gives us a (topological) Hopf algebra homomorphism:

$$(4.24) \quad \check{\text{Fr}}: \tilde{U}_\epsilon(\mathfrak{g}, P) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, Q^*).$$

On the other hand, we have the following Hopf algebras morphisms:

$$\psi_1: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P), \quad \phi_Q: \dot{U}_R(\mathfrak{g}^d) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, Q^*),$$

---

<sup>5</sup>There is no such proposition with the usual coproduct of quantum groups.

in which  $\phi_Q$  is an inclusion, see Section 4.4. Furthermore, the images of  $\psi_1(\tilde{E}_i^{(n)})$ ,  $\psi_1(\tilde{F}_i^{(n)})$ ,  $\psi_1(K^{2\lambda})$  under (4.24) are contained in  $\phi_Q(\dot{U}_R(\mathfrak{g}^d))$ <sup>6</sup>. Therefore, the Hopf algebra homomorphism

$$\dot{U}_\epsilon(\mathfrak{g}) \xrightarrow{\psi_1} \tilde{U}(\mathfrak{g}, P) \xrightarrow{\check{\text{Fr}}} \tilde{U}(\mathfrak{g}, Q^*)$$

gives a rise to the desired Hopf algebra homomorphism  $\dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$ .  $\square$

**Lemma 4.18.** *The kernel of  $\check{\text{Fr}}: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  is the two-sided ideal generated by*

$$(4.25) \quad \{K^\lambda - 1 \mid \lambda \in 2P\} \cup \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}.$$

*Proof.* We have

$$\check{\text{Fr}} \circ \psi_1 \left( \binom{K_i; 0}{t_i} \right) = \prod_{\nu \in Q^*} \binom{(\nu, \alpha_i^\vee)}{t_i}_{\epsilon_i} 1_\nu = \begin{cases} 0 & \text{if } \ell_i \nmid t_i \\ \prod_{\nu \in Q^*} \binom{(\nu, \alpha_i^{*\vee})}{t_i/\ell_i} 1_\nu & \text{if } \ell_i \mid t_i \end{cases} = \phi_Q \left( \binom{h_i; 0}{t_i/\ell_i} \right),$$

therefore,

$$\phi \left( \binom{K_i; 0}{t_i} \right) = \begin{cases} 0 & \text{if } \ell_i \nmid t_i \\ \binom{h_i; 0}{t_i/\ell_i} & \text{if } \ell_i \mid t_i \end{cases}.$$

*Step 1:* We will show that  $\text{Ker}(\check{\text{Fr}})$  is the two-sided ideal generated by

$$\{K^\lambda - 1 \mid \lambda \in 2P\} \cup \left\{ \binom{K_i; 0}{t_i} \mid \ell_i \nmid t_i \right\} \cup \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}.$$

The map  $\check{\text{Fr}}$  is compatible with triangular decomposition, that is  $\check{\text{Fr}} \cong \check{\text{Fr}}^< \otimes \check{\text{Fr}}^0 \otimes \check{\text{Fr}}^>$  with

$$\check{\text{Fr}}^>: \dot{U}_\epsilon^> \rightarrow \dot{U}_R^>(\mathfrak{g}^d), \quad \check{\text{Fr}}^0: \dot{U}_\epsilon^0 \rightarrow \dot{U}_R^0(\mathfrak{g}^d), \quad \check{\text{Fr}}^<: \dot{U}_\epsilon^< \rightarrow \dot{U}_R^<(\mathfrak{g}^d).$$

By Remark 4.15,  $\text{Ker}(\check{\text{Fr}}^>)$  is the two side ideal generated by  $\{\tilde{E}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}$ , meanwhile,  $\text{ker}(\check{\text{Fr}}^<)$  is the two sided ideal generated by  $\{\tilde{F}_i^{(n)} \mid \ell_i \geq 2, \ell_i \nmid n\}$ . On the other hand, using the PBW-basis in Lemma 2.10(c3), one can see that  $\text{Ker}(\check{\text{Fr}}^0)$  is the two-sided ideal generated by

$$\{K^\lambda - 1 \mid \lambda \in 2P\} \cup \left\{ \binom{K_i; 0}{t_i} \mid \ell_i \nmid t_i \right\}.$$

All of  $\check{\text{Fr}}^>, \check{\text{Fr}}^<, \check{\text{Fr}}^0$  are surjective. All of  $\dot{U}_R^>(\mathfrak{g}^d), \dot{U}_R^0(\mathfrak{g}^d), \dot{U}_R^<(\mathfrak{g}^d)$  are free  $R$ -modules. Hence  $\check{\text{Fr}}^<, \check{\text{Fr}}^0, \check{\text{Fr}}^>$  are split morphisms of  $R$ -modules. Therefore, the compatibility of  $\check{\text{Fr}}$  with triangular decompositions implies that

$$\text{Ker}(\check{\text{Fr}}) = \text{Ker}(\check{\text{Fr}}^<) \otimes \check{\text{Fr}}^0 \otimes \check{\text{Fr}}^> + \check{\text{Fr}}^< \otimes \text{Ker}(\check{\text{Fr}}^0) \otimes \check{\text{Fr}}^> + \check{\text{Fr}}^< \otimes \check{\text{Fr}}^0 \otimes \text{Ker}(\check{\text{Fr}}^>).$$

This finishes the first step .

*Step 2:* The lemma follows if we can show that  $\binom{K_i; -c}{t_i}, \ell_i \nmid t_i, \ell_i \mid c \geq 0$  belongs to the two-sided ideal  $\mathcal{J}$  generated by (4.25). We have

$$(4.26) \quad \binom{K_i; -c}{t} = \sum_{0 \leq k \leq t} (-1)^k \epsilon_i^{2t(c+k)-k(k+1)} \binom{c+k-1}{k} \binom{K_i; 0}{t-k} \quad (t \geq 0, c \geq 1)$$

Since  $\binom{c+k-1}{k} = 0$  for  $\ell_i \mid c$  and  $\ell_i \nmid k$ , we have

$$\binom{K_i; -c}{t} = \sum_{0 \leq k \leq t, \ell_i \nmid k} (-1)^k \epsilon_i^{2t(c+k)-k(k+1)} \binom{c+k-1}{k} \binom{K_i; 0}{t-k}, \quad (t \geq 0, c \geq 1, \ell_i \mid c).$$

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<sup>6</sup>We want to note that  $\check{\text{Fr}}(\psi_1(K^{2\lambda})) = \prod_{\nu \in Q^*} 1_\nu$ , only happen when  $2\lambda \in 2P$  and  $\nu$  runs over  $Q^*$ , another justification for the lattice restrictions in our arguments.

This implies that  $\binom{K_i; -c}{t}$  with  $c \geq 1, \ell_i \mid c, \ell_i \nmid t$  is a linear combination of  $\binom{K_i; 0}{t'}$  with  $\ell_i \nmid t' \leq t$ . Now the second step is proved by induction on  $t_i$  and the following equation

$$(4.27) \quad \tilde{E}_i^{(p)} \tilde{F}_i^{(p)} = \sum_{c=0}^{\min(p,s)} \epsilon_i^{2ps-c^2} \tilde{F}_i^{(p-c)} \binom{K_i; 2c-2p}{c} \tilde{E}_i^{(p-c)}.$$

By using (4.27) for  $0 < p < \ell_i$ , we have  $\binom{K_i; 0}{t_i} \in \mathcal{J}$  for  $0 < t_i < \ell_i$ , hence  $\binom{K_i; -c}{t_i}$  with  $c \geq 0, \ell_i \mid c, 0 < t_i < \ell_i$  belongs to  $\mathcal{J}$ . By induction and (4.27),  $\binom{K_i; 0}{t} \in \mathcal{J}$  for  $\ell_i \nmid t$  hence  $\binom{K_i; -c}{t_i} \in \mathcal{J}$  for all  $c \geq 0, \ell_i \mid c, \ell_i \nmid t_i \leq t$ .  $\square$

**Proposition 4.19.** (a) There is an  $R$ -Hopf algebra homomorphism  $\phi: \dot{U}_\epsilon^*(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  defined by

$$(4.28) \quad \tilde{e}_i^{(n)} \mapsto (\epsilon_i^*)^{-n} e_i^{(n)}, \quad \tilde{f}_i^{(n)} \mapsto f_i^{(n)}, \quad K^\lambda \mapsto 1 \ (\lambda \in 2P^*), \quad \binom{K_i^*; 0}{t} \mapsto \binom{h_i; 0}{t}.$$

(b) The kernel of  $\phi$  is the Hopf ideal generated (as an ideal) by  $\{K^\lambda - 1\}_{\lambda \in 2P^*}$ .

*Proof.* (a) We have Hopf algebra homomorphisms:

$$\dot{U}_\epsilon^*(\mathfrak{g}) \xrightarrow{\psi_1^*} \tilde{U}_\epsilon(\mathfrak{g}, P^*) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, Q^*) \xleftarrow{\phi_Q} \dot{U}_R(\mathfrak{g}^d).$$

The images of  $\tilde{e}_i^{(n)}, \tilde{f}_i^{(n)}, K^\lambda$  in  $\tilde{U}_\epsilon(\mathfrak{g}, Q^*)$  are contained in  $\phi_Q(\dot{U}_R(\mathfrak{g}^d))$ . Hence we obtain the desired  $R$ -Hopf algebra homomorphism  $\phi: \dot{U}_\epsilon^*(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$ .

(b) The map  $\phi$  is compatible with triangular decompositions, that is  $\phi \cong \phi^< \otimes \phi^0 \otimes \phi^>$  with

$$\phi^<: \dot{U}_\epsilon^{*<} \rightarrow \dot{U}_R^{<}(\mathfrak{g}^d), \quad \phi^0: \dot{U}_\epsilon^{*0} \rightarrow \dot{U}_R^0(\mathfrak{g}^d), \quad \phi^>: \dot{U}_\epsilon^{*>} \rightarrow \dot{U}_R^{>}(\mathfrak{g}^d).$$

We note that  $\phi^<, \phi^>$  are isomorphisms. By (4.28) and the PBW-basis of  $\dot{U}_\epsilon^*(\mathfrak{g})$  in Lemma 4.4, one see that  $\text{Ker}(\phi^0)$  is the two-sided ideal generated by  $\{K^\lambda - 1 | \lambda \in 2P^*\}$ . On the other hand,  $\phi^0$  is a split morphism of  $R$ -modules since  $\dot{U}_R^0(\mathfrak{g}^d)$  is free over  $R$ , therefore,

$$\text{Ker}(\phi) = \dot{U}_\epsilon^{*<} \otimes \text{Ker}(\phi^0) \otimes \dot{U}_\epsilon^{*>}.$$

The lemma follows.  $\square$

**Remark 4.20.** The left adjoint action of  $\dot{U}_\epsilon^*(\mathfrak{g})$  on itself and the adjoint action of  $\dot{U}_\epsilon^*(\mathfrak{g})$  on  $\tilde{U}_\epsilon(\mathfrak{g}, X^*)$  factor through  $\phi: \dot{U}_\epsilon^*(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  of Proposition 4.19. This is because the adjoint action of the elements  $K^\lambda, \lambda \in 2P^*$ , is trivial. Therefore,  $\phi: \dot{U}_\epsilon^*(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  and  $\phi_Q: \dot{U}_R(\mathfrak{g}^d) \hookrightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, X^*)$ , where  $X$  is either  $P$  or  $Q$ , are morphisms of  $\dot{U}_R(\mathfrak{g}^d)$ -modules.

We now state some technical results that will be needed in Section 5. Let  $A_{Q^*}$  be the  $R$ -subalgebra of  $\dot{U}_\epsilon^*(\mathfrak{g})$  generated by  $\tilde{e}_i^{(n)} K^{n\gamma(\alpha_i^*)}, \tilde{f}_i^{(n)} K^{n\kappa(\alpha_i^*)}, \binom{K_i^*; 0}{t_i}$  for  $1 \leq i \leq r; t_i, n \geq 1$ .

**Lemma 4.21.** (a)  $A_{Q^*}$  is a  $\dot{U}_R(\mathfrak{g}^d)$ -submodule of  $\dot{U}_\epsilon^*(\mathfrak{g})$  and the restriction map  $\phi: A_{Q^*} \rightarrow \dot{U}_R(\mathfrak{g}^d)$  is an  $R$ -algebra isomorphism.

(b) The restriction map  $\psi_1^*: A_{Q^*} \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*)$  is an inclusion of  $\dot{U}_R(\mathfrak{g}^d)$ -modules with dense image in the codomain.

*Proof.* (a) By Lemma 4.4, we can choose  $Q^*$ -weight  $R$ -bases  $\{u^<\}$  and  $\{u^>\}$  of  $\dot{U}_\epsilon^{*<}$  and  $\dot{U}_\epsilon^{*>}$ , respectively, so that  $A_{Q^*}$  has an  $R$ -basis of the form

$$u^< K^{\kappa(-\deg(u^<))} \cdot \prod_i \binom{K_i^*; 0}{t_i} \cdot u^> K^{\gamma(\deg(u^>))}.$$

Combining this with Lemmas 4.5 and 4.6, we see that  $\phi: A_{Q^*} \rightarrow \dot{U}_R(\mathfrak{g}^d)$  is an isomorphism. It is straightforward to see that  $A_{Q^*}$  is stable under the left adjoint action of  $\dot{U}_\epsilon^*(\mathfrak{g})$  by checking the action on the set of generators  $\{\tilde{e}_i^{(n)}, \tilde{f}_i^{(n)}, K^\lambda \mid \lambda \in 2P^*\}$ , hence  $A_{Q^*}$  is a  $\dot{U}_R(\mathfrak{g}^d)$ -submodule of  $\dot{U}_\epsilon^*(\mathfrak{g})$ .

(b) The proof is the same as that of Lemma 4.10(b).  $\square$

**Remark 4.22.** We have two inclusions of  $\dot{U}_R(\mathfrak{g}^d)$ -modules with dense image in the codomains:

$$\dot{U}_R(\mathfrak{g}^d) \xrightarrow{\phi^{-1}} A_{Q^*} \xrightarrow{\psi_1^*} \dot{U}_\epsilon^*(\mathfrak{g}, P^*), \quad \dot{U}_R(\mathfrak{g}^d) \xrightarrow{\phi_P} \tilde{U}_\epsilon^*(\mathfrak{g}, P^*).$$

which coincide after composing with the map  $\tilde{U}_\epsilon^*(\mathfrak{g}, P) \rightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, Q^*)$ . The first inclusion and the algebra  $A_{Q^*}$  are used in the construction of the pairing  $Z_{Fr} \times \dot{U}_R(\mathfrak{g}^d) \rightarrow R$  in Section 5.

**Remark 4.23.** Let  $A^>$  (resp.  $A^<$ ) be the  $R$ -subalgebra of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\dot{U}_\epsilon^>(\text{resp. } \dot{U}_\epsilon^<)$  and  $K^\lambda$  with all  $\lambda \in 2P$ . Then  $A^>$  and  $A^<$  are  $R$ -Hopf subalgebras of  $\dot{U}_\epsilon(\mathfrak{g})$ .

**Lemma 4.24.** *Let us recall the algebra homomorphism  $\tilde{\text{Fr}}: \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  of (4.12). For any  $x \in \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\mu - 1 \mid \ell_i \nmid n, \mu \in 2P\}$  and  $y \in \widehat{U}_\epsilon(\mathfrak{g}, P)$ , we have  $\text{ad}'(x)(y) \in \text{Ker}(\tilde{\text{Fr}})$ .*

*Proof.* First, we note that for  $y = \tilde{F}_{i_1}^{(k_1)} \dots \tilde{F}_{i_m}^{(k_m)} 1_\lambda \tilde{E}_{j_1}^{(r_1)} \dots \tilde{E}_{j_n}^{(r_n)}$  such that  $\ell_{i_t} \mid k_t$  for all  $1 \leq t \leq m$  and  $\ell_{j_t} \mid r_t$  for all  $1 \leq t \leq n$ , we clearly have  $\text{ad}'(K^\mu - 1)(y) = 0$ . Hence, for any  $y \in \widehat{U}_\epsilon(\mathfrak{g}, P)$ , the image  $\text{ad}'(K^\mu - 1)(y)$  is an  $R$ -linear combination of elements of the form  $\tilde{F}_{i_1}^{(k_1)} \dots \tilde{F}_{i_m}^{(k_m)} 1_\lambda \tilde{E}_{j_1}^{(r_1)} \dots \tilde{E}_{j_n}^{(r_n)}$  such that there is either  $t \leq m$  such that  $\ell_{i_t} \geq 2, \ell_{i_t} \nmid k_t$  or  $t \leq n$  such that  $\ell_{j_t} \geq 2, \ell_{j_t} \nmid r_t$ . This implies that  $\text{ad}'(K^\mu - 1)(y) \in \text{Ker}(\tilde{\text{Fr}})$  for any  $y \in \widehat{U}_\epsilon(\mathfrak{g}, P)$ .

Any element of  $\widehat{U}_\epsilon(\mathfrak{g}, P)$  is an  $R$ -linear combination of elements  $\{y1_\lambda \mid y \in \dot{U}_\epsilon(\mathfrak{g}, P), \lambda \in P\}$ . According to (2.20), we have:

$$\begin{aligned} \text{ad}'(\tilde{E}_i^{(n)})(y1_\lambda) &= \sum_{c=0}^n \tilde{E}_i^{(n-c)} y1_\lambda (-1)^c K^{(n-c)\zeta_i^>} \epsilon_i^{c(c-1)} \tilde{E}_i^{(c)} K^{c\zeta_i^>} \\ &= \sum_{c=0}^n (-1)^c \epsilon_i^{n(\lambda, \zeta_i^>)} \epsilon_i^{-c(c+1)} \tilde{E}_i^{(n-c)} y1_\lambda \tilde{E}_i^{(c)}. \end{aligned}$$

If  $\ell_i \nmid n$ , then either  $n - c$  or  $c$  is not divisible by  $\ell_i$ , and so  $\tilde{E}_i^{(n-c)} y1_\lambda \tilde{E}_i^{(c)} \in \text{Ker}(\tilde{\text{Fr}})$ . This proves  $\text{ad}'(\tilde{E}_i^{(n)})(y) \in \text{Ker}(\tilde{\text{Fr}})$  for any  $y \in \widehat{U}_\epsilon(\mathfrak{g}, P)$  if  $\ell_i \nmid n$ . The proof for  $\tilde{F}_i^{(n)}$  is similar.  $\square$

#### 4.6. Comparison with Lusztig's quantum Frobenius map.

Let  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$  be the corresponding idempotented Lusztig forms of [L5]. Let  $\dot{\mathcal{U}}_\epsilon^>$ ,  $\dot{\mathcal{U}}_\epsilon^<$  be the  $R$ -subalgebras of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\{E_i^{[n]}\}_{1 \leq i \leq r}^{n \geq 1}$ ,  $\{F_i^{[n]}\}_{1 \leq i \leq r}^{n \geq 1}$ , respectively. Let  $\dot{\mathcal{U}}_\epsilon^{*>}$ ,  $\dot{\mathcal{U}}_\epsilon^{*<}$  be the  $R$ -subalgebras of  $\dot{U}_\epsilon^*(\mathfrak{g})$  generated by  $\{\hat{e}_i^{[n]}\}_{1 \leq i \leq r}^{n \geq 1}$ ,  $\{\hat{f}_i^{[n]}\}_{1 \leq i \leq r}^{n \geq 1}$ , respectively. According to [L5], we have the quantum Frobenius homomorphisms:

$$(4.29) \quad \text{Fr}: \widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P) \longrightarrow \widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*), \quad \text{Fr}^>: \dot{\mathcal{U}}_\epsilon^> \longrightarrow \dot{\mathcal{U}}_\epsilon^{*>}, \quad \text{Fr}^<: \dot{\mathcal{U}}_\epsilon^< \longrightarrow \dot{\mathcal{U}}_\epsilon^{*<}.$$

The twists  $\mathsf{F}$  and  $\mathsf{F}^*$  equip  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$  and  $\widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$  new (topological) Hopf algebra structures so that  $\text{Fr}$  is still a morphism of Hopf algebras.

Assume  $R$  contains an element  $\epsilon^{1/2}$ . Then we can identify  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$  with (twisted Hopf structure)  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$  via

$$\tilde{E}_i^{(n)} 1_\lambda = \epsilon_i^{\frac{n(1-n)}{2}} \epsilon^{(n\nu_i^>, \lambda)} E_i^{[n]} 1_\lambda, \quad F_i^{(n)} 1_\lambda = \epsilon_i^{\frac{n(n-1)}{2}} \epsilon^{-(n\nu_i^<, \lambda)} F_i^{[n]} 1_\lambda.$$

Similarly, we can identify  $\widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  with (twisted Hopf structure via the twist  $F^*$ )  $\widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$  via

$$\tilde{e}_i^{(n)} 1_\lambda = (\epsilon_i^*)^{\frac{n(1-n)}{2}} \epsilon^{(n\nu_i^*, \lambda)} \hat{e}_i^{[n]} 1_\lambda, \quad \tilde{f}_i^{(n)} 1_\lambda = (\epsilon_i^*)^{\frac{n(n-1)}{2}} \epsilon^{-(n\nu_i^*, \lambda)} \hat{f}_i^{[n]} 1_\lambda.$$

Let us define the map  $\Phi : \widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*) \rightarrow \widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$  by

$$e_i^{[n]} 1_\lambda \mapsto \left(\epsilon_i^{\frac{\ell_i^2 - \ell_i}{2}}\right)^n e_i^{[n]} 1_\lambda, \quad f_i^{[n]} 1_\lambda \mapsto \left(\epsilon_i^{\frac{\ell_i - \ell_i^2}{2}}\right)^n f_i^{[n]} 1_\lambda,$$

The map  $\Phi$  is an automorphism of Hopf algebras with either the usual or twisted Hopf structure (via the twist  $F^*$ ) on  $\widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$ .

**Lemma 4.25.** *Recall the map  $\tilde{\text{Fr}} : \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$ . Under the above identifications of Hopf algebras,  $\tilde{\text{Fr}} = \Phi \circ \text{Fr}$ .*

*Proof.* This follows by straightforwardly checking the equality on the generators  $\tilde{E}_i^{(n)} 1_\lambda, \tilde{F}_i^{(n)} 1_\lambda$ .  $\square$

We now state some technical results that will be needed in Section 5.

**Lemma 4.26.** *Assume that  $\ell_i \geq \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$  for all  $i$ . Let  $\mathcal{K}$  be a subset of  $\mathbb{Z}_{\geq 0}^N$  consisting of all  $\vec{k} = (k_1, \dots, k_N)$  with some  $k_i$  not divisible by  $\ell_{\beta_i}$ .*

- (a) *The sets  $\{F^{[\vec{k}]}\}_{\vec{k} \in \mathcal{K}}$ ,  $\{F^{[\vec{k}]}\}_{\vec{k} \in \mathcal{K}}$  are  $R$ -bases of  $\text{Ker}(\text{Fr}^<)$ . Similarly, the sets  $\{E^{[\vec{k}]}\}_{\vec{k} \in \mathcal{K}}$ ,  $\{E^{[\vec{k}]}\}_{\vec{k} \in \mathcal{K}}$  are  $R$ -bases of  $\text{Ker}(\text{Fr}^>)$ .*
- (b) *The sets  $\{\tilde{F}^{(\vec{k})}\}_{\vec{k} \in \mathcal{K}}$ ,  $\{\tilde{F}^{(\vec{k})}\}_{\vec{k} \in \mathcal{K}}$  are  $R$ -bases of  $\text{Ker}(\tilde{\text{Fr}}^<)$ . Similarly, the sets  $\{\tilde{E}^{(\vec{k})}\}_{\vec{k} \in \mathcal{K}}$ ,  $\{\tilde{E}^{(\vec{k})}\}_{\vec{k} \in \mathcal{K}}$  are  $R$ -bases of  $\text{Ker}(\tilde{\text{Fr}}^>)$ .*

*Proof.* (a) Let us fix  $\lambda \in P^*$ . We have that  $\dot{\mathcal{U}}_\epsilon^< 1_\lambda$  is a free  $\dot{\mathcal{U}}_\epsilon^<$ -module and  $\dot{\mathcal{U}}_\epsilon^{*<} 1_\lambda$  is a free  $\dot{\mathcal{U}}_\epsilon^{*<}$ -module. Since the homomorphism  $\text{Fr} : \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  is compatible with the braid group action by [L5, Remark 41.1.9], we see that  $\text{Fr}^<(F^{[\vec{k}]} 1_\lambda) = \text{Fr}^<(F^{[\vec{k}]}) 1_\lambda = 0$  iff  $\vec{k} \in \mathcal{K}$ . Furthermore, the images  $\text{Fr}^<(F^{[\vec{k}]} 1_\lambda)$  with  $\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus \mathcal{K}$  form an  $R$ -basis for  $\dot{\mathcal{U}}_\epsilon^< 1_\lambda$ . Thus, the set  $\{F^{[\vec{k}]}\}_{\vec{k} \in \mathcal{K}}$  is an  $R$ -basis of  $\text{Ker}(\text{Fr}^<)$ . The proofs for the other statements are analogous.

(b) Follows from part (a) and Lemma 4.25.  $\square$

**Corollary 4.27.** (a) *The kernel of  $\tilde{\text{Fr}}$  has an  $R$ -basis consisting of elements*

- $\tilde{F}^{(\vec{k})} 1_\lambda \tilde{E}^{(\vec{r})}$  with  $\lambda \in P \setminus P^*$  and  $\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N$ .
- $\tilde{F}^{(\vec{k})} 1_\lambda \tilde{E}^{(\vec{r})}$  with  $\lambda \in P^*$  and either  $\vec{k}$  or  $\vec{r}$  contained in  $\mathcal{K}$ .

(b) *The kernel of  $\tilde{\text{Fr}}$  is the two-sided ideal generated by  $\{1_\mu, \tilde{E}_i^{(n)} 1_\lambda, \tilde{F}_i^{(n)} 1_\lambda\}$  with  $1 \leq i \leq r, \lambda \in P, \mu \notin P^*, \ell_i \nmid n$  for  $\ell_i \geq 2$ .*

*Proof.* (a) Follows immediately from Lemma 4.26. Then (b) follows by (a) and Remark 4.15.  $\square$

We shall now relate the root generators in the PBW-bases of  $\dot{U}_\epsilon(\mathfrak{g}), \dot{U}_\epsilon^*(\mathfrak{g}), \dot{U}_R(\mathfrak{g}^d)$  under the morphisms  $\tilde{\text{Fr}}$  in Proposition 4.17 and  $\phi$  in Proposition 4.19.

Let us introduce a notation: in an  $R$ -module  $M$ , we write  $m_1 \overset{\mathcal{S}}{\sim} m_2$  if  $m_1 = am_2$  for some  $a \in R^\times$ . We start with the following technical lemma, see (4.11) for definitions of some terms:

**Lemma 4.28.** *Let  $\mu \in Q_+^*$  and suppose  $s_i^*(\mu) \in Q_+^*$ .*

- (a) *Let  $x \in \dot{U}_{\epsilon, \mu}^>$  and  $\hat{x} := \epsilon^{b_{s_i^*(\mu)}} T_i^* (\epsilon^{-b_\mu^>} x K^{-\nu_\mu^>}) K^{\nu_{s_i^*(\mu)}^>}$ . Then  $\phi(\hat{x}) \overset{\mathcal{S}}{\sim} T_i^*(\phi(x))$ .*
- (b) *Suppose  $y \in \dot{U}_{\epsilon, -\mu}^*<$  and  $\hat{y} := \epsilon^{b_{s_i^*(\mu)}} K^{-\nu_{s_i^*(\mu)}^<} T_i^* (\epsilon^{-b_\mu^<} K^{\nu_\mu^<}) y$ . Then  $\phi(\hat{y}) \overset{\mathcal{S}}{\sim} T_i^*(\phi(y))$*

*Proof.* Let us prove (a) only. Let  $\mu = \sum_j u_j \alpha_j^*$ . Arguing as in Lemma 4.6, we can assume that  $R = \mathcal{K}$  the field of characteristics 0 which contains an element  $\epsilon^{1/2}$ . Let

$$x = \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^* = \mu} p_{i_1, \dots, i_m} \tilde{e}_{i_1} \dots \tilde{e}_{i_m}$$

with  $p_{i_1, \dots, i_m} \in \mathcal{K}$ . Then

$$\epsilon^{-b_\mu^>} x K^{-\nu_\mu^>} = \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^* = \mu} p_{i_1, \dots, i_m} \epsilon^{-b_\mu^> + \sum_{1 \leq j < l \leq m} (\nu_{i_j}^*, \alpha_{i_l}^*)} \hat{e}_{i_1} \dots \hat{e}_{i_m}.$$

Hence

$$\begin{aligned} \hat{x} &= \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^* = \mu} p_{i_1, \dots, i_m} \epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + \sum_{j < l} (\nu_{i_j}^*, \alpha_{i_l}^*)} \left( \overrightarrow{\prod}_{1 \leq j \leq m} T_i^*(\hat{e}_{i_j}) \right) K^{\nu_{s_i^*(\mu)}^>} \\ &= \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^* = \mu} p_{i_1, \dots, i_m} \epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m}} \overrightarrow{\prod}_{1 \leq j \leq m} \left( T_i^*(\hat{e}_{i_j}) K^{\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*} \right), \end{aligned}$$

in which

$$B_{i_1, \dots, i_m} = \sum_{j < l} (\nu_{i_j}^*, \alpha_{i_l}^*) - \sum_{j < l} (\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*, s_i^*(\alpha_{i_l}^*)).$$

*Step 1:* We claim that for different  $(i'_1, \dots, i'_m)$  we have  $B_{i_1, \dots, i_m} - B_{i'_1, \dots, i'_m} \in \mathbb{Z}$  and

$$\epsilon^{B_{i_1, \dots, i_m} - B_{i'_1, \dots, i'_m}} = 1.$$

It is enough to prove it in the case when  $(i'_1, \dots, i'_m)$  is obtained from  $(i_1, \dots, i_m)$  by permuting two consecutive indices  $i_j$  and  $i_{j+1}$ . Then  $B_{i_1, \dots, i_m} - B_{i'_1, \dots, i'_m}$  equals

$$\begin{aligned} &(\nu_{i_j}^*, \alpha_{i_{j+1}}^*) - (\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*, s_i^*(\alpha_{i_{j+1}}^*)) - (\nu_{i_{j+1}}^*, \alpha_{i_j}^*) + (\nu_{i_{j+1}}^* - a_{i, i_{j+1}}^* \nu_i^*, s_i^*(\alpha_{i_j}^*)) \\ &= \mathbf{d}_i^* a_{i, i_{j+1}}^* a_{i, i_j}^* (\epsilon_{i, i_{j+1}} - \epsilon_{i, i_j}) \in 2\mathbf{d}_i^* \mathbb{Z}. \end{aligned}$$

*Step 2:* We compute  $T_i^*(\hat{e}_{i_j}) K^{\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*}$ .

If  $i_j = i$  then

$$T_i^*(\hat{e}_{i_j}) K^{\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*} = -\hat{f}_i K^{\alpha_i^*} K^{-\nu_i^*} = -\tilde{f}_i K^{2\alpha_i^*}.$$

If  $i_j \neq i$  then

$$T_i^*(\hat{e}_{i_j}) K^{\nu_{i_j}^* - a_{i, i_j}^* \nu_i^*} = (\epsilon_i^*)^{a_{i, i_j}^* (\alpha_{i, i_j}^* + 1)/2} \epsilon^{a_{i, i_j}^* (\nu_i^*, \alpha_{i_j}^*)} \sum_{k=0}^{-a_{i, i_j}^*} (-1)^k \hat{e}_i^{(-a_{i, i_j}^* - k)} \tilde{e}_{i_j} \tilde{e}_i^{(k)},$$

where we use  $(\epsilon_i^*)^2 = 1$ .

*Step 3:* Therefore, we have

$$\phi(\hat{x}) = \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^* = \mu} p_{i_1, \dots, i_m} \epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m} + C_\mu} \overrightarrow{\prod}_{1 \leq j \leq m} T_i^*(e_{i_j}),$$

in which

$$C_\mu = \sum_{j \neq i} u_j \left( -\mathbf{d}_j^* a_{ij}^* (1 - a_{ij}^*) + a_{ij}^* (\nu_i^*, \alpha_j^*) \right).$$

Since  $\epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m} + C_\mu}$  does not depend on the choice of  $(i_1, \dots, i_m)$  by Step 1, let us fix such  $(i_1, \dots, i_m)$ . Then we have:

$$\begin{aligned}\phi(\hat{x}) &= \epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m} + C_\mu} \sum_{\alpha_{i_1}^* + \dots + \alpha_{i_m}^*} p_{i_1, \dots, i_m} \prod_j T_i^*(e_{i_j}) \\ &= \epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m} + C_\mu} T_i^*(\phi(x)).\end{aligned}$$

This finises the proof since  $\epsilon^{b_{s_i^*(\mu)}^> - b_\mu^> + B_{i_1, \dots, i_m} + C_\mu} \in R^\times$ .  $\square$

**Lemma 4.29.** (a) We have  $\phi(\tilde{e}_{\beta_k^*}^{(n)}) \stackrel{\text{S}}{\sim} e_{\beta_k^*}^{(n)}$  and  $\phi(\tilde{f}_{\beta_k^*}^{(n)}) \stackrel{\text{S}}{\sim} f_{\beta_k^*}^{(n)}$ .

(b) Let us recall the homomorphism  $\check{\text{Fr}}: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  from Proposition 4.17. Then

$$\check{\text{Fr}}(\tilde{E}_{\beta_k}^{(n)}) \stackrel{\text{S}}{\sim} \begin{cases} 0 & \text{if } \ell_{\beta_k} \nmid n \\ e_{\beta_k^*}^{(n/\ell_{\beta_k})} & \text{if } \ell_{\beta_k} \mid n \end{cases}, \quad \check{\text{Fr}}(\tilde{F}_{\beta_k}^{(n)}) \stackrel{\text{S}}{\sim} \begin{cases} 0 & \text{if } \ell_{\beta_k} \nmid n \\ f_{\beta_k^*}^{(n/\ell_{\beta_k})} & \text{if } \ell_{\beta_k} \mid n \end{cases}.$$

*Proof.* (a) Let us prove the first statement only. According to (4.12), we have

$$\tilde{e}_{\beta_k^*}^{(n)} = \epsilon^{nb_{\beta_k^*}^>} \epsilon^{(\nu_{\beta_k^*}, \beta_k^*)n(n-1)/2} T_{i_1}^* \dots T_{i_{k-1}}^* (\hat{e}_{i_k}^{(n)}) K^{n\nu_{\beta_k^*}^>}.$$

Let  $\beta'_j = s_{i_j}^* \dots s_{i_{k-1}}^* \alpha_{i_k}$ . Then  $\beta'_j \in Q_+^*$  and  $s_{j-1}^*(\beta'_j) = \beta'_{j-1} \in Q_+^*$ . For  $1 \leq i \leq k$ , let

$$x_j = \epsilon^{nb_{\beta'_j}^>} T_{i_j}^* \dots T_{i_{k-1}}^* (\hat{e}_{i_k}^{(n)}) K^{n\nu_{\beta'_j}^>} \in \dot{U}_\epsilon^>, \quad e_{\beta'_j}^{(n)} = T_{i_j}^* \dots T_{i_{k-1}}^* (e_{i_k}^{(n)}) \in \dot{U}_R^>(\mathfrak{g}^d).$$

Let us prove by a decreasing induction on  $1 \leq j \leq k$  that  $\phi(x_j) \stackrel{\text{S}}{\sim} e_{\beta'_j}^{(n)}$ . The base case  $j = k$  is obvious as  $x_k = \hat{e}_{i_k}^{(n)}$ . Let us now prove the induction step. We have  $x_j \in \dot{U}_{\epsilon, \beta'_j}^*>$ , so that  $\phi(x_{j-1}) \stackrel{\text{S}}{\sim} T_{i_{j-1}}^*(\phi(x_j))$ , due to Lemma 4.28. By the induction hypothesis, we have  $\phi(x_j) \stackrel{\text{S}}{\sim} e_{\beta'_j}^{(n)}$ . Therefore  $\phi(x_{j-1}) \stackrel{\text{S}}{\sim} T_{i_{j-1}}^*(e_{\beta'_j}^{(n)}) = e_{\beta'_{j-1}}^{(n)}$ , completing the step of induction. In particular, for  $j = 1$  we get  $\phi(\tilde{e}_{\beta_k^*}^{(n)}) \stackrel{\text{S}}{\sim} e_{\beta_k^*}^{(n)}$ . The claim follows.

(b) Let us show the first statement only. By Lemma 4.25, we have:

$$\check{\text{Fr}}^>(\tilde{E}_{\beta_k}^{(n)}) \stackrel{\text{S}}{\sim} \begin{cases} 0 & \text{if } \ell_{\beta_k} \nmid n \\ \tilde{e}_{\beta_k^*}^{(n/\ell_{\beta_k})} & \text{if } \ell_{\beta_k} \mid n \end{cases}.$$

Note that the restriction of  $\check{\text{Fr}}: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  on  $\dot{U}_\epsilon^>(\mathfrak{g})$  is a composition of  $\check{\text{Fr}}^>$  and the restriction of  $\phi$  on  $\dot{U}_\epsilon^*>(\mathfrak{g})$ . Therefore, combining the above formula with the part (a), we get the first statement of part (b).  $\square$

#### 4.7. *R*-matrix.

Following [J1, §8.30] and using the divided powers  $\{F_{\beta_i}^{[n]}, E_{\beta_i}^{[n]}\}_{1 \leq i \leq N}^{n \geq 1}$ , we have the following *R*-matrix of  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$ :

$$(4.30) \quad \mathcal{R} = \sum_{\lambda, \mu \in P} \epsilon^{-(\lambda, \mu)} 1_\lambda \otimes 1_\mu + \left( \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus (0, \dots, 0)} c_{\vec{k}} F^{[\vec{k}]} \otimes E^{[\vec{k}]} \right) \left( \sum_{\lambda, \mu \in P} \epsilon^{-(\lambda, \mu)} 1_\lambda \otimes 1_\mu \right),$$

where  $c_{\vec{k}} = \prod_{t=1}^N \left( \epsilon_{i_t}^{k_t(1-k_t)/2} (\epsilon_{i_t}^{-1} - \epsilon_{i_t})^{k_t} [k_t]_{i_t}! \right)$ .

Evoking (2.6) and  $F$  of (2.7), we get the following  $R$ -matrix for  $\widehat{U}_\epsilon(\mathfrak{g}, P)$ :

$$(4.31) \quad \mathcal{R}^F = F^{-1} \mathcal{R} F^{-1} = \sum_{\lambda, \mu \in P} \epsilon^{-(\lambda, \mu) - \sum_{i,j} 2\phi_{ij}(\omega_i^\vee, \lambda)(\omega_j^\vee, \mu)} 1_\lambda \otimes 1_\mu + F^{-1} \left( \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus (0, \dots, 0)} \sum_{\lambda, \mu \in P} d_{\lambda, \mu, \vec{k}} \tilde{F}^{(\vec{k})} 1_\lambda \otimes \tilde{E}^{(\vec{k})} 1_\mu \right) F^{-1}$$

with  $d_{\lambda, \mu, \vec{k}} = \epsilon^{a_{\lambda, \mu, \vec{k}}} \cdot \prod_{t=1}^N (\epsilon_{i_t}^{-1} - \epsilon_{i_t})^{k_t} (k_t)_{i_t}!$  for some  $a_{\lambda, \mu, \vec{k}} \in \mathbb{Z}[\frac{1}{2}]$ .

**Lemma 4.30.** *Assume that  $\ell_i \geq \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$  for all  $i$ . Then the image of  $\mathcal{R}^F$  under the homomorphism (under suitable completions)  $\tilde{F}r^{\otimes 2}: \widehat{U}_\epsilon(\mathfrak{g}, P)^{\otimes 2} \rightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*)^{\otimes 2}$  is*

$$(4.32) \quad \mathcal{R}^* = \sum_{\lambda, \mu \in P^*} \epsilon^{-(\lambda, \mu) - \sum_{i,j} 2\phi_{ij}(\omega_i^\vee, \lambda)(\omega_j^\vee, \mu)} 1_\lambda \otimes 1_\mu.$$

*Proof.* If  $\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus (0, \dots, 0)$  is such that  $k_t \geq \ell_{i_t}$  for some  $1 \leq t \leq N$ , then  $(k_t)_{i_t}! = 0$  and so  $d_{\lambda, \mu, \vec{k}} = 0$  for all  $\lambda, \mu \in P$ . If  $\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus (0, \dots, 0)$  is such that  $0 \leq k_t < \ell_{i_t}$  for all  $1 \leq t \leq N$ , then there is  $t$  such that  $0 < k_t < \ell_{i_t}$  and so  $\tilde{F}^{(\vec{k})} \in \text{Ker}(\tilde{F}r^<)$ ,  $\tilde{E}^{(\vec{k})} \in \text{Ker}(\tilde{F}r^>)$  by Lemma 4.26. Therefore, for any  $\vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus (0, \dots, 0)$  and  $\lambda, \mu \in P$ , we have

$$\tilde{F}r^{\otimes 2}(d_{\lambda, \mu, \vec{k}} \tilde{F}^{(\vec{k})} 1_\lambda \otimes \tilde{E}^{(\vec{k})} 1_\mu) = 0.$$

This implies the lemma.  $\square$

**Remark 4.31.** For  $\lambda, \mu \in Q^*$ , the coefficient of  $1_\lambda \otimes 1_\mu$  in  $\mathcal{R}^*$  is 1. Coefficients of other  $1_\lambda \otimes 1_\mu$  may be equal to 1 depending on the root of unity  $\epsilon$ .

**Remark 4.32.** We will take this  $\mathcal{R}^*$  as an  $R$ -matrix of  $\widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  henceforth.

## 5. THE FROBENIUS CENTER $Z_{Fr}$

In this section, we study the so-called Frobenius center of  $U_\epsilon^{ev}(\mathfrak{g})$ . The condition we will impose on  $\ell$  is as follows:  $\ell_i \geq \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$ . Under this condition, the results of Section 4 hold, and furthermore in Remarks 4.15 and 4.18, the index  $i$  runs over all  $\{1, \dots, r\}$ . We assume that  $R$  has an element  $\epsilon^{1/N}$ . Let  $\mathcal{A}_N := \mathbb{Z}[v^{\pm 1/N}] \left[ \left\{ \frac{1}{v^{2k}-1} \right\}_{1 \leq k \leq \max\{d_i\}} \right]$ . Then with the choice of  $\epsilon^{1/N}$ , the ring homomorphism  $\sigma: \mathcal{A} \rightarrow R$  factors through  $\mathcal{A}_N \rightarrow R$  which maps  $v^{\pm 1/N}$  to  $\epsilon^{\pm 1/N}$ .

### 5.1. The pairing for the idempotent version.

Let us consider the  $R$ -linear pairing  $\langle \cdot, \cdot \rangle': U_\epsilon^{ev}(\mathfrak{g}) \times \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$  defined by

$$(5.1) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), \dot{y} \underset{\lambda}{\cdot} \dot{x} \right\rangle' = \delta_{-\lambda/2, \dot{\lambda}} (y, \dot{x})' (\dot{y}, x)' \epsilon^{(2\rho, \nu)} \epsilon^{(\dot{\lambda}, \kappa(\dot{\nu})) - (\dot{\lambda}, \kappa(\dot{\nu}) + \gamma(\dot{\mu}))}$$

for any  $\lambda \in 2P$ ,  $\dot{\lambda} \in P$ ,  $y \in U_{-\nu}^{ev}<$ ,  $x \in U_\mu^{ev}>$ ,  $\dot{y} \in \dot{U}_{-\dot{\nu}}^{<}$ ,  $\dot{x} \in \dot{U}_\mu^{>}$  and  $\mu, \nu, \dot{\mu}, \dot{\nu} \in Q_+$ .

**Lemma 5.1.** *The pairing (5.1) is  $\widehat{U}_\epsilon(\mathfrak{g})$ -adjoint invariant and non-degenerate in the first argument.*

*Proof.* The pairing (5.1) can be clearly extended to a pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$ :

$$\left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), \dot{y} \prod a_{\dot{\lambda}} 1_{\dot{\lambda}} \dot{x} \right\rangle' = a_{-\lambda/2} (y, \dot{x})' (\dot{y}, x)' \epsilon^{(2\rho, \nu)} \epsilon^{(\dot{\lambda}, \kappa(\dot{\nu})) - (\dot{\lambda}, \kappa(\dot{\nu}) + \gamma(\dot{\mu}))}.$$

This pairing is continuous, whereas  $R$  and  $U_\epsilon^{ev}(\mathfrak{g})$  are equipped with the discrete topology and topology of  $\tilde{U}_\epsilon(\mathfrak{g}, P)$  is defined in Section 4.4. The left adjoint  $\dot{U}_\epsilon(\mathfrak{g})$ -action on  $\tilde{U}_\epsilon(\mathfrak{g}, P)$  is continuous. Furthermore, the pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \dot{U}_\epsilon(\mathfrak{g}) \rightarrow R$  of Proposition 3.19 factors through the above pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \tilde{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$  via the homomorphism  $\psi_1: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P)$ . Since  $U_\epsilon^{ev}(\mathfrak{g}) \times \dot{U}_\epsilon(\mathfrak{g}) \rightarrow R$  is non-degenerate in the first argument, so is  $U_\epsilon^{ev}(\mathfrak{g}) \times \tilde{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$ . Moreover, since the pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \dot{U}_\epsilon(\mathfrak{g}) \rightarrow R$  is adjoint  $\dot{U}_\epsilon(\mathfrak{g})$ -invariant and the map  $\psi_1: \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P)$  is a  $\dot{U}_\epsilon(\mathfrak{g})$ -module homomorphism with a dense image, the pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \tilde{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$  must be adjoint  $\dot{U}_\epsilon(\mathfrak{g})$ -invariant.

Since  $\psi_2: \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \tilde{U}_\epsilon(\mathfrak{g}, P)$  is a  $\dot{U}_\epsilon(\mathfrak{g})$ -module embedding with a dense image, the pairing  $U_\epsilon^{ev}(\mathfrak{g}) \times \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow R$  must be adjoint  $\dot{U}_\epsilon(\mathfrak{g})$ -invariant and non-degenerate in the first argument.  $\square$

## 5.2. The Frobenius center $Z_{Fr}$ .

Recall the following homomorphisms from Section 4:

$$\tilde{\text{Fr}}: \widehat{U}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*), \quad \tilde{\text{Fr}}^>: \dot{U}_\epsilon^> \rightarrow \dot{U}_\epsilon^{*>}, \quad \tilde{\text{Fr}}^<: \dot{U}_\epsilon^< \rightarrow \dot{U}_\epsilon^{*<}.$$

By Corollary 4.27, the kernel of the homomorphism  $\tilde{\text{Fr}}$  is spanned over  $R$  by the following elements:

- $\dot{y}1_{\dot{\lambda}}\dot{x}$ , with  $\dot{y} \in \dot{U}_\epsilon^<$ ,  $\dot{x} \in \dot{U}_\epsilon^>$  and  $\dot{\lambda} \in P \setminus P^*$ ;
- $\dot{y}1_{\dot{\lambda}}\dot{x}$ , with  $\dot{\lambda} \in P^*$  and either  $\dot{x} \in \text{Ker}(\tilde{\text{Fr}}^>)$  or  $\dot{y} \in \text{Ker}(\tilde{\text{Fr}}^<)$ .

**Definition 5.2.** The *Frobenius center*, denoted by  $Z_{Fr}$ , is the orthogonal complement in  $U_\epsilon^{ev}$  of  $\text{Ker}(\tilde{\text{Fr}})$  under the adjoint  $\dot{U}_\epsilon(\mathfrak{g})$ -invariant pairing (5.1).

Let  $Z_{Fr}^> := Z_{Fr} \cap U_\epsilon^{ev>}$ ,  $Z_{Fr}^0 := Z_{Fr} \cap U_\epsilon^{ev0}$ ,  $Z_{Fr}^< := Z_{Fr} \cap U_\epsilon^{ev<}$ . We have the following properties:

**Lemma 5.3.** (a)  $Z_{Fr}$  is central in  $U_\epsilon^{ev}$ . Moreover,  $\text{ad}'(x)(z) = 0$  for any  $z \in Z_{Fr}$  and  $x \in \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\lambda - 1 \mid \ell_i \nmid n, \lambda \in 2P\}$ .

(b)  $Z_{Fr}^0 = \bigoplus_{\lambda \in 2P^*} R \cdot K^\lambda$ ,  $Z_{Fr}^> = R[\tilde{E}_\alpha^{\ell_\alpha}]_{\alpha \in \Delta_+}$ , and  $Z_{Fr}^< = R[\tilde{F}_\alpha^{\ell_\alpha}]_{\alpha \in \Delta_+}$ . In particular,  $Z_{Fr}^<, Z_{Fr}^0, Z_{Fr}^>$  are  $R$ -subalgebras of  $U_\epsilon^{ev}$ .

(c)  $Z_{Fr} = Z_{Fr}^< \otimes_R Z_{Fr}^0 \otimes_R Z_{Fr}^>$ .

*Proof.* (a) Let  $z \in Z_{Fr}$ . By Lemma 4.24, we have  $\text{ad}'(S'(x))(y) \in \text{Ker}(\tilde{\text{Fr}})$  for any  $y \in \widehat{U}_\epsilon(\mathfrak{g}, P)$  and  $x \in \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\lambda - 1 \mid \ell_i \nmid n, \lambda \in 2P\}$ . Therefore, with these  $x$  and  $y$ , we get:

$$\langle \text{ad}'(x)z, y \rangle' = \langle z, \text{ad}'(S'(x))y \rangle' = 0.$$

Since the pairing (5.1) is non-degenerate in the first argument, it follows that  $\text{ad}'(x)(z) = 0$  for all  $x \in \{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\lambda - 1 \mid \ell_i \nmid n, \lambda \in 2P\}$ . In particular, we can take  $n = 1$ . This implies that  $z$  is central in  $U_\epsilon^{ev}$ , due to (2.34).

(b) From equation (5.1), it is easy to see that  $Z_{Fr}^0 = \bigoplus_{\lambda \in 2P^*} R \cdot K^\lambda$ . For  $x \in U_\epsilon^{ev>}$ , we have

$$\langle x, \dot{y}1_{\dot{\lambda}}\dot{x} \rangle' = \delta_{\gamma(\deg(x))/2, \dot{\lambda}} (\dot{y}, x)' (1, \dot{x})' \epsilon^{(\dot{\mu}, \gamma(\dot{\mu})) - (\dot{\lambda}, \kappa(\dot{\nu}) + \gamma(\dot{\mu}))} \epsilon^{(\gamma(\deg(x)), \deg(x))},$$

which implies that

$$Z_{Fr}^> = \{x \in U_\epsilon^{ev>} \mid (\dot{y}, x)' = 0 \text{ for all } \dot{y} \in \text{Ker}(\tilde{\text{Fr}}^<) \}.$$

By Corollary 3.9 and Lemma 4.26,  $Z_{Fr}^>$  is a free  $R$ -module with a basis  $\{\tilde{E}^{\vec{k}} \mid \vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus \mathcal{K}\}$ , see Lemma 4.26 for the definition of  $\mathcal{K}$ . This implies that  $Z_{Fr}^> = R[\tilde{E}_\alpha^{\ell_\alpha}]_{\alpha \in \Delta_+}$ . The proof for  $Z_{Fr}^<$  is the same.

(c) Due to Lemma 2.10, any  $u \in U_\epsilon^{ev}$  can be written as

$$u = \sum_{\substack{\lambda \in P \\ \vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N}} a_{\vec{k}, \vec{r}, \lambda} \tilde{F}^{\vec{k}} K^{2\lambda} \tilde{E}^{\vec{r}}$$

for finitely many nonzero  $a_{\vec{k}, \vec{r}, \lambda} \in R$ . By Corollary 3.9, we have

$$\langle u, \tilde{F}^{(\vec{r})} 1_{-\lambda} \tilde{E}^{(\vec{k})} \rangle' = a_{\vec{k}, \vec{r}, \lambda} b_{\vec{k}, \vec{r}, \lambda}$$

for some invertible element  $b_{\vec{k}, \vec{r}, \lambda} \in R$ . Therefore, if  $u \in Z_{Fr}$  then we must have  $a_{\vec{k}, \vec{r}, \lambda} = 0$  if either  $\lambda \in P \setminus P^*$  or  $\vec{k} \in \mathcal{K}$ , or  $\vec{r} \in \mathcal{K}$ . Thus, if  $u \in Z_{Fr}$  then

$$u = \sum_{\substack{\lambda \in P^* \\ \vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N \setminus \mathcal{K}}} a_{\vec{k}, \vec{r}, \lambda} \tilde{F}^{\vec{k}} K^{2\lambda} \tilde{E}^{\vec{r}},$$

so that  $u \in Z_{Fr}^< \otimes_R Z_{Fr}^0 \otimes_R Z_{Fr}^>$ .  $\square$

**Corollary 5.4.**  $U_\epsilon^{ev}$  is a free module over  $Z_{Fr}$  of rank  $d = \prod_{\alpha \in \Delta_+} \ell_\alpha^2 \cdot \prod_{1 \leq i \leq r} \ell_i$  with  $\ell_\alpha$  defined in (4.2).

**Remark 5.5.** Since  $\text{Ker}(\check{F}\check{r})$  is a two-side ideal in  $\widehat{U}_\epsilon(\mathfrak{g}, P)$  hence closed under the adjoint action of  $\dot{U}_\epsilon(\mathfrak{g})$ , the subalgebra  $Z_{Fr}$  is also closed under the adjoint action of  $\dot{U}_\epsilon(\mathfrak{g})$ . The Hopf algebra  $\dot{U}_\epsilon(\mathfrak{g})$  also naturally acts on  $\widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  as in Remark 4.16. Furthermore, by the description of  $\text{Ker}(\check{F}\check{r})$  in Lemma 4.18 and computations in Lemma 4.24 and Lemma 5.3(a), it follows that both  $\dot{U}_\epsilon(\mathfrak{g})$ -actions on  $Z_{Fr}$  and  $\widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  factor through the morphism  $\check{F}\check{r} : \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$ .

By definition of  $Z_{Fr}$ , we have an induced pairing

$$(5.2) \quad \langle \cdot, \cdot \rangle' : Z_{Fr} \times \widehat{U}_\epsilon^*(\mathfrak{g}, P^*) \longrightarrow R,$$

which is adjoint  $\dot{U}_R(\mathfrak{g}^d)$ -invariant, see Remark 5.5, and is non-degenerate in the first argument.

Let  $G^d$  be the simply connected algebraic group with the Lie algebra  $\mathfrak{g}^d$ . Note that  $\mathfrak{g}^d$  has the root lattice  $Q^*$  and the weight lattice  $P^*$ . Let  $G_0^d$  be the open Bruhat cell in  $G^d$ , that is  $G_0^d = U_-^d \times T^d \times U_+^d$ , where  $T^d$  is the maximal torus and  $U_\pm^d$  are opposite maximal unipotent subgroups (so that  $U_+^d$  corresponds to  $\Delta_+$ ). Everything is defined over  $\mathbb{Z}$ , hence makes sense over  $R$  by base change. We are now going to identify  $Z_{Fr}$  with  $R[G_0^d]$ , see Proposition 5.10.<sup>7</sup> To do so, we start with some technical results.

