QUANTUM HARISH-CHANDRA BIMODULES AT ROOTS OF UNITY

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Contents

1. Set up

In this section, we establish notations and recall various results in [14].

Let $\mathfrak{g}$ be a semisimple Lie algebra with simple roots $\{\alpha_1, \ldots, \alpha_r\}$ and fundamental weights $\omega_1, \ldots, \omega_r$. Let $P := \bigoplus_{i=1}^r \mathbb{Z}\omega_i$ be the weight lattice and $Q := \bigoplus_{i=1}^r \mathbb{Z}\alpha_i$ be the root lattice. We fix a non-degenerate invariant bilinear form $(\, , \,)$ on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using $(\, , \,)$. We set $\mathsf{d}_i := \frac{(\alpha_i,\alpha_i)}{2}$. The choice of $(\, , \,)$ is such that $\mathsf{d}_i = 1$ for short roots α_i, in partcular, $\mathsf{d}_i \in \{1,2,3\}$ for any i. Define $\omega_i^\vee := \frac{\omega_i}{\mathsf{d}_i}$ and $\alpha_i^\vee := \frac{\alpha_i}{\mathsf{d}_i}$ the fundamental coweights and coroots. We have the Cartan matrix $(a_{ij})_{i,j=1}^n$ and the symmetrized Cartan matrix $(b_{ij})_{i,j=1}^r$:

$$a_{ij} = (\alpha_i^\vee, \alpha_j) = 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i), \qquad b_{ij} := (\alpha_i, \alpha_j).$$

Let v be a formal variable and

$$\mathcal{A} := \mathbb{Z}[v, v^{-1}] \left[\left\{ \frac{1}{v^{2k}-1} \right\} \right]_{1 \le k \le \max\{\mathsf{d}_i\}}$$

Let us define the elements in $\mathcal{A}$:

$$[s]_v := \frac{v^s - v^{-s}}{v - v^{-1}}, \qquad [s]_v! = [1]_v \ldots [s]_v, \qquad \begin{bmatrix} m \\ s \end{bmatrix}_v := \prod_{c=1}^s \frac{v^{m-c+1} - v^{-m+c-1}}{v^c - v^{-c}},$$

$$(s)_v := \frac{1 - v^{-2s}}{1 - v^{-2}}, \qquad (s)_v := (1)_v \ldots (s)_v, \qquad \binom{m}{s}_v := \prod_{c=1}^s \frac{1 - v^{-2(m+1-c)}}{1 - v^{-2c}}.$$

The quantum group $\mathbf{U}_v(\mathfrak{g})$ is the Hopf algebra over $\mathbb{Q}(v)$ generated by generators $\{E_i, F_i, K_i := K^{\alpha_i}\}_{1 \le i \le r}$ subject to relations:

$$\begin{aligned}
&K^\mu K^{\mu'} = K^{\mu+\mu'}, \qquad K^0 = 1, \\
&K^\mu E_i K^{-\mu} = v^{(\mu,\alpha_i)} E_i, \qquad K^\mu F_i K^{-\mu} = v^{-(\mu,\alpha_i)} F_i, \\
&[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{v_i - v_i^{-1}}, \\
&\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} E_i^{1-a_{ij}-m} E_j E_i^m = 0 \quad (i \neq j) \\
&\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} F_i^{1-a_{ij}-m} F_j F_i^m = 0 \quad (i \neq j)
\end{aligned}$$

here $v_i := v^{\mathsf{d}_i}$, with the Hopf structure as follows:

$$\begin{aligned}
&\Delta : E_i \mapsto E_i \otimes 1 + K_i \otimes E_i, \quad F_i \mapsto F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad K^\mu \mapsto K^\mu \otimes K^\mu, \\
&S : E_i \mapsto -K_i^{-1} E_i, \quad F_i \mapsto -F_i K_i, \quad K^\mu \mapsto K^{-\mu}, \\
&\varepsilon : E_i \mapsto 0, \quad F_i \mapsto 0, \quad K^\mu \mapsto 1.
\end{aligned}$$

There is a left adjoint action of $\mathbf{U}_v(\mathfrak{g})$ on itself defined by

$$\mathrm{ad}(x)(u) = \sum x_{(1)} u S(x_{(2)}), \qquad \forall x, u \in \mathbf{U}_v(\mathfrak{g}), \tag{1.1}$$

here we use the Sweedler's notation for coproduct.

Let $E_i^{(n)} := \dfrac{E_i^n}{(n)_{v_i}!}$, $F_i^{(n)} := \dfrac{F_i^n}{(n)_{v_i}!}$. In $\mathbf{U}_v(\mathfrak{g})$, there are two $\mathcal{A}$-integral forms: the Lusztig form $\check{\mathcal{U}}_v(\mathfrak{g})$ and the De Concini-Kac form $\mathcal{U}_v(\mathfrak{g})$. The Lusztig form $\check{\mathcal{U}}_v(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra generated by $\{E_i^{(n)}, F_i^{(n)}, K^{\alpha_i}\}$ while the De Concini-Kac form $\mathcal{U}_v(\mathfrak{g})$ is the $\mathcal{A}$-algebra generated by $\{E_i, F_i, K^{\alpha_i}\}$. Both are Hopf $\mathcal{A}$-subalgebras of $\mathbf{U}_v(\mathfrak{g})$. However the adjoint action (1.1) does not restrict to an action of $\check{\mathcal{U}}_v(\mathfrak{g})$ on $\mathcal{U}_v(\mathfrak{g})$. One of the main construction in [14] is to remedy this issue. Roughly speaking, we will twist the coproduct of $\mathbf{U}_v(\mathfrak{g})$ so that the left adjoint action give a rise to an action of *(twisted) Lusztig form* $\check{U}_v(\mathfrak{g})$ on the *even subaglebra* $U_v^{ev}(\mathfrak{g})$, which is a suitable alternative to the De Concini-Kac form.

1.1. **Twisted construction.**

Let us recall the construction in [14]. It starts with the standard twist construction in [11, Theorem 1]

Proposition 1.1. *(a) For a (topological) Hopf algebra $(A, m, \Delta, S, \varepsilon)$ and $F \in A \widehat{\otimes} A$ satisfying*

$$(\Delta \otimes \mathrm{Id})(F) = F_{13}F_{23}, \quad (\mathrm{Id} \otimes \Delta)(F) = F_{13}F_{12}, \quad F_{12}F_{13}F_{23} = F_{23}F_{13}F_{12}, \quad F_{12}F_{21} = 1, \tag{1.2}$$

the formulas

$$\Delta^{(F)}(a) = F\Delta(a)F^{-1}, \qquad S^{(F)}(a) = uS(a)u^{-1}, \qquad \varepsilon^{(F)}(a) = \varepsilon(a)$$

with $u := m(\mathrm{Id} \otimes S)(F)$, endow A with a new Hopf algebra structure $(A, m, \Delta^{(F)}, S^{(F)}, \varepsilon^{(F)})$.
(b) If $(A, m, \Delta, S, \varepsilon)$ is a quasitriangular Hopf algebra with universal R-matrix $R \in A \otimes A$, then $(A, m, \Delta^{(F)}, S^{(F)}, \varepsilon^{(F)})$ is also a quasitriangular Hopf algebra with universal R-matrix:

$$R^{(F)} = F^{-1}RF^{-1}.$$

Let $\mathbf{U}_v(\mathfrak{g}, P/2)$ be an Hopf algebra over $\mathbb{Q}(v^{1/2})$ obtained from $\mathbf{U}_v(\mathfrak{g})$ by extending the base ring to $\mathbb{Q}(v^{1/2})$ and adding elements $\{K^\lambda\}_{\lambda \in P/2}$.

Let $\mathrm{Dyn}(\mathfrak{g})$ denote the graph obtained from the Dynkin diagram of $\mathfrak{g}$ by replacing all multiple edges by simple ones, e.g., $\mathrm{Dyn}(\mathfrak{sp}_{2r}) = \mathrm{Dyn}(\mathfrak{so}_{2r+1}) = \mathrm{Dyn}(\mathfrak{sl}_{r+1}) = A_r$. Let us fix an orientation Or of Dynkin diagram of $\mathfrak{g}$. We associate to such orientation a skew-symmetric matrix $(\epsilon_{ij})_{i,j=1}^r$ via

$$\epsilon_{ij} = \begin{cases} 0 & \text{if } a_{ij} \geq 0 \\ 1 & \text{if } a_{ij} < 0 \text{ and } \mathsf{Or} \text{ contains an oriented edge } i \to j \\ -1 & \text{if } a_{ij} < 0 \text{ and } \mathsf{Or} \text{ contains an oriented edge } i \leftarrow j \end{cases}$$

Let us consider the skew-symmetric matrix $(\phi_{ij})_{i,j=1}^r$ in which

$$\phi_{ij} = \epsilon_{ij}\frac{(\alpha_i, \alpha_j)}{2}.$$

The twist

$$\mathsf{F} = v^{\sum_{ij} \phi_{ij} \omega_i^\vee \otimes \omega_j^\vee} \tag{1.3}$$

satisfies the condition (1.2). To be more precise , this twist belongs to a topological completion of $\mathbf{U}_v(\mathfrak{g}, P/2)$ (May describe this topological completion). Nevertheless, it still gives a new coproduct on $\mathbf{U}_v(\mathfrak{g}, P/2)$ as follows:

$$\begin{aligned}
&\Delta'(K^\mu) = K^\mu \otimes K^\mu\,, \\
&\Delta'(E_i) = E_i \otimes K^{\sum_{j=1}^r \phi_{ij}\omega_j^\vee} + K^{\alpha_i - \sum_{j=1}^r \phi_{ij}\omega_j^\vee} \otimes E_i\,, \\
&\Delta'(F_i) = F_i \otimes K^{-\alpha_i - \sum_{j=1}^r \phi_{ij}\omega_j^\vee} + K^{\sum_{j=1}^r \phi_{ij}\omega_j^\vee} \otimes F_i\,, \\
&S'(K^\mu) = K^{-\mu}\,, \qquad S'(E_i) = -K^{-\alpha_i}E_i\,, \qquad S'(F_i) = -F_iK^{\alpha_i}\,, \\
&\varepsilon'(K^\mu) = 1\,, \qquad \varepsilon'(E_i) = \varepsilon'(F_i) = 0\,,
\end{aligned} \tag{1.4}$$

Let

$$\nu_i^> := -\alpha_i + \sum_{j=1}^r \phi_{ij}\omega_j^\vee, \qquad \nu_i^< := \sum_{j=1}^r \phi_{ij}\omega_j^\vee,$$
$$\zeta_i^> := \alpha_i - 2\sum_{j=1}^r \phi_{ij}\omega_j^\vee, \qquad \zeta_i^< := -\alpha_i - 2\sum_{j=1}^r \phi_{ij}\omega_j^\vee.$$

Then set

$$\tilde{E}_i := E_i K^{\nu_i^>}, \qquad \tilde{F}_i := K^{-\nu_i^<} F_i.$$

Remark 1.2. These elements $\zeta_i^<, \zeta_i^>$ belong to $2P$.

One can show that $\mathbf{U}_v(\mathfrak{g}, P/2)$ is generated by $\{\tilde{E}_i, \tilde{F}_i, K^\lambda\}_{1\le i\le r}^{\lambda\in P/2}$ subjects to relations:

$$\begin{aligned}
&K^\mu K^{\mu'} = K^{\mu+\mu'}, \qquad K^0 = 1, \\
&K^\mu \tilde{E}_i K^{-\mu} = v^{(\alpha_i,\mu)}\tilde{E}_i, \qquad K^\mu \tilde{F}_i K^{-\mu} = v^{-(\alpha_i,\mu)}\tilde{F}_i, \\
&\tilde{E}_i\tilde{F}_j = v^{(\alpha_i,-\zeta_j^<)}\tilde{F}_j\tilde{E}_i \quad (i\neq j), \qquad \tilde{E}_i\tilde{F}_i - v_i^2\tilde{F}_i\tilde{E}_i = v_i\frac{1-K_i^{-2}}{1-v_i^{-2}}, \\
&\sum_{m=0}^{1-a_{ij}} (-1)^m v^{m\epsilon_{ij}b_{ij}} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} \tilde{E}_i^{1-a_{ij}-m}\tilde{E}_j\tilde{E}_i^m = 0 \quad (i\neq j), \\
&\sum_{m=0}^{1-a_{ij}} (-1)^m v^{m\epsilon_{ij}b_{ij}} \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{v_i} \tilde{F}_i^{1-a_{ij}-m}\tilde{F}_j\tilde{F}_i^m = 0 \quad (i\neq j),
\end{aligned} \tag{1.5}$$

here $K_i := K^{\alpha_i}$, $v_i = v^{\mathsf{d}_i}$ as usual. Moreover, we have

$$\begin{aligned}
&\Delta'(K^\mu) = K^\mu\otimes K^\mu, \quad \Delta'(\tilde{E}_i) = 1\otimes\tilde{E}_i + \tilde{E}_i\otimes K^{-\zeta_i^>}, \quad \Delta'(\tilde{F}_i) = 1\otimes\tilde{F}_i + \tilde{F}_i\otimes K^{\zeta_i^<}, \\
&S'(K^\mu) = K^{-\mu}, \qquad S'(\tilde{E}_i) = -\tilde{E}_i K^{\zeta_i^>}, \qquad S'(\tilde{F}_i) = -\tilde{F}_i K^{-\zeta_i^<}.
\end{aligned} \tag{1.6}$$

Definition 1.3. The *(twist) Lusztig form* $\check{U}_v(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra of $\mathbf{U}_v(\mathfrak{g}, P/2)$ generated by $\{\tilde{E}_i^{(n)}, \tilde{F}_i^{(n)}, K^\lambda\}_{\lambda\in 2P}^{1\le i\le r}$ with $\tilde{E}_i^{(n)} := \frac{\tilde{E}_i^n}{(n)_{v_i}!}$ and $\tilde{F}_i^{(n)} := \frac{\tilde{F}_i^n}{(n)_{v_i}!}$. The *even subalgebra* $U_v^{ev}(\mathfrak{g})$ is the $\mathcal{A}$-subalgebra of $\mathbf{U}_v(\mathfrak{g}, P/2)$ generated by $\{\tilde{E}_i, \tilde{F}_i, K^\lambda\}_{\lambda\in 2P}^{1\le i\le r}$.

These algebras are Hopf $\mathcal{A}$-subalgebras of $\mathbf{U}_v(\mathfrak{g}, P/2)$ with the twisted Hopf structure. The name *even subalgebra* comes from the fact that we only use the lattice $2P$ for the Cartan part in the set of generators of $U_v^{ev}(\mathfrak{g})$. We have the left adjoint action ad'_l of $\mathbf{U}_v(\mathfrak{g}, P/2)$ on itself similar to (1.1). By Proposition in [14], we have

Proposition 1.4. *The left adjoint action ad'_l of $\mathbf{U}_v(\mathfrak{g}, P/2)$ on itself restricts to an adjoint action of $\check{U}_v(\mathfrak{g})$ on $U_v^{ev}(\mathfrak{g})$.*

To any algebra homomorphism $\mathcal{A}\to R$, $v\mapsto q\in R^\times$, we define the specalizations

$$\check{U}_q(\mathfrak{g}) := \check{U}_v(\mathfrak{g})\otimes_{\mathcal{A}} R, \qquad U_q^{ev}(\mathfrak{g}) := U_v^{ev}(\mathfrak{g})\otimes_{\mathcal{A}} R.$$

Then we has the adjoint action $\mathrm{ad}'_l : \check{U}_q(\mathfrak{g}) \curvearrowright U_q^{ev}(\mathfrak{g})$. Furtheremore, the inclusion $\iota : U_v^{ev}(\mathfrak{g}) \hookrightarrow \check{U}_v(\mathfrak{g})$ induces the morphism $\iota : U_q^{ev}(\mathfrak{g}) \to \check{U}_q(\mathfrak{g})$.

Let $\dot{U}_q(\mathfrak{g}, P)$ denote **the idempotented Lusztig form** defined similarly to [9, Chapter 23] with generators

$$\{\tilde{E}_i^{(n)}1_\lambda, \tilde{F}_i^{(n)}1_\lambda | 1\le i\le r, n\ge 0, \lambda\in P\}$$

We record the coproduct of $\dot{U}_q(\mathfrak{g}, P)$:

$$\begin{aligned} \Delta(\tilde{E}_i^{(r)} 1_\lambda) &= \sum_{c=0}^{r} \prod_{\lambda'+\lambda''=\lambda} q^{-(r-c)(\zeta_i^{\geqslant},\lambda'')} \tilde{E}_i^{(r-c)} 1_{\lambda'} \otimes \tilde{E}_i^{(c)} 1_{\lambda''}\,, \\ \Delta(\tilde{F}_i^{(r)} 1_\lambda) &= \sum_{c=0}^{r} \prod_{\lambda'+\lambda''=\lambda} q_i^{2c(r-c)} q^{c(\zeta_i^{\leqslant},\lambda'')} \tilde{F}_i^{(c)} 1_{\lambda'} \otimes \tilde{F}_i^{(r-c)} 1_{\lambda''}\,, \end{aligned} \tag{1.7}$$

2. Rational Representation of $\check{U}_q(\mathfrak{g})$

2.1. Rational representations of $\check{U}_q(\mathfrak{g})$.

Let N be such that $\mathsf{N}(\frac{P}{2}, \frac{P}{2}) \in \mathbb{Z}$. Fix element $q^{1/\mathsf{N}}$ in R such that $(q^{1/\mathsf{N}})^{\mathsf{N}} = q$. For any $\lambda \in P$, let $\chi_\lambda : \check{U}_q^0(\mathfrak{g}) \to R$ defined by

$$\chi_\lambda(K^\mu) = q^{(\mu,\lambda)}, \qquad \chi_\lambda\left(\binom{K_i;0}{m}\right) = \binom{(\lambda,\alpha_i^\vee)}{m}_{q_i}, \tag{2.1}$$

for $\mu \in P$ and $m \in \mathbb{N}$.

Definition 2.1. A $\check{U}_q(\mathfrak{g})$-module M is a rational representation (of type 1) if it satisfies the following:

(i) M is a weight module meaning that there is a decomposition $M = \bigoplus_{\lambda \in P} M_\lambda$, where $u_0 m = \chi_\lambda(u_0)m$ for all $u_0 \in \check{U}_q^0, m \in M_\lambda$.

(ii) For any $m \in M$, there is $k > 0$ such that $\tilde{E}_i^{(s)} m = 0$ for all $s > k$ and all $1 \le i \le r$.

(iii) For any $m \in M$, there is $k > 0$ such that $\tilde{F}_i^{(s)} m = 0$ for all $s > k$ and all $1 \le i \le r$.

Let $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ denote the category of rational representations of $\check{U}_q(\mathfrak{g})$. Let $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ denote the full subcategory of $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ consisting of objects which are finitely generated over R.

Definition 2.2. (a) For any $\lambda \in P$, let R_λ denote the representation of $\dot{U}_q^{\geqslant}$ defined via $\dot{U}_q^{\geqslant} \to \dot{U}_q^0 \xrightarrow{\chi_\lambda} R$. Then the Verma module $\Delta_q(\lambda) := \dot{U}_q(\mathfrak{g}) \otimes_{\dot{U}_q^{\geqslant}} R_\lambda$.
(b) For any $\lambda \in P_+$, the Weyl module $W_q(\lambda)$ is the maximal rational representaion of the Verma module $\Delta_q(\lambda)$. Let 1_λ be the image of $1 \in \dot{U}_q(\mathfrak{g})$ in $\Delta_q(\lambda)$ then $W_q(\lambda)$ is the quotient of $\Delta_q(\lambda)$ by the left $\dot{U}_q(\mathfrak{g})$-submodules generated by $\tilde{F}_i^{(s)} 1_\lambda$ for $s > (\lambda, \alpha_i^\vee)$ and $1 \le i \le r$.

The existence of the maximal rational quotient of $\Delta_q(\lambda)$ and the description of $W_q(\lambda)$, see [reference to the draft, APW)]

Definition 2.3. A $\dot{U}_q(\mathfrak{g}, P)$-module M is a unital rational representation if it satisfies

(i) For any $m \in M$ then $1_\lambda m = 0$ for all but finitely many $\lambda \in P$.

(ii) For any $m \in M$, there is $k > 0$ such that $\tilde{E}_i^{(s)} 1_\lambda m = 0$ for all $s > k$ and all $1 \le i \le r$.

(iii) For any $m \in M$, there is $k > 0$ such that $\tilde{F}_1^{(s)} 1_\lambda m = 0$ for all $s > k$ and all $1 \le i \le r$.

Let $\mathrm{Rep}(\dot{U}_q(\mathfrak{g}, P))$ denote the category of unital rational representations of $\dot{U}_q(\mathfrak{g}, P)$.

There is a natural equivalence of monoidal categories

$$\mathrm{Rep}(\check{U}_q(\mathfrak{g})) \cong \mathrm{Rep}(\dot{U}_q(\mathfrak{g}, P)).$$

Remark 2.4. Relate to the usual $\dot{\mathcal{U}}_q(\mathfrak{g}, P)$.

2.2. **Quantum Frobenious morphism.**
Let $\ell_i := \gcd(2d_i, \ell)$ for $1 \leq i \leq r$. Let $\ell_\alpha := \gcd((\alpha,\alpha),\ell)$ for all positive roots $\alpha \in \Delta_+$ of $\mathfrak{g}$. Let us consider the following data:

- The lattices $P^* = \bigoplus_{i=1}^r \mathbb{Z}\omega_i^*$ and $Q^* = \bigoplus_{i=1}^r \mathbb{Z}\alpha_i^*$ in which $\omega_i^* := \ell_i\omega_i$, $\alpha_i^* := \ell_i\alpha_i$. Then set $\omega_i^{*\vee} := \omega_i^\vee/\ell_i$, $\alpha_i^{*\vee} := \alpha_i^\vee/\ell_i$.
- The new Cartan matrix with (i,j)-entry

$$a_{ij}^* = 2(\alpha_i^*, \alpha_j^*)/(\alpha_i^*, \alpha_i^*) = 2\ell_j(\alpha_i,\alpha_j)/\ell_i(\alpha_i,\alpha_i)\,. \tag{2.2}$$

- The bilinear form on P^* induced from the bilinear form on P via the inclusion $P^* \subset P$.

So that (a_{ij}^*) is the Cartan matrix of a semisimple Lie algebra $\mathfrak{g}^d$, see [9, §2.2.4]. Furtheremore, $\mathfrak{g}^d$ is either $\mathfrak{g}$ or the Langland dual $\mathfrak{g}^\vee$ of $\mathfrak{g}$. Hence, $\mathrm{Dyn}(\mathfrak{g}^d)$ is the same graph as $\mathrm{Dyn}(\mathfrak{g})$. Let us fix the same orientation Or for $\mathrm{Dyn}(\mathfrak{g}^d)$ as one of $\mathrm{Dyn}(\mathfrak{g})$.

We form the $\mathbb{Q}(v^{1/2})$-Hopf algebra $\mathbf{U}^*(\mathfrak{g}, P^*/2)$ with generators $\{\hat{e}_i, \hat{f}_i, K^\mu\}_{1\leq i\leq r}^{\mu\in P^*/2}$ by the above data. We have the following twist with respect to the orientation Or of $\mathrm{Dyn}(\mathfrak{g}^d)$:

$$\mathsf{F}^* := v^{\sum_{i,j}\phi_{ij}^*\omega_i^{*\vee}\otimes\omega_j^{*\vee}}, \qquad \text{here} \qquad \phi_{ij}^* = \epsilon_{ij}\frac{(\alpha_i^*,\alpha_j^*)}{2}.$$

As in Section 1.1, we consider the following twisted generators:

$$\tilde{e}_i := \hat{e}_i K^{\nu_i^{*>}}, \qquad \tilde{f}_i := K^{-\nu_i^{*<}}\hat{f}_i,$$

in which

$$\nu_i^{*>} := -\alpha_i^* + \sum_{1\leq j\leq r}\phi_{ij}^*\omega_j^{*\vee} = \ell_i\nu_i^>, \qquad \nu_i^{*<} := \sum_{1\leq j\leq r}\phi_{ij}^*\omega_j^{*\vee} = \ell_i\nu_i^<.$$

Finally, we obtain the idempotented Lusztig form $\dot{U}_q^*(\mathfrak{g}, P^*)$ (after base change $\mathcal{A} \to R, v \mapsto q \in R^\times$) with generators:

$$\{\tilde{e}_i^{(n)}1_\lambda, \tilde{f}_i^{(n)}1_\lambda | 1 \leq i \leq r, \lambda \in P^*\}.$$

The next proposition is a twisted version of [9, Theorem]. We refer the dicussion about it to [14, §4].

Proposition 2.5. *There is a unique R-homomorphism*

$$\tilde{\mathrm{Fr}} : \dot{U}_\epsilon(\mathfrak{g}, P) \to \dot{U}_\epsilon^*(\mathfrak{g}, P^*)$$

such that

- $\tilde{\mathrm{Fr}}(\tilde{E}_i^{(n)}1_\lambda)$ *equals* $\tilde{e}_i^{(n/\ell_i)}1_\lambda$ *if* $\lambda in P^*$ *and n is divisible by ℓ_i, and is zero otherwise.*
- $\tilde{\mathrm{Fr}}(\tilde{F}_i^{(n)}1_\lambda$ *equals* $\tilde{f}_i^{(n/\ell_i)}1_\lambda$ *if* $\lambda \in P^*$ *and n is divisible by ℓ_i, and is zero otherwise.*

Furthermore, this homomorphism is compatible with comultiplications.

Remark 2.6. This morphism gives a rise to a functor of monoidal categories:

$$\tilde{\mathrm{Fr}}^* : \mathrm{Rep}(\dot{U}_\epsilon^*(\mathfrak{g}, P^*) \to \mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}, P)). \tag{2.3}$$

To the Lie algebra $\mathfrak{g}^d$, we have the Kostant $\mathbb{Z}$-form $\check{U}_\mathbb{Z}(\mathfrak{g}^d)$ of the universal enveloping algebra $\mathbf{U}_\mathbb{Q}(\mathfrak{g}^d)$ with generators $\{e_i, f_i, h_i\}_{1\leq i\leq r}$, for details see [4] or [14, §4.3]. Let

$$\check{U}_R(\mathfrak{g}^d) := \check{U}_\mathbb{Z}(\mathfrak{g}^d) \otimes_\mathbb{Z} R.$$

The next proposition is [14, Proposition 4.17]:

Proposition 2.7. *There is a unique R-homomorphism of Hopf algebras*
$$\tilde{\mathrm{Fr}} : \check{U}_\epsilon(\mathfrak{g}) \to \check{U}_R(\mathfrak{g}^d)$$
such that
$$\tilde{E}_i^{(n)} \mapsto (\epsilon_i^*)^{-n/\ell_i} e_i^{(n/\ell_i)}, \qquad \tilde{F}_i^{(n)} \mapsto f_i^{(n/\ell_i)}, \qquad K^\lambda = 1,$$
where $\lambda \in 2P$ and we set $e_i^{(n/\ell_i)} = f_i^{(n/\ell_i)} = 0$ if ℓ_i does not divide n.

2.3. More on rational representations.

In this section, we consider the following two cases: (Trung: May move it into the introduction. Say some results can be defined over general rings but we are mostly interested in the following cases)

- (**A**) $q = \epsilon \in \mathbb{C}$ a root of unity of order ℓ. We assume that $\ell_i \geq \max\{2, a_{ij}\}_{1\leq i\leq r}$ for all $1 \leq i \leq r$. We use $\check{U}_\epsilon(\mathfrak{g})$ to denote the (twisted) Lusztig form.
- (**B**) $q = \epsilon e^\hbar \in \mathbb{C}[[\hbar]]$. We will still use $\check{U}_q(\mathfrak{g})$ to denote the (twisted) Lusztig form in this case.

So we have a natural short exact sequence: $0 \to \check{U}_q(\mathfrak{g}) \xrightarrow{\cdot\hbar} \check{U}_q(\mathfrak{g}) \xrightarrow{/\hbar} \check{U}_\epsilon(\mathfrak{g}) \to 0$.

Definition 2.8. Let $\lambda_{\mathbf{St}} := \sum_i(\ell_i - 1)\omega_i$. The Steinberg representation $\mathbf{St}_\epsilon \in \mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ is the Weyl module $W_\epsilon(\lambda_{\mathbf{St}})$. The Steinberg representation $\mathbf{St}_q \in \mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ is the Weyl module $W_q(\lambda_{\mathbf{St}})$.

We have the equivalence of braided monoidal categories $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g})) \cong \mathrm{Rep}(\dot{\mathcal{U}}_\epsilon(\mathfrak{g}, P))$, see Remark 2.4. Hence by [10], we have the following proposition:

Proposition 2.9. *(a) The module $\boldsymbol{St}_\epsilon$ is projective and injective in $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$.*
(b) The category $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ has enough projectives and injectives. Any object M in $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$ admits a surjective morphism from a projective object of the form $\boldsymbol{St}_\epsilon \otimes_{\mathbb{C}} N$ with $N \in \mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$.

Proposition 2.10. *(a) For any $N_q \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ which is a free module of finite rank over $\mathbb{C}[[\hbar]]$, the object $\boldsymbol{St}_q \otimes_{\mathbb{C}[[\hbar]]} N_q$ is projective in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$.*
(b) Any object in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ admits a surjective morphism from some projective object of the form $\boldsymbol{St}_q \otimes_{\mathbb{C}[[\hbar]]} N_q$ as in part (a).
(c) The category $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ has enough projectives.

Proof. (a) Since any object in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ is a union of objects in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$, it is enough to prove the statement in the category $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Then the statement follows by Proposition 2.9 and the following claim:

> *Claim:* Suppose $V_q \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ such that V_q is a free module over finite rank over $\mathbb{C}[[\hbar]]$ and $V_q/\hbar V_q$ is a projective object in $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$, then V_q is projective in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$.

Let us prove the claim.
Step 1: For $N \in \mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$, we will show that $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) = 0$.

Let $0 \to N \to M \to V_q \to 0$ be a short exact sequence in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Since V_q is free over $\mathbb{C}[[\hbar]]$, there is a short exact sequence in $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$:
$$0 \to N \to M/\hbar M \to V_q/\hbar V_q \to 0,$$
which is split since $V_q/\hbar V_q$ is projective in $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$. Let $V_q/\hbar V_q \to M/\hbar M$ be a splitting and let V_1 denote the image of $V_q/\hbar V_q$ under that splitting map. Let M_1 denote the preimage of V_1 under the quotient map $M \twoheadrightarrow M/\hbar M$. One can show that the composition map $M_1 \hookrightarrow$

$M \to V_q$ is surjective, the kernel N_1 is a submodule of N. So we have another short exact sequence in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g})) : 0 \to N_1 \to M_1 \to V_q \to 0$. Since V_q is free over $\mathbb{C}[[\hbar]]$, if N is nonzero then M_1 is a proper submodule of M and N_1 is also a proper submodule of N.

Proceed this procedure iteriately, we get a decreasing sequence $M \supset M_1 \supset M_2 \dots$ and $N \supset N_1 \supset N_2 \dots$ such that $0 \to N_i \to M_i \to V_q \to 0$ is a short exact seqence for any i and N_{i+1} is always a proper submodule of N_i if N_i is nonzero. Because N is a finite dimensional vector space, the decreasing sequence $N \supset N_1 \supset N_2 \dots$ must terminate. Hence, we can find some submodule $M' \subset M$ such that the composition $M' \hookrightarrow M \to V_q$ is an isomorphism in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. As a result, the short exact sequence $0 \to N \to M \to V_q \to 0$ splits in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$.

Step 2: For $N \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ flat over $\mathbb{C}[[\hbar]]$, we will show that $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) = 0$.

We have a short exact sequence $0 \to N \xrightarrow{\cdot\hbar} N \to N/\hbar N \to 0$ in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$, which give a long exact sequence of $\mathbb{C}[[\hbar]]$-modules

$$\dots \to \mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) \xrightarrow{\cdot\hbar} \mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) \to \mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N/\hbar N) \dots$$

This gives a surjective map $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) \xrightarrow{\cdot\hbar} \mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N)$. On the other hand, $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N)$ is finitely generated over $\mathbb{C}[[\hbar]]$, one way to prove it is in [15, Proposition 5.15]. Therefore, by Nakayama lemma, $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) = 0$.

Step 3: We will show that $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) = 0$ for any $N \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Let $N_{\mathrm{tor}} := \{n \in N | \hbar^k n = 0 \text{ for some } k > 0\}$. Then N_{tor} is a subobject of N in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g})$. Since N_{tor} is finitely generated over $\mathbb{C}[[\hbar]]$, it admits a finite filtration whose subquotients are objects in $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$. On the other hand N/N_{tor} is an object in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ which is flat over $\mathbb{C}[[\hbar]]$. Therefore by *Step 1* and *Step 2*, we have $\mathrm{Ext}^1_{\mathrm{Rep}(\check{U}_q(\mathfrak{g}))}(V_q, N) = 0$.

This completes the proof.

(b) For any $N \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$, the quotient $N/\hbar N$ belongs to $\mathrm{Rep}^{fd}(\check{U}_\epsilon(\mathfrak{g}))$. Then there is a surjective map:

$$\mathbf{St}_\epsilon \otimes_{\mathbb{C}} \Big(\bigoplus_{\lambda_i} W_\epsilon(\lambda_i) \Big) \twoheadrightarrow N/\hbar N,$$

for a finite collection of dominant weights $\{\lambda_i\}$. Since $\mathbf{St}_q \otimes_{\mathbb{C}[[\hbar]]} W_q(\lambda_i)$ is projective in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ by part (a), we have the following commutative diagram in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$:

$$\begin{tikzcd}
\mathbf{St}_q \otimes_{\mathbb{C}[[\hbar]]} \big(\bigoplus_{\lambda_i} W_q(\lambda_i)\big) \arrow[r, dashed] \arrow[d, "/\hbar"] & N \arrow[d, "/\hbar"'] \\
\mathbf{St}_\epsilon \otimes_{\mathbb{C}} \big(\bigoplus_{\lambda_i} W_\epsilon(\lambda_i)\big) \arrow[r, two heads] & N/\hbar N
\end{tikzcd}$$

By Nakayama lemma, the upper horizontal arrow is surjective. This finishes the proof.

(c) Since any object in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ is a union of objects in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$, this part follows by part (b). $\square$

Definition 2.11. Let $\mathfrak{u}_\epsilon$ be the Hopf subalgebra of $\check{U}_\epsilon(\mathfrak{g})$ generated by $\{\tilde{E}_i, \tilde{F}_i, K^\lambda\}^{\lambda \in 2P}_{1 \le i \le r}$.

2.4. **Tilting modules.** Discuss about tilting modules on $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ and $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$.

- Learn about the highest weight category over complete local rings, for example $\mathbb{C}[[\hbar]]$.

- Show one-to-one correspondence between indecomposable tilting objects in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ and $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$: use the lemma that for any V_q free of finite rank over $\mathbb{C}[[\hbar]]$, V_q has a good filtration iff $V_q/\hbar V_q$ has a good filtration, see [14, Lemma 6.25].
- Mention that projective objects in $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ are tilting. Deduce that projective objects in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ are tilting: Any projective object is the direct summand of $\mathbf{St} \otimes_{\mathbb{C}[[\hbar]]} W_q$ for some free of finite rank W_q. Then use the lemma in the second bullet point to conclude that $\mathbf{St} \otimes_{\mathbb{C}[[\hbar]]} W_q$ is tilting, hence so is its direct summands.

3. The even subalgebra $U_q^{ev}(\mathfrak{g})$

(Trung: various terminologies are needed to be defined. Need to think of how to present the results over various base changes)

In this section, $U_\epsilon^{ev}(\mathfrak{g})$ and $U_q^{ev}(\mathfrak{g})$ are referred to the base changes of $U_v^{ev}(\mathfrak{g})$ with respect to case **(A)** and **(B)**, respectively.

There is the Lusztig's braided group action on $\mathbf{U}_v(\mathfrak{g}, P/2)$ defined as follows , see [9, Part VI] and also [3, §4.9]

$$
\begin{aligned}
&T_i(K^\mu) = K^{s_{\alpha_i}\mu}\,, \qquad T_i(E_i) = -F_iK^{\alpha_i}\,, \qquad T_i(F_i) = -K^{-\alpha_i}E_i\,,\\
&T_i(E_j) = \sum_{k=0}^{-a_{ij}} (-1)^k \frac{v_i^{-k}}{[-a_{ij}-k]_{v_i}![k]_{v_i}!} E_i^{-a_{ij}-k}E_jE_i^k\,,\\
&T_i(F_j) = \sum_{k=0}^{-a_{ij}} (-1)^k \frac{v_i^{k}}{[-a_{ij}-k]_{v_i}![k]_{v_i}!} F_i^{k}F_jF_i^{-a_{ij}-k}\,.
\end{aligned}
\tag{3.1}
$$

Let us pick a reduce expression of the longest element $w_0 = s_{i_1}s_{i_2}\dots s_{i_N}$ in the Weyl group W, here N is the cardinality of the positive root system Δ_+. Then the set of roots $\beta_k = s_{i_1}\dots s_{i_{k-1}}\alpha_{i_k} (1 \le k \le N)$ provides a labelling of all positive roots in Δ_+. Using Lusztig's braid group action, we define the root vectors $\{E_\beta, F_\beta\}_{\beta\in\Delta_+}$ in a standard way:

$$
E_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}E_{i_k}\,, \qquad F_{\beta_k} = T_{i_1}\cdots T_{i_{k-1}}F_{i_k} = \tau(E_{\beta_k}) \qquad \forall\, 1 \le k \le N\,. \tag{3.2}
$$

For positive root $\beta = \sum a_i\alpha_i$, let

$$
\nu_\beta^> := \sum_i a_i\nu_i^>\,, \qquad \nu_i^< := \sum_i a_i\nu_i^<.
$$

and

$$
\tilde{E}_{\beta_k} := v^{b^>_{\beta_k}}E_{\beta_k}K^{\nu^>_{\beta_k}}\,, \qquad \tilde{F}_{\beta_k} := v^{b^<_{\beta_k}}K^{-\nu^<_{\beta_k}}F_{\beta_k}\,, \tag{3.3}
$$

here $b^>_{\beta_k}, \beta^<_{\beta_k} \in \mathbb{Z}/2$ are normalizer, see [14, §2.6], so that $\tilde{E}_{\beta_k}, \tilde{F}_{\beta_k} \in U_{\mathcal{A}}^{ev}(\mathfrak{g})$. We have the PBW-basis for $U_{\mathcal{A}}^{ev}(\mathfrak{g})$ by [14, Lemma 2.10], let

$$
\tilde{E}^{\vec{k}} := \tilde{E}^{k_1}_{\beta_1}\dots\tilde{E}^{k_N}_{\beta_N}\,, \qquad \tilde{F}^{\vec{k}} := \tilde{F}^{k_1}_{\beta_1}\dots\tilde{F}^{k_N}_{\beta_N}\,, \quad \tilde{E}^{\overleftarrow{k}} := \tilde{E}^{k_N}_{\beta_N}\dots\tilde{E}^{k_1}_{\beta_1}\,, \qquad \tilde{F}^{\overleftarrow{k}} := \tilde{F}^{k_N}_{\beta_N}\dots\tilde{F}^{k_1}_{\beta_1}\,,
$$

Lemma 3.1. *(a) The sets $\{\tilde{E}^{\vec{k}}\}_{\vec{k}\in\mathbb{Z}^N_{\ge 0}}, \{\tilde{E}^{\overleftarrow{k}}\}_{\vec{k}\in\mathbb{Z}^N_{\ge 0}}$ are $\mathcal{A}$-bases of $U_{\mathcal{A}}^{ev>}$.*

(b) The sets $\{\tilde{F}^{\vec{k}}\}_{\vec{k}\in\mathbb{Z}^N_{\ge 0}}, \{\tilde{F}^{\overleftarrow{k}}\}_{\vec{k}\in\mathbb{Z}^N_{\ge 0}}$ are $\mathcal{A}$-bases of $U_{\mathcal{A}}^{ev<}$.

(c) The set $\{K^\mu\}_{\mu\in 2P}$ is a $\mathcal{A}$-basis of $U_{\mathcal{A}}^{ev0}$.

Via base change, we have the corresponding PBW-bases for $U_\epsilon^{ev}(\mathfrak{g})$.

3.1. **Harish-Chandra center Z_{HC}.**
(Trung: the base ring is general R)

Definition 3.2. The Harish-Chandra center Z_{HC} of $U_q^{ev}(\mathfrak{g})$ is the $\check{U}_q(\mathfrak{g})$-invariant part of $U_q^{ev}(\mathfrak{g})$.

It is not hard to show that Z_{HC} is central in $U_q^{ev}(\mathfrak{g})$, see [14, ...].
Let us consider the natural map:

$$\pi : Z_{HC} \hookrightarrow U_q^{ev}(\mathfrak{g}) \cong U_q^{ev<}(\mathfrak{g}) \otimes_R U_q^{ev0}(\mathfrak{g}) \otimes_R U_q^{ev>}(\mathfrak{g}) \to U_q^{ev0}(\mathfrak{g}) \cong R\langle K^{2\lambda}\rangle_{\lambda\in P},$$

here $R\langle K^{2\lambda}\rangle_{\lambda\in P}$ is the polynomial algebra of the lattice $2P$. The following result is [14, Theorem 8.29]:

Proposition 3.3. *The morphism π gives a rise to an isomorphism of algebras*

$$\pi : Z_{HC} \xrightarrow{\sim} R\langle K^{2\lambda}\rangle_{\lambda\in P}^{W_\bullet},$$

where the dot-action of the Weyl group W on $R\langle K^{2\lambda}\rangle_{\lambda\in P}$ is defined via:

$$w_\bullet(K^\mu) = q^{(w^{-1}\rho-\rho,\mu)} K^{w(\mu)} \qquad \text{for all } x \in W,\ \mu \in 2P.$$

Remark 3.4. Let $\gamma_{-\rho} : R\langle K^{2\lambda}\rangle_{\lambda\in P} \to R\langle K^{2\lambda}\rangle_{\lambda\in P}$ is defined by $\gamma_{-\rho}(K^{2\lambda}) = \epsilon^{(-\rho,2\lambda)}K^{2\lambda}$ for all $\lambda \in P$. Then we have an isomorphism

$$\gamma_{-\rho}\circ\pi : Z_{HC} \xrightarrow{\sim} R\langle K^{2\lambda}\rangle_{\lambda\in P}^{W}.$$

3.2. **The Frobenious center Z_{Fr}.**

Definition 3.5. The Frobenious center Z_{Fr} of $U_\epsilon^{ev}(\mathfrak{g})$ is the subalgebra generated by

$$\{\tilde{E}_\alpha^{\ell_\alpha}, \tilde{F}_\alpha^{\ell_\alpha}, K^\mu\}_{\alpha\in\Delta_+}^{\mu\in 2P^*}.$$

Remark 3.6. In [14, §5], we gave a conceptual definition of Z_{Fr}. That definition requires a $\check{U}_\epsilon(\mathfrak{g})$-adjoint invariant pairing $U_\epsilon^{ev}(\mathfrak{g}) \times \dot{U}_\epsilon(\mathfrak{g}, P) \to \mathbb{C}$ which is non-degenerate in the first argument. Then Z_{Fr} is defined to be the orthogonal complement of $\mathrm{Ker}(\tilde{\mathrm{Fr}})$ under this pairing. Since the construction of the pairing is involved and we will not need it in this paper, we refer the details to [14, §5] and decide to provide more hand-on definition of Z_{Fr}.

We will give some properties of Z_{Fr}.

Proposition 3.7. *(a) The Frobenious center Z_{Fr} is stable under the adjoint action of $\check{U}_\epsilon(\mathfrak{g})$ on $U_\epsilon^{ev}(\mathfrak{g})$. Furthermore, the action of $\check{U}_\epsilon(\mathfrak{g})$ on Z_{Fr} factors through the morphism $\tilde{\mathrm{Fr}} : \check{U}_\epsilon(\mathfrak{g}) \to \check{U}_\mathbb{C}(\mathfrak{g}^d)$.*
(b) We have an isomorphism $Z_{Fr} \cong \mathbb{C}[\tilde{E}_\alpha^{\ell_\alpha}]_{\alpha\in\Delta_+} \otimes_\mathbb{C} \left(\bigoplus_{\lambda\in 2P^*}\mathbb{C}K^\lambda\right) \otimes_\mathbb{C} \mathbb{C}[\tilde{F}_\alpha^{\ell_\alpha}]_{\alpha\in\Delta_+}$. Here $\mathbb{C}[\tilde{E}_\alpha^{\ell_\alpha}]_{\alpha\in\Delta_+}, \mathbb{C}[\tilde{F}_\alpha^{\ell_\alpha}]_{\alpha\in\Delta_+}$ are polynomial algebras in the corresponding variables.

Let us define two linear morphisms $\kappa, \gamma \in \mathrm{End}(\mathfrak{h}^*)$ as follows:

$$\kappa(\alpha_i) := \alpha_i + \sum_{j=1}^r 2\phi_{ij}\omega_j^\vee = -\zeta_i^< \qquad \text{and} \qquad \gamma(\alpha_i) := \alpha_i - \sum_{j=1}^r 2\phi_{ij}\omega_j^\vee = \zeta_i^>\,, \tag{3.4}$$

Let

$$\tilde{Z}_{Fr}^> := \mathbb{C}[\tilde{E}_\alpha^{\ell_\alpha}K^{\ell_\alpha\gamma(\alpha)}]_{\alpha\in\Delta_+}, \qquad \tilde{Z}_{Fr}^< := \mathbb{C}[\tilde{F}_\alpha^{\ell_\alpha}K^{\ell_\alpha\kappa(\alpha)}]_{\alpha\in\Delta_+}, \qquad \tilde{Z}_{Fr}^0 := \bigoplus_{\lambda\in 2P^*}\mathbb{C}K^\lambda.$$

The next proposition is Proposition 5.10 in [14]:

Proposition 3.8. *There is a $\check{U}_\epsilon(\mathfrak{g}^d)$-linear algebra homomorphism:*

$$\varphi\colon Z_{Fr} \xrightarrow{\sim} \mathbb{C}[G_0^d] \simeq \mathbb{C}[U_-^d] \otimes_{\mathbb{C}} \mathbb{C}[T^d] \otimes_{\mathbb{C}} \mathbb{C}[U_+^d]\,.$$

Furthermore, under this isomorphism, we have $\tilde{Z}_{Fr}^{>} \cong \mathbb{C}[U_-^d]$, $\tilde{Z}_{Fr}^{0} \cong \mathbb{C}[U_0^d]$, $\tilde{Z}_{Fr}^{<} \cong \mathbb{C}[U_+^d]$.

3.3. Center Z.

In this section, we will study the whole center Z of $U_\epsilon^{ev}(\mathfrak{g})$. Let

$$\mathcal{A}' := \mathbb{C}[v, v^{-1}]\left[\frac{1}{v^{2k}-1}\right]_{1\le k\le \max\{d_i\}},$$

and we consider the algebra $U_{\mathcal{A}'}^{ev}(\mathfrak{g})$ via the base change $\mathcal{A} \to \mathcal{A}'$. We have the surjection map $\varphi_\epsilon : U_{\mathcal{A}'}^{ev}(\mathfrak{g}) \to U_\epsilon^{ev}(\mathfrak{g})$ corresponding the algebra homomorphism $\sigma : \mathcal{A}' \to \mathbb{C}$ sending v to ϵ. For any $a \in Z$, pick arbitrary lifts $\hat{a}$ in $U_{\mathcal{A}'}^{ev}(\mathfrak{g})$, then define

$$\{a, b\} := \varphi_\epsilon\left(\frac{[\hat{a}, \hat{b}]}{v-\epsilon}\right), \qquad \text{for all } a, b \in Z.$$

The following results are in [14, §9]

Proposition 3.9. *(a) The Frobenius center Z_{Fr} is closed under the Poisson bracket. Moreover, Z_{Fr} is generated by $\{\tilde{E}_i^{\ell_i}, \tilde{F}_i^{\ell_i}, K^\lambda\}_{\lambda\in 2P^*}^{1\le i\le r}$ as a Poisson algebra.*
(b) Recall the $\check{U}_{\mathbb{C}}(\mathfrak{g}^d)$-equivariant isomorphism $Z_{Fr} \cong \mathbb{C}[G_0^d]$. Then the symplective leaves of $\operatorname{Spec} Z_{Fr}$ *coincides with the intersection of conjugacy classes of G^d with the open Bruhat cell G_0^d.*
(c) The fiber of $U_\epsilon^{ev}(\mathfrak{g})$ over points in the same symplectic leave are isomorphic as algebras.