The perfect pairings (the "perfect" follows from PBW bases in Lemma 2.10 and Corollary 3.9)

$$\langle \cdot, \cdot \rangle' : U_\epsilon^{ev<} \times \dot{U}_\epsilon^> \longrightarrow R, \quad \langle \cdot, \cdot \rangle' : \dot{U}_\epsilon^< \times U_\epsilon^{ev>} \longrightarrow R$$

give us the pairings

$$(5.3) \quad \langle \cdot, \cdot \rangle' : Z_{Fr}^< \times \dot{U}_R^>(\mathfrak{g}^d) \longrightarrow R, \quad \langle \cdot, \cdot \rangle' : \dot{U}_R^<(\mathfrak{g}^d) \times Z_{Fr}^> \longrightarrow R,$$

through restricting to a subalgebra  $Z_{Fr}^< \subset U_\epsilon^{ev<} \cap \dot{U}_R^>(\mathfrak{g}^d)$  (resp.  $Z_{Fr}^> \subset U_\epsilon^{ev>} \cap \dot{U}_R^<(\mathfrak{g}^d)$ ) and passing through the quotient map  $\check{F}\check{r}^> : \dot{U}_\epsilon^> \rightarrow \dot{U}_R^>(\mathfrak{g}^d)$  (resp.  $\check{F}\check{r}^< : \dot{U}_\epsilon^< \rightarrow \dot{U}_R^<(\mathfrak{g}^d)$ ), see Lemma 4.17.

<sup>7</sup>It would be good to summarize what is known about similar identifications. For odd roots of unity (and coprime to 3 in the presence of  $G_2$  factors, this follows from DCKP. What is known for even orders?

**Lemma 5.6.** *Two pairings in (5.3) are perfect. In particular,*

$$(5.4) \quad \begin{aligned} (\tilde{F}^{\vec{k}}, e^{(\vec{r}^*)})' &= \begin{cases} \in R^\times & \text{if } \vec{k} = \vec{r} \in \mathbb{Z}_{\geq 0}^N / \mathcal{K} \\ 0 & \text{if } \vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N / \mathcal{K} \text{ and } \vec{k} \neq \vec{r}, \end{cases} \\ (f^{(\vec{r}^*)}, \tilde{E}^{\vec{k}})' &= \begin{cases} \in R^\times & \text{if } \vec{k} = \vec{r} \in \mathbb{Z}_{\geq 0}^N / \mathcal{K} \\ 0 & \text{if } \vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N / \mathcal{K} \text{ and } \vec{k} \neq \vec{r}, \end{cases} \end{aligned}$$

here  $\vec{r}^* = (r_1/\ell_{\beta_1}, \dots, r_N/\ell_{\beta_N})$  for  $\vec{r} \in \mathcal{K}$ . The subset  $\mathcal{K}$  is defined in Lemma 4.26.

*Proof.* By Lemma 5.3,  $Z_{Fr}^<$  has a  $R$ -basis  $\{\tilde{F}^{(\vec{k})} | \vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus \mathcal{K}\}$ . By Lemma 4.26, the elements  $\{\check{F}^{<}(\tilde{E}^{(\vec{k})}) | \vec{k} \in \mathbb{Z}_{\geq 0}^N \setminus \mathcal{K}\}$  form a  $R$ -basis of  $\dot{U}_R^>(\mathfrak{g}^d)$ . Moreover, by Lemma 4.29

$$\check{F}^{<}(\tilde{E}^{(\vec{k})}) = a_{\vec{k}} e^{(\vec{k}^*)}$$

for some  $a_{\vec{k}} \in R^\times$ . Combining these results with computations of pairings in Corollary 3.9, we get the first computation in (5.4), hence the first pairing in (5.3) is perfect. The proof for the second pairing is the same.  $\square$

For any  $\lambda \in P^*$ , we have the corresponding homomorphism  $\chi_\lambda: \dot{U}_R^0(\mathfrak{g}^d) \rightarrow R$ . In other words, we have a pairing

$$R[T^d] \times \dot{U}_R^0(\mathfrak{g}^d) \longrightarrow R.$$

This is just the Hopf algebra pairing between the ring of regular functions  $R[T^d]$  and the distribution algebra  $\text{Dist}_R(T^d) \simeq \dot{U}_R^0(\mathfrak{g}^d)$ . In particular, we have

$$\chi_{\lambda_1 + \lambda_2}(u_0) = \chi_{\lambda_1}(u_{0(1)}) \chi_{\lambda_2}(u_{0(2)}) \quad \forall u_0 \in \dot{U}_R^0(\mathfrak{g}^d).$$

**Lemma 5.7.** *The pairing (5.2) gives rise to a  $\dot{U}_R(\mathfrak{g}^d)$ -adjoint invariant pairing*

$$\langle \cdot, \cdot \rangle': Z_{Fr} \times \dot{U}_R(\mathfrak{g}^d) \longrightarrow R,$$

which is non-degenerate in the first argument. The pairing is given by

$$(5.5) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), \dot{y} u_0 \dot{x} \right\rangle' = \chi_{-\lambda/2}(u_0) (y, \dot{x})' (\dot{y}, x)',$$

where  $\lambda \in 2P^*$ ,  $u_0 \in \dot{U}_R^0(\mathfrak{g}^d)$ ,  $y \in Z_{Fr, -\nu}^<$ ,  $x \in Z_{Fr, \mu}^>$ ,  $\dot{y} \in \dot{U}_R^<(\mathfrak{g}^d)_{-\nu}$ ,  $\dot{x} \in \dot{U}_R^>(\mathfrak{g}^d)_{\mu}$  with  $\nu, \mu, \dot{\nu}, \dot{\mu} \in Q_+^*$ . The pairings  $(y, \dot{x})'$ ,  $(\dot{y}, x)'$  come from (5.3).

The construction is given in the proof.

*Proof.* We have the following inclusions of  $\dot{U}_R(\mathfrak{g}^d)$ -modules:

$$(5.6) \quad \dot{U}_R(\mathfrak{g}^d) \xrightarrow{\phi^{-1}} A_{Q^*} \xrightarrow{\psi_1^*} \tilde{U}_\epsilon^*(\mathfrak{g}, P^*) \hookrightarrow \widehat{U}_\epsilon^*(\mathfrak{g}, P^*).$$

The pairing  $Z_{Fr} \times \widehat{U}_\epsilon^*(\mathfrak{g}, P^*) \rightarrow R$  naturally extends to a pairing  $Z_{Fr} \times \tilde{U}_\epsilon^*(\mathfrak{g}, P^*) \rightarrow R$ . Since  $\widehat{U}_\epsilon^*(\mathfrak{g}, P^*)$  is a dense  $\dot{U}_R(\mathfrak{g}^d)$ -submodule of  $\tilde{U}_\epsilon^*(\mathfrak{g}, P^*)$ , the pairing  $Z_{Fr} \times \tilde{U}_\epsilon^* \rightarrow R$  is adjoint  $\dot{U}_R(\mathfrak{g}^d)$ -invariant and non-degenerate in the first argument. Since the map  $\psi_1^*: A_{Q^*} \hookrightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*)$  is an injective morphism of  $\dot{U}_R(\mathfrak{g}^d)$ -modules with a dense image by Lemma 4.21, the pairing  $Z_{Fr} \times A_{Q^*} \rightarrow R$  obtained by restriction is adjoint  $\dot{U}_R(\mathfrak{g}^d)$ -invariant and non-degenerate in the first argument.

The pairing  $Z_{Fr} \times A_{Q^*} \rightarrow R$  is explicitly given by (cf. (5.1)):

$$\left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), (\dot{y} K^{\kappa(\dot{\nu})}) u_0 (\dot{x} K^{\gamma(\dot{\mu})}) \right\rangle' = (y, \dot{x})' (\dot{y}, x)' \hat{\chi}_{-\lambda/2}(u_0) \epsilon^{(2\rho, \nu)}$$

with  $\hat{\chi}_{-\lambda/2}$  of (3.28), for any  $\nu, \mu, \dot{\nu}, \dot{\mu} \in Q_+^*$ ,  $\lambda \in 2P^*$ ,  $u_0 \in A_{Q^*}^0$ ,  $y \in Z_{Fr, -\nu}^<$ ,  $x \in Z_{Fr, \mu}^>$ ,  $\dot{y} \in \dot{U}_\epsilon^{*<}(\mathfrak{g})_{-\dot{\nu}}$ ,  $\dot{x} \in \dot{U}_\epsilon^{*>}(\mathfrak{g})_{\dot{\mu}}$ . Note that the coefficient  $\epsilon^{(\dot{\mu}, \gamma(\dot{\mu})) - (\dot{\lambda}, \kappa(\dot{\nu}) + \gamma(\dot{\mu}))}$  in (5.1) does not show up because we go back to the non-idempotent version, compare to (3.37). Moreover,  $\epsilon^{(2\rho, \nu)} = 1$  for  $\nu \in Q_+^*$ , hence

$$(5.7) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), (\dot{y} K^{\kappa(\dot{\nu})}) u_0 (\dot{x} K^{\gamma(\dot{\mu})}) \right\rangle' = (y, \dot{x})' (\dot{y}, x)' \hat{\chi}_{-\lambda/2}(u_0),$$

with indices as above.

Now using the isomorphism of  $\dot{U}_R(\mathfrak{g}^d)$ -modules  $A_{Q^*} \xrightarrow{\phi} \dot{U}_R(\mathfrak{g}^d)$  in Lemma 4.21, we obtain the desired  $\dot{U}_R(\mathfrak{g}^d)$ -invariant pairing  $Z_{Fr} \times \dot{U}_R(\mathfrak{g}^d) \rightarrow R$ . The equation (5.5) follows from (5.7) by the following observations:

- $\hat{\chi}_{-\lambda/2} : \dot{U}_R^0(\mathfrak{g}^d) \cong A_{Q^*}^0 \rightarrow R$  coincides with  $\chi_{-\lambda/2} : \dot{U}_R^0(\mathfrak{g}^d) \rightarrow R$ .
- The pairing  $(y, \dot{x})' : Z_{Fr}^< \times \dot{U}_\epsilon^{*>} \rightarrow R$  in (5.7) is obtained from the pairing  $Z_{Fr}^< \times \dot{U}_\epsilon^> \rightarrow R$  by passing through the quotient  $\check{Fr}^> : \dot{U}_\epsilon^> \rightarrow \dot{U}_\epsilon^{*>}$ . On the other hand, the morphism  $\check{Fr}^> : \dot{U}_\epsilon^> \rightarrow \dot{U}_R^>(\mathfrak{g}^d)$  is equal to the composition  $\dot{U}_\epsilon^> \xrightarrow{\check{Fr}^>} \dot{U}_\epsilon^{*>} \xrightarrow{\phi} \dot{U}_R^>(\mathfrak{g}^d)$ .  $\square$

**Lemma 5.8.** (a) For any  $x_1, x_2 \in Z_{Fr}^>$  and  $\dot{y} \in \dot{U}_R^<(\mathfrak{g}^d)$ , we have

$$(5.8) \quad (\dot{y}, x_1 x_2)' = (\dot{y}_{(1)}, x_1)' (\dot{y}_{(2)}, x_2)'.$$

(b) For any  $y_1, y_2 \in Z_{Fr}^<$  and  $\dot{x} \in \dot{U}_R^>(\mathfrak{g}^d)$ , we have

$$(5.9) \quad (y_1 y_2, \dot{x})' = (y_1, \dot{x}_{(1)})' (y_2, \dot{x}_{(2)})'.$$

*Proof.* Let us recall the  $R$ -Hopf subalgebras  $A^>$  and  $A^<$  from Remark 4.23. The homomorphism of Hopf algebras  $\check{Fr} : \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_R(\mathfrak{g}^d)$  restricts to homomorphisms of Hopf algebras:

$$\check{Fr}^{A, >} : A^> \longrightarrow \dot{U}_R^>(\mathfrak{g}^d), \quad \check{Fr}^{A, <} : A^< \longrightarrow \dot{U}_R^<(\mathfrak{g}^d).$$

(a) The Hopf pairing  $(\cdot, \cdot)' : \dot{U}_\epsilon^{\leqslant} \times U_\epsilon^{ev \geqslant} \rightarrow R$  restricts to the pairing  $(\cdot, \cdot)' : A^< \times Z_{Fr}^> \rightarrow R$  which satisfies

$$(5.10) \quad (\dot{y}, x_1 x_2)' = (\dot{y}_{(2)}, x_1)' (\dot{y}_{(1)}, x_2)'$$

for  $\dot{y} \in A^<$  and  $x_1, x_2 \in Z_{Fr}^>$ . The kernel of the map  $\check{Fr}^{A, <}$  is spanned over  $R$  by  $\ker(\check{Fr}^<)$  and  $\dot{U}_\epsilon^<(K^\lambda - 1)$  with  $\lambda \in 2P$ . Moreover,  $Z_{Fr}^>$  is orthogonal to  $\dot{U}_\epsilon^<(K^\lambda - 1)$ , indeed, for any  $x \in Z_{Fr, \mu}^>$  for any  $\mu \in Q_+^*$  and  $\dot{y} \in \dot{U}_\epsilon^<$ , we have

$$(\dot{y}(K^\lambda - 1), x)' = (\dot{y}, x)' (\epsilon^{(\lambda, \mu)} - 1) = 0,$$

since  $\mu \in Q_+^*$  and  $\lambda \in 2P$ . Therefore,  $Z_{Fr}^>$  is orthogonal to  $\text{Ker}(\check{Fr}^{A, <})$ . Hence the pairing  $A^< \times Z_{Fr}^> \rightarrow R$  gives a rise to a pairing  $\dot{U}_R^<(\mathfrak{g}^d) \times Z_{Fr}^> \rightarrow R$  by descending to the quotient, which coincides with the pairing  $\dot{U}_R^<(\mathfrak{g}^d) \times Z_{Fr}^> \rightarrow R$  in (5.3). By (5.10), it follows that the pairing  $\dot{U}_R^<(\mathfrak{g}^d) \times Z_{Fr}^> \rightarrow R$  satisfies:

$$(\dot{y}, x_1 x_2)' = (\dot{y}_{(2)}, x_1)' (\dot{y}_{(1)}, x_2)'$$

for  $\dot{y} \in \dot{U}_R^<(\mathfrak{g}^d)$  and  $x_1, x_2 \in Z_{Fr}^>$ . However,  $\dot{U}_R^<(\mathfrak{g}^d)$  is cocommutative, therefore, part (a) follows.

(b) The proof is similar.  $\square$

Let us consider the embedding  $Z_{Fr} \hookrightarrow \text{Hom}_R(\dot{U}_R(\mathfrak{g}^d), R)$  arising from Lemma 5.7. Note that the algebra structure on  $\text{Hom}_R(\dot{U}_R(\mathfrak{g}^d), R)$  comes from the coalgebra structure on  $\dot{U}_R(\mathfrak{g}^d)$ .

**Lemma 5.9.** The above embedding  $Z_{Fr} \hookrightarrow \text{Hom}_R(\dot{U}_R(\mathfrak{g}^d), R)$  is an  $R$ -algebra homomorphism.

*Proof.* Let  $a_i = (y_i K^{\kappa(\nu_i)}) K^{\lambda_i} (x_i K^{\gamma(\mu_i)})$  with  $y_i \in Z_{Fr,-\nu_i}^<, x_i \in Z_{Fr,\mu_i}^>, \lambda_i \in 2P^*, \nu_i, \mu_i \in Q_+^*$  for  $i = 1, 2$ . Then, for any  $u = \dot{y}u_0\dot{x} \in \dot{U}_R(\mathfrak{g}^d)$ , we need to verify the following equality:

$$\langle a_1 a_2, u \rangle' = \langle a_1, u_{(1)} \rangle' \langle a_2, u_{(2)} \rangle'.$$

Indeed, using the Sweedler's notation (2.2), we have:

$$\begin{aligned} \langle a_1, u_{(1)} \rangle' \langle a_2, u_{(2)} \rangle' &= \chi_{-\lambda_1/2}(u_{0(1)})(y_1, \dot{x}_{(1)})' (\dot{y}_{(1)}, x_1)' \cdot \chi_{-\lambda_2/2}(u_{0(2)})(y_2, \dot{x}_{(2)})' (\dot{y}_{(2)}, x_2)' \\ &= \chi_{-\lambda_1/2}(u_{0(1)}) \chi_{-\lambda_2/2}(u_{0(2)}) \cdot (y_1, \dot{x}_{(1)})' (y_2, \dot{x}_{(2)})' \cdot (\dot{y}_{(1)}, x_1)' (\dot{y}_{(2)}, x_2)' \\ &= \chi_{(-\lambda_1-\lambda_2)/2}(u_0)(y_1 y_2, \dot{x})' (\dot{y}, x_1 x_2)' = \langle a_1 a_2, u \rangle', \end{aligned}$$

where we used that  $K^{\gamma(\mu_i)}, K^{\kappa(\nu_i)}$  are central, due to Remark 4.2. This completes the proof.  $\square$

We are now ready to identify  $Z_{Fr}$  with  $R[G_0^d]$ . There are direct sum decompositions into weight components:  $Z_{Fr}^> = \bigoplus_{\mu \in Q_+^*} Z_{Fr,\mu}^>$  and  $Z_{Fr}^< = \bigoplus_{\nu \in Q_+^*} Z_{Fr,-\nu}^<$ . Let

$$\tilde{Z}_{Fr}^< = \bigoplus_{\nu \in Q_+^*} \tilde{Z}_{Fr,-\nu}^<, \quad \tilde{Z}_{Fr}^> = \bigoplus_{\mu \in Q_+^*} \tilde{Z}_{Fr,\mu}^> \quad \text{with} \quad \tilde{Z}_{Fr,-\nu}^< = Z_{Fr,-\nu}^< K^{\kappa(\nu)}, \quad \tilde{Z}_{Fr,\mu}^> = Z_{Fr,\mu}^> K^{\gamma(\mu)}.$$

Then we still have a decomposition  $Z_{Fr} = \tilde{Z}_{Fr}^< \otimes_R Z_{Fr}^0 \otimes_R \tilde{Z}_{Fr}^>$ , cf. Lemma 5.3(c).

**Proposition 5.10.** *There is a  $\dot{U}_R(\mathfrak{g}^d)$ -linear algebra isomorphism*

$$\varphi: Z_{Fr} \xrightarrow{\sim} R[G_0^d] \simeq R[U_-^d] \otimes_R R[T^d] \otimes_R R[U_+^d].$$

Furthermore, under this isomorphism, we have:  $\tilde{Z}_{Fr}^< \simeq R[U_+^d], \tilde{Z}_{Fr}^> \simeq R[U_-^d], Z_{Fr}^0 \simeq R[T^d]$ .

*Proof.* We have two adjoint  $\dot{U}_R(\mathfrak{g}^d)$ -invariant pairings which are non-degenerate in the first arguments:

$$Z_{Fr} \times \dot{U}_R(\mathfrak{g}^d) \longrightarrow R \quad \text{and} \quad R[G_0^d] \times \dot{U}_R(\mathfrak{g}^d) \longrightarrow R.$$

They give us  $R$ -algebra embeddings

$$Z_{Fr} \hookrightarrow \text{Hom}_R(\dot{U}_R(\mathfrak{g}^d), R) \hookleftarrow R[G_0^d],$$

which intertwine the  $\dot{U}_R(\mathfrak{g}^d)$ -actions. So it is enough to show that the images of  $\tilde{Z}_{Fr}^<, \tilde{Z}_{Fr}^>, Z_{Fr}^0$  coincide with the images of  $R[U_+^d], R[U_-^d], R[T^d]$  as  $R$ -submodules, respectively.

We will use the notations of Lemma 5.9. Due to Lemma 5.9(a), we see that  $K^\lambda$  is identified with  $\chi_{-\lambda/2} \in R[T^d]$ , hence  $Z_{Fr}^0$  is identified with  $R[T^d]$  viewed as  $R$ -subalgebras of  $\text{Hom}_R(\dot{U}_R(\mathfrak{g}^d), R)$ .

Let  $\dot{U}_R^0(\mathfrak{g}^d)_+, \dot{U}_R^<(\mathfrak{g}^d)_+$  be the kernels of the counit maps restricted to  $\dot{U}_R^0(\mathfrak{g}^d), \dot{U}_R^<(\mathfrak{g}^d)$ , respectively. For any  $z \in \tilde{Z}_{Fr,\mu}^>$ , we have

$$\langle z, \dot{y}u_0\dot{x} \rangle' = 0,$$

if either  $u_0 \in \dot{U}_R^0(\mathfrak{g}^d)_+$  or  $\dot{x} \in \dot{U}_R^>(\mathfrak{g}^d)_+$ . Moreover, the pairing  $\tilde{Z}_{Fr,\mu}^> \times \dot{U}_R^<(\mathfrak{g}^d)_{-\mu} \rightarrow R$  is a perfect pairing by Lemma 5.6. Therefore,  $\tilde{Z}_{Fr,\mu}^>$  must be identified with  $R[U_-^d]_{-\mu}$ , and so  $\tilde{Z}_{Fr}^> \simeq R[U_-^d]$ .

The proof of the coincidence of the images  $\tilde{Z}_{Fr}^< \simeq R[U_+^d]$  is similar.  $\square$

### 5.3. The locally finite part of the Frobenious center when $R = \mathbb{F}$ algebraically closed field.

Let us define

$$(5.11) \quad Z_{Fr}^{fin} := \{z \in Z_{Fr} \mid \dim_{\mathbb{F}}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)z) < \infty\}$$

This is a subalgebra of  $Z_{Fr}$  since  $Z_{Fr}$  is a  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ -module algebra.

**Lemma 5.11.** (a)  $Z_{Fr}^{fin} \cong \mathbb{F}[G^d]$  under the isomorphism  $\varphi$  in Proposition 5.10.

(b) Let  $\lambda_0 = \sum_i \omega_i^* = \sum_i \ell_i \omega_i$ , where  $\omega_i$  are the fundamental weights and  $\ell_i$  are as in Section 4. Then  $Z_{Fr}^{fin}[K^{2\lambda_0}] = Z_{Fr}$ , where the left hand side is the localization of  $Z_{Fr}^{fin}$  by  $K^{2\lambda_0}$ .

(c) Suppose  $\mathbb{F}$  has characteristic 0 then  $Z_{Fr}^{fin} = \bigoplus_{\lambda \in P_+^*} \dot{U}_{\mathbb{F}}(\mathfrak{g}^d) K^{-2\lambda}$ .

*Proof.* (a) We only need to show that the  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ -locally finite part of  $\mathbb{F}[G_0^d]$  is  $\mathbb{F}[G^d]$ . First, all conjugacy classes of  $G^d(\mathbb{F})$  intersect  $G_0^d(\mathbb{F})$  because  $G_0^d(\mathbb{F})$  contains a Borel subgroup  $B^d(\mathbb{F})$  and any element of  $G^d(\mathbb{F})$  can be conjugated to some element in  $B^d(\mathbb{F})$ .

Let  $V^d(\lambda_0)$  be the Weyl representation of  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ . Let  $v_{\lambda_0}^d$  be the highest weight vector of  $V^d(\lambda_0)$  and  $(v_{\lambda_0}^d)^*$  be the dual weight basis of  $v_{\lambda_0}^d$  in  $V^d(\lambda_0)$ . By using the Bruhat cell decomposition  $G^d(\mathbb{F}) = \bigsqcup_{w \in W} U_-^d(\mathbb{F}) w T^d(\mathbb{F}) U_+^d(\mathbb{F})$ , one can show that the vanishing locus of  $f = c_{(v_{\lambda_0}^d)^*, v_{\lambda_0}^d}$  is  $G^d(\mathbb{F}) \setminus G_0^d(\mathbb{F})$ , the complement of the open Bruhat cell  $G_0^d(\mathbb{F})$  in  $G^d(\mathbb{F})$ .

Let  $V$  be a finite dimensional  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ -submodule of  $\mathbb{F}[G_0^d]$ . Let  $C$  be a subalgebra of  $\mathbb{F}[G_0^d]$  generated by  $\mathbb{F}[G^d]$  and  $V$ . Then  $C$  is a  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ -locally finite and finitely generated as an algebra. Let  $X = \text{Spec}(C)$ . We have an inclusion of reduced  $G^d(\mathbb{F})$ -algebras  $\mathbb{F}[G^d] \hookrightarrow C$ , inducing an isomorphism of localizations  $\mathbb{F}[G^d]_f \cong C_f \cong \mathbb{F}[G_0^d]$ . Hence a map of  $G^d(\mathbb{F})$ -varieties  $\psi : X \rightarrow G^d(\mathbb{F})$  such that  $X_f \rightarrow G_0^d(\mathbb{F})$  is an isomorphism, where  $X_f = \text{Spec}(C_f)$ . Let  $G^d X_f$  be the minimal  $G^d$ -subvariety of  $X$  containing  $X_f$ . Since  $G_0^d(\mathbb{F})$  intersects all  $G^d(\mathbb{F})$ -orbits, we have that  $\psi : G^d X_f \rightarrow G^d$  is an isomorphism. On the other hand, since  $C$  is an integral domain, it follows that  $X$  is an irreducible variety and  $G^d X_f$  is an open dense subset of  $X$ . Therefore, a composition of the inclusion map  $\mathbb{F}[G_0^d] \hookrightarrow C \rightarrow \mathbb{F}[G^d X_f]$  is an isomorphism, so that  $\mathbb{F}[G^d] = C$ . This implies that any finite dimensional  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ -submodule  $V$  of  $\mathbb{F}[G_0^d]$  is contained in  $\mathbb{F}[G^d]$ .

(b) Using the equation (5.5), it is easy to show that  $\varphi(K^{-2\lambda_0}) = c_{(v_{\lambda_0}^d)^*, v_{\lambda_0}^d}$ . Then part (b) follows by part (a).

(c) For any  $\lambda \in P_+^*$ , let  $V^d(\lambda)$  be the Weyl module of  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ . Let  $v_{\lambda}^d$  be the nonzero highest weight vector of  $V^d(\lambda)$  and  $(v_{\lambda}^d)^*$  be the dual weight vector. When  $\mathbb{F}$  has characteristics 0,  $\mathbb{F}[G^d] = \bigoplus \dot{U}_{\mathbb{F}}(\mathfrak{g}^d) c_{(v_{\lambda}^d)^*, v_{\lambda}^d}$  by the Peter-Weyl theorem. Using the equation (5.5) again,  $\varphi(K^{-2\lambda}) = c_{v_{\lambda}^d, v_{\lambda}}$  for  $\lambda \in P_+^*$ . Hence part (c) follows.  $\square$

## 6. RATIONAL REPRESENTATIONS OF $\dot{\mathcal{U}}_q(\mathfrak{g})$ AND $\dot{U}_q(\mathfrak{g})$

Consider the ring  $\underline{\mathcal{A}} := \mathbb{Z}[v, v^{-1}]$  so that  $\mathcal{A}$  is a finite localization of  $\underline{\mathcal{A}}$ . Let  $R$  be an  $\underline{\mathcal{A}}$ -algebra and  $q$  denote the image of  $v$  in  $R$ . Then we can form the Lusztig form  $\dot{\mathcal{U}}_q(\mathfrak{g})$  over  $R$  as above.

The goal of this section is to review several known results and constructions related to the rational representations of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ . We then carry these results to rational representations of  $\dot{U}_q(\mathfrak{g})$ .

**Definition 6.1.** [ Let  $M$  be a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module. We say that it is *rational of type 1* (in what follows, just rational) if the following conditions hold:

- (i)  $M$  is a (type 1) weight module meaning that there is a decomposition  $M = \bigoplus_{\lambda \in P} M_\lambda$ , where  $K_i$  acts on  $M_\lambda$  by  $q^{(\lambda, \alpha_i)}$  and  $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$  acts by  $\begin{bmatrix} (\lambda, \alpha_i^\vee) \\ t \end{bmatrix}_{q_i}$  for all  $i = 1, \dots, r$  and all  $t > 0$ .
- (ii) For any  $m \in M$  there is  $k > 0$  such that  $E_i^{(s)}m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .
- (iii) For any  $m \in M$  there is  $k > 0$  such that  $F_i^{(s)}m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .

**Definition 6.2.** Let  $\dot{\mathcal{U}}_q(\mathfrak{g})\text{-Lmod}$  be the category of left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules. Let  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  be the category of rational representations of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ . The morphisms in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  are morphisms of left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules which preserve the weight decompositions.

**Proposition 6.3.** *The natural functor  $\mathcal{I} : \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})) \rightarrow \dot{\mathcal{U}}_q(\mathfrak{g})\text{-Lmod}$  is fully faithful. Moreover, the image is closed under taking subquotients.*

Before going to the proof, we need the following result. For any  $\lambda \in P$ , let  $\chi_\lambda : \dot{\mathcal{U}}_q^0(\mathfrak{g}) \rightarrow R$  be the  $R$ -algebra homomorphism defined by

$$(6.1) \quad \chi_\lambda : \quad K_i \mapsto q^{(\lambda, \alpha_i)}, \quad \begin{bmatrix} K_i; 0 \\ t \end{bmatrix}_{q_i} \mapsto \begin{bmatrix} (\lambda, \alpha_i^\vee) \\ t \end{bmatrix}_{q_i}.$$

For any  $N \in \dot{\mathcal{U}}_q(\mathfrak{g})\text{-Lmod}$ , let  $N_{(\lambda)} := \{n \in N \mid xn = \chi_\lambda(x)n \ \forall x \in \dot{\mathcal{U}}_q^0(\mathfrak{g})\}$ .

**Lemma 6.4.** *The sum  $\sum_{\lambda \in P} N_{(\lambda)}$  in  $N$  is a direct sum  $\bigoplus N_{(\lambda)}$ . As a result, for any  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ , the weight space  $M_\lambda$  is uniquely characterized by  $M_\lambda = \{m \in M \mid xm = \chi_\lambda(x)m \ \forall x \in \dot{\mathcal{U}}_q^0(\mathfrak{g})\}$ .*

*Proof.* Suppose  $\sum_{i=1}^m n_{\lambda_i} = 0$  with  $n_{\lambda_i} \neq 0 \in N_{(\lambda_i)}$ . Then  $\chi_{\lambda_1}(x)n_{\lambda_1} + \dots + \chi_{\lambda_m}(x)n_{\lambda_m} = 0$  for all  $x \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$ . Then  $\sum_{i=2}^m (\chi_{\lambda_i}(x) - \chi_{\lambda_1}(x))n_{\lambda_i} = 0$ . Repeating this process, we have  $\prod_{i=1}^{m-1} (\chi_{\lambda_m}(x_i) - \chi_{\lambda_i}(x_i))n_{\lambda_m} = 0$  for all  $x_1, \dots, x_{m-1} \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$ . Let  $I$  be the annihilator of  $n_{\lambda_m}$  in  $R$ . Since  $n_{\lambda_m} \neq 0$ , there is a maximal ideal  $\mathfrak{m}$  of  $R$  containing  $I$ . Consider the characters  $\underline{\chi}_{\lambda_i} : \dot{\mathcal{U}}_q^0(\mathfrak{g}) \rightarrow R \rightarrow R/\mathfrak{m}$ . These characters  $\underline{\chi}_{\lambda_i}$  are still pairwise distinct, hence we can find  $x_i \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$  such that  $\underline{\chi}_{\lambda_m}(x_i) - \underline{\chi}_{\lambda_i}(x_i) \neq 0$ . In the other word, there is  $x_i \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$  such that  $\chi_{\lambda_m}(x_i) - \chi_{\lambda_i}(x_i) \notin \mathfrak{m}$ . So there are  $x_1, \dots, x_{m-1} \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$  such that  $\prod_{i=1}^{m-1} (\chi_{\lambda_m}(x_i) - \chi_{\lambda_i}(x_i)) \notin \mathfrak{m}$  since  $\mathfrak{m}$  is a maximal ideal. Therefore, there are  $x_1, \dots, x_{m-1} \in \dot{\mathcal{U}}_q^0(\mathfrak{g})$  such that  $\prod_{i=1}^{m-1} (\chi_{\lambda_m}(x_i) - \chi_{\lambda_i}(x_i)) \notin I$ , the annihilator of  $n_{\lambda_m}$ , contradiction.  $\square$

*Proof of Proposition 6.3.* Let  $M_1, M_2 \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ . To show the fully faithfulness it is enough to show that any morphism of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules  $M_1 \rightarrow M_2$  preserves the weight decompositions. This follows by Lemma 6.4. Let us prove the image is closed under taking subquotients. Let  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ . Suppose we have a short exact sequence in  $\dot{\mathcal{U}}_q(\mathfrak{g})\text{-Lmod}$ :  $0 \rightarrow M_1 \xrightarrow{\iota} M \xrightarrow{\pi} M_2$

$M_2 \rightarrow 0$ . We have  $M_2 = \sum_{\lambda} \pi(M_{\lambda})$  and  $\pi(M_{\lambda}) \subset M_{2,(\lambda)}$ . By Lemma 6.4, it follows that  $M_2 = \bigoplus_{\lambda} \pi(M_{\lambda})$ , hence,  $M_2 \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  and then  $M_1 \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ .  $\square$

**Remark 6.5.** Let  $N \in \dot{\mathcal{U}}_q(\mathfrak{g})\text{-Lmod}$ . By Proposition 6.3 and Lemma 6.4, we can talk about the maximal rational subrepresentation  $N^{ral}$  of  $N$ . Furthermore, for any  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ , we have

$$\text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(M, N^{ral}) = \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(M, N).$$

We will need the following result.

**Proposition 6.6.** *Let  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  and  $m \in M$ . Then  $\dot{\mathcal{U}}_q(\mathfrak{g})m$  is finitely generated over  $R$ .*

*Proof.* Pick any reduced decomposition  $w_0 = s_{i_1}s_{i_2}\cdots s_{i_N}$  of the longest element  $w_0$  of the Weyl group  $W$ . The desired result follows immediately from the following lemma:  $\square$

**Lemma 6.7.** (a) *Let  $\dot{\mathcal{U}}_i^>$  denote the  $R$ -subalgebra generated by  $\{E_i^{(s)}\}_{s \in \mathbb{N}}$ . Then:*

$$\dot{\mathcal{U}}^> = \dot{\mathcal{U}}_{i_1}^>\dot{\mathcal{U}}_{i_2}^>\cdots\dot{\mathcal{U}}_{i_N}^>.$$

(b) *Let  $\dot{\mathcal{U}}_i^<$  denote the  $R$ -subalgebra generated by  $\{F_i^{(s)}\}_{s \in \mathbb{N}}$ . Then:*

$$\dot{\mathcal{U}}^< = \dot{\mathcal{U}}_{i_1}^<\dot{\mathcal{U}}_{i_2}^<\cdots\dot{\mathcal{U}}_{i_N}^<.$$

*Proof.* For the case of  $R = k[v, v^{-1}]$  (where  $k$  is any field), this was proved in [JL1, §5.8-5.11].

The case of general  $R$  is handled below in several cases.

*Case 1:*  $R$  is a field. This follows from the case of  $R = k[v^{\pm 1}]$  by base change.

*Case 2:*  $R$  is a local ring with residue field  $k$ . Consider the situation of (a), (b) is similar. Pick  $\mu \in Q^+$ . Then the  $\mu$ -weight components in  $\dot{\mathcal{U}}^>$  and its submodule  $\dot{\mathcal{U}}_{i_1}^>\dot{\mathcal{U}}_{i_2}^>\cdots\dot{\mathcal{U}}_{i_N}^>$  are finitely generated  $R$ -module. We note that the following diagram is commutative:

$$\begin{array}{ccc} (\dot{\mathcal{U}}_{R,i_1}^>\dot{\mathcal{U}}_{R,i_2}^>\cdots\dot{\mathcal{U}}_{R,i_N}^>)_{\mu} & \longrightarrow & \dot{\mathcal{U}}_{R,\mu}^> \\ \downarrow & & \downarrow \\ (\dot{\mathcal{U}}_{k,i_1}^>\dot{\mathcal{U}}_{k,i_2}^>\cdots\dot{\mathcal{U}}_{k,i_N}^>)_{\mu} & \longrightarrow & \dot{\mathcal{U}}_{k,\mu}^> \end{array}.$$

The vertical arrows are base changes from  $R$  to  $k$ . By Case 1, the bottom horizontal arrow is surjective. We apply the Nakayama lemma and deduce that the top horizontal arrow is surjective.

*Case 3:*  $R$  is general. By Case 2, the homomorphism  $(\dot{\mathcal{U}}_{R,i_1}^>\dot{\mathcal{U}}_{R,i_2}^>\cdots\dot{\mathcal{U}}_{R,i_N}^>)_{\mu} \rightarrow \dot{\mathcal{U}}_{R,\mu}^{>0}$  becomes surjective after localizing at every maximal ideal. Hence it is surjective.  $\square$

### 6.1. Weyl and dual Weyl modules.

Let  $\text{Rep}(\dot{\mathcal{U}}_q^{\geqslant})$  (resp.  $\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$ ) denote the category of rational representations of  $\dot{\mathcal{U}}_q^{\geqslant}$  (resp.  $\dot{\mathcal{U}}_q^{\leqslant}$ ), those satisfying the condition (i) and (ii) (resp. (iii)) of Definition 6.1. Let  $\text{Rep}^{fd}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ ,  $\text{Rep}^{fd}(\dot{\mathcal{U}}_q^{\geqslant})$ ,  $\text{Rep}^{fd}(\dot{\mathcal{U}}_q^{\leqslant})$  denote the subcategories of the corresponding categories consisting of all objects which are finitely generated over  $R$ .

Following [APW, §1.1.14], we define Joseph's induction functor  $\mathfrak{J} : \text{Rep}^{fd}(\dot{\mathcal{U}}_q^{\geqslant}) \rightarrow \text{Rep}^{fd}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  as follows: for any  $N \in \text{Rep}(\dot{\mathcal{U}}_q^{\geqslant})$ , let  $\mathfrak{J}(N)$  be the maximal rational quotient of  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}_q^{\geqslant}} N$ . The existence of such maximal rational quotient is in [APW, Proposition 1.14].

**Remark 6.8.** The algebras considered in [APW] are defined over the localization  $A_{\mathfrak{m}_0}$  at the maximal ideal  $\mathfrak{m}_0 = (v - 1, p)$  for some odd prime  $p \in \mathbb{Z}$ . However, at the moment, the results about Joseph's induction functor in [APW] up to Section 1.20 hold for general  $R$ . We note

that Lemma 1.13 in [APW] follows by using Lusztig's braid group action in arbitrary rational  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules constructed in [L5, §41.2], so that we do not use the proof in [APW] which does not apply to general ring  $R$ . The proof of Lemma 1.11 in [APW] also holds with care.

The following result is in [APW, Proposition 1.17]

**Proposition 6.9.** *The functor  $\mathfrak{J}$  is left adjoint to the restriction functor  $\mathfrak{F} : \text{Rep}^{fd}(\dot{\mathcal{U}}_q(\mathfrak{g})) \rightarrow \text{Rep}^{fd}(\dot{\mathcal{U}}_q^{\geqslant})$ .*

Any  $\lambda \in P$  gives a rise to the homomorphism  $\dot{\mathcal{U}}_q^{\geqslant} \rightarrow R$  given by the character  $\chi_{\lambda} : \dot{\mathcal{U}}_q^0(\mathfrak{g}) \rightarrow R$  inflated to  $\dot{\mathcal{U}}_q^{\geqslant}(\mathfrak{g})$ . We write  $R_{\lambda}$  for the rank 1 module over  $\dot{\mathcal{U}}_q^{\geqslant}$  with the action corresponding to this homomorphism.

**Definition 6.10.** The  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module  $\Delta_q(\lambda) := \dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}_q^{\geqslant}} R_{\lambda}$  is called *the Verma module*. For  $\lambda \in P_+$ , the module  $W_q(\lambda) := \mathfrak{J}(R_{\lambda})$  is called *the Weyl module*.

Proposition 6.9 yields natural isomorphism:

$$(6.2) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda), ?) \xrightarrow{\sim} \text{Hom}_{\dot{\mathcal{U}}_q^{\geqslant}}(R_{\lambda}, ?).$$

Follow [APW, Proposition 1.20], we have

**Proposition 6.11.**  *$W_q(\lambda)$  is the quotient of  $\Delta_q(\lambda)$  by the relations  $F_i^{[k]}v_{\lambda} = 0$  for all  $k > (\lambda, \alpha_i^{\vee})$  and all  $i = 1, \dots, r$ . Here,  $v_{\lambda}$  denotes the element in  $R_{\lambda}$  corresponding to 1.*

Proposition 6.11 implies that  $W_q(\lambda) = W_{\mathcal{A}}(\lambda) \otimes_{\mathcal{A}} R$ . Then by Equation 2.6.2 in [RH, §2.6], we have an important property of the Weyl modules:

**Lemma 6.12.** *The module  $W_q(\lambda)$  is free over  $R$ , admits a  $R$ -basis consisting of weight vectors and its character is given by the Weyl character formula.*

**Remark 6.13.** Thus, the results of [APW, §1-4] over the ring  $\mathcal{A}_{m_0}$  also hold over any  $\mathcal{A}$ -algebra  $R$  where one should replace the condition of freeness of finitely generated  $\mathcal{A}_{m_0}$ -modules with the condition of projectivity of finitely generated  $R$ -modules (except some places)

Furthermore, we have the usual induction functor  $H^0 : \text{Rep}(\dot{\mathcal{U}}_q^{\leqslant}) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  defined as follows. For any  $M \in \text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$ , the space  $\text{Hom}_{\dot{\mathcal{U}}_q^{\leqslant}}(\dot{\mathcal{U}}_q(\mathfrak{g}), M)$  carries a left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module structure:  $(xf)(u) = f(ux)$  for any  $x, u \in \dot{\mathcal{U}}_q(\mathfrak{g}), f \in \text{Hom}_{\dot{\mathcal{U}}_q^{\leqslant}}(\dot{\mathcal{U}}_q(\mathfrak{g}), M)$ . Then  $H^0(M)$  is defined as the maximal rational  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -subrepresentation of  $\text{Hom}_{\dot{\mathcal{U}}_q^{\leqslant}}(\dot{\mathcal{U}}_q(\mathfrak{g}), M)$ . The following result is in [APW, Proposition 2.12]

**Proposition 6.14.** *The functor  $H^0 : \text{Rep}(\dot{\mathcal{U}}_q^{\leqslant}) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  is right adjoint to the restriction functor  $\mathfrak{F} : \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$ .*

**Definition 6.15.** For any dominant  $\lambda$ , the module  $H_q^0(\lambda) := H^0(R_{\lambda})$  is called *the dual Weyl module*.

By Proposition 6.14, we have

$$(6.3) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(?, H_q^0(\lambda)) \xrightarrow{\sim} \text{Hom}_{\dot{\mathcal{U}}_q^{\leqslant}}(?, R_{\lambda}).$$

**Remark 6.16.** We follow the definition of the induction functor  $H^0$  in [APW] in contrast to [RH], see Section 2.9 [RH]. This is just a matter of convention so we can apply the results in [RH] to  $H^0$ .

### 6.2. Some homological properties.

Replacing  $\dot{\mathcal{U}}_q^{\leqslant}$  with  $R$  and  $\dot{\mathcal{U}}_q^0$  we have the following induction functors

$$\begin{aligned} H^0(\dot{\mathcal{U}}/R, -) : R\text{-Lmod} &\rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})), \\ H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, -) : \text{Rep}(\dot{\mathcal{U}}_q^0) &\rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})). \end{aligned}$$

**Lemma 6.17.** (a) The functor  $H^0(\dot{\mathcal{U}}/R, -)$  is exact, sends injective objects to injective objects and  $H^0(\dot{\mathcal{U}}/R, M) \cong H^0(\dot{\mathcal{U}}/R, R) \otimes_R M$  for any  $R$ -module  $M$ . In particular,  $H^0(\dot{\mathcal{U}}/R, -)$  commutes with direct limits.

(b) The functor  $H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, -)$  is exact and sends injective objects to injective objects.

(c)  $H^0(\dot{\mathcal{U}}/\underline{\mathcal{A}}, \underline{\mathcal{A}}) \otimes_{\underline{\mathcal{A}}} R \cong H^0(\dot{\mathcal{U}}/R, R)$  for any  $\underline{\mathcal{A}}$ -algebra  $R$ . As a result, for any  $R$ -algebra  $R'$ , we have  $H^0(\dot{\mathcal{U}}/R, R) \otimes_R R' \cong H^0(\dot{\mathcal{U}}/R', R')$ .

*Proof.* Part (a) follows by [APW, §1.31] and Remark 6.13. Part (b) follows by [APW, §2.11, 2.13] and Remark 6.13. Part(c) is in [AW, §4.2](thanks to the results explained in Sections 6.1 and 6.3 we can work with the ring  $\underline{\mathcal{A}}$  instead of the ring  $\mathcal{A}_1$  from [AW, §4.1]).  $\square$

Let us construct an injective resolution for  $N \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ . First, we embed  $N$  into an injective  $R$ -module  $I^0$ . Then  $N$  is a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -submodule of  $N^0 := H^0(\dot{\mathcal{U}}/R, I^0)$ . Repeat the process with  $N^0/N$ . We then obtain an exact sequence  $\dots 0 \rightarrow N \rightarrow N^0 \rightarrow N^1 \rightarrow \dots$  such that  $N^i = H^0(\dot{\mathcal{U}}/R, I^i)$  for some injective  $R$ -module  $I^i$ , and then  $N^0 \rightarrow N^1 \dots$  is an injective resolution of  $N$ .

There is a **big standard resolution** of  $N$  constructed as follows:  $N \hookrightarrow H^0(\dot{\mathcal{U}}/R, N)$  makes  $N$  into a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -submodule and a direct  $R$ -module summand of  $H^0(\dot{\mathcal{U}}/R, N)$ . Let  $Q^0 = H^0(\dot{\mathcal{U}}/R, N)$ . Repeat the process with  $Q^0/N$  then we obtain a resolution  $N \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots$  in which  $Q^i = H^0(\dot{\mathcal{U}}/R, Q^{i-1}/Q^{i-2})$ .

Moreover, there is a **standard resolution** of  $N$  constructed as follows:  $N \hookrightarrow H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, N)$  makes  $N$  into a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -submodule and a direct  $\dot{\mathcal{U}}_q^0$ -module summand of  $H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, N)$ . Let  $Q^0 := H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, N)$ . Repeat the process with  $Q^0/N$  then we obtain a resolution  $N \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots$  in which  $Q^i = H^0(\dot{\mathcal{U}}/\dot{\mathcal{U}}^0, Q^{i-1}/Q^{i-2})$ . We abuse the notation  $Q^i$  here. This standard resolution is the one considered in [APW, §2.17].

We now establish some homological properties of  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ , which are not automatic since  $\dot{\mathcal{U}}_q(\mathfrak{g})$  is neither Noetherian nor finitely generated as an algebra. The point is that the induction functor  $H^0(\dot{\mathcal{U}}/R, -)$  allows us to translate those homological statements to the ones in  $R\text{-Lmod}$ . To emphasize the base ring of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ , we will denote  $\dot{\mathcal{U}}_q(\mathfrak{g})$  by  $\dot{\mathcal{U}}_R(\mathfrak{g})$ .

**Lemma 6.18.** Let  $R$  be a Noetherian ring. Let  $M \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$  such that  $M$  is a finitely generated over  $R$ . Then the functor  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))}^i(M, -)$  commutes with filtered direct limits.

*Proof.* First, we recall a functorial way to embed  $R$ -modules into injective  $R$ -modules. Let  $M$  be an  $R$ -module. Set  $M^\vee := \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ . We have an embedding of  $R$ -modules  $M \hookrightarrow (M^\vee)^\vee$ . Let  $F_{M^\vee} := \bigoplus_{m \in M^\vee} R$  be a free  $R$ -module where the index runs over all elements  $m \in M^\vee$  then  $F_{M^\vee}^\vee = \text{Hom}_{\mathbb{Z}}(F_{M^\vee}, \mathbb{Q}/\mathbb{Z})$  is an injective  $R$ -module. There is a natural surjective morphism of  $R$ -modules  $F_{M^\vee} \twoheadrightarrow M^\vee$ , which gives us an injective morphism of  $R$ -modules  $(M^\vee)^\vee \hookrightarrow F_{M^\vee}^\vee$ . Then the embedding  $M \hookrightarrow F_{M^\vee}^\vee$  is functorial in  $M$ . In particular, for any direct system of  $R$ -modules  $\{M_i\}_I$ , we can construct a direct system of embedding  $\{M_i \hookrightarrow F_{M_i^\vee}^\vee\}_I$ .

Let  $\{N_i\}$  be a direct system in  $\text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$  and  $\varinjlim N_i = N \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ . By the above paragraph, we can construct a direct system of embeddings of  $R$ -modules  $\{N_i \hookrightarrow I_i^0\}_I$ , here  $I_i^0$  are injective  $R$ -modules. This gives us a direct system of embeddings  $\{N_i \hookrightarrow N_i^0\}_I$ , where

$N_i^0 = H^0(\dot{\mathcal{U}}/R, I_i^0)$ . Repeating this process, we can construct a direct system of injective resolution  $\{N \rightarrow N_i^0 \rightarrow N_i^1 \dots\}_I$  in  $\text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ , in which  $N_i^j = H^0(\dot{\mathcal{U}}/R, I_i^j)$  for some injective  $R$ -module  $I_i^j$ .

Filtered direct limits preserve exact sequences in the category of modules and the direct limit of injective  $R$ -modules is again an injective  $R$ -module when  $R$  is Noetherian. Therefore, the direct limit of  $\{N_i \rightarrow N_i^0 \rightarrow N_i^1 \dots\}$  gives us an injective resolution  $N \rightarrow N^0 \rightarrow N^1 \rightarrow \dots$  in which  $N^j = \varinjlim H^0(\dot{\mathcal{U}}/R, I_i^j) = H^0(\dot{\mathcal{U}}/R, \varinjlim I_i^j)$  ( $\varinjlim I_i^j$  is an injective  $R$ -module), the last equality holds since  $H^0(\dot{\mathcal{U}}/R, -)$  commutes with direct limits by Lemma 6.17. Now the lemma follows from

$$\varinjlim \text{Hom}_{\dot{\mathcal{U}}_R(\mathfrak{g})}(M, N_i^j) = \varinjlim \text{Hom}_R(M, I_i^j) = \text{Hom}_R(M, \varinjlim I_i^j) = \text{Hom}_{\dot{\mathcal{U}}_R(\mathfrak{g})}(M, N^j),$$

the second equality holds since  $R$  is Noetherian and  $M$  is a finitely generated  $R$ -module.  $\square$

**Lemma 6.19.** *Let  $R$  be a Noetherian ring. Let  $M \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$  such that  $M$  is a finitely generated  $R$ -module. Then for any  $N \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ , we have an isomorphism*

$$(6.4) \quad \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))}^i(M, N) \otimes_R R_{\mathfrak{p}} \cong \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_{R_{\mathfrak{p}}}(\mathfrak{g}))}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

for  $i \geq 0$ , here  $\dot{\mathcal{U}}_{R_{\mathfrak{p}}}(\mathfrak{g}) := \dot{\mathcal{U}}_R(\mathfrak{g}) \otimes_R R_{\mathfrak{p}}$ ,  $M_{\mathfrak{p}} := M \otimes_R R_{\mathfrak{p}}$  and  $N_{\mathfrak{p}} := N \otimes_R R_{\mathfrak{p}}$ .

*Proof.* *Step 1.* We will prove that for any  $N \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ , the natural map  $N^{\dot{\mathcal{U}}} \otimes_R R_{\mathfrak{p}} \rightarrow (N_{\mathfrak{p}})^{\dot{\mathcal{U}}}$  is an isomorphism. By Proposition 6.6,  $N$  is a union of modules in  $\text{Rep}(\dot{\mathcal{U}}_R)$  which are finitely generated over  $R$ . So by Lemma 6.18, we can assume  $N$  is finitely generated over  $R$ . Then there is  $m_0$  such that  $E_i^{(m)}n = F_i^{(m)}n = 0$  for all  $m \geq m_0, n \in N$  and  $1 \leq i \leq r$ . Hence,

$$\begin{aligned} N^{\dot{\mathcal{U}}} &= \{n \in N_0 \mid E_i^{(m)}n = F_i^{(m)}n = 0 \ \forall 1 \leq m < m_0, 1 \leq i \leq r\} \\ N_{\mathfrak{p}}^{\dot{\mathcal{U}}} &= \{n \in (N_{\mathfrak{p}})_0 \mid E_i^{(m)}n = F_i^{(m)}n = 0 \ \forall 1 \leq m < m_0, 1 \leq i \leq r\} \end{aligned}$$

Since localization commutes with finite intersections, it follows that  $N^{\dot{\mathcal{U}}} \otimes_R R_{\mathfrak{p}} \cong (N_{\mathfrak{p}})^{\dot{\mathcal{U}}}$ .

*Step 2.* For any  $N \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ , the natural map

$$(6.5) \quad \text{Hom}_{\dot{\mathcal{U}}_R}(M, N) \otimes_R R_{\mathfrak{p}} \rightarrow \text{Hom}_{\dot{\mathcal{U}}_{R_{\mathfrak{p}}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

is an isomorphism. Indeed,

$$\text{Hom}_{\dot{\mathcal{U}}_R}(M, N) \otimes_R R_{\mathfrak{p}} = \text{Hom}_R(M, N)^{\dot{\mathcal{U}}} \otimes_R R_{\mathfrak{p}}, \quad \text{Hom}_{\dot{\mathcal{U}}_{R_{\mathfrak{p}}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})^{\dot{\mathcal{U}}}.$$

Since  $M$  is a finitely generated  $R$ -module, it follows that  $\text{Hom}_R(M, N) \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$  and  $\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \in \text{Rep}(\dot{\mathcal{U}}_{R_{\mathfrak{p}}}(\mathfrak{g}))$  and  $\text{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \text{Hom}_R(M, N) \otimes_R R_{\mathfrak{p}}$ . Therefore, (6.5) holds by Step 1.

*Step 3.* Let  $N \rightarrow N^0 \rightarrow N^1 \rightarrow \dots$  be an injective resolution of  $N$ , in which  $N^i = H^0(\dot{\mathcal{U}}/R, I^i) \cong H^0(\dot{\mathcal{U}}/R, R) \otimes_R I^i$  for some injective  $R$ -module  $I^i$ , the isomorphism follows by Lemma 6.17. We have  $N^i \otimes_R R_{\mathfrak{p}} = H^0(\dot{\mathcal{U}}/R_{\mathfrak{p}}, R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} I_{\mathfrak{p}}^i$  and  $I_{\mathfrak{p}}^i$  is an injective  $R_{\mathfrak{p}}$ -module because  $R$  is Noetherian. Therefore, applying  $- \otimes_R R_{\mathfrak{p}}$  to the injective resolution of  $N$ , we get an injective resolution of  $N_{\mathfrak{p}}$ . By using Step 2 and computing Ext groups with the injective resolutions of  $N$  and  $N_{\mathfrak{p}}$ , we obtain (6.4).  $\square$

Let  $I$  be an ideal of  $R$  and  $R_I := R/I$ . Let  $N \in \text{Rep}(\dot{\mathcal{U}}_{R_I}(\mathfrak{g}))$  then  $N$  can be viewed as an object in  $\text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ . Let  $M \in \text{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$ . Then  $M_{R_I} := M \otimes_R R_I \in \text{Rep}(\dot{\mathcal{U}}_{R_I}(\mathfrak{g}))$ .

**Lemma 6.20.** *Let  $R$  be a Noetherian ring. Assume  $M$  is finitely generated projective module over  $R$  then*

$$(6.6) \quad \mathrm{Ext}_{\mathrm{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))}^i(M_R, N_{R_I}) \cong \mathrm{Ext}_{\mathrm{Rep}(\dot{\mathcal{U}}_{R_I}(\mathfrak{g}))}^i(M_{R_I}, N_{R_I}).$$

for all  $i \geq 0$ .

*Proof.* Since  $M_R$  is a finitely generated projective  $R$ -module,  $M_{R_I}$  is a finitely generated projective  $R_I$ -module. Therefore, we can use the big standard resolutions of  $N_{R_I}$  in  $\mathrm{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))$  and  $\mathrm{Rep}(\dot{\mathcal{U}}_{R_I}(\mathfrak{g}))$  to compute the Ext groups in (6.6), see the argument in last paragraph in the proof of Lemma 6.19. These two big standard resolutions of  $N_{R_I}$  are as follows:

$$N_{R_I} \rightarrow Q_R^0 \rightarrow Q_R^1 \rightarrow \dots, \quad N_{R_I} \rightarrow Q_{R_I}^0 \rightarrow Q_{R_I}^1 \rightarrow \dots$$

in which  $Q_R^i = H^0(\dot{\mathcal{U}}/R, Q_R^{i-1}/Q_R^{i-2})$  and  $Q_{R_I}^i = H^0(\dot{\mathcal{U}}/R_I, Q_{R_I}^{i-1}/Q_{R_I}^{i-2})$ .

By Lemma 6.17,

$$Q_R^0 = H^0(\dot{\mathcal{U}}/R, N_{R_I}) \cong H^0(\dot{\mathcal{U}}/R, R) \otimes_R N_{R_I} \cong H^0(\dot{\mathcal{U}}/R_I, R_I) \otimes_{R_I} N_{R_I} \cong H^0(\dot{\mathcal{U}}/R_I, N_{R_I}) = Q_{R_I}^0$$

By induction,  $Q_R^i \cong Q_{R_I}^i$ . On the other hand,

$$\mathrm{Hom}_{\mathrm{Rep}(\dot{\mathcal{U}}_R(\mathfrak{g}))}(M_R, Q_{R_I}^i) \cong \mathrm{Hom}_{\mathrm{Rep}(\dot{\mathcal{U}}_{R_I}(\mathfrak{g}))}(M_{R_I}, Q_{R_I}^i).$$

Therefore, (6.6) holds.  $\square$

### 6.3. Kempf vanishing.

The category  $\mathrm{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$  has enough injectives, see [APW, §2.13]. So we can define the higher derived functors  $H^i$  of the induction functor  $H^0$ .

We write  $H_q^i(\lambda)$  for  $H^i(R_\lambda)$ . Let  $\lambda^* := -w_0\lambda$ .

**Proposition 6.21.** *Suppose  $R$  is Noetherian and  $\lambda$  is dominant. The following claims are true:*

- (i) *We have  $H_q^0(\lambda) \cong \mathrm{Hom}_R(W_q(\lambda^*), R)$ .*
- (ii) *We have  $H_q^i(\lambda) = \{0\}$  for all  $i > 0$ .*

The second part of the proposition is known as the Kempf vanishing theorem. In [APW], specializing  $\mathcal{A}_{\mathfrak{m}_0}$  to its residue field  $k$  with  $q = 1 \in k$  allowed the authors to use some results about modular representations in the proof of the Kempf vanishing theorem. Let us now consider the case when the base ring is  $\underline{\mathcal{A}}$ . For any maximal ideal  $\mathfrak{m} \in \underline{\mathcal{A}}$ , the field  $\underline{\mathcal{A}}/\mathfrak{m}$  is a finite field and hence  $q \in \underline{\mathcal{A}}/\mathfrak{m}$  is a root of unity. Some results in [RH] in the case when the field  $k$  of positive characteristics and  $q \in k$  is a root of unity can replace the references to modular representations in the proof of the Kempf vanishing theorem in [APW]. This allows us to prove the Kempf vanishing theorem over  $\underline{\mathcal{A}}$  and then over the  $\underline{\mathcal{A}}$ -algebra  $R$  via base change.

*Proof.* (i) Following Remark 6.13, this part is proved in [APW, Proposition 3.3].

(ii) *Step 1:* First, we consider the case when the base ring is  $\underline{\mathcal{A}}$ . This is shown by generalizing results from [APW, §5.5] as follows. For any maximal ideal  $\mathfrak{m} \in \underline{\mathcal{A}}$ , let  $\underline{\mathcal{A}}_{\mathfrak{m}}$  be the localization and  $k := \underline{\mathcal{A}}_{\mathfrak{m}}/\mathfrak{m}\underline{\mathcal{A}}_{\mathfrak{m}} \cong \underline{\mathcal{A}}/\mathfrak{m}$  be the residue field of  $\underline{\mathcal{A}}_{\mathfrak{m}}$ . Then  $k$  is a finite field of positive characteristics. Let  $\bar{q}$  denote the image of  $q$  in  $k$  then  $\bar{q}$  is a root of unity in  $k$ .

- [APW, Theorem 5.1] holds over  $\underline{\mathcal{A}}$ . To show that the natural homomorphism  $H_q^0(\lambda) \rightarrow H_q^0(s, \lambda)$  (the target is introduced before the theorem) is surjective, it is enough to show the surjectivity over the localization  $\underline{\mathcal{A}}_{\mathfrak{m}}$  at any maximal ideal  $\mathfrak{m} \subset \underline{\mathcal{A}}$ . The later is done as in *loc.cit.* In the proof of (ii), to show the surjectivity of the natural homomorphism  $H_{\bar{q}}^0(\lambda) \rightarrow H_{\bar{q}}^0(s, \lambda)$  one replaces the references there with the reference to [RH, Lemma

5.3]: that lemma combined the isomorphism  $\mathcal{D}_{w_0}(\lambda) \cong \mathcal{D}(\lambda)$  in [RH, Theorem 5.4] imply that the dual of  $H_q^0(\lambda) \rightarrow H_q^0(s, \lambda)$  is injective.

- [APW, Lemma 5.3] holds over  $\underline{\mathcal{A}}$ . As in the proof in *loc. cit.*, inside  $H^0(\dot{\mathcal{U}}^{\leq}/\dot{\mathcal{U}}^0, \mu)$ , we produce a union of submodules isomorphic to  $H_{\underline{\mathcal{A}}}^0(m\rho) \otimes \underline{\mathcal{A}}_{m\rho+\mu}$  over  $m \geq 0$ . But then this ascending chain of submodules is equal to  $H^0(\dot{\mathcal{U}}^{\leq}/\dot{\mathcal{U}}^0, \mu)$  after localizing at any maximal ideal  $\mathfrak{m} \subset \underline{\mathcal{A}}$  by the same proof in *loc. cit.*. Therefore, [APW, Lemma 5.3] holds over  $\underline{\mathcal{A}}$ .
- Using this we see that [APW, Theorem 5.4] holds over  $\underline{\mathcal{A}}$ .
- [APW, Proposition 5.6] holds over  $\underline{\mathcal{A}}$ . Part (i) of that proposition holds over localization  $\underline{\mathcal{A}}_{\mathfrak{m}}$  at any maximal ideal  $\mathfrak{m}$ : in the proof of  $H_q^0(\lambda) \xrightarrow{\sim} H_q^0(w_0, \lambda)$  one replaces the reference there with [RH, Theorem 5.4]. Therefore part(i) holds over  $\underline{\mathcal{A}}$  and then parts (ii), (iii) follow.

Then the claims that  $H_{\underline{\mathcal{A}}}^i(\lambda) = 0$  for  $i \geq 1$  follow by combining [APW, Theorem 5.4(i)] and [APW, Proposition 5.6 (iii)].

*Step 2:* For a general  $\underline{\mathcal{A}}$ -algebra  $R$ , we now can use the spectral sequence from [APW, §3.4 – 3.5] (with  $\underline{\mathcal{A}}_{\mathfrak{m}}$  is replaced by  $\underline{\mathcal{A}}$  and  $\Gamma = R$ ), see also [APW, Remark 3.5], to see that  $H_q^i(\lambda) = \{0\}$  for all  $i > 0$ , while

$$H_q^0(\lambda) = R \otimes_{\underline{\mathcal{A}}} H_{\underline{\mathcal{A}}}^0(\lambda).$$

□

We remark that for two dominant weights  $\lambda, \lambda'$ , we have

$$(6.7) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda), H_q^0(\lambda')) \simeq R^{\oplus \delta_{\lambda, \lambda'}}.$$

When  $R$  is a field, the image of a nonzero homomorphism  $W_q(\lambda) \rightarrow H_q^0(\lambda)$  is simple. We denote it by  $L_q(\lambda)$ . The assignment  $\lambda \mapsto L_q(\lambda)$  is a bijection between the dominant weights and the simple modules in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ .

#### 6.4. Good filtrations.

**Definition 6.22.** Let  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ . An exhaustive  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module filtration  $\{0\} = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots$  is called *good* if, for each  $i$  we have,  $M_i/M_{i-1} \simeq H_q^0(\lambda_i) \otimes_R P_i$  for some dominant weight  $\lambda_i$  and some finitely generated projective  $R$ -module  $P_i$ .

**Lemma 6.23.** Let  $R$  be a field. Suppose that  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  satisfies the following condition

$$(6.8) \quad \dim_R \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda), M) < \infty, \quad \forall \lambda \in P_+,$$

where  $P_+$  denotes the set of dominant weights. Then the following two conditions are equivalent.

- (1)  $M$  admits a good filtration.
- (2)  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M) = 0$ , for all  $i > 0$  and dominant weights  $\lambda$ .

*Proof.* The proof is in several steps.

*Step 1.* Let us show (1)  $\Rightarrow$  (2). The submodule  $M_j$  has a good filtration and is finite dimensional. Hence  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M_j) = 0$  for all  $i > 0$  and dominant weights  $\lambda$  by Theorem 3.1 in [P]. By Lemma 6.18,

$$\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M) = \varinjlim \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M_j) = 0,$$

for all  $i > 0$  and dominant weights  $\lambda$ .

*Step 2.* It remains to show (2)  $\Rightarrow$  (1). We refine the order of  $P_+$  to a total order:  $\lambda_1 < \lambda_2 < \dots$ . We will construct inductively a filtration  $\{0\} = M_0 \subset M_1 \subset \dots$  on  $M$  that is exhaustive and satisfies the following conditions

- (3)  $M_i/M_{i-1}$  is the direct sum of several (finitely many) copies of  $H_q^0(\lambda_i)$ ,
- (4)  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^n(N, M/M_i) = 0$  for all  $n \geq 0$  and finite dimensional module  $N$  contained in the Serre span of  $W_q(\lambda_j)$  with  $j \leq i$ .

*Step 2.1. The base case*  $i = 0$ .  $M_0 = \{0\}$  and (3), (4) are vacuously true.

*Step 2.2. The induction step.* Assume we have constructed  $M_0 \subset M_1 \subset \dots M_i$  satisfying (3) and (4). Then  $M/M_i$  admits no nonzero homomorphism from objects filtered by  $L_q(\lambda_j)$  with  $j \leq i$ . Therefore, the following map is injective

$$(6.9) \quad \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(H_q^0(\lambda_{i+1}), M/M_i) \otimes H_q^0(\lambda_{i+1}) \rightarrow M/M_i.$$

Let  $M_{i+1}$  be the preimage of the image of (6.9) under the projection  $M \rightarrow M/M_i$ . Since the kernel and cokernel of the map  $W_q(\lambda_{i+1}) \rightarrow H_q^0(\lambda_{i+1})$  are contained in the Serre span of  $W_q(\lambda_j)$  with  $j \leq i$ , we have by (4) that

$$\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(H_q^0(\lambda_{i+1}), M/M_i) \cong \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda_{i+1}), M/M_i).$$

Then  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^n(W_q(\lambda_j), M/M_{i+1}) = 0$  for all  $n \geq 0$  and  $j \leq i+1$ . This implies that  $M_{i+1}$  satisfies (4).

*Step 2.3.* It is left to show that the filtration we have constructed is exhaustive. By (4), for any  $N \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  where all weights are  $\leq \lambda_i$ , we have

$$0 \rightarrow \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(N, M_i) \rightarrow \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(N, M) \rightarrow \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(N, M/M_i) = 0,$$

This implies that  $M_i$  is the maximal submodule of  $M$  where all weights are  $\leq \lambda_i$ . Thanks to Proposition 6.6,  $\dot{\mathcal{U}}_q(\mathfrak{g})m$  is finite dimensional for all  $m \in M$ . Suppose that all weights in this submodule are less than or equal to  $\lambda_i$ . Then  $m \in M_i$ .  $\square$

The following is the main result of [P].

**Proposition 6.24.** *Let  $R$  be a field. Then the tensor product of two finite dimensional modules in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  with good filtrations also has a good filtration.*

We will need analogs of Lemma 6.23 and Proposition 6.24 over regular Noetherian domains  $R$  (we will be interested in the completions of localizations of  $\mathcal{A}$ ). A technical result we are going to use is the following claim from [APW, §5.13], see also [P, §4.2].

**Lemma 6.25.** *Let  $M$  be an object in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  that is a finitely generated projective  $R$ -module. Then  $M$  admits a good filtration if and only if  $k \otimes_R M \in \text{Rep}(\dot{\mathcal{U}}_k(\mathfrak{g}))$  admits a good filtration for every epimorphism  $R \twoheadrightarrow k$  onto a field.*

*Proof.* The proof follows [P, §4.2]. We only write down needed modifications. Let  $\lambda \in P_+$  be maximal among the weights of  $M$ . By the assumption on  $M$ , it follows that  $M_\lambda$  is a finitely generated projective  $R$ -module. The  $\dot{\mathcal{U}}_q^\leq$ -modules homomorphism  $M \twoheadrightarrow M_\lambda$  gives rise to a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -homomorphism  $M \rightarrow H_q^0(M_\lambda)$ . We note that  $H_q^0(M_\lambda) \cong H_q^0(\lambda) \otimes_R M_\lambda$ . This is because  $H_q^0(-)$  commutes with direct sums,  $H_q^0(R_\lambda^{\oplus n}) \cong H_q^0(\lambda)^{\oplus n}$  and  $M_\lambda$  is a finitely generated projective  $R$ -module. So we have a  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules homomorphism  $M \rightarrow H_q^0(\lambda) \otimes_R M_\lambda$ . The rest of the proof follows [P, §4.2].  $\square$

The direct analog of Proposition 6.24 follows:

**Corollary 6.26.** *Tensor product of two finitely generated  $R$ -modules in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  with good filtrations also has a good filtration.*

We will now get to an analog of Lemma 6.23. Let us start with the following lemma.

**Lemma 6.27.** *Let  $R$  be a Noetherian ring. Then  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), H_q^0(\lambda')) = 0$  for all  $i > 0$  and dominant weights  $\lambda, \lambda'$ .*

*Proof.* Since  $W_q(\lambda)$  is free over  $R$  and  $\text{Hom}_R(W_q(\lambda), R) \cong H_q^0(\lambda^*)$ , we have

$$\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), H_q^0(\lambda')) \cong \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(R_0, H_q^0(\lambda^*) \otimes_R H_q^0(\lambda')),$$

here  $R_0$  is the  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module via the counit  $\varepsilon : \dot{\mathcal{U}}_q(\mathfrak{g}) \rightarrow R$ . By Corollary 6.26,  $H_q^0(\lambda^*) \otimes_R H_q^0(\lambda')$  has a good filtration. Therefore, it is enough to show that

$$(6.10) \quad \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(R_0, H_q^0(\lambda)) = 0,$$

for  $i > 0$  and dominant weights  $\lambda$ . Since  $H_q^i(\lambda) = 0$  for  $i > 0$  and dominant  $\lambda$ , it follows that

$$\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(R_0, H_q^0(\lambda)) \cong \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})}^i(R_0, R_\lambda).$$

Let consider the induction functor  $H^0(\dot{\mathcal{U}}^{\leqslant}/\dot{\mathcal{U}}^0, -) : \text{Rep}(\dot{\mathcal{U}}_q^0) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$ . We then form a standard resolution for  $R_\lambda$  in  $\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$  as in Section 6.2:

$$(6.11) \quad R_\lambda \rightarrow Q^0 \rightarrow Q^1 \rightarrow \dots,$$

where  $Q^0 = H^0(\dot{\mathcal{U}}^{\leqslant}/\dot{\mathcal{U}}^0, R)$ ,  $Q^i = H^0(\dot{\mathcal{U}}^{\leqslant}/\dot{\mathcal{U}}^0, Q^{i-1}/Q^{i-2})$  for  $i > 0$ ; here we set  $Q^{-1} := R_\lambda$ . Then the space  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})}^i(R_0, R_\lambda)$  can be computed by this standard resolution. We note the following: all weights of  $Q^i$  are contained in  $\lambda + Q_+$ . Hence,  $Q^i$  does not have weights smaller than 0. Therefore, applying  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})}(R_0, -)$  to (6.11), we get  $(R_\lambda)_0 \rightarrow Q_0^0 \rightarrow Q_0^1 \rightarrow \dots$  which is exact. It follows that  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})}^i(R_0, R_\lambda) = 0$  for  $i > 0$ , hence (6.10) holds.  $\square$

**Lemma 6.28.** *Let  $R$  be a regular Noetherian domain. Suppose that  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  satisfies the following condition*

$$(6.12) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda), M) \text{ is a finitely generated } R\text{-module}, \forall \lambda \in P_+,$$

*Then the following two conditions are equivalent.*

- (1)  $M$  admits a good filtration.
- (2) The  $R$ -module  $M$  is isomorphic to the direct sum of finitely generated projective  $R$ -modules, and

$$\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M) = 0,$$

for all  $i > 0$  and dominant weights  $\lambda$ .

*Proof.* The proof is in several steps. We fix some notations: For any  $R$ -algebra  $R'$  and  $R$ -module  $M$ , we write  $W_{R'}(\lambda) := W_q(\lambda) \otimes_R R'$ ,  $H_{R'}^0(\lambda) := H_q^0(\lambda) \otimes_R R'$  and  $M_{R'} := M \otimes_R R'$ .

*Step 1.* Let us show (1)  $\Rightarrow$  (2). By Lemma 6.27,  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M_j) = 0$  for  $i > 0$  and all  $j$ . By Lemma 6.18, we have

$$\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M) = \varinjlim \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M_j) = 0.$$

*Step 2.* Assume  $M$  satisfies (2) and (6.12). We will show that  $M$  satisfies the following:

- (a)  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M)$  is a finitely generated projective  $R$ -module,

- (b)  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_k(\mathfrak{g}))}(W_k(\lambda), M_k)$  is finite dimensional over  $k$ , and  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_k(\mathfrak{g}))}^i(W_k(\lambda), M_k) = 0$  for all  $i > 0$ , all dominant  $\lambda$ , and all epimorphisms  $R \twoheadrightarrow k$  onto a field.

Lemma 6.19 allows us to reduce to the local case. So we can assume  $R$  is a local regular Noetherian domain. Let  $\mathfrak{m} = (x_1, \dots, x_n)$  be the maximal ideal of  $R$ , where  $x_1, \dots, x_n$  is a maximal regular sequence. Let  $R_i = R/(x_1, \dots, x_i)$  and  $M_{R_i} = M/(x_1M + \dots + x_iM)$ . We see that  $M_{R_i} \in \text{Rep}(\dot{\mathcal{U}}_{R_i}(\mathfrak{g}))$ . Since  $M$  is projective over  $R$ , we have a short exact sequence

$$0 \rightarrow M \xrightarrow{\cdot x_1} M \rightarrow M_{R_1} \rightarrow 0.$$

Applying  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), -)$  to get long exact sequence of Ext's group, we obtain

$$\begin{aligned} \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M) &\xhookrightarrow{\cdot x_1} \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), W) \rightarrow \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M_{R_1}), \\ \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), M_{R_1}) &= 0 \quad \forall i > 0. \end{aligned}$$

Combining these with Lemma 6.20, we have

$$\begin{aligned} \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M) &\xhookrightarrow{\cdot x_1} \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), W) \rightarrow \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_{R_1}(\mathfrak{g}))}(W_{R_1}(\lambda), M_{R_1}), \\ \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_{R_1}(\mathfrak{g}))}^i(W_{R_1}(\lambda), M_{R_1}) &= 0 \quad \forall i > 0. \end{aligned}$$

Since  $M$  is a direct sum of finitely generated projective  $R$ -modules,  $M_{R_1}$  is a direct sum of finitely generated projective  $R_1$ -modules. So we can proceed inductively to get (b) and also the claim that  $x_1, \dots, x_n$  is a regular sequence of  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M)$ . So  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M)$  is a finitely generated Cohen-Macaulay module over a local regular Noetherian domain  $R$ . Hence it is free over  $R$ , equivalently, (a) holds.

*Step 3.* Let us show (2)  $\Rightarrow$  (1). We will construct inductively a filtration  $\{0\} = M_0 \subset M_1 \subset M_2 \subset \dots$  on  $M$  that is exhaustive and satisfies

- (3)  $M_i/M_{i-1} \cong H_q^0(\lambda_i) \otimes_R P_i$  for some finitely generated projective  $R$ -module  $P_i$ .
- (4)  $M_i$  is a direct summand of  $M$  as an  $R$ -module.
- (5) For any epimorphism  $R \twoheadrightarrow k$  onto a field,  $M_{k,i} := M_i \otimes_R k$  is the  $i$ -th component of  $M_k := M \otimes_R k$  constructed in Step 2 of the proof of Lemma 6.23.
- (6)  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda), M/M_{i-1})$  is a finitely generated  $R$ -module for all dominant  $\lambda$ .
- (7)  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^n(N, M/M_i) = 0$  for all  $n \geq 0$  and finitely generated  $R$ -module  $N$  contained in Serre span of  $W_q(\lambda_j)$  with  $j \leq i$ .

*Step 3.1. The base case  $i = 0$ .*  $M_0 = \{0\}$  and (3)-(7) are vacuously true.

*Step 3.2. The induction step.* Suppose we have already constructed  $M_0 \subset M_1 \subset \dots \subset M_{i-1}$ . Since the kernel and cokernel of the natural homomorphism  $W_q(\lambda_i) \rightarrow H_q^0(\lambda_i)$  have weights strictly less than  $\lambda_i$ , it follows that they lie in the Serre span of the objects  $W_q(\lambda_j)$  with  $j < i$ . By (7), we have

$$(6.13) \quad \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(H_q^0(\lambda_i), M/M_{i-1}) \cong \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda_i), M/M_{i-1}).$$

By (6)-(7) and Step 2, (6.13) is a finitely generated projective  $R$ -module. Let us consider the homomorphism

$$(6.14) \quad \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(H_q^0(\lambda_i), M/M_{i-1}) \otimes_R H_q^0(\lambda_i) \rightarrow M/M_{i-1}.$$

In Step 2, we saw that

$$\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda_i), M/M_{i-1}) \otimes_R k \cong \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_k(\mathfrak{g}))}(W_k(\lambda_i), (M/M_{i-1})_k).$$

Combining this with (6.13) and its version over  $k$ , we have

$$\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(H_q^0(\lambda_i), M/M_{i-1}) \otimes_R k \cong \text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_k(\mathfrak{g}))}(H_k^0(\lambda_i), (M/M_{i-1})_k).$$

Therefore, the functor  $\bullet \otimes_R k$  sends (6.14) to the similarly defined homomorphism over  $k$ . The latter is injective. Combining this with the fact that  $M/M_{i-1}$  is a direct sum of finitely generated projective  $R$ -modules, we conclude that (6.14) is injective and the cokernel is a direct sum of finitely generated projective  $R$ -modules.

We now let  $M_i$  be the preimage of the left hand side of (6.14) under the projection  $M \rightarrow M/M_{i-1}$ . Then  $M_i$  satisfies (3)-(6). The cokernel of (6.14) is  $M/M_i$ . Applying the functor  $\text{Hom}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}(W_q(\lambda_j), -)$  with  $j \leq i$  to (6.14), one can show that  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^n(W_q(\lambda_j), M/M_i) = 0$  for all  $n \geq 0$  and  $j \leq i$ . This implies that  $M_i$  satisfies (7).

*Step 3.3.* By using (7) we see that  $M_i$  is the maximal submodule where all weights are  $\leq \lambda_i$ . Thanks to Proposition 6.6,  $\dot{\mathcal{U}}_q(\mathfrak{g})m$  is finitely generated over  $R$  for any  $m \in M$ . Suppose that all weights in this submodule are less than or equal to  $\lambda_i$  then  $\dot{\mathcal{U}}_q(\mathfrak{g})m \subset M_i$ . So the filtration we constructed is exhaustive.  $\square$

### 6.5. Quantized coordinate algebra.

Let  $R$  be a Noetherian ring. Following [APW, Section 1] we define the *quantized coordinate algebra*  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  as the maximal rational subrepresentation in  $\text{Hom}_R(\dot{\mathcal{U}}_q(\mathfrak{g}), R)$  (with respect to either left or right action of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ , the result does not depend on which action we choose by [APW, Section 1.30]). By the definition, for  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  we have a natural isomorphism

$$(6.15) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(M, R[\dot{\mathcal{U}}_q(\mathfrak{g})]) = \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(M, \text{Hom}_R(\dot{\mathcal{U}}_q(\mathfrak{g}), R)) \xrightarrow{\sim} \text{Hom}_R(M, R).$$

So,  $R[\dot{\mathcal{U}}_q(\mathfrak{g})] = H^0(\dot{\mathcal{U}}/R, R)$  in Section 6.2. If  $M$  is projective over  $R$ , we have

$$(6.16) \quad \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(M, R[\dot{\mathcal{U}}_q(\mathfrak{g})]) = 0, \quad \forall i > 0.$$

**Remark 6.29.** On  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ , we have two left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -actions  $\gamma, \delta$  as follows:

$$(\delta(x) \cdot f)(u) = f(ux), \quad (\gamma(x) \cdot f)(u) = f(S(x)u),$$

for  $x, u \in \dot{\mathcal{U}}_q(\mathfrak{g})$  and  $f \in R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ . So  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  is a  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g})$ -module (the first action is  $\delta$  and the second action is  $\gamma$ ). The left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -action on  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  defined via  $\delta$  is what we considered in the induction functors in the previous sections as well as the one used in (6.15)-(6.16). Moreover, (6.15) is an isomorphism of  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -modules with  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module structures on both sides as follows: the left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -action on  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  defined via  $\gamma$  gives the left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module structure on  $\text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(M, R[\dot{\mathcal{U}}_q(\mathfrak{g})])$ , meanwhile, the left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module structure on  $\text{Hom}_R(M, R)$  is defined as usual.

**Lemma 6.30.** *The following claims are true:*

- (1)  $\underline{\mathcal{A}}[\dot{\mathcal{U}}_q(\mathfrak{g})]$  is a free over  $\underline{\mathcal{A}}$ .
- (2)  $R[\dot{\mathcal{U}}_q(\mathfrak{g})] \simeq R \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{A}}[\dot{\mathcal{U}}_q(\mathfrak{g})]$ .

*Proof.* (1) is essentially in [AW, Section 4.4], while (2) is in [AW, Section 4.2] (thanks to the results explained in Sections 6.1 and 6.3 we can work with the ring  $\underline{\mathcal{A}}$  instead of the ring  $\mathcal{A}_1$  from [AW, Section 4.1]).  $\square$

**Remark 6.31.** We have a natural morphism

$$(6.17) \quad R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R R[\dot{\mathcal{U}}_q(\mathfrak{g})] \rightarrow R[\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})]$$

arising from  $\text{Hom}_R(\dot{\mathcal{U}}_q(\mathfrak{g}), R) \otimes_R \text{Hom}_R(\dot{\mathcal{U}}_q(\mathfrak{g}), R) \rightarrow \text{Hom}_R(\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g}), R)$ . This morphism (6.17) is an isomorphism. Indeed, thanks to Proposition 6.6, it is enough to show that

$$\text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes \dot{\mathcal{U}}_q(\mathfrak{g})}(M, R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R R[\dot{\mathcal{U}}_q(\mathfrak{g})]) \xrightarrow{\sim} \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes \dot{\mathcal{U}}_q(\mathfrak{g})}(M, R[\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})])$$

for all rational  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g})$ -modules  $M$  that are finitely generated over  $R$ . This is an easy consequence of (6.15) and its analog for  $\mathfrak{g} \times \mathfrak{g}$ .

Here is our main result about the  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g})$ -module structure of  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ .

**Proposition 6.32.** *Equip  $P_+$  with a total order refining the usual partial order and let  $\lambda_1 < \lambda_2 < \dots$  be the elements. There is an exhaustive filtration  $\{0\} = M_0 \subset M_1 \subset \dots$  on  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  such that  $M_i/M_{i-1} \simeq H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*)$ .*

*Proof.* Thanks to (2) of Lemma 6.30, it is enough to prove the claim when  $\mathcal{A} \rightarrow R$  is an isomorphism. From (6.15) combined with Proposition 6.21 we conclude that

$$(6.18) \quad \text{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda), R[\dot{\mathcal{U}}_q(\mathfrak{g})]) \simeq H_q^0(\lambda^*), \quad \forall \lambda \in P_+,$$

here the  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module structure on  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  is defined via  $\delta$ .

We are now going to check condition (2) of Lemma 6.28 for  $\mathfrak{g}$  replaced with  $\mathfrak{g} \times \mathfrak{g}$  – note that  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g}) \simeq \dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})$  – and  $M = R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ . The lemma is applicable when  $R \cong \mathcal{A}$ . The claim that the  $R$ -module  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  is isomorphic to the direct sum of finitely generated projective  $R$ -modules follows from (1) of Lemma 6.30. So we need to show that

$$(6.19) \quad \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))}^i(W_q(\lambda) \otimes_R W_q(\mu), R[\dot{\mathcal{U}}_q(\mathfrak{g})]) = 0,$$

for all  $i > 0$  and dominant weights  $\lambda, \mu$ .

We are going to check (6.19). By Lemma 6.18, any object of the form  $R[\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})] \otimes_R Q$ , where  $Q$  is an injective  $R$ -module, is injective in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$ .

Every object in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$  admits a resolution by injective objects of this form. By (6.17), in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$ ,

$$R[\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})] \otimes Q = R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R Q.$$

Thanks to the isomorphism  $\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}) \cong \dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g})$ , any  $M \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$  admits two left  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -actions. We have the functor

$$\text{Hom}_{\text{Rep}(\delta(\dot{\mathcal{U}}_q))}(W_q(\mu), \bullet) : \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})),$$

where we consider the first  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -action on modules in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$ . We use  $\delta(\dot{\mathcal{U}}_q)$  in the subscript since the first  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -action on  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  is defined via  $\delta$ . This functor sends  $R[\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g})] \otimes_R Q$  to

$$\text{Hom}_{\delta(\dot{\mathcal{U}}_q)}(W_q(\mu), R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R Q) = R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R \text{Hom}_R(W_q(\lambda), Q).$$

The  $R$ -module  $W_q(\mu)$  is free of finite rank hence  $\text{Hom}_R(W_q(\mu), Q)$  is injective. Therefore,  $R[\dot{\mathcal{U}}_q(\mathfrak{g})] \otimes_R \text{Hom}_R(W_q(\mu), Q)$  is an injective object in  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  (where  $\text{Hom}_R(W_q(\mu), Q)$  has the trivial action).

Hence we have a spectral sequence with second page equal to

$$(6.20) \quad \text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), \text{Ext}_{\text{Rep}(\delta(\dot{\mathcal{U}}_q))}^j(W_q(\mu), R[\dot{\mathcal{U}}_q(\mathfrak{g})]))$$

converging to  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))}^{i+j}(W_q(\lambda) \otimes_R W_q(\mu), R[\dot{\mathcal{U}}_q(\mathfrak{g})])$ . Thanks to (6.15)-(6.16), we have  $\text{Ext}_{\text{Rep}(\delta(\dot{\mathcal{U}}_q))}^j(W_q(\mu), R[\dot{\mathcal{U}}_q(\mathfrak{g})])) = H_q^0(\mu^*)$  for  $j = 0$  and  $\{0\}$  for  $j > 0$ . So (6.20) becomes  $\text{Ext}_{\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))}^i(W_q(\lambda), H_q^0(\mu^*))$  for  $j = 0$  and zero else. This is  $R^{\oplus \delta_{\lambda, \mu^*}}$  if  $i = 0$  and zero else. In particular, (6.19) follows and  $R[\dot{\mathcal{U}}_q(\mathfrak{g})] \in \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g} \times \mathfrak{g}))$  admits a good filtration.

The previous paragraph implies also that the only dual Weyl modules that appear in a good filtration of  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  are of the form  $H_q^0(\lambda) \otimes_R H_q^0(\lambda^*)$ . Furthermore,

$$\mathrm{Hom}_{\dot{\mathcal{U}}_q(\mathfrak{g}) \times \dot{\mathcal{U}}_q(\mathfrak{g})}(W_q(\lambda) \otimes_R W_q(\lambda^*), R[\dot{\mathcal{U}}_q(\mathfrak{g})]) \cong R.$$

Therefore, our claim on the filtration holds.  $\square$

**Remark 6.33.** Let us define a total order on  $P_+ \times P_+$  as follows. Firstly, we equip the first component  $P_+$  with the total order used in Proposition 6.32. Secondly, applying  $-w_0$ , where  $w_0$  is the longest element in  $W$ , to the total order used in Proposition 6.32, we obtain another total order on  $P_+$ . We then equip the second component  $P_+$  with this new total order. Finally, we consider  $P_+ \times P_+$  with the lexicographic order  $(\lambda, 0) > (0, \mu)$ . For this order on  $P_+ \times P_+$ , the module  $M_i$  in Proposition 6.32 will be the maximal  $\dot{\mathcal{U}}_q(\mathfrak{g}) \otimes_R \dot{\mathcal{U}}_q(\mathfrak{g})$ -subrepresentation of  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$  whose weights are bounded by  $(\lambda_i, \lambda_i^*)$ .

### 6.6. Rational representations of $\dot{\mathcal{U}}_q(\mathfrak{g})$ at roots of unity.

In this section, we will work over a field  $\mathbb{F}$ . Let  $q := \epsilon \in \mathbb{F}$  be a root of unity of order  $\ell$ . We assume that  $\ell_i \geq \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$ . We will review some results about the rational representations of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  at roots of unity. Many arguments follow [N] and [APW]. To simplify the notations, tensor products without subscript in this section are over  $\mathbb{F}$ .

We will need a technical lemma. Let  $\widehat{\mathcal{U}}_\epsilon^{\geq}(\mathfrak{g}, P), \widehat{\mathcal{U}}_\epsilon^{*\geq}(\mathfrak{g}, P^*)$  be the idempotent versions of  $\dot{\mathcal{U}}_\epsilon^{\geq}, \dot{\mathcal{U}}_\epsilon^{*\geq}$ , respectively, defined as in Section 4.6. Let  $\widehat{\mathcal{U}}_\epsilon^{\leq}(\mathfrak{g}, P), \widehat{\mathcal{U}}_\epsilon^{*\leq}(\mathfrak{g}, P^*)$  be the idempotent versions of  $\dot{\mathcal{U}}_\epsilon^{\leq}, \dot{\mathcal{U}}_\epsilon^{*\leq}$ , respectively.

**Lemma 6.34.** (a) *The kernel of  $\mathrm{Fr} : \widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$  is equal to the left ideal generated by  $\{1_\mu, E_i 1_\lambda, F_i 1_\lambda | \mu \notin P^*, \lambda \in P, 1 \leq i \leq r\}$  and also equal to the right ideal generated by the same set of elements.*

(b) *The kernel of  $\mathrm{Fr}^{\geq} : \widehat{\mathcal{U}}_\epsilon^{\geq}(\mathfrak{g}, P) \rightarrow \widehat{\mathcal{U}}_\epsilon^{*\geq}(\mathfrak{g}, P^*)$  is equal to the left ideal generated by  $\{1_\mu, E_i 1_\lambda | \mu \notin P^*, \lambda \in P, 1 \leq i \leq r\}$  and also equal to the right ideal generated by the same set of elements. There is the similar statement for  $\mathrm{Fr}^{\leq} : \widehat{\mathcal{U}}_\epsilon^{\leq}(\mathfrak{g}, P) \rightarrow \widehat{\mathcal{U}}_\epsilon^{*\leq}(\mathfrak{g}, P^*)$ .*

*Proof.* (a) *Step 1:* We have a result similar to Corollary 4.27(b) for  $\mathrm{Fr}$  so that  $\mathrm{Ker}(\mathrm{Fr})$  is a two-sided ideal generated by

$$\{1_\mu, E_i^{(n)} 1_\lambda, F_i^{(n)} 1_\lambda | \mu \notin P^*, \lambda \in P, \ell_i \nmid n\}.$$

For  $\ell_i \nmid n$ , let  $n = \ell_i n_1 + n_0$  with  $0 < n_0 < \ell_i$  then

$$E_i^{(n)} = \frac{1}{(n_0)_{\epsilon_i}} E_i^{(\ell_i n_1)} E_i^{n_0}, \quad F_i^{(n)} = \frac{1}{(n_0)_{\epsilon_i}} F_i^{(\ell_i n_1)} F_i^{n_0}.$$

Therefore,  $\mathrm{Ker}(\mathrm{Fr})$  is the two-sided ideal generated by

$$S = \{1_\mu, E_i 1_\lambda, F_i 1_\lambda | \mu \notin P^*, \lambda \in P, 1 \leq i \leq r\}.$$

*Step 2:* We will prove that  $\mathrm{Ker}(\mathrm{Fr})$  is the left ideal generated by  $S$ . The proof that  $\mathrm{Ker}(\mathrm{Fr})$  is the right ideal generated by  $S$  is similar. We write  $\widehat{\mathcal{U}}$  for  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$ . Let  $\widehat{\mathcal{U}}S$  denote the left ideal of  $\widehat{\mathcal{U}}$  generated by  $S$ . Let  $1_\mu \widehat{\mathcal{U}}, E_i 1_\lambda \widehat{\mathcal{U}}, F_i 1_\lambda \widehat{\mathcal{U}}$  be the right ideals generated by the corresponding elements in  $S$ . Then it is enough to prove that these right ideals are contained in  $\widehat{\mathcal{U}}S$ .

Let  $\mu \notin P^*$ . We have:

$$1_\mu E_j^{(n)} 1_\nu = \delta_{\nu, \mu - n\alpha_j} E_j^{(n)} 1_{\mu - n\alpha_j}, \quad 1_\mu F_j^{(n)} 1_\nu = \delta_{\nu, \mu + n\alpha_j} F_j^{(n)} 1_{\mu + n\alpha_j}.$$

If  $\ell_j \nmid n$  then  $1_\mu E_j^{(n)} 1_\nu, 1_\mu F_j^{(n)} 1_\nu \in \widehat{\mathcal{U}}S$ . On the other hand, if  $\ell_j \mid n$  then  $\mu \pm n\alpha_j \notin P^*$ , hence,  $E_j^{(n)} 1_{\mu - n\alpha_j}, F_j^{(n)} 1_{\mu + n\alpha_j} \in \widehat{\mathcal{U}}S$ . Therefore,  $1_\mu \widehat{\mathcal{U}} \subset \widehat{\mathcal{U}}S$  for  $\mu \notin P^*$ .

Let us consider  $E_i 1_\lambda$ . We have

$$(6.21) \quad \begin{aligned} E_i 1_\lambda F_j^{(n)} 1_\nu &= \delta_{\lambda+n\alpha_j} F_j^{(n)} E_i 1_\nu \quad (i \neq j), & E_i 1_\lambda E_i^{(n)} 1_\nu &= \delta_{\lambda-n\alpha_i, \nu} E_i^{(n)} E_i 1_\nu \\ E_i 1_\lambda F_i^{(n)} 1_\nu &= \delta_{\lambda+n\alpha_i, \nu} \left( F_i^{(n)} E_i 1_\nu + b_n F_i^{(n-1)} 1_\nu \right) \\ E_i 1_\lambda E_j^{(n)} 1_\nu &= \delta_{\lambda-n\alpha_j, \nu} \sum_{0 \leq s \leq -a_{ij} < \ell_i} a_s E_i^{(n-s)} E_j E_i^{(s)} 1_\nu \quad (i \neq j; n \geq 1 - a_{ij}) \end{aligned}$$

The last equality follows from *higher order quantum Serre relations* in [L5, §7.1.6] for some  $b_n, a_s \in \mathbb{F}$ .

If  $\ell_i \nmid n$  then  $E_i 1_\lambda F_i^{(n)} 1_\nu \in \widehat{\mathcal{U}}S$ . On the other hand, if  $\ell_i \mid n$  then  $F_i^{(n)} E_i 1_\nu + b_n F_i^{(n-1)} 1_\nu \in \widehat{\mathcal{U}}S$ . So we always have  $E_i 1_\lambda F_i^{(n)} 1_\nu \in \widehat{\mathcal{U}}S$ . So (6.21) implies that  $E_i 1_\lambda \widehat{\mathcal{U}} \subset \widehat{\mathcal{U}}S$ . Similarly, we have  $F_i 1_\lambda \widehat{\mathcal{U}} \subset \widehat{\mathcal{U}}S$ . This finishes the proof of part (a).

(b) The proof is similar to part (a).  $\square$

**Remark 6.35.** By the same analysis in Lemma 6.34, one can show that the left ideal of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $\{E_i, F_i\}_{1 \leq i \leq r}$  is equal to the right ideal of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $\{E_i, F_i\}_{1 \leq i \leq r}$ . Therefore, the weight space  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})_\nu$  with  $\nu \notin Q^*$  is contained in this left ideal of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ .

**Definition 6.36.** Let  $\mathfrak{u}\dot{\mathcal{U}}^0$  be the Hopf subalgebra of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $E_i, F_i, \dot{\mathcal{U}}_\epsilon^0$ . Let  $\mathfrak{u}^>$  (resp.  $\mathfrak{u}^<$ ) be the subalgebras of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $E_i$  (resp.  $F_i$ ). Let  $\mathfrak{u}$  be the Hopf subalgebra of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $E_i, F_i, K^{\alpha_i}$ . Here  $1 \leq i \leq r$ .

**Lemma 6.37.** (a) We have a triangular decomposition via the multiplication map:

$$\mathfrak{u}^< \otimes \dot{\mathcal{U}}_\epsilon^0 \otimes \mathfrak{u}^> \xrightarrow{\mathbf{m}} \mathfrak{u} \dot{\mathcal{U}}^0$$

(b) The algebra  $\mathfrak{u}^<$  has a  $\mathbb{F}$ -basis  $\{F^{\vec{k}} := F_{\beta_1}^{k_1} \dots F_{\beta_N}^{k_N} \mid 1 \leq k_i \leq \ell_{\beta_i} - 1\}$ . The algebra  $\mathfrak{u}^>$  has a  $\mathbb{F}$ -basis  $\{E^{\vec{k}} := E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_N} \mid 0 \leq k_i \leq \ell_{\beta_i} - 1\}$ . As a result,  $\dim_{\mathbb{F}} \mathfrak{u}^< = \dim_{\mathbb{F}} \mathfrak{u}^> = \prod_{\alpha \in \Delta_+} \ell_\alpha$ .

*Proof.* By using commutations between  $E_i, F_i, \dot{\mathcal{U}}_\epsilon^0$ , it is easy to show that the image of  $\mathbf{m}$  is a subalgebra containing all generators of  $\mathfrak{u}\dot{\mathcal{U}}^0$  hence  $\mathbf{m}$  is surjective. Using the triangular decomposition of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  in Lemma 1.6, we see that  $\mathbf{m}$  is injective. Hence part (a) follows. Let us prove part (b). The algebra  $\mathfrak{u}^<$  is the image of the algebra morphism  $\iota : \dot{\mathcal{U}}_\epsilon^< \rightarrow \dot{\mathcal{U}}_\epsilon^<$ . Recall the PBW-bases of  $\dot{\mathcal{U}}_\epsilon^<$  and  $\dot{\mathcal{U}}_\epsilon^>$  in Lemma 1.4 and Lemma 1.6. We have  $\iota(F^{\vec{k}}) = \prod_{j=1}^N [k_j]_{\epsilon_{i_j}} ! F^{[\vec{k}]}$ . On the other hand,  $[k_j]_{\epsilon_{i_j}} = 0$  if  $k_j \geq \ell_{\beta_j}$  and  $\neq 0$  if  $0 \leq k_j \leq \ell_{\beta_j} - 1$ . This implies the statement of part (b) for  $\mathfrak{u}^<$ . The proof for  $\mathfrak{u}^>$  is similar.  $\square$

**Definition 6.38.** A  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module  $M$  is called a rational representation of type 1 if there is a decomposition  $M = \bigoplus_{\lambda \in P} M_\lambda$ , where  $\dot{\mathcal{U}}_\epsilon^0$  acts on  $M_\lambda$  via the character  $\chi_\lambda$ .

Let us consider the idempotent version of  $\mathfrak{u}\dot{\mathcal{U}}^0$  denoted by  $\hat{\mathfrak{u}}$ . The  $R$ -matrix  $\mathcal{R}$  in (4.30) is an element in  $\hat{\mathfrak{u}}$ : if there is any entry  $k_t$  of  $\vec{k}$  such that  $k_t \geq \ell_t$  then  $c_{\vec{k}} = 0$ , meanwhile if all  $k_t < \ell_t$  then  $F^{[\vec{k}]}$  and  $E^{[\vec{k}]}$  are elements in  $\mathfrak{u}\dot{\mathcal{U}}^0$ .

We can define the following induction functors:

$$\begin{aligned} H^0(\mathfrak{u}\dot{\mathcal{U}}^0 / \mathbb{F}, -) : \mathbb{F}\text{-mod} &\rightarrow \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0), \\ H^0(\mathfrak{u}\dot{\mathcal{U}}^0 / \dot{\mathcal{U}}^0, -) : \text{Rep}(\dot{\mathcal{U}}^0) &\rightarrow \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0), \\ H^0(\dot{\mathcal{U}} / \mathfrak{u}\dot{\mathcal{U}}^0, -) : \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0) &\rightarrow \text{Rep}(\dot{\mathcal{U}}). \end{aligned}$$

The following lemma is proved as [APW, Proposition 2.16]

**Lemma 6.39.** (a) Let  $V \in \text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$  and  $M \in \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ . Then there is a natural isomorphism of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -modules:  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M \otimes V) \cong H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M) \otimes V$ .

(b) Let  $V \in \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$  and  $N \in \text{Rep}(\dot{\mathcal{U}}_\epsilon^0)$ . Then there is a natural isomorphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules:  $H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, N \otimes V) \cong H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, N) \otimes V$ .

For any  $\lambda \in P$ , let us recall the character  $\chi_\lambda : \dot{\mathcal{U}}_\epsilon^0 \rightarrow \mathbb{F}$  defined in (6.1). Then we have a  $\mathfrak{u}^>\dot{\mathcal{U}}^0$ -module  $\mathbb{F}_\lambda$  via the homomorphism  $\mathfrak{u}^>\dot{\mathcal{U}}^0 \rightarrow \dot{\mathcal{U}}^0 \xrightarrow{\chi_\lambda} \mathbb{F}$ . Set

$$\widehat{W}_\epsilon(\lambda) := \mathfrak{u}\dot{\mathcal{U}}^0 \otimes_{\mathfrak{u}^>\dot{\mathcal{U}}^0} \mathbb{F}_\lambda.$$

Let  $\tau$  be the anti-involution on  $\mathfrak{u}\dot{\mathcal{U}}^0$  defined by  $\tau(E_i) = F_i, \tau(F_i) = E_i, \tau(u_0) = u_0$  for all  $u_0 \in \dot{\mathcal{U}}^0, 1 \leq i \leq r$ . For any finite dimensional module  $M$  in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ , let  $M^\tau := \text{Hom}_{\mathbb{F}}(M, \mathbb{F})$  as vector spaces, where  $\mathfrak{u}\dot{\mathcal{U}}^0$  acts via  $\tau$ .

The following lemma is standard:

**Lemma 6.40.** (a)  $\widehat{W}_\epsilon(\lambda)$  has a unique simple quotient  $\widehat{L}_\epsilon(\lambda)$ . The assignment  $\lambda \mapsto \widehat{L}_\epsilon(\lambda)$  is one-to-one correspondence between  $P$  and simple modules in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ .

(b)  $\widehat{L}_\epsilon(\lambda)^\tau \cong \widehat{L}_\epsilon(\lambda)$ . For any finite dimensional modules  $M, N$  in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ ,

$$\text{Ext}_{\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)}^1(M, N) \cong \text{Ext}_{\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)}^1(N^\tau, M^\tau).$$

Let  $\lambda_{\mathbf{St}} := \sum_i (\ell_i - 1)\omega_i$  and  $\mathbf{St} := L_\epsilon(\lambda_{\mathbf{St}})$ , the simple  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module with highest weight  $\lambda_{\mathbf{St}}$ . We are now going to study properties of  $\mathbf{St}$  as a  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module and show that the functor  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, -)$  is exact.

The proof of the next two lemmas follow [N, §8.2], where similar results are proved over complex number  $\mathbb{C}$ .

**Lemma 6.41.** For any  $\lambda \in P_\ell := \{\sum_i a_i \omega_i \mid 0 \leq a_i \leq \ell_i - 1, \forall 1 \leq i \leq r\}$ , the  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module  $L_\epsilon(\lambda)$  is still a simple  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module.

*Proof.* Step 1: We will prove that  $(L_\epsilon(\lambda))^{\mathfrak{u}^>}$  is one dimensional. The proof follows [N, Lemma 8.4]. Since  $\lambda$  is a highest weight of  $L_\epsilon(\lambda)$ , we have  $\dim \text{Hom}_{\mathfrak{u}^>\dot{\mathcal{U}}^0}(\mathbb{F}_\lambda, L_\epsilon(\lambda)) = 1$ . It remains to show that  $\text{Hom}_{\mathfrak{u}^>\dot{\mathcal{U}}^0}(\mathbb{F}_\mu, L_\epsilon(\lambda)) = 0$  for all  $\mu \neq \lambda$ , equivalently,  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^{\mathfrak{u}^>\dot{\mathcal{U}}^0} = 0$  for  $\mu \neq \lambda$ . Note that  $\mathbb{F}_{-\mu}$  is naturally a  $\dot{\mathcal{U}}_\epsilon^{\geq}$ -module.

For any  $M \in \text{Rep}(\dot{\mathcal{U}}_\epsilon^{\geq}(\mathfrak{g})) = \text{Rep}(\widehat{\dot{\mathcal{U}}}_\epsilon^{\geq}(\mathfrak{g}, P))$ , let

$$M^\bullet := \{m \in M \mid 1_\mu m = E_i 1_\lambda m = 0 \ \forall 1 \leq i \leq r, \lambda \in P, \mu \notin P^*\}.$$

By Lemma 6.34(b),  $M^\bullet$  is a  $\widehat{\dot{\mathcal{U}}}_\epsilon^{\geq}(\mathfrak{g}, P)$ -submodule and then  $M^\bullet$  is a module over  $\widehat{\dot{\mathcal{U}}}_\epsilon^{*\geq}(\mathfrak{g}, P^*)$ .

Let us assume  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^{\mathfrak{u}^>\dot{\mathcal{U}}^0} \neq 0$  for  $\mu \neq \lambda$ . Since this space is contained in  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^\bullet$ , the latter is nonzero. Since  $L_\epsilon(\lambda)$  is a simple  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module, both  $L_\epsilon(\lambda)$  and  $L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu}$  have one dimensional highest weight spaces which are  $L_\epsilon(\lambda)_\lambda$  and  $L_\epsilon(\lambda)_\lambda \otimes \mathbb{F}_{-\mu}$ , respectively. Therefore,  $L_\epsilon(\lambda)_\lambda \otimes \mathbb{F}_{-\mu}$  is the highest weight vector space of  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^\bullet$ .

Since  $\lambda \neq \mu$ , there is a simple root  $\alpha_i$  such that  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^\bullet$  has a nonzero vector of a weight  $\lambda - \mu - m\ell_i \alpha_i$  for some  $m > 0$  (if  $\text{char } \mathbb{F} = 0$  the we can choose  $m = 1$ ). Then the weight space of  $L_\epsilon(\lambda)$  contains the  $W$ -orbit of  $\lambda - m\ell_i \alpha_i$ . However,  $s_i(\lambda - m\ell_i \alpha_i) = \lambda + (m\ell_i - a_i) \alpha_i > \lambda$ , contradiction. Therefore, if  $\lambda \neq \mu$  then  $(L_\epsilon(\lambda) \otimes \mathbb{F}_{-\mu})^{\mathfrak{u}^>\dot{\mathcal{U}}^0} = 0$ .

Step 2: If  $L_\epsilon(\lambda)$  is not simple as a  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module, then  $L_\epsilon(\lambda)$  must contain a non-zero  $\mathfrak{u}\dot{\mathcal{U}}^0$ -submodule  $N \subset \bigoplus_{\mu < \lambda} L_\epsilon(\lambda)_\mu$ . Note that  $N^{\mathfrak{u}^>} \neq 0$ , hence,  $L_\epsilon(\lambda)^{\mathfrak{u}^>}$  must be at least two-dimensional, contradiction. Hence  $L_\epsilon(\lambda)$  is simple over  $\mathfrak{u}\dot{\mathcal{U}}^0$ .  $\square$

**Lemma 6.42.** *The socle of  $\widehat{W}_\epsilon(\lambda)$  in the category of  $\text{Rep}(\mathfrak{u}^<\dot{\mathcal{U}}^0)$  is one-dimensional and is generated by the lowest weight space.*

*Proof.* The proof follows [N, Lemma 9.5]. We have  $\widehat{W}_\epsilon(\lambda) \cong \mathfrak{u}^<$  as  $\mathfrak{u}^<$ -modules. It is enough to show that the space of invariants  $(\mathfrak{u}^<)^{\mathfrak{u}^<} := \{u \in \mathfrak{u}^< | xu = 0 \forall x \in \mathfrak{u}^<\}$  is one dimensional. This space of invariants is at least one-dimensional because it contains the one-dimensional lowest weight space of  $\mathfrak{u}^<$ . Note that  $\mathfrak{u}^<$  is  $Q$ -graded. So, any invariant vector decomposed into the sum of homogeneous invariant vectors.