The last part follows by the general results about Poisson order in [2].

Let $Z_\cap := Z_{Fr} \cap Z_{HC}$.

Lemma 3.10. *Under the isomorphism $\gamma_{-\rho} \circ \pi : Z_{HC} \xrightarrow{\sim} \mathbb{C}\langle K^{2\lambda}\rangle_{\lambda\in P}^W$, the algebra $Z_\cap$ is identified with $\mathbb{C}\langle K^{2\lambda}\rangle_{\lambda\in P^*}^W$.*

So the inclusion $Z_\cap \hookrightarrow Z_{HC}$ gives a rise to the finite morphism $\bullet^l : T/W \to T^d/W$. On the other hand, the inclusion $Z_\cap \hookrightarrow Z_{Fr}$ corresponding to the morphism $G_0^d \hookrightarrow G^d \to G^d /\!\!/ G^d \cong T^d/W$. So we can form the fibered produce $G_0^d \times_{T^d/W} T/W$.

Proposition 3.11. *$Z \cong Z_{Fr} \otimes_{Z_\cap} Z_{HC}$ so that* $\operatorname{Spec} Z \cong G_0^d \times_{T^d/W} T/W$.

Let $G^{d,reg}$ be the set of regular element in G^d. Let us consider the projection

$$\mathsf{p}_1 : \operatorname{Spec} Z \cong G_0^d \times_{T^d/W} T/W \to G_0^d \hookrightarrow G^d.$$

Theorem 3.12. *The Azumaya locus of $U_\epsilon^{ev}(\mathfrak{g})$ over Z contains $\mathsf{p}_1^{-1}(G^{d,reg})$. In the other word, the restriction of $U_\epsilon^{ev}(\mathfrak{g})$ on the open set $\mathsf{p}_q^{-1}(G^{d,reg}) \subset \operatorname{Spec} Z$ is a sheaf of Azumaya algebras.*

Remark 3.13. (Check) The Poisson structure on Z_{HC} is trivial. Moreover, $Z_\cap$ is the Poisson center of Z_{Fr}.

3.4. The locally finite parts $U_\epsilon^{fin}, U_q^{fin}$.

Let

$$U_q^{fin} : \{u \in U_q^{ev}(\mathfrak{g}) \,|\, \operatorname{ad}'(\check{U}_q(\mathfrak{g}))(u) \text{ is a finitely generated } R\text{-module}\} \tag{3.5}$$

then U_q^{fin} is a subalgebra of $U_q^{ev}(\mathfrak{g})$.

Let $O_q[G] \subset \operatorname{Hom}_R(\check{U}_q(\mathfrak{g}), R)$ consisting of all matrix coefficients of representation $V \in \operatorname{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. On $O_q[G]$, we can equip an algebra structure so called *reflection equation algebra*, see [14, §7], so that we have:

Proposition 3.14. *There is an isomorphism of $\check{U}_q(\mathfrak{g})$-module algebras $O_q[G] \xrightarrow{\sim} U_q^{fin}$.*

The proof is based on the adjoint invariant pairing $U_q^{ev}(\mathfrak{g}) \times \check{U}_q(\mathfrak{g}) \to R$, see [14, §8]. So the $\check{U}_q(\mathfrak{g})$-module structure of U_q^{fin} can be understood via $\check{U}_q(\mathfrak{g})$-module $O_q[G]$, which can be studied via properties of the category $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$.

Now we focus on the case $U_\epsilon^{ev}(\mathfrak{g})$ with respect to (**A**). Let

$$Z_{Fr}^{fin} := U_\epsilon^{fin} \cap Z_{Fr}$$

Proposition 3.15. *(a) Under the $\check{U}_{\mathbb{C}}(\mathfrak{g}^d)$-equivariant isomorphism $Z_{Fr} \xrightarrow{\sim} \mathbb{C}[G_0^d]$, we have $Z_{Fr}^{fin} \cong \mathbb{C}[G^d]$.*
(b) The algebra U_ϵ^{fin} is a finitely generated projective Z_{Fr}^{fin}-module. Moreover, $U_\epsilon^{ev} \cong U_\epsilon^{fin} \otimes_{Z_{Fr}^{fin}} Z_{Fr}$, i.e., U_ϵ^{ev} is obtained from U_ϵ^{fin} by pulling back via open embedding $G_0^d \hookrightarrow G^d$.
(c) The center Z^{fin} of U_ϵ^{fin} is $Z_{Fr}^{fin} \otimes_{Z_\cap} Z_{HC}$.

Let us comment on the proof of part (b) which is [14, §8.3, 8.5]. The proof is based on the isomorphism $O_\epsilon[G] \cong U_\epsilon^{fin}$. Recall the Frobenious functor $\tilde{\mathrm{Fr}}^* : \mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}, P^*) \to \mathrm{Rep}(\dot{U}_\epsilon(\mathfrak{g}, P))$. This functor induces an inclusion $O_\epsilon[G^d] \hookrightarrow O_\epsilon[G]$, here $O_\epsilon[G^d]$ is the reflection equation algebra for $\dot{U}_\epsilon(\mathfrak{g}, P^*)$. It turns out that $O_\epsilon[G^d] \xrightarrow{\sim} Z_{Fr}^{fin}$ under the isomorphism $O_\epsilon[G] \xrightarrow{\sim} U_\epsilon^{fin}$. We then prove $O_\epsilon[G]$ is a finitely generated projective $O_\epsilon[G^d]$-module.

4. Completion

(For simplicity, we will assume the order of ϵ is an odd number ℓ which is coprime with the determinant of Cartan matrix. In this case, $\ell_i = \ell_\alpha = \ell$ for $1 \leq i \leq r$ and $\alpha \in \Delta_+$). We will comment about the general ℓ in Section 13.

To simplify the notations, we will denote $\mathcal{W}_q := Z_{q,HC}$, the Harish-Chandra center of $U_q^{ev}(\mathfrak{g})$.

Lemma 4.1. *The algebra $U_q^{ev}(\mathfrak{g})$ is Noetherian. Let $\mathcal{W}_q^\wedge$ be any completion of $\mathcal{W}_q$ then $U_q^{ev}(\mathfrak{g}) \otimes_{\mathcal{W}_q} \mathcal{W}_q^\wedge$ is also Noetherian.*

Proof. $U_q^{ev}(\mathfrak{g})$ admits a $\mathbb{Z}_{\geq 0}^{2N+1}$ filtration so that the associated graded algebra is the twisted polynomial over the Noetherian ring R. The latter is Noetherian hence so is $U_q^{ev}(\mathfrak{g})$.

Since we have a surjective map $U_q^{ev}(\mathfrak{g}) \otimes_R \mathcal{W}_q^\wedge \to U_q^{ev}(\mathfrak{g}) \otimes_{\mathcal{W}_q} \mathcal{W}^\wedge$, it is enough to show that $U_q^{ev}(\mathfrak{g}) \otimes_R \mathcal{W}^\wedge$ is Noetherian. Tensoring $- \otimes_R \mathcal{W}^\wedge$ with the mentioned filtration of $U_q^{ev}(\mathfrak{g})$ gives us a filtration on $U_q^{ev}(\mathfrak{g}) \otimes_R \mathcal{W}_q^\wedge$ so that the associated graded algebra is a twisted polynomial over $\mathcal{W}_q^\wedge$, which is Noetherian. □

Consider the naturap map $\varphi_\epsilon : U_q^{ev}(\mathfrak{g}) \to U_\epsilon^{ev}(\mathfrak{g})$. We now describe the procedure to produce various completions of $U_q^{ev}(\mathfrak{g})$. Suppose I be an ideal of the center Z of $U_\epsilon^{ev}(\mathfrak{g})$. Let $J = \varphi^{-1}(I)$ then $U_q^{ev}(\mathfrak{g})J = JU_q^{ev}(\mathfrak{g}) = \phi^{-1}(U_\epsilon^{ev}(\mathfrak{g})I)$ which implies $U_q^{ev}(\mathfrak{g})J^k = J^k U_q^{ev}(\mathfrak{g}) = (U_q^{ev}(\mathfrak{g})J)^k$, particularly, $U_q^{ev}(\mathfrak{g})J^k$ is a two-sided ideal for any $k \geq 1$. Let $U_q^{ev\wedge_J}$ denote the completion of $U_q^{ev}(\mathfrak{g})$ with respect to the two-sided ideal $U_q^{ev}(\mathfrak{g})J$.

The following lemma summarizes some properties of $U_q^{ev\wedge_J}$, some of them are proved via arguments to prove Artin-Ree lemma in commutative algebra.

Lemma 4.2. *(a) $U_q^{ev\wedge_J}$ is a flat (left and right) $U_q^{ev}(\mathfrak{g})$-module.*
(b) $U_q^{ev\wedge_J}$ is Noetherian.
(c) $U_q^{ev\wedge_J}$ is complete and seperated in the $U_q^{ev}J$-adic topology. In particular, $U_q^{ev\wedge_J}$ is complete and seperated in the $\hbar$-topology.
(d) The completion functor $M \mapsto M^\wedge := \varprojlim M/(U_q^{ev}J)^k M$ from the category of finitely generated left $U_q^{ev}(\mathfrak{g})$-modules to the category of left $U_q^{ev\wedge_J}$-modules is exact. Moreover, $M^\wedge$ is canonically isomorphic to $U_q^{ev\wedge_J} \otimes_{U_q^{ev}(\mathfrak{g})} M$.
(e) We have a natural short exact sequence $0 \to U_q^{ev\wedge_J} \xrightarrow{\cdot\hbar} U_q^{ev\wedge_J} \to U_\epsilon^{ev\wedge_I} \to 0$, here $U_\epsilon^{ev\wedge_I} = U_\epsilon^{ev}(\mathfrak{g}) \otimes_Z Z^{\wedge_I}$ is the completion of $U_\epsilon^{ev}(\mathfrak{g})$ at the two-sided ideal $U_\epsilon^{ev}(\mathfrak{g})I$. So $U_q^{ev\wedge_J}$ is a $\mathbb{C}[[\hbar]]$-flat deformation of $U_\epsilon^{ev\wedge_I}$.

Proof. Write down the proof following [6] □

Lemma 4.3. *(a) Let $U_q^{fin\wedge_\hbar} := \varprojlim U_q^{fin}/\hbar^k U_q^{fin}$. Then $U_q^{fin\wedge_\hbar}$ is complete and seperated in the $\hbar$-adic topology. Furtheremore, $U_q^{fin\wedge_\hbar}$ is a $\mathbb{C}[[\hbar]]$-flat deformation of U_ϵ^{fin}, hence $U_q^{fin\wedge_\hbar}$ is Noetherian.*
(b) Let $\phi_{f,\hbar} : U_q^{fin\wedge_\hbar} \to U_\epsilon^{fin}$. For any two-sided ideal $I \subset Z(U_\epsilon^{fin})$, let $J = \phi_{f,\hbar}^{-1}(I)$. Let $U_q^{fin\wedge_J}$ be the completion of $U_q^{fin\wedge_\hbar}$ with respect to the two-sided ideal $U_q^{fin\wedge_\hbar}J$. Then $U_q^{fin\wedge_J}$ has the desired properties as in Lemma 4.2. In particular $U_q^{fin\wedge_J}$ is a $\mathbb{C}[[\hbar]]$-flat deformation of $U_\epsilon^{fin} \otimes_{Z^{fin}} Z^{fin\wedge_I}$.

Remark 4.4. (More details) Since the action of $\check{U}_\epsilon(\mathfrak{g})$ on Z factors through $\mathrm{Fr} : \check{U}_\epsilon(\mathfrak{g}) \to \check{U}_{\mathbb{C}}(\mathfrak{g}^d)$, the equivariant $\check{U}_\epsilon(\mathfrak{g})$-action on $U_\epsilon^{ev}(\mathfrak{g})$ extends to an equivariant $\check{U}_\epsilon(\mathfrak{g})$-action on $U_\epsilon^{ev\wedge_I}$. Moreover, the equivariant $\check{U}_q(\mathfrak{g})$-action on $U_q^{ev}(\mathfrak{g})$ extends to an equivariant $\check{U}_q(\mathfrak{g})$-action on $U_q^{ev\wedge_J}$.

Lemma 4.5. *For any $\mathbb{C}$-finite dimensional $V \in \operatorname{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ then $\operatorname{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V, U_\epsilon^{fin})$ is finitely generated over $Z_\cap$ and hence finitely generated over $Z_{\epsilon,HC}$.*

Proof. Since U_ϵ^{fin} is finitely projective over Z_{Fr}^{fin}, it is an projective object in the category of $Z_{Fr}^{fin}\text{-mod}^{G_\epsilon}$. Therefore, there is a finite dimensional $V_1 \in \operatorname{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ such that U_ϵ^{fin} is a direct summand of $V_1 \otimes_{\mathbb{C}} Z_{Fr}^{fin}$ in the category $Z_{Fr}^{fin}\text{-mod}^{G_\epsilon}$. So it is enough to show that $\operatorname{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V, Z_{Fr}^{fin})$ is finitely generated over $Z_\cap$. Note that

$$\operatorname{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V, Z_{Fr}^{fin}) = \Big((V^*)^{\mathfrak{u}_\epsilon} \otimes_{\mathbb{C}} Z_{Fr}^{fin} \Big)^{\check{U}_{\mathbb{C}}(\mathfrak{g}^d)}.$$

By standard argument in invariant theory, the right hand side is finitely generated over $(Z_{Fr}^{fin})^{\check{U}_{\mathbb{C}}(\mathfrak{g}^d)}$, which is just $Z_\cap$. □

4.1. **Completion of Poisson algebra Z.**

Recall the structure $\operatorname{Spec} Z \cong G_0^d \times_{T^d/W} T/W$. Let χ be the regular point in G_0^d and $\xi = (\chi, \vartheta)$ be a point in $\operatorname{Spec} Z$. Let us consider the following ideals in Z:

- $\mathfrak{m}_\xi$ the maximal ideal of Z at ξ.
- $Z\mathfrak{m}_\chi$, here $\mathfrak{m}_\chi$ is the maximal ideal of Z_{Fr} at χ.

The completion $Z_{Fr}^{\wedge_\chi}$ is a naturally Poisson algebra. Let V be the tangent space at χ of the conjugacy class at χ, which is the symplectic leaf containning χ. Then V carries a natural non-degenerate 2-form. This 2-form induces a Poisson structure on $\mathbb{C}[[V]]$. Let $\underline{\chi}$ be the image of χ under the natural map $\operatorname{Spec} Z_{Fr} \to \operatorname{Spec} Z_\cap$, then we form the completion $Z_\cap^{\wedge_{\underline{\chi}}}$.

Lemma 4.6. *There is an isomorphism of Poisson algebras $Z_{Fr}^{\wedge_\chi} \cong \mathbb{C}[[V]] \widehat{\otimes} Z_\cap^{\wedge_{\underline{\chi}}}$.*

Proof. □

Let $Z^{\wedge_\chi}$ and $Z^{\wedge_\xi}$ be the completions of Z at the ideals $\mathfrak{m}_\xi$ and $Z\mathfrak{m}_\chi$, respectively. Then there is a natural isomorphism of Poisson algebras:

$$Z^{\wedge_\chi} \cong \prod_{\xi = (\chi, \vartheta)} Z^{\wedge_\xi},$$

note that there are only finitely many ξ of the form (χ, ϑ). Let $\mathfrak{m}_{\underline{\chi}}$ be the maximal ideal of $Z_\cap$ at $\underline{\chi}$ while $\mathfrak{m}_\vartheta$ be the maximal ideal of $\mathcal{W}_\epsilon$ at the point ϑ. Let $\mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$ and $\mathcal{W}_\epsilon^{\wedge_\vartheta}$ be the completions of Harish-Chandra center $\mathcal{W}_\epsilon$ at the ideal $\mathcal{W}_\epsilon \mathfrak{m}_{\underline{\chi}}$ and $\mathfrak{m}_\vartheta$, respectively. We have the following isomorphism of algebras:

$$Z^{\wedge_\chi} \cong Z_{Fr}^{\wedge_\chi} \widehat{\otimes}_{Z_\cap^{\wedge_{\underline{\chi}}}} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}, \qquad Z^{\wedge_\xi} \cong Z_{Fr}^{\wedge_\chi} \widehat{\otimes}_{Z_\cap^{\wedge_{\underline{\chi}}}} \mathcal{W}_\epsilon^{\wedge_\vartheta}. \tag{4.1}$$

Lemma 4.6 implies the following corollary:

Corollary 4.7. *There is a decomposition of Poisson algebras $Z^{\wedge_\chi} \cong \mathbb{C}[[V]] \widehat{\otimes} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$. Fix one such decomposition, it gives rise a family of inclusions of Poisson algebras $\mathbb{C}[[V]] \hookrightarrow Z^{\wedge_\xi}$ and then a family of isomorphisms of Poisson algebras $Z^{\wedge_\xi} \cong \mathbb{C}[[V]] \widehat{\otimes} \mathcal{W}_\epsilon^{\wedge_\vartheta}$.*

(More details)

4.2. **Structural results of $U_q^{ev\wedge_\chi}, U_q^{ev\wedge_\xi}$.**

Let $U_\epsilon^{ev\wedge_\chi}$ and $U_\epsilon^{ev\wedge_\xi}$ be the completions of $U_\epsilon^{ev}(\mathfrak{g})$ with respect to the ideal $Z\mathfrak{m}_\chi$ and $\mathfrak{m}_\xi$, respectively. Then we have a natural isomorphisms:

$$U_\epsilon^{ev\wedge_\chi} \cong \prod_{\xi=(\chi,\vartheta)} U_\epsilon^{\wedge_\xi}.$$

Let us consider $\chi \in G_0^{d,reg} \subset \operatorname{Spec} Z_{Fr}$. By Theorem 3.12, there is an isomorphism of algebras $U_\epsilon^{ev\wedge_\xi} \cong \operatorname{Mat}_{\ell^N}(Z^{\wedge_\xi})$. Therefore, $U_\epsilon^{ev\wedge_\chi} \cong \operatorname{Mat}_{\ell^N}(Z^{\wedge_\chi})$. Moreover, if we fix an isomorphism $U_\epsilon^{ev\wedge_\chi} \cong \operatorname{Mat}_{\ell^N}(Z^{\wedge_\chi})$ then it induces a family of algebra isomorphisms $U_\epsilon^{ev\wedge_\xi} \cong \operatorname{Mat}_{\ell^N}(Z^{\wedge_\xi})$.

We will now describe the structures of completion algebras $U_q^{ev\wedge_\xi}$ and $U_q^{ev\wedge_\chi}$. Consider the natural map $\psi : \mathcal{W}_q \xrightarrow{/\hbar} \mathcal{W}_\epsilon$. Let

$$\mathfrak{J}_\vartheta : \psi^{-1}(\mathfrak{m}_\vartheta) \qquad \text{and} \qquad \mathfrak{J}_{\underline{\chi}} := \psi^{-1}(\mathcal{W}_\epsilon \mathfrak{m}_{\underline{\chi}}).$$

Let $\mathcal{W}_q^{\wedge_\vartheta}$ and $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$ denote the completions of $\mathcal{W}_q$ with respect to the ideal $\mathfrak{J}_\vartheta$ and $\mathfrak{J}_{\underline{\chi}}$, respectively.

Lemma 4.8. *The following natural maps are isomorphisms:*

$$U_q^{ev\wedge_\chi} \xrightarrow{\sim} \prod_{\xi=(\chi,\vartheta)} U_q^{ev\wedge_\xi}, \qquad \mathcal{W}_q^{\wedge_{\underline{\chi}}} \xrightarrow{\sim} \prod_{\vartheta} \mathcal{W}_q^{\wedge_\vartheta},$$

here ϑ runs over all preimages of $\underline{\chi}$ under the map $\operatorname{Spec} \mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$.

Proof. □

With the non-degenerate 2-form on the cotangent space V at χ of sympletic leaf containing χ, we can form the formal Weyl algebra $\mathbb{C}[V,\hbar]$. Let $\mathcal{A}_q^\wedge := \mathbb{C}[[V,\hbar]]$ the completion of $\mathbb{C}[V,\hbar]$ at the maximal ideal generated by V and $\hbar$.

Proposition 4.9. *Let $\chi \in G_0^{d,reg} \subset \operatorname{Spec} Z_{Fr}$. There are isomorphisms:*

$$U_q^{ev\wedge_{\underline{\chi}}} \xrightarrow{\sim} \operatorname{Mat}_{\ell^N}(\mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{A}_q^\wedge \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathcal{W}_q^{\wedge_{\underline{\chi}}})$$
$$U_q^{ev\wedge_\xi} \xrightarrow{\sim} \operatorname{Mat}_{\ell^N}(\mathbb{C}) \otimes_{\mathbb{C}} (\mathcal{A}_q^\wedge \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathcal{W}_q^{\wedge_\vartheta}).$$

Proof. □

5. Quantum Harish-Chandra bimodules

To simplify the exposition, we will consider either (A) or (B). For any $\check{U}_q(\mathfrak{g})$-module M, let M^{rat} be the maximal rational subrepresentation of $\check{U}_q(\mathfrak{g})$ in M.

5.1. Non-complete version.

Since $U_q^{ev}(\mathfrak{g})$ is a $\check{U}_q(\mathfrak{g})$-module algebra, we can define:

Definition 5.1. Let U_q^{ev}-$\mathrm{Rmod}^{\check{U}_q}$, U_q^{ev}-$\mathrm{Lmod}^{\check{U}_q}$ and U_q^{ev}-$\mathrm{Bimod}^{\check{U}_q}$ be the categories of right $U_q^{ev}(\mathfrak{g})$-modules, left $U_q^{ev}(\mathfrak{g})$-modules and $U_q^{ev}(\mathfrak{g})$-bimodules in the category of $\check{U}_q(\mathfrak{g})$-mod, respectively.

Remark 5.2. Recall the natural map $\iota : U_q^{ev}(\mathfrak{g}) \to \check{U}_q(\mathfrak{g})$. For any $M \in U_q^{ev}$-$\mathrm{Rmod}^{\check{U}_q}$, there is a natural left $U_q^{ev}(\mathfrak{g})$-module defined by

$$hm = \sum (\iota(h_{(1)}) \cdot m) h_{(2)}, \tag{5.1}$$

here $\cdot$ represents the action of $\check{U}_q(\mathfrak{g})$ on M. With this left $U_q^{ev}(\mathfrak{g})$-action, M becomes an object in U_q^{ev}-$\mathrm{Bimod}^{\check{U}_q}$. Similarly, any $N \in U_q^{ev}$-$\mathrm{Lmod}^{\check{U}_q}$ is naturally an object in U_q^{ev}-$\mathrm{Bimod}^{\check{U}_q}$ with the right U_q^{ev}-module defined by:

$$mh = \sum h_{(2)} (\iota(S^{-1} h_{(1)}) \cdot m). \tag{5.2}$$

Similarly, since U_q^{fin} is an algebra object in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$, we can define:

Definition 5.3. Let U_q^{fin}-Rmod^{G_q}, U_q^{fin}-Lmod^{G_q} and U_q^{fin}-Bimod^{G_q} be the categories of right U_q^{fin}-modules, left U_q^{fin}-modules and U_q^{fin}-bimodules in the category $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$, respectively.

Example 5.4. For any $V \in \mathrm{Rep}(\check{U}_q(\mathfrak{g}))$, the object $V \otimes_R U_q^{fin}$ is naturally an object in U_q^{fin}-Rmod^{G_q}: the right U_q^{fin}-module structure comes from the right U_q^{fin}-action on U_q^{fin} while $\check{U}_q(\mathfrak{g})$ acts on $V \otimes_R U_q^{fin}$ via tensor product. Similarly, $U_q^{fin} \otimes_R V$ is naturally an object in U_q^{fin}-Lmod^{G_q}.