Let  $u$  be an invariant vector of weight  $\lambda$  in  $\mathfrak{u}^<$ , then  $u(\sum_\mu \epsilon^{(\lambda, \mu)} K^\lambda)$  is a homogeneous invariant vector in  $\mathfrak{u}^<$ . This gives us an embedding of vector space  $(\mathfrak{u}^<)^{\mathfrak{u}^<} \hookrightarrow (\mathfrak{u}^<)^{\mathfrak{u}^<}$ . But  $\mathfrak{u}^<$  is a finite dimensional Hopf algebra and the invariants in any finite dimensional Hopf algebra are 1-dimensional [Su, Theorem 2.1.3]. Therefore the dimension of  $(\mathfrak{u}^<)^{\mathfrak{u}^<}$  must be one.  $\square$

**Lemma 6.43.** *The natural morphism  $\widehat{W}_\epsilon(\lambda_{\mathbf{St}}) \rightarrow \mathbf{St}$  is an isomorphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules. As a result,  $\mathbf{St} \cong W_\epsilon(\lambda_{\mathbf{St}})$  as  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -modules.*

*Proof.* Since  $\mathbf{St}$  is a simple  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module by Lemma 6.41, the natural morphism  $\widehat{W}_\epsilon(\lambda_{\mathbf{St}}) \rightarrow \mathbf{St}$  is surjective. The lowest weight of  $\mathbf{St}$  is  $w_0(\sum_i (\ell_i - 1)\omega_i) = \sum_i (\ell_i - 1)w_0(\omega_i)$  and the dimension of this lowest weight subspace is 1. On the other hand,  $\widehat{W}_\epsilon(\lambda_{\mathbf{St}})$  has a unique one-dimensional lowest weight subspace of the weight

$$\sum_i (\ell_i - 1)\omega_i - \sum_{\alpha \in \Delta_+} (\ell_\alpha - 1)\alpha = \sum_i (\ell_i - 1)w_0(\omega_i),$$

where the equality follows by properties of the root system in Lemma 9.23.d') and the equality  $\ell_\alpha = \ell_i$  if  $\alpha = w(\alpha_i)$  for some  $w \in W$ . Hence the lowest weight space of  $\widehat{W}_\epsilon(\lambda)$  maps into  $\mathbf{St}$ . By Lemma 6.42, the morphism  $\widehat{W}_\epsilon(\lambda_{\mathbf{St}}) \rightarrow \mathbf{St}$  is injective.

Hence  $\dim_{\mathbb{F}}(\mathbf{St}) = \dim_{\mathbb{F}}(\widehat{W}_\epsilon(\lambda_{\mathbf{St}})) = \prod_{\alpha \in \Delta_+} \ell_\alpha$ . On the other hand, by the Weyl character formula,  $\dim_{\mathbb{F}}(W_\epsilon(\lambda_{\mathbf{St}})) = \prod_{\alpha \in \Delta_+} \ell_\alpha$ . Therefore, the surjective morphism of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -modules  $W_\epsilon(\lambda_{\mathbf{St}}) \rightarrow \mathbf{St}$  must be an isomorphism.  $\square$

**Lemma 6.44.**  *$\mathbf{St}$  is projective in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ .*

*Proof.* With Lemma 6.40, 6.41 and 6.43, the proof is the same as in [J2, Proposition 10.2]  $\square$

**Lemma 6.45.** *The functor  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, -)$  is exact.*

*Proof.* Let  $0 \rightarrow M \rightarrow N \rightarrow P \rightarrow 0$  be an exact sequence in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ . Let  $\mathbf{St}^*$  be the dual of  $\mathbf{St}$  in  $\text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$ . Note that

$$\begin{aligned} H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M \otimes \mathbf{St}^*) &\cong (M \otimes \mathbf{St}^* \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})])^{\mathfrak{u}\dot{\mathcal{U}}^0} \\ &\cong \text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\mathbb{F}, M \otimes \mathbf{St}^* \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]) \\ &\cong \text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\mathbb{F}, M \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})] \otimes \mathbf{St}^*) \\ &\cong \text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\mathbf{St}, M \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]) \end{aligned}$$

Since  $\mathbf{St}$  is projective in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ , it follows that we have an exact sequence

$$0 \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M \otimes \mathbf{St}^*) \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, N \otimes \mathbf{St}^*) \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, P \otimes \mathbf{St}^*) \rightarrow 0.$$

Since  $\mathbf{St}^* \in \text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$ , by Lemma 6.39,  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M \otimes \mathbf{St}^*) \cong H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M) \otimes \mathbf{St}^*$ . Therefore, the sequence

$$0 \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, M) \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, N) \rightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, P) \rightarrow 0$$

is exact.  $\square$

Let  $\mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0] := H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\mathbb{F}, \mathbb{F})$ . By restriction, we have a natural morphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules:  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})] \rightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]$ .

**Lemma 6.46.** *The morphism  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})] \rightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]$  is surjective.*

*Proof.* Recall the characters  $\chi_\nu : \dot{\mathcal{U}}_\epsilon^0 \rightarrow \mathbb{F}$  for  $\nu \in P$ . Set  $D(\nu) := \mathfrak{u}\dot{\mathcal{U}}^0/\mathfrak{u}\dot{\mathcal{U}}^0\text{Ker}(\chi_\nu)$ , then  $\mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]_\nu \cong \text{Hom}_{\mathbb{F}}(D(\nu), \mathbb{F})$  as vector spaces. The natural map  $\mathfrak{u}^{<\mathfrak{u}>} \rightarrow D(\nu)$  is an isomorphism of vector spaces, here  $\mathfrak{u}^{<\mathfrak{u}>}$  is the linear span of elements  $u_1 u_2$  with  $u_1 \in \mathfrak{u}^<, u_2 \in \mathfrak{u}^>$ .

For any dominant weight  $\lambda \in P_+$ , let  $W'_\epsilon(-\lambda)$  be the maximal rational quotient of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g}) \otimes_{\dot{\mathcal{U}}_\epsilon^0} \mathbb{F}_{-\lambda}$ . Then  $W'_\epsilon(-\lambda)$  is the lowest weight analog of the Weyl module  $W_\epsilon(\lambda)$ . The following results in this paragraph are in [APW, §1]. Let  $\lambda, \mu$  be dominant weights such that  $\lambda - \mu = \nu$ . Let  $x_\lambda$  be the highest weight vector of  $W_\epsilon(\lambda)$  and  $x_{-\mu}$  be the lowest weight vector of  $W'_\epsilon(-\mu)$ . Then  $x_{-\mu} \otimes x_\lambda$  generates  $W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$  as a  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module. Let  $J(\mu, \lambda)$  be the left ideal of  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  generated by  $\text{Ker}(\chi_\mu), \text{Ann}_{\dot{\mathcal{U}}_\epsilon^0} x_\lambda$  and  $\text{Ann}_{\dot{\mathcal{U}}_\epsilon^0} (x_{-\mu})$ . Set  $D(\mu, \lambda) := \dot{\mathcal{U}}_\epsilon(\mathfrak{g})/J(\mu, \lambda)$ . Then the map  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g}) \rightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$  defined by  $u \mapsto u(x_{-\mu} \otimes x_\lambda)$  factors through an isomorphism of vector spaces  $D(\mu, \lambda) \cong W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$ . Furthermore,  $\text{Hom}_{\mathbb{F}}(D(\mu, \lambda), \mathbb{F}) \subset \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]_\nu$ .

Let us consider the following maps:

$$\mathfrak{u}^< \rightarrow W_\epsilon(\lambda), \quad u \mapsto ux_\lambda; \quad \mathfrak{u}^> \rightarrow W'_\epsilon(-\mu), \quad u \mapsto ux_{-\mu}.$$

One can show that for sufficiently large  $\lambda, \mu$ , the above two maps will be injective. From this, for sufficiently large  $\lambda, \mu$  with  $\lambda - \mu = \nu$ , the map  $\mathfrak{u}^{<\mathfrak{u}>} \rightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$  defined by  $u \mapsto u(x_{-\mu} \otimes x_\lambda)$  will be injective. On the other hand, the map  $\mathfrak{u}\dot{\mathcal{U}}^0 \rightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$  defined by  $u \mapsto u(x_{-\mu} \otimes x_\lambda)$  factors through the map  $D(\nu) \rightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$ . Therefore, for sufficiently large  $\lambda, \mu$  with  $\lambda - \mu = \nu$ , the natural map  $\mathfrak{u}\dot{\mathcal{U}}^0 \rightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$  factors through an injection  $D(\nu) \hookrightarrow W'_\epsilon(-\mu) \otimes W_\epsilon(\lambda)$ .

For any  $\lambda, \mu$  such that  $\lambda - \mu = \nu$ , the inclusion  $\mathfrak{u}\dot{\mathcal{U}}^0 \hookrightarrow \dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  induces the natural map  $D(\nu) \rightarrow D(\mu, \lambda)$ . The previous two paragraphs imply that for sufficiently large  $\lambda, \mu$  with  $\lambda - \mu = \nu$ , the map  $D(\nu) \rightarrow D(\mu, \lambda)$  becomes injective, then  $\text{Hom}_{\mathbb{F}}(D(\mu, \lambda), \mathbb{F}) \rightarrow \text{Hom}_{\mathbb{F}}(D(\nu), \mathbb{F})$  is surjective. This implies that the map  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]_\nu \rightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]_\nu$  is surjective. Hence  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})] \rightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]$  is surjective.  $\square$

Let us consider the idempotent version of  $\mathfrak{u}\dot{\mathcal{U}}^0$  to be denoted by  $\hat{\mathfrak{u}}$ . By restriction, the Frobenius homomorphism  $\text{Fr} : \widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{\dot{\mathcal{U}}}_\epsilon^*(\mathfrak{g}, P^*)$  gives rise to a homomorphism  $\text{fr} : \hat{\mathfrak{u}} \rightarrow \bigoplus_{\lambda \in P^*} \mathbb{F}1_\lambda$ . The latter gives us the pullback functor  $\text{fr}^* : \text{Rep}(\widehat{\dot{\mathcal{U}}}_\epsilon^{*0}) \rightarrow \text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ .

**Lemma 6.47.** *Let  $M \in \text{Rep}(\widehat{\dot{\mathcal{U}}}_\epsilon^{*0})$ . Then  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \text{fr}^*(M))$  is contained in the image of the functor  $\text{Fr}^* : \text{Rep}(\widehat{\dot{\mathcal{U}}}_\epsilon^*(\mathfrak{g})) \rightarrow \text{Rep}(\widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}))$ .*

*Proof.* Recall  $\widehat{\mathcal{U}} := \widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}, P) = \bigoplus_\lambda \dot{\mathcal{U}}_\epsilon(\mathfrak{g})1_\lambda$ . Observe that  $\widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}, P)$  is a bimodule over  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ .

Let  $\text{hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M)) \subset \text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$  denote the subspace consisting of all homomorphisms  $f$  that vanish on all except finitely many summands  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})1_\lambda$ . We see that  $\text{hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$  is a  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -submodule of  $\text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$ . Furthermore,

$$\text{hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M)) = \bigoplus_\lambda \text{Hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g})1_\lambda, \text{fr}^*(M)).$$

So  $\text{hom}_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$  is also a module over  $\widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}, P)$ .

By Lemma 6.34, the kernel of  $\text{Fr} : \widehat{\dot{\mathcal{U}}}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{\dot{\mathcal{U}}}_\epsilon^*(\mathfrak{g}, P^*)$  is equal to the left ideal generated by  $\{E_i 1_\lambda, F_i 1_\lambda, 1_\mu | \lambda \in P, \mu \in P/P^*\}$  as well as the right ideal generated by the same set

of elements. Therefore, the action of  $\widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P)$  on  $\hom_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$  factors through Frobenius morphism  $\text{Fr} : \widehat{\mathcal{U}}_\epsilon(\mathfrak{g}, P) \rightarrow \widehat{\mathcal{U}}_\epsilon^*(\mathfrak{g}, P^*)$ . On the other hand, we can interpret the module  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \text{fr}^*(M))$  as the maximal rational  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -submodule of  $\hom_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$ . Therefore,  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \text{fr}^*(M))$  viewed as the maximal rational  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -submodule of  $\hom_{\mathfrak{u}\dot{\mathcal{U}}^0}(\widehat{\mathcal{U}}, \text{fr}^*(M))$  will be contained in the image of the functor  $\text{Fr}^* : \text{Rep}(\dot{\mathcal{U}}_\epsilon^*(\mathfrak{g})) \rightarrow \text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$ .  $\square$

**Lemma 6.48.** *There is a finite dimensional module  $V \in \text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$  with the following property. For any finite dimensional module  $M$  in  $\text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$  there is a finite dimensional module  $N$  in  $\text{Rep}(\dot{\mathcal{U}}_\epsilon^*(\mathfrak{g}))$  such that  $M$  is isomorphic to a subquotient of  $\text{Fr}^*(N) \otimes V$ .*

*Proof.* The proof is in several steps.

*Step 1.* We have an identification  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}] \cong \bigoplus_{\lambda \in P^*} \mathbb{F}_\lambda$ . Thanks to this identification,  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]$  becomes a  $\mathfrak{u}\dot{\mathcal{U}}^0$ -module. There is a finite dimensional  $\dot{\mathcal{U}}_\epsilon^0$ -module  $V_1$  such that  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^0] \cong V_1 \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]$  as  $\dot{\mathcal{U}}_\epsilon^0$ -modules. By Lemma 6.39,

$$\mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0] \cong H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1) \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}].$$

Note that  $H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1)$  is finite dimensional.

*Step 2.* We will show that there is a finite dimensional  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module  $V$  in  $\text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$  such that  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}] \otimes V \twoheadrightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0]$  in  $\text{Rep}(\mathfrak{u}\dot{\mathcal{U}}^0)$ . Indeed, since the trivial representation is a direct summand of  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]$ , we have a surjective morphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}] \rightarrow \mathbb{F}$ . This gives us a surjective morphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules

$$\mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0] \cong H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1) \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}] \rightarrow H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1).$$

Combining this with Lemma 6.46, we have a surjective morphism of  $\mathfrak{u}\dot{\mathcal{U}}^0$ -modules

$$\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})] \twoheadrightarrow \mathbb{F}[\mathfrak{u}\dot{\mathcal{U}}^0] \rightarrow H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1).$$

Then any finite dimensional  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -submodule  $V$  of  $\mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]$  that maps onto  $H^0(\mathfrak{u}\dot{\mathcal{U}}^0/\dot{\mathcal{U}}^0, V_1)$  will work.

*Step 3.* The exactness of  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, -)$  yields a surjective morphism

$$H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]) \otimes V \twoheadrightarrow H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]) \cong \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})].$$

Note that  $V$  is a  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -module hence  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}] \otimes V) \cong H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]) \otimes V$ .

*Step 4.* Let us fix a module  $V$  as in Step 2. Let  $M$  be a finite dimensional module in  $\text{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}))$ . Then we have an embedding  $M \hookrightarrow H^0(\dot{\mathcal{U}}/\mathbb{F}, M) \cong M \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})]$ , here  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$  acts trivially on the copy of  $M$  in the codomain. Consider the surjective homomorphism

$$\pi : M_{triv} \otimes H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}]) \otimes V \rightarrow M \otimes \mathbb{F}[\dot{\mathcal{U}}_\epsilon(\mathfrak{g})],$$

where  $M_{triv}$  is the same space as  $M$  but with a trivial  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -action. We can find a finite dimensional  $\dot{\mathcal{U}}_\epsilon(\mathfrak{g})$ -submodule  $N$  of  $M_{triv} \otimes H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}])$  such that the image of  $N \otimes V$  under  $\pi$  contains  $M$ . By Lemma 6.47,  $H^0(\dot{\mathcal{U}}/\mathfrak{u}\dot{\mathcal{U}}^0, \mathbb{F}[\dot{\mathcal{U}}_\epsilon^{*0}])$  is contained in the image of the functor  $\text{Fr}^*$ , hence so is  $N$ . Therefore,  $M$  is a subquotient of  $\text{Fr}^*(N) \otimes V$ . This completes the proof.  $\square$

## 6.7. Rational representations of $\widehat{\mathcal{U}}_q(\mathfrak{g}, P)$ .

Let us recall the idempotent versions  $\widehat{\mathcal{U}}_q(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^>(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^<(\mathfrak{g}, P)$ .

**Definition 6.49.** A  $\widehat{\mathcal{U}}_q(\mathfrak{g}, P)$ -module  $M$  is called *rational* if for any  $m \in M$  we have

- (i) There are only finitely many  $\lambda \in P$  such that  $1_\lambda m \neq 0$ .
- (ii) There is  $k > 0$  such that  $E_i^{[s]} 1_\lambda m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .

(iii) There is  $k > 0$  such that  $F_i^{[s]} 1_\lambda m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .

Then we define  $\text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$  to be the category of all rational  $\widehat{\mathcal{U}}_q(\mathfrak{g}, P)$ -modules. The categories  $\text{Rep}(\widehat{\mathcal{U}}_q^{\geqslant}(\mathfrak{g}, P))$ ,  $\text{Rep}(\widehat{\mathcal{U}}_q^{\leqslant}(\mathfrak{g}, P))$  are defined similarly. The categories  $\text{Rep}^{fd}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$ ,  $\text{Rep}^{fd}(\widehat{\mathcal{U}}_q^{\geqslant}(\mathfrak{g}, P))$  and  $\text{Rep}^{fd}(\widehat{\mathcal{U}}_q^{\leqslant}(\mathfrak{g}, P))$  are the full subcategories of the corresponding categories consisting of all objects which are finitely generated over  $R$ .

There is a natural equivalence of braided monoidal categories  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g})) \rightarrow \text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$ , here we equip  $\text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$  with the braided structure via the  $R$ -matrix  $\mathcal{R}$  in (4.30). Therefore, the following constructions and results in Sections 6.1-6.4 carry over to  $\text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$ :

- Joseph's induction functor  $\mathfrak{J} : \text{Rep}^{fd}(\widehat{\mathcal{U}}_q^{\geqslant}(\mathfrak{g}, P)) \rightarrow \text{Rep}^{fd}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$
- The induction functor  $H^0 : \text{Rep}(\widehat{\mathcal{U}}_q^{\leqslant}(\mathfrak{g}, P)) \rightarrow \text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$  and various invariants in Section 6.2
- The Weyl module  $W_q(\lambda) := \mathfrak{J}(R_\lambda)$ , the dual Weyl module  $H_q^0(\lambda) := H^0(R_\lambda)$ , the quantized coordinate algebra  $R[\dot{\mathcal{U}}_q(\mathfrak{g})] := H^0(\dot{\mathcal{U}}_q/R, R)$ .
- The notion of good filtrations and the good filtration on the  $\dot{\mathcal{U}}_q(\mathfrak{g}) \times \dot{\mathcal{U}}_q(\mathfrak{g})$ -module  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ .

Let  $\widehat{\mathcal{U}}_q^F(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^{\geqslant, F}(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^{\leqslant, F}(\mathfrak{g}, P)$  denote the new Hopf algebras obtained from  $\widehat{\mathcal{U}}_q(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^{\geqslant}(\mathfrak{g}, P)$ ,  $\widehat{\mathcal{U}}_q^{\leqslant}(\mathfrak{g}, P)$  via the twist

$$F = \prod_{\lambda, \mu \in P} q^{\sum_{ij} \phi_{ij}(\omega_i^\vee, \lambda)(\omega_j^\vee, \mu)} 1_\lambda \otimes 1_\mu.$$

Then the category  $\text{Rep}(\widehat{\mathcal{U}}_q^F(\mathfrak{g}, P))$  is just the abelian category  $\text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$  with new monoidal structure, similarly with  $\text{Rep}(\widehat{\mathcal{U}}_q^{\geqslant, F}(\mathfrak{g}, P))$ ,  $\text{Rep}(\widehat{\mathcal{U}}_q^{\leqslant, F}(\mathfrak{g}, P))$ . Furthermore,  $\text{Rep}(\widehat{\mathcal{U}}_q^F(\mathfrak{g}, P))$  is braided with the  $R$ -matrix  $\mathcal{R}^F$  in (4.31).

**Remark 6.50.** The categories  $\text{Rep}(\widehat{\mathcal{U}}_q^F(\mathfrak{g}, P))$  and  $\text{Rep}(\widehat{\mathcal{U}}_q(\mathfrak{g}, P))$  are equivalent as braided monoidal categories.

### 6.8. Rational representations of $\dot{\mathcal{U}}_q(\mathfrak{g})$ .

**Definition 6.51.** A  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -module  $M$  is called *rational* if the following conditions hold:

- (i) There is a weight decomposition  $M = \bigoplus_{\lambda \in P} M_\lambda$ , where  $\dot{\mathcal{U}}_q^0$  acts on  $M_\lambda$  via the character  $\hat{\chi}_\lambda$ .
- (ii) For any  $m \in M$  there is  $k > 0$  such that  $\tilde{E}_i^{(s)} m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .
- (iii) For any  $m \in M$  there is  $k > 0$  such that  $\tilde{F}_i^{(s)} m = 0$  for all  $s > k$  and all  $i = 1, \dots, r$ .

We define  $\text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  to be the category of all rational  $\dot{\mathcal{U}}_q(\mathfrak{g})$ -representations. Similarly, we can define the categories  $\text{Rep}(\dot{\mathcal{U}}_q^{\geqslant})$ ,  $\text{Rep}(\dot{\mathcal{U}}_q^{\leqslant})$ ,  $\text{Rep}(\dot{\mathcal{U}}_q^0)$ .

Then similarly to Sections 6.1-6.5, we can define the following objects

- Joseph's induction functor  $\mathfrak{J} : \text{Rep}^{fd}(\dot{\mathcal{U}}_q^{\geqslant}) \rightarrow \text{Rep}^{fd}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ .
- The induction functor  $H^0(\dot{\mathcal{U}}_q/\mathcal{B}, -) : \text{Rep}(\mathcal{B}) \rightarrow \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$ , here  $\mathcal{B}$  is one of these algebras:  $R, \dot{\mathcal{U}}_q^0, \dot{\mathcal{U}}_q^{\geqslant}, \dot{\mathcal{U}}_q^{\leqslant}$ .
- The Weyl module  $W_q(\lambda) := \mathfrak{J}(R_\lambda)$ . The dual Weyl module  $H_q^0(\lambda) := H^0(\dot{\mathcal{U}}_q/\dot{\mathcal{U}}_q^{\leqslant}, R_\lambda)$ .
- The quantized coordinate algebra  $R[\dot{\mathcal{U}}_q(\mathfrak{g})]$ .
- The notion of good filtration.

The results in Sections 6.1-6.6 carry over to similar results in  $\text{Rep}(\dot{U}_q(\mathfrak{g}))$ . The justification for this is as follows: we have an equivalence of braided monoidal categories

$$\text{Rep}(\dot{U}_q(\mathfrak{g})) \rightarrow \text{Rep}(\widehat{\mathcal{U}}_q^F(\mathfrak{g}, P))$$

and equivalences of monoidal categories

$$\text{Rep}(\dot{U}_q^{\geqslant}(\mathfrak{g})) \rightarrow \text{Rep}(\widehat{\mathcal{U}}_q^{\geqslant, F}(\mathfrak{g}, P)), \quad \text{Rep}(\dot{U}_q^{\leqslant}(\mathfrak{g})) \rightarrow \text{Rep}(\widehat{\mathcal{U}}_q^{\leqslant, F}(\mathfrak{g}, P)),$$

combining with the discussions in Section 6.7.

We will need an analog of Lemma 6.48 for  $\text{Rep}(\dot{U}_q(\mathfrak{g}))$

**Lemma 6.52.** *Let  $R = \mathbb{F}$  and  $q = \epsilon \in \mathbb{F}$  be a root of unity of order  $\ell$  such that  $\ell_i \geq \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$ . There is a finite dimensional module  $V \in \text{Rep}(\dot{U}_q(\mathfrak{g}))$  with the following property. For any finite dimensional module  $M$  in  $\text{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$ , there is a finite dimensional module  $N$  in  $\text{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$  such that  $M$  is isomorphic to a subquotient of  $\tilde{\text{Fr}}^*(V) \otimes V$ .*

The only difference here is that we use the morphism  $\tilde{\text{Fr}}^*$  instead of  $\text{Fr}^*$ , but this nuisance does not cause issues by Lemma 4.25 that says  $\tilde{\text{Fr}}^*$  is  $\text{Fr}^*$  composed with a self-equivalence  $\Phi^*$  of the category  $\text{Rep}(\dot{U}_\epsilon^*(\mathfrak{g}))$ .

We end this section with a result the submodule of  $\dot{U}_q(\mathfrak{g})$ -invariants for the adjoint action on  $R[\dot{U}_q(\mathfrak{g})]$ . Let us restate a result about the good filtration of  $\dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$ -module  $R[\dot{U}_q(\mathfrak{g})]$ .

**Lemma 6.53.** *Equip  $P_+$  with a total order refining the usual partial order and let  $\lambda_1 < \lambda_2 < \dots$  be the elements so that we have a total order on  $P_+ \times P_+$  as in Remark 6.33. There is an exhaustive filtration  $\{0\} = M_0 \subset M_1 \subset \dots$  on  $R[\dot{U}_q(\mathfrak{g})]$  such that  $M_i/M_{i-1} \cong H_q^0(\lambda_i) \otimes H_q^0(\lambda_i^*)$ . The module  $M_i$  is the maximal  $\dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$ -subrepresentation of  $R[\dot{U}_q(\mathfrak{g})]$  whose weights are bounded from the above by  $(\lambda_i, \lambda_i^*)$ .*

The  $\dot{U}_q(\mathfrak{g}) \otimes \dot{U}_q(\mathfrak{g})$ -module structure on  $R[\dot{U}_q(\mathfrak{g})]$  is as follows:

$$(x \otimes y)f(u) = f(S(y)ux) \quad \text{for all } x, y, u \in \dot{U}_q(\mathfrak{g}) \text{ and } f \in R[\dot{U}_q(\mathfrak{g})].$$

Let us make a small modification. Lemma 6.53 is the same if we consider the following  $\dot{U}_q(\mathfrak{g}) \otimes \dot{U}_q(\mathfrak{g})$ -module structure on  $R[\dot{U}_q(\mathfrak{g})]$ :

$$(x \otimes y)f(u) = f(S(y)uS^2(x)) \quad \text{for all } x, y, u \in \dot{U}_q(\mathfrak{g}) \text{ and } f \in R[\dot{U}_q(\mathfrak{g})].$$

Using this modified action and the coproduct map  $\Delta' : \dot{U}_q(\mathfrak{g}) \rightarrow \dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$ , we get the adjoint action of  $\dot{U}_q(\mathfrak{g})$  on  $R[\dot{U}_q(\mathfrak{g})]$  given by  $(xf)(u) = f(\text{ad}'_l(S(x))u)$  for all  $x, u \in \dot{U}_q(\mathfrak{g})$  and  $f \in R[\dot{U}_q(\mathfrak{g})]$ .

For any  $V \in \text{Rep}(\dot{U}_q(\mathfrak{g}))$ , we define an  $R$ -linear map

$$V \otimes_R V^* \rightarrow R[\dot{U}_q(\mathfrak{g})], \quad v \otimes f \mapsto c_{f, K^{-2\rho}v}, \forall v \in V, f \in V^*.$$

This is a homomorphism of  $\dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$ -modules (note that we use the modified action of  $\dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$  on  $R[\dot{U}_q(\mathfrak{g})]$  here).

The Weyl module  $W_q(\lambda)$  is free over  $R$  with a weight basis to be denoted by  $\{v_i\}$ . Hence  $\text{End}_R(W_q(\lambda)) \cong W_q(\lambda) \otimes_R W_q(\lambda)^*$  as  $\dot{U}_q(\mathfrak{g})$ -modules. The image of  $\text{Id} : W_q(\lambda) \rightarrow W_q(\lambda)$  in  $W_q(\lambda) \otimes_R W_q(\lambda)^*$  is  $\sum_i v_i \otimes v_i^*$ , here  $v_i^*$  is the dual weight basis of  $W_q(\lambda)^*$ . Hence  $\sum_i v_i \otimes v_i^* \in (W_q(\lambda) \otimes_R W_q(\lambda)^*)^{\dot{U}_q}$ .

Let  $c_\lambda$  be the image of  $\sum_i v_i \otimes v_i^*$  under the map  $W_q(\lambda) \otimes_R W_q(\lambda)^* \rightarrow R[\dot{U}_q(\mathfrak{g})]$ . Then  $c_\lambda \in R[\dot{U}_q(\mathfrak{g})]^{\dot{U}_q}$ .

**Lemma 6.54.** *Let us consider the adjoint action of  $\dot{U}_q(\mathfrak{g})$  on  $R[\dot{U}_q(\mathfrak{g})]$ . The  $\dot{U}_q(\mathfrak{g})$ -invariant part  $R[\dot{U}_q(\mathfrak{g})]^{\dot{U}_q}$  is a free  $R$ -module with a basis  $\{c_\lambda | \lambda \in P_+\}$ .*

*Proof.* *Step 1.* Since the dominant weights of the  $\dot{U}_q(\mathfrak{g}) \otimes_R \dot{U}_q(\mathfrak{g})$ -module  $W_q(\lambda_i) \otimes_R W_q(\lambda_i)^*$  are bounded by  $(\lambda_i, \lambda_i^*)$ , the image of  $W_q(\lambda_i) \otimes_R W_q(\lambda_i)^* \rightarrow R[\dot{U}_q(\mathfrak{g})]$  is contained in  $M_i$ .

*Step 2.* Under the first  $\dot{U}_q(\mathfrak{g})$ -module structure on  $R[\dot{U}_q(\mathfrak{g})]$  (that is defined via  $\delta$ ), the natural map

$$\text{Hom}_{\dot{U}_q(\mathfrak{g})}(W_q(\lambda_i), W_q(\lambda_i) \otimes_R W_q(\lambda_i)^*) \rightarrow \text{Hom}_{\dot{U}_q(\mathfrak{g})}(W_q(\lambda_i), R[\dot{U}_q(\mathfrak{g})]),$$

is just the identity morphism  $W_q(\lambda_i)^* \xrightarrow{\text{Id}} W_q(\lambda_i)^* \cong H_q^0(\lambda_i^*)$ . Therefore, the composition

$$W_q(\lambda_i) \otimes_R W_q(\lambda_i)^* \rightarrow M_i \rightarrow H_q(\lambda_i) \otimes_R H_q^0(\lambda_i^*)$$

just comes from the natural homomorphism  $W_q(\lambda_i) \rightarrow H_q^0(\lambda_i)$ .

Note that  $(H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*))^{\dot{U}_q} \cong \text{Hom}_{\dot{U}_q(\mathfrak{g})}(W_q(\lambda_i), H_q^0(\lambda_i)) \cong R$ , here the first isomorphism uses the  $R$ -freeness of  $W_q(\lambda_i)$  and the isomorphism  $H_q^0(\lambda_i^*) \cong W_q(\lambda_i)^*$ . Hence we see that the image of  $c_{\lambda_i}$  in  $H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*)$  spans the free  $R$ -module of rank one  $(H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*))^{\dot{U}_q}$ .

*Step 3.* We now proceed by induction to show that  $M_i^{\dot{U}_q}$  is a free  $R$ -module with basis  $\{c_{\lambda_j} | 0 \leq j \leq i\}$ . The base case  $i = 1$  holds by Step 2. Now we do the induction step. Assume that  $M_{i-1}^{\dot{U}_q}$  is a free  $R$ -module with the basis  $\{c_{\lambda_j} | 0 \leq j \leq i-1\}$ . The module  $M_{i-1}$  has a good filtration and the trivial  $\dot{U}_q(\mathfrak{g})$ -module  $R$  is just the Weyl module  $W_q(0)$ . Therefore, the short exact sequence

$$0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*) \rightarrow 0$$

gives us a short exact sequence

$$0 \rightarrow (M_{i-1})^{\dot{U}_q} \rightarrow (M_i)^{\dot{U}_q} \rightarrow (H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*))^{\dot{U}_q} \rightarrow 0.$$

Since  $(H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*))^{\dot{U}_q}$  is a free  $R$ -module of rank one, by induction hypothesis,  $(M_i)^{\dot{U}_q}$  is a free  $R$ -module of finite rank. Furthermore, the image of  $c_{\lambda_i} \in (M_i)^{\dot{U}_q}$  spans  $(H_q^0(\lambda_i) \otimes_R H_q^0(\lambda_i^*))^{\dot{U}_q}$  by Step 2. Combining this with the induction hypothesis again, it follows that the elements  $\gamma_{\lambda_j}$  for  $0 \leq j \leq i$  form an  $R$ -basis in  $(M_i)^{\dot{U}_q}$ .

*Step 4.* Since the filtration  $M_0 \subset M_1 \subset M_2 \dots$  of  $R[\dot{U}_q(\mathfrak{g})]$  is exhaustive, the lemma follows by Step 3.  $\square$

## 7. REFLECTION EQUATION ALGEBRAS

In this section we recall an algebra closely related to both the quantized coordinate algebra and the De Concini-Kac form, the *reflection equation algebra*.

Recall that an  $H$ -module algebra  $A$  over a Hopf algebra  $H$  is a unital algebra  $A$  with a left  $H$ -module structure such that

$$(7.1) \quad h1_A = \varepsilon(h)1_A, \quad h \cdot ab = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b) \quad \forall a, b \in A, h \in H.$$

In this section, we show that one can twist an algebra structure on  $R[\dot{U}_q(\mathfrak{g})]$  with respect to the  $R$ -matrix so that  $R[\dot{U}_q(\mathfrak{g})]$  becomes an  $\dot{U}_q(\mathfrak{g})$ -module algebra with the  $\dot{U}_q(\mathfrak{g})$ -module structure given by

$$(xf)(y) = f(\text{ad}'_l(S'(x))(y)), \quad \forall x, y \in \dot{U}_q(\mathfrak{g}), f \in R[\dot{U}_q(\mathfrak{g})].$$

The new algebra, denoted by  $O_q[G]$ , is often called the *reflection equation algebra* in the literature.

### 7.1. Reflection equation algebra $O_q[G]$ .

For a technical reason, see Remark 7.8, we will work with  $\dot{U}_q := \dot{U}_q^{\text{op},\text{cop}}(\mathfrak{g})$  the Hopf algebra obtained from  $\dot{U}_q(\mathfrak{g})$  by taking the opposites of both product and coproduct structures. Let  $\text{ad}_l^\circ, \text{ad}_r^\circ$  denote the left and right adjoint actions of  $\dot{U}_q$  on itself. We will use  $\cdot^{\text{op}}$  to denote the product on  $\dot{U}_q$  as well as the action of  $\dot{U}_q$  on its modules, to distinguish from the corresponding constructions for  $\dot{U}_q(\mathfrak{g})$ .

We record Proposition-Definition 2.5 in [KS]:

**Proposition-Definition 7.1.** (a) Let  $(H, 1_H, \mu, \varepsilon, \Delta)$  be a bialgebra and let  $F \in H \otimes H$  be an element such that

$$(\Delta \otimes \text{Id})(F)F_{12} = (\text{Id} \otimes \Delta)(F)F_{23}, \quad (\varepsilon \otimes \text{Id})(F) = 1_H = (\text{Id} \otimes \varepsilon)(F).$$

For all  $x \in H$  define

$$\Delta^F(x) = F^{-1}\Delta(x)F.$$

Then  $H_F := (H, 1_H, \mu, \varepsilon, \Delta_F)$  is a bialgebra. We say  $F$  is a twist for  $H$ .

(b) Let  $H$  be a bialgebra and  $F \in H \otimes H$  a twist for  $H$ . Let  $(A, 1_A, m)$  be a unital left  $H$ -module algebra with multiplication  $m : A \otimes A \rightarrow A$ . Define a linear map

$$m_F : A \otimes A \rightarrow A, \quad m_F(a \otimes b) = m(F(a \otimes b)).$$

Then  $A_F := (A, 1_A, m_F)$  is an algebra. The left  $H$ -module structure on  $A$  turns  $A_F$  into a left  $H_F$ -module algebra.

(c) Let  $(U, 1, \mu, \varepsilon, \Delta, \mathcal{R})$  be a braided bialgebra with universal  $R$ -matrix  $\mathcal{R}$  and  $H := U^{\text{cop}} \otimes U$  the product bialgebra. Then the two elements

$$F := \mathcal{R}_{13}\mathcal{R}_{23} \in H \otimes H, \quad \bar{F} := \mathcal{R}_{24}^{-1}\mathcal{R}_{14}^{-1} \in H \otimes H$$

are twists for  $H$ .

**Lemma 7.2.** With notations from Proposition 7.1.c), the coproduct of  $U$  defines a bialgebra homomorphism  $\Delta : U \rightarrow (U^{\text{cop}} \otimes U)_F$  with  $F = \mathcal{R}_{13}\mathcal{R}_{23}$ .

*Proof.* Since the algebra structure of  $(U^{\text{cop}} \otimes U)_F$  is the same as of  $U \otimes U$ , we only need to check that the map is a coalgebra homomorphism. Let  $\tilde{\Delta}, \check{\Delta}$  denote the coproducts of

$(U^{\text{cop}} \otimes U)_F, U^{\text{cop}} \otimes U$ , respectively. Then

$$\begin{aligned}\tilde{\Delta}(x \otimes y) &= \mathcal{R}_{23}^{-1} \mathcal{R}_{13}^{-1} \tilde{\Delta}(x \otimes y) \mathcal{R}_{13} \mathcal{R}_{23} \\ &= \mathcal{R}_{23}^{-1} \mathcal{R}_{13}^{-1} (x_{(2)} \otimes y_{(1)} \otimes x_{(1)} \otimes y_{(2)}) \mathcal{R}_{13} \mathcal{R}_{23} \\ &= \mathcal{R}_{23}^{-1} (x_{(1)} \otimes y_{(1)} \otimes x_{(2)} \otimes y_{(2)}) \mathcal{R}_{23}\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\Delta}(\Delta(x)) &= \tilde{\Delta}(x_{(1)} \otimes x_{(2)}) = \mathcal{R}_{23}^{-1} (x_{(1)} \otimes x_{(3)} \otimes x_{(2)} \otimes x_{(4)}) \mathcal{R}_{23} = x_{(1)} \otimes \mathcal{R}^{-1} \Delta^{\text{op}}(x_{(2)}) \mathcal{R} \otimes x_{(3)} \\ &= x_{(1)} \otimes \Delta(x_{(2)}) \otimes x_{(3)} = (\Delta \otimes \Delta)\Delta(x)\end{aligned}$$

□

There is a direct analog of the category of rational representations for the algebra  $\text{Rep}(\mathring{U}_q)$ . We also have the quantized coordinate algebra  $R[\mathring{U}_q]$  as in Section 6.5. Moreover, by viewing  $\mathring{U}_q$  and  $\mathring{U}_q(\mathfrak{g})$  as the same  $R$ -module, we can identify the  $R$ -modules  $R[\mathring{U}_q]$  and  $R[\mathring{U}_q(\mathfrak{g})]$ .

The quantized coordinate algebra  $R[\mathring{U}_q]$  naturally carries an algebra structure coming from the coproduct of  $\mathring{U}_q$ . Furthermore,  $R[\mathring{U}_q]$  is a  $\mathring{U}_q^{\text{cop}} \otimes \mathring{U}_q$ -module algebra with the  $\mathring{U}_q^{\text{cop}} \otimes \mathring{U}_q$ -module structure defined by

$$(x \otimes y)f(u) = f(S'(x) \cdot^{\text{op}} u \cdot^{\text{op}} y),$$

for any  $x, y, u \in \mathring{U}_q$  and  $f \in R[\mathring{U}_q]$ .

We consider the following  $R$ -matrix for  $\mathring{U}_q$

$$(7.2) \quad \mathring{\mathcal{R}} := (\mathcal{R}_{21}^F)^{-1} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N}^{v, \mu \in P} q^{(\nu, \mu) - (\nu + \mu, \kappa(\text{wt}(\tilde{E}^{(\vec{k})})))} d_{\vec{k}}(\tilde{E}^{(\vec{k})} \cdot^{\text{op}} 1_\mu) \otimes (S'(\tilde{F}^{(\vec{k})}) \cdot^{\text{op}} 1_\nu),$$

with the coefficients  $d_{\vec{0}} \in \mathbb{F}$  in which  $d_{\vec{0}} = 1$

Applying the construction in Proposition 7.1 to

$$H = \mathring{U}_q^{\text{cop}} \otimes \mathring{U}_q, \quad A = R[\mathring{U}_q], \quad F = \mathring{\mathcal{R}}_{13} \mathring{\mathcal{R}}_{23},$$

we have

**Lemma 7.3.**  $R[\mathring{U}_q]_F$  is a left  $(\mathring{U}_q^{\text{cop}} \otimes \mathring{U}_q)_F$ -module algebra.

**Remark 7.4.** We note again that  $\mathring{\mathcal{R}}$  is not an element in  $\mathring{U}_q \otimes \mathring{U}_q$  and  $\mathring{U}_q$  is not quasi-triangular. However the algebra  $R[\mathring{U}_q]_F$  is still well-defined. This is because any element of  $R[\mathring{U}_q]$  is a matrix coefficient of some finitely generated  $R$ -module object in  $\text{Rep}(\mathring{U}_q)$ . Hence, in the formula of Proposition 7.1.b applying to  $R[\mathring{U}_q]$ , all but finitely many components of  $F = \mathring{\mathcal{R}}_{13} \mathring{\mathcal{R}}_{23}$  act by zero.

By Lemma 7.2, the coproduct  $\Delta^{\text{op}}$  of  $\mathring{U}_q$  is a Hopf algebra homomorphism  $\Delta^{\text{op}} : \mathring{U}_q \rightarrow (\mathring{U}_q^{\text{cop}} \otimes \mathring{U}_q)_F$ . So that we obtain the following

**Lemma 7.5.**  $R[\mathring{U}_q]_F$  is a left  $\mathring{U}_q$ -module algebra with the  $\mathring{U}_q$ -module structure defined by

$$(x \cdot^{\text{op}} f)(u) = f(\text{ad}_r^\circ(x)(u)),$$

for  $x, u \in \mathring{U}_q$  and  $f \in R[\mathring{U}_q]_F$ .

We arrive at the definition of  $O_q[G]$ .

**Definition 7.6.** The reflection equation algebra  $O_q[G]$  is the  $R$ -module  $R[\mathring{U}_q(\mathfrak{g})]$  with the algebra structure of  $R[\mathring{U}_q]_F$ . Here we identify the  $R$ -modules  $R[\mathring{U}_q(\mathfrak{g})]$  and  $R[\mathring{U}_q]_F$  as above.

**Lemma 7.7.**  $O_q[G]$  is a left  $\dot{U}_q(\mathfrak{g})$ -module algebra with the  $\dot{U}_q(\mathfrak{g})$ -module structure given by

$$(xf)(u) = f(\text{ad}'_l(S'(x))(u)),$$

for  $x, u \in \dot{U}_q(\mathfrak{g})$  and  $f \in O_q[G]$

*Proof.* We have that

$$(xf)(u) = f(\text{ad}'_l(S'(x))(u)) = f(\text{ad}'_r(S'(x))(u)) = (S'(x) \cdot^{\text{op}} f)(u),$$

for  $x, u \in \dot{U}_q(\mathfrak{g})$  and  $f \in O_q[G]$ . In the expression  $S'(x) \cdot^{\text{op}} f$ , we view  $S'(x)$  as an element in  $\mathring{U}_q$ . Therefore, we have

$$x(fg) = S'(x) \cdot^{\text{op}} (fg) = (S'(x_{(1)}) \cdot^{\text{op}} f)(S'(x_{(2)}) \cdot^{\text{op}} g) = (x_{(1)}f)(x_{(2)}g),$$

for  $x \in \dot{U}_q(\mathfrak{g})$  and  $f, g \in O_q[G]$ .  $\square$

**Remark 7.8.** One can apply the construction in Proposition 7.1.b) directly with  $\dot{U}_q(\mathfrak{g})$ , however, the algebra  $O_q[G]$  obtained will be a left  $\dot{U}_q(\mathfrak{g})$ -module structure with the  $\dot{U}_q(\mathfrak{g})$ -module structure defined by  $(xf)(u) = f(\text{ad}'_r(x))(u)$  for  $x, u \in \dot{U}_q(\mathfrak{g})$  and  $f \in O_q[G]$ . Later, we want the  $\dot{U}_q(\mathfrak{g})$ -module structure as in Lemma 7.7, therefore, which is why we need to work with  $\mathring{U}_q$ .

**Remark 7.9.** The construction of  $O_q[G]$  is functorial with respect to the base ring. Namely, for any  $\mathcal{A}_N$ -algebra  $R$ , the algebra  $O_{\mathcal{A}_N}[G] \otimes_{\mathcal{A}_N} R$  is the same as  $O_R[G]$  (we already know the  $R$ -module isomorphism  $O_{\mathcal{A}_N}[G] \otimes_{\mathcal{A}_N} R \cong O_R[G]$  by Lemma 6.30).

7.1.1. *Categorical construction of  $O_q[G]$ .* We end this subsection by interpreting the algebra structure of  $O_q[G]$  using the categorical language. For any  $X \in \text{Rep}(\mathring{U}_q)$ , set  $X^* := \text{Hom}_R(X, R) \in \text{Rep}(\mathring{U}_q)$ . Then

$$(7.3) \quad O_q[G] \simeq \bigsqcup X^* \otimes X / \sim,$$

where  $X$  runs over all modules in  $\text{Rep}(\mathring{U}_q)$  that are finitely generated over  $R$ . The equivalence relation  $\sim$  is given by

$$\phi^* y^* \otimes x \sim y^* \otimes \phi x \quad \forall \phi: X \rightarrow Y, x \in X, y^* \in Y^*.$$

Let  $\sigma_{-, -}$  be the braiding on  $\text{Rep}(\mathring{U}_q)$ . Then the multiplication  $m$  on  $O_q[G]$  can be described as follows:

$$(7.4) \quad \begin{array}{ccc} X^* \otimes X \otimes Y^* \otimes Y & \xrightarrow{\sigma_{X^*, Y^*}} & Y^* \otimes X^* \otimes X \otimes Y \longrightarrow (X \otimes Y)^* \otimes (X \otimes Y) \\ \downarrow & & \downarrow \\ O_q[G] \otimes O_q[G] & \xrightarrow{m} & O_q[G] \end{array}$$

One routinely recovers the algebraic description of algebra structure on  $O_q[G]$  constructed above from (7.4), see [Jo, §2.3].

## 7.2. The case when $R = \mathbb{F}$ is a field.

Let  $\text{Rep}^{fd}(\mathring{U}_q)$  be the full subcategory of  $\text{Rep}(\mathring{U}_q)$  consisting of all finite dimensional representations. Recall that any element of  $\mathbb{F}[\mathring{U}_q]$  can be represented as a linear combination of matrix coefficients of some object in  $\text{Rep}^{fd}(\mathring{U}_q)$ . Let  $X \in \text{Rep}^{fd}(\mathring{U}_q)$  and  $X^* = \text{Hom}_{\mathbb{F}}(X, \mathbb{F})$ . Elements  $v \in X$  and  $f \in X^*$  give rise to the matrix coefficient  $\mathring{c}_{f,v}$ . We use the notation  $\mathring{c}_{-, -}$  to distinguish these matrix coefficients from the matrix coefficients  $c_{-, -}$  of  $\dot{U}_q(\mathfrak{g})$ .

The Hopf algebra structure of  $\mathbb{F}[\mathring{U}_q]$  can be described as follows:

$$\begin{aligned}\mathring{c}_{f,v} \cdot \mathring{c}_{g,w} &= \mathring{c}_{f \otimes g, v \otimes w}, \quad \Delta(\mathring{c}_{f,v}) = \sum \mathring{c}_{f,v_i} \otimes \mathring{c}_{v_i^*,v}, \\ \varepsilon(\mathring{c}_{f,v}) &= f(v), \quad [S^{\pm 1}(\mathring{c}_{f,v})](u) = f((S^{\pm 1}u)v),\end{aligned}$$

where  $v \in V, f \in V^*$  and  $w \in W$  with  $V, W \in \text{Rep}^{fd}(\mathring{U}_q)$ ;  $u \in \mathring{U}_q$  and  $\{v_i\}, \{v_i^*\}$  are dual bases of  $V, V^*$ .

Let us define a bilinear form  $\mathbf{r} : \mathbb{F}[\mathring{U}_q] \otimes \mathbb{F}[\mathring{U}_q] \rightarrow \mathbb{F}$  by

$$\mathbf{r}(\mathring{c}_{f,v}, \mathring{c}_{g,w}) = \langle f \otimes g, \mathcal{R}(v \otimes w) \rangle,$$

with  $\mathcal{R}$  from (7.2). This bilinear form makes  $\mathbb{F}[\mathring{U}_q]$  a *coquasi-triangular Hopf algebra* according to the following definition, see [KSc, §10.1.1] or [KS, Definition 1.1].

**Definition 7.10.** A coquasi-triangular bialgebra/Hopf algebra  $(A, \mathbf{r})$  over  $\mathbf{k}$  is a pair consisting of a bialgebra/Hopf algebra  $A$  over  $\mathbf{k}$  with a convolution invertible linear map  $\mathbf{r} : A \otimes A \rightarrow \mathbf{k}$  which satisfies the following relations:

$$\begin{aligned}\mathbf{r}(a_{(1)} \otimes b_{(1)})a_{(2)}b_{(2)} &= b_{(1)}a_{(1)}\mathbf{r}(a_{(2)} \otimes b_{(2)}), \\ \mathbf{r}(ab \otimes c) &= \mathbf{r}(a \otimes c_{(1)})\mathbf{r}(b \otimes c_{(2)}), \\ \mathbf{r}(a \otimes bc) &= \mathbf{r}(a_{(1)} \otimes c)\mathbf{r}(a_{(2)} \otimes b).\end{aligned}$$

Then we can twist the algebra structure on  $\mathbb{F}[\mathring{U}_q]$  as follows, see [KS, Proposition-Definition 2.1]

$$(7.5) \quad a \cdot_{\mathbf{r}} b := \mathbf{r}(a_{(1)}, b_{(2)})\mathbf{r}(a_{(3)}, Sb_{(1)})a_{(2)}b_{(3)} = \mathbf{r}(a_{(2)}, b_{(3)})\mathbf{r}(a_{(3)}, Sb_{(1)})b_{(2)}a_{(1)},$$

with  $a, b \in \mathbb{F}[\mathring{U}_q]$ . The resulting algebra structure coincides with one from Proposition 7.1, see [KS, Lemma 2.6].

**Lemma 7.11.** *The algebra  $O_q[G]$  can be obtained via the twist (7.5).*

### 7.3. The case when $\mathbb{F} = \mathbb{Q}(v^{1/\mathbb{N}})$ and $q = v$ .

Set  $\mathbb{F}[\mathring{U}_q]^* := \text{Hom}_{\mathbb{F}}(\mathbb{F}[\mathring{U}_q], \mathbb{F})$ . This is naturally an algebra. Let us define a bilinear map  $\mathbf{q}(\cdot, \cdot) : \mathbb{F}[\mathring{U}_q] \otimes \mathbb{F}[\mathring{U}_q] \rightarrow \mathbb{F}$  via:

$$\mathbf{q}(a \otimes b) = \mathbf{r}(b_{(1)}, a_{(1)})\mathbf{r}(a_{(2)}, b_{(2)}),$$

for  $a, b \in \mathbb{F}[\mathring{U}_q]$ . This gives rise to a linear map  $\tilde{l}_{\mathbf{r}} : \mathbb{F}[\mathring{U}_q] \rightarrow \mathbb{F}[\mathring{U}_q]^*$  via

$$\tilde{l}_{\mathbf{r}}(a) = \mathbf{q}(a, \cdot)$$

for  $a \in \mathbb{F}[\mathring{U}_q]$ .

When  $\mathbb{F} = \mathbb{Q}(v^{1/\mathbb{N}})$ , the natural Hopf pairing  $\mathbb{F}[\mathring{U}_q] \times \mathring{U}_q \rightarrow \mathbb{F}$  is non-degenerate in the second argument, a proof can be found in [J1, Proposition 5.11]. So we have an algebra embedding  $\mathring{U}_q \hookrightarrow \mathbb{F}[\mathring{U}_q]^*$ .

**Lemma 7.12.** (a)  $\tilde{l}_{\mathbf{r}}(\mathbb{F}[\mathring{U}_q]) \subset \mathring{U}_q$ .

(b) *The map  $\tilde{l}_{\mathbf{r}} : O_q[G] \rightarrow \mathring{U}_q$  is a homomorphism of left  $\mathring{U}_q$ -module algebras, where the  $\mathring{U}_q$ -action on  $O_q[G]$  is in Lemma 7.5 and the  $\mathring{U}_q$ -action on  $\mathring{U}_q$  is the left adjoint action  $\text{ad}_l^\circ$ .*

*Proof.* (a) Let  $\mathring{\mathcal{R}} = \sum_{i \in I} s_i \otimes t_i$ . Let  $V, W \in \text{Rep}^{fd}(\mathring{U}_q)$ . Let  $v, w, f, g$  be weight vectors in  $V, W, V^*, W^*$ , respectively. Let  $\{v_l\}, \{w_k\}$  be weight bases of  $V, W$ , respectively. Let  $\{v_l^*\}, \{w_k^*\}$

be the corresponding dual bases of  $V^*, W^*$ . Then we have

$$\begin{aligned}
 \mathbf{q}(\dot{c}_{f,v}, \dot{c}_{g,w}) &= \sum_{l,k} \left\langle g \otimes f, \mathring{\mathcal{R}}(w_k \otimes v_l) \right\rangle \left\langle v_l^* \otimes w_k^*, \mathring{\mathcal{R}}(v \otimes w) \right\rangle \\
 (7.6) \quad &= \sum_{l,k,i,j} g(s_i w_k) f(t_i v_l) v_l^*(s_j v) w_k^*(t_j w) \\
 &= g\left(\sum_{i,j} s_i t_j f(t_i s_j v) w\right)
 \end{aligned}$$

Using (7.6) and (7.2), we get

$$\begin{aligned}
 \tilde{l}_{\mathbf{r}}(\dot{c}_{f,v}) &= \sum_{\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N} d_{\vec{k}} d_{\vec{r}} q^{-(\text{wt}(v), \kappa(\text{wt}(\tilde{E}^{(\vec{k})})) + \kappa(\text{wt}(\tilde{E}^{(\vec{r})})))} f(S'(\tilde{F}^{(\vec{k})}) \cdot^{\text{op}} \tilde{E}^{(\vec{r})}(v)) \\
 &\quad \tilde{E}^{(\vec{k})} \cdot^{\text{op}} S'(\tilde{F}^{(\vec{r})}) \cdot^{\text{op}} K^{2\text{wt}(v) - \kappa(\text{wt}(\tilde{E}^{(\vec{k})})) - \kappa(\text{wt}(\tilde{E}^{(\vec{r})}))}.
 \end{aligned}$$

Then we see that  $\tilde{l}_{\mathbf{r}}(\dot{c}_{f,v}) \in \dot{U}_q$  since  $\kappa(\text{wt}(\tilde{E}^{(\vec{k})})) \in 2P$  for all  $\vec{k} \in \mathbb{Z}_{\geq 0}^N$

(b) This follows by Proposition 11 (equations 10.1.3(28)-(30)) in [KS, §10.1.3], see also Proposition 1.7 and 2.8 in [KS]. Note that in equation 10.1.3(28) [KS], the term  $g(a_{(1)}S(a_{(3)}))$  should be replaced by  $g(S(a_{(1)})a_{(3)})$ . This misprint has been noticed in [KS, Proposition 1.7].  $\square$

Let us consider the following composition:

$$(7.7) \quad l_{\mathbf{r}} : O_q[G] \xrightarrow{\tilde{l}_{\mathbf{r}}} \dot{U}_q \xrightarrow{(S')^{-1}} \dot{U}_q(\mathfrak{g}).$$

For each  $\lambda \in P_+$ , let  $L_q(\lambda)$  be the simple module of highest weight  $\lambda$  in  $\text{Rep}(\dot{U}_q(\mathfrak{g}))$ . Let  $v_\lambda$  be a nonzero highest weight vector in  $L_q(\lambda)$  and  $v_\lambda^*$  be the dual weight vector in  $(L_q(\lambda))^*$ . So we have the matrix coefficient  $c_{v_\lambda^*, v_\lambda} \in O_q[G]$ .

**Lemma 7.13.** (a) *The map  $l_{\mathbf{r}}$  is a morphism of left  $\dot{U}_q(\mathfrak{g})$ -module algebras, where the  $\dot{U}_q(\mathfrak{g})$ -action on  $O_q[G]$  is in Lemma 7.7 and the  $\dot{U}_q(\mathfrak{g})$ -action on  $\dot{U}_q(\mathfrak{g})$  is the left adjoint action  $\text{ad}'_l$ .*  
(b)  $l_{\mathbf{r}}(c_{v_\lambda^*, v_\lambda}) = K^{-2\lambda}$ .

*Proof.* (a) Both  $\tilde{l}_{\mathbf{r}}$  and  $(S')^{-1}$  are algebra homomorphisms hence so is  $l_{\mathbf{r}}$ . It remains to check that  $l_{\mathbf{r}}$  is  $\dot{U}_q(\mathfrak{g})$ -linear:

$$\begin{aligned}
 l_{\mathbf{r}}(xf) &= (S')^{-1}[\tilde{l}_{\mathbf{r}}(xf)] = (S')^{-1}[\tilde{l}_{\mathbf{r}}(S'(x) \cdot^{\text{op}} f)] = (S')^{-1}[\text{ad}_l^\circ(S'(x))(\tilde{l}_{\mathbf{r}}(f))] = \\
 &\quad \text{ad}'_l(x)((S')^{-1}[\tilde{l}_{\mathbf{r}}(f)]) = \text{ad}'_l(x)(l_{\mathbf{r}}(f)),
 \end{aligned}$$

for  $x \in \dot{U}_q(\mathfrak{g})$  and  $f \in O_q[G]$ . Again, in the expression  $S'(x) \cdot^{\text{op}} f$ , we view  $S'(x)$  as an element in  $\dot{U}_q$ .

(b) Any left  $\dot{U}_q(\mathfrak{g})$ -module  $V$  acquires the left  $\dot{U}_q$ -module structure as follows:  $x \cdot^{\text{op}} v = S'(x)v$  for all  $x \in \dot{U}_q$  and  $v \in V$ . Let  $\mathring{L}_q(\lambda)$  be the left  $\dot{U}_q$ -module obtained from  $L_q(\lambda)$ . So we have a matrix coefficient  $\dot{c}_{v_\lambda^*, v_\lambda}$  of  $\dot{U}_q$ . Note that  $v_\lambda$  has the weight  $-\lambda$  in  $\mathring{L}_q(\lambda)$ .

It turns out that  $c_{v_\lambda^*, v_\lambda} = (S')^{-1}\dot{c}_{v_\lambda^*, v_\lambda}$  in  $O_q[G]$ . Indeed,

$$(S')^{-1}\dot{c}_{v_\lambda^*, v_\lambda}(x) = v_\lambda^*((S')^{-1}x \cdot^{\text{op}} v_\lambda) = v_\lambda^*(xv_\lambda) = c_{v_\lambda^*, v_\lambda}(x),$$

for all  $x \in \dot{U}_q$ , where we identify  $\dot{U}_q$  with  $\dot{U}_q(\mathfrak{g})$  as  $\mathbb{F}$ -vector spaces.

On the other hand, notice that  $v_\lambda$  has the weight  $-\lambda$  in  $\mathring{L}_q(\lambda)$ . Then one argues as in the proof of part a) of Lemma 7.12 to get

$$\mathbf{q}(S'^{-1}\mathring{c}_{v_\lambda^*, v_\lambda}, \mathring{c}_{g, w}) = g(q^{2(\lambda, \text{wt}(w))}w) = g(K^{2\lambda}w),$$

where  $w, g$  are weight vectors in  $W, W^*$  for some  $W \in \text{Rep}^{fd}(\mathring{U}_q)$ . It follows that

$$\tilde{l}_{\mathbf{r}}((S')^{-1}\mathring{c}_{v_\lambda^*, v_\lambda}) = K^{2\lambda}.$$

Therefore,

$$l_{\mathbf{r}}(c_{v_\lambda^*, v_\lambda}) = (S')^{-1}(\tilde{l}_{\mathbf{r}}((S')^{-1}\mathring{c}_{v_\lambda^*, v_\lambda})) = (S')^{-1}(K^{2\lambda}) = K^{-2\lambda}.$$

□

**Lemma 7.14.**  $O_q[G] \cong \bigoplus_{\lambda \in P_+} \dot{U}_q(\mathfrak{g}) c_{v_\lambda^*, v_\lambda}$ .

*Proof.* For any  $V \in \text{Rep}^{fd}(\dot{U}_q(\mathfrak{g}))$ , the map  $V \otimes V^* \rightarrow O_q[G]$  defined by  $v \otimes f \mapsto c_{f, K^{-2\rho}v}$  is  $\dot{U}_q(\mathfrak{g})$ -linear, where the  $\dot{U}_q(\mathfrak{g})$ -action on  $O_q[G]$  is from Lemma 7.7. Then we have

$$O_q[G] \cong \bigsqcup_{X \in \text{Rep}^{fd}(\dot{U}_q(\mathfrak{g}))} X \otimes X^*/\sim,$$

where the equivalence relation  $\sim$  is given by

$$x \otimes \phi^*y^* \sim \phi x \otimes y^* \quad \forall \phi : X \rightarrow Y, x \in X, y^* \in Y^*.$$

Since we are working over  $\mathbb{F} = \mathbb{Q}(v^{1/\mathbb{N}})$ , the category  $\text{Rep}^{fd}(\dot{U}_q(\mathfrak{g}))$  is semisimple with simple objects  $\{L_q(\lambda) | \lambda \in P_+\}$ . Therefore,

$$O_q[G] \cong \bigoplus_{\lambda \in P_+} L_q(\lambda) \otimes L_q(\lambda)^*.$$

On the other hand, it is easy to show that  $L_q(\lambda) \otimes L_q(\lambda)^*$  is generated by  $(K^{2\rho}v_\lambda) \otimes v_\lambda^* = q^{(2\rho, \lambda)}v_\lambda \otimes v_\lambda^*$  over  $\dot{U}_q(\mathfrak{g})$ . Moreover,  $(K^{2\rho}v_\lambda) \otimes v_\lambda^*$  is mapped to  $c_{v_\lambda^*, v_\lambda}$  under the morphism  $L_q(\lambda) \otimes L_q(\lambda)^* \rightarrow O_q[G]$  defined in the beginning of the proof. Therefore,

$$O_q[G] \cong \bigoplus_{\lambda \in P_+} \dot{U}_q(\mathfrak{g}) c_{v_\lambda^*, v_\lambda}.$$

□

## 8. ISOMORPHISM BETWEEN REA AND THE LOCALLY FINITE PART

In this section, let  $R$  be an  $\mathcal{A}_N$ -algebra where  $\mathcal{A}_N = \mathcal{A}[v^{\pm 1/N}]$ .

### 8.1. Ad-invariant bilinear form and the induced embedding.

Let us recall the pairing  $\langle \cdot, \cdot \rangle' : U_q^{ev}(\mathfrak{g}) \times \dot{U}_q(\mathfrak{g}) \rightarrow R$  given by

$$(8.1) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), (\dot{y} K^{\kappa(\dot{\nu})}) K^{\dot{\lambda}} \prod_{i=1}^r \binom{K_i; 0}{s_i} (\dot{x} K^{\gamma(\dot{\mu})}) \right\rangle' = \\ (y, \dot{x})' \cdot (\dot{y}, x)' \cdot q^{-\frac{(\lambda, \dot{\lambda})}{2} + (2\rho, \nu)} \cdot \prod_{i=1}^r \binom{-\frac{(\alpha_i^\vee, \lambda)}{2}}{s_i}_{q_i}$$

for  $y \in U_{-\nu}^{ev<} , x \in U_\mu^{ev>} , \dot{y} \in \dot{U}_{-\dot{\nu}}^<, \dot{x} \in \dot{U}_{\dot{\mu}}^>, \nu, \mu, \dot{\nu}, \dot{\mu} \in Q_+$ , and  $\lambda, \dot{\lambda} \in 2P$ . Proposition 3.19. Evoking the characters  $\hat{\chi}_\lambda : \dot{U}_q^0 \rightarrow R$  of (3.28), the above pairing can be written as:

$$(8.2) \quad \left\langle (y K^{\kappa(\nu)}) K^\lambda (x K^{\gamma(\mu)}), (\dot{y} K^{\kappa(\dot{\nu})}) u_0 (\dot{x} K^{\gamma(\dot{\mu})}) \right\rangle' = (y, \dot{x})' \cdot (\dot{y}, x)' \cdot \hat{\chi}_{-\lambda/2}(u_0) \cdot q^{(2\rho, \nu)},$$

for  $x, y, \dot{x}, \dot{y}$  as above and  $u_0 \in \dot{U}_q^0$ . This pairing induces an embedding of  $\dot{U}_q(\mathfrak{g})$ -modules:

$$(8.3) \quad \iota : U_q^{ev}(\mathfrak{g}) \hookrightarrow \left( \dot{U}_q(\mathfrak{g}) \right)^* := \text{Hom}_R(\dot{U}_q(\mathfrak{g}), R).$$

**Lemma 8.1.** (a) Let  $\phi : \dot{U}_{-\nu}^< \times \dot{U}_\mu^> \rightarrow R$  be a bilinear homomorphism and  $\lambda \in P$ . Then there is a unique element  $u \in (U_{-\mu}^{ev<} K^{\kappa(\mu)}) K^{2\lambda} (U_\nu^{ev>} K^{\gamma(\nu)})$  such that

$$(8.4) \quad \langle u, (\dot{y} K^{\kappa(\nu)}) u_0 (\dot{x} K^{\gamma(\mu)}) \rangle' = \phi(\dot{y}, \dot{x}) \hat{\chi}_{-\lambda}(u_0) \quad \forall \dot{x} \in \dot{U}_\mu^>, \dot{y} \in \dot{U}_{-\nu}^<, u_0 \in \dot{U}_q^0.$$

(b) The image of  $\iota$  from (8.3) contains  $R[\dot{U}_q(\mathfrak{g})]$ .

*Proof.* (a) Let us consider the pairings  $\langle \cdot, \cdot \rangle' : U_{-\mu}^{ev<} \times \dot{U}_\mu^> \rightarrow R$  and  $\langle \cdot, \cdot \rangle' : \dot{U}_{-\nu}^< \times U_\nu^{ev>} \rightarrow R$ . Following the proof of Theorem 3.14, we can choose a basis  $u_1^\mu, \dots, u_{r(\mu)}^\mu$  for  $\dot{U}_\mu^>$  and the dual basis  $v_1^{-\mu}, \dots, v_{r(\mu)}^{-\mu}$  of  $U_{-\mu}^{ev<}$ . Similarly, we can choose a basis  $u_1^{-\nu}, \dots, u_{r(\nu)}^{-\nu}$  for  $\dot{U}_{-\nu}^<$  and the dual basis  $v_1^\nu, \dots, v_{r(\nu)}^\nu$  of  $U_\nu^{ev>}$ . Then the element

$$u = \sum \phi(u_j^{-\nu}, u_i^\mu) q^{-(2\rho, \nu)} (v_i^{-\mu} K^{\kappa(\mu)}) K^{2\lambda} (v_j^\nu K^{\gamma(\nu)})$$

satisfies (8.4). The uniqueness of  $u$  is due to the non-degeneracy of  $\langle \cdot, \cdot \rangle'$  in the first argument by Proposition 3.19.

(b) Let  $V$  be a representation in  $\text{Rep}(\dot{U}_q(\mathfrak{g}))$  such that  $V$  is finitely generated over  $R$ . Let  $m \in V$  be a homogeneous element of weight  $\lambda$  and  $f \in V^* := \text{Hom}_R(V, R)$  be a homogeneous element of weight  $\lambda'$ . It suffices to show that  $c_{f,m}$  is contained in the image of  $\iota$  since  $R[\dot{U}_q(\mathfrak{g})]$  is spanned by these matrix coefficients over  $R$ . We note that  $c_{f,m}(\dot{U}_{-\nu}^< \dot{U}_q^0 \dot{U}_\mu^>) \neq 0$  for only finitely many pairs  $(\nu, \mu) \in Q_+ \times Q_+$ . For any such  $\nu, \mu$  and all  $\dot{x} \in \dot{U}_\mu^>, \dot{y} \in \dot{U}_{-\nu}^<, u_0 \in \dot{U}_q^0$ , we have

$$c_{f,m}((\dot{y} K^{\kappa(\nu)}) u_0 (\dot{x} K^{\gamma(\mu)})) = f((\dot{y} K^{\kappa(\nu)}) u_0 (\dot{x} K^{\gamma(\mu)}) m) = \hat{\chi}_{\mu+\lambda}(u_0) f((\dot{y} K^{\kappa(\nu)}) (\dot{x} K^{\gamma(\mu)}) m).$$

By part (a), there is  $u_{\mu\nu} \in U_{-\mu}^{ev<} U_\nu^{ev,0} U_\nu^{ev>}$  such that  $\langle u_{\mu\nu}, v \rangle' = c_{f,m}(v)$  for all  $v \in \dot{U}_{-\nu}^< \dot{U}_q^0 \dot{U}_\mu^>$ . Hence, the element  $u = \sum_{\mu, \nu} u_{\mu\nu}$  (where all but finitely many  $u_{\mu\nu}$  are 0) satisfies  $\iota(u) = c_{f,m}$ .  $\square$

For any  $\lambda \in P_+$ , we recall the Weyl module  $W_q(\lambda)$  from Section 6. By Lemma 6.12,  $W_q(\lambda)$  is free of finite rank over  $R$  and has a basis consisting of weight vectors. Let  $v_\lambda$  be a non-zero vector spanning the highest weight space  $W_q(\lambda)_\lambda$ . Then we can find a weight vector  $v_\lambda^* \in W_q(\lambda)^*$  such that  $v_\lambda^*(v_\lambda) = 1$  and  $v_\lambda^*(v_i) = 0$  for other weight vectors in the  $R$ -basis of  $W_q(\lambda)$ .

**Lemma 8.2.** *We have  $\iota(K^{-2\lambda}) = c_{v_\lambda^*, v_\lambda}$ .*

*Proof.* The pairing  $\langle K^\lambda, u \rangle'$  equals to  $\hat{\chi}_{-\lambda/2}(u)$  for any  $u \in \dot{U}_q^0$  and vanishes for all elements  $u \in \bigoplus_{(\nu, \mu) \in Q_+ \times Q_+ \setminus (0, 0)} \dot{U}_{-\nu}^0 \dot{U}_q^0 \dot{U}_\mu^>$ . On the other hand,  $c_{v_\lambda^*, v_\lambda}(u)$  equals  $\hat{\chi}_\lambda(u)$  for any  $u \in \dot{U}_q^0$  and vanishes for all  $u \in \bigoplus_{(\nu, \mu) \in Q_+ \times Q_+ \setminus (0, 0)} \dot{U}_{-\nu}^0 \dot{U}_q^0 \dot{U}_\mu^>$ . This immediately implies the result.  $\square$

Thus, we obtain the embedding of left  $\dot{U}_q(\mathfrak{g})$ -modules:

$$\hat{\iota} : R[\dot{U}_q(\mathfrak{g})] \hookrightarrow U_q^{ev}(\mathfrak{g}).$$

such that  $\hat{\iota}(c_{v_\lambda^*, v_\lambda}) = K^{-2\lambda}$ . Here we equip  $R[\dot{U}_q(\mathfrak{g})]$  with a left  $\dot{U}_q(\mathfrak{g})$ -module structure via  $(xf)(y) = f(\text{ad}'_l(S'(x))y)$  for all  $x, y \in \dot{U}_q(\mathfrak{g})$  and  $f \in R[\dot{U}_q(\mathfrak{g})]$ . Let us recall the algebra  $O_q[G]$  in Section 7.1.

**Lemma 8.3.** *The embedding  $\hat{\iota} : O_q[G] \hookrightarrow U_q^{ev}(\mathfrak{g})$  is a homomorphism of  $\dot{U}_q(\mathfrak{g})$ -module algebras.*

*Proof.* Since  $\hat{\iota}$  is a homomorphism of  $\dot{U}_q(\mathfrak{g})$ -modules, it is enough to show that  $\hat{\iota}$  is an algebra homomorphism. In the rest of this proof, we shall replace the subscript  $q$  by the base ring that  $q$  is contained in. Let  $\mathbb{K} = \mathbb{Q}(v^{1/\mathbb{N}})$ . Consider the following diagram

$$\begin{array}{ccc} O_{\mathbb{K}}[G] & \xrightarrow{\hat{\iota}_{\mathbb{K}}} & U_{\mathbb{K}}^{ev}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ O_{\mathcal{A}_N}[G] & \xrightarrow{\hat{\iota}_{\mathcal{A}_N}} & U_{\mathcal{A}_N}^{ev}(\mathfrak{g}) \\ \downarrow & & \downarrow \\ O_R[G] & \xrightarrow{\hat{\iota}_R} & U_R^{ev}(\mathfrak{g}) \end{array}$$

The lowest horizontal map is obtained from the middle map by base change. The two upper vertical arrows are inclusions since  $O_{\mathcal{A}_N}[G]$  and  $U_{\mathcal{A}_N}^{ev}(\mathfrak{g})$  are torsion free over  $\mathcal{A}_N$  by Lemma 6.30 and Lemma 2.10. Therefore, it is enough to show  $\hat{\iota}_{\mathbb{K}} : O_{\mathbb{K}}[G] \rightarrow U_{\mathbb{K}}^{ev}(\mathfrak{g})$  is an algebra homomorphism.

Let us recall the homomorphism of  $\dot{U}_{\mathbb{K}}(\mathfrak{g})$ -module algebras  $l_r : O_{\mathbb{K}}[G] \rightarrow \dot{U}_{\mathbb{K}}(\mathfrak{g})$  in (7.7). On the other hand, we consider the morphism  $\hat{\iota}_{\mathbb{K}} : O_{\mathbb{K}}[G] \rightarrow U_{\mathbb{K}}^{ev}(\mathfrak{g}) \cong \dot{U}_{\mathbb{K}}(\mathfrak{g})$ . Both  $\hat{\iota}_{\mathbb{K}}$  and  $l_r$  are morphisms of  $\dot{U}_q(\mathfrak{g})$ -modules such that  $\hat{\iota}_{\mathbb{K}}(c_{v_\lambda^*, v_\lambda}) = l_r(c_{v_\lambda^*, v_\lambda}) = K^{-2\lambda}$ , see Lemma 7.13 and Lemma 8.2. Moreover,  $O_{\mathbb{K}}[G]$  is generated by  $c_{v_\lambda^*, v_\lambda}$  over  $\dot{U}_{\mathbb{K}}(\mathfrak{g})$  by Lemma 7.14. Therefore,  $l_r$  and  $\hat{\iota}_{\mathbb{K}}$  coincide. This implies that  $\hat{\iota}_{\mathbb{K}}$  is an algebra homomorphism.  $\square$

## 8.2. The locally finite part $U_q^{fin}$ .

Let us define

$$U_q^{fin} := \left\{ a \in U_q^{ev}(\mathfrak{g}) \mid \text{ad}'(\dot{U}_q(\mathfrak{g}))(a) \text{ is a finitely generated } R\text{-module} \right\}$$

Since  $U_q^{ev}(\mathfrak{g})$  is a weight module,  $U_q^{fin}$  is also the maximal rational subrepresentation of  $U_q^{ev}(\mathfrak{g})$  and then  $\hat{\iota}(O_q[G]) \subset U_q^{fin}$ . It is obvious that  $U_q^{fin}$  is an  $R$ -subalgebra of  $U_q^{ev}(\mathfrak{g})$  by (2.3). It is this subalgebra that we call the *locally finite part*. We will show that  $\hat{\iota}(O_q[G]) \xrightarrow{\sim} U_q^{fin}$  for any Noetherian  $\mathcal{A}_N$ -algebra  $R$ , and identify  $O_q[G]$  with a subalgebra of  $U_q^{ev}$  via  $\hat{\iota}$ .

The key result of this section is similar to [JL1, Theorem 6.4].

**Proposition 8.4.** (a)  $K^{2\beta} \in U_q^{fin}$  iff  $\beta \in -P_+$ .

(b)  $U_q^{ev}(\mathfrak{g})$  is obtained from  $U_q^{fin}$  (resp.  $O_q[G]$ ) by localizing the elements  $\{K^{-2\omega_i}\}_{i=1}^r$ .

*Proof.* (a) We claim that  $K^\beta \in U_q^{fin} \cap U_q^{ev,0}(\mathfrak{g})$  if and only if  $\frac{(\alpha_i, \beta)}{2d_i} \in \mathbb{Z}_{\leq 0}$  for all  $i$ . The if part follows since  $\hat{i}(c_{v_\lambda^*, v_\lambda}) = K^{-2\lambda}$  for all  $\lambda \in P_+$  and  $O_q[G]$  is a rational representation. Let us prove the only if part.

Assuming the contradiction, pick  $\beta \in 2P$  such that  $K^\beta \in U_q^{fin}$  and there is  $i \in \{1, \dots, r\}$  with  $c := \frac{(\alpha_i, \beta)}{2d_i} \in \mathbb{Z}_{>0}$ . According to (2.39), we have

$$(8.5) \quad \text{ad}'(\tilde{E}_i^{(m)})(K^\beta) = \frac{\prod_{s=0}^{m-1} (1 - q_i^{2(c+s)})}{(m)_{q_i}!} \cdot \tilde{E}_i^m K^{\beta + m\zeta_i^>} ,$$

where, as always, the coefficient  $\frac{\prod_{s=0}^{m-1} (1 - q_i^{2(c+s)})}{(m)_{q_i}!}$  denotes the image of  $\frac{\prod_{s=0}^{m-1} (1 - v_i^{2(c+s)})}{(m)_{v_i}!} \in \mathcal{A}$  under the algebra homomorphism  $\sigma : \mathcal{A} \rightarrow R$  of (1.2). We will show that

$$(8.6) \quad \text{there are infinitely many } m \geq 1 \text{ such that } c_m := \frac{\prod_{s=0}^{m-1} (1 - q_i^{2(c+s)})}{(m)_{q_i}} \neq 0 \text{ in } R.$$

This implies that  $\text{ad}'(\dot{U}_q(\mathfrak{g}))(K^\beta)$  contains nonzero multiples of infinitely many elements of  $\{\tilde{E}_i^m K^{\beta + m\zeta_i^>}\}_{m \geq 1}$ . These elements are  $R$ -linearly independent by PBW-bases in Lemma 2.10. Hence,  $\text{ad}'(\dot{U}_q(\mathfrak{g}))(K^\beta)$  is not finitely generated over  $R$ , which leads to contradiction.

Let  $\mathfrak{m}$  be any maximal ideal of  $R$  and let  $\mathbb{F} = R/\mathfrak{m}$ . Let  $\bar{q}$  be the image of  $q$  in  $\mathbb{F}$  then it is enough to show (8.6) when we replace  $(R, q)$  with  $(\mathbb{F}, \bar{q})$ . But we have

$$c_m = (-1)^m (1 - \bar{q}_i^{-2})^m \bar{q}_i^{(2c+m-1)m} \binom{c+m-1}{m}_{\bar{q}_i} .$$

From this one can see that there are infinitely many  $m \geq 1$  such that  $c_m \neq 0$  in  $\mathbb{F}$  regardless of characteristics of  $\mathbb{F}$  and whether  $\bar{q}$  is a root of unity in  $\mathbb{F}$  or not.

(b) Since  $K^{-2\omega_i} \in U_q^{fin}$  and  $K^{-2\omega_i} \in O_q[G]$ , we get

$$\begin{aligned} (1 - q_i^{-2}) \tilde{E}_i K^{\zeta_i^>} K^{-2\omega_i} &= \text{ad}'(\tilde{E}_i)(K^{-2\omega_i}) \in U_q^{fin}, \quad O_q[G], \\ (1 - q_i^{-2}) \tilde{F}_i K^{-\zeta_i^<} K^{-2\omega_i} &= \text{ad}'(\tilde{F}_i)(K^{-2\omega_i}) \in U_q^{fin}, \quad O_q[G]. \end{aligned}$$

As  $1 - q_i^{-2} \in R$  is assumed to be invertible, we then obtain

$$\tilde{E}_i K^{\zeta_i^>} , \tilde{F}_i K^{-\zeta_i^<} \in U_q^{fin} K^{2\omega_i}, \quad O_q[G] K^{2\omega_i} \quad \forall 1 \leq i \leq r.$$

This completes our proof since  $U_q^{ev}(\mathfrak{g})$  is generated by  $\{\tilde{E}_i K^{\zeta_i^>} , \tilde{F}_i K^{-\zeta_i^<} , K^{\pm 2\omega_i}\}_{i=1}^r$ .  $\square$

### 8.3. $O_\epsilon[G] \cong U_\epsilon^{fin}$ when $R = \mathbb{F}$ a field and $q$ is not a root of unity.

The content of this section follows [JL2]. For reader convenience, we sketch main ideas in [JL2] along with some modifications accounting for the twists in this paper.

Let  $R = \mathbb{F}$  be a field and  $q \in \mathbb{F}$  be not a root of unity. In this case, we have  $U_q^{ev}(\mathfrak{g}) \cong \dot{U}_q(\mathfrak{g})$ . The category  $\text{Rep}(\dot{U}_q(\mathfrak{g})) \cong \text{Rep}(\dot{\mathcal{U}}_q(\mathfrak{g}))$  is semisimple, see [APW, Corollary 7.7]. Combining this observation with the twisted pairing (3.37) and arguments in [J1, Section 6], we obtain

**Proposition 8.5.** Let  $Z$  be the center of  $U_\epsilon^{ev}(\mathfrak{g}) \cong \dot{U}_q(\mathfrak{g})$ . The Harish Chandra homomorphism (for the construction, see [J1, §6.2] or Section 8.7 below) gives an isomorphism  $Z \cong \mathbb{F}[K^{2\lambda}]_{\lambda \in P}^{W_\bullet}$ .

Let us define the category  $O$  for  $\dot{U}_q(\mathfrak{g})$ . The objects in  $O$  are  $\dot{U}_q(\mathfrak{g})$ -modules  $M$  with weight decompositions  $M = \bigoplus_\lambda M_\lambda$  such that

- (i) The set  $\{\lambda \mid M_\lambda \neq \{0\}\}$  is bounded from above with respect to the usual order on the weight lattice and
- (ii)  $\dim_{\mathbb{F}} M_\lambda < \infty$  for all  $\lambda$ .

The morphisms in the category  $O$  are homomorphisms of  $\dot{U}_q(\mathfrak{g})$ -modules. The *Verma module*  $M_q(\lambda) = \dot{U}_q(\mathfrak{g}) \otimes_{\dot{U}_q^{\geqslant}} \mathbb{F}_\lambda$  is an object of  $O$ .

We want to introduce a duality  $\tau$  on the category  $O$ . Let us consider the inclusion  $\dot{U}_q(\mathfrak{g}) \subset \dot{U}_q(\mathfrak{g}, P/2)$ . Then each object in  $O$  is naturally a module over  $\dot{U}_q(\mathfrak{g}, P/2)$ . On  $\dot{U}_q(\mathfrak{g}, P/2)$ , we have the anti-automorphism  $\tau$  defined by

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K^\lambda) = K^\lambda \quad \text{for } \lambda \in P/2.$$

For any  $M \in O$ , set  $M^\tau := \bigoplus_\lambda (M_\lambda)^* \subset \text{Hom}_{\mathbb{F}}(M, \mathbb{F})$ , as vector spaces. The  $\dot{U}_q(\mathfrak{g}, P/2)$ -module structure on  $M^\tau$  is defined by

$$xf(m) = f(\tau(x)m) \quad \text{for } x \in \dot{U}_q(\mathfrak{g}, P/2), f \in M^\tau, m \in M.$$

Then  $M^\tau$  becomes an object in  $O$  by restricting the action to  $\dot{U}_q(\mathfrak{g})$ .

Once we have the result about the center of  $\dot{U}_q(\mathfrak{g})$  in Proposition 8.5, the following is proved by standard arguments [REF]

**Lemma 8.6.** (a) *Each object in  $O$  has finite length.*

(b) *The module  $M_q(\lambda)$  has a unique simple quotient  $L_q(\lambda)$ . The module  $M_q(\lambda)^\tau$  has a unique simple submodule  $L_q(\lambda)$ . The module  $L_q(\lambda)$  is finite dimensional if and only if  $\lambda \in P_+$ .*

We now define a filtration  $\mathcal{F}$  on  $U_q^{ev}(\mathfrak{g})$  which is stable under the adjoint action of  $\dot{U}_q(\mathfrak{g})$  on  $U_q^{ev}(\mathfrak{g})$ . This filtration is labeled by the weight lattice  $P$  with the usual dominance order. On the free associative algebra generated by  $\tilde{E}_i, \tilde{F}_i, K^\lambda$ , we put

$$\deg(K^{2\lambda}) = -\lambda, \quad \deg(\tilde{E}_i) = \zeta_i^>/2, \quad \deg(\tilde{F}_i) = -\zeta_i^</2.$$

This gives a filtration labeled by the weight lattice  $P$  with the dominant order. This induces the filtration  $\mathcal{F}$  on  $U_q^{ev}(\mathfrak{g})$ . Recall the adjoint action of  $\dot{U}_q(\mathfrak{g})$  on  $U_q^{ev}(\mathfrak{g})$ :

$$\text{ad}'(K^\mu)(x) = K^\mu x K^{-\mu}, \quad \text{ad}'(\tilde{E}_i)(x) = [\tilde{E}_i, x] K^{\zeta_i^>}, \quad \text{ad}'(\tilde{F}_i)(x) = [\tilde{F}_i, x] K^{-\zeta_i^<}.$$

We see that the  $\dot{U}_q(\mathfrak{g})$ -action preserves the filtration  $\mathcal{F}$  on  $U_q^{ev}(\mathfrak{g})$ .

Let  $G^-$  and  $G^+$  be the subalgebras of  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$  generated by  $\{\tilde{F}_i K^{-\zeta_i^<}\}$  and  $\{\tilde{E}_i K^{\zeta_i^>}\}$ , respectively. Then

$$\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g})) = \bigoplus_{\lambda} G(\lambda), \quad \text{here } G(\lambda) = G^- \otimes_{\mathbb{F}} \mathbb{F} K^{-2\lambda} \otimes_{\mathbb{F}} G^+.$$

We have  $G(\lambda)G(\mu) \subset G(\lambda + \mu)$  and  $G^-, G^+ \subset G(0)$ . The isomorphism  $G^- \otimes_{\mathbb{F}} \mathbb{F} K^{-2\lambda} \otimes_{\mathbb{F}} G^+ \rightarrow G(\lambda)$  is given by the multiplication map (we view  $G^-, G^+$  as subsets of  $G(0)$  and  $K^{-2\lambda}$  as an element in  $G(\lambda)$ ). The action of  $U_q(\mathfrak{g})$  on  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$  satisfies

$$(8.7) \quad \text{ad}'(x)(ab) = \text{ad}'(x_{(1)})(a) \text{ad}'(x_{(2)})(b),$$

for all  $x \in \dot{U}_q(\mathfrak{g})$  and  $a, b \in \text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ . Furthermore, the  $Q$ -grading on  $U_q^{ev}(\mathfrak{g})$  induces  $Q$ -gradings on  $G(\lambda), G^-, G^+$ .

The first observation in [JL2] is that  $G^-, G^+$  are closed under the action of  $\dot{U}_q(\mathfrak{g})$  on  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ .

**Lemma 8.7.**  $G^-, G^+$  are  $\dot{U}_q(\mathfrak{g})$ -submodules of  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ .

*Proof.* Let us prove this in the case of  $G^-$ . The actions of the elements  $K^\lambda, \tilde{F}_i \in \dot{U}_q(\mathfrak{g})$  obviously preserve  $G^-$ . On the other hand, in  $U_q^{ev}(\mathfrak{g})$ , we have

$$\text{ad}'(\tilde{E}_i)(\tilde{F}_j K^{-\zeta_i^<}) = 0, \quad \text{ad}'(\tilde{E}_i)(\tilde{F}_i K^{-\zeta_i^<}) = \frac{q_i}{1 - q_i^{-2}}(K^{2\alpha_i} - 1),$$

for  $i \neq j$ . Hence in  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ , we have

$$\text{ad}'(\tilde{E}_i)(\tilde{F}_j K^{-\zeta_i^<}) = -\delta_{ij} \frac{q_i}{1 - q_i^{-2}}.$$

Combining this with (8.7), we can see that  $G^-$  is a  $\dot{U}_q(\mathfrak{g})$ -submodule of  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ .  $\square$

Let  $G^{fin}(\lambda) := \{m \in G(\lambda) \mid \dim \text{ad}'(\dot{U}_q(\mathfrak{g}))m < \infty\}$ . Here is another important result from [JL2].

**Lemma 8.8.**  $G^{fin}(\lambda) = 0$  unless  $\lambda \in P^+$ . When  $\lambda \in P^+$  then  $G^{fin}(\lambda) = \text{ad}'(\dot{U}_q(\mathfrak{g}))(K^{-2\lambda})$ .

*Proof.* Step 1. Here is the key observation:  $G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda}$  can be equipped with a  $\dot{U}_q(\mathfrak{g})$ -module structure as follows:

$$\begin{aligned} \tilde{E}_i(u_- \otimes K^{-2\lambda}) &= q^{-(\lambda, \zeta_i^>)} \text{ad}'(\tilde{E}_i)(u_-) \otimes K^{-2\lambda} \\ K^\mu(u_- \otimes K^{-2\lambda}) &= q^{(\lambda, \mu)} \text{ad}'(K^\mu)(u_- \otimes K^{-2\lambda}) \\ \tilde{F}_i(u_- \otimes K^{-2\lambda}) &= q^{(\lambda, \zeta_i^<)} \text{ad}'(\tilde{F}_i)(u_- \otimes K^{-2\lambda}), \end{aligned}$$

for  $u_- \in G^-$ , where we use that  $G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda}$  is a  $\dot{U}_q^<$ -submodule and  $G^-$  is a  $\dot{U}_q(\mathfrak{g})$ -submodule of  $\text{gr}_{\mathcal{F}}(U_q^{ev}(\mathfrak{g}))$ . Then we see that

$$(8.8) \quad \{m \in G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda} \mid \dim \text{ad}'(\dot{U}_q^<)(m) < \infty\} = \{m \in G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda} \mid \dim \dot{U}_q(\mathfrak{g})(m) < \infty\}$$

Step 2. Arguing as in [JL2, §4.8], where we use Lemma 8.6, we have  $G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda} \cong M_q(\lambda)^\tau$ . Combining this with (8.8), we have

$$L(\lambda)^- := \{m \in G^- \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda} \mid \dim \text{ad}'(\dot{U}^<)(m) < \infty\} = \begin{cases} 0 & \text{if } \lambda \text{ is not dominant,} \\ \text{ad}'(\dot{U}^<)(K^{-2\lambda}) & \text{if } \lambda \text{ is dominant.} \end{cases}$$

If  $\lambda$  is dominant, then  $L(\lambda)^- = K(\lambda)^- \otimes \mathbb{F}K^{-2\lambda}$  for some  $K(\lambda)^- \subset G^-$ . By (8.8),  $L(\lambda)^-$  is a  $\dot{U}_q(\mathfrak{g})$ -submodule of  $G^- \otimes \mathbb{F}K^{-2\lambda}$ . By looking at the action of  $\tilde{E}_i$  on  $G^- \otimes \mathbb{F}K^{-2\lambda}$  defined in Step 1, it follows that  $K(\lambda)^-$  is a  $\dot{U}_q^>$ -submodule of  $G^-$ .

Step 3. Similarly, we have

$$L(\lambda)^+ := \{m \in G^+ \otimes_{\mathbb{F}} \mathbb{F}K^{-2\lambda} \mid \dim \text{ad}'(\dot{U}^>)(m) < \infty\} = \begin{cases} 0 & \text{if } \lambda \text{ is not dominant,} \\ \text{ad}'(\dot{U}^>)(K^{-2\lambda}) & \text{if } \lambda \text{ is dominant.} \end{cases}$$

When  $\lambda$  is dominant,  $L(\lambda)^+ = K(\lambda)^+ \otimes \mathbb{F}K^{-2\lambda}$  for some  $\dot{U}_q^<$ -submodule  $K(\lambda)^+$  of  $G^+$ .

Step 4. Using Steps 2 and 3, we proceed exactly the same as in [JL2, §4.9] to finish the proof of the lemma.  $\square$

**Corollary 8.9.**  $U_q^{fin}$  is spanned by  $\{K^{-2\lambda}\}_{\lambda \in P_+}$  as a  $\dot{U}_q(\mathfrak{g})$ -module.

**Proposition 8.10.** The inclusion  $O_q[G] \hookrightarrow U_q^{fin}$  is an isomorphism of  $\dot{U}_q(\mathfrak{g})$ -module algebras.

*Proof.* It is enough to show that this is an epimorphism of  $\dot{U}_q(\mathfrak{g})$ -modules. Lemma 8.2 implies that  $K^{-2\lambda}, \lambda \in P_+$  is contained in the image. Therefore, by Corollary 8.9, the inclusion  $O_q[G] \hookrightarrow U_q^{fin}$  is surjective.  $\square$

#### 8.4. $O_q[G] \cong U_q^{fin}$ when $R = \mathbb{F}$ a field and $q = \epsilon$ is a root of unity.

Let us first assume that  $\mathbb{F}$  is algebraically closed so that  $\mathbb{F}[G^d]$  is spanned by matrix coefficients of  $\text{Rep}^{fd}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d))$ . The condition is also used in Proposition 8.13 below.

Recall the Hopf algebra  $\dot{U}_q$  from Section 7. Similarly, let  $\dot{U}_q^*$  be the Hopf algebra obtained from  $\dot{U}_q^*(\mathfrak{g})$  by taking the opposite of both product and coproduct structures. Then the category of rational representations  $\text{Rep}(\dot{U}_q^*)$  is defined in an obvious way. Let  $\text{Rep}^{fd}(\dot{U}_q)$ ,  $\text{Rep}^{fd}(\dot{U}_q^*)$  be the full subcategories of  $\text{Rep}(\dot{U}_q)$ ,  $\text{Rep}(\dot{U}_q^*)$  consisting of finite dimensional representations. We then can define the algebra  $O_\epsilon[G^d]$  for  $\dot{U}_\epsilon^*(\mathfrak{g})$  as in Section 7.

The Hopf algebra homomorphism  $\tilde{\text{Fr}} : \dot{U}_\epsilon(\mathfrak{g}, P) \rightarrow \dot{U}_\epsilon^*(\mathfrak{g}, P^*)$  from (4.21) gives rise to braided monoidal fully faithful functor

$$\tilde{\text{Fr}}^* : \text{Rep}(\dot{U}_q^*) \rightarrow \text{Rep}(\dot{U}_q).$$

By using the categorical realization of the REA from Section 7.1.1, we obtain an inclusion

$$O_\epsilon[G^d] \hookrightarrow O_\epsilon[G].$$

The categorical realization

$$O_\epsilon[G^d] \cong \bigsqcup_{X \in \text{Rep}^{fd}(\dot{U}_q^*)} X^* \otimes X / \sim$$

also tells us that  $O_\epsilon[G^d]$  is a  $\dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $O_\epsilon[G]$ . Furthermore, the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $O_\epsilon[G^d]$  factors through  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$ : the elements  $\tilde{E}_i, \tilde{F}_i$  act by zero; the weights of  $O_\epsilon[G^d]$  are contained in  $Q^*$ , hence  $K^\lambda$  acts as the identity on  $O_\epsilon[G^d]$  for all  $\lambda \in 2P$ .

So we have the composition of embedding

$$\varphi : O_\epsilon[G^d] \hookrightarrow O_\epsilon[G] \xrightarrow{\hat{i}} U_\epsilon^{ev}(\mathfrak{g}),$$

where  $\hat{i}$  comes from Lemma 8.3. Our next result describes the image of  $O_\epsilon[G^d]$  under  $\varphi$ . To this end, we recall  $Z_{Fr}^{fin}$  in (5.11):

$$Z_{Fr}^{fin} = \{z \in Z_{Fr} \mid \dim_{\mathbb{F}}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)z) < \infty\}.$$

**Lemma 8.11.** *Evoking the algebra homomorphism  $Z_{Fr} \cong \mathbb{F}[G_0^d]$  from Proposition 5.10, we have  $\varphi(O_\epsilon[G^d]) = Z_{Fr}^{fin} \cong \mathbb{F}[G^d]$ .*

*Proof.* The  $\mathbb{F}$ -vector space  $O_\epsilon[G^d]$  is spanned by the matrix coefficients of the  $\dot{U}_\epsilon(\mathfrak{g}, P)$ -modules that are objects in  $\text{Rep}^{fd}(\dot{U}_\epsilon^*)$ . These matrix coefficients vanish on the kernel of  $\tilde{\text{Fr}} : \dot{U}_\epsilon(\mathfrak{g}, P) \rightarrow \dot{U}_\epsilon^*(\mathfrak{g}, P^*)$ . By definition of  $Z_\epsilon^{ev}$ , we have  $\varphi(O_\epsilon[G^d]) \subset Z_\epsilon^{ev}$ .

Recall homomorphisms (5.6):

$$(8.9) \quad \dot{U}_{\mathbb{F}}(\mathfrak{g}^d) \xrightarrow{\phi^{-1}} A_{Q^*} \hookrightarrow \tilde{U}_\epsilon^*(\mathfrak{g}, P^*) \hookleftarrow \dot{U}_\epsilon^*(\mathfrak{g}, P^*).$$

All representations in  $\text{Rep}(\dot{U}_\epsilon^*)$  and  $\text{Rep}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d))$  can be naturally extended to representations of  $\tilde{U}_\epsilon^*(\mathfrak{g}, P^*)$ . By taking these extensions and then restricting the actions to the other algebras, we obtain the following functors:

$$(8.10) \quad \mathfrak{T} : \text{Rep}(\dot{U}_\epsilon^*(\mathfrak{g})) \rightarrow \text{Rep}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)), \quad \mathfrak{T}' : \text{Rep}(\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)) \rightarrow \text{Rep}(\dot{U}_\epsilon^*(\mathfrak{g})),$$

These functors are mutually quasi-inverse. In particular,  $\mathfrak{T}$  is an equivalence of categories.

By analyzing the construction of  $Z_{Fr} \cong \mathbb{F}[G_0^d]$ , one sees that  $\varphi(O_\epsilon[G^d]) \subset Z_\epsilon^{ev} \cong \mathbb{F}[G_0^d]$  is spanned by the matrix coefficients of the representations in  $\text{Rep}^{fd}(\dot{U}_\epsilon^*(\mathfrak{g}))$  viewed as modules over  $\dot{U}_{\mathbb{F}}(\mathfrak{g}^d)$  via the functor  $\mathfrak{T}$ . However,  $\mathfrak{T}$  is an equivalence of categories. Therefore,

$\varphi(O_\epsilon[G^d]) \subset \mathbb{F}[G_0^d]$  contains all matrix coefficients of the representations in  $\text{Rep}^{fd}(\dot{U}_\mathbb{F}(\mathfrak{g}^d))$ . These matrix coefficients span  $\mathbb{F}[G^d]$ . Hence  $\varphi(O_\epsilon[G^d]) = \mathbb{F}[G^d]$   $\square$

**Remark 8.12.** By Lemma 8.11,  $O_\epsilon[G^d]$  is central in  $O_\epsilon[G]$ , so the left and right actions of  $O_\epsilon[G^d]$  on  $O_\epsilon[G]$  coincide.

The following technical result will be established in Section 8.5:

**Proposition 8.13.**  $O_\epsilon[G]$  is a finitely generated projective left and right  $O_\epsilon[G^d]$ -module.

Recall that  $\dot{U}_\epsilon(\mathfrak{g})$  acts on  $O_\epsilon[G^d]$  making  $O_\epsilon[G^d]$  into a  $\dot{U}_\epsilon(\mathfrak{g})$ -module algebra. Thus we can form the following categories:  $O_\epsilon[G^d]\text{-Lmod}^{\dot{U}_\epsilon}$ ,  $O_\epsilon[G^d]\text{-Rmod}^{\dot{U}_\epsilon}$ ,  $O_\epsilon[G^d]\text{-Bimod}^{\dot{U}_\epsilon}$  of  $\dot{U}_\epsilon(\mathfrak{g})$ -equivariant left modules, right modules, and bimodules over  $O_\epsilon[G^d]$ , respectively. We refer the reader to Appendix A for details.

**Definition 8.14.** Let  $O_\epsilon[G^d]\text{-bimod}^{G_\epsilon}$  be the full subcategory of the category  $O_\epsilon[G^d]\text{-Bimod}^{\dot{U}_\epsilon}$  consisting of all objects which are finitely generated as left and right  $O_\epsilon[G^d]$ -modules, and rational as  $\dot{U}_\epsilon(\mathfrak{g})$ -representations. The categories  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$  and  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$  are defined similarly.

Since  $O_\epsilon[G^d] \cong \mathbb{F}[G^d]$  is Noetherian and the category  $\text{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$  is abelian, all three categories  $O_\epsilon[G^d]\text{-bimod}^{G_\epsilon}$ ,  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ ,  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$  are abelian. We note that  $O_\epsilon[G]$  belongs to  $O_\epsilon[G^d]\text{-bimod}^{G_\epsilon}$ , thanks to Proposition 8.13.

For any  $\dot{U}_\epsilon(\mathfrak{g})$ -representation  $M$ , let

$$(8.11) \quad M^{fin} := \{m \in M \mid \dim_{\mathbb{F}}(\dot{U}_\epsilon(\mathfrak{g})m) < \infty\}.$$

Since the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $O_\epsilon[G^d]$  is locally finite as  $O_\epsilon[G^d]$  is a rational  $\dot{U}_\epsilon(\mathfrak{g})$ -representation, for any  $M \in O_\epsilon[G^d]\text{-Rmod}^{\dot{U}_\epsilon}$ , the locally finite part  $M^{fin}$  is an object in  $O_\epsilon[G^d]\text{-Rmod}^{\dot{U}_\epsilon}$ . Note that the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $M^{fin}$  is not required to be rational.

**Lemma 8.15.** Let  $M \in O_\epsilon[G^d]\text{-Rmod}^{\dot{U}_\epsilon}$  and  $P \in O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ , where  $P$  is a finitely generated projective left  $O_\epsilon[G^d]$ -module. Then the natural map

$$(8.12) \quad M^{fin} \otimes_{O_\epsilon[G^d]} P \rightarrow M \otimes_{O_\epsilon[G^d]} P$$

is injective with image  $(M \otimes_{O_\epsilon[G^d]} P)^{fin}$ .

*Proof.* Since  $P$  is a rational  $\dot{U}_\epsilon(\mathfrak{g})$ -representation, the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $P$  is locally finite. Therefore, the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $M^{fin} \otimes_{O_\epsilon[G^d]} P$  is locally finite and the image of (8.12) is contained in  $(M \otimes_{O_\epsilon[G^d]} P)^{fin}$ .

It suffices to show that the induced map

$$(8.13) \quad \text{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M^{fin} \otimes_{O_\epsilon[G^d]} P) \rightarrow \text{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M \otimes_{O_\epsilon[G^d]} P)$$

is bijective for any finite dimensional  $\dot{U}_\epsilon(\mathfrak{g})$ -module  $V$ .

Let  $Q = \text{Hom}_{O_\epsilon[G^*]\text{-Lmod}}(P, O_\epsilon[G^*])$ . Since  $P$  is a finitely generated projective left  $O_\epsilon[G^*]$ -module,  $Q$  is a finitely generated projective right  $O_\epsilon[G^*]$ -module and we have an isomorphism  $P \simeq \text{Hom}_{O_\epsilon[G^*]\text{-Rmod}}(Q, O_\epsilon[G^*])$  in the category  $O_\epsilon[G^*]\text{-lmod}^{G_\epsilon}$  by Lemma A.7. Using the tensor-hom adjunction of Lemma A.7, we have:

$$(8.14) \quad \begin{aligned} \text{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M \otimes_{O_\epsilon[G^*]} P) &= \text{Hom}_{\mathbb{F}}(V, M \otimes_{O_\epsilon[G^*]} P)^{\dot{U}_\epsilon(\mathfrak{g})} \\ &= \text{Hom}_{O_\epsilon[G^*]\text{-Rmod}}(V \otimes_{\mathbb{C}} Q, M)^{\dot{U}_\epsilon(\mathfrak{g})} \\ &= \text{Hom}_{O_\epsilon[G^*]\text{-Rmod}^{\dot{U}_\epsilon(\mathfrak{g})}}(V \otimes_{\mathbb{C}} Q, M). \end{aligned}$$

Since  $Q \in O_\epsilon[G^*]\text{-Rmod}^{G_\epsilon}$ , the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $Q$  is locally finite, and thus we have:

$$\begin{aligned}\mathrm{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M \otimes_{O_\epsilon[G^*]} P) &= \mathrm{Hom}_{O_\epsilon[G^*]\text{-Rmod}^{\dot{U}_\epsilon(\mathfrak{g})}}(V \otimes_{\mathbb{C}} Q, M^{fin}) \\ &= \mathrm{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M^{fin} \otimes_{O_\epsilon[G^*]} P).\end{aligned}$$

This completes the proof.  $\square$

We are now ready to prove the main result of this section

**Proposition 8.16.** *The inclusion  $\hat{i} : O_\epsilon[G] \hookrightarrow U_\epsilon^{fin}$  is an isomorphism of  $\dot{U}_\epsilon(\mathfrak{g})$ -module algebras*

*Proof.* First, we assume  $\mathbb{F}$  is algebraically closed. According to Lemma 8.11 and Lemma 5.11, we have  $O_\epsilon[G^d][K^{2\lambda_0}] = Z_\epsilon^{ev}$ . By Proposition 8.4,  $O_\epsilon[G][K^{2\lambda_0}] = U_\epsilon^{ev}$ . As  $O_\epsilon[G]$  is finitely generated over  $O_\epsilon[G^d]$  by Proposition 8.13,

$$Z_{Fr} \otimes_{O_\epsilon[G^d]} O_\epsilon[G] \cong O_\epsilon[G^d][K^{2\lambda_0}] \otimes_{O_\epsilon[G^d]} O_\epsilon[G] \cong O_\epsilon[G][K^{2\lambda_0}] = U_\epsilon^{ev}.$$

Since  $O_\epsilon[G] \in O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ , and  $O_\epsilon[G]$  is finitely generated projective over  $O_\epsilon[G^d]$ , and  $Z_{Fr} \in O_\epsilon[G^d]\text{-Rmod}^{\dot{U}_\epsilon}$ , it follows that

$$(Z_{Fr} \otimes_{O_\epsilon[G^d]} O_\epsilon[G])^{fin} \cong Z_{Fr}^{fin} \otimes_{O_\epsilon[G^d]} O_\epsilon[G] \cong O_\epsilon[G^d] \otimes_{O_\epsilon[G^d]} O_\epsilon[G] \cong O_\epsilon[G],$$

thanks to Lemma 8.11 and Lemma 8.15. This implies the isomorphism  $O_\epsilon[G] \cong U_\epsilon^{fin}$  when  $\mathbb{F}$  is algebraically closed.

In general, let  $\bar{\mathbb{F}}$  be an algebraically closed field of  $\mathbb{F}$ . We have the following commutative diagram

$$\begin{array}{ccc} O_{\mathbb{F}}[G] \otimes_{\mathbb{F}} \bar{\mathbb{F}} & \xhookrightarrow{\quad} & U_{\mathbb{F}}^{fin} \otimes_{\mathbb{F}} \bar{\mathbb{F}} \\ \downarrow \cong & & \downarrow \\ O_{\bar{\mathbb{F}}}[G] & \xrightarrow{\cong} & U_{\bar{\mathbb{F}}}^{fin} \end{array}$$

This diagram implies the top horizontal map is bijective. Hence  $O_{\mathbb{F}}[G] \rightarrow U_{\mathbb{F}}^{fin}$  is an isomorphism.  $\square$

### 8.5. Proof of Proposition 8.13.

Here  $\mathbb{F}$  is algebraically closed. Recall the functor

$$\tilde{\mathrm{Fr}}^* : \mathrm{Rep}(\dot{U}_q^*) \rightarrow \mathrm{Rep}(\dot{U}_q).$$

**Lemma 8.17.**  *$O_\epsilon[G]$  is finitely generated over  $O_\epsilon[G^d]$ .*

*Proof.* By Lemma 6.48, there is a finite dimensional module  $V \in \mathrm{Rep}(\dot{U}_q)$  such that the following holds. For any finite dimensional module  $M \in \mathrm{Rep}(\dot{U}_q)$  there is a finite dimensional module  $N \in \mathrm{Rep}(\dot{U}_q^*)$  such that  $M$  is a subquotient of  $\tilde{\mathrm{Fr}}^*(N) \otimes V$ . Therefore,  $O_\epsilon[G]$  is spanned by matrix coefficients of the modules of the form  $\tilde{\mathrm{Fr}}^*(N) \otimes V$  with  $N \in \mathrm{Rep}(\dot{U}_q^*)$ . Equivalently,  $O_\epsilon[G]$  is linearly spanned by the images of the maps

$$(\tilde{\mathrm{Fr}}^*(N) \otimes V)^* \otimes (\tilde{\mathrm{Fr}}^*(N) \otimes V) \rightarrow O_\epsilon[G].$$

Evoking the multiplication in (7.4), we have

$$\begin{array}{ccc} (\tilde{\mathrm{Fr}}^*(N))^* \otimes \tilde{\mathrm{Fr}}^*(N) \otimes V^* \otimes V & \xrightarrow{\cong} & (\tilde{\mathrm{Fr}}^*(N) \otimes V)^* \otimes \tilde{\mathrm{Fr}}^*(N) \otimes V \\ \downarrow & & \downarrow \\ O_\epsilon[G] \otimes O_\epsilon[G] & \xrightarrow{\mathrm{m}} & O_\epsilon[G] \end{array}$$

On the other hand, the image of  $(\tilde{\text{Fr}}(N))^* \otimes \tilde{\text{Fr}}^*(N) \rightarrow O_\epsilon[G]$  is contained in  $O_\epsilon[G^d]$ . From these observations, we see that  $O_\epsilon[G]$  is spanned by the image of  $V^* \otimes V \rightarrow O_\epsilon[G]$  over  $O_\epsilon[G^d]$ . Since  $V$  is finite dimensional,  $O_\epsilon[G]$  is finitely generated over  $O_\epsilon[G^d]$ .  $\square$

To show that  $O_\epsilon[G]$  is projective over  $O_\epsilon[G^d]$ , we need two technical lemmas. Recall the categories  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ ,  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ ,  $O_\epsilon[G^d]\text{-bimod}^{G_\epsilon}$  from Definition 8.14.