Lemma 5.5. *(a) There are (fully faithful?) functors:*

$$U_q^{fin}\text{-}\mathrm{Rmod}^{G_q} \to U_q^{fin}\text{-}\mathrm{Bimod}^{G_q}, \qquad U_q^{fin}\text{-}\mathrm{Lmod}^{G_q} \to U_q^{fin}\text{-}\mathrm{Bimod}^{G_q}.$$

(b) If $M \in U_q^{fin}$-Rmod^{G_q} is finitely generated as a right U_q^{fin}-module then M is also finitely generated as a left U_q^{fin}-module. Similarly, if $M \in U_q^{fin}$-Lmod^{G_q} is finitely generated as a left U_q^{fin}-module then M is also finitely generated as a right U_q^{fin}-module.

Proof. (a) *Step 1:* We define the left action of $V \otimes_R U_q^{fin}$, in which $V \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ projective over R. First, we have

$$V \otimes_R U_q^{fin} = (V \otimes_R U_q^{ev}(\mathfrak{g}))^{rat}.$$

Since V is finitely generated projective over R, we have

$$\begin{aligned}
\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_1, V \otimes_R U_q^{ev}(\mathfrak{g})) &\cong \mathrm{Hom}_R(V_1, V \otimes_R U_q^{ev}(\mathfrak{g}))^{\check{U}_q(\mathfrak{g}))} \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(V, R) \otimes_R V_1, U_q^{ev}(\mathfrak{g}))^{\check{U}_q(\mathfrak{g})} \\
&\cong \mathrm{Hom}_R(\mathrm{Hom}_R(V, R) \otimes_R V_1, U_q^{fin})^{\check{U}_q(\mathfrak{g})} \\
&\cong \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_1, V \otimes_R U_q^{fin}),
\end{aligned}$$

for all $V_1 \in \operatorname{Rep}(\check{U}_q(\mathfrak{g}))$. Here $\check{U}_q(\mathfrak{g})$ acts on $\operatorname{Hom}_R(M,N)$ by unusual action: $(uf)(m) = \sum u_{(2)} f(S^{-1}(u_{(1)})m)$ for all $m \in M, f \in \operatorname{Hom}_R(M,N)$ and $u \in \check{U}_q(\mathfrak{g})$.

By Remark 5.2, there is a left $U_q^{ev}(\mathfrak{g})$-action on $V \otimes_R U_q^{ev}(\mathfrak{g})$ so that $V \otimes_R U_q^{ev}(\mathfrak{g}) \in U_q^{ev}\text{-Bimod}^{\check{U}_q(\mathfrak{g})}$. The left U_q^{fin}-action on $V \otimes_R U_q^{ev}(\mathfrak{g})$ will preserves $(V \otimes_R U_q^{ev}(\mathfrak{g}))^{rat}$, hence we have a natural left U_q^{fin}-action on $V \otimes_R U_q^{fin}$ so that the object belongs to $U_q^{fin}\text{-Bimod}^{G_q}$.
Step 2: For any $M \in U_q^{fin}\text{-Rmod}^{G_q}$, there is a set of objects $\{V_i\}_{i \in I}$ consisting of projective R-module objects in $\operatorname{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ such that we have a surjective map $(\bigoplus V_i) \otimes_R U_q^{fin} \twoheadrightarrow M$. The case **(A)** is obvious while the case **(B)** follows by Proposition 2.10.

Hence any object M in $U_q^{fin}\text{-Rmod}^{G_q}$ are presentable by objects of the form $(\bigoplus V_i) \otimes_R U_q^{fin}$ as above. Therefore, the construction in Step 1 extends to a functor $U_q^{fin}\text{-Rmod}^{G_q} \to U_q^{fin}\text{-Bimod}^{G_q}$.
(b) We will prove the first statement only since the proof for the second statement is the same. For any $V \in \operatorname{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ then $V \otimes_R U_q^{fin}$ and $U_q^{fin} \otimes_R V$ are objects in $U_q^{fin}\text{-Bimod}^{G_q}$ by part (a). The morphism $p_1 : V \to V \otimes_R U_q^{fin}$ and $p_2 : V \to U_q^{fin} \otimes_R V$ give a rise to morphism:

$$p_1 : U_q^{fin} \otimes_R V \to V \otimes_R U_q^{fin}, \qquad p_2 : V \otimes_R U_q^{fin} \to U_q^{fin} \otimes_R V.$$

One can see that p_1 and p_2 are mutually inverse hence $V \otimes_R U_q^{fin} \cong U_q^{fin} \otimes_R V$ in $U_q^{fin}\text{-Bimod}^{G_q}$.

If $M \in U_q^{fin}\text{-Rmod}^{G_q}$ such that M is finitely generated as a right U_q^{fin}-module, then there is $V \in \operatorname{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ with a surjective map $V \otimes U_q^{fin} \twoheadrightarrow M$ in $U_q^{fin}\text{-Rmod}^{G_q}$. By above paragraph, $V \otimes_R U_q^{fin}$ is finitely generated as a left U_q^{fin}-module, hence M is also finitely generated as a left U_q^{fin}-module. ☐

Definition 5.6. The category of quantum Harish-Chandra bimodules is the full subcategory $U_q^{fin}\text{-rmod}^{G_q}$ of the category $U_q^{fin}\text{-Rmod}^{G_q}$ consisting of all objects which are finitely generated over U_q^{fin}. We denote this category by HC_q.

Remark 5.7. It is not clear in the case **(B)** that U_q^{fin} is Noetherian so we are not sure if HC_q is an abelian category. Nevertheless, we will later be interested in some complete versions of HC_q which will be proved to be abelian categories.

Remark 5.8. Lemma 5.5 equips the category HC_q with a monoidal structure.

Let us consider the case **(A)**.

Lemma 5.9. *The left and right action of Z_{Fr}^{fin} on any object of $U_\epsilon^{fin}\text{-Rmod}^{G_\epsilon}$ coincide.* [1]

Proof. Note that $Z_{Fr}^{fin} = \bigoplus_{\lambda \in P_+^*} \operatorname{ad}'(\check{U}_{\mathbb{C}}(\mathfrak{g}^d))K^{-2\lambda}$. Let $M \in U_\epsilon^{fin}\text{-Rmod}^{G_\epsilon}$. Let $\cdot$ denote the action of $\check{U}_\epsilon(\mathfrak{g})$ on corresponding spaces.

Step 1: For any $m \in M$, by construction (5.1)

$$K^{-2\lambda}m = (K^{-2\lambda} \cdot m)K^{-2\lambda} = mK^{-2\lambda},$$

here by assumption on ℓ, for all $\lambda \in P_+^*$, the action of $K^{-2\lambda}$ on any rational representation in $\operatorname{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ is trivial.

Step 2: Let $u \in Z_{Fr}^{fin}$ such that $um = mu$ for all $m \in M$. We will show that

$$(\tilde{E}_i^{(\ell_i)} \cdot u)m = m(\tilde{E}_i^{(\ell_i)} \cdot u), \qquad (\tilde{F}_i^{(\ell_i)} \cdot u)m = m(\tilde{F}_i^{(\ell_i)} \cdot u), \tag{5.3}$$

[1]The same is true for weight modules in $U_\epsilon^{ev}\text{-rmod}^{\check{U}_\epsilon(\mathfrak{g})}$: the left and right Z_{Fr}-action coincide. By the proof the left and right action of $Z_{Fr}^{fin}, K^\lambda (\lambda \in 2P^*)$ coincide but $Z_{Fr} = Z_{Fr}^{fin}[K^{\lambda_0}]$ where $\lambda_0 = 2\sum \ell_i \omega_i$.

for all $1 \le i \le r$. Indeed, we have

$$\tilde{E}_i^{(\ell_i)} \cdot (um) = (\tilde{E}^{(\ell_i)} \cdot u)(K^{\ell_i \zeta_i^>} \cdot m) + u(\tilde{E}_i^{(\ell_i)} \cdot m) = (\tilde{E}_i^{(\ell_i)} \cdot u)m + u(\tilde{E}^{(\ell_i)} \cdot m)$$
$$\tilde{E}_i^{(\ell_i)} \cdot (mu) = (\tilde{E}_i^{(\ell_i)} \cdot m)(K^{\ell_i \zeta_i^<} \cdot m) + m(\tilde{E}_i^{(\ell_i)} \cdot u) = (\tilde{E}_i^{(\ell_i)} \cdot m)u + m(\tilde{E}_i^{(\ell_i)} \cdot u)$$

then the first equality of (5.3) follows. The proof for the second equality is the same.

The lemma follows by using both steps and the decomposition $Z_{Fr}^{fin} = \bigoplus_{\lambda \in P_+^*} \mathrm{ad}'(\check{U}_{\mathbb{C}}(\mathfrak{g}^d))K^{-2\lambda}$. □

5.2. Complete version.

Let us define the following algebras:

$$U_\epsilon^{fin,\underline{\chi}} := U_\epsilon^{fin} \otimes_{\mathcal{W}_\epsilon} \mathcal{W}_\epsilon^{\wedge\underline{\chi}}, \qquad U_q^{fin,\underline{\chi}} := U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\underline{\chi}} / \cap \hbar^k U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\underline{\chi}},$$
$$U_\epsilon^{fin,\vartheta} := U_\epsilon^{fin} \otimes_{\mathcal{W}_\epsilon} \mathcal{W}_\epsilon^{\wedge\vartheta} \qquad U_q^{fin,\vartheta} := U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\vartheta} / \cap \hbar^k U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\vartheta}$$

Remark 5.10. We expect $U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}^{\wedge ?}$ to be seperated in the $\hbar$-adic topology but cannot prove it. The quotient terms in the definition $U_q^{fin,\underline{\chi}}$ and $U_q^{fin,\vartheta}$ is to make sure that these algebras are seperated in the $\hbar$-adic topology. These two algebras are flat over $\mathbb{C}[[\hbar]]$ and the maximal rational subrepresentations of the completions $U_q^{fin\wedge\underline{\chi}}$ and $U_q^{fin\wedge\vartheta}$ (to be shown), respectively.

Then

$$U_\epsilon^{fin,\underline{\chi}} = \prod_\vartheta U_\epsilon^{fin,\vartheta} \qquad \text{and} \qquad U_q^{fin,\underline{\chi}} = \prod_\vartheta U_q^{fin,\vartheta}, \tag{5.4}$$

where ϑ runs over the preimages of $\underline{\chi}$ under the map $\operatorname{Spec} \mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$. Note that $U_\epsilon^{fin,\underline{\chi}} := U_\epsilon^{fin} \otimes_{Z_\cap} Z_\cap^{\wedge\underline{\chi}}$.

Let us introduce several categories of interest.

Definition 5.11. Let $\mathsf{HC}_\epsilon(\vartheta, \vartheta')$ be the category consisting of all objects in $U_\epsilon^{fin,\vartheta'}\text{-rmod}^{G_\epsilon}$ such that the left U_ϵ^{fin}-action factors through a left $U_\epsilon^{fin,\vartheta}$-action.

Remark 5.12. Let $\underline{\chi}'$ be the image of ϑ' under the map $\operatorname{Spec} \mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$. By Lemma 5.9, there is natural left $U_\epsilon^{fin,\underline{\chi}'}$-action on any object of $U_\epsilon^{fin,\vartheta'}\text{-rmod}^{G_\epsilon}$. By decomposition (5.4), we have a natural functor

$$U_\epsilon^{fin,\vartheta'}\text{-rmod}^{G_\epsilon} \to \mathsf{HC}_\epsilon(\vartheta, \vartheta'), \tag{5.5}$$

by projecting to the direct summand corresponding to the left $U_\epsilon^{fin,\vartheta}$-action. In particular, if the images of ϑ and ϑ' in $\operatorname{Spec} Z_\cap$ are different then the category $\mathsf{HC}_\epsilon(\vartheta, \vartheta')$ is zero.

Definition 5.13. Let $\mathsf{HC}_q(\vartheta, \vartheta')$ be the category consisting of all objects in $U_q^{fin,\vartheta'}\text{-rmod}^{G_q}$ such that the left U_q^{fin}-action factors through a left $U_q^{fin,\vartheta}$-action.

Lemma 5.14. *Let $\underline{\chi}'$ be the image of ϑ' under the map $\operatorname{Spec} \mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$. Let $M \in U_q^{fin,\underline{\chi}'}\text{-rmod}^{G_q}$.*
(a) For any $V_q \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ then $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, M)$ is finitely generated over $\mathcal{W}^{\wedge\underline{\chi}'}$.
(b) Any $M \in U_q^{fin,\underline{\chi}'}\text{-rmod}^{G_q}$ is seperated in the $\hbar$-adic topology.
(c) The left action of $\mathcal{W}_q$ on M extends uniquely into a left action of $\mathcal{W}_q^{\wedge\underline{\chi}'}$. So that M is a naturaly an object in $U_q^{fin,\underline{\chi}'}\text{-bimod}^{G_q}$.

Proof. (a) Since $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ has enough projectives, we can assume V_q is projective in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ and M is of the form $V_q' \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}'}$ for some V_q' is a free of finite rank over $\mathbb{C}[[\hbar]]$ in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Note that

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, V_q' \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}'}) = \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}((V_q')^t \otimes_{\mathbb{C}[[\hbar]]} V_q, U_q^{fin,\underline{\chi}'}),$$

here $(V_q')^t$ is the right dual of V_q'. Therefore, we reduce to prove that $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'})$ is finitely generated over $\mathcal{W}_q^{\wedge\underline{\chi}'}$ for any $V_q \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$.

We have a short exact sequence

$$0 \to \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'}) \xrightarrow{\cdot\hbar} \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'}) \to \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_\epsilon^{fin,\underline{\chi}'}).$$

Note that

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_\epsilon^{fin,\underline{\chi}'}) = \mathrm{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V_\epsilon, U_\epsilon^{fin,\underline{\chi}'}) \cong \mathrm{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V_\epsilon, U_\epsilon^{fin}) \otimes_{\mathcal{W}_\epsilon} \mathcal{W}_\epsilon^{\wedge\underline{\chi}'},$$

here $V_\epsilon := V_q/\hbar V_q$. By Lemma 4.5, $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_\epsilon^{fin,\underline{\chi}'})$ is finitely generated over $\mathcal{W}_\epsilon^{\wedge\underline{\chi}'}$. Therefore

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'})/\hbar\,\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'})$$

is finitely generated over $\mathcal{W}_\epsilon^{\wedge\underline{\chi}'}$. On the other hand, $\mathcal{W}_q^{\wedge\underline{\chi}'}$ is complete in the $\hbar$-adic topology and $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'})$ is separated in the $\hbar$-adic topology (since $U_q^{fin,\underline{\chi}'}$ is separated). Therefore, $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}'})$ is finitely generated over $\mathcal{W}_q^{\wedge\underline{\chi}'}$.
(b) Let V_q be a projective object in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Then

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, \cap\hbar^k M) \xrightarrow{\sim} \cap\,\hbar^k\,\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, M).$$

The right hand side is zero since by part (a), it is finitely generated over $\mathcal{W}_q^{\wedge\underline{\chi}'}$ hence complete and separated in the $\hbar$-adic topology.

Since $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, \cap\hbar^k M) = 0$ for all projective objects in $\mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$ and the later category has enough projectives, it follows that $\cap\hbar^k M = 0$.
(c) Recall $\mathfrak{J}_{\underline{\chi}'} = \phi_\epsilon^{-1}(\mathcal{W}_\epsilon \mathfrak{m}_{\underline{\chi}'})$. Since $M/\hbar M \in U_\epsilon^{fin,\underline{\chi}'}\text{-rmod}^{G_\epsilon}$, it follows that $(\mathfrak{J}_{\underline{\chi}'})^k M = M(\mathfrak{J}_{\underline{\chi}'})^k$ for all k.

Let $m \in M$ and let $V_q = \check{U}_q(\mathfrak{g})m \subset M$. We note that $\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, M)$ is a module over $\mathcal{W}_q \otimes \mathcal{W}_q^{\wedge\underline{\chi}'}$ and the natural map

$$V_q \otimes_{\mathbb{C}[[\hbar]]} \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, M) \to M,$$

is a morphism of $\mathcal{W}_q \otimes \mathcal{W}_q^{\wedge\underline{\chi}'}$-module with the image containing m. Let the image of this morphism to be M'.

Any element of $\mathcal{W}_q^{\wedge\underline{\chi}'}$ is of the form $\sum_k x_k$ with $x_k \in (\mathfrak{J}_{\underline{\chi}'})^k$. By the first paragraph, we have $x_k m \in M(\mathfrak{J}_{\underline{\chi}'})^k$ for all k. On the other hand, by part (a), the subspace M' is finitely generated as a right $\mathcal{W}_q^{\wedge\underline{\chi}'}$-module, hence complete in the $\mathfrak{J}_{\underline{\chi}'}$-topology. Therefore, $\sum_k x_k m$ is a well-defined element in M'. This implies the first half of part (c). So M has a left $U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\underline{\chi}'}$-action, which then factors through $U_q^{fin,\underline{\chi}'}$ since M is seperated in the $\hbar$-adic topology. This implies the second half of part (c). $\square$

Remark 5.15. By decomposition (5.4), we have a natural functor

$$U_q^{fin,\vartheta'}\text{-rmod}^{G_q} \to \mathsf{HC}_q(\vartheta,\vartheta'), \tag{5.6}$$

by projecting to the direct summand corresponding to the left $U_q^{fin,\vartheta}$-action. In particular, if the image of ϑ and ϑ' in $\operatorname{Spec} Z_\cap$ are different then the category $\mathsf{HC}_q(\vartheta,\vartheta')$ is zero.

We see that

$$U_\epsilon^{fin,\underline{\chi}}\text{-rmod}^{G_\epsilon} \cong \prod_{(\vartheta,\vartheta')} \mathsf{HC}_\epsilon(\vartheta,\vartheta'), \qquad U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q} \cong \prod_{(\vartheta,\vartheta')} \mathsf{HC}_q(\vartheta,\vartheta'), \tag{5.7}$$

where (ϑ,ϑ') runs over all pairs such that images of ϑ,ϑ' under the map $\operatorname{Spec}\mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$ are $\underline{\chi}$

Lemma 5.16. *The categories* $\mathsf{HC}_\epsilon(\vartheta,\vartheta')$ *and* $\mathsf{HC}_q(\vartheta,\vartheta')$ *are abelian.*

Proof. It is obvious that $\mathsf{HC}_\epsilon(\vartheta,\vartheta')$ is abelian since $U_\epsilon^{fin,\vartheta'}$ is Noetherian. The proof for $\mathsf{HC}_q(\vartheta,\vartheta')$ is in several steps: □

Definition 5.17. Any $\lambda \in P$ defines a point (by abuse notation) $\lambda \in \operatorname{Spec}(\mathcal{W}_\epsilon)$. The integral blocks are $\mathsf{HC}_\epsilon(\lambda,\lambda')$ and $\mathsf{HC}_q(\lambda,\lambda')$.

Remark 5.18. By the asumption on ℓ, the image of $\lambda \in \operatorname{Spec}\mathcal{W}_\epsilon$ under the map $\operatorname{Spec}\mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$ is the point $1 \in T/W \cong \operatorname{Spec} Z_\cap$.

Definition 5.19. For any $V_q \in \operatorname{Rep}(\check{U}_q(\mathfrak{g}))$ which is free of finite rank over $\mathbb{C}[[\hbar]]$, let $P^{\vartheta,\vartheta'}(V_q)$ be the direct summand of $V_q \otimes_R U_q^{fin,\vartheta'}$ in $\mathsf{HC}_q(\vartheta,\vartheta')$. We call $P^{\vartheta,\vartheta'}(V_q)$, their direct sums and direct summands the *diagonal bimodules*.

Here (q,R) is either the case **(A)** or the case **(B)**.

Remark 5.20. Let $\underline{\chi}$ be the image of ϑ and ϑ' in $\operatorname{Spec} Z_\cap$. The $P^{\vartheta,\vartheta'}(V_q)$ is also the direct summand of $U_q^{fin,\vartheta} \otimes_R V_q$ in $\mathsf{HC}_q(\vartheta,\vartheta')$, and also the direct summand of $V_q \otimes_R U_q^{fin,\underline{\chi}}$ in $\mathsf{HC}_q(\vartheta,\vartheta')$.

Let consider the $\bullet_\ell$-action of the affine Weyl group $W_{aff} := W \ltimes Q$ on $\mathfrak{h}^*$ and the corresponding alcoves on the real form $P \otimes_{\mathbb{Z}} \mathbb{R}$ of $\mathfrak{h}^*$. For $\lambda \in P$, let W_λ be the stabilizer of λ under the $\bullet_\ell$-action of W_{aff}.

For any $\lambda,\mu \in P$, let $W_q(\mu-\lambda)$ be the Weyl module in $\operatorname{Rep}(\check{U}_q(\mathfrak{g}))$ such that the highest weight of $W_q(\mu-\lambda)$ is in the W-orbit of $\mu-\lambda$.

Definition 5.21. For λ,μ in the closure of the fundamental alcove, let $P_q^{\mu,\lambda} := P^{\mu,\lambda}(W_q(\mu-\lambda))$. These are *translation bimodules.*

Remark 5.22. We get the same translation bimodule if we replace $W_q(\mu-\lambda)$ by the indecomposable tilting modules with the highest weight contained in the W-orbit of $\mu-\lambda$. (Need to check whether we need to assume $W_\lambda \subset W_\mu$, here W_λ is the stabilizer of λ under the W_{aff}-action.

Definition 5.23. The *Hilting bimodules* in $\mathsf{HC}_q(\mu,\lambda)$ are direct summands of direct sums of objects of the form $P^{\mu,\lambda}(V_q)$ for tilting modules V_q in $\operatorname{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$. Let $\operatorname{Hilt}_q(\mu,\lambda)$ denote the full additive subcategories of hilting modules.

Remark 5.24. We avoid to call *tilting bimodules* since $\mathsf{HC}_q(\mu,\lambda)$ has no highest weight structure. The projective objects in $\mathsf{HC}_q(\mu,\lambda)$ are hilting bimodules. Tensors of hilting bimodules are hilting.

Discuss the Krull-Schmidt property of $\mathsf{HC}_q(\mu,\lambda)$: later we will talk about the projective covers of simple modules in $\mathsf{HC}_q(\mu,\lambda)$.

6. Poisson bimodules

Definition 6.1. Let $\mathcal{B}$ be an associative $\mathbb{C}[[\hbar]]$-algebra. By the noncommutative Poisson structure on $\mathcal{B}$ we mean a pair $(\mathcal{B}, \mathcal{P})$, here $\mathcal{P}$ is a $\mathbb{C}[[\hbar]]$-subalgebra of $\mathcal{B}$, along with a $\mathbb{C}[[\hbar]]$-bilinear map $\mathcal{P} \otimes \mathcal{B} \to \mathcal{B}$ such that $\mathcal{P}$ is closed with respect to $\{\ ,\ \}$ and

(1) $\{z, z\} = 0$,
(2) $\{hz, b\} = [z, b]$,
(3) $\{z, ab\} = \{z, a\}b + a\{z, b\}$,
(4) $\{z_1 z_2, a\} = \{z_1, a\}z_2 + z_1\{z_2, a\}$,
(5) $\{\{z_1, z_2\}, a\} = \{z_1, \{z_2, a\}\} - \{z_2, \{z_1, a\}\}$,

for all $z, z_1, z_2 \in \mathcal{P}$ and $z, b \in \mathcal{B}$.

Remark 6.2. By the condition $\{hz, b\} = [z, b]$, $\mathcal{P}$ must satisfy that $[\mathcal{P}, \mathcal{B}] \subset \hbar\mathcal{B}$. Moreover, if $\mathcal{B}$ is flat over $\mathbb{C}[[\hbar]]$, the Poisson bracket is uniquely recovered by $\{z, a\} = \hbar^{-1}[z, a]$.