**Lemma 8.18.** *For any  $M \in O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ , all Ext-groups  $\text{Ext}_{O_\epsilon[G^d]\text{-rmod}}^i(M, O_\epsilon[G^d])$  are objects of  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ .*

*Proof.* Since  $M$  is finitely generated over  $O_\epsilon[G^d]$ , there is a finite dimensional rational  $\dot{U}_\epsilon(\mathfrak{g})$ -module  $V_1 \subset M$  such that  $V_1$  generates  $M$  over  $O_\epsilon[G^d]$  as a right  $O_\epsilon[G^d]$ -module. Hence we have a surjective morphism  $V_1 \otimes O_\epsilon[G^d] \twoheadrightarrow M$  in the category  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ . Applying the above process to the kernel of this surjective morphism and so on, we obtain a long exact sequence in  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ :

$$\cdots \rightarrow V_2 \otimes O_\epsilon[G^d] \rightarrow V_1 \otimes O_\epsilon[G^d] \rightarrow M \rightarrow 0,$$

where  $V_i$  are finite dimensional rational  $\dot{U}_\epsilon(\mathfrak{g})$ -modules. This is a free resolution of the right  $O_\epsilon[G^d]$ -module  $M$ . On the other hand, by Lemma A.4,  $\text{Hom}_{O_\epsilon[G^d]\text{-rmod}}(N, O_\epsilon[G^d])$  is an object in  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$  for any  $N \in O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ . Hence,  $\text{Ext}_{O_\epsilon[G^d]\text{-rmod}}^i(M, O_\epsilon[G^d])$  are indeed objects in  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ .  $\square$

Recall that the vanishing locus of  $f = K^{-2\lambda_0} \in O_\epsilon[G^d] \cong \mathbb{F}[G^d]$  in  $G^d(\mathbb{F})$  is  $G^d(\mathbb{F}) \setminus G_0^d(\mathbb{F})$ . For any  $M \in O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ , let  $M_f$  be the localization of  $M$  by  $f \in O_\epsilon[G^d]$ .

**Lemma 8.19.** *If  $M \in O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$  is such that  $M_f = 0$ , then  $M = 0$ .*

This result is proved using its classical counterpart for equivariant sheaves under an action of an algebraic group. Let  $\mathfrak{u}_\epsilon$  be the Hopf subalgebra of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\{\tilde{E}_i, \tilde{F}_i, K^\lambda\}_{1 \leq i \leq r}^{\lambda \in 2P}$ .

**Lemma 8.20.** (a) *For any rational  $\dot{U}_\epsilon(\mathfrak{g})$ -module  $M$ , the  $\mathfrak{u}_\epsilon$ -invariant part  $M^{\mathfrak{u}_\epsilon}$  is a rational  $\dot{U}_\mathbb{F}(\mathfrak{g}^d)$ -module.*

(b) *For any  $M \in O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$ , the  $\mathfrak{u}_\epsilon$ -invariant part  $M^{\mathfrak{u}_\epsilon}$  belongs to  $O_\epsilon[G^d]\text{-mod}^{G^d}$ , the category of  $G^d$ -equivariant coherent sheaves on  $\text{Spec}(O_\epsilon[G^d]) = G^d$ .*

*Proof.* (a) This follows if we can show that  $M^{\mathfrak{u}_\epsilon}$  is a  $\dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $M$ . Because by that,  $M^{\mathfrak{u}_\epsilon}$  is a rational  $\dot{U}_\epsilon(\mathfrak{g})$ -module and the action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $M^{\mathfrak{u}_\epsilon}$  will factor through the morphism  $\check{\text{Fr}} : \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_\mathbb{F}(\mathfrak{g}^d)$  by Lemma 4.18.

As in Remark 6.35, the left ideal of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\{\tilde{E}_i, \tilde{F}_i\}_{1 \leq i \leq r}$  is equal to the right ideal of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\{\tilde{E}_i, \tilde{F}_i\}_{1 \leq i \leq r}$ ; and the weight space  $\dot{U}_\epsilon(\mathfrak{g})_\nu$  with  $\nu \notin Q^*$  is contained in this left ideal of  $\dot{U}_\epsilon(\mathfrak{g})$ .

Let  $u$  be a weight element in  $\dot{U}_\epsilon(\mathfrak{g})$  and  $m \in M^{\mathfrak{u}_\epsilon}$ . We will show that  $um \in M^{\mathfrak{u}_\epsilon}$ . If the weight of  $u$  is not contained in  $Q^*$ , then by the second paragraph,  $um = 0$ . Therefore, we can assume that the weight of  $u$  is contained in  $Q^*$ . Then  $K^{2\lambda}u = uK^{2\lambda}$  for all  $\lambda \in P$ , hence  $K^{2\lambda}um = uK^{2\lambda}m = um$ . Meanwhile, the weights of  $\tilde{E}_i u$  and  $\tilde{F}_i u$  are not contained in  $Q^*$ , therefore, by the second paragraph,  $\tilde{E}_i u$  and  $\tilde{F}_i u$  are contained in the left ideal of  $\dot{U}_\epsilon(\mathfrak{g})$  generated by  $\{\tilde{E}_i, \tilde{F}_i\}_{1 \leq i \leq r}$ . This implies that  $\tilde{E}_i um = \tilde{F}_i um = 0$ .

(b) Since  $\mathfrak{u}_\epsilon$  acts on  $O_\epsilon[G^d]$  via the counit and using part (a),  $M^{\mathfrak{u}_\epsilon}$  is a  $\dot{U}_\mathbb{F}(\mathfrak{g}^d)$ -equivariant  $O_\epsilon[G^d]$ -module. Since  $O_\epsilon[G^d]$  is Noetherian and  $M$  is finitely generated over  $O_\epsilon[G^d]$ , it follows that  $M^{\mathfrak{u}_\epsilon}$  is finitely generated over  $O_\epsilon[G^d]$ .  $\square$

*Proof.* Assume that  $M \neq 0$ . Then there is a finite dimensional rational  $\dot{U}_\epsilon(\mathfrak{g})$ -module  $V$  such that  $\text{Hom}_{\dot{U}_\epsilon(\mathfrak{g})}(V, M) \neq 0$ . This implies that  $(M \otimes V^*)^{\mathbf{u}_\epsilon} = \text{Hom}_{\mathbf{u}_\epsilon}(V, M) \neq 0$ . On the other hand,  $M \otimes V^* \in O_\epsilon[G^d]\text{-mod}^{G^d}$  and  $(M \otimes V^*)_f = 0$ , where  $O_\epsilon[G^d]$  acts only on the first factor in  $M \otimes V^*$ . Therefore,  $(M \otimes V^*)^{\mathbf{u}_\epsilon} \neq 0$  belongs to  $O_\epsilon[G^d]\text{-mod}^{G^d}$ , and  $(M \otimes V^*)_f^{\mathbf{u}_\epsilon} = 0$ . However, the latter is impossible. Indeed, the support of  $(M \otimes V^*)^{\mathbf{u}_\epsilon}$  must be a  $G^d(\mathbb{F})$ -invariant closed subvariety of  $G^d = \text{Spec}(O_\epsilon[G^d])$ , while  $(M \otimes V^*)_f^{\mathbf{u}_\epsilon} = 0$  implies that this support is contained in  $G^d(\mathbb{F}) \setminus G_0^d(\mathbb{F})$ . However, the open Bruhat cell intersects nontrivially any conjugacy classes in  $G^d(\mathbb{F})$ , see part (a) of Lemma 5.11. Therefore, the support of  $(M \otimes V^*)^{\mathbf{u}_\epsilon}$  must be empty. Thus,  $(M \otimes V^*)^{\mathbf{u}_\epsilon} = 0$ , contradiction.  $\square$

We are now ready to finish the proof of Proposition 8.13

**Proposition 8.21.**  *$O_\epsilon[G]$  is a projective left and right module over  $O_\epsilon[G^d]$ .*

*Proof.* As  $O_\epsilon[G^d]$  is a central subalgebra of  $O_\epsilon[G]$  by Remark 8.12, it is enough to show that  $O_\epsilon[G]$  is projective as a right  $O_\epsilon[G^d]$ -module. Since  $(O_\epsilon[G])_f \cong U_\epsilon^{\text{fin}}[K^{2\lambda_0}] \cong U_\epsilon^{\text{ev}}$  by Proposition 8.4 and  $O_\epsilon[G^d]_f \cong Z_{Fr}$ , it follows by Corollary 5.4 that  $(O_\epsilon[G])_f$  is free as a right  $O_\epsilon[G^d]_f$ -module. By viewing  $O_\epsilon[G]$  as an object in  $O_\epsilon[G^d]\text{-rmod}^{G_\epsilon}$ , we know that all  $\text{Ext}_{O_\epsilon[G^d]\text{-rmod}}^i(O_\epsilon[G], O_\epsilon[G^d])$  are objects in  $O_\epsilon[G^d]\text{-lmod}^{G_\epsilon}$  by Lemma 8.18.

Furthermore, since  $(O_\epsilon[G])_f$  is a free right  $(O_\epsilon[G^d])_f$ -module, we must have

$$(\text{Ext}_{O_\epsilon[G^d]\text{-rmod}}^i(O_\epsilon[G], O_\epsilon[G^d]))_f = 0, \quad , \forall i > 0$$

By Lemma 8.19, it follows that

$$(8.15) \quad \text{Ext}_{O_\epsilon[G^d]\text{-rmod}}^i(O_\epsilon[G], O_\epsilon[G^d]) = 0, \forall i > 0.$$

Since the variety  $G^d = \text{Spec}(O_\epsilon[G^d])$  is smooth, hence Cohen-Macaulay, and  $O_\epsilon[G]$  is finitely generated over  $O_\epsilon[G^d]$ , (8.15) implies that  $O_\epsilon[G]$  is a projective right  $O_\epsilon[G^d]$ -module.  $\square$

## 8.6. $O_q[G] \cong U_q^{\text{fin}}$ for Noetherian $\mathcal{A}_N$ -algebra $R$ .

### 8.6.1. A technical result about filtrations on $O_\epsilon[G]$ and $U_\epsilon^{\text{ev}}(\mathfrak{g})$ .

In this section, we assume that  $R = \mathbb{F}$  is a field. Let us refine  $P$  into a linearly ordered set, in particular,  $P_+$  has elements  $\lambda_1 < \lambda_2 < \dots$ . We note that the results in this section depend on the refined order on  $P$ . By Lemma 6.53, there is an exhaustive good  $\dot{U}_\epsilon(\mathfrak{g}) \otimes_{\mathbb{F}} \dot{U}_\epsilon(\mathfrak{g})$ -module filtration

$$M_1 \subset M_2 \subset M_3 \subset \dots$$

on  $O_\epsilon[G]$  such that  $M_i$  is the maximal  $\dot{U}_\epsilon(\mathfrak{g}) \otimes_{\mathbb{F}} \dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $O_\epsilon[G]$  whose weights are bounded by  $(\lambda_i, \lambda_i^*)$ .

The  $\dot{U}_\epsilon(\mathfrak{g}) \otimes \dot{U}_\epsilon(\mathfrak{g})$ -action on  $O_\epsilon[G]$  is given by  $(x \otimes y)f(u) = f(S(y)uS^2(x))$  for all  $x, y, u \in \dot{U}_\epsilon(\mathfrak{g})$  and  $f \in O_\epsilon[G]$ . Let  $c_{f,v}$  be a matrix coefficient of  $V \in \text{Rep}^{fd}(\dot{U}_\epsilon(\mathfrak{g}))$ , then the action reads  $(x \otimes y)c_{f,v} = c_{yf, S^2(x)v}$ . From this description of the action and the property that the weights of  $M_i$  are bounded by  $(\lambda_i, \lambda_i^*)$ , it follows that

**Lemma 8.22.**  *$M_i$  is spanned by the matrix coefficients of all  $V \in \text{Rep}^{fd}(\dot{U}_\epsilon(\mathfrak{g}))$  such that the weights of  $V$  are bounded by  $\lambda_i$ .*

*Proof.* Matrix coefficients of  $V \in \text{Rep}^{fd}(\dot{U}_\epsilon(\mathfrak{g}))$  is contained in  $M_i$  since  $M_i$  is the maximal  $\dot{U}_\epsilon(\mathfrak{g}) \otimes_{\mathbb{F}} \dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $O_\epsilon[G]$  with the weight space bounded by  $(\lambda_i, \lambda_i^*)$ . Let us prove the other direction.

Let  $\phi \in M_i$ . Let  $V := \dot{U}_\epsilon(\mathfrak{g})\phi \subset M_i$  denote the  $\dot{U}_\epsilon(\mathfrak{g})$ -submodule under the first  $\dot{U}_\epsilon(\mathfrak{g})$ -action. It follows that the weights of  $V$  are bounded by  $\lambda_i$  and the weights of  $V^*$  are bounded by  $\lambda_i^*$ . Let  $f \in V^*$  be defined by  $f(v) = v(1)$  for all  $v \in V \subset O_\epsilon[G]$ .

Recall the  $\dot{U}_\epsilon(\mathfrak{g}) \otimes_{\mathbb{F}} \dot{U}_\epsilon(\mathfrak{g})$ -linear map  $V \otimes V^* \rightarrow O_\epsilon[G]$  defined by  $v \otimes f \mapsto c_{f,K^{-2\rho}v}$ . Its image is contained in  $M_i$ . On the other hand, the image of  $K^{2\rho}\phi \otimes f$  is  $c_{f,\phi}$ . This element coincides with  $\phi$ , indeed,

$$c_{f,\phi}(u) = f(u\phi) = u\phi(1) = \phi(u), \quad \text{for all } u \in \dot{U}_\epsilon(\mathfrak{g}).$$

This finishes the proof.  $\square$

On the other hand, as in Section 8.3,  $U_\epsilon^{ev}(\mathfrak{g})$  has a  $\dot{U}_\epsilon(\mathfrak{g})$ -module filtration labeled by the weight lattice  $P$  such that  $\deg(\tilde{E}_i K^{\zeta_i^>}) = 0$ ,  $\deg(\tilde{F}_i K^{-\zeta_i^<}) = 0$ ,  $\deg(K^{2\lambda}) = -\lambda$  for  $1 \leq i \leq r, \lambda \in P$ . Set

$$\tilde{U}_\epsilon^{ev>} =: \bigoplus_{\mu \in Q_+} U_{\epsilon,\mu}^{ev>} K^{\gamma(\mu)}, \quad \tilde{U}_\epsilon^{ev<} =: \bigoplus_{\mu \in Q_+} U_{\epsilon,-\mu}^{ev<}(\mathfrak{g}) K^{\kappa(\mu)}.$$

Then

$$U_{\epsilon,\leq \lambda}^{ev} = \bigoplus_{\mu \leq \lambda} \tilde{U}_\epsilon^{ev<} \otimes K^{-2\mu} \otimes \tilde{U}_\epsilon^{ev>}.$$

**Remark 8.23.**  $U_\epsilon^{ev}(\mathfrak{g}) = \bigcup_{\lambda_i \in P_+} U_{\epsilon,\leq \lambda_i}^{ev}$ .

**Lemma 8.24.** *Let  $R = \mathbb{F}$  a field. Then for all  $\lambda_i \in P_+$  we have*

$$(U_{\epsilon,\leq \lambda_i}^{ev})^{fin} = U_{\epsilon,\leq \lambda_i}^{ev} \cap O_\epsilon[G] = M_i.$$

*Proof.* The equality  $(U_{\epsilon,\leq \lambda_i}^{ev})^{fin} = U_{\epsilon,\leq \lambda_i}^{ev} \cap O_\epsilon[G]$  holds since  $O_\epsilon[G] \cong U_\epsilon^{fin}$ .

Let  $c_{f,v}$  be a matrix coefficient of  $V \in \text{Rep}^{fd}(\dot{U}_\epsilon(\mathfrak{g}))$  such that  $c_{f,v} \in U_{\epsilon,\leq \lambda_i}^{ev}$ . By Lemma 8.22, it is enough to show that  $c_{f,v}$  can be regarded as a matrix coefficient of some module  $V' \in \text{Rep}^{fd}(\dot{U}_\epsilon(\mathfrak{g}))$  such that the weights of  $V'$  are bounded by  $\lambda_i$ .

Note that we can assume  $f, v$  be weight vectors of  $V^*, V$ , respectively. Furthermore, we can assume  $V = \dot{U}_\epsilon(\mathfrak{g})v$ . Let  $N$  be the  $\dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $\dot{U}_\epsilon(\mathfrak{g})v$  generated by all weight vectors  $v_i$  of weight bigger than  $\lambda_i$ . Then we will show that  $V'$  can be chosen to be  $V/N$ .

In Lemma 8.1, we constructed the element  $c_{f,v} \in U_\epsilon^{ev}(\mathfrak{g})$ . That is  $c_{f,v} = \sum_{\mu,\nu} u_{\mu\nu}$  for  $u_{\mu,\nu} \in (U_{\epsilon,-\mu}^{ev<} K^{\kappa(\mu)}) K^{-2\text{wt}(v)-2\mu} (U_{\epsilon,\nu}^{ev>} K^{\gamma(\nu)})$  such that

$$\langle u_{\mu\nu}, (\dot{y} K^{\kappa(\nu)}) u_0 (\dot{x} K^{\gamma(\mu)}) \rangle' = \hat{\chi}_{\text{wt}(v)+\mu}(u_0) f((\dot{y} K^{\kappa(\nu)}) (\dot{x} K^{\gamma(\mu)}) v),$$

for all  $\dot{y} \in \dot{U}_{\epsilon,-\nu}^{<}, \dot{x} \in \dot{U}_{\epsilon,\mu}^{>}$  and  $u_0 \in \dot{U}_\epsilon^0$ . In particular,  $u_{\mu\nu} = 0$  if and only if  $f((\dot{y} K^{\kappa(\nu)}) (\dot{x} K^{\gamma(\mu)}) v) = 0$  for all  $\dot{x}, \dot{y}, u_0$  as above.

Since  $c_{f,v} \in U_{\epsilon,\leq \lambda_i}^{ev}$ , it follows that  $u_{\mu,\nu} = 0$  for all  $\mu > \lambda_i - \text{wt}(v)$ . Hence  $f((\dot{y} K^{\kappa(\nu)}) (\dot{x} K^{\gamma(\mu)}) v) = 0$  for all above  $\dot{y}, \dot{x}, u_0$  such that  $\mu > \lambda_i - \text{wt}(v)$ . It follows that  $f(N) = 0$ , hence  $c_{f,v}$  can be regarded as a matrix coefficient of  $V' = V/N$ .  $\square$

### 8.6.2. $O_q[G] \cong U_q^{fin}$ .

We make a remark that for any weight  $\dot{U}_\epsilon(\mathfrak{g})$ -module  $M = \bigoplus_\lambda M_\lambda$ , the  $\dot{U}_\epsilon(\mathfrak{g})$ -locally finite part  $M^{fin}$  is also the maximal rational  $\dot{U}_\epsilon(\mathfrak{g})$ -submodule of  $M$ .

**Lemma 8.25.** *Let  $R = \mathbb{F}$ , a field. Then the  $\dot{U}_\epsilon(\mathfrak{g})$ -locally finite part  $(U_\epsilon^{ev}(\mathfrak{g})/O_\mathbb{F}[G])^{fin}$  is zero.*

*Proof.* We have a short exact sequence

$$(8.16) \quad 0 \rightarrow O_\mathbb{F}[G] \rightarrow U_\mathbb{F}^{ev}(\mathfrak{g}) \xrightarrow{\pi} U_\mathbb{F}^{ev}(\mathfrak{g})/O_\mathbb{F}[G] \rightarrow 0.$$

Assume that  $(U_{\mathbb{F}}^{ev}(\mathfrak{g})/O_{\mathbb{F}}[G])^{fin} \neq 0$  then there is a finite dimensional rational  $\dot{U}_{\epsilon}(\mathfrak{g})$ -submodule  $V \neq 0$  in  $U_{\mathbb{F}}^{ev}(\mathfrak{g})/O_{\mathbb{F}}[G]$ . By Remark 8.23, there  $\lambda_i$  such that the image of  $M := U_{\mathbb{F}, \leq \lambda_i}^{ev}$  under the quotient  $\pi : U_{\mathbb{F}}^{ev}(\mathfrak{g}) \rightarrow U_{\mathbb{F}}^{ev}/O_{\mathbb{F}}[G]$  contains  $V$ . By Lemma 8.24,  $M^{fin}$  is a finite dimensional rational  $\dot{U}_{\epsilon}(\mathfrak{g})$ -module.

The short exact sequence (8.16) gives a rise to a short exact sequence

$$0 \rightarrow M^{fin} \rightarrow M \rightarrow \pi(M) \rightarrow 0.$$

We note that  $M^{fin}$  and  $V$  are finite dimensional rational  $\dot{U}_{\epsilon}(\mathfrak{g})$ -modules and  $M$  is a weight  $\dot{U}_{\epsilon}(\mathfrak{g})$ -module. Therefore, the preimage  $\pi^{-1}(V) \subset M$  under the map  $\pi : M \rightarrow \pi(M)$  must be a rational  $\dot{U}_{\epsilon}(\mathfrak{g})$ -submodule properly containing  $M^{fin}$ . This leads to a contradiction since  $M^{fin}$  is the maximal rational  $\dot{U}_{\epsilon}(\mathfrak{g})$ -submodule of  $M$ .  $\square$

**Proposition 8.26.** *Assume  $R$  is a domain. Then the inclusion  $O_R[G] \hookrightarrow U_R^{fin}$  is an isomorphism.*

*Proof.* For any nonzero  $x \in R$ , we have the following commutative diagram, where the rows are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_R[G] & \xrightarrow{\cdot x} & O_R[G] & \longrightarrow & O_{R/(x)}[G] \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U_R^{ev}(\mathfrak{g}) & \xrightarrow{\cdot x} & U_R^{ev}(\mathfrak{g}) & \longrightarrow & U_{R/(x)}^{ev}(\mathfrak{g}) \longrightarrow 0 \end{array}$$

where all vertical morphisms are injective. By the Snake Lemma, the morphism

$$U_R^{ev}(\mathfrak{g})/O_R[G] \xrightarrow{\cdot x} U_R^{ev}(\mathfrak{g})/O_R[G]$$

is injective for all nonzero  $x \in R$ . In other words,  $U_R^{ev}(\mathfrak{g})/O_R[G]$  is torsion free over  $R$ . Let  $\mathbb{K}$  be the fraction field of  $R$ . Then we have the inclusion

$$U_R^{ev}(\mathfrak{g})/O_R[G] \hookrightarrow (U_R^{ev}(\mathfrak{g})/O_R[G]) \otimes_R \mathbb{K} \cong U_{\mathbb{K}}^{ev}(\mathfrak{g})/O_{\mathbb{K}}[G].$$

Since the  $\dot{U}_{\mathbb{K}}(\mathfrak{g})$ -locally finite part of  $U_{\mathbb{K}}^{ev}(\mathfrak{g})/O_{\mathbb{K}}[G]$  is zero by Lemma 8.25, it follows that  $(U_R^{ev}(\mathfrak{g})/O_R[G])^{fin} = 0$ . On the other hand, we have a left exact sequence

$$0 \rightarrow O_R[G] \rightarrow U_R^{fin} \rightarrow (U_R^{ev}/O_R[G])^{fin}.$$

Therefore,  $O_R[G] \cong U_R^{fin}$ .  $\square$

The following is the main result of this section.

**Theorem 8.27.** *For any Noetherian  $\mathcal{A}_N$ -algebra  $R$ , the inclusion  $O_R[G] \rightarrow U_R^{fin}$  is an isomorphism.*

*Proof.* Let  $\mathfrak{p}$  be a prime ideal of  $R$ . Let  $\bullet^{fin, \dot{U}_{R/\mathfrak{p}}}$  denote the  $\dot{U}_{R/\mathfrak{p}}(\mathfrak{g})$ -locally finite part. We reserve the notation  $\bullet^{fin}$  for the  $\dot{U}_R(\mathfrak{g})$ -locally finite part. Then one see that

$$(U_{R/\mathfrak{p}}^{ev}(\mathfrak{g}))^{fin} \cong (U_{R/\mathfrak{p}}^{ev}(\mathfrak{g}))^{fin, \dot{U}_{R/\mathfrak{p}}}.$$

By Proposition 8.26, we have the isomorphism

$$O_{R/\mathfrak{p}}[G] \rightarrow (U_{R/\mathfrak{p}}^{ev}(\mathfrak{g}))^{fin, \dot{U}_{R/\mathfrak{p}}}.$$

Note that  $O_R[G] \otimes_R R/\mathfrak{p} = O_{R/\mathfrak{p}}[G]$  and  $U_R^{ev}(\mathfrak{g}) \otimes R/\mathfrak{p} = U_{R/\mathfrak{p}}^{ev}(\mathfrak{g})$ . Therefore, the natural morphism

$$(8.17) \quad O_R[G] \otimes_R R/\mathfrak{p} \rightarrow (U_R^{ev}(\mathfrak{g}) \otimes_R R/\mathfrak{p})^{fin}$$

is an isomorphism for any prime ideal  $\mathfrak{p}$  of  $R$ .

Since  $R$  is Noetherian, there is a finite  $R$ -module filtration  $0 = M_0 \subset M_1 \subset \cdots \subset M_n = R$  such that  $M_i/M_{i-1} \cong R/\mathfrak{p}_i$  with some prime ideal  $\mathfrak{p}_i$  for all  $1 \leq i \leq n$ . We will prove by induction that the homomorphisms

$$O_R[G] \otimes_R M_i \rightarrow (U_R^{ev} \otimes_R M_i)^{fin}$$

are isomorphisms for all  $1 \leq i \leq n$ . The base case  $i = 1$  holds by (8.17). Let us do the induction step. Since  $O_R[G]$  and  $U_R^{ev}(\mathfrak{g})$  are free over  $R$ , we have the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_R[G] \otimes_R M_{i-1} & \longrightarrow & O_R[G] \otimes_R M_i & \longrightarrow & O_R[G] \otimes_R R/\mathfrak{p}_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U_R^{ev} \otimes_R M_{i-1} & \longrightarrow & U_R^{ev}(\mathfrak{g}) \otimes_R M_i & \longrightarrow & U_R^{ev}(\mathfrak{g}) \otimes_R R/\mathfrak{p}_i \longrightarrow 0 \end{array}$$

which gives us the commutative diagram

$$(8.18) \quad \begin{array}{ccccccc} 0 & \longrightarrow & O_R[G] \otimes_R M_{i-1} & \longrightarrow & O_R[G] \otimes_R M_i & \longrightarrow & O_R[G] \otimes_R R/\mathfrak{p}_i \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & (U_R^{ev} \otimes_R M_{i-1})^{fin} & \longrightarrow & (U_R^{ev}(\mathfrak{g}) \otimes_R M_i)^{fin} & \longrightarrow & (U_R^{ev}(\mathfrak{g}) \otimes_R R/\mathfrak{p}_i)^{fin} \end{array}$$

The first vertical arrow in (8.18) is an isomorphism by induction hypothesis while the third vertical arrow in (8.18) is an isomorphism by (8.17). Therefore, the middle vertical morphism is an isomorphism. This finishes the induction. In particular, when  $i = n$ , we have the desired isomorphism  $O_R[G] \cong U_R^{fin}$ .  $\square$

### 8.7. Harish-Chandra center.

**Definition 8.28.** The Harish-Chandra center  $Z_{HC} \subset U_q^{ev}(\mathfrak{g})$  is defined to be the  $\dot{U}_q(\mathfrak{g})$ -invariant part  $U_q^{ev}(\mathfrak{g})^{\dot{U}_q}$ .

**Lemma 8.29.**  $Z_{HC}$  is central in  $U_q^{ev}(\mathfrak{g})$ . Moreover,  $Z_{HC} \subset U_q^{ev}(\mathfrak{g})_0$ , the zero weight space of  $U_q^{ev}(\mathfrak{g})$ .

*Proof.* Since  $Z_{HC} = \text{Hom}_{\dot{U}_q(\mathfrak{g})}(W_q(0), U_q^{ev}(\mathfrak{g}))$ , the second statement holds. On the other hand, recall the following formulas for the action of  $\dot{U}_q(\mathfrak{g})$  on  $U_q^{ev}(\mathfrak{g})$ :

$$\text{ad}'(K^\lambda)(x) = K^\lambda x K^{-\lambda}, \quad \text{ad}'(\tilde{E}_i)(x) = [\tilde{E}_i, x] K^{\zeta_i^>}, \quad \text{ad}'(\tilde{F}_i)(x) = [\tilde{F}_i, x] K^{-\zeta_i^<},$$

for  $x \in U_q^{ev}(\mathfrak{g})$ . Therefore, for  $z \in Z_{HC}$ , we must have  $z$  commutes with  $\tilde{E}_i, \tilde{F}_i, K^\lambda$  for  $1 \leq i \leq r$  and  $\lambda \in 2P$ , hence,  $z$  is central in  $U_q^{ev}(\mathfrak{g})$ .  $\square$

The zero weight space  $U_q^{ev}(\mathfrak{g})_0$  has a decomposition:

$$U_q^{ev}(\mathfrak{g})_0 = U_q^{ev0} \oplus \bigoplus_{\mu \in Q_+} U_{q,-\mu}^{ev<} U_q^{ev0} U_{q,\mu}^{ev>}.$$

Using the commutation relations between  $\tilde{E}_i, \tilde{F}_i$  and  $K^\lambda$ , one can easily see that  $\bigoplus_{\mu \in Q_+} U_{q,-\mu}^{ev<} U_q^{ev0} U_{q,\mu}^{ev>}$  is a two-sided ideal in  $U_q^{ev}(\mathfrak{g})_0$ . Therefore, we have an algebra homomorphism

$$\pi : Z_{HC} \hookrightarrow U_q^{ev}(\mathfrak{g})_0 \rightarrow U_q^{ev0}(\mathfrak{g}).$$

**Theorem 8.30.** The homomorphism  $\pi$  defines an algebra isomorphism between  $Z_{HC}$  and the following subalgebra of  $U_q^{ev0}(\mathfrak{g})$ :

$$(8.19) \quad R \left[ K^{\pm 2\omega_1}, \dots, K^{\pm 2\omega_r} \right]^{W_\bullet},$$

where we use the dot-action of the Weyl group  $W$  on  $U_q^{ev0}(\mathfrak{g})$  extended by linearity from:

$$w_\bullet(K^\mu) = q^{(w^{-1}\rho - \rho, \mu)} K^{w(\mu)} \quad \text{for all } x \in W, \mu \in 2P.$$

<sup>8</sup>

*Proof.* By Lemma 6.54,  $O_q[G]^{\dot{U}_q}$  is a free  $R$ -module with a basis  $c_\lambda = \sum_i c_{v_i^*, K^{-2\rho} v_i}$  for all  $\lambda \in P_+$ . Here,  $\{v_i\}$  is a weight basis of the Weyl module  $W_q(\lambda)$ .

Since we have the isomorphism  $\hat{i} : O_q[G] \cong U_q^{fin}$ , it follows that  $Z_{HC}$  is a free  $R$ -module with basis  $\{\hat{i}(c_\lambda) | \lambda \in P_+\}$ .

Let us compute  $\pi \circ \hat{i}(c_\lambda)$ . This element is uniquely determined by

$$\langle \pi \circ \hat{i}(c_\lambda), u \rangle = c_\lambda(u),$$

for all  $u \in \dot{U}_q^0$ . On the other hand, let  $P_{+,\lambda}$  be the set of dominant weights in  $W_q(\lambda)$ , then it is straightforward to see that

$$c_\lambda(u) = \sum_{\mu \in P_{+,\lambda}} \text{rank}(W_q(\lambda)_\mu) \sum_{\mu' \in W\mu} q^{(-2\rho, \mu')} \hat{\chi}_{\mu'}(u),$$

for all  $u \in \dot{U}_q^0$ . Therefore,

$$c_\lambda(u) = \left\langle \sum_{\mu \in P_{+,\lambda}} \text{rank}(W_q(\lambda)_\mu) \sum_{\mu' \in W\mu} q^{(2\rho, \mu')} K^{2\mu'}, u \right\rangle$$

for all  $u \in \dot{U}_q^0$ . It follows that

$$(8.20) \quad \pi \circ \hat{i}(c_\lambda) = \sum_{\mu \in P_{+,\lambda}} \text{rank}(W_q(\lambda)_\mu) \sum_{\mu' \in W\mu} q^{(\rho, 2\mu')} K^{2\mu'}.$$

It is an easy exercise to show that the collection of elements in the right hand side of (8.20) over all  $\lambda \in P_+$  forms a  $R$ -basis of the free  $R$ -module (8.19). This implies that  $\pi$  defines an algebra homomorphism between  $Z_{HC}$  and algebra (8.19).  $\square$

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<sup>8</sup>Put a reference to the usual statement about the quantum HC isomorphism in Jantzen. Then discuss what's known for roots of unity.

## 9. POISSON GEOMETRY OF CENTER $Z$ OF $U_\epsilon^{ev}(\mathfrak{g})$ OVER $\mathbb{C}$

The goal of this section is to generalize the results about the center and Azumaya locus of De Concini-Kac quantum groups at odd order roots of unity to the even part algebra  $U_\epsilon^{ev}(\mathfrak{g})$  at both odd and even order roots of unity. Most of the arguments follow [DCK], [DCKP], [DCKP2].

### 9.1. Poisson bracket on $Z$ and $Z_{Fr}$ .

Let  $q = \epsilon \in \mathbb{C}$  be a root of unity of order  $\ell$  such that  $\ell_i > \max\{2, 1 - a_{ij}\}_{1 \leq j \leq r}$ . The center  $Z$  of  $U_\epsilon^{ev}(\mathfrak{g})$  naturally carries a Poisson structure via quantization.

Let  $\mathcal{A}' = \mathbb{C}[v^\pm] \left[ \frac{1}{v^{2d_i} - 1} \right]_{1 \leq i \leq r}$ . Let  $\varphi_\epsilon : U_{\mathcal{A}'}^{ev}(\mathfrak{g}) \rightarrow U_\epsilon^{ev}(\mathfrak{g})$  denote the natural epimorphism sending  $v$  to  $\epsilon$ . Then, any  $x \in \varphi_\epsilon^{-1}(Z)$  naturally gives rise to a derivation of the algebra  $U_\epsilon^{ev}(\mathfrak{g})$  as follows: Given  $u \in U_\epsilon^{ev}(\mathfrak{g})$ , we pick its arbitrary lift  $\hat{u}$  in  $U_{\mathcal{A}'}^{ev}(\mathfrak{g})$  then define

$$(9.1) \quad \{x, u\} := \varphi_\epsilon \left( \frac{[x, \hat{u}]}{v - \epsilon} \right).$$

**Remark 9.1.** (a) We note that  $\{x, u\}$  is clearly independent on the choice of a lift  $\hat{u} \in \varphi_\epsilon^{-1}(u)$ .  
(b) The above construction also gives rise to a natural Poisson bracket in  $Z$  as follows: Given  $u, w \in Z$ , pick any lifts  $\hat{u} \in \varphi_\epsilon^{-1}(u)$  and  $\hat{w} \in \varphi_\epsilon^{-1}(w)$ , and define:

$$(9.2) \quad \{u, w\} := \varphi_\epsilon \left( \frac{[\hat{u}, \hat{w}]}{v - \epsilon} \right).$$

This bracket makes  $Z$  into a Poisson algebra.

(c) We note that the derivation  $\{x, -\}$  of the entire algebra  $U_\epsilon^{ev}(\mathfrak{g})$ , defined in (9.1), does actually depend on  $x$  and not just on its specialization  $\varphi_\epsilon(x) \in Z$ , in contrast to (b).

Evoking the algebra homomorphism  $\sigma : \mathcal{A}' \rightarrow \mathbb{C}$  maps  $v$  to  $\epsilon$ . We define a nonzero scalar  $p_i \in \mathbb{C}$  via:

$$(9.3) \quad p_i := \sigma \left( \frac{(\ell_i)_{v_i}!}{v - \epsilon} \right)$$

**Proposition 9.2.** *For any  $u \in Z$ , we have the following equalities:*

$$(9.4) \quad \text{ad}' \left( p_i \tilde{E}_i^{(\ell_i)} \right)(u) = \{\tilde{E}_i^{\ell_i}, u\} K^{\ell_i \zeta_i^>} \quad \text{and} \quad \text{ad}' \left( p_i \tilde{F}_i^{(\ell_i)} \right) = \{\tilde{F}_i^{\ell_i}, u\} K^{-\ell_i \zeta_i^<} ,$$

where the Poisson bracket  $\{\tilde{E}_i^{\ell_i}, u\}$  and  $\{\tilde{F}_i^{\ell_i}, u\}$  are defined in (9.2) in Remark 9.1.

*Proof.* We shall prove only the first equality. According to (2.20), we have in  $U_{\mathcal{A}'}^{ev}(\mathfrak{g})$ :

$$\begin{aligned} \Delta'(\tilde{E}_i^r) &= \sum_{c=0}^r v_i^{2(r-c)c} \binom{r}{c}_{v_i} \tilde{E}_i^{r-c} \otimes K^{-(r-c)\zeta_i^>} \tilde{E}_i^c, \\ S'(K^{-(r-c)\zeta_i^>} \tilde{E}_i^c) &= (-1)^c v_i^{c(c-1)} \tilde{E}_i^c K^{r\zeta_i^>}. \end{aligned}$$

Therefore, for any  $r \in \mathbb{N}$  and  $y \in U_{\mathcal{A}'}^{ev}(\mathfrak{g})$ , we have:

$$\begin{aligned} \text{ad}'(\tilde{E}_i^r)(y) &= \sum_{c=0}^r (-1)^c v_i^{c(2r-c-1)} \binom{r}{c}_{v_i} \tilde{E}_i^{r-c} y \tilde{E}_i^c K^{r\zeta_i^>} = \\ &\quad \left( \tilde{E}_i^r y + \sum_{c=1}^r (-1)^c v_i^{c(2r-c-1)} \binom{r}{c}_{v_i} y \tilde{E}_i^r + \sum_{c=1}^{r-1} (-1)^c v_i^{c(2r-c-1)} \binom{r}{c}_{v_i} [\tilde{E}_i^{r-c}, y] \tilde{E}_i^c \right) K^{r\zeta_i^>}. \end{aligned}$$

Evoking the identity

$$\sum_{c=0}^r (-1)^c v_i^{c(2r-c-1)} \binom{r}{c}_{v_i} = 0,$$

we thus obtain:

$$(9.5) \quad \text{ad}'(\tilde{E}_i^r)(y) = [\tilde{E}_i^r, y] K^{r\zeta_i^>} + \sum_{c=0}^{r-1} (-1)^c v_i^{c(2r-c-1)} \binom{r}{c}_{v_i} [\tilde{E}_i^{r-c}, y] \tilde{E}_i^c K^{r\zeta_i^>}. \quad \text{[Note: } r = \ell_i \text{]}$$

Applying the formula (9.5) for  $r = \ell_i$  and  $y \in \varphi_\epsilon^{-1}(u)$  with  $u \in Z$ , we obtain:

$$\text{ad}'(\mathfrak{p}_i \tilde{E}_i^{(\ell_i)})(u) = \varphi_\epsilon \left( \frac{[\tilde{E}_i^{\ell_i}, y]}{v - \epsilon} K^{\ell_i \zeta_i^>} + \sum_{c=1}^{\ell_i-1} (-1)^c v_i^{c(2\ell_i-c-1)} \frac{\binom{\ell_i}{c}_{v_i}}{v - \epsilon} [\tilde{E}_i^{\ell_i-c}, y] \tilde{E}_i^c K^{\ell_i \zeta_i^>} \right).$$

Note that  $\varphi_\epsilon([\tilde{E}_i^k, y]) = [\tilde{E}_i^k, u] = 0$ , we finally get

$$\text{ad}'(\mathfrak{p}_i \tilde{E}_i^{(\ell_i)})(u) = \{\tilde{E}_i^{\ell_i}, u\} K^{\ell_i \zeta_i^>} \quad \text{for any } u \in Z.$$

This completes our proof of the first equality of (9.4).  $\square$

It turns out that the Frobenius central  $Z_{Fr}$  is a Poisson subalgebra of  $Z$ .

**Proposition 9.3.** (a) The center  $Z_{Fr}$  is closed under the Poisson bracket of  $Z$ .

(b)  $Z_{Fr}$  is generated by  $\{K^\lambda, \tilde{E}_i^{\ell_i}, \tilde{F}_i^{\ell_i}\}$  for  $1 \leq i \leq r$  and  $\lambda \in 2P^*$  as a Poisson algebra.

(c) Let us recall the identification  $\text{Spec}(Z_{Fr}) \cong G_0^d$ . The symplectic leaves of  $\text{Spec}Z_{Fr} \cong G_0^d$  are the intersections of conjugacy classes of  $G^d$  with  $G_0^d$ .

*Proof.* Firstly, we will prove part (a) and (b) simultaneously.

*Step 1:* We show that  $Z_{Fr}$  is closed under derivations  $\{\tilde{E}_i^{\ell_i}, -\}$ ,  $\{\tilde{F}_i^{\ell_i}, -\}$  and  $\{K^\lambda, -\}$  for all  $1 \leq i \leq r$  and  $\lambda \in 2P^*$ . By Proposition 9.2,

$$(9.6) \quad \text{ad}'(\mathfrak{p}_i \tilde{E}^{(\ell_i)})(u) = \{\tilde{E}_i^{\ell_i}, u\} K^{\ell_i \zeta_i^>}, \quad \text{ad}'(\mathfrak{p}_i \tilde{F}^{(\ell_i)})(u) = \{\tilde{F}_i^{\ell_i}, u\} K^{-\ell_i \zeta_i^<}$$

for any  $u \in Z$ . Since  $Z_{Fr}$  is closed under the adjoint action of  $\dot{U}_\epsilon(\mathfrak{g})$  and  $K^{\ell_i \zeta_i^>}, K^{-\ell_i \zeta_i^<} \in Z_{Fr}$ , it follows that  $Z_{Fr}$  is closed under the derivations  $\{\tilde{E}_i^{\ell_i}, -\}$  and  $\{\tilde{F}_i^{\ell_i}, -\}$ .

Let  $\theta := \frac{v^\ell - 1}{v - \epsilon} \Big|_\epsilon$ . Then for any homogenous element  $u \in Z_{Fr}$  and  $\lambda \in 2P^*$ , we have

$$(9.7) \quad \{K^\lambda, u\} = \frac{\theta(\lambda, \text{wt}(u))}{\ell} u K^\lambda.$$

Therefore,  $Z_{Fr}$  is also closed under the derivations  $\{K^\lambda, -\}$  for any  $\lambda \in 2P^*$ .

*Step 2:* Let  $\mathcal{P}$  denote the Poisson subalgebra of  $Z$  generated by  $\{\tilde{E}_i^{\ell_i}, \tilde{F}_i^{\ell_i}, K^\lambda\}$  for  $1 \leq i \leq r$  and  $\lambda \in 2P^*$ . Then by Step 1,  $Z_{Fr}$  contains  $\mathcal{P}$ . We will show that  $\mathcal{P}$  contains  $Z_{Fr}$ , hence prove part (a) and (b). We have

$$Z_{Fr} = Z_{Fr}^{fin}[K^{\lambda_0}], \quad Z_{Fr}^{fin} = \bigoplus_{\lambda \in P_+^*} \text{ad}'(\dot{U}_\epsilon(\mathfrak{g})) K^{-2\lambda},$$

by Lemma 5.11. Therefore, it is enough to show that  $\text{ad}'(\dot{U}_\epsilon(\mathfrak{g})) K^{-2\lambda} \in \mathcal{P}$ . The action of  $\dot{U}_\epsilon(\mathfrak{g})$  on  $Z_{Fr}$  factor through  $\check{F}_\epsilon : \dot{U}_\epsilon(\mathfrak{g}) \rightarrow \dot{U}_\mathbb{C}(\mathfrak{g}^d)$  and  $\dot{U}_\mathbb{C}(\mathfrak{g}^d)$  is generated by  $e_i, f_i$  as an algebra. Therefore,  $\text{ad}'(\dot{U}_\epsilon(\mathfrak{g})) K^{-2\lambda}$  is  $\mathbb{C}$ -linealy spanned by elements of the form

$$\text{ad}'(x_1 \dots x_n) K^{-2\lambda} \quad \text{with } x_j \in \{\tilde{E}_i^{(\ell_i)}, \tilde{F}_i^{(\ell_i)}\}_{1 \leq i \leq r},$$

but these elements are contained in  $\mathcal{P}$ .

c) Let  $\mathcal{V}_{\tilde{E}_i}, \mathcal{V}_{\tilde{F}_i}, \mathcal{V}_{K^\lambda}$  be the vector fields on  $\text{Spec}Z_{Fr} \cong G_0^d$  generated by derivations  $\{\tilde{E}_i^{\ell_i}, -\}$ ,  $\{\tilde{F}_i^{\ell_i}, -\}$ ,  $\{K^\lambda, -\}$ , respectively. Let  $e_i, f_i, \lambda$  be the vector fields on  $G_0^d$  coming from the adjoint action of  $G^d$  on itself. Therefore, from (9.6) and (9.7), we have that

$$\mathcal{V}_{\tilde{E}_i} = \frac{\mathsf{p}_i}{\epsilon_i^*} K^{-\ell_i} \zeta_i^* e_i, \quad \mathcal{V}_{\tilde{F}_i} = \mathsf{p}_i K^{\ell_i} \zeta_i^* f_i, \quad \mathcal{V}_{K^\lambda} = \frac{\theta}{\ell} K^\lambda \lambda.$$

Since  $Z_{Fr}$  is a Poisson algebra generated by  $\{\tilde{E}_i^{\ell_i}, \tilde{F}_i^{\ell_i}, K^\lambda\}$ , we see that the Hamiltonian vector fields at any points on  $\text{Spec}Z_{Fr} \cong G_0^d$  is contained in the space of vector fields from  $\mathfrak{g}^d$ .

On the other hand, for any function  $f, g$  and vector field  $V_1, V_2$  we have

$$[fV_1, gV_2] = fg[V_1, V_2] + fV_1(g)V_2 - gV_2(f)V_1$$

Hence we have

$$[\mathcal{V}_{\tilde{E}_{i_1}}, \dots, [\mathcal{V}_{\tilde{E}_{i_{k-1}}}, \mathcal{V}_{\tilde{E}_{i_k}}], \dots] = aK^b[e_{i_1}, \dots, [e_{i_{k-1}}, e_{i_k}], \dots] + f$$

for some  $b \in 2P^*$ ,  $a \in \mathbb{C}^\times$  and  $f$  is a linear combination of vector fields (with functions coefficients) corresponding to root vectors attached to roots  $\alpha \in \Delta_+$  such that  $\text{ht}(\alpha) < \alpha_{i_1} + \dots + \alpha_{i_k}$ . There are similar formulas for  $\mathcal{V}_{\tilde{F}_i}$ . From this we see that all vector fields generated by root vectors in  $\mathfrak{g}^d$  are contained in the Hamiltonian vector fields. Hence at any point on  $\text{Spec}Z_{Fr}$ , the Hamiltonian vector fields contains all vector fields from  $\mathfrak{g}^d$ .

So at any point on  $\text{Spec}(Z_{Fr})$ , the Hamiltonian vector fields and the vector fields from  $\mathfrak{g}^d$  coincide. This implies part (c).  $\square$

We now pick a  $\mathbb{C}$ -basis  $\{z_i\}$  of  $Z_{Fr}$  and lift it to elements  $\{\hat{z}_i\}$  with  $\hat{z}_i \in \varphi^{-1}(z_i)$ . Then by  $\mathbb{C}$ -linearly extending the collection of derivations  $\{\hat{z}_i, -\}$  of  $U_\epsilon^{ev}(\mathfrak{g})$ , we have a  $\mathbb{C}$ -linear map

$$D : Z_{Fr} \rightarrow \text{Dev}_{\mathbb{C}}(U_\epsilon^{ev}(\mathfrak{g})).$$

This makes  $(Z_{Fr}, U_\epsilon^{ev}(\mathfrak{g}))$  into a Poisson order defined in [BG, §2.2]. By Theorem 4.2 in [BG], we have

**Proposition 9.4.** *The fibers of  $U_\epsilon^{ev}(\mathfrak{g})$  over two points in same conjugacy classes of  $\text{Spec}(Z_{Fr})$ , equivalently, in the same symplectic leaves, are isomorphic as  $\mathbb{C}$ -algebras.*

## 9.2. Degeneration of $U_q^{ev}(\mathfrak{g})$ .

One of the main tool in [DCKP2] is the remarkable degeneration of the form. In this section, we establish the similar degeneration for  $U_q^{ev}(\mathfrak{g})$  for  $q \in \text{Spec}(\mathcal{A}')$ .

Recall that we fixed a reduced expression of the longest element  $w_0 = s_{i_1} \dots s_{i_N}$ . This defines the convex ordering on  $\Delta_+$

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \quad \dots, \quad \beta_N = s_{i_1} \dots s_{i_{N-1}}(\alpha_{i_N}).$$

(convex mean if  $\alpha < \beta \in \Delta_+$  such that  $\alpha + \beta \in \Delta_+$  then  $\alpha < \alpha + \beta < \beta$ )

With this reduced expression, we constructed PBW bases of  $U_q^{ev}(\mathfrak{g})$  in Lemma 2.10.

**Lemma 9.5.** (a) For  $i < j$  one has:

$$F_{\beta_i} F_{\beta_j} - q^{(\beta_i, \beta_j)} F_{\beta_j} F_{\beta_i} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} a_{\vec{k}} F^{\vec{k}},$$

where  $a_{\vec{k}} \in \mathbb{C}[q, q^{-1}]$  and  $a_{\vec{k}} \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  is such that  $k_s = 0$  for  $s \leq i$  and  $s \geq j$ .

(b) For  $i < j$  one has:

$$E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} b_{\vec{k}} E^{\vec{k}},$$

where  $b_{\vec{k}} \in \mathbb{C}[q, q^{-1}]$  and  $b_k \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  such that  $k_s = 0$  for  $s \leq i$  and  $s \geq j$ .

*Proof.* By [DCKP2, Lemma 4.2], for  $i < j$  one has

$$(9.8) \quad E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} c_{\vec{k}} E^{\vec{k}},$$

where  $c_{\vec{k}} \in \mathbb{C}[q, q^{-1}]$  and  $c_{\vec{k}} \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  is such that  $k_s = 0$  for  $s \leq i$  and  $s \geq j$ . Note that the right hand side of (9.8) is  $E^{\vec{k}}$ , not  $E^{\vec{k}}$ .

(a) Follows by applying the anti-automorphism  $\tau$  (1.11).

(b) By (9.8), one obtain

$$E^{\vec{r}} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} d_{\vec{k}} E^{\vec{k}},$$

Here  $\vec{r} = (0, \dots, 0, r_i, \dots, r_j, 0, \dots, 0)$  and  $d_{\vec{k}} \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  is such that  $k_s = 0$  for  $s < i$  and  $s > j$ . Then (9.8) implies part (b).  $\square$

**Corollary 9.6.** (a) For  $i < j$  one has:

$$q^{-(\beta_j, \kappa(\beta_i))} \tilde{F}_{\beta_i} \tilde{F}_{\beta_j} - \tilde{F}_{\beta_j} \tilde{F}_{\beta_i} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} a'_{\vec{k}} \tilde{F}^{\vec{k}},$$

where  $a'_{\vec{k}} \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  is such that  $k_s = 0$  for  $s \leq i$  and  $s \geq j$ .

(b) For  $i < j$  one has:

$$\tilde{E}_{\beta_j} \tilde{E}_{\beta_i} - q^{-(\beta_j, \kappa(\beta_i))} \tilde{E}_{\beta_i} \tilde{E}_{\beta_j} = \sum_{\vec{k} \in \mathbb{Z}_{\geq 0}^N} b'_{\vec{k}} \tilde{E}^{\vec{k}},$$

where  $b'_{\vec{k}} \neq 0$  only when  $\vec{k} = (k_1, \dots, k_N)$  is such that  $k_s = 0$  for  $s \leq i$  and  $s \geq j$ .

For any  $\vec{k}, \vec{r} \in \mathbb{Z}_{\geq 0}^N$  and  $u \in U_q^{ev0}(\mathfrak{g})$ , let

$$M_{\vec{k}, \vec{r}, u} = \tilde{F}^{\vec{k}} u \tilde{E}^{\vec{r}},$$

in which  $\tilde{F}^{\vec{k}} = \tilde{F}_{\beta_N}^{k_N} \dots \tilde{F}_{\beta_1}^{k_1}$  and  $\tilde{E}^{\vec{r}} = \tilde{E}_{\beta_N}^{r_N} \dots \tilde{E}_{\beta_1}^{r_1}$ . For any positive roots  $\beta = \sum_i b_i \alpha_i$ , we define the height  $\text{ht}\beta = \sum_i b_i$ . Let

$$\text{ht}(M_{\vec{k}, \vec{r}, u}) = \sum_i (k_i + r_i) \text{ht}(\beta_i).$$

Then we define degree of  $M_{\vec{k}, \vec{r}, u}$  as follows:

$$\deg(M_{\vec{k}, \vec{r}, u}) = (r_N, \dots, r_1, k_N, \dots, k_1, \text{ht}(M_{\vec{k}, \vec{r}, u})) \in \mathbb{Z}_{\geq 0}^{2N+1}.$$

By Corollary 9.6 and commutator relations between  $\tilde{E}_i$  and  $\tilde{F}_j$  in (2.14), we obtain the following

**Proposition 9.7.** *The associated graded algebra  $\text{gr}U_q^{ev}$  of the  $\mathbb{Z}_{\geq 0}^{2N+1}$ -filtered algebra  $U_q^{ev}(\mathfrak{g})$  is an algebra over  $\mathbb{C}$  on the generators*

$$X_\alpha = \tilde{E}_\alpha K^{\gamma(\alpha)}, \quad Y_\alpha = \tilde{F}_\alpha, \quad K^\lambda \quad (\lambda \in 2P)$$

subject to the following relations:

$$\begin{aligned} K^{\lambda_1}K^{\lambda_2} &= K^{\lambda_1+\lambda_2}, \quad K^0 = 1 \\ K^\lambda X_\alpha &= q^{(\lambda,\alpha)}X_\alpha K^\lambda, \quad K^\lambda Y_\alpha = q^{-(\lambda,\alpha)}Y_\alpha K^\lambda, \\ X_\alpha Y_\beta &= Y_\beta X_\alpha, \\ X_\alpha X_\beta &= q^{-(\kappa(\alpha),\beta)}X_\beta X_\alpha \quad \text{if } \alpha > \beta \text{ in the convex ordering} \\ Y_\alpha Y_\beta &= q^{-(\alpha,\kappa(\beta))}Y_\beta Y_\alpha \quad \text{if } \alpha > \beta \text{ in the convex ordering} \end{aligned}$$

in which  $\lambda_1, \lambda_2 \in 2P$  and  $\alpha, \beta \in \Delta_+$  the set of positive roots of  $\mathfrak{g}$ .