Definition 6.3. Let M be a $\mathcal{B}$-bimodule such that the left and right actions of $\mathbb{C}[[\hbar]]$ coincide. We say that M is a Poisson $\mathcal{B}$-bimodule if it is equipped with a $\mathbb{C}[[\hbar]]$-bilinear map $\mathcal{P} \otimes M \to M$ satisfying the following equalities:

- $\{\hbar z, m\} = [z, m]$,
- $\{z, am\} = \{z, a\}m + a\{z, m\}$, $\{z, ma\} = \{z, m\}a + m\{z, a\}$,
- $\{z_1 z_2, m\} = \{z_1, m\}z_2 + z_1\{z_2, m\}$,
- $\{\{z_1, z_2\}, m\} = \{z_1, \{z_2, m\}\} - \{z_2, \{z_1, m\}\}$

for all $z, z_1, z_2 \in \mathcal{P}$ and $a \in \mathcal{B}, m \in M$.

Remark 6.4. If M is flat over $\mathbb{C}[[\hbar]]$ then the Poisson bracket is uniquely recovered by $\{z, m\} = \hbar^{-1}[z, m]$. Morphisms between Poisson bimodules M_1, M_2 are morphisms of bimodules $f : M_1 \to M_2$ such that $f\{z, m\} = \{z, f(m)\}$ for any $z \in \mathcal{P}$ and $m \in M_1$. The tensor product $M_1 \otimes_{\mathcal{B}} M_2$ of two Poisson bimodules is naturally a Poisson bimodule with the bracket defined by $\{z, m \otimes n\} = \{z, m\} \otimes n + m \otimes \{z, n\}$. Denote $\mathrm{Pbim}(\mathcal{B})$ the category of Poisson bimodules with respect to the pair $(\mathcal{B}, \mathcal{P})$. Then $\mathrm{Pbim}(\mathcal{B})$ is a monoidal category.

Remark 6.5. For $M, N \in \mathrm{Pbim}(\mathcal{B})$ such that M, N are flat over $\mathbb{C}[[\hbar]]$, the forgetful map

$$\mathrm{Hom}_{\mathrm{Pbim}(\mathcal{B})}(M, N) \to \mathrm{Hom}_{\mathcal{B}\text{--bimod}}(M, N)$$

is an isomorphism.

Let $\chi \in G_0^{d,reg} \subset \mathrm{Spec}\, Z_{Fr}$. Recall the formal $\hbar$-deformations:

$$\phi_\chi : U_q^{ev\wedge_\chi} \cong \mathrm{Mat}_{\ell^N}(R_\hbar) \twoheadrightarrow U_\epsilon^{ev\wedge_\chi} \cong \mathrm{Mat}_{\ell^N}(Z^{\wedge_\chi}), \qquad \psi_\chi : R_\hbar \to Z^{\wedge_\chi}.$$

here $R_\hbar \cong \mathcal{A}_q^\wedge \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathcal{W}_q^{\wedge_{\underline{\chi}}}$ is a formal $\hbar$-deformation of $Z^{\wedge_\chi} \cong Z_{Fr}^{\wedge_\chi} \widehat{\otimes}_{Z_\cap^{\wedge_{\underline{\chi}}}} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}} \cong \mathbb{C}[[V]] \widehat{\otimes}_{\mathbb{C}} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$.

Let

$$P_\hbar := \phi_\chi^{-1}(Z_{Fr}^{\wedge_\chi}), \qquad C_\hbar := \psi_\chi^{-1}(Z_{Fr}^{\wedge_\chi})$$

then $P_\hbar = C_\hbar + \hbar U_q^{ev\wedge_\chi}$, here $C_\hbar$ is embedded into $U_q^{\wedge_\chi}$ via the diagonal matrices.

Let consider the map $\pi : \mathcal{W}_q^{\wedge_{\underline{\chi}}} \to \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$ and let $B_\hbar := \pi^{-1}(Z_\cap^{\wedge_{\underline{\chi}}})$. Then $C_\hbar = \mathcal{A}_q^\wedge \widehat{\otimes}_{\mathbb{C}[[\hbar]]} B_\hbar$. It is easy to see that

Lemma 6.6. $(U_q^{ev\wedge_\chi}, P_\hbar)$, $(R_\hbar, C_\hbar)$, $(\mathcal{W}_q^{\wedge_{\underline{\chi}}}, B_\hbar)$ *are noncommutative Poisson structures on the corresponding* $\mathbb{C}[[\hbar]]$*-algebras.*

Definition 6.7. Let $U_q^{ev\wedge_\chi}$-Pbim be the category of left and right finitely generated Poisson $U_q^{ev\wedge_\chi}$-bimodules. The categories $R_\hbar$-Pbim and $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$-Pbim are similarly defined.

Remark 6.8. Since $U_q^{ev\wedge_\chi}$, $R_\hbar$, and $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$ are Noetherian, all these three categories are abelian.

Let $e = E_{11}$ be the idempotent of $U_q^{ev\wedge_\chi} \cong \mathrm{Mat}_N(R_\hbar)$ then $eU_q^{\wedge_\chi}e \cong R_\hbar$ and $eP_\hbar e \cong C_\hbar$. For $M \in U_q^{ev\wedge_\chi}$-Pbim, the space eMe is naturally a Poisson $R_\hbar$-bimodule with Poisson structure: $\{epe, eme\} = e\{p, eme\}e$ for $p \in P_\hbar$ and $m \in M$. One need to check that this Poisson bracket is well-defined but it is not hard. This construction gives us an equivalence of monoidal abelian categories:

$$\mathfrak{P}_1 : U_q^{ev\wedge_\chi}\text{-Pbim} \xrightarrow{\sim} R_\hbar\text{-Pbim}. \tag{6.1}$$

Using arguments in [6], we have an equivalence of monoidal abelian categories:

$$\mathfrak{P}_2 : R_\hbar\text{-Pbim} \xrightarrow{\sim} \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-Pbim}. \tag{6.2}$$

We recall the decompositions:

$$U_q^{ev\wedge_\chi} \cong \prod_{\xi=(\chi,\vartheta)} U_q^{ev\wedge_\xi}, \qquad \mathcal{W}_q^{\wedge_{\underline{\chi}}} = \prod_{\vartheta \mapsto \underline{\chi}} \mathcal{W}_q^{\wedge_\vartheta}, \tag{6.3}$$

here $\vartheta \mapsto \underline{\chi}$ means the image of ϑ under the map $\mathrm{Spec}\, \mathcal{W}_\epsilon \to \mathrm{Spec}\, Z_\cap$ is $\underline{\chi}$. These two decompositions are resemble from the natural surjections $U_q^{ev\wedge_\chi} \twoheadrightarrow U_q^{ev\wedge_\xi}$ and $\mathcal{W}_q^{\wedge_{\underline{\chi}}} \twoheadrightarrow \mathcal{W}_q^{\wedge_\xi}$.

Let $\xi = (\chi, \vartheta)$ and $\xi' = (\chi, \vartheta')$ then any $(U_q^{ev\wedge_\xi}, U_q^{ev\wedge_{\xi'}})$-bimodule can be viewed as a $U_q^{ev\wedge_\chi}$-bimodule, while any $(\mathcal{W}_q^{\wedge_\vartheta}, \mathcal{W}_q^{\wedge_{\vartheta'}})$-bimodule can be viewed as a $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$-bimodule.

Definition 6.9. A $(U_q^{ev\wedge_\xi}, U_q^{ev\wedge_{\xi'}})$-bimodule is called Poisson if it is a Poisson $U_q^{ev\wedge_\chi}$-bimodule. A $(\mathcal{W}_q^{\wedge_\vartheta}, \mathcal{W}_q^{\wedge_{\vartheta'}})$-bimodule is called Poisson if it is a Poisson $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$-bimodule.

Definition 6.10. Let $\mathrm{Pbim}(U_q^{\xi,xi'})$ denote the categories of left and right finitely generated Poisson $(U_q^{ev\wedge_\xi}, U_q^{ev\wedge_{\xi'}})$-bimodules. Let $\mathrm{Pbim}(\mathcal{W}_q^{\vartheta,\vartheta'})$ denote the categories of left and right finitely generated $(\mathcal{W}_q^{\wedge_\vartheta}, \mathcal{W}_q^{\wedge_{\vartheta'}})$-bimodules.

The decompositions (6.3) come with complete systems of idempotents so that one maps into the other under the morphism $\mathcal{W}_q^{\wedge_{\underline{\chi}}} \hookrightarrow U_q^{ev\wedge_\chi}$. Furthermore, $\mathfrak{P}_2 \circ \mathfrak{P}_1$ is also compatible with these two systems of complete idempotents, so that $\mathfrak{P}_2 \circ \mathfrak{P}_1$ gives a rise to a family of equivalences of monoidal abelian categories:

$$\mathfrak{P} : \mathrm{Pbim}(U_q^{\xi,\xi'}) \xrightarrow{\sim} \mathrm{Pbim}(\mathcal{W}_q^{\vartheta,\vartheta'}) \tag{6.4}$$

Let $\Lambda := P/Q$. Then we have the Λ-grading version of the above discussions: the categories $\mathrm{Pbim}^\Lambda(U_q^{\xi,\xi'})$, $\mathrm{Pbim}^\Lambda(\mathcal{W}_q^{\vartheta,\vartheta'})$ and the functors between them.

7. Quantum category O

Let $U_q^{mix} := U_q^{ev}\check{U}_q^{\geqslant}$. Let $\mathsf{R} := \mathbb{C}[[\hbar, \mathfrak{h}^*]]$ with the maximal ideal $\mathfrak{m}$. As in [7], introduce

- the category O_ϵ over the mixed quantum group U_ϵ^{mix},
- the category O_q over U_q^{mix} ,
- the deformed category $O_{q,\mathsf{R}}$ over $U_q^{mix} \otimes_{\mathbb{C}[[\hbar]]} \mathsf{R}$.

The category O_ϵ is embedded into O_q via the quotient $U_q^{mix} \twoheadrightarrow U_\epsilon^{mix}$.

Definition 7.1. The Verma modules $\Delta_\epsilon(\lambda)$, $\Delta_q(\lambda)$ and $\Delta_{q,\mathsf{R}}(\lambda)$ for $\lambda \in P$.

Lemma 7.2. *The ℓ-shifted dot action of extended affine Weyl group $W_{ext} := W \ltimes P$ on P gives us block decomposition:*

$$O_\epsilon = \bigoplus_{[\lambda]\in P/(W_{ext},\bullet_\ell)} O_\epsilon^{[\lambda]}, \qquad O_q = \bigoplus_{\lambda\in P/(W_{ext},\bullet_\ell)} O_q^{[\lambda]}, \qquad O_{q,\mathsf{R}} = \bigoplus_{[\lambda]\in P/(W_{ext},\bullet_\ell)} O_{q,\mathsf{R}}^{[\lambda]}$$

Let $\mathsf{pr}_{[\lambda]} : O_\epsilon \to O_\epsilon^{[\lambda]}$ be the natural projection. We use the same notations for the projections of the other two decompositions.

Recall that each $\lambda \in P$ gives a point $\lambda \in \operatorname{Spec} \mathcal{W}_\epsilon$.

Lemma 7.3. *Any object in $O_{q,\mathsf{R}}^{[\lambda]}$ carries a natural action of $U_q^{ev} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge_\lambda}$, which gives a natural action of $U_q^{fin,\lambda}$ and $U_q^{ev,\lambda} := U_q^{ev} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge_\lambda} / \cap \hbar^k U_q^{ev} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge_\lambda}$.*

Proof. Step 1: Let $\mathfrak{m}_\lambda$ be the maximal ideal of $\lambda \in \operatorname{Spec} \mathcal{W}_\epsilon$. We will show that any object $M \in O_\epsilon^{[0]}$ is killed by some power of $\mathfrak{m}_\lambda$. Since $\tilde{E}_\alpha^{\ell_\alpha}$ acts as zero and $K^\lambda (\lambda \in 2P^*)$ acts as 1 on any object in O_ϵ, any object in O_ϵ is kill by maximal ideal of $1 \in \operatorname{Spec} Z_\cap$. So $M = \oplus_{\vartheta \mapsto 1} M_\vartheta$, where $\vartheta \mapsto 1$ means the image of ϑ under the map $\operatorname{Spec} \mathcal{W}_\epsilon \to \operatorname{Spec} Z_\cap$ is 1, and M_ϑ is the component support at $\vartheta \in \operatorname{Spec} \mathcal{W}_\epsilon$. Since $O_\epsilon^{[\lambda]}$ is Serre spanned by Verma module $\Delta_\epsilon(\mu)$ with $\mu \in W_{ext} \bullet_\ell \lambda$, we must have $M = M_\lambda$.

Step 2: Let $M \in O_q^{[\lambda]}$. Let $\mathfrak{J}_\lambda : \phi^{-1}(\mathfrak{m}_\lambda)$ in which $\phi : \mathcal{W}_q \to \mathcal{W}_\epsilon$. The Harish-Chandra center $\mathcal{W}_q$ acts on each weight space M_μ of M. Since $M/M\mathfrak{m}^k$ has a finite filtration of subquotient contained in $O_\epsilon^{[0]}$, by Step 1, there is $s_k > 0$ such that $\mathfrak{J}_\lambda^{s_k} M \subset M\mathfrak{m}^k$. On the other hand, each weight space M_μ is finitely generated over R, hence complete and separated in the $\mathfrak{m}$-adic topology. Therefore, the action of $\mathcal{W}_q$ on M_μ extends uniquely to an action of $\mathcal{W}_q^{\wedge_\lambda}$.

Step 3: Since each weight space M_μ is separated in the $\hbar$-adic topology, hence the second part of the lemma follows. □

Let λ, μ in the closure of the fundamental alcove such that $W_\lambda \subset W_\mu$. Recall the Weyl module $W_q(\mu - \lambda)$.

Lemma 7.4. $\mathsf{pr}_{[\mu]}(W_q(\mu-\lambda) \otimes_{\mathbb{C}[[\hbar]]} \Delta_\epsilon(\lambda)) = \Delta_\epsilon(\mu)$, $\mathsf{pr}_{[\mu]}(\Delta_q(\lambda)) = \Delta_q(\mu)$ *and* $\mathsf{pr}_{[\mu]}(\Delta_{q,\mathsf{R}}(\lambda)) = \Delta_{q,\mathsf{R}}(\mu)$.

Proof. The functor $\mathsf{pr}_{[\mu]}$ is exact. Construct the filtration of Verma modules for $W_q(\mu-\lambda) \otimes_{\mathbb{C}[[\hbar]]} \Delta_\epsilon(\lambda) \cong W_\epsilon(\mu - \lambda) \otimes_{\mathbb{C}} \Delta_\epsilon(\lambda) \ldots$.

□

Remark 7.5. Note that $\mathsf{pr}_{[\mu]}\Big(W_q(\mu - \lambda) \otimes_{\mathbb{C}[[\hbar]]} \Delta_q(\lambda)\Big) \cong P^{\mu,\lambda} \otimes_{U_q^{fin,\lambda}} \Delta_q(\lambda)$. Same with the other two cases.

Remark 7.6. We also consider the right module versions O_ϵ^r, O_q^r and $O_{q,\mathsf{R}}^r$.

8. Restriction functor

8.1. The functor $\bullet_\dagger : \mathsf{HC}_q(\vartheta,\vartheta') \to \mathrm{Pbim}^\Lambda(\mathcal{W}_q^{\xi,\xi'})$.

Let us define a functor:

$$\mathfrak{C} : U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q} \to U_q^{ev\wedge_\chi}\text{-Pbim}^\Lambda.$$

Let $M \in U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q}$.

- Construct the tensor $M_{loc} := M \otimes_{U_q^{fin,\underline{\chi}}} U_q^{ev,\underline{\chi}}$. We have a left $U_q^{ev} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge_{\underline{\chi}}}$-action on M_{loc}, this action factors through $U_q^{ev,\underline{\chi}}$. Indeed we have a surjective map $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}} \twoheadrightarrow M$ for some finite $\mathbb{C}[[\hbar]]$-free $V_q \in \mathrm{Rep}(\check{U}_q(\mathfrak{g}))$. This give us a surjective map $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev,\underline{\chi}} \twoheadrightarrow M_{loc}$. The left action of $U_q^{ev} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge_{\underline{\chi}}}$ on the domain factors through $U_q^{ev,\underline{\chi}}$ since $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev,\underline{\chi}}$ is separated in the $\hbar$-adic topology.
- Let $\phi_{1,\hbar} : U_q^{ev,\underline{\chi}} \to U_q^{ev} \otimes_{\mathcal{W}_\epsilon} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$. Let $P_{1,\hbar} = \phi_{1,\hbar}^{-1}(Z_{Fr}^{\wedge_{\underline{\chi}}})$. Then M_{loc} carries a natural structure of $U_q^{ev,\underline{\chi}}$-Pbim$^\Lambda$. Then need to check that we have a right exact functor $\bullet_{loc} : U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q} \to U_q^{ev,\underline{\chi}}\text{-Pbim}^\Lambda$.
- The completion functor $\bullet^{\wedge_\chi} : U_q^{ev,\underline{\chi}}\text{-Pbim}^\Lambda \to U_q^{ev\wedge_\chi}\text{-Pbim}^\Lambda$.[2] [3]
- The Λ-grading comes as follows: Any object in $U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q}$ comes with the default P-grading, which gives a $\Lambda = P/Q$-grading. This Λ-grading passes through the construction of functors.

Then $\mathfrak{C} := \bullet^{\wedge_\chi} \circ \bullet_{loc}$.

Composing with the equivalence $\mathfrak{P} : U_q^{ev\wedge_\chi}\text{-Pbim}^\Lambda \xrightarrow{\sim} \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-Pbim}^\Lambda$, we obtain

$$\bullet_\dagger : U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q} \xrightarrow{\mathfrak{C}} U_q^{ev,\underline{\chi}}\text{-Pbim}^\Lambda \xrightarrow{\mathfrak{P}} \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-Pbim}^\Lambda. \tag{8.1}$$

Decomposition of categories give us the functor $\bullet_\dagger : \mathsf{HC}_q(\vartheta,\vartheta') \to \mathrm{Pbim}^\Lambda(\mathcal{W}_q^{\xi,\xi'})$.

Proposition 8.1. *The functor $\bullet_\dagger$ in (8.1) is right exact, monoidal and $(\mathcal{W}_q^{\wedge_{\underline{\chi}}}, \mathcal{W}_q^{\wedge_{\underline{\chi}}})$-linear.*

Proposition 8.2. *Assume the condition of χ as in Lemma 8.6. Then the functor $\bullet_\dagger$ in (8.1) is fully faithful on the diagonal modules.*

Proof. It is enough to show the following map is bijective

$$\begin{aligned}\mathrm{Hom}_{U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q}}(V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}}, W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}}) \\ \to \mathrm{Hom}_{U_q^{ev\wedge_\chi}\text{-Pbim}^\Lambda}(V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}, W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi})\end{aligned} \tag{8.2}$$

for $V_q, W_q \in \mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ which are free of finite rank over $\mathbb{C}[[\hbar]]$.

[2] Since $U_q^{ev,\underline{\chi}}$ is Noetherian and $\mathfrak{J}_\chi$ is center and generated by finitely many elements (ensure the Noetherian of blow up algebra), hence the completion $U_q^{ev\wedge_{\underline{\chi}}}$ satisfies the properties of Lemma 4.2. Then further complete $U_q^{ev\wedge_{\underline{\chi}}}$ to get the completion $U_q^{ev\wedge_\chi}$. Let $J_\chi := \phi_{1,\hbar}^{-1}(\mathfrak{m}_\chi)$ under the map $\phi_{1,\hbar} : U_q^{ev,\underline{\chi}} \to U_\epsilon^{ev} \otimes_{\mathcal{W}_\epsilon} \mathcal{W}_\epsilon^{\wedge_{\underline{\chi}}}$. Then $J_\chi^k U_q^{ev,\underline{\chi}}$ is finitely generated two-side ideal for all k (Need $J_\chi U = U J_\chi$). Since $U_q^{ev\wedge_{\underline{\chi}}}$ satisfies the properties in Lemma 4.2, we have $U_q^{ev\wedge_{\underline{\chi}}}/J_\chi^k U_q^{ev\wedge_{\underline{\chi}}} \cong U_q^{ev,\underline{\chi}}/U_q^{ev,\underline{\chi}} J_\chi^k$. So two-step completion is the same as one step completion.

[3] One need to check for any $M \in U_q^{ev,\underline{\chi}}$-Pbim$^\Lambda$, the completion $M^{\wedge_\chi}$ naturally carries a Poisson bimodule structure in $U_q^{ev\wedge_\chi}$-Pbim$^\Lambda$. One need to show that $P_{1,\hbar}$ is dense in $P_\hbar$ in the J_χ-adic topology: the closure of $P_{1,\hbar}$ contains $\hbar U_q^{ev\wedge_\chi}$ and one show that its image under $\phi_\hbar : U_q^{ev\wedge_\chi} \to U_\epsilon^{ev\wedge_\chi}$ is $Z_{Fr}^{\wedge_\chi}$.

• Since both $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}$ and $W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}$ are flat over $\mathbb{C}[[\hbar]]$, hence the right hand side of (8.2) is equal to

$$\mathrm{Hom}_{U_q^{ev\wedge_\chi}\text{-bimod}^\Lambda}(V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}, W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}). \tag{8.3}$$

• $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}$ is also an object in $U_q^{ev\wedge_\chi}\text{-rmod}^{\check{U}_q,\Lambda}$. We claim that (8.3) is equal to

$$\mathrm{Hom}_{U_q^{ev\wedge_\chi}\text{-rmod}^{\check{U}_q,\Lambda}}(V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}, W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}) \tag{8.4}$$

We have a forgeful map from (8.4) $\to$ (8.3) which is injective. This map is also surjective as follows: Any bimodule map is equivariant under the adjoint $U_q^{ev}(\mathfrak{g})$-action. Since both $V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}$ and $W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{ev\wedge_\chi}$ are flat over $\mathbb{C}[[\hbar]]$, the adjoint $\check{U}_q(\mathfrak{g})$-actions on these two modules are uniquely recovered from the adjoint $U_q^{ev}(\mathfrak{g})$-actions.

• Now we transform the isomorphism (8.2) into

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}}) \to \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{ev\wedge_\chi}) \tag{8.5}$$

for $V_q \in \mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ which is free of finite rank and has the weight space contained in the root lattice Q (the Λ-grading in $U_q^{ev\wedge_\chi}\text{-rmod}^{\check{U}_q,\Lambda}$ is used here where $U_q^{ev\wedge_\chi}$ is viewed as in the degree $0 \in \Lambda$.)

• We assume Lemma 8.3 below whose proof is in the next section. Again since $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$ has enough projective, we can assume V_q is projective. Since both $U_q^{fin,\underline{\chi}}$ and $U_q^{ev\wedge_\chi}$ are flat over $\mathbb{C}[[\hbar]]$, we have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\check{U}_q}(V_q, U_q^{fin,\underline{\chi}}) & \xrightarrow{\cdot\hbar} & \mathrm{Hom}_{\check{U}_q}(V_q, U_q^{fin,\underline{\chi}}) & \longrightarrow & \mathrm{Hom}_{\check{U}_q}(V_q, U_\epsilon^{fin,\underline{\chi}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Hom}_{\check{U}_q}(V_q, U_q^{ev\wedge_\chi}) & \xrightarrow{\cdot\hbar} & \mathrm{Hom}_{\check{U}_q}(V_q, U_q^{ev\wedge_\chi}) & \longrightarrow & \mathrm{Hom}_{\check{U}_q}(V_q, U_\epsilon^{ev\wedge_\chi}) & &
\end{array}$$

The last vertical arrow is the same as

$$\mathrm{Hom}_{\check{U}_\epsilon}(V_q/\hbar V_q, U_\epsilon^{fin,\underline{\chi}}) \to \mathrm{Hom}_{\check{U}_\epsilon}(V_q/\hbar V_q, U_\epsilon^{ev\wedge_\chi})$$

which is an isomorphism by Lemma 8.3. Therefore, the second row is also a short exact sequence.