**Remark 9.8** (cf. Remark 4.2 [DCKP2]). Considering the degree by total height  $d_0$ , we obtain a  $\mathbb{Z}_{\geq 0}$ -filtration of  $U_q^{ev}(\mathfrak{g})$ ; let  $U_q^{(0)}$  be the associated graded algebra. Letting  $d_1(M_{\vec{k}, \vec{r}, u}) = k_1$ , we obtain a  $\mathbb{Z}_{\geq 0}$ -filtration of  $U_q^{(0)}$ , let  $U_q^{(1)}$  be the associated graded algebra. One keeps going. It is clear that at the last step we get the algebra  $\text{gr}U_q^{ev} \cong U_q^{(2N)}$ .

The following corollary is proved as same as in Corollary 1.8 in [DCK]

**Corollary 9.9.**  $\text{gr}U_q^{ev}$  has no zero divisor. Hence all algebras  $U^{(i)}$  in Remark 9.8 have no zero divisor, in particular,  $U_q^{ev}(\mathfrak{g})$  has no zero divisor.

Now we go back to the case then  $q = \epsilon$  is the order  $\ell$  root of unity. Let  $A$  be an algebra with no zero divisors. Let  $Z$  be the center of  $A$  and  $Q(Z)$  be the quotient field of  $Z$ , and let  $Q(A) = Q(Z) \otimes_Z A$ . The algebra  $A$  is called *integrally closed* if for any subring  $B$  of  $Q(A)$  such that  $A \subset B \subset z^{-1}A$  for some  $z \neq 0 \in Z$ , we have  $B = A$ . The following proposition is proved by Theorem 6.5 [DP]

**Proposition 9.10.** All algebras  $U_\epsilon^{(i)}$  in Remark 9.8 are integrally closed. In particular, both  $U_\epsilon^{ev}(\mathfrak{g})$  and  $\text{gr}U_\epsilon^{ev}$  are integrally closed.

### 9.3. Finite dimensional representations of $U_\epsilon^{ev}(\mathfrak{g})$ .

All of the following facts about orders are in [DCKP2, §1].

**Definition 9.11.** (a) Let  $R$  be a Noetherian domain over  $\mathbb{C}$  and  $K$  be its fraction field. An  $R$ -algebra  $A$  is called *an order* if  $A$  is finitely generated torsion free over  $R$  and  $A_K := A \otimes_R K$  is a finite dimensional algebra over  $K$ .

(b) An  $R$ -order  $A$  is a *maximal order* if there is no finitely generated  $R$ -algebra  $B$  of  $A_K$  that strictly contains  $A$  (Automatically, if such  $B$  exists then it is an  $R$ -order).

We will be interested in the  $R$ -orders  $A$  such that  $R = Z$  the center of  $A$  and  $A_K$  is a central simple algebra over  $K$ . Note that in that case  $K = Q(Z)$ , the fraction field of  $Z$ . We will call such  $A$  *an order in a central simple algebra*. Let  $\overline{Q(Z)}$  be the algebraic closure of  $Q(Z)$  then  $A \otimes_Z \overline{Q(Z)} \cong \text{Mat}_d(\overline{Q(Z)})$  for some  $d \in \mathbb{N}$ .

**Definition 9.12.** Let  $A$  be an order in a central simple algebra. The number  $d$  as above is called *the degree* of  $A$ .

**Remark 9.13.** (see [Re]) We have on  $A \otimes_Z Q(Z)$  (and on  $A$ ) the reduced trace map  $\text{tr} : A \otimes_Z Q(Z) \rightarrow Q(Z)$ . If  $Z$  is normal then  $\text{tr}(A) \subset Z$ . In particular, this holds for maximal order since the center of a maximal order is normal.

**Theorem 9.14** (Theorem 1.1 [DCKP2]). Suppose  $A$  is a finitely generated algebra over  $\mathbb{C}$  and  $A$  is an order closed under the trace (i.e.,  $\text{tr}(A) \subset Z$ ) in a central simple algebra then

(a)  $Z$  is a finitely generated algebra over  $\mathbb{C}$ .

(b) The points of  $\text{Spec}(Z)$  parametrize equivalence classes of  $d$ -dimensional semisimple representations of  $A$ .

(c) Let us consider the natural map  $\chi : \text{Irr}(A) \rightarrow \text{Spec}(Z)$  from the set of irreducible representations of  $A$  to the corresponding characters of  $Z$ . This map is surjective. Each fiber consists of all those representations which are irreducible components of the corresponding semisimple representation. In particular, each irreducible representation of  $A$  has dimension at most  $d$ .

(d) The set

$$\Omega_A = \{a \in \text{Spec}Z \mid \text{the corresponding semisimple representation is irreducible}\}$$

is a nonempty Zariski open subset of  $\text{Spec}Z$ , which is dense since  $Z$  is an integral domain.

(e) Suppose  $Z$  is a finitely generated module over a subalgebra  $Z_0$ . Let consider the map

$$\text{Irr}(A) \xrightarrow{\chi} \text{Spec}Z \xrightarrow{\tau} \text{Spec}Z_0.$$

Then the set

$$\Omega_A^0 = \{a \in \text{Spec}Z_0 \mid (\tau \circ \chi)^{-1}(a) \text{ contains all irreducible representations of dimension } d\}.$$

is a nonempty Zariski open subset of  $\text{Spec}(Z_0)$ , which is dense since  $Z_0$  is an integral domain.

Since  $U_\epsilon^{ev}(\mathfrak{g})$  has no zero divisor and finitely generated over  $Z$ ,  $U_\epsilon^{ev}(\mathfrak{g})$  is an order in a central simple algebra. Using the fact that  $U_\epsilon^{ev}(\mathfrak{g})$  is integrally closed in Proposition 9.10, one can easily show that  $U_\epsilon^{ev}(\mathfrak{g})$  is a maximal order so that  $U_\epsilon^{ev}(\mathfrak{g})$  is closed under the trace. Therefore, Theorem 9.14 can be applied to the collection  $(U_\epsilon^{ev}(\mathfrak{g}), Z, Z_0 = Z_{Fr})$ . Combining with Proposition 9.4, we have

**Proposition 9.15.**  $\Omega_{U_\epsilon^{ev}}^0$  is a union of some symplectic leaves of  $\text{Spec}Z_{Fr}$ .

We now will determine the degree  $d$  of  $U_\epsilon^{ev}(\mathfrak{g})$ . The character of  $Z_{Fr}$  on any irreducible representations of  $U_\epsilon^{ev}$  will be called the *Frobenius central character*. We need the following proposition which will be proved in the next section

**Proposition 9.16.** The degree of  $\text{gr}U_\epsilon^{ev}$  is  $\prod_{\alpha \in \Delta_+} \ell_\alpha$ .

For any point  $p \in \text{Spec}\mathbb{C}[K^{2\lambda}]_{\lambda \in P} \cong T$ , we define the diagonal module  $M_{p,0}$  as follows:

$$M_{p,0} = U_\epsilon^{ev}/U_\epsilon^{ev}(\tilde{E}_i, K^{2\lambda} - p(K^{2\lambda}), \tilde{F}_\alpha^{\ell_\alpha}),$$

with  $\alpha \in \Delta_+$ ,  $1 \leq i \leq r$  and  $q$  viewed as the algebra morphism  $\mathbb{C}[K^{2\lambda}]_{\lambda \in P} \rightarrow \mathbb{C}$ .

We recall the identification in Proposition 5.10

$$Z_{Fr} = \mathbb{C}[\tilde{E}_\alpha^{\ell_\alpha} K^{\ell_\alpha \gamma(\alpha)}]_{\alpha \in \Delta_+} \otimes \mathbb{C}[K^{2\lambda}]_{\lambda \in P^*} \otimes \mathbb{C}[\tilde{F}_\alpha^{\ell_\alpha} K^{\ell_\alpha \kappa(\alpha)}]_{\alpha \in \Delta_+} \cong \mathbb{C}[U_-^d] \otimes \mathbb{C}[T^d] \otimes \mathbb{C}[U_+^d]$$

Let  $p_\ell$  denote the point in  $T^d$  defined by  $K^{2\lambda} \mapsto p(K^{2\lambda})$  for all  $\lambda \in P^*$ . The following lemma is straightforward:

**Lemma 9.17.** Dimension of  $M_{p,0}$  is  $\prod_{\alpha \in \Delta_+} \ell_\alpha$ . The Frobenius central character of  $M_{p,0}$  corresponds to the point  $(1, p_\ell, 1) \in U_-^d \times T^d \times U_+^d$ .

**Corollary 9.18.** At any point  $\xi \in 1 \times T^d \times 1 \subset \text{Spec}Z_{Fr}$ , there are modules  $M_{p,0}$  with the Frobenius central character corresponding to the point  $\xi$ .

Now we can compute the degree of  $U_\epsilon^{ev}(\mathfrak{g})$

**Proposition 9.19.** (a)  $d = \prod_{\alpha \in \Delta_+} \ell_\alpha$ .

(b)  $\dim_{Q(Z_{Fr})} Q(Z) = \prod_{i=1}^r \ell_i$ .

*Proof.* (a) Since degeneration can not increase the degree, from Proposition 9.16, we have

$$d = \deg(U_\epsilon^{ev}(g)) \geq \prod_{\alpha \in \Delta_+} \ell_\alpha.$$

Let  $G^d T^d$  be the union of intersections of all conjugacy classes of  $T^d$  with  $\text{Spec } Z_{Fr}$  then  $G^d T^d \cap \Omega_{U_\epsilon^{ev}}^0$  is non-empty since both are open Zariski dense. Since  $\Omega_{U_\epsilon^{ev}}^0$  is the union of symplectic leaves, we must have

$$T^d \cap \Omega_{U_\epsilon^{ev}}^0 \neq \emptyset.$$

Let  $\xi \in T^d \cap \Omega_{U_\epsilon^{ev}}^0$ . Since  $\xi \in \Omega_{U_\epsilon^{ev}}^0$ , all irreducible representations with Frobenius central character  $\xi$  must have dimension  $d$ . On the other hand, by Corollary 9.18, there are some irreducible representations with Frobenius central character  $\xi$  and dimension smaller than  $\prod_{\alpha \in \Delta_+} \ell_\alpha$ . Hence

$$d \leq \prod_{\alpha \in \Delta_+} \ell_\alpha.$$

Therefore,  $d = \prod_{\alpha \in \Delta_+} \ell_\alpha$ .

(b) Since  $Z$  is a finitely generated module over  $Z_{Fr}$  and both are integral domains, it follows that  $Q(Z) = Z \otimes_{Z_{Fr}} Q(Z_{Fr})$  and then  $Q(U_\epsilon^{ev}(\mathfrak{g})) = U_\epsilon^{ev}(\mathfrak{g}) \otimes_{Z_{Fr}} Q(Z_{Fr})$ . Then part (b) holds by the following equalities

$$\begin{aligned} \dim_{Q(Z)} Q(U_\epsilon^{ev}(\mathfrak{g})) &= d^2 = \left( \prod_{\alpha \in \Delta_+} \ell_\alpha \right)^2 \\ \dim_{Q(Z_{Fr})} Q(U_\epsilon^{ev}(\mathfrak{g})) &= \left( \prod_{\alpha \in \Delta_+} \ell_\alpha \right)^2 \cdot \prod_{1 \leq i \leq r} \ell_i \\ \dim_{Q(Z_{Fr})} Q(U_\epsilon^{ev}(\mathfrak{g})) &= \dim_{Q(Z_{Fr})} Q(Z) \cdot \dim_{Q(Z)} Q(U_\epsilon^{ev}(\mathfrak{g})). \end{aligned}$$

The first equality follows by the definition 9.12 applying to  $U_\epsilon^{ev}(\mathfrak{g})$ . The second equality follows by Corollary 5.4.  $\square$

#### 9.4. The degree of $\text{gr } U_\epsilon^{ev}$ .

The algebra  $\text{gr } U_\epsilon^{ev}$  belongs to the family of *twisted (Laurent) polynomial algebras* in [DCKP2, §2]. Let  $q \in \mathbb{C}^\times$ . Given an  $n \times n$  skew-symmetric matrix  $H = (h_{ij})$  over  $\mathbb{Z}$ , the twisted polynomial algebra  $\mathbb{C}_H[x_1, \dots, x_n]$  is the algebra on generators  $x_1, \dots, x_n$  and the following defining relations:

$$x_i x_j = q^{h_{ij}} x_j x_i \quad (i, j = 1, \dots, n).$$

The twisted Laurent polynomial algebra  $\mathbb{C}_H[x_1^\pm, \dots, x_n^\pm]$  is similarly defined.

**Lemma 9.20** (Lemma 2.2 [DCKP2]). *In any irreducible  $\mathbb{C}_H[x_1, \dots, x_n]$ -module each element  $x_i$  is either 0 or invertible.*

Let  $q = \epsilon$  be a root of unity of order  $\ell$ . The matrix  $H$  defines a homomorphism

$$\underline{H} : \mathbb{Z}^n \rightarrow (\mathbb{Z}/\ell\mathbb{Z})^n.$$

Let  $h$  be the cardinality of the image of the homomorphism  $\underline{H}$ . Let  $K$  be the kernel of  $\underline{H}$ . We will need the following result in [DCKP2, Proposition 2] (for more discussions about twisted polynomials and its representation, see [DCKP2, §2]).

**Proposition 9.21.** (a) *The elements  $x^a = x_1^{a_1} \dots x_n^{a_n}$  with  $a \in K \cap \mathbb{Z}_{\geq 0}^n$  (resp.  $a \in K$ ) form a basis of the center of  $\mathbb{C}_H[x_1, \dots, x_n]$  (resp.  $\mathbb{C}_H[x_1^\pm, \dots, x_n^\pm]$ ).*

(b)  $\deg \mathbb{C}_H[x_1, \dots, x_n] = \deg \mathbb{C}_H[x_1^\pm, \dots, x_n^\pm] = \sqrt{h}$ .

(c) *The algebra  $\mathbb{C}_H[x_1^\pm, \dots, x_n^\pm]$  is Azumaya over its center.*

Let  $S \subset \{1, \dots, n\}$  and  $H_S$  be the matrix obtained from  $H$  by removing all rows and columns labeled by indices not in  $S$ .

**Lemma 9.22.** *Let  $Z$  be the center of  $\mathbb{C}_H[x_1, \dots, x_n]$ . Assume  $Z$  is a finitely generated module over a subalgebra  $Z_0$  and there is  $\{x_i^{m_i} | 1 \leq i \leq n\} \subset Z_0$ . Let  $\mathcal{O}_0$  be the set of zeroes in  $\text{Spec}Z_0$  defined by equations  $x_i^{m_i} = 0, i \notin S$ . Assume  $\deg \mathbb{C}_{H_S}[x_i]_{i \in S} = \deg \mathbb{C}_H[x_1, \dots, x_n]$  then*

$$(9.9) \quad \Omega_{\mathbb{C}_H[x_1, \dots, x_n]}^0 \cap \mathcal{O}_0 \text{ is a nonempty Zariski open subset in } \mathcal{O}_0.$$

*Proof.* Since  $\Omega_{\mathbb{C}_H[x_1, \dots, x_n]}^0$  is a Zariski open subset of  $\text{Spec}Z_0$ , it is enough to show that

$$\Omega_{\mathbb{C}_H[x_1, \dots, x_n]}^0 \cap \mathcal{O}_0 \neq \emptyset.$$

Let  $Z_0[x_j^{-m_j}]_{j \in S}$  be the localization of  $Z_0$  at  $\{x_j^{m_j} | j \in S\}$ , then  $\text{Spec}(Z_0[x_j^{-m_j}]_{j \in S})$  is an nonempty open subset in  $\text{Spec}Z_0$ . On any irreducible modules of  $\mathbb{C}_H[x_1, \dots, x_n]$  whose characters contained in  $\text{Spec}(Z_0[x_j^{-m_j}]_{j \in S})$ , the action of  $x_j, j \in S$ , will be invertible by Lemma 9.20. Therefore, irreducible modules of  $\mathbb{C}_H[x_1, \dots, x_n]$  whose characters contained in  $\text{Spec}(Z_0[x_j^{-m_j}]_{j \in S})$  are one-to-one correspondent to irreducible modules of  $\mathbb{C}_H[x_i, x_j^\pm]_{i \notin S, j \in S}$ .

Since there is  $\{x_i^{m_i} | 1 \leq i \leq n\}$  contained in  $Z_0$ , the intersection  $\mathcal{O}_0 \cap \text{Spec}(Z_0[x_j^{-m_j}]_{j \in S})$  is nonempty. By Lemma 9.20, on any irreducible modules over  $\mathbb{C}_H[x_1, \dots, x_n]$  whose characters contained in  $\mathcal{O}_0$ , the actions of  $x_i, i \notin S$ , are zero. Therefore, irreducible modules of  $\mathbb{C}_H[x_1, \dots, x_n]$  whose characters contained in  $\mathcal{O}_0 \cap \text{Spec}(Z_0[x_j^{-m_j}]_{j \in S})$  are one-to-one correspondent with irreducible modules of  $\mathbb{C}_{H_S}[x_j^\pm]_{j \in S}$ . Then by the assumption  $\deg \mathbb{C}_{H_S}[x_j]_{j \in S} = \deg \mathbb{C}_H[x_1, \dots, x_n]$  and Lemma 9.21(c), it follows that all irreducible modules of  $\mathbb{C}_H[x_1, \dots, x_n]$  whose characters contained in  $\mathcal{O}_0 \cap \text{Spec}Z_0[x_j^{-m_j}]_{j \in S}$  are of the maximal dimension. Therefore,  $\emptyset \neq \mathcal{O}_0 \cap \text{Spec}(Z_0[x_j^{-m_j}]_{j \in S} \subset \mathcal{O}_0 \cap \Omega_{\mathbb{C}_H[x_1, \dots, x_n]}^0)$ . This finishes the proof.  $\square$

By Proposition 9.7,  $\text{gr}U_\epsilon^{ev}$  is the algebra

$$\mathbb{C}_H[K^{\pm 2\omega_1}, \dots, K^{\pm 2\omega_r}, X_{\beta_1}, \dots, X_{\beta_N}, Y_{\beta_1}, \dots, Y_{\beta_N}]$$

with the skew-symmetric matrix  $H$ :

$$(9.10) \quad H = \begin{bmatrix} B & 0 & -A^T \\ 0 & C & A^T \\ A & -A & 0 \end{bmatrix}$$

Let us describe  $H$ :

- The columns from left to right are labeled by  $X_{\beta_1}, \dots, X_{\beta_N}, Y_{\beta_1}, \dots, Y_{\beta_N}, K^{2\omega_1}, \dots, K^{2\omega_r}$ . The rows from top to bottom are labeled by the same set.
- The matrix  $B = (b_{ij})$  is a  $N \times N$  skew symmetry with  $b_{ij} = (\beta_i, \kappa(\beta_j))$  for  $i < j$ .
- The matrix  $C = (c_{ij})$  is a  $N \times N$  skew-symmetry with  $c_{ij} = (\kappa(\beta_i), \beta_j)$  for  $i < j$
- The matrix  $A = (a_{ij})$  is of the size  $r \times N$  with  $a_{ij} = (2\omega_i, \beta_j)$ .

To compute the cardinality of image of  $\underline{H}$ , we need the results about the root systems in [DCKP2, §3]. Given a simple root  $\alpha$ , let

$$(9.11) \quad I_\alpha = \{1 \leq j \leq N | s_{i_j} = s_\alpha\}, \quad \mathcal{I}_\alpha = \{\beta_{k_1}, \dots, \beta_{k_t} | k_j \in I_\alpha\},$$

**Lemma 9.23** (Lemma 3.2, [DCKP2]). *Fix a simple root  $\alpha$  and let  $\omega$  be the corresponding fundamental weight. Let  $k_1 < \dots < k_r$  be all elements of the set  $I_\alpha$ . For  $t \in \mathbb{Z}$ ,  $1 \leq t \leq r$ , let*

$$\lambda_t = s_{\beta_{k_t}} \dots s_{\beta_{k_1}}(\omega), \quad \mu_t = -s_{\beta_{k_{r-t+1}}} \dots s_{\beta_{k_r}}(\omega^*),$$

here  $\omega^* = -w_0(\omega)$ . Let  $\beta_j^\vee := 2\beta_j/(\beta_j, \beta_j)$  for all  $1 \leq j \leq N$ . Then:

(a)  $\lambda_t = s_{i_1} \dots s_{i_{k_t}}(\omega)$ ; in particular  $\lambda_r = -\omega^*$ .

- (b) If  $k_t < j < k_{t+1}$ , then  $(\lambda_t, \beta_j^\vee) = 0$ .
  - (c)  $(\lambda_t, \beta_{k_{t+1}}^\vee) = 1$ .
  - (d)  $\lambda_t = \omega - \sum_{i=1}^t \beta_{k_i}$ .
- Similarly:
- (a')  $\mu_t = \lambda_{r-t}$ ; in particular  $\mu_r = \omega$ .
  - (b') If  $k_{r-t} < j < k_{r-t+1}$ , then  $(\mu_t, \beta_j^\vee) = 0$ .
  - (c')  $(\mu_t, \beta_{k_{r-t}}^\vee) = -1$ .
  - (d')  $-s_{\beta_{k_1}} \dots s_{\beta_{k_r}}(w^*) = -\omega^* + \sum_{i=1}^r \beta_{k_i} = \omega$ .

**Corollary 9.24** (Corollary 3.2, [DCKP2]). (a)  $(\beta_l, \omega) = \sum_{k_i < l} (\beta_l, \beta_{k_i})$  if  $l \notin I_\alpha$ .  
(b)  $(\beta_{k_t}, 2\omega) = (\beta_{k_t}, \beta_{k_t}) + \sum_{i < t} (2\beta_{k_1}, \beta_{k_i})$  for  $k_t \in I_\alpha$ .

For each set  $I_{\alpha_i}$ , pick  $\beta_{m_i} \in \mathcal{I}_{\alpha_i}$ . Let

$$(9.12) \quad S = \begin{cases} \{K^{2\omega_1}, \dots, K^{2\omega_r}, Y_{\beta_i}, X_{\beta_i}\} / \{X_{\beta_{m_1}}, \dots, X_{\beta_{m_r}}\} \\ \{K^{2\omega_1}, \dots, K^{2\omega_r}, Y_{\beta_i}, X_{\beta_i}\} / \{Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}\} \end{cases}$$

Let  $H_S$  be the  $2N \times 2N$  submatrix of  $H$  consisting of columns and rows labeled by the set  $S$ . So we have the following morphisms

$$\underline{H} : \mathbb{Z}^{2N+r} \rightarrow (\mathbb{Z}/\ell\mathbb{Z})^{2N+r}, \quad \underline{H}_S : \mathbb{Z}^{2N} \rightarrow (\mathbb{Z}/\ell\mathbb{Z})^{2N}.$$

**Proposition 9.25.** The cardinalities of  $\text{Im } \underline{H}$  and  $\text{Im } \underline{H}_S$  are equal to  $\left(\prod_{\alpha \in \Delta_+} \ell_\alpha\right)^2$ .

*Proof.* Note that the following operators do not change the cardinalities of  $\text{Im } \underline{H}$  and  $\text{Im } \underline{H}_S$ :

- Swapping rows or columns.
- Adding integral multiply of one row (resp. column) into other row (resp. column).
- Multiply rows (columns) by  $\pm 1$ .

Using these operators, we will transform  $H$  and  $H_S$  into diagonal matrices with non zero entries are  $(\beta_i, \beta_i)$ ,  $1 \leq i \leq N$ , each occurs twice. This will imply the lemma. Indeed

$$\text{Im } \underline{H} \cong \bigoplus_{i=1}^N \left( ((\beta_i, \beta_i)\mathbb{Z} + \ell\mathbb{Z})/\ell\mathbb{Z} \right)^{\oplus 2} \cong \bigoplus_{i=1}^N \left( \text{gcd}((\beta_i, \beta_i), \ell)\mathbb{Z}/\ell\mathbb{Z} \right)^{\oplus 2}.$$

Hence  $\#\text{Im } \underline{H} = \left(\prod_{\alpha \in \Delta_+} \ell_\alpha\right)^2$ . The case of  $H_S$  is the same.

Now let us describe how to transform  $H, H_S$  into such diagonal matrices. Recall the conventions in (9.10).

**Computations on  $H$ .** We will do the case of  $H$  first. The following equalities are frequently used

$$(9.13) \quad (\gamma(\alpha), \beta) = (\alpha, \kappa(\beta)), \quad \gamma(\alpha) + \kappa(\alpha) = 2\alpha, \quad (\gamma(\alpha), \alpha) = (\alpha, \kappa(\alpha)) = (\alpha, \alpha).$$

*Step 1:* Use the last  $r$  columns to modify  $\begin{bmatrix} B \\ 0 \\ A \end{bmatrix}$  and use the last  $r$  rows to modify  $\begin{bmatrix} 0 & C & A^T \end{bmatrix}$ .

( $\star$ ) The column of  $\begin{bmatrix} B \\ 0 \end{bmatrix}$  at  $X_{\beta_j}$  is

$$[(\beta_1, \kappa(\beta_j)) \quad \dots \quad (\beta_{j-1}, \kappa(\beta_j)) \quad 0 \quad -(\kappa(\beta_{j+1}), \beta_j) \quad \dots \quad -(\kappa(\beta_N), \beta_j) \quad 0 \quad \dots \quad 0]^T.$$

By adding the following vector, which is an integral linear combinations of columns in  $\begin{bmatrix} -A^T \\ A^T \end{bmatrix}$

$$[-(\beta_1, \kappa(\beta_j)) \quad \dots \quad -(\beta_N, \kappa(\beta_j)) \quad (\beta_1, \kappa(\beta_j)) \quad \dots \quad (\beta_N, \kappa(\beta_j))]^T,$$

we get

$$(9.14) \quad [0 \ \dots \ 0 \ -(\beta_j, \beta_j) \ -2(\beta_{j+1}, \beta_j) \ \dots \ -2(\beta_N, \beta_j) \ (\beta_1, \kappa(\beta_j) \ \dots \ (\beta_N, \kappa(\beta_j))]^T,$$

here the entries  $-2(\beta_k, \beta_j)$  and  $(\beta_j, \beta_j)$  show up because by (9.13)

$$-(\kappa(\beta_k), \beta_j) - (\beta_k, \kappa(\beta_j)) = -(\kappa(\beta_k) + \gamma(\beta_k), \beta_j) = -2(\beta_k, \beta_j), \quad (\beta_j, \kappa(\beta_j)) = (\beta_j, \beta_j).$$

( $\star$ ) Similarly, the row of  $[0 \ C]$  at  $Y_{\beta_i}$  is

$$[0 \ -( \beta_i, \kappa(\beta_1)) \ \dots \ -( \beta_i, \kappa(\beta_{i-1})) \ 0 \ (\kappa(\beta_i), \beta_{i+1}) \ \dots \ (\kappa(\beta_i), \beta_N)].$$

By adding with vector,

$$[-(\gamma(\beta_i), \beta_1) \ \dots \ -(\gamma(\beta_i), \beta_N) \ (\gamma(\beta_i), \beta_1) \ \dots \ (\gamma(\beta_i), \beta_N)]$$

we get

$$(9.15) \quad [-(\gamma(\beta_i), \beta_1) \ \dots \ -(\gamma(\beta_i), \beta_N) \ 0 \ (\beta_i, \beta_i) \ 2(\beta_i, \beta_{i+1}) \ \dots \ 2(\beta_i, \beta_N)].$$

Note that the last entries of columns (9.14) will cancel the first entries of rows:  $(\beta_i, \kappa(\beta_j)) - (\gamma(\beta_i), \beta_j) = 0$ . Therefore, we get

$$H_1 = \begin{bmatrix} B_1 & 0 & -A^T \\ 0 & C_1 & A^T \\ A & -A & 0 \end{bmatrix},$$

with

•  $B_1$  is a lower triangular matrix with the column at  $X_{\beta_j}$  is

$$[0 \ \dots \ 0 \ -(\beta_j, \beta_j) \ -2(\beta_{j+1}, \beta_j) \ \dots \ -2(\beta_N, \beta_j)]^T.$$

•  $C_1$  is an upper triangular matrix with row at  $Y_{\beta_i}$  is

$$[0 \ \dots \ 0 \ (\beta_i, \beta_i) \ 2(\beta_i, \beta_{i+1}) \ \dots \ 2(\beta_i, \beta_N)].$$

Let

$$\begin{aligned} v_{\beta_i} &= [0 \ \dots \ 0 \ (\beta_i, \beta_i) \ 2(\beta_i, \beta_{i+1}) \ \dots \ 2(\beta_i, \beta_N)] \\ V_{\beta_j} &= [2(\beta_1, \beta_j) \ \dots \ 2(\beta_{j-1}, \beta_j) \ (\beta_j, \beta_j) \ 0 \ \dots \ 0]^T \end{aligned}$$

Then

$$B_1 = [-v_{\beta_1}^T \ \dots \ -v_{\beta_N}^T], \quad C_1 = [V_{\beta_1} \ \dots \ V_{\beta_N}] = \begin{bmatrix} v_{\beta_1} \\ \vdots \\ v_{\beta_N} \end{bmatrix} = -B_1^T.$$

*Step 2:* Let recall  $\mathcal{I}_{\alpha_l} = \{\beta_{k_1}^{\omega_l}, \dots, \beta_{k_{s_l}}^{\omega_l}\}$ , here the cardinality of  $\mathcal{I}_{\alpha_l}$  is  $s_l$  and we put the superscript to emphasize the elements in this set. Then

$$\begin{aligned} [(2\omega_l, \beta_1) \ \dots \ (2\omega_l, \beta_N)] &= v_{\beta_{k_1}^{\omega_l}} + \dots + v_{\beta_{k_{s_l}}^{\omega_l}} \\ [(2\omega_l^*, \beta_1) \ \dots \ (2\omega_l^*, \beta_N)]^T &= V_{\beta_{k_1}^{\omega_l}} + \dots + V_{\beta_{k_{s_l}}^{\omega_l}}. \end{aligned}$$

Indeed, by Corollary 9.24, we have

$$\begin{aligned} (2\omega_l, \beta_j) &= \begin{cases} \sum_{i=1}^{t-1} 2(\beta_{k_i}^{\omega_l}, \beta_j), & k_{t-1} < j < k_t \\ \sum_{i=1}^{t-1} 2(\beta_{k_i}^{\omega_l}, \beta_{k_t}) + (\beta_{k_t}, \beta_{k_t}), & j = k_t \end{cases} \\ (2\omega_l^*, \beta_j) &= (2 \sum_{i=1}^{s_l} \beta_{\beta_{k_i}}^{\omega_l}, \beta_j) - (2\omega_l, \beta_j) = \begin{cases} \sum_{i=t}^{s_l} 2(\beta_{k_i}^{\omega_l}, \beta_j), & k_{t-1} < j < k_t \\ \sum_{i=t+1}^{s_l} 2(\beta_{k_i}^{\omega_l}, \beta_{k_t}) + (\beta_{k_t}, \beta_{k_t}), & j = k_t \end{cases} \end{aligned}$$

*Step 3:* Using Step 2, we can use the columns of  $B_1$  to eliminate top right submatrix  $-A^T$ . Meanwhile, we can use the rows of  $C_1$  to eliminate the middle submatrix  $-A$  in the last row of  $H_1$ . We will get the new matrix

$$H_2 = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & C_1 & A^T \\ A & 0 & D + E \end{bmatrix},$$

in which

- $D = (d_{ij})$  is a  $r \times r$  matrix coming from eliminating  $-A^T$ , and  $d_{ij} = -(2\omega_i, \sum_{l=1}^{s_j} \beta_{k_l}^{\omega_j})$

- $E = (e_{ij})$  is a  $r \times r$  matrix coming from eliminating  $-A$ , and  $e_{ij} = (\sum_{l=1}^{s_i} \beta_{k_l}^{\omega_i}, 2\omega_j) = -d_{ji}$

On the other hand,  $D$  is a symmetric matrix. Indeed, note that  $\omega_i + \omega_i^* = \sum_{l=1}^{s_i} \beta_{k_l}^{\omega_i}$ , hence

$$d_{ij} = -2(\omega_i, \omega_j + \omega_i^*) = -2(\omega_i + \omega_i^*, \omega_j) = d_{ji}.$$

Therefore,  $D + E = 0$

*Step 4:* We observe that in  $H_2$ , in any row or column with nonzero diagonal entries, all other entries on those rows or columns are divided by the diagonal entries. So we can use the diagonal entries of  $B_1$  to eliminate other entries in  $\begin{bmatrix} B_1 \\ 0 \\ A \end{bmatrix}$ , meanwhile, we can use the diagonal entries of  $C_1$  to eliminate other entries in  $\begin{bmatrix} 0 & C_1 & A^T \end{bmatrix}$ . So we can transform  $H_2$ , hence  $H$ , into the desired diagonal matrix.

**Computations on  $H_S$ .** Now let us do  $H_S$  with

$$S = \{X_{\beta_1}, \dots, X_{\beta_N}, Y_{\beta_1}, \dots, Y_{\beta_N}, K^{2\omega_1}, \dots, K^{2\omega_r}\} / \{Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}\}.$$

We still perform as in Step 1-2 but in Step 3, since some rows and columns of  $\begin{bmatrix} 0 & C & A^T \end{bmatrix}$  has been deleted, we will get the following matrix

$$H_{S,1} = \begin{bmatrix} B_1 & 0 & 0 \\ 0 & \underline{C}_1 & \underline{A}^T \\ A & \underline{F}_1 & \underline{F}_2 \end{bmatrix},$$

in which

- $\begin{bmatrix} \underline{C}_1 & \underline{A}^T \end{bmatrix}$  is obtained from  $\begin{bmatrix} C_1 & A^T \end{bmatrix}$  by deleting rows and columns at  $Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}$ .

- Row at  $K^{2\omega_i}$  of  $\begin{bmatrix} \underline{F}_1 & \underline{F}_2 \end{bmatrix}$  is  $-$  row at  $Y_{\beta_{m_i}}$  of  $\begin{bmatrix} C_1 & A^T \end{bmatrix}$  after deleting columns at  $Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}$

So we can multiply rows of  $\begin{bmatrix} A & \underline{F}_1 & \underline{F}_2 \end{bmatrix}$  with  $-1$  then permute rows of  $H_{S,1}$  to get

$$H_{S,2} = \begin{bmatrix} B_1 & 0 & 0 \\ * & C_2 & A^T \end{bmatrix},$$

in which  $C_2$  is obtained from  $C_1$  by deleting columns at  $Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}$ .

Now by using Step 2:

$$[(2\omega_l^*, \beta_1) \quad \dots \quad (2\omega_l^*, \beta_N)]^T = V_{\beta_{k_1}^{\omega_l}} + \dots + V_{\beta_{k_{s_l}}^{\omega_l}},$$

we can use  $C_2$  to transform  $A^T$  into a matrix whose columns are all missing columns of  $C_1$  in  $C_2$ . Then permuting columns and rows, we obtain

$$H_{S,3} = \begin{bmatrix} B_1 & 0 \\ * & C_1 \end{bmatrix},$$

which can be transformed into the desired diagonal matrix as in Step 4.  $\square$

**Corollary 9.26.** *Proposition 9.16 holds.*

**Proposition 9.27.** *The center of  $\text{gr}U_\epsilon^{ev}$  is generated by elements  $K^{\pm 2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}$  and  $u_i$  for  $1 \leq i \leq r, \alpha \in \Delta_+$ , in which*

$$u_i = K^{-2\omega_i + \kappa(\omega_i + \omega_i^*)} \prod X_{\beta_{k_i}} \prod Y_{\beta_{l_j}},$$

here  $\mathcal{I}_{\alpha_i} = \{\beta_{k_1}, \dots, \beta_{k_r}\}$  and  $\mathcal{I}_{\alpha_i^*} = \{\beta_{l_1}, \dots, \beta_{l_r}\}$ ;  $\alpha_i^* = -w_0(\alpha_i)$ ,  $\omega_i^* = -w_0(\omega_i)$ .

*Proof.* *Step 1.* We show that  $u_i$  is central in  $\text{gr}U_\epsilon^{ev}$ . By abuse of notations, we will use  $u_i$  to denote the coordinate of  $u_i$  in the lattice  $\mathbb{Z}^{2N+r}$  labeled by  $(X_{\beta_N}, \dots, X_{\beta_1}, Y_{\beta_N}, \dots, Y_{\beta_1}, K^{2\omega_1}, \dots, K^{2\omega_r})$ . Then it is enough to show that  $u_i H = 0$  with the matrix  $H$  in (9.10).

The matrix multiplication of  $u_i$  with vector column of  $H$  labeled by  $X_{\beta_j}$  is

$$\begin{aligned} & \sum_{1 \leq k \leq r} (-2\omega_i + \kappa(\omega_i + \omega_i^*), \alpha_k^\vee/2)(2\omega_k, \beta_j) + \sum_{k_m < j} (\beta_{k_m}, \kappa(\beta_j)) - \sum_{k_m > j} (\kappa(\beta_{k_m}), \beta_j) \\ &= -(2\omega_i, \beta_j) + (\kappa(\omega_i + \omega_i^*), \beta_j) + \sum_{k_m < j} (\beta_{k_m}, \kappa(\beta_j)) - \sum_{k_m > j} (\kappa(\beta_{k_m}), \beta_j) \\ &= -(2\omega_i, \beta_j) + \sum_{k_m < j} (\beta_{k_m}, 2\beta_j) + \begin{cases} (\beta_j, \beta_j) & \text{if } j \in I_{\alpha_i} \\ 0 & \text{otherwise} \end{cases} \\ &= 0 \quad \text{by Corollary 9.24} \end{aligned}$$

In the second equality we use  $\omega_i + \omega_i^* = \beta_{k_1} + \dots + \beta_{k_r}$  by Lemma 9.23,  $(\beta_{k_m}, \kappa(\beta_j)) = (\gamma(\beta_{k_m}), \beta_j) = (\beta_{k_m}, 2\beta_j) - (\kappa(\beta_{k_m}), \beta_j)$  and  $(\kappa(\beta_j), \beta_j) = (\beta_j, \beta_j)$ .

The matrix multiplication of  $u_i$  with column vector of  $H$  labeled by  $Y_{\beta_j}$  is similarly computed. The matrix multiplication of  $u_i$  with the column vector of  $H$  labeled by  $K^{2\omega_j}$  is

$$\sum_{k_m} (-2\omega_j, \beta_{k_m}) + \sum_{l_n} (2\omega_j, \beta_{l_n}) = (-2\omega_j, \omega_i + \omega_i^*) + (2\omega_j, \omega_i + \omega_i^*) = 0,$$

here we use  $\omega_i + \omega_i^* = \beta_{k_1} + \dots + \beta_{k_r} = \beta_{l_1} + \dots + \beta_{l_r}$  by Lemma 9.23.

*Step 2.* We will show that  $K^{\pm 2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}$  are central in  $\text{gr}U_\epsilon^{ev}$ . Let us do it for  $X_\alpha^{\ell_\alpha}$  only. Let  $X_\alpha^{\ell_\alpha}$  also denote the coordinate of itself in  $\mathbb{Z}^{2N+r}$  labelled by  $(X_{\beta_N}, \dots, X_{\beta_1}, Y_{\beta_N}, \dots, Y_{\beta_1}, K^{2\omega_1}, \dots, K^{2\omega_r})$ .

The matrix multiplication of  $X_\alpha^{\ell_\alpha}$  with column vector labeled by  $X_{\beta_j}$  of  $H$  in (9.10) is

$$\begin{cases} \ell_\alpha(\beta_k, \kappa(\alpha)) = 2\ell_\alpha d_\alpha(\gamma(\beta_k)/2, \alpha^\vee) & \text{if } \beta_k < \alpha \\ 0 & \text{if } \beta_k = \alpha \\ -\ell_\alpha(\kappa(\beta_k), \alpha) = 2\ell_\alpha d_\alpha(\kappa(\beta_k)/2, \alpha^\vee) & \text{if } \beta_k > \alpha \end{cases}$$

here  $d_\alpha = (\alpha, \alpha)/2$ . All these integer values are divided by  $\ell$  since  $\gamma(\beta_k), \kappa(\beta_k) \in 2P$ . Do the same computations with the rest of column vectors in  $H$ .

*Step 3.* Let us use  $K^{2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}, u_i$  to denote the coordinates of corresponding elements in  $\mathbb{Z}^{2N+r}$  labeled by  $(X_{\beta_N}, \dots, X_{\beta_1}, Y_{\beta_N}, \dots, Y_{\beta_1}, K^{2\omega_1}, \dots, K^{2\omega_r})$ . Let  $K'$  denote the sublattice generated by  $K^{2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}, u_i$  in  $\mathbb{Z}^{2N+r}$ .

In each set  $\mathcal{I}_{\alpha_i}$  choose one elements  $\beta'_i$  then replace the coordinate  $X_{\beta'_i}$  by  $u_i$  then we still get a basis for the lattice  $\mathbb{Z}^{2N+r}$ . Moreover,  $\ell_{\beta'_i} = \ell_i$ . By these observations, one can easily show that the cardinality of  $\mathbb{Z}^{2N+r}/K'$  is less than or equal to  $(\prod_{\alpha \in \Delta_+} \ell_\alpha)^2$ . On the other hand,  $K'$  is contained in the kernel  $K$  of  $\underline{H}$  and the cardinality of  $\mathbb{Z}^{2N+r}/K$  is  $(\prod_{\alpha \in \Delta_+} \ell_\alpha)^2$  by Proposition 9.25. Therefore  $K' = K$ .

*Step 4.* Then the proposition follows by the description of center of twisted polynomials in Proposition 9.21.  $\square$

### 9.5. The center $Z$ of $U_\epsilon^{ev}(\mathfrak{g})$ .

This section shows that the center  $Z$  is generated by  $Z_{HC}$  and  $Z_{Fr}$ .

Recall the Harish-Chandra isomorphism in Theorem 8.30:

$$\pi : Z_{HC} \rightarrow \mathbb{C}[K^{2\lambda}]_{\lambda \in P}^W,$$

here  $\mathbb{C}[K^{2\lambda}]_{\lambda \in P}$  means the group algebra generated by the lattice  $2P$ .

The algebra automorphism  $\gamma_{-\rho} : \mathbb{C}[K^{2\lambda}]_{\lambda \in P} \rightarrow \mathbb{C}[K^{2\lambda}]_{\lambda \in P}$  is defined by  $\gamma(K^{2\lambda}) = e^{(-\rho, 2\lambda)} K^{2\lambda}$  for all  $\lambda \in P$  (here we fix an element  $e^{1/\mathbb{N}}$ ). Then we have

$$\gamma_{-\rho} \circ \pi : Z_{HC} \cong \mathbb{C}[K^{2\lambda}]_{\lambda \in P}^W,$$

in which  $W$ -action is as follows:  $w(K^\mu) = K^{w(\mu)}$  for all  $w \in W, \mu \in 2P$ .

Let us define

$$Z_\cap := Z_{Fr} \cap Z_{HC} = (Z_{Fr})^{\dot{U}_\epsilon}$$

**Lemma 9.28.**  $Z_\cap \cong \mathbb{C}[K^{2\lambda}]_{\lambda \in P^*}^W \subset \mathbb{C}[K^{2\lambda}]_{\lambda \in P}^W \cong Z_{HC}$ .

*Proof.* Under the isomorphism  $\varphi : O_\epsilon[G] \cong U_\epsilon^{fin}$ , by Lemma 8.11,  $O_\epsilon[G^d] \cong Z_{Fr}^{fin}$ . Hence  $O_\epsilon[G^d]^{\dot{U}_\epsilon} \cong (Z_{Fr})^{\dot{U}_\epsilon} = Z_\cap$ .

For any  $\lambda \in P^*$ , let  $W_\epsilon^d(\lambda)$  be the Weyl module in  $\text{Rep}(\dot{U}_\epsilon^*)$ . Let  $c_\lambda^d = \sum c_{(v_i^d)^*, K - 2\rho v_i^d} \in O_\epsilon[G^d]$ , here  $\{v_i^d\}$  is a weight basis of  $W_\epsilon^d(\lambda)$ . Then  $\{c_\lambda^d\}_{\lambda \in P_+^*}$  form a  $\mathbb{C}$ -basis of  $O_\epsilon[G^d]^{\dot{U}_\epsilon}$ . The reason is as follows:  $\text{Rep}(\dot{U}_\epsilon^*)$  is equivalent to  $\text{Rep}(\dot{U}_\mathbb{C}(\mathfrak{g}^d))$  (cf. (8.10)), hence semisimple, and then  $O_\epsilon[G^d] \cong \bigoplus_{\lambda \in P_+^*} W_\epsilon^d(\lambda) \otimes (W_\epsilon^d(\lambda))^*$ . Now compute as in the proof of Theorem 8.30, one can see that  $\{\gamma_{-\rho} \circ \pi \circ \varphi(c_\lambda^d)\}_{\lambda \in P_+^*}$  forms a basis of  $\mathbb{C}[K^{2\lambda}]_{\lambda \in P^*}^W$ . This implies the lemma.  $\square$

By arguing as in Proposition 6.4 in [DCKP] (but we replace the lattice  $\ell P$  by  $P^*$ ), we obtain

**Proposition 9.29.**  $Z_{HC}$  is a complete intersection over its subring  $Z_\cap$ .

**Remark 9.30.** The inclusion  $Z_\cap \hookrightarrow Z_{HC}$  gives a rise to a map  $\bullet^\ell : T/W \rightarrow T^d/W$ . Under the identification  $Z_{Fr} \cong \mathbb{C}[G_0^d]$ , the inclusion  $Z_\cap \hookrightarrow Z_{Fr}$  corresponds to the categorical quotient  $G_0^d \hookrightarrow G^d \rightarrow G^d // G^d \cong T^d/W$ . Hence

$$\text{Spec } Z_{Fr} \otimes_{Z_\cap} Z_{HC} \cong G_0^d \times_{T^d/W} T/W,$$

the fiber product of the above two morphisms.

**Theorem 9.31** (cf. Theorem 6.4 [DCKP]). *The natural map  $Z_{Fr} \otimes_{Z_\cap} Z_{HC} \rightarrow Z$  is an isomorphism.*

*Proof.* The proof is the same as in [DCKP, Theorem 6.4]. Let  $\tilde{Z} := Z_{Fr} \otimes_{Z_\cap} Z_{HC}$

*Step 1:* We will show that the algebra  $Z_{Fr} \otimes_{Z_\cap} Z_{HC}$  is normal. By Proposition 9.29,  $\tilde{Z}$  is a complete intersection of  $Z_{Fr}$ . On the other hand,  $Z_{Fr}$  is regular. Hence  $\tilde{Z}$  is local complete intersection over  $\mathbb{C}$ . By Serre's criterion of normality, it is now enough to show that  $\tilde{Z}$  is regular in codimension 1.

The simply connectedness of  $G^d$  allows us to apply the results in [St, §3.8]. By [St, Theorem 1, §3.8],  $G_0^d = G_1 \cup G_2$ , where  $G_1$  is the open set of regular elements and  $G_2$  is the closed subvariety of codimension at least 2. Furthermore, By [St, Theorem 3, §3.8], the map  $G_1 \rightarrow T^d/W$  is smooth, hence the map  $G_1 \times_{T^d/W} T/W \rightarrow T/W$  is smooth. Meanwhile  $T/W$  is smooth, hence  $G_1 \times_{T^d/W} T/W$  is smooth. On the other hand,  $\bullet^\ell : T/W \rightarrow T^d/W$  is a finite map, hence  $G_2 \times_{T^d/W} T/W$  has codimension at least 2. So, we have proved that  $\tilde{Z}$  is regular in codimension 1.

*Step 2:* We have the following diagram

$$\begin{array}{ccc} \mathrm{Spec} Z & \xrightarrow{\quad} & \mathrm{Spec} \tilde{Z} \\ & \searrow & \swarrow \\ & \mathrm{Spec} Z_{Fr} & \end{array}$$

Two diagonal maps are finite and dominant and  $\mathrm{Spec} \tilde{Z}$  is normal by Step 1. Therefore, the horizontal map is dominant, equivalently, the map  $\tilde{Z} \rightarrow Z$  is injective.

*Step 3:* One can see that  $\dim_{Q(Z_{Fr})} Q(\tilde{Z}) = \prod_{i=1}^r \ell_i$ , hence  $\dim_{Q(Z_{Fr})} Q(\tilde{Z}) = \dim_{Q(Z_{Fr})} Q(Z)$  by Proposition 9.19. Therefore, the map  $Q(\tilde{Z}) \hookrightarrow Q(Z)$  is an isomorphism, equivalently,  $\vartheta : \mathrm{Spec} \tilde{Z} \rightarrow \mathrm{Spec} Z$  is birational. So  $\vartheta$  is a finite birational map to a normal variety, hence,  $\vartheta$  is an isomorphism. This completes the proof.  $\square$

**Proposition 9.32.** *Let  $Z^{(i)}$  be the center of the algebra  $U_\epsilon^{(i)}$  in Remark 9.8. Then  $\mathrm{gr} Z^{(i)} = Z^{(i+1)}$ .*

*Proof.* The proposition follows if we can show that  $\mathrm{gr} Z = Z_{\mathrm{gr} U_\epsilon^{ev}}$ . But then it is enough to prove the generators  $K^{\pm 2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}, u_i$  of  $Z_{\mathrm{gr} U_\epsilon^{ev}}$  in Proposition 9.27 are contained in  $\mathrm{gr} Z$ . On the other hand, each elements  $K^{\pm 2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}$  in  $U_\epsilon^{ev}(\mathfrak{g})$  are contained in the Frobenius center  $Z_{Fr}$ . Furthermore, their images in  $\mathrm{gr} U_\epsilon^{ev}$  are the same representing elements in  $\mathrm{gr} U_\epsilon^{ev}$ . That says  $K^{\pm 2\ell_i \omega_i}, X_\alpha^{\ell_\alpha}, Y_\alpha^{\ell_\alpha}$  are contained in  $\mathrm{gr} Z$ . Therefore, it is left to show that  $u_i \in \mathrm{gr} Z$ . This follows by Lemma 9.33 below.  $\square$

Let us recall the Weyl module  $W_\epsilon(\omega_i)$  for the fundamental weight  $\omega_i$ . Let  $v_1, \dots, v_t$  be the weight basis of  $W_\epsilon(\omega_i)$  such that if  $\mathrm{wt}(v_i) > \mathrm{wt}(v_j)$  then  $i > j$ . In particular,  $v_1$  is the lowest weight vector of  $W_\epsilon(\omega_i)$  and  $v_t$  is the highest weight vector of  $W_\epsilon(\omega_i)$ . The element  $c_{\omega_i} = \sum_j c_{v_j^*, K^{-2\rho} v_j} \in O_\epsilon[G] \xrightarrow{\hat{i}} U_\epsilon^{ev}$  is contained in the Harish-Chandra center  $Z_{HC}$  of  $U_\epsilon^{ev}(\mathfrak{g})$ , see Section 8.7.

**Lemma 9.33.**  $\mathrm{gr}(c_{\omega_i}) = au_i$  for some  $a \in \mathbb{C}^\times$ .

To prove Lemma 9.33, we need some technical results.

**Lemma 9.34.** *Let  $M \in \mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$ . Let  $0 \neq m \in M_\lambda$ . For each  $\alpha \in \Delta_+$ , set  $r_\alpha = \max\{r | \tilde{E}_\alpha^{(r)} m \neq 0\}$  and  $s_\alpha = \max\{s | \tilde{F}_\alpha^{(s)} m \neq 0\}$ . Then  $s_\alpha - r_\alpha = (\lambda, \alpha^\vee)$ , here  $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ .*

*Proof.* We will prove the similar statement where  $\tilde{E}_\alpha^{(n)}, \tilde{F}_\alpha^{(n)}$  are replaced by the standard roots generators  $E_\alpha^{(n)}, F_\alpha^{(n)}$  because we want to use the Lusztig's braided group actions on rational representations in  $\mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g})) \cong \mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$ . The result is the same since the twisted root generators  $\tilde{E}_\alpha^{(n)}, \tilde{F}_\alpha^{(n)}$  of  $\dot{U}_\epsilon(\mathfrak{g})$  are different from the standard root generators by only some element  $K^?$ .

When  $\alpha = \alpha_i$  for some simple root  $\alpha_i$ , the statement is proved in [APW, Lemma 1.11]. Note that in *loc. cit.* they proved over the localization  $\mathbb{Z}[v, v^{-1}]_{\mathfrak{m}_0}$  of  $\mathbb{Z}[v, v^{-1}]$  at the ideal  $\mathfrak{m}_0 = (p, v-1)$ . However, their proof works over arbitrary ring  $R$ .

In general,  $\alpha = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$  so that  $E_\alpha^{(n)} = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}^{(n)}), F_\alpha^{(n)} = T_{i_1} \dots T_{i_{k-1}}(F_{i_k}^{(n)})$ . There are corresponding braided group actions on any rational representation in  $\mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}))$  [L5, §41.2]. Let  $m' = T_{i_{k-1}}^{-1} \dots T_{i_1}^{-1} m$ , a weight vector of weight  $s_{i_{k-1}} \dots s_{i_1}(\lambda)$ , then

$$E_\alpha^{(n)} m = T_{i_1} \dots T_{i_{k-1}}(E_{i_k}^{(n)} m'), \quad F_\alpha^{(n)} m = T_{i_1} \dots T_{i_{k-1}}(F_{i_k}^{(n)} m').$$

Hence the statement for general  $\alpha$  follows by the case when  $\alpha = \alpha_i$  a simple root.  $\square$

Recall the set  $\mathcal{I}_{\alpha_i} = \{\beta_{k_1}, \dots, \beta_{k_r}\}, \mathcal{I}_{\alpha_i^*} = \{\beta_{l_1}, \dots, \beta_{l_r}\}$ . Let

$$\lambda_t = s_{\beta_{k_t}} \dots s_{\beta_{k_1}}(\omega_i), \quad \lambda'_t = s_{\beta_{k_t}} \dots s_{\beta_{k_1}}(-\omega_i^*) \quad \forall 1 \leq t \leq r.$$

In the weight basis  $v_1, \dots, v_m$  of  $W_\epsilon(\omega_i)$  as above, let  $v_{\lambda_t}, v_{\lambda'_t}$  be the unique weight vectors of weights  $\lambda_t, \lambda'_t$ , respectively.