Both $\mathrm{Hom}_{\check{U}_q}(V_q, U_q^{fin,\underline{\chi}})$ and $\mathrm{Hom}_{\check{U}_q}(V_q, U_q^{ev\wedge_\chi}$ are finitely generated over $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$. Hence (8.5) is surjective. Since $\mathrm{Hom}_{\check{U}_q}(V_q, U_q^{ev\wedge_\chi})$ is flat over $\mathbb{C}[[\hbar]]$, it then follows that (8.5) is injective. $\square$

Lemma 8.3. *For any $V_\epsilon \in \mathrm{Rep}^{fd}(\check{U}_\epsilon(g))$ which has the weight space in the root lattice Q, the following natural map is bijective*

$$\mathrm{Hom}_{\check{U}_\epsilon}(V_\epsilon, U_\epsilon^{fin,\underline{\chi}}) \to \mathrm{Hom}_{\check{U}_\epsilon}(V_\epsilon, U_\epsilon^{ev\wedge_\chi}). \tag{8.6}$$

8.2. **Proof of Lemma 8.3.**

Conjugacy classes in algebraic groups. Let G be a simply connected semisimple algebraic group. We give a non-exhaustive list of some geometric facts about the conjugacy action of G on itself from [13], note that some of those facts hold without simply connectedness.

Proposition 8.4. *(recheck which parts require simply connectedness) Let G acts on itself via conjugation and consider the categorical quotient map $\pi : G \to G/\!\!/G$. Let F be the (reduced) fiber of any closed point p in $G/\!\!/G$.*

a) Let T be a maximal torus of G then there is a natural isomorphism $\mathbb{C}[G]^G \cong \mathbb{C}[T]^W$. Furthermore, if G is simply connected then $G/\!\!/G \cong \mathbb{A}^r$, the affine space with dimension r equal to the rank of Lie algebra $\mathfrak{g}$.

b) *F is a closed, irreducible and normal subvariety of codimension r in G. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathbb{C}[G/\!\!/G]$ corresponding to p then the defining ideal of F is $\mathfrak{m}_p\mathbb{C}[G]$.*
c) *F contains a unique class of regular elements. This class is open and dense in F and its complement has codimension ≥ 2.*
d) *There is a cross section S, which is called Steinberg section, that paramerizes the classes of regular elements. Moreover, S is contained in the regular locus of G, and the natural map $\pi(S) \to G/\!\!/G$ is an isomorphism of varieties.*

We need the following technical result from [6] : Let G be a simply connected semisimple algebraic group. Let H be a subgroup of G such that G/H is a quasi-affine and $\mathbb{C}[G/H]$ is finitely generated. Let $x \in G/H$ whose stablizer in G is H. Let H^0 be the identity component of H and denote by $C(x) = H/H^0$ the component group of H. Consider the natural map $\phi : G/H^0 \to G/H$. Let M be a G-equivariant vector bundle on G/H. The completion $M^{\wedge_x}$ carries a natural actions of $\mathfrak{g}$ and H. Let denote this H-action by ρ. Integrating the $\mathfrak{g}$-action on the locally $\mathfrak{g}$-finite part $M^{\wedge_x}_{\mathfrak{g}\text{-lf}}$ into G-action then restrict to H we get another H-action on $M^{\wedge_x}_{\mathfrak{g}\text{-lf}}$. We denote this action by ρ'. One can show that $\sigma(h) = \rho(h)\rho'(h^{-1})$ defines a new H-action on $M^{\wedge_x}_{\mathfrak{g}\text{-lf}}$ which commutes with G-action. Furthermore, $\sigma(H^0)$ acts trivially so that we have an action of $C(x)$ on $M^{\wedge_x}_{\mathfrak{g}\text{-lf}}$, hence we can define the $C(x)$-invariant part $M^{\wedge_x, C(x)}_{\mathfrak{g}\text{-lf}}$. The following result is in [6, Proposition 3.2.3]

Lemma 8.5. $M^{\wedge_x}_{\mathfrak{g}\text{-lf}} \cong \Gamma(G/H^0, \phi^*M), \qquad M^{\wedge_x, C(x)}_{\mathfrak{g}\text{-lf}} \cong \Gamma(G/H, M)$

Let χ be a regular element in G and $\underline{\chi}$ be the image of χ under the quotient map $\pi : G \to G/\!\!/G$. Let $\mathbb{C}[G]^{\wedge_\chi}, \mathbb{C}[G/\!\!/G]^{\wedge_{\underline{\chi}}}$ be the completion of $\mathbb{C}[G], \mathbb{C}[G/\!\!/G]$ at the closed points $\chi, \underline{\chi}$, respectively. Denote $I_\chi := \mathfrak{m}_{\underline{\chi}}\mathbb{C}[G]$ and $C[G]^{\wedge_{I_\chi}}$ the completions of $C[G]$ with respect to the ideal I_χ.

The action of $\mathfrak{g}$ on $\mathbb{C}[G]$ extends to an $\mathfrak{g}$-action on $\mathbb{C}[G]^{\wedge_\chi}$. Let $\mathbb{C}[G]^{\wedge_\chi}_{\mathfrak{g}\text{-lf}}$ be the locally finite part of this $\mathfrak{g}$-action. Integrate the $\mathfrak{g}$-action into the G-action on $\mathbb{C}[G]^{\wedge_\chi}_{\mathfrak{g}\text{-lf}}$, and let $\mathbb{C}[G]^{\wedge_\chi, Z}_{\mathfrak{g}\text{-lf}}$ denote the $Z(G)$-invariant part. We have a natural map $\mathbb{C}[G] \otimes_{\mathbb{C}[G/\!\!/G]} \mathbb{C}[G/\!\!/G]^{\wedge_{\underline{\chi}}} \to \mathbb{C}[G]^{\wedge_\chi, Z}_{\mathfrak{g}\text{-lf}}$.

Lemma 8.6. *Suppose the natural map $Z(G) \to C(\chi)$ is surjective. Then the natural map $\mathbb{C}[G] \otimes_{\mathbb{C}[G/\!\!/G]} \mathbb{C}[G/\!\!/G]^{\wedge_{\underline{\chi}}} \to \mathbb{C}[G]^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}}$ is an isomorphism.*

The surjectivity condition holds in the case of regular unipotent elements, indeed, in that case the natural map $Z(G) \to C(\chi)$ becomes an isomorphism

Proof. Since G is Cohen-Macaulay (indeed regular) and I_χ is generated by $\mathrm{codim}(I_\chi)$ elements, I^k_χ/I^{k+1}_χ is a free module of finite rank over $\mathbb{C}[G]/I_\chi = \mathbb{C}[\overline{G\chi}]$.

Let M be a G-equivariant coherent sheaf on $\overline{G\chi}$ such that $Z(G)$ acts on M trivially. If the natural map $Z(G) \to C(\chi)$ is surjective then we have an isomorphism $M^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}} \cong M^{\wedge_\chi, C(\chi)}_{\mathfrak{g}\text{-lf}}$. By Lemma 8.5, we have $M^{\wedge_\chi, C(\chi)}_{\mathfrak{g}\text{-lf}} \cong \Gamma(G\chi, M|_{G\chi})$. Note that $\overline{G\chi}$ is a normal variety. Therefore, if we assume further that M is a free sheaf then $M^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}} \cong \Gamma(\overline{G\chi}, M)$. Applying this analysis to the free $\mathbb{C}[\overline{G\chi}]$-module I^k_χ/I^{k+1}_χ whose $Z(G)$-action is trivial we have

$$I^k_\chi/I^{k+1}_\chi \to (I^k_\chi/I^{k+1}_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}}$$

is an isomorphism of $\mathfrak{g}$-modules for all $k \geq 0$, in which $I^0_\chi = \mathbb{C}[G]$. Now consider the following commutative diagram in the category of $\mathfrak{g}$-modules:

$$\begin{tikzcd}
0 \arrow[r] & I^k_\chi/I^{k+1}_\chi \arrow[r] \arrow[d] & \mathbb{C}[G]/I^{k+1}_\chi \arrow[r] \arrow[d] & \mathbb{C}[G]/I^k_\chi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I^k_\chi/I^{k+1}_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}} \arrow[r] & (\mathbb{C}[G]/I^{k+1}_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}} \arrow[r] & (C[G]/I^k_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}} &
\end{tikzcd}$$

This diagram allows us to inductively prove that the natural map

$$\mathbb{C}[G]/I^k_\chi \to (\mathbb{C}[G]/I^k_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}}$$

is an isomorphism of $\mathfrak{g}$-modules for all $k \geq 1$.

Let V be any finite dimensional representation of $\mathfrak{g}$ which is also representation of the adjoint group G_{ad}. Since $\mathfrak{m}_{\underline{\chi}} \subset \mathfrak{m}_\chi$, $\mathbb{C}[G]^{\wedge_\chi}$ is $\mathfrak{m}_{\underline{\chi}}$-adically complete. Moreover, $\mathbb{C}[G]^{\wedge_\chi}/\mathfrak{m}^k_{\underline{\chi}}\mathbb{C}[G]^{\wedge_\chi} \cong (\mathbb{C}[G]/I^k_\chi)^{\wedge_\chi}$. Therefore, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}}(V, \mathbb{C}[G]^{\wedge_\chi}) &\cong \varprojlim \mathrm{Hom}_{\mathfrak{g}}(V, (\mathbb{C}[G]/I^k_\chi)^{\wedge_\chi}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathfrak{g}}(V, (\mathbb{C}[G]/I^k_\chi)^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}}) \\
&\cong \varprojlim \mathrm{Hom}_{\mathfrak{g}}(V, \mathbb{C}[G]/I^k_\chi) \\
&\cong \mathrm{Hom}_{\mathfrak{g}}(V, \mathbb{C}[G]^{\wedge_{I_\chi}}),
\end{aligned}$$

for any $\mathfrak{g}$-representation V as above. Hence, we have

$$\mathbb{C}[G]^{\wedge_{I_\chi}, Z(G)}_{\mathfrak{g}\text{-lf}} \cong \mathbb{C}[G]^{\wedge_\chi, Z(G)}_{\mathfrak{g}\text{-lf}},$$

but the former is just $\mathbb{C}[G] \otimes_{\mathbb{C}[G/\!\!/G]} \mathbb{C}[G /\!\!/ G]^{\wedge_{\underline{\chi}}}$ since $\mathrm{Hom}_{\mathfrak{g}}(V, \mathbb{C}[G])$ is finitely generated over $\mathbb{C}[G /\!\!/ G]$. □

Proof of Lemma 8.3. We note that $U^{ev\wedge_\chi}_\epsilon = U^{fin\wedge_\chi}_\epsilon$. So we need to show the following map is bijective:

$$(V^t_\epsilon \otimes U^{fin,\underline{\chi}}_\epsilon)^{\check{U}_\epsilon} \to (V^t_\epsilon \otimes U^{fin\wedge_\chi}_\epsilon)^{\check{U}_\epsilon}, \tag{8.7}$$

here V^t_ϵ is the right dual. Since $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ has enough projective, we can assume V_ϵ is projective. Moreover, projective objects are also injective objects in $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$, so V^t_ϵ is also projective.

Let $Z^{fin}_{Fr}\text{-mod}^{G_\epsilon,Q}$ be the category of finitely generated Z^{fin}_{Fr}-modules in the category $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ whose weight spaces are contained in the weight lattice Q.

Step 1: Since V^t_ϵ is projective in $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ and U^{fin}_ϵ is projective over Z^{fin}_{Fr}, one show that $V^t_\epsilon \otimes U^{fin}_\epsilon$ is a projective object in $Z^{fin}_{Fr}\text{-mod}^{G_\epsilon,Q}$. So $V^t_\epsilon \otimes U^{fin}_\epsilon$ is a direct summand of some object of the form $W_\epsilon \otimes Z^{fin}_{Fr}$ with $W_\epsilon \in \mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ whose weight space is contained in Q.

Step 2: We consider the following two functors:

$$A: \ Z^{fin}_{Fr}\text{-mod}^{G_\epsilon,Q} \to \mathrm{Vect}_{\mathbb{C}}, \qquad B: \ Z^{fin}_{Fr}\text{-mod}^{G_\epsilon,Q} \to \mathrm{Vect}_{\mathbb{C}}$$

Let $M \in Z^{fin}_{Fr}\text{-mod}^{G_\epsilon}$. Then $B(M) := (M \otimes_{Z_\cap} Z^{\wedge_{\underline{\chi}}}_\cap)^{\mathfrak{u}_\epsilon}$.

Let us define $A(M)$. First we taking the completion $M^{\wedge_\chi} = M \otimes_{Z^{fin}_{Fr}} Z^{fin\wedge_\chi}_{Fr}$. Then take the $\mathfrak{u}_\epsilon$-invariant part of $M^{\wedge_\chi}$ so that $(M^{\wedge_\chi})^{\mathfrak{u}_\epsilon}$ is a module over $\check{U}_{\mathbb{C}}(\mathfrak{g}^d)$. Then take the $\mathfrak{g}^d$-locally finite part whose weight space is contained in the root lattice Q^* of $\mathfrak{g}^d$ and define $A(M) := (M^{\wedge_\chi})^{\mathfrak{u}_\epsilon}_{\mathfrak{g}^d\text{-}fin, Z(G^d)}$.

So we have a natural transformation $B(M) \to A(M)$.

Step 3: Let W_ϵ be a finite dimensional module in $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ whose weight space is contained in the root lattice Q. We will show that $B(W_\epsilon \otimes Z_{Fr}^{fin}) \to A(W_\epsilon \otimes Z_{Fr}^{fin})$. Indeed we have

$$B(W_\epsilon \otimes Z_{Fr}^{fin}) = (W_\epsilon)^{\mathfrak{u}_\epsilon} \otimes Z_{Fr}^{fin,\underline{\chi}}, \qquad A(W_\epsilon \otimes Z_{Fr}^{fin}) = (W_\epsilon)^{\mathfrak{u}_\epsilon} \otimes (Z_{Fr}^{fin\wedge_\chi})^{Z(G^d)}_{\mathfrak{g}^d\text{-}fin},$$

here since the weight space of W_ϵ is contained in Q, hence the $\mathfrak{u}_\epsilon$-invariant part $(W_\epsilon)^{\mathfrak{u}_\epsilon}$ is a rational representation of $\mathfrak{g}^d$ with a trivial action of $Z(G^d)$. Therefore,

$$((W_\epsilon)^{\mathfrak{u}_\epsilon} \otimes (Z_{Fr}^{fin\wedge_\chi})_{\mathfrak{g}^d\text{-}fin})^{Z(G^d)} \cong (W_\epsilon)^{\mathfrak{u}_\epsilon} \otimes (Z_{Fr}^{fin\wedge_\chi})^{Z(G^d)}_{\mathfrak{g}^d\text{-}fin}.$$

By Lemma 8.6, we have $Z_{Fr}^{fin,\underline{\chi}} \xrightarrow{\sim} (Z_{Fr}^{fin\wedge_\chi})^{Z(G^d)}_{\mathfrak{g}^d\text{-}fin}$. Therefore $B(W_\epsilon \otimes Z_{Fr}^{fin}) \xrightarrow{\sim} A(W_\epsilon \otimes Z_{Fr}^{fin})$.

Step 4: By Step 1 and 3, we have

$$B(V_\epsilon^t \otimes U_\epsilon^{fin,\underline{\chi}}) \xrightarrow{\sim} A(V_\epsilon^t \otimes U_\epsilon^{fin,\underline{\chi}}).$$

Note that (8.7) is obtained from the above isomorphism via taking $\check{U}_{\mathbb{C}}(\mathfrak{g}^d)$-invariant part, hence (8.7) is an isomorphism. This completes the lemma. □

8.3. **The functor $\bullet_\dagger : O_q^{[0]} \to \mathcal{W}_q^{\wedge_0}\text{-mod}^\Lambda$.** Here $\mathcal{W}_q^{\wedge_0}\text{-mod}^\Lambda$ is the category of Λ-grading $\mathcal{W}_q^{\wedge_0}$-modules.

Let us construct the functor $\bullet_\dagger : O_q \to \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-mod}^\Lambda$. Then this gives us a family of functors $\bullet_\dagger : O_q^{[\lambda]} \to \mathcal{W}_q^{\wedge_\lambda}\text{-mod}^\Lambda$.

Let us recall the identification:

$$Z_{Fr} = \mathbb{C}[\tilde{E}_\alpha^\ell K^{\ell\gamma(\alpha)}]_{\alpha\in\Delta_+} \otimes_{\mathbb{C}} \bigoplus_{\lambda\in\ell P} \mathbb{C}K^{2\lambda} \otimes_{\mathbb{C}} \mathbb{C}[\tilde{F}_\alpha^\ell K^{\ell\kappa(\alpha)}]_{\alpha\in\Delta_+} \cong \mathbb{C}[U_-] \otimes_{\mathbb{C}} \mathbb{C}[T] \otimes_{\mathbb{C}} \mathbb{C}[U_+]$$

Let χ be a regular unipotent element in $U_+ \subset G_0$ then $\{K^{2\ell\lambda-1}, \tilde{E}_\alpha^\ell\} \subset \mathfrak{m}_\chi$.

Let us consider the map

$$\iota : \mathfrak{I} := Z_{Fr}\langle K^{2\ell\lambda} - 1, \tilde{E}_\alpha^\ell\rangle \subset \mathfrak{m}_\chi \to \mathfrak{m}_\chi/\mathfrak{m}_\chi^2.$$

On $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2$, we have the skew-symmetric form $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2 \times \mathfrak{m}_\chi/\mathfrak{m}_\chi^2 \to Z_{Fr}/\mathfrak{m}_\chi \cong \mathbb{C}$ as follows: $\{f + \mathfrak{m}_\chi^2, g + \mathfrak{m}_\chi^2\} = \{f, g\} + \mathfrak{m}_\chi$.

Lemma 8.7. *$\{K^{2\ell\lambda} - 1, \tilde{E}_\alpha^\ell\} = ?K^{2\ell\lambda}\tilde{E}_\alpha^\ell \in \mathfrak{I}$ and $\{\tilde{E}_\alpha^\ell, \tilde{E}_\beta^\ell\} \subset Z_{Fr,\alpha+\beta}^{>} \subset \mathfrak{I}$.*

Proof. □

From this lemma, we see that the image of ι in $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2$ is an isotopic subspace. Note that this space has at least of dimension $N + r$. On the other hand $\dim_{\mathbb{C}} \mathfrak{m}_\chi/\mathfrak{m}_\chi^2 = 2N + r$. Let V be a maximal symplectic subspace in $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2$ which is a lift of the cotangent space of conjugacy classes at χ. Since χ is a regular element, the maximal symplectic subspace of $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2$ is of dimension $2N$. Therefore, the image of ι is the maximal isotropic subspace of $\mathfrak{m}_\chi/\mathfrak{m}_\chi^2$, and the intersection $\mathfrak{u} : \mathrm{Im}(\iota) \cap V$ is a Lagrangian subspace of V.

Proposition 8.8. *Let $(A, \mathfrak{m})$ be a complete local Poisson algebra. Let the ideal $\mathfrak{n} \subset \mathfrak{m}$ such that*

- *$\mathfrak{n}$ is closed under the Poisson bracket.*
- *The image of the map $\mathfrak{n} \to \mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ is a maximal isotropic subspace of $\mathfrak{m}/\mathfrak{m}^2$. Denote this image by $\mathfrak{b}$*

Let $V \subset \mathfrak{m}/\mathfrak{m}^2$ be a maximal symplectic subspace of $\mathfrak{m}/\mathfrak{m}^2$ and $\mathfrak{u} := V \cap \mathfrak{b}$ is a Lagrangian subspace of V. Then we can lift $V \to A$ so that $u \hookrightarrow \mathfrak{n}$ and extends to a Poisson embedding $\mathbb{C}[[V^*]] \hookrightarrow A$.

Recall the $\mathbb{C}[[\hbar]]$-flat deformation $\phi_\hbar : R_\hbar \to Z^{\wedge_\chi}$.

Lemma 8.9. *We can find a lift $V \to R_\hbar$ such that $\mathfrak{u} \subset \phi_\hbar^{-1}(\mathfrak{I})$ and the lift gives us the decomposition $R_\hbar \cong \mathcal{A}_q \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathcal{W}_q^{\wedge_{\underline{\chi}}}$.*

Recall the decompositions $U_q^{ev \wedge_\chi} \cong \mathrm{Mat}_N(\mathbb{C}) \otimes R_\hbar$ and the idempotent element $e := E_{11}$. For any $M \in O_q$, let $M^{\wedge_\chi} := U_q^{ev\wedge_\chi} \otimes_{U_q^{ev}} M$ then $eM^{\wedge_\chi}$ is a finitely generated module over $R_\hbar$.

Lemma 8.10. *(a) For any $M \in O_q$, we can define the natural bilinear map $\{\ ,\ \} : \mathfrak{u} \times eM^{\wedge_\chi} \to eM^{\wedge_\chi}$ such that $u \cdot m$ such that*

- *$\hbar\{u, m\} = um$ for $u \in \mathfrak{u}, m \in eM^{\wedge_\chi}$.*
- *$\{u, xm\} = \{u, x\}m + x\{u, m\}$ for $u \in \mathfrak{u}, x \in R_\hbar, m \in eM^{\wedge_\chi}$.*

(b) For any $M \in O_q$, we have a decomposition of $R_\hbar := \mathcal{A}_q^{\wedge} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} \mathcal{W}_q^{\wedge_{\underline{\chi}}}$-module

$$eM^{\wedge_\chi} := \mathbb{C}[[\mathfrak{u}, \hbar]] \widehat{\otimes}_{\mathbb{C}[[\hbar]]} M_\dagger,$$

here $M_\dagger := \{m \in eM^{\wedge_\chi} | \{u, m\} = 0 \ \forall\ u \in \mathfrak{u}\}$. [4]

Proof. (a) For any $M \in O_q$, then $ueM^{\wedge_\chi} \in \hbar eM^{\wedge_\chi}$ for all $u \in \mathfrak{u}$.

For any $M \in O_q$, we can find a $\mathbb{C}[[\hbar]]$-objects $N_1, N_2 \in O_q$ with an exact sequence $N_2 \to N_1 \to M \to 0$. This gives us an exact sequence

$$eN_2^{\wedge_\chi} \xrightarrow{\phi} eN_1^{\wedge_\chi} \xrightarrow{\pi} eM^{\wedge_\chi} \to 0.$$

Let us define $\{\ ,\ \} : \mathfrak{u} \times eM^{\wedge_\chi} \to eM^{\wedge_\chi}$ by $\{u, m\} = \pi(\hbar^{-1}un)$ for any $n \in eN^{\wedge_\chi}$ such that $\pi(n) = m$.

- This definition is well-defined: for n_1, n_2 such that $\pi(n_1) = \pi(n_2)$ then $\pi(\hbar^{-1}un_1) - \pi(\hbar^{-1}un_2) = \pi(\hbar^{-1}u\phi(n')) = \pi(\phi(\hbar^{-1}un')) = 0$ for some $n' \in eN_2^{\wedge_\chi}$.
- This does not depend on the choice of the surjective map $N_1 \twoheadrightarrow M$. Indeed let us consider the other surjective map $N_1' \twoheadrightarrow M$. The fiber product $N_1 \times_M N_1' \subset N_1 \oplus N_1'$ is an object in O_q (it is equal to $\bigoplus_\lambda N_{1,\lambda} \times_{M_\lambda} N'_{1,\lambda}$). It is flat over $\mathbb{C}[[\hbar]]$. So both bilinear forms defined over N_1, N_1' can be obtained from $N_1 \times_M N_1'$, hence are identical.
- For any $f : M \to N$, then the map $f : eM^{\wedge_\chi} \to eN^{\wedge_\chi}$ satisfies $f\{u, m\} = \{u, f(m)\}$. The truncated category $O_q^{\leq \nu}$ has enough projectives, see [12, §2.3.2]. So in the category $O_q^{\leq \nu}$ with ν large enough, we can find a commutative diagram

$$\begin{tikzcd} M' \arrow[r] \arrow[d, two heads] & N' \arrow[d, two heads] \\ M \arrow[r] & N \end{tikzcd}$$

  with M', N' are flat over $\mathbb{C}[[\hbar]]$. The the claim follows.

(b) Follows the proof of [5, Lemma 4.2] [5] $\square$

Then we define the restriction functor $\bullet_\dagger : O_q \to \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-mod}^\Lambda$ by $M \mapsto M_\dagger$. The Λ-grading comes from the Λ-grading on M.

Proposition 8.11. *(a) The functor $\bullet_\dagger : O_q \to \mathcal{W}_q^{\wedge_{\underline{\chi}}}\text{-mod}^\Lambda$ is exact and $\mathcal{W}_q^{\wedge_{\underline{\chi}}}$-linear.*
(b) For $(\Delta_\epsilon(\lambda))_\dagger \cong \mathbb{C}$ for all $\lambda \in P$.

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[4] $\mathbb{C}[[\mathfrak{u}, \hbar]]$, should $\mathfrak{u}$ be replaced by the Lagrangian complement $\mathfrak{u}^*$?

[5] Since $\mathbb{C}[[\mathfrak{u}, \hbar]]$ is topological free over $\mathbb{C}[[\hbar]]$, the complete tensor product $\mathbb{C}[[\mathfrak{u}, \hbar]] \widehat{\otimes}_{\mathbb{C}[[\hbar]]} -$ is exact and map nonzero object to nonzero object.

Proof. (a) Follows by the construction.
(b) For any $M \in O_\epsilon \subset O_q$, we see that $\ell^N \dim_{\mathbb{C}} M_\dagger$ is equal to the dimension of fiber of M at the point $\chi \in \operatorname{Spec} Z_{Fr}$. On the other hand, $\Delta_\epsilon(\lambda)$ is a free sheaf of rank ℓ^N. □

9. Soergel Bimodules

The extend Weyl group $W_{aff} := W \ltimes P$ acts on $\mathsf{R} := \mathbb{C}[[\hbar, \mathfrak{h}^*]]$ and let $\Lambda := P/Q$, here P and Q are the weight and root lattice of $\mathfrak{g}$. It acts on $\mathfrak{h}^* \oplus \mathbb{C}\hbar$ as follows: $w.(\mu, z) = (w\mu, z)$ and $t_\lambda(\mu, z) = (\mu + z\lambda, z)$ for $wt_\lambda \in W_{aff}$ and $\mu \in \mathfrak{h}^*, z \in \mathbb{C}$.

Let us consider the category R--bimod$^\Lambda$ of Λ-grading R-bimodules. The (extended) affine Soergel bimodules $\mathsf{SB}_\hbar$ is the full Kroubian subcategory of R--bimod$^\Lambda$ generated by Bott-Samelson bimodules $\mathsf{R} \otimes_{\mathsf{R}^s} \mathsf{R}$ for $s \in I_a$ (in the degree 0) and the graph modules R_x for $x \in \Lambda$ (in the degree x).

Let SB be the category with the same set of objects as $\mathsf{SB}_\hbar$ but the set of morphism is replaced by

$$\operatorname{Hom}_{\mathsf{SB}}(M, N) = \operatorname{Hom}_{\mathsf{SB}_\hbar}(M, N)/\hbar \operatorname{Hom}_{\mathsf{SB}_\hbar}(M, N)$$

(Will need Abe's realization of SB later.)

- Talk about cells and the smallest two-sided cells in $\mathsf{SB}_\hbar$ and SB.

10. Some images of $\bullet_\dagger$, the first main result

Let us define the map $\iota : \mathfrak{h}^* \to \mathsf{R}$ via $\iota(\lambda) = (\lambda, \ ) \in \mathfrak{h} \subset \mathsf{R}$. For each $\lambda \in P$, we define the map $\varepsilon_\lambda : \mathcal{W}_q = Z_{q,HC} \subset \mathbb{C}[[\hbar]][K^{2\lambda}]_{\lambda \in P} \to \mathsf{R}$ via $K^{2\nu} \mapsto q^{(2\lambda,\nu)} e^{2\pi\sqrt{-1}(\iota(2\lambda)}$

Lemma 10.1. *The map ϵ_λ extends to an isomorphism $\mathcal{W}_q^{\wedge_\lambda} \to \mathsf{R}^{W_\lambda}$*

Proof. See [7, Lemma 5.5.9]. □

Under identifications $\varepsilon_\mu : \mathcal{W}_q^{\wedge_\mu} \xrightarrow{\sim} \mathsf{R}^{W_\mu}$ and $\varepsilon_\lambda : \mathcal{W}_q^{\wedge_\lambda} \xrightarrow{\sim} \mathsf{R}^{W_\lambda}$, we have the functor $\bullet_\dagger$: $\mathsf{HC}_q(\mu, \lambda) \to (\mathsf{R}^{W_\mu}, \mathsf{R}^{W_\lambda})$-bimod$^\Lambda$.

Proposition 10.2. *Let μ, λ be in the closure of the fundamental alcove. Assume $W_\lambda \subset W_\mu$. Then*
(a) We have isomorphisms in the category $(\mathsf{R}^{W_\mu}, \mathsf{R}^{W_\lambda})$-bimod$^\Lambda$: $P_{q,\dagger}^{\mu,\lambda} \cong \mathsf{R}^{W_\lambda}$, here the left R^{W_μ}-action comes from the inclusion $\mathsf{R}^{W_\mu} \hookrightarrow \mathsf{R}^{W_\lambda}$ while the right R^{W_λ}-action comes from the right multiplication.
(b) We have isomorphism in the category $(\mathsf{R}^{W_\lambda}, \mathsf{R}^{W_\mu})$-bimod$^\Lambda$: $P_{q,\dagger}^{\lambda,\mu} \cong \mathsf{R}^{W_\lambda}$, here the left R^{W_λ}-action comes from the left multiplication while the right R^{W_μ}-action comes from the inclusion $\mathsf{R}^{W_\mu} \hookrightarrow \mathsf{R}^{W_\lambda}$.

Proof. (a) *Step 1:* Since $\bullet_\dagger : \mathsf{HC}_q(\mu, \lambda) \to (\mathsf{R}^{W_\mu}, \mathsf{R}^{W_\lambda})$-bimod$^\Lambda$ is right exact and $(\mathsf{R}^{W_\mu}, \mathsf{R}^{W_\lambda})$-linear, we have

$$(P_{q,\dagger}^{\mu,\lambda}/\mathfrak{J}_\lambda \cong (P_q^{\mu,\lambda}/P_q^{\mu,\lambda}\mathfrak{J}_\lambda)_\dagger,$$

here $\mathfrak{J}_\lambda$ is the maximal ideal of $\mathcal{W}_q^{\wedge_\lambda} \xrightarrow{\sim} \mathsf{R}^{W_\lambda}$.

- Show that $(P_q^{\mu,\lambda}/P_q^{\mu,\lambda}\mathfrak{J}_\lambda)_\dagger \otimes_{\mathbb{C}} \Delta_\epsilon(\lambda)_\dagger \cong \Delta_\epsilon(\mu)_\dagger$. Hence $(P_q^{\mu,\lambda}/P_q^{\mu,\lambda}\mathfrak{J}_\lambda)_\dagger \cong \mathbb{C}$.

Since $P_{q,\dagger}^{\mu,\lambda}$ is torsion free over R^{W_λ} and finitely generated as a right R^{W_λ}-module, we have $P_{q,\dagger}^{\mu,\lambda} \cong \mathsf{R}^{W_\lambda}$ as right R^{W_λ}-modules.

Step 2: Since

$$\operatorname{Hom}_{\mathsf{HC}_q(\mu,\lambda)}(P_q^{\mu,\lambda}, P_q^{\mu,\lambda}) \cong \operatorname{Hom}_{(\mathsf{R}^{W_\mu},\mathsf{R}^{W_\lambda})\text{-bimod}^\Lambda}(P_{q,\dagger}^{\mu,\lambda}, P_{q,\dagger}^{\mu,\lambda}),$$

The right multiplication of $\mathcal{W}_q^{\wedge\lambda} \cong \mathsf{R}^{W_\lambda}$ must induce an isomorphism

$$\mathsf{R}^{W_\lambda} \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{HC}_q(\mu,\lambda)}(P_q^{\mu,\lambda}, P_q^{\mu,\lambda}).$$

Step 3: From isomorphism $P_q^{\mu,\lambda} \otimes_{U_q^{fin,\lambda}} \Delta_{\mathsf{R}}(\lambda) \cong \Delta_{\mathsf{R}}(\mu)$, we have a $(\mathcal{W}_q^{\wedge\mu}, \mathcal{W}_q^{\wedge\lambda})$-linear morphism:

$$\mathrm{Hom}_{\mathsf{HC}_q(\mu,\lambda)}(P_q^{\mu,\lambda}, P_q^{\mu,\lambda}) \to \mathrm{Hom}_{O_{q,\mathsf{R}}}(\Delta_{\mathsf{R}}(\mu), \Delta_{\mathsf{R}}(\mu)). \tag{10.1}$$

The right multiplication of $\mathcal{W}_q^{\wedge\lambda} \xrightarrow{\sim} \mathsf{R}^{W_\lambda}$ on $P^{\mu,\lambda}$ maps to the right multiplication R^{W_λ} in $\Delta_{\mathsf{R}}(\mu)$. Combining with Step 2, we see that (10.1) is injective.

Therefore, the left multiplication of $\mathcal{W}^{\wedge\mu}$ on $P_q^{\mu,\lambda}$ is maps to the left multiplication of $\mathcal{W}_q^{\wedge\mu}$ on $\Delta_{\mathsf{R}}(\mu)$. Note that $\varepsilon_\mu : \mathcal{W}_q^{\wedge\mu} \to \mathsf{R}^{W_\mu}$ identifies the left $\mathcal{W}_q^{\wedge\mu}$-action on $\Delta_{\mathsf{R}}(\mu)$ with the right R^{W_μ}-action on $\Delta_{\mathsf{R}}(\mu)$. This impliest the left and right R^{W_μ}-actions on $P_q^{\mu,\lambda}$ coincides, hence we obtain the isomorphism $P_{q,\dagger}^{\mu,\lambda} \cong \mathsf{R}^{W_\lambda}$ in $(\mathsf{R}^{W_\mu}, \mathsf{R}^{W_\lambda})$-bimod$^\Lambda$.
(b) The same proof as in part (a) but involving the right versions $O_\epsilon^r, O_q^r.O_{q,\mathsf{R}}^r$. □

Theorem 10.3. *There is a full embedding of additive categories* $\mathsf{SB}_\hbar \to \mathrm{Hilt}_q(0,0)$.

Proof. For each $s \in I_a$, let λ_s be contained in the facet of the closure of the fundamental alcove associated to s. For each $x \in \Lambda$ which is in the decomposition $W_{ext} = \lambda \ltimes W_{aff}$, then $x \bullet_\ell 0$ is contained in the fundamental alcove. The following bimodules are contained in $\mathrm{Hilt}_q(0,0)$:

$$P_q^{x,0} := P_q^{x\bullet_\ell 0,0}, \quad P_q^{0,x} := P_q^{0,x\bullet_\ell 0}, \quad P_q^{0,\lambda_s} \otimes_{U_q^{fin,\lambda_s}} P_q^{\lambda_s,0}.$$

By Proposition 10.2, the images of these bimodules under $\bullet_\dagger : \mathsf{HC}_q(0,0) \to \mathsf{R}\text{-bimod}^\Lambda$ are

$$\mathsf{R}_x, \quad \mathsf{R}_{x^{-1}}, \quad \mathsf{R} \otimes_{\mathsf{R}^s} \mathsf{R},$$

in the degree $x, x^{-1}, 0$, respectively.

Combining with Proposition 8.1, we obtain the full embedding $\mathsf{SB}_\hbar \to \mathrm{Hilt}_q(0,0)$. □

Corollary 10.4. *There is a full embedding of additive categories* $\mathsf{SB} \to \mathrm{Hilt}_\epsilon(0,0)$.

Proof. • There is one-to-one correspondence between hilting bimodules in $\mathrm{Hilt}_q(0,0)$ and hilting bimodules in $\mathrm{Hilt}_\epsilon(0,0)$ via $M_q \mapsto M_e := M_q/\hbar M_q$.

• We will show that for any hilting bimodules $M_q, N_q \in \mathrm{Hilt}_q(0,0)$ then

$$\mathrm{Hom}_{\mathsf{HC}_\epsilon(0,0)}(M_\epsilon, N_\epsilon) \cong \mathrm{Hom}_{\mathsf{HC}_q(0,0)}(M_q, N_q)/\hbar\, \mathrm{Hom}_{\mathsf{HC}_q(0,0)}(M_q, N_q).$$

It is enough to prove that for any V_q, W_q tilting modules in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$, then

$$\mathrm{Hom}_{U_\epsilon^{fin,\underline{\chi}}\text{-rmod}^{G_\epsilon}}(V_\epsilon \otimes_{\mathbb{C}} U_\epsilon^{fin,\underline{\chi}}, W_\epsilon \otimes_{\mathbb{C}} U_\epsilon^{fin,\underline{\chi}}) \cong$$
$$\mathrm{Hom}_{U_q^{fin,\underline{\chi}}\text{-rmod}^{G_q}}(V_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}}, W_q \otimes_{\mathbb{C}[[\hbar]]} U_q^{fin,\underline{\chi}})/\hbar \text{ the object}$$

which is amount to prove the following for tilting module V_q

$$\mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_\epsilon, U_\epsilon^{fin,\underline{\chi}}) \cong \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}})/\hbar\, \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}})$$

Let consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\underline{\chi}}) \arrow[r, "\hbar"] \arrow[d] & \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin} \otimes_{\mathcal{W}_q} \mathcal{W}_q^{\wedge\underline{\chi}}) \arrow[r] \arrow[d] & \mathrm{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V_\epsilon, U_\epsilon^{fin,\underline{\chi}}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}}) \arrow[r, "\hbar"] & \mathrm{Hom}_{\check{U}_q(\mathfrak{g})}(V_q, U_q^{fin,\underline{\chi}}) \arrow[r] & \mathrm{Hom}_{\check{U}_\epsilon(\mathfrak{g})}(V_\epsilon, U_\epsilon^{fin,\underline{\chi}}) &
\end{tikzcd}$$

The first row is exact, i.e., the surjective map holds since U_q^{fin} has an exhausted good filtration and V_q is tilting. Therefore the second row is also exact, i.e., the surjective map holds. $\square$

Definition 10.5. Let $\mathcal{H}_q$ denote the image of $\mathsf{SB}_\hbar$ in $\mathrm{Hilt}_q(0,0)$. Let $\mathcal{H}_\epsilon$ be the image of SB in $\mathrm{Hilt}_\epsilon(0,0)$.

11. Simple objects in HC_P

Let HC_P denote the category $U_\epsilon^{fin,\underline{1}}\text{-rmod}^{G_\epsilon}$, here $\underline{1} \in \mathrm{Spec}\, Z_\cap$. We are going to classifiy the simple objects in HC_P. To simplify the notation, we replace the tensor product $- \otimes_{U_\epsilon^{fin,?}} -$ by $- \star -$. We denote $\mathrm{Rep}(\check{U}_\epsilon^*(\mathfrak{g}))$ and $\mathrm{Rep}(\check{U}_\epsilon(\mathfrak{g}))$ by $\mathrm{Rep}(G_\epsilon^*)$ and $\mathrm{Rep}(G_\epsilon)$, respectively

Lemma 11.1. *$P_\epsilon^{\mu,\lambda} \star V_\epsilon = \mathsf{pr}_{[\mu]}\Big(W_q(\mu-\lambda)) \otimes_{\mathbb{C}} V_\epsilon\Big)$ as functors from $O_\epsilon^{[\lambda]} \to O_\epsilon^{[\mu]}$.*

Proof. May need assumption on ℓ here. $\square$

Recall the Frobenious functor $\mathrm{Fr}^* : \mathrm{Rep}(G_\epsilon^*) \to \mathrm{Rep}(G_\epsilon)$. Under the assumption on ℓ, $\mathrm{Rep}(\check{U}_\epsilon^*(\mathfrak{g}))$ is just $\mathrm{Rep}(G)$.

Lemma 11.2. *For any diagonal bimodule $D \in \mathsf{HC}_\epsilon(\mu,\lambda)$ and $M \in \mathrm{Rep}_{[\lambda]}(\check{U}_\epsilon(\mathfrak{g}))$, $V \in \mathrm{Rep}(G)$, we have*
$$D \star (\mathrm{Fr}^*(V) \otimes M) \cong \mathrm{Fr}^*(V) \otimes (D \star M)$$

Proof. It is enough to prove for $P^{\mu,\lambda}(N)$ for some $N \in \mathrm{Rep}(G_\epsilon)$. $\square$

11.1. **Left (right)-trivial Harish-Chandra bimodules.** The action of $-w_0$ on P induce an action of w_0 on $P/(W_{ext}, \bullet_\ell)$ since if $\lambda = wt_\mu \bullet_\ell \lambda'$ then $-w_0\lambda = w_0ww_0t_{-w_0\mu} \bullet_\ell (-w_0\lambda')$. For any $\lambda \in P$ let $\lambda^* = -w_0\lambda$

Let $\iota : U_\epsilon^{ev} \to \check{U}_\epsilon$ be the natural map from the even part to the Lusztig form. Let $\varepsilon : U_\epsilon^{ev} \to \mathbb{C}$ be the counit of U_ϵ^{ev}.

Definition 11.3 (Definition/Lemma)**.** Let $V \in \mathrm{Rep}_{[\lambda]}(G_\epsilon)$
- V can be viewed as an object in $\mathsf{HC}_\epsilon(\lambda, 0)$ as follows:
$$uv = \iota(u)v, \qquad vu - \varepsilon(u)v,$$
here $u \in U_\epsilon^{fin} \subset U_\epsilon^{ev}$ and $v \in V$. We call this bimodules structure on V the *right-trivial Harish-Chandra bimodule* and denote it by V^r.
- V can be viewed as an object in $\mathsf{HC}_\epsilon(0,\lambda)$ as follows:
$$uv = \varepsilon(u)v, \qquad vu = \iota(S^{-1}(u))v,$$
here $u \in U_\epsilon^{fin}$ and $v \in V$. We call this bimodule structure on V is the *left-trivial Harish-Chandra bimodule* and denote it by V^l.

Lemma 11.4. *(a) The following functors are fully faithful:*
$$\bullet^r \ ; \ \mathrm{Rep}_{[\lambda]}(G_\epsilon) \to \mathsf{HC}_\epsilon(\lambda,0) \qquad V \mapsto V^r,$$
$$\bullet^l \ : \ \mathrm{Rep}_{[\lambda]}(G_\epsilon) \to \mathsf{HC}_\epsilon(0,\lambda^*) \qquad V \mapsto V^l$$
(b) For $V_1^r \in \mathsf{HC}_\epsilon(\lambda,0)$ and $V_2^l \in \mathsf{HC}_\epsilon(0,\mu)$ then $V_1^r \star V_2^l \in \mathsf{HC}_\epsilon(\lambda,\mu)$.
(c) For any $V \in \mathrm{Rep}(G)$ then $\mathrm{Fr}^*(V)^r \cong \mathrm{Fr}^*(V)^l$.

The next lemma consider concerns the action of translation bimodules on left (right)-trivial bimodules.

Lemma 11.5. *(a) For any $V \in \mathrm{Rep}(G_\epsilon)$ and $V_1, V_2 \in \mathrm{Rep}_{[\lambda]}(G_\epsilon)$ we have*

$$P_\epsilon^{\mu,\lambda}(V) \star V_i^r \cong (\mathsf{pr}_{[\mu]}(V \otimes V_1))^r, \qquad V_2^l \star P_\epsilon^{\lambda^*,\mu} \cong \mathsf{pr}_{[\lambda^*]}(V \otimes V_2))^l.$$

(b) For any $V \in \mathrm{Rep}_{[\lambda]}(G_\epsilon)$, we have

$$P_\epsilon^{\mu,\lambda} \star V^r \cong (P_\epsilon^{\mu,\lambda} \star V)^r, \qquad V^l \star P_\epsilon^{\lambda^*,\mu^*} \cong (P_\epsilon^{\mu,\lambda} \star V)^l.$$

Proof. □

11.2. **Heck action on $\mathrm{Rep}(G_\epsilon)$.** [6]

Let H_e be the extended affine Hecke algebra. Let M^{asph} be the antispherical module of H_e. Let $\mathrm{Tilt}_{[0]}(G_\epsilon)$ be the subcategory of tilting modules in $\mathrm{Rep}_{[0]}(G_\epsilon)$. Recall the Steinberg module $\mathbf{St}_\epsilon = W_\epsilon((\ell-1)\rho)$.

Remark 11.6. (a) Any simple modules $L_\epsilon(\lambda)$ with $\lambda \in (P/\ell P)_+$ is tilting.
(b) The block $\mathrm{Rep}_{[-\rho]}(G_\epsilon)$ is semisimple with simple objects $\mathbf{St}_\epsilon \otimes \tilde{\mathrm{Fr}}^*(V)$ for any irreducible module $V \in \mathrm{Rep}(G)$.

Lemma 11.7. *We have $K_0(\mathrm{Tilt}_{[0]}(G_\epsilon)) \cong M^{asph}$ as H_e-modules.*

Proof. Known folklore but can not find the reference. □

Corollary 11.8. *(a) For any simple object $L_\epsilon(\lambda) \in \mathrm{Rep}_{[0]}(G_\epsilon)$, there $P \in \mathcal{H}_\epsilon$ such that $P_\epsilon^{-\rho,0} \star P \star L_\epsilon(\lambda) \neq 0$.*
(b) For any simple object $L_\epsilon(\lambda) \in \mathrm{Rep}_{[0]}(G_\epsilon)$, there $P \in \mathcal{H}_\epsilon$ such that $L_\epsilon(\lambda)$ is an composition factor of $P \star \mathbb{C}$, here $\mathbb{C}$ is the trivial module.
(c) For any $V \in \mathrm{Rep}(G)$, there $P \in \mathcal{H}_\epsilon$ such that $\mathbb{C}$ is a composition factor of $P \star \tilde{\mathrm{Fr}}^*(V)$.

Proof. (a) First, we assume $\lambda \in W_{ext} \bullet_\ell 0 \cap (P/\ell P)_+$ then $L_\epsilon(\lambda)$ is tilting. By [16, §1.3], the lowest canonical right cell in M^{asph} contain elements of the form $\Omega_0 := \{\rho x w | x \in P_+, w \in W, R(w) \subset L(x)\}$ [7], here $R(x) = \{s \in W | xs \leq x\}$ and $L(x) := \{s \in W | sx \leq x\}$. Therefore we can find $P \in \mathcal{H}_\epsilon$ such that $P \star L_\epsilon(\lambda)$ contain tilting direct summand $T_\epsilon(\lambda')$ with $\lambda' = \rho x w \bullet_\ell 0$ for some $\rho x w \in \Omega_0$. Then $P_\epsilon^{-\rho,0} \star P \star L_\epsilon(\lambda)$ contains a composition factor $T_\epsilon(\rho x w \bullet_\ell (-\rho)) = T_\epsilon((\ell-1)\rho + \ell x) \neq 0$, hence $P_\epsilon^{-\rho,0} \star P \star L_\epsilon(\lambda) \neq 0$.

For general λ, let $\lambda = \lambda_0 + \ell\lambda_1$ with $\lambda_0 \in (P/\ell P)_+ \cap W_{ext} \bullet_\ell 0$, then $L_\epsilon(\lambda) \cong L_\epsilon(\lambda_0) \otimes \tilde{\mathrm{Fr}}^*(L(\lambda_1))$. Then choose P such that $P \star L_\epsilon(\lambda_0) \neq 0$ will work. (b) Since $[\mathbb{C}]$ generates H_e-module M^{asph}, there is $p \in \mathcal{H}_\epsilon$ such that $P \star \mathbb{C}$ contains a tilting summand $T_\epsilon(\lambda)$. This implies part (b).
(c) We can assume V is simple, then by part (b), there is $P \in \mathcal{H}_\epsilon$ such that $\tilde{\mathrm{Fr}}^*(V^*)$ is a composition factor of $P \star \mathbb{C}$. Then we see that $\tilde{\mathrm{Fr}}^*(V^*) \otimes \tilde{\mathrm{Fr}}^*(V)$ is a composition factor of $P \star \tilde{\mathrm{Fr}}^*(V)$. This implies that $\mathbb{C}$ is a composition factor of $P \star \tilde{\mathrm{Fr}}^*(V)$. □

11.3. **Simple Harish-Chandra bimodules.**

Recall the small quantum group $\dot{\mathfrak{u}}$ (notation conflict: use $\mathfrak{u}$ for small quantum group above) as the Hopf subalgebra of $\check{U}_\epsilon(\mathfrak{g})$ generated by $\{\tilde{E}_i, \tilde{F}_i, K^{2\lambda}\}_{1 \leq i \leq r}^{\lambda \in P}$. Let $\mathfrak{u}$ be the quotient algebra of $\check{U}_\epsilon^{ev}$ at the point $1 \in G_0^d \cong \mathrm{Spec}\, Z_{Fr}$, which is the same as the quotient of U_ϵ^{fin} at the point $1 \in G_0^d \cong \mathrm{Spec}\, Z_{Fr}$. Then we see that $\mathfrak{u}$ is the quotient algebra of $\dot{\mathfrak{u}}$ by the two-sided ideal generated by $I; = \{K^{2\ell\lambda} - 1\}_{\lambda \in P}$. Since I is also a Hopf ideal in $\dot{\mathfrak{u}}$, we see that $\mathfrak{u}$ is also a Hopf algebra.