**Lemma 9.35.** (a)  $\tilde{E}_{\beta_j}^{(n)}(v_{\lambda_t}) = 0$  for all  $n \geq 1$  and  $k_t < j \leq k_{t+1}$ .

(b)  $\tilde{F}_{\beta_j}^{(n)} v_{\lambda'_t} = 0$  for all  $n \geq 1$  and  $l_t < j \leq l_{t+1}$ .

*Proof.* Again, we will prove the statements where  $\tilde{E}_\alpha^{(n)}, \tilde{F}_\alpha^{(n)}$  are replaced by the standard root generators  $E_\alpha^{(n)}, F_\alpha^{(n)}$ .

We will prove part (a) only since the proof of part (b) is similar. By the Lusztig's braided action, we have

$$E_{\beta_j}^{(n)} = T_{s_{i_1}} \dots T_{s_{i_{j-1}}} (E_{i_k}^{(n)}), \quad v_{\lambda_t} = a T_{s_{i_1}} \dots T_{s_{i_{k_t}}} (v_{\omega_i}), \quad a \in \mathbb{C}^\times.$$

here we use  $\lambda_t = s_{i_1} \dots s_{i_{k_t}}(\omega_i)$  by Lemma 9.23. Hence

$$E_{\beta_j}^{(n)} v_{\lambda_t} = a T_{s_{i_1}} \dots T_{s_{i_{j-1}}} (E_{i_k}^{(n)} T_{s_{i_{j-1}}}^{-1} \dots T_{s_{i_{k_t+1}}}^{-1} (v_{\omega_i})).$$

Since  $s_{i_m}(\omega_i) = \omega_i$  for all  $k_t < m < k_{t+1}$  then  $T_{s_{i_m}}^{-1}(v_{\omega_i})$  is a nonzero weight vector of weight  $w_i$  in  $W_\epsilon(\omega_i)$ . Therefore,

$$T_{s_{i_{j-1}}}^{-1} \dots T_{s_{i_{k_t+1}}}^{-1} (v_{\omega_i}) = b v_{\omega_i}, \quad b \in \mathbb{C}^\times$$

Hence

$$E_{\beta_j}^{(n)}(v_{\lambda_t}) = ab T_{s_{i_1}} \dots T_{s_{i_{j-1}}} (E_{i_k}^{(n)} v_{\omega_i}) = 0.$$

□

*Proof of Lemma 9.33.* We note that  $c_{\omega_i} = \sum_j \epsilon^{-(2\rho, \text{wt}(v_j))} c_{v_j^*, v_j}$ . We need to show that  $u_i$  is the highest monomial of  $c_{\omega_i}$  in the  $\mathbb{Z}^{2N+1}$ -filtration. This holds by the following two steps.

*Step 1.* We will show that the highest degree monomial in  $c_{v_1^*, v_1}$  is

$$(\tilde{F}_{\beta_{l_s}} \dots \tilde{F}_{\beta_{l_1}} K^{\kappa(\omega_i + \omega_i^*)}) K^{-2\omega_i} (\tilde{E}_{\beta_{k_t}} \dots \tilde{E}_{\beta_{k_1}} K^{\gamma(\omega_i + \omega_i^*)}),$$

which is equal to  $b u_i$  for some  $b \in \mathbb{C}^\times$ . The height of this monomial is  $2\omega_i + 2\omega_i^*$ .

By Lemma 8.1,  $c_{v_1^*, v_1}$  contains the monomial

$$(9.16) \quad (\tilde{F}^{\bar{r}} K^{\kappa(-\text{wt}(\tilde{F}^{\bar{r}}))}) K^{-2\text{wt}(\tilde{E}^{\bar{r}}) - 2\text{wt}(v_1)} (\tilde{E}^{\bar{k}} K^{\gamma(\text{wt}(\tilde{E}^{\bar{k}}))})$$

if and only if

$$v_1^* (\tilde{F}^{\bar{k}} \tilde{E}^{\bar{r}} v_1) \neq 0.$$

- The height of monomial (9.16) is  $\text{wt}(\tilde{E}^{\bar{k}}) - \text{wt}(\tilde{F}^{\bar{r}}) = \text{wt}(\tilde{E}^{\bar{r}}) - \text{wt}(\tilde{F}^{\bar{k}})$ . This height is largest and equal to  $2\omega_i + 2\omega_i^*$  only when  $\tilde{E}^{\bar{r}} v_1 = a v_t$  and  $\tilde{F}^{\bar{k}} v_t = b v_1$  for some  $a, b \in \mathbb{C}^\times$ .

- To keep track the degree of (9.16), we put the degree  $(k_N, \dots, k_1)$  on  $\tilde{F}^{\bar{k}}$  and the degree  $(r_N, \dots, r_1)$  on  $\tilde{E}^{\bar{r}}$ . The lexicographical orders are  $(k_N, 0, \dots, 0) < \dots < (0, \dots, 0, k_1)$  and  $(r_N, 0, \dots, 0) < \dots < (0, \dots, 0, r_1)$ .

- We will show that the highest degree monomial  $\tilde{E}^{\bar{r}}$  such that  $\tilde{E}^{\bar{r}} v_1 = a v_t$  for some  $a \in \mathbb{C}^\times$  is  $\tilde{E}_{\beta_{l_s}} \dots \tilde{E}_{\beta_{l_1}}$ . To reduce the number of references, all computational results used below are in Lemma 9.23 and Lemma 9.35. Note that  $v_1 = v_{-\omega_i^*}, v_t = v_{\omega_i}$ .

For  $1 \leq j < l_1$ ,  $\tilde{F}_{\beta_j}^{(n)} v_{-\omega_i^*} = 0$  for all  $n \geq 1$ , and  $(-\omega_i^*, \beta_j^\vee) = 0$ . Then by Lemma 9.34,  $\tilde{E}_{\beta_j}^{(n)} v_{-\omega_i^*} = 0$  for all  $n \geq 1$  and  $1 \leq j < l_1$ .

Moreover,  $\tilde{F}_{\beta_{l_1}}^{(n)} v_{-\omega_i^*} = 0$  for all  $n \geq 1$ , and  $(-\omega_i^*, \beta_{l_1}^\vee) = 1$ . Then by Lemma 9.34,

$$\begin{cases} \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} \neq 0 \Rightarrow \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} = av_{\lambda'_1}, & a \neq 0 \\ \tilde{E}_{\beta_{l_1}}^{(n)} v_{-\omega_i^*} = 0 & \forall n \geq 2. \end{cases}$$

So we can assume  $\tilde{E}^{(\tilde{r})} = \tilde{E}_{\beta_N}^{(r_N)} \dots \tilde{E}_{\beta_{l_1+1}}^{(r_{l_1+1})} \tilde{E}_{\beta_{l_1}}$ .

For  $l_1 < j < l_2$ ,  $\tilde{F}_{\beta_j}^{(n)} v_{\lambda'_1} = 0$ , and  $(\lambda'_1, \beta_j^\vee) = 0$ . Then by Lemma 9.34,  $\tilde{E}_{\beta_j}^{(n)} \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} = 0$ .

Moreover,  $\tilde{E}_{\beta_{l_2}}^{(n)} v_{\lambda'_1} = 0$  for  $n \geq 1$ , and  $(\lambda'_1, \beta_{l_2}^\vee) = 1$  then by Lemma 9.34

$$\begin{cases} \tilde{E}_{\beta_{l_2}} \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} \neq 0 \Rightarrow \tilde{E}_{\beta_{l_2}} \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} = bv_{\lambda'_2}, & b \neq 0 \\ \tilde{E}_{\beta_{l_2}}^{(n)} \tilde{E}_{\beta_{l_1}} v_{-\omega_i^*} = 0 & \forall n \geq 2 \end{cases}$$

So we can assume  $\tilde{E}^{(\tilde{r})} = \tilde{E}_{\beta_N}^{(r_N)} \dots \tilde{E}_{\beta_{l_2+1}}^{(r_{l_2+1})} \tilde{E}_{\beta_{l_2}} \tilde{E}_{\beta_{l_1}}$ .

Keep doing, we see that the highest degree monomial is  $\tilde{E}_{\beta_{l_r}} \dots \tilde{E}_{\beta_{l_1}}$ .

• Similarly, the highest degree monomial  $\tilde{F}^{(\tilde{k})}$  such that  $\tilde{F}^{(\tilde{k})} v_t = bv_1$  for some  $b \in \mathbb{C}^\times$  is  $\tilde{F}_{\beta_{k_t}} \dots \tilde{F}_{\beta_{k_1}}$ .

*Step 2.* All monomials in  $c_{v_i^*, v_i}$  with  $i > 1$  have lower degree than  $u_i$ . Indeed, all monomials in  $c_{v_i^*, v_i}$  have the height at most  $2\omega_i - 2\text{wt}(v_i) < 2\omega_i + 2\omega_i^*$ .  $\square$

## 9.6. The Azumaya locus of $U_\epsilon^{ev}(\mathfrak{g})$ over $\text{Spec } Z$ .

Let us recall the following set in Theorem 9.14:

$$\Omega_{U_\epsilon^{ev}} = \{p \in \text{Spec } Z \mid \text{the corresponding semisimple representation is irreducible}\}.$$

Since  $U_\epsilon^{ev}(\mathfrak{g})$  is a prime (i.e, has no zero divisors) Noetherian algebra, by [BG2, Proposition 3.1],

$$\Omega_{U_\epsilon^{ev}} = \{p \in \text{Spec } Z \mid U_\epsilon^{ev} \otimes_Z Z_{\mathfrak{m}_p} \text{ is Azumaya over } Z_{\mathfrak{m}_p}\}.$$

Hence,  $\Omega_{U_\epsilon^{ev}}$  is called the *Azumaya locus* of  $U_\epsilon^{ev}(\mathfrak{g})$ . In this section, we construct a large open set contained in  $\Omega_{U_\epsilon^{ev}}$ . We follow the arguments in [DCKP2]. Let us state two key lemmas.

For a filtered algebra  $A$ , let  $\text{gr}A = \bigoplus_i A_{i+1}/A_i$  be the associated graded algebra and  $\mathcal{R}(A) = \bigoplus_i A_i t^i$  be the Ree algebra of  $A$ .

**Lemma 9.36** (Proposition 1.4 [DCKP2]). *Let  $A$  be a commutative filtered algebra and let  $a_1, \dots, a_n \in A$  be such that  $\bar{a}_1, \dots, \bar{a}_n$  is a regular sequence of  $\text{gr}A$ . Let  $I = (a_1, \dots, a_n)$  be the ideal of  $A$  generated by  $a_1, \dots, a_n$ . Then*

- a)  $a_1, \dots, a_n$  is a regular sequence in  $A$ .
- b) The ideal  $\text{gr}I$  of  $\text{gr}A$  is generated by the elements  $\bar{a}_1, \dots, \bar{a}_n$ .

**Lemma 9.37** (Lemma 1.5 [DCKP2]). *Let  $A$  be a finitely generated (algebra over  $\mathbb{C}$ ) equipped with a filtration such that*

- $A$ ,  $\text{gr}A$  and  $\mathcal{R}(A)$  are orders closed under trace in central simple algebras
- $\text{gr}A$  is a finitely generated order of the same degree as  $A$

*Let  $Z_0$  be the central subalgebra of  $A$  such that  $\text{gr}Z_0$  is a finitely generated algebra over  $\mathbb{C}$  and  $\text{gr}A$  is a finitely generated module over  $\text{gr}Z_0$ . Let  $I$  be an ideal of  $Z_0$  and  $\text{gr}I$  be the associated graded ideal of  $\text{gr}Z_0$ . Let  $\mathcal{O}$  (resp.  $\mathcal{O}_1$ ) be the set of zeroes of  $I$  (resp.  $\text{gr}I$ ) in  $\text{Spec } Z_0$  (resp.  $\text{Spec } (\text{gr}Z_0)$ ). Suppose that  $\mathcal{O}_1 \cap \Omega_{\text{gr}A}^0 \neq 0$ . Then  $\mathcal{O} \cap \Omega_A^0 \neq 0$ .*

**Remark 9.38.** The condition "closed under trace" on  $\mathcal{R}(A)$  is not mentioned in [DCKP2, Lemma 1.5]. But we think this condition is important so that the set  $\Omega_{\mathcal{R}(A)}^0$  is open dense in  $\text{Spec}\mathcal{R}(Z_0)$  by Theorem 9.14. The openness of  $\Omega_{\mathcal{R}(A)}^0$  is crucial in the last argument of the proof of [DCKP2, Lemma 1.5].

Recall the  $\dot{U}_{\mathbb{C}}(\mathfrak{g}^d)$ -equivariant isomorphism  $Z_{Fr} \cong \mathbb{C}[G_0^d]$ . For any  $\lambda \in P_+^*$ , let  $V^d(\lambda)$  be the irreducible representation of the highest weight  $\lambda$  of  $G^d$ . Consider the morphism

$$\pi_\lambda : \text{Spec}Z_{Fr} \cong G_0^d \subset G^d \rightarrow GL(V^d(\lambda)).$$

Let

$$\phi_\lambda(u) = \text{tr}_{V^d(\lambda)}(\pi_\lambda(u)), \quad \forall p \in \text{Spec}Z_{Fr}.$$

Then the following lemma is standard

**Lemma 9.39.** (a)  $\{\phi_\lambda | \lambda \in P_+^*\}$  forms a  $\mathbb{C}$ -basis of  $Z_{\cap} = \mathbb{C}[G_0^d]^{\dot{U}_{\mathbb{C}}(\mathfrak{g}^d)}$ .

(b)  $Z_{\cap}$  is a polynomial algebra over  $\mathbb{C}$  on the generators  $\{\phi_{\omega_1^d}, \dots, \phi_{w_r^d}\}$ , here  $\{\omega_i^d = \ell_i \omega_i\}_{1 \leq i \leq r}$  are fundamental weights of  $\mathfrak{g}^d$ .

The set  $\{\alpha_i^d = \ell_i \alpha_i\}_{1 \leq i \leq r}$  is the set of simple roots of  $\mathfrak{g}^d$ . Recall that the Weyl group  $W^d$  of  $G^d$  is the same as  $W$  via identification  $s_{\alpha_i} \mapsto s_{\alpha_i^d}$ . Recall the fixed reduced expression  $w_0 = s_{i_1} \dots s_{i_N}$  so that we have the set of positive roots  $\{\beta_k^d = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}^d) = \ell_{i_k} \beta_k\}$ . For each simple root  $\alpha^d$ , let

$$(9.17) \quad I_{\alpha^d} := \{1 \leq j \leq N | s_{i_j} = s_{\alpha^d}\}, \quad \mathcal{I}_{\alpha^d} := \{\beta_{k_1}^d, \dots, \beta_{k_t}^d | k_j \in I_{\alpha^d}\}.$$

It is not hard to see that  $I_{\alpha^d} = I_\alpha$ , the set in (9.11). Set

$$x_{\beta^d} = \tilde{E}_\beta^{\ell_\beta} K^{\ell_\beta \gamma(\beta)}, \quad y_{\beta^d} = \tilde{F}_\beta^{\ell_\beta} K^{\ell_\beta \kappa(\beta)}, \quad z_{\omega_i^d} = K^{2\ell_i \omega_i}.$$

in which  $\beta$  is a positive root of  $\mathfrak{g}$  and  $\beta^d = \ell_\beta \beta$  is the positive root of  $\mathfrak{g}^d$ .

**Lemma 9.40** (cf. Lemma 4.6 [DCKP2]). *Recall the element  $\phi_{\omega_i^d} \in Z_{Fr} \cong \mathbb{C}[G_0^d]$ . Then the image  $\bar{\phi}_{\omega_i^d} \in \text{gr}Z_{Fr}$  is*

$$(9.18) \quad \bar{\phi}_{\omega_i^d} = z_{-\omega_i^d} x_{\beta_{k_r}^d} \dots x_{\beta_{k_1}^d} y_{\beta_{l_r}^d} \dots y_{\beta_{l_1}^d},$$

here  $\mathcal{I}_{\alpha_i^d} = \{\beta_{k_1}^d, \dots, \beta_{k_r}^d\}$  and  $\mathcal{I}_{-w_0(\alpha_i^d)} = \{\beta_{l_1}^d, \dots, \beta_{l_r}^d\}$  as in (9.17).

*Proof.* The proof is exactly the same as the proof of Lemma 9.33. We just want to make a remark. In the proof of Lemma 9.33, the orthogonality of pairings  $(\tilde{F}^{\bar{k}}, \tilde{E}^{(\bar{r})})'$  and  $(\tilde{F}^{(\bar{k})}, \tilde{E}^{\bar{r}})'$  allow us to detect the monomial  $\tilde{F}^{\bar{k}} K^? \tilde{E}^{\bar{r}}$  of the matrix coefficient  $c_{f,v} \in U_{\epsilon}^{ev}(\mathfrak{g})$  by looking at the value  $f(\tilde{F}^{(\bar{r})} \tilde{E}^{(\bar{k})} v)$ . One will need the same orthogonality of pairing  $(\tilde{F}^{\bar{k}}, e^{(\bar{r})})'$  in  $Z_{Fr}^< \times \dot{U}^>(\mathfrak{g}^d) \rightarrow \mathbb{C}$  and pairing  $(f^{(\bar{k})}, \tilde{E}^{\bar{r}})'$  in  $\dot{U}^<(\mathfrak{g}^d) \times Z_{Fr}^> \rightarrow \mathbb{C}$  in Lemma 5.6.  $\square$

We are now going to construct the promised open subset of  $\Omega_{U_{\epsilon}^{ev}}$ . Let us consider the projection

$$\mathbf{p}_1 : \text{Spec}Z \cong G_0^d \times_{T^d/W} T/W \rightarrow G_0^d \hookrightarrow G^d.$$

Let  $G^{d,\text{reg}}$  be the set of regular elements in  $G^d$ .

**Theorem 9.41** (cf. Theorem 5.1 [DCKP2]). *The set  $\Omega_{U_{\epsilon}^{ev}}$  contains the open set  $\mathbf{p}_1^{-1}(G^{d,\text{reg}})$ .*

Before proceeding to prove this theorem, we need to show that Lemma 9.37 can be applied to the chain of filtered algebras  $U_{\epsilon}^{(i)}$  in Remark 9.8. To this end, this is enough to prove that  $\mathcal{R}(U^{(i)})$  is closed under the trace for all  $0 \leq i \leq 2N$ . By Remark 9.13, it is sufficient to prove

**Lemma 9.42.** *Each Ree algebra  $\mathcal{R}(Z^{(i)})$  is normal.*

*Proof.* Since  $\mathcal{R}(Z^{(i)})/(t) = \text{gr}Z^{(i)} \cong Z^{(i+1)}$ , a finitely generated algebra over  $\mathbb{C}$ ,  $\mathcal{R}(Z^{(i)})$  is a finitely generated algebra over  $\mathbb{C}$ .

The closed fiber at 0 is  $\text{gr}Z^{(i)} \cong Z^{(i+1)}$  and the closed fiber at  $a \in \mathbb{C}^\times$  is  $Z^{(i)}$ . Since  $U_\epsilon^{(i)}$  and  $U_\epsilon^{(i+1)}$  are integrally closed by Proposition 9.10, both  $Z^{(i)}$  and  $Z^{(i+1)}$  are normal. Hence all closed fibers of  $\mathcal{R}(Z^{(i)})$  over  $\mathbb{C}[t]$  are normal. On the other hand,  $\mathcal{R}(Z^{(i)})$  is faithfully flat over  $\mathbb{C}[t]$  and  $\mathbb{C}[t]$  is normal. Hence by [Ma, Corollary 22.E],  $\mathcal{R}(Z^{(i)})$  is normal.  $\square$

Let us now prove Theorem 9.41.

*Proof. Step 1:* Let  $\chi_0 : \text{Irr}(U_\epsilon^{ev}(\mathfrak{g})) \rightarrow \text{Spec}Z_{Fr}$  be the map that sends irreducible representations of  $U_\epsilon^{ev}$  to their characters of  $Z_{Fr}$ . Let us recall

$$\Omega_{U_\epsilon^{ev}}^0 := \{p \in \text{Spec}Z_{Fr} \mid \text{all representations of } \chi_0^{-1}(p) \text{ have the maximal dimension}\}.$$

Then it is enough to show that  $G_0^{d,reg} := G_0^d \cap G^{d,reg} \subset \Omega_{U_\epsilon^{ev}}^0$ .

Let  $\mathcal{O}$  be a conjugacy class of regular elements in  $G^d$  and  $\bar{\mathcal{O}}$  be the Zariski closure of  $\mathcal{O}$  in  $G^d$ . So we need to show that  $\mathcal{O} \cap G_0^d \subset \Omega_{U_\epsilon^{ev}}^0$ . By Proposition 9.15,  $\Omega_{U_\epsilon^{ev}}^0$  is a union of symplectic leaves, meanwhile symplectic leaves of  $\text{Spec}Z_{Fr} = G_0^d$  are intersections of conjugacy classes of  $G^d$  with it by Proposition 9.3. Therefore, it is enough to show that  $\mathcal{O} \cap \Omega_{U_\epsilon^{ev}}^0 \neq 0$ . Since  $\Omega_{U_\epsilon^{ev}}^0$  is open in  $G_0^d$ , it is then enough to show that  $\bar{\mathcal{O}} \cap \Omega_{U_\epsilon^{ev}}^0 \neq 0$ .

*Step 2:* It is well-known that  $\bar{\mathcal{O}} \subset G^d$ , a closure of a conjugacy class of regular elements, is given by equations

$$\text{tr}_{V^d(\omega_i^d)}(g) = c_i, \quad i = 1, \dots, n, \text{ and } g \in G^d.$$

hence the set  $\bar{\mathcal{O}} \cap \text{Spec}Z_{Fr}$  is given by the equations

$$\phi_{\omega_i^d} = c_i, \quad i = 1, \dots, n.$$

We want to apply Lemma 9.37 to the ideal  $I$  of  $Z_{Fr}$  generated by  $\{\phi_{\omega_i^d} - c_i\}$ . For this first of all, we need to show that each of the algebra  $U^{(j)}$  introduced in Remark 9.8 has degree  $d = \prod_{\alpha \in \Delta_+} \ell_\alpha$ . Since the degree can only decrease in each step, it is enough to show that  $d$  is the degree of  $U^{(2N)} = \text{gr}U_\epsilon^{ev}$ , but this follows by Proposition 9.16. By easy induction with Lemma 9.37, we are then reduced to show that if  $\text{gr}I$  is the associated graded ideal of  $I$  in  $\text{gr}Z_{Fr}$ ,  $\mathcal{O}_1$  be its set of zeros, then

$$(9.19) \quad \mathcal{O}_1 \cap \Omega_{\text{gr}U_\epsilon^{ev}}^0 \neq 0$$

*Step 3:* The elements  $\bar{\phi}_{\omega_i^d}, i = 1, \dots, r$  in (9.18) form a regular sequence of  $\text{gr}Z_{Fr}$  since they are monomials in disjoint sets of indeterminates. Hence, by Lemma 9.36,  $\text{gr}I$  is generated by  $\{\bar{\phi}_{\omega_i^d}, i = 1, \dots, r\}$ . So  $\mathcal{O}_1$  is a union of subvarieties given as follows: we choose from each monomial  $\bar{\phi}_{\omega_i^d}$  a factor in  $\{x_{\beta_{k_r}^d}, \dots, x_{\beta_{k_1}^d}, y_{\beta_{l_r}^d}, \dots, y_{\beta_{l_1}^d}\}$  and letting them 0 we define a component of  $\mathcal{O}_1$ . So it is enough to show that some of its components intersect  $\Omega_{U_\epsilon^{ev}}^0$  nontrivially.

Let us recall the set  $\mathcal{I}_{\alpha_i}$  and pick  $\beta_{m_i} \in \mathcal{I}_{\alpha_i}$ . As in (9.12), let

$$S = \{K^{2\omega_1}, \dots, K^{2\omega_r}, Y_{\beta_i}, X_{\beta_i}\} / \{Y_{\beta_{m_1}}, \dots, Y_{\beta_{m_r}}\}.$$

Then  $Y_{\beta_{m_i}}^{\ell_{\beta_{m_i}}} \in \text{gr}Z_{Fr}$ . The closed subvariety  $\mathcal{Y}$  defined by equations  $Y_{\beta_{m_i}}^{\ell_{\beta_{m_i}}} = 0, i = 1, \dots, r$  is a component of  $\mathcal{O}_1$  (note that  $y_{\beta_{m_i}} = Y_{\beta_{m_i}}^{\ell_{\beta_{m_i}}} K^{\ell_{\beta_{m_i}} \gamma(\beta_{m_i})}$  and  $K^{\ell_{\beta_{m_i}} \gamma(\beta_{m_i})}$  is invertible in  $\text{gr}Z_{Fr}$ ). By Proposition 9.25, we can apply Lemma 9.22 to obtain  $\mathcal{Y} \cap \Omega_{\text{gr}U_\epsilon^{ev}}^0 \neq 0$ . Therefore, (9.19) holds. This finishes the proof.  $\square$

### APPENDIX A. EQUIVARIANT MODULES

This Appendix is devoted to setting the basic properties of (left, right, or bi) modules over an algebra which are additionally also modules over a Hopf algebra, in a compatible way.

**A.1.  $H$ -invariants.** Let  $H$  be a Hopf algebra over a field  $k$ . For any left  $H$ -module  $V$ , the  **$H$ -invariant part** of  $V$  is defined by

$$V^H = \{v \in V \mid hv = \varepsilon(h)v \text{ for all } h \in H\}.$$

For any left  $H$ -modules  $M$  and  $N$ , we have the following two actions of  $H \otimes H$  on  $\text{Hom}_k(M, N)$ :

$$\begin{aligned} (x \otimes y)(m) &= xf(S(y)m), \\ (x \otimes y)(m) &= yf(S^{-1}(x)m). \end{aligned}$$

Combining with the coproduct  $\Delta: H \rightarrow H \otimes H$ , we obtain two left actions of  $H$  on  $\text{Hom}_k(M, N)$ , which shall be denoted by  $(H, \cdot_1)$  and  $(H, \cdot_2)$ , respectively.

**Lemma A.1.**  $\text{Hom}_H(M, N) = \text{Hom}_k(M, N)^{(H, \cdot_1)} = \text{Hom}_k(M, N)^{(H, \cdot_2)}$ , where the upper indices mean taking the  $H$ -invariant parts with respect to the corresponding  $H$ -actions.

*Proof.* Since  $(h \cdot_1 f)(m) = \sum h_{(1)}f(S(h_{(2)}))m$  for all  $h \in H, f \in \text{Hom}_k(M, N), m \in M$ , and  $\sum h_{(1)}S(h_{(2)}) = \varepsilon(h)$ , it is clear that  $\text{Hom}_H(M, N) \subseteq \text{Hom}_k(M, N)^{(H, \cdot_1)}$ . Let us now prove the reverse inclusion. Let  $f \in \text{Hom}_k(M, N)^{(H, \cdot_1)}$ . We also recall that  $\varepsilon(h) = \varepsilon(S(h)) = \varepsilon(S^{-1}(h))$ ,  $h = \sum h_{(1)}\varepsilon(h_{(2)}) = \sum h_{(2)}\varepsilon(h_{(1)}), \sum h_{(1)}S(h_{(2)}) = \sum S(h_{(1)})h_{(2)} = \varepsilon(h)$ . Hence, we have:

$$\begin{aligned} hf(m) &= \sum h_{(2)}\varepsilon(S^{-1}(h_{(1)}))f(m) = \sum h_{(3)}S^{-1}(h_{(2)})f(S(S^{-1}(h_{(1)}))m) \\ &= \sum \varepsilon(S^{-1}(h_{(2)}))f(h_{(1)}m) = f\left(\sum \varepsilon(h_{(2)})h_{(1)}m\right) = f(hm). \end{aligned}$$

Similarly, as  $(h \cdot_2 f)(m) = \sum h_{(2)}f(S^{-1}(h_{(1)}))m$  for all  $h \in H, f \in \text{Hom}_k(M, N), m \in M$  and  $\sum h_{(2)}S^{-1}(h_{(1)}) = \varepsilon(S^{-1}(h)) = \varepsilon(h)$ , we obtain  $\text{Hom}_H(M, N) \subseteq \text{Hom}_k(M, N)^{(H, \cdot_2)}$ . To prove the reserve inclusion, pick  $f \in \text{Hom}_k(M, N)^{(H, \cdot_2)}$ . Similarly to above, we get:

$$\begin{aligned} hf(m) &= \sum h_{(1)}\varepsilon(S(h_{(2)}))f(m) = \sum h_{(1)}S(h_{(2)})f(S^{-1}S(h_{(3)}))m \\ &= \sum \varepsilon(h_{(1)})f(h_{(2)}m) = f\left(\sum \varepsilon(h_{(1)})h_{(2)}m\right) = f(hm). \end{aligned}$$

This completes the proof.  $\square$

**A.2.  $H$ -invariants  $A$ -modules.** We shall now also assume that  $A$  is an algebra over  $k$ . We start with the standard definition:

**Definition A.2.** An  $H$ -module algebra  $A$  (assumed to be unital, with  $1_A$  the unit) is an algebra  $A$  with a left  $H$ -module structure such that

$$h(1_A) = \varepsilon(h)1_A, \quad h(ab) = \sum (h_{(1)}a)(h_{(2)}b) \quad \forall h \in H, a, b \in A.$$

In this setup, one naturally introduces the categories of  $H$ -equivariant left/right  $A$ -modules:

**Definition A.3.** Let  $A$  be an  $H$ -module algebra.

(a) An  $H$ -equivariant left  $A$ -module  $M$  is a left  $A$ -module  $M$  with a left  $H$ -module structure such that

$$(A.1) \quad h(am) = \sum (h_{(1)}a)(h_{(2)}m) \quad \forall h \in H, a \in A, m \in M.$$

Let  $A\text{-Rmod}^H$  denote the category of  $H$ -equivariant left  $A$ -modules. A morphism  $\eta: M \rightarrow N$  in  $A\text{-Lmod}^H$  is a morphism of left  $A$ -modules which is also a morphism of left  $H$ -modules.

(b) An  $H$ -equivariant right  $A$ -module  $M$  is a right  $A$ -module  $M$  with a left  $H$ -module structure such that

$$(A.2) \quad h(mb) = \sum (h_{(1)}m)(h_{(2)}b) \quad \forall h \in H, b \in A, m \in M.$$

Let  $A\text{-Rmod}^H$  denote the category of  $H$ -equivariant right  $A$ -modules. A morphism  $\eta: M \rightarrow N$  in  $A\text{-Rmod}^H$  is a morphism of right  $A$ -modules which is also a morphism of left  $H$ -modules.

(c) An  $H$ -equivariant  $A$ -bimodule  $M$  is a  $A$ -bimodule  $M$  with a left  $H$ -module structure such that

$$(A.3) \quad h(amb) = \sum (h_{(1)}a)(h_{(2)}m)(h_{(3)}b) \quad \forall h \in H, a, b \in A, m \in M.$$

Let  $A\text{-Bimod}^H$  denote the category of  $H$ -equivariant  $A$ -bimodules. A morphism  $\eta: M \rightarrow N$  in  $A\text{-Bimod}^H$  is a morphism of  $A$ -bimodules which is also a morphism of left  $H$ -modules.

We note that the condition (A.3) is equivalent to the combination of (A.1) and (A.2).

**Lemma A.4.** (a) For any  $M, N \in A\text{-Rmod}^H$ , the space  $\text{Hom}_{A\text{-Rmod}}(M, N)$  carries a natural structure of a left  $H$ -module via

$$(A.4) \quad (hf)(m) = \sum h_{(1)}f(S(h_{(2)})m)$$

for  $h \in H, f \in \text{Hom}_{A\text{-Rmod}}(M, N), m \in M$ . Furthermore, we have

$$\text{Hom}_{A\text{-Rmod}^H}(M, N) = \text{Hom}_{A\text{-Rmod}}(M, N)^{(H, \cdot_1)}.$$

(b) For any  $M \in A\text{-Bimod}^H$  and  $N \in A\text{-Rmod}^H$ , the space  $\text{Hom}_{A\text{-Rmod}}(M, N)$  is naturally an object in  $A\text{-Rmod}^H$  with the left  $H$ -module structure as in (A.4) and the right  $A$ -module structure defined by  $(fa)(m) = f(am)$  for all  $a \in A, f \in \text{Hom}_{A\text{-Rmod}}(M, N), m \in M$ .

(c) For any  $M \in A\text{-Rmod}^H$  and  $N \in A\text{-Bimod}^H$ , the space  $\text{Hom}_{A\text{-Rmod}}(M, N)$  is naturally an object in  $A\text{-Lmod}^H$  with the left  $H$ -module structure as in (A.4) and the left  $A$ -module structure defined by  $(af)(m) = af(m)$  for all  $a \in A, f \in \text{Hom}_{A\text{-Rmod}}(M, N), m \in M$ .

(d) For any  $M, N \in A\text{-Bimod}^H$ , the space  $\text{Hom}_{A\text{-Rmod}}(M, N)$  is naturally an object in  $A\text{-Bimod}^H$  with the left  $H$ -module structure as in (A.4) and the  $A$ -bimodule structure defined by  $(afb)(m) = af(bm)$  for  $a, b \in A, f \in \text{Hom}_{A\text{-Rmod}}(M, N), m \in M$ .

*Proof.* (a) Recall that (A.4) defines an  $H$ -action  $\cdot_1$  on  $\text{Hom}_k(M, N)$  from Subsection A.1. We shall now verify that  $hf \in \text{Hom}_{A\text{-Rmod}}(M, N)$  if  $f \in \text{Hom}_{A\text{-Rmod}}(M, N)$ . This follows from:

$$\begin{aligned} (hf)(m.a) &= \sum h_{(1)}f(S(h_{(2)})(m.a)) = \sum h_{(1)}f((S(h_{(3)})m).(S(h_{(2)})a)) \\ &= \sum h_{(1)}(f(S(h_{(3)})m).(S(h_{(2)})a)) = \sum h_{(1)}f((S(h_{(4)})m).h_{(2)}S(h_{(3)}))a \\ &= \sum h_{(1)}f(S(h_{(3)})m).\varepsilon(h_{(2)})a = \sum h_{(1)}f(S(h_{(2)})m).a = (hf)(m).a. \end{aligned}$$

Viewing  $\text{Hom}_{A\text{-Rmod}^H}(M, N), \text{Hom}_{A\text{-Rmod}}(M, N), \text{Hom}_H(M, N)$  as subspaces of  $\text{Hom}_k(M, N)$ , we have

$$\text{Hom}_{A\text{-Rmod}^H}(M, N) = \text{Hom}_{A\text{-Rmod}}(M, N) \cap \text{Hom}_H(M, N).$$

Since  $\text{Hom}_H(M, N) = \text{Hom}_k(M, N)^{(H, \cdot_1)}$  by Lemma A.1 and  $\text{Hom}_{A\text{-Rmod}}(M, N)$  is a left  $H$ -submodule of  $\text{Hom}_k(M, N)$ , we get  $\text{Hom}_{A\text{-Rmod}^H}(M, N) = \text{Hom}_{A\text{-Rmod}}(M, N)^{(H, \cdot_1)}$ .

(b) First,  $(fa)(m) = f(am)$  for all  $a \in A, f \in \text{Hom}_k(M, N), m \in M$  defines a right  $A$ -action on  $\text{Hom}_k(M, N)$ . To verify the compatibility of these left  $H$ -action and right  $A$ -action, we note:

$$(h(fa))(m) = \sum h_{(1)}(fa)(S(h_{(2)})m) = \sum h_{(1)}f(a.S(h_{(2)})m)$$

while

$$\begin{aligned} \left( \sum (h_{(1)}f)(h_{(2)}a) \right) (m) &= \sum (h_{(1)}f)(h_{(2)}a.m) = \sum h_{(1)}f \left( S(h_{(2)})(h_{(3)}a.m) \right) = \\ &\sum h_{(1)}f \left( S(h_{(3)})h_{(4)}a.S(h_{(2)})m \right) = \sum h_{(1)}f \left( a.S(h_{(2)})\varepsilon(h_{(3)})m \right) = \sum h_{(1)}f(a.S(h_{(2)})m), \end{aligned}$$

so that  $h(fa) = \sum (h_{(1)}f)(h_{(2)}a)$ .

(c) First,  $(af)(m) = af(m)$  for all  $a \in A, f \in \text{Hom}_k(M, N), m \in M$  defines a left  $A$ -action on  $\text{Hom}_k(M, N)$ . To verify the compatibility of these left  $H$ -action and left  $A$ -action, we note:

$$\begin{aligned} (h(af))(m) &= \sum h_{(1)}(af)(S(h_{(2)})m) = \sum h_{(1)} \left( a.f(S(h_{(2)})m) \right) \\ &= \sum h_{(1)}a.h_{(2)}f(S(h_{(3)})m) = \sum h_{(1)}a.(h_{(2)}f)(m) = \left( \sum (h_{(1)}a)(h_{(2)}f) \right) (m), \end{aligned}$$

so that  $h(af) = \sum (h_{(1)}a)(h_{(2)}f)$ .

(d) Follows by combining parts (b) and (c).  $\square$

**Remark A.5.** (a) If  $A$  is an  $H$ -module algebra, then  $A^{\text{op}}$  is an  $H^{\text{cop}}$ -module algebra with the same action, whereas  $H^{\text{cop}} \simeq H$  as algebras.

(b) If  $M, N \in A\text{-Lmod}^H$  then (A.4) does not equip  $\text{Hom}_{A\text{-Lmod}}(M, N)$  with an  $H$ -action in general. However, evoking (a), we note that any  $M \in A\text{-Lmod}^H$  is naturally an object in  $A^{\text{op}}\text{-Rmod}^{H^{\text{cop}}}$ , where the left  $H^{\text{cop}}$ -module structure on  $M$  is unchanged and the right  $A^{\text{op}}$ -module structure coincides with the left  $A$ -module structure. Thus, we have  $\text{Hom}_{A\text{-Lmod}}(M, N) = \text{Hom}_{A^{\text{op}}\text{-Rmod}}(M, N)$ . By Lemma A.4(a), the right-hand side has a left  $H^{\text{cop}}$ -action defined by  $(hf)(m) = \sum h_{(2)}f(S^{-1}(h_{(1)})m)$  for  $h \in H^{\text{cop}}, f \in \text{Hom}_{A^{\text{op}}\text{-Rmod}}(M, N), m \in M$ . This implies that for any  $M, N \in A\text{-Lmod}^H$ , we have a left  $H$ -action on  $\text{Hom}_k(M, N)$  given by

$$(A.5) \quad (hf)(m) = \sum h_{(2)}f(S^{-1}(h_{(1)})m) \quad \forall h \in H, f \in \text{Hom}_{A\text{-Lmod}}(M, N), m \in M.$$

The following result follows from Lemma A.4 and Remark A.5 (though we also provide a direct proof for clarity):

**Lemma A.6.** (a) For any  $M, N \in A\text{-Lmod}^H$ , the space  $\text{Hom}_{A\text{-Lmod}}(M, N)$  carries a natural structure of a left  $H$ -module via (A.5). Furthermore, we have

$$\text{Hom}_{A\text{-Lmod}^H}(M, N) = \text{Hom}_{A\text{-Lmod}}(M, N)^{(H, \cdot_2)}.$$

(b) For any  $M \in A\text{-Bimod}^H$  and  $N \in A\text{-Lmod}^H$ , the space  $\text{Hom}_{A\text{-Lmod}}(M, N)$  is naturally an object in  $A\text{-Lmod}^H$  with the left  $H$ -module structure as in (A.5) and the left  $A$ -module structure defined by  $(af)(m) = f(ma)$  for  $a \in A, f \in \text{Hom}_{A\text{-Lmod}}(M, N), m \in M$ .

(c) For any  $M \in A\text{-Lmod}^H$  and  $N \in A\text{-Bimod}^H$ , the space  $\text{Hom}_{A\text{-Lmod}}(M, N)$  is naturally an object in  $A\text{-Rmod}^H$  with the left  $H$ -module structure as in (A.5) and the right  $A$ -module structure defined by  $(fa)(m) = f(m)a$  for  $a \in A, f \in \text{Hom}_{A\text{-Lmod}}(M, N), m \in M$ .

(d) For any  $M, N \in A\text{-Bimod}^H$ , the space  $\text{Hom}_{A\text{-Lmod}}(M, N)$  is a naturally an object in  $A\text{-Bimod}^H$  with the left  $H$ -module structure as in (A.5) and the  $A$ -bimodule structure defined by  $(afb)(m) = f(ma)b$  for  $a, b \in A, f \in \text{Hom}_{A\text{-Lmod}}(M, N), m \in M$ .

*Proof.* (a) By Lemma A.1, we have  $\text{Hom}_H(M, N) = \text{Hom}_k(M, N)^{(H, \cdot_2)}$ . As with Lemma A.4(a), it is thus enough to show that  $H$  acts on  $\text{Hom}_{A\text{-Lmod}}(M, N)$  via  $\cdot_2$ . This follows from:

$$\begin{aligned}
(hf)(am) &= \sum h_{(2)} f(S^{-1}(h_{(1)})(am)) = \sum h_{(3)} f(S^{-1}(h_{(2)})a.S^{-1}(h_{(1)})m) \\
&= \sum h_{(3)} (S^{-1}(h_{(2)})a.f(S^{-1}(h_{(1)})m)) = \sum h_{(3)} S^{-1}(h_{(2)})a.h_{(4)} f(S^{-1}(h_{(1)})m) \\
&= \sum \varepsilon(h_{(2)})a.h_{(3)} f(S^{-1}(h_{(1)})m) = \sum a.h_{(2)} f(S^{-1}(h_{(1)})m) = a.(hf)(m).
\end{aligned}$$

(b) We have

$$(h(af))(m) = \sum h_{(2)}(af)(S^{-1}(h_{(1)})m) = \sum h_{(2)} f(S^{-1}(h_{(1)})m.a)$$

and

$$\begin{aligned}
(\sum (h_{(1)}a)(h_{(2)}f))(m) &= \sum (h_{(2)}f)(m.h_{(1)}a) = \sum h_{(3)} f(S^{-1}(h_{(2)})(m.h_{(1)}a)) \\
&= \sum h_{(4)} f(S^{-1}(h_{(3)})m.S^{-1}(h_{(2)})h_{(1)}a) = \sum h_{(3)} f(S^{-1}(h_{(2)})m.\varepsilon(h_{(1)})a) \\
&= \sum h_{(3)} f(\varepsilon(h_{(1)})S^{-1}(h_{(2)})m.a) = \sum h_{(2)} f(S^{-1}(h_{(1)})m.a),
\end{aligned}$$

so that  $h(af) = \sum (h_{(1)}a)(h_{(2)}f)$ .

(c) We have

$$\begin{aligned}
(h(fa))(m) &= \sum h_{(2)}(fa)(S^{-1}(h_{(1)})m) = \sum h_{(2)} (f(S^{-1}(h_{(1)})m).a) \\
&= \sum h_{(2)} f(S^{-1}(h_{(1)})m).h_{(3)}a = \sum (h_{(1)}f)(m).h_{(2)}a = (\sum (h_{(1)}f)(h_{(2)}a))(m),
\end{aligned}$$

so that  $h(fa) = \sum (h_{(1)}f)(h_{(2)}a)$ .

(d) Follows by combining parts (b) and (c).  $\square$

In the next lemma, we shall equip  $\text{Hom}_{A\text{-Rmod}}(\cdot, \cdot)$  with the left  $H$ -action via (A.4), while  $\text{Hom}_{A\text{-Lmod}}(\cdot, \cdot)$  shall be equipped with the left  $H$ -action via (A.5).

**Lemma A.7** (Tensor-Hom Adjunction). (a) For any  $M, N \in A\text{-Rmod}^H$  and  $P \in A\text{-Bimod}^H$ , we have a natural isomorphism of left  $H$ -modules:

$$(A.6) \quad \text{Hom}_{A\text{-Rmod}}(N \otimes_A P, M) \simeq \text{Hom}_{A\text{-Rmod}}(N, \text{Hom}_{A\text{-Rmod}}(P, M)).$$

Here,  $\text{Hom}_{A\text{-Rmod}}(P, M) \in A\text{-Rmod}^H$  by Lemma A.4(b) and  $N \otimes_A P$  is naturally in  $A\text{-Rmod}^H$ .

(b) Let  $P, M \in A\text{-Rmod}^H$  such that  $P$  is a finitely generated projective right  $A$ -module. Note that  $A$  is naturally an object in  $A\text{-Bimod}^H$ . Then we have an isomorphism of left  $H$ -modules:

$$(A.7) \quad \text{Hom}_{A\text{-Rmod}}(P, M) \simeq M \otimes_A \text{Hom}_{A\text{-Rmod}}(P, A).$$

If we assume further that  $P \in A\text{-Bimod}^H$ , then (A.7) is an isomorphism in  $A\text{-Rmod}^H$ .

(c) For any  $P \in A\text{-Lmod}^H$ ,  $\text{Hom}_{A\text{-Lmod}}(P, A)$  is an object in  $A\text{-Rmod}^H$  by Lemma A.5(c), and  $\text{Hom}_{A\text{-Rmod}}(\text{Hom}_{A\text{-Lmod}}(P, A), A)$  is an object in  $A\text{-Lmod}^H$  by Lemma A.4(c). Define the map

$$(A.8) \quad \pi: P \longrightarrow \text{Hom}_{A\text{-Rmod}}(\text{Hom}_{A\text{-Lmod}}(P, A), A)$$

via  $\pi(p)(f) = f(p)$  for  $p \in P, f \in \text{Hom}_{A\text{-Lmod}}(P, A)$ . Then  $\pi$  is a morphism in  $A\text{-Lmod}^H$ . If additionally  $P$  is a finitely generated projective left  $A$ -module, then  $\pi$  is an isomorphism.

*Proof.* (a) If we forget about the  $H$ -module structures, then (A.6) is the usual tensor-hom adjunction. Thus, we only need to check that the  $H$ -actions on both sides are compatible. Let  $\psi$  be the natural vector space isomorphism from the right-hand to the left-hand sides of (A.6):

$$\psi(f)(n \otimes p) = f(n)(p) \quad \text{for } n \in N, p \in P, f \in \text{Hom}_{A\text{-Rmod}}(N, \text{Hom}_{A\text{-Rmod}}(P, M)).$$

Then, we have:

$$(\psi(hf))(n \otimes p) = ((hf)(n))(p) = \left( h_{(1)} f(S(h_{(2)})n) \right)(p) = h_{(1)} f(S(h_{(3)})n)(S(h_{(2)})p),$$

and

$$\begin{aligned} (h\psi(f))(n \otimes p) &= h_{(1)}\psi(f)\left(S(h_{(2)})(n \otimes p)\right) = h_{(1)}\psi(f)\left(S(h_{(3)})n \otimes S(h_{(2)})p\right) \\ &= h_{(1)}f(S(h_{(3)})n)(S(h_{(2)})p), \end{aligned}$$

so that  $\psi(hf) = h\psi(f)$ .

(b) It is standard that if we forget about the  $H$ -module structures, then (A.7) is an isomorphism of  $k$ -modules when  $P$  is a finitely generated projective right  $A$ -module. Thus, we only need to check that the  $H$ -actions on both sides are compatible. Let  $\phi$  be the natural vector space isomorphism from the right-hand to the left-hand sides of (A.7):

$$\phi(m \otimes f)(p) = mf(p) \quad \text{for } m \in M, p \in P, f \in \text{Hom}_{A\text{-Rmod}}(P, A).$$

Then, we have:

$$\begin{aligned} \phi(h(m \otimes f))(p) &= \sum h_{(1)}m.(h_{(2)}f)(p) = \sum h_{(1)}m.h_{(2)}f(S(h_{(3)})p) \\ &= \sum h_{(1)}\left(m.f(S(h_{(2)})p)\right) = \sum h_{(1)}\phi(m \otimes f)(S(h_{(2)})p) = \left(h\phi(m \otimes f)\right)(p), \end{aligned}$$

so that  $\phi(h(m \otimes f)) = h\phi(m \otimes f)$ .

(c) First, we note that the map  $\pi$  is a morphism of left  $A$ -modules, due to:

$$\pi(ap)(f) = f(ap) = af(p) = (a\pi(p))(f).$$

Let us now show that  $\pi$  is a morphism of left  $H$ -modules:

$$\begin{aligned} (h\pi(p))(f) &= \sum h_{(1)}\pi(p)(S(h_{(2)})f) = \sum h_{(1)}(S(h_{(2)})f)(p) \\ &= \sum h_{(1)}S(h_{(2)})f(S^{-1}(S(h_{(3)}))p) = f\left(\sum \varepsilon(h_{(1)})h_{(2)}p\right) = f(hp) = \pi(hp)(f). \end{aligned}$$

Since  $\pi$  is an isomorphism if  $P$  is a free left  $A$ -module of finite rank and  $\pi$  commutes with taking direct sums, it is also an isomorphism if  $P$  is a finitely generated projective left  $A$ -module.  $\square$

## REFERENCES

- [AP] H. Andersen, J. Paradowski, *Fusion categories arising from semisimple Lie algebras*, Comm. Math. Phys. **169** (1995), no. 3, 563–588.
- [APW] H. Andersen, P. Polo, K. Wen, *Representations of quantum algebras*, Invent. Math. **104** (1991), no. 1, 1–59.
- [AW] H. Andersen, K. Wen, *Representations of quantum algebras. The mixed case*, J. Reine Angew. Math. **427** (1992), 35–50.
- [B] J. Beck, *Representations of quantum groups at even roots of unity*, J. Algebra **167** (1994), no. 1, 29–56.
- [BG] K. A. Brown, I. Gordon, *Poisson orders, symplectic reflection algebras and representation theory*, J. Reine Angew. Math. , **559** (2003), 193–216.
- [BG2] K. A. Brown, K. R. Goodearl, *Homological Aspects of Noetherian PI Hopf algebras and Irreducible Modules of Maximal Dimension*, J. Algebra., **198** (1997), no. 1, 240–265.
- [CLT] K. Chan, J. Lu, S. To, *On intersections of conjugacy classes and Bruhat cells*, Transform. Groups **15** (2010), no. 2, 243–260.
- [CP] V. Chari, A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge (1994).
- [DP] C. De Concini, C. Procesi, *Quantum group*, D-modules, representation theory, and quantum groups (Venice, 1992), 31–140. Lecture Notes in Math., 1565, Springer-Verlag, Berlin, (1993).
- [DCK] C. De Concini, V. Kac, *Representations of quantum groups at roots of 1*, Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989), 471–506, Progr. Math., **92**, Birkhäuser Boston, Boston, MA (1990).
- [DCKP] C. De Concini, V. Kac, C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. **5** (1992), no. 1, 151–189.
- [DCKP2] C. De Concini, V. Kac, C. Procesi, *Remarkable degenerations of quantum algebras*
- [J1] J. Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, American Mathematical Society, Providence, RI (1996).
- [J2] J. Jantzen, *Representations of algebraic groups*, Math. Surveys Monogr., 107, American Mathematical Society, Providence, RI, (2003).
- [Jo] D. Jordan, *Quantum D-modules, elliptic braided groups, and double affine Hecke algebras*, Int. Math. Res. Not. (2009), no. 11, 2081–2105.
- [JL1] A. Joseph, G. Letzter, *Local finiteness of the adjoint action for quantized enveloping algebras*, J. Algebra **153** (1992), no. 2, 289–318.
- [JL2] A. Joseph, G. Letzter, *Separation of variables for quantized enveloping algebras*, Amer. J. Math. **116** (1994), no. 1, 127–177.
- [JL3] A. Joseph, G. Letzter, *Rosso’s form and quantized Kac Moody algebras*, Math. Z. **222** (1996), no. 4, 543–571.
- [Ka] C. Kassel, *Quantum groups*, Grad. Texts in Math. **155**, Springer-Verlag, New York (1995).
- [Ko] B. Kostant, *Groups over  $\mathbb{Z}$* , 1966 Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I. (1966), 90–98.
- [KS] S. Kolb, J. Stokman, *Reflection equation algebras, coideal subalgebras, and their centres*, Selecta Math. (N.S.) **15** (2009), no. 4, 621–664.
- [KSc] A. Klimyk, K. Schmüdgen, *Quantum groups and Their Representations*
- [L1] G. Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988), no. 2, 237–249.
- [L2] G. Lusztig, *Modular representations and quantum groups*, Classical groups and related topics (Beijing, 1987), 59–77, Contemp. Math. **82**, Amer. Math. Soc., Providence, RI (1989).
- [L3] G. Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata **35** (1990), no. 1–3, 89–113.
- [L4] G. Lusztig, *Finite-dimensional Hopf algebras arising from quantized universal enveloping algebra*, J. Amer. Math. Soc. **3** (1990), no. 1, 257–296.
- [L5] G. Lusztig, *Introduction to quantum groups*, Progr. Math. **110** (1993).
- [Ma] Matsumura, *Commutative algebra*, Math. Lecture Note Ser., 56, Benjamin/Cummings Publishing Co., Inc., Reading, MA, (1980).
- [N] C. Negron, *Revisiting the Steinberg representation at arbitrary roots of 1*, preprint, arXiv:2306.14453 (2023).
- [P] J. Paradowski, *Filtrations of modules over the quantum algebra*, Algebraic groups and their generalizations: quantum and infinite-dimensional methods (University Park, PA, 1991), 93–108, Proc. Sympos. Pure Math., 56, Part 2, Amer. Math. Soc., Providence, RI (1994).
- [R] N. Reshetikhin, *Multiparameter quantum groups and twisted quasitriangular Hopf algebras*, Lett. Math. Phys. **20** (1990), no. 4, 331–335.

- [Re] I. Reiner, *Maximal orders*, London Math. Soc. Monogr., No. 5, Academic Press[Harcourt Brace Jovanovich, Publishers], London-New York, (1975).
- [RH] S. Ryom-Hansen, *A  $q$ -analogue of Kempf's vanishing theorem*, Mosc. Math. J. **3** (2003), no. 1, 173–187.
- [S1] A. Sevostyanov, *Regular nilpotent elements and quantum groups*, Comm. Math. Phys. **204** (1999), no. 1, 1–16.
- [S2] A. Sevostyanov, *Quantum deformation of Whittaker modules and the Toda lattice*, Duke Math. J. **105** (2000), no. 2, 211–238.
- [St] R. Steinberg. *Conjugacy classes in Algebraic Groups*, Lecture Notes in Math., Vol 366, Springer-Verlag, Berlin-New York (1974).
- [Su] M. Susan, *Hopf algebras and Their Action on Rings*, CBMS Regional Conf. Ser. in Math., 82, Publishing for the Conference Board of the Mathematical Science, Washington, DC, by the American Mathematical Society, Providence, RI, (1993).
- [SV] P. Shan, E. Vasserot, *Heisenberg algebras and rational double affine Hecke algebras*, J. Amer. Math. Soc. **25** (2012), no. 4, 959–1031.
- [T] T. Tanisaki, *Killing forms, Harish-Chandra isomorphisms, and universal  $R$ -matrices for quantum algebras*, Infinite analysis, Part A, B (Kyoto, 1991), 941–961, Adv. Ser. Math. Phys., **16**, World Sci. Publ., River Edge, NJ (1992).