[6] Need to be a bit more careful with the extended affine Hecke algebra

[7] Need to double check it. Think about the equivariant sheaves on the Springer resolution of nilpotent cone.

Following [8, Proposition 5.11], and the assumption on ℓ, the irreducible $\mathfrak{u}$-modules are paramatrized by $P/\ell P$. Let

$$(P/\ell P)_+ := \{\lambda \in P \mid 0 \leq (\lambda, \alpha_i^\vee) \leq \ell - 1 \;\forall\; 1 \leq i \leq r\}$$

the bu restriction to $\mathfrak{u}$-action, irreducible $\check{U}_\epsilon(\mathfrak{g})$-modules $L_\epsilon(\lambda), \lambda \in (P/\ell P)_+$ gives all irreducible $\mathfrak{u}$-modules.

Lemma 11.9. *The Harish-Chandra center gives a decomposition of categories*

$$\mathrm{Rep}(\mathfrak{u}) = \bigoplus_{[\lambda] \in P/(W_{ext}, \bullet_\ell)} \mathrm{Rep}_{[\lambda]}(\mathfrak{u}).$$

Furthermore, all simple modules in $\mathrm{Rep}_{[\lambda]}(\mathfrak{u})$ *can be obtained from simpe modules* $L_\epsilon(\lambda), \lambda \in (P/\ell P)_+$, *in* $\mathrm{Rep}_{[\lambda]}(G_\epsilon)$.

Definition 11.10. Let $\mathfrak{u} \otimes \mathfrak{u}^{op}\text{-mod}^{G_\epsilon}$ be the category of $\check{U}_\epsilon$-equivariant $\mathfrak{u} \otimes \mathfrak{u}^{op}$-modules with the rational $\check{U}_\epsilon(\mathfrak{g})$-action.

Note that we have the algebra morphism $\dot{\mathfrak{u}} \twoheadrightarrow \mathfrak{u}$ so any $\mathfrak{u} \otimes \mathfrak{u}^{op}$-modules carries an adjoint action of $\dot{\mathfrak{u}}$.

Definition 11.11. Let $\mathsf{HC}(\mathfrak{u})$ be the full subcategories of $\mathfrak{u} \otimes \mathfrak{u}^{op}\text{-mod}^{G_\epsilon}$ consisting of all objects on which the adjoint action of $\dot{\mathfrak{u}}$ coincides with the adjoint action of $\dot{\mathfrak{u}}$ in $\check{U}_\epsilon$.

Remark 11.12. Simple Harish-Chandra bimodules in HC_P are one-to-one correspondent to simple objects in $\mathsf{HC}(\mathfrak{u})$. Let λ_1, λ_2 be two weights in $(P/\ell P)_+$. Let $V \in \mathrm{Rep}(G)$. Then the following objects belongs to $\mathsf{HC}(\mathfrak{u})$:

$$L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \mathrm{Fr}^*(V)$$

with $\check{U}_\epsilon$-equivariant $\mathfrak{u} \otimes \mathfrak{u}^{op}$-modules as follows:

- $\check{U}_e$ acts via the action on tensor products.
- for $x_1, x_2 \in \mathfrak{u}$ and $v_1 \in L_\epsilon(\lambda_1)^l$, $v_2 \in L_\epsilon(\lambda_2)^l$, $v \in \mathrm{Fr}*(V)$ then
$$x_1(v_1 \otimes v_2 \otimes v)x_2 = (x_1 v_1) \otimes (v_2 x_2) \otimes v.$$

Lemma 11.13. *The set* $\{L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \mid (\lambda_1, \lambda_2) \in (P/\ell P)_+^{\oplus 2}\}$ *classifies all simple* $\mathfrak{u} \otimes \mathfrak{u}^{op}$*-modules up to isomorphism.*

Lemma 11.14. *The set*

$$S := \left\{ L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_1)^l \otimes \tilde{\mathrm{Fr}}^*(V) \mid \lambda_1, \lambda_2 \in (P/\ell P)_+, V \in \mathrm{Irr}(G) \right\}$$

contains pair-wise non-isomorphism simple objects in $\mathsf{HC}(\mathfrak{u})$. *Any simple object in* $\mathsf{HC}(\mathfrak{u})$ *is isomorphic to one objects in* S.

Proof. *Step* 1*:* The object $L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)$ is simple. Let M be its non-zero subobject and consider the injective map

$$M \hookrightarrow L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)$$

This gives us an injective map of vector spaces:

(11.1)
$$\mathrm{Hom}_{\mathfrak{u} \otimes \mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M) \hookrightarrow \mathrm{Hom}_{\mathfrak{u} \otimes \mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V))$$

Both components of (11.1) are naturally $\check{U}_\epsilon$-modules so that (11.1) is a morphism of $\check{U}_\epsilon$-modules. Furthermore, the actions of $\check{U}_\epsilon$ on both components factor through the Frobenious morphism $\tilde{\mathrm{Fr}} : \check{U}_\epsilon \to \check{U}_{\mathbb{C}}(\mathfrak{g})$.

Since $L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l$ is a simple $\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}$-module, we have

$$M \cong L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M) \text{ as } \mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}\text{-modules} \tag{11.2}$$

and

$$\mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M) \neq 0$$
$$\mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)) \cong \tilde{\mathrm{Fr}}^*(V) \text{ as } \check{U}_\epsilon\text{-modules}$$

Moreover, V is an irreducible representation of G. Therefore, (11.1) is bijective. Combining with (11.2), we see that $M = L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)$. So $L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)$ is a simple object in $\mathsf{HC}(\mathfrak{u})$.

Step 2: The objects in S are pair-wise non-isomorphic. Let $(\lambda_1,\lambda_2) \neq (\lambda_1^{'},\lambda_2^{'}) \in (P/\ell P)_+^{\oplus 2}$ and $V, V' \in \mathrm{Irr}(G)$. We have

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{HC}(\mathfrak{u})}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V), L(\lambda_1^{'})^r \otimes L(\lambda_2^{'})^l \otimes \tilde{\mathrm{Fr}}^*(V'))\\
&= \mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V), L(\lambda_1^{'})^r \otimes L(\lambda_2^{'})^l \otimes \tilde{\mathrm{Fr}}^*(V'))^{\check{U}_\epsilon}\\
&= \begin{cases} 0 & \text{If } (\lambda_1,\lambda_2) \neq (\lambda_1^{'},\lambda_2^{'}) \\ \mathrm{Hom}_{\mathbb{C}}(\tilde{\mathrm{Fr}}^*(V), \tilde{\mathrm{Fr}}^*(V'))^{\check{U}_\epsilon} & \text{If } (\lambda_1,\lambda_2) = (\lambda_1^{'},\lambda_2^{'}) \end{cases}
\end{aligned}$$

So $L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V) \cong L(\lambda_1^{'})^r \otimes L(\lambda_2^{'}) \otimes \tilde{\mathrm{Fr}}^*(V')$ iff $(\lambda_1,\lambda_2) = (\lambda_1^{'},\lambda_2^{'})$ and $V \cong V'$ as G-representations.

Step 3: Any simple object in $\mathsf{HC}(\mathfrak{u})$ is isomorphic to one object in S. Let M be a simple object in $\mathsf{HC}(\mathfrak{u})$. Any simple $\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}$-modules is isomorphic to simple object of the form

$$L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l,$$

for $(\lambda_1,\lambda_2) \in (P/\ell P)_+^{\oplus 2}$. Therefore, we can find $(\lambda_1,\lambda_2) \in (P/\ell P)_+^{\oplus 2}$ such that

$$\mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M) \neq 0.$$

Note that $\mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M)$ is naturally $\check{U}_\epsilon$-modules, furthermore, the $\check{U}_\epsilon$-action factors through an action of $\check{U}(\mathfrak{g})$. Let consider the following object in $\mathsf{HC}(\mathfrak{u})$

$$A = L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M).$$

We have

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{HC}(\mathfrak{u})}(A, M)\\
&= \mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(A, M)^{\check{U}_\epsilon}\\
&= \Big(\mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M) \otimes \mathrm{Hom}_{\mathfrak{u}\otimes\mathfrak{u}^{\mathrm{op}}}(L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l, M)^*\Big)^{\check{U}_\epsilon}\\
&\neq 0
\end{aligned}$$

Therefore, there is a nonzero morphism $A \to M$, which must be surjective since M is simple. On the other hand, as in Step 1, we see that any quotient objects of A is of the form

$$L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_1)^l \otimes \tilde{\mathrm{Fr}}^*(V)$$

For some G-representation V. Therefore, M is isomorphic to an object in S. □

Definition 11.15. Let M_1, M_2 be two simple objects in $\mathsf{HC}_\epsilon(0,0)$. We say $M_1 \prec M_2$ if there are bimodules in $\mathcal{H}_\epsilon$ such that M_1 is the composition factor of the object $P_1 \star M_2 \star P_2$. We say that $M_1 \sim M_2$ if $M_1 \prec M_2$ and $M_2 \prec M_1$.

Lemma 11.16. *All simple objects in* $\mathsf{HC}_\epsilon(0,0)$ *are equivalent to each other.*

Proof. We will prove that any simple object M are equivalent to the trivial bimodules $\mathbb{C}$.

Step 1: We will show that $M \prec \mathbb{C}$. By Lemma 11.14, $M = L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l$ for some $\lambda_1, \lambda_2 \in W_{ext} \bullet_\ell 0$. Note that $\mathbb{C} \cong \mathbb{C}^r \otimes \mathbb{C}^l$. There $P_1, P_2 \in \mathcal{H}_\epsilon$ such that $L_\epsilon(\lambda_1)^r$ is a composition factor of $P_1 \star \mathbb{C}^r$ and $L_\epsilon(\lambda_2)^l$ is a composition factor of $\mathbb{C}^l \star P_2$. Therefore, M is a composition factor of $P_1 \star \mathbb{C} \star P_2$, equivalently, $M \prec \mathbb{C}$.

Step 2: We will show that $\mathbb{C} \prec M$. By Lemma 11.14, $M = L_\epsilon(\lambda_1)^r \otimes L_\epsilon(\lambda_2)^l \otimes \tilde{\mathrm{Fr}}^*(V)$ for $\lambda_1, \lambda_2 \in (P/\ell P)_+ \cap W_{ext} \bullet_\ell 0$ and $V \in \mathrm{Irr}(G)$. By Corollary 11.8, there are $P_1, P_2 \in \mathcal{H}_\epsilon$ such that

$$P_\epsilon^{-\rho,0} \star P_1 \star L_\epsilon(\lambda_1)^r, \qquad L_\epsilon(\lambda_2)^l \star P-2 \star P_\epsilon^{0,-\rho} \neq 0.$$

Hence,

$$P_\epsilon^{\rho,0} \star P_1 \star M \star P_2 \star P_\epsilon^{0,-\rho} \neq 0. \tag{11.3}$$

Step 2': We will show that for any simple object N in $\mathsf{HC}_\epsilon(-\rho,\rho)$, there are $P_3, P_4 \in \mathcal{H}_\epsilon$ such that $\mathbb{C}$ is the composition factor of $P_3 \star P_\epsilon^{0,\rho} \star N \star P_\epsilon^{-\rho,0} \star P_4$. By Corollary 11.8, we can assume $N := \mathbf{St}_\epsilon^r \otimes \mathbf{St}_\epsilon^l \otimes \tilde{\mathrm{Fr}}^*(V)$ for some $V \in \mathrm{Irr}(G)$. Then

$$P_\epsilon^{0,\rho} \star N \star P_\epsilon^{-\rho,0} \cong (T_{\rho\to 0}\mathbf{St}_\epsilon)^r \otimes (T_{-\rho\to 0}\mathbf{St}_\epsilon)^l \otimes \tilde{\mathrm{Fr}}^*(V)$$

Since $T_{-\rho\to 0}\mathbf{St}_\epsilon$ has a composition factor $L_\epsilon(\ell\rho)$, it follows that $L_\epsilon(\ell\rho)^r \otimes L_\epsilon(\ell\rho)^l \otimes \tilde{\mathrm{Fr}}^*(V)$ is a composition factor of $P_\epsilon^{0,-\rho} \star N \star P_\epsilon^{-\rho,0}$.

On the other hand,

$$L_\epsilon(\ell\rho)^r \otimes L_\epsilon(\ell\rho)^l \otimes \tilde{\mathrm{Fr}}^*(V) \cong \Big(\tilde{\mathrm{Fr}}^*(L(\rho) \otimes L(\rho) \otimes V)\Big)^r$$

By Corollary 11.8, there is $P_2 \in \mathcal{H}_\epsilon$ such that $\mathbb{C}$ is a composition factor of $P_3 \star \tilde{\mathrm{Fr}}^*(L(\rho) \otimes L(\rho) \otimes V)$. Hence $\mathbb{C}$ is a composition factor of $P_2 \star P_\epsilon^{0,-\rho} \star N \star P_\epsilon^{-\rho,0}$.

Now we can finish Step 2. Pick a composition factor N in (11.3) and P_3 as in Step $2'$, we see that $\mathbb{C}$ is a composition factor of

$$P_3 \star P_\epsilon^{0,-\rho} \star P_\epsilon^{-\rho,0} \star P_1 \star M \star P_2 \star P_\epsilon^{0,-\rho} \star P_\epsilon^{-\rho,0}.$$

This implies that $\mathbb{C} \prec M$. $\square$

11.4. **The second main result.**

Theorem 11.17. *Under the full embedding*

$$\mathsf{SB}_\hbar \to \mathrm{Hilt}_q(0,0), \qquad \mathsf{SB} \to \mathrm{Hilt}_\epsilon(0,0),$$

the smallest two-sided cell of $\mathsf{SB}_\hbar$ *and* SB *maps to the full subcategories of projective objects in* $\mathsf{HC}_q(0,0)$ *and* $\mathsf{HC}_\epsilon(0,0)$*, respectively.*

Proof. Step 1: Discuss about the Krull-Schmidt properties of $\mathsf{HC}_q(0,0)$ and $\mathsf{HC}_\epsilon(0,0)$. The goal: so that we can talk about projective cover of simple objects in $\mathsf{HC}_q(0,0)$ and $\mathsf{HC}_\epsilon(0,0)$. Note that the simple objects in $\mathsf{HC}_q(0,0)$ are simple objects in $\mathsf{HC}_\epsilon(0,0)$.

Step 2: Show that for any projective object Q_1, Q_2 in $\mathsf{HC}_q(0,0)$, there is $P_1, P_2 \in \mathcal{H}_q$ such that Q_2 is a direct summand of $P_1 \star Q_1 \star P_2$ by using Lemma 11.16. Same for projective objects in $\mathsf{HC}_\epsilon(0,0)$.

Step 3: Show that there is a projective object Q in $\mathsf{HC}_q(0,0)$ such that $Q_\dagger \in \mathsf{SB}_\hbar$. This is done by showing that $U_q^{fin,-\rho}$ is projective in $\mathsf{HC}_q(-\rho,-\rho)$ therefore $P_q^{0,-\rho} \star P_q^{-\rho,0}$ is projective in $\mathsf{HC}_q(0,0)$. On the other hand, $(P_q^{0,-\rho} \star P_q^{-\rho,0})_\dagger \cong \mathsf{R} \otimes_{\mathsf{R}^W} \mathsf{R}$ is contained in the smallest two-sided cell of $\mathsf{SB}_\hbar$.

We show that $U_q^{fin,-\rho}$ is projective as follows. Let $R := \mathbb{C}[[\hbar]]$.

- Recall the isomorphism $U_q^{fin} \cong O_q[G]$. For any $V \in \mathrm{Rep}^{fd}(\check{U}_q(\mathfrak{g}))$, we have a morphism $V \otimes_R V^* \to O_q[G]$ in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$, here $V^* := \mathrm{Hom}_R(V, R)$, defined by $v \otimes f \mapsto c_{f, K^{-2\rho}v}$. For V free of finite rank over R with a basis $\{v_i\}$ and dual basis $\{v_i^*\}$, then the element $c_V := \sum_i c_{v_i^*, K^{-2\rho}v_i} \in O_q[G]^{\check{U}_q} \xrightarrow{\sim} \mathcal{W}_q \subset U_q^{fin}$.

  Let $V = W_q(\lambda)$, then the representation of $c_\lambda := c_{W_q(\lambda)}$ under the Harish-Chandra morphism $\mathcal{W}_q \cong R\Big[K^{\pm 2w_1}, \dots K^{\pm 2\omega_r}\Big]^{W_\bullet}$ is
$$\sum_{\mu \in P_{+,\lambda}} \mathrm{rank}\big(W_q(\lambda)_\mu\big) \sum_{\mu' \in W\mu} q^{(\rho, 2\mu')} K^{2\mu'},$$
  here $P_{+,\lambda}$ is the set of dominant weights in $W_q(\lambda)$.

  We see that the evaluation of c_λ at the point $-\rho \in \mathrm{Spec}\, \mathcal{W}_\epsilon$ is equal to $\mathrm{rank}(W_q(\lambda)) \neq 0$. Hence c_λ is an invetible element in $\mathcal{W}_q^{\wedge -\rho}$.
- Let $\{v_i\}$ be a basis of $\mathbf{St}_q$, then $\{v_i^*\}$ and $\{v_i^{**}\}$ be the dual basis in $\mathbf{St}_q^*$ and $\mathbf{St}_q^{**}$, respectively. We have the following morphisms in $\mathrm{Rep}(\check{U}_q(\mathfrak{g}))$:
$$\mathbf{St}_q^* \otimes_R \mathbf{St}_q^{**} \to O_q[G] \cong U_q^{fin}$$
$$R \xrightarrow{\mathrm{coev}_{\mathbf{St}_q^* \otimes_R \mathbf{St}_q}} (\mathbf{St}_q^* \otimes_R \mathbf{St}_q) \otimes_R (\mathbf{St}_q^* \otimes_R \mathbf{St}_q)^* \cong (\mathbf{St}_q^* \otimes_R \mathbf{St}_q) \otimes_R (\mathbf{St}_q^* \otimes_R \mathbf{St}_q^{**})$$
  Combining these two morphisms then using the evaluation map $\mathbf{St}_q^* \otimes_R \mathbf{St}_q \to R$, we get
$$R \to \mathbf{St}_q^* \otimes_R \mathbf{St}_q \otimes_R U_q^{fin} \xrightarrow{\mathrm{ev}_{\mathbf{St}_q} \otimes \mathrm{Id}} U_q^{fin},$$
  with the image of $1 \in R$ is $c_{\mathbf{St}_q^*} \in \mathcal{W}_q \subset U_q^{fin}$. So we have a composition in $U_q^{fin}\text{-rmod}^{G_q}$
$$U_q^{fin} \to \mathbf{St}_q^* \otimes \mathbf{St}_q \otimes_R U_q^{fin} \to U_q^{fin},$$
  which then gives us a composition in $U_q^{fin,-\rho}\text{-rmod}^{G_q}$
$$U_q^{fin,-\rho} \to \mathbf{St}_q^* \otimes_R \mathbf{St}_q \otimes_R U_q^{fin,-\rho} \to U_q^{fin,-\rho}, \tag{11.4}$$
  with the image of $1 \in U_q^{fin,-\rho}$ is $c_{\mathbf{St}_q^*} \in \mathcal{W}_q^{\wedge -\rho} \subset U_q^{fin,-\rho}$. Since $c_{\mathbf{St}_q^*}$ is invertible in $\mathcal{W}_q^{\wedge -\rho}$, the composition (11.4) is identity, hence $U_q^{fin,-\rho}$ is a direct summand of $\mathbf{St}_q^* \otimes_R \mathbf{St}_q \otimes_R U_q^{fin,-\rho}$, but the latter is projective in $U_q^{fin,-\rho}\text{-rmod}^{G_q}$, hence $U_q^{fin,-\rho}$ is projective in $U_q^{fin,-\rho}\text{-rmod}^{G_q}$, hence projective in $\mathsf{HC}_q(-\rho, -\rho)$.

$\square$

12. Non-commutative Springer resolution

Introduce the Non-commutative Springer resolution A. Let us recall the Grothendieck-Springer resolution

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl] \arrow[dr] & \\ \mathfrak{g} \arrow[dr] & & \mathfrak{h} \arrow[dl] \\ & \mathfrak{h}/W & \end{tikzcd}$$

Then $\tilde{\mathfrak{g}}$ is a resolution of $\mathfrak{g} \times_{\mathfrak{h}/W} \mathfrak{h}$, which is an isomorphism over $\mathfrak{g}^{reg} \times_{\mathfrak{h}/W} \mathfrak{h}$.

Introduce the tilting bundle $\mathcal{E}$ on $\tilde{\mathfrak{g}}$: $\mathcal{E} = \mathcal{O}_{\tilde{\mathfrak{g}}} \oplus \bigoplus \mathcal{R}_{i_1} \dots \mathcal{R}_{i_k} \mathcal{O}_{\tilde{\mathfrak{g}}}$, here $\mathcal{R}_i$ are the reflection functors constructed in [1, §2.3]. Then $\mathsf{A} := \mathrm{End}_{\mathcal{O}_{\tilde{\mathfrak{g}}}}(\mathcal{E})$, so A is an algebra over $\mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}[\mathfrak{h}]$. Let

$$\mathsf{A}^{\wedge_0} := \mathsf{A} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}]^{\wedge_0}, \qquad \mathbb{C}[\mathfrak{g}[^{\wedge_0} := \mathbb{C}[\mathfrak{g}] \otimes_{\mathbb{C}[\mathfrak{h}/W]} \mathbb{C}[\mathfrak{h}/W]^{\wedge_0}.$$

Restricting to $\mathfrak{g}^{reg}$, then $\mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}$ is a sheaf of Azumaya algebras over

$$X := (\mathfrak{h} \times_{\mathfrak{h}/W} \times \mathfrak{g}^{reg} \times_{\mathfrak{h}/W} \mathfrak{h})^{\wedge_{0,0}}$$

with the splitting bundle $\mathcal{E} \otimes \mathcal{E}^\vee|_X$. Let $\mathcal{S}$ be the Kostant section in $\mathfrak{g}^{reg}$. Let $\mathbb{J}$ be the group scheme of centralizer of G on $\mathfrak{g}$ and let $\mathbb{I} := \mathbb{J} \times_{\mathfrak{g}} \mathcal{S}$ be the restriction of $\mathbb{J}$ on the Kostant section $\mathcal{S}$. We denote the pull back of $\mathbb{I}$ under $(\mathfrak{h} \times_{\mathfrak{h}/W} \mathcal{S} \times_{\mathfrak{h}/W} \mathfrak{h})^{\wedge_{0,0}} \to \mathcal{S}$ by the same notation. So we have a composition of functors:

$$\begin{aligned} \mathfrak{R} : \mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}\text{-mod}^G \to \mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}|_X\text{-mod}^G \xrightarrow{\sim} X\text{-mod}^G \\ \xrightarrow{\sim} (\mathfrak{h} \times_{\mathfrak{h}/W} \mathcal{S} \times_{\mathfrak{h}/W} \mathfrak{h})^{\wedge_{0,0}}\text{-mod}^{\mathbb{I}} \end{aligned} \tag{12.1}$$

- The first functor is the restriction to the open subset X.
- The second functor is obtained via the splitting of Azumaya algebras, hence is an equivalence.
- The third functor is obtained via restriction of the Kostant slice. Need to check/understand why it is an equivalence

Lemma 12.1. *The functor $\mathfrak{R}$ is fully faithful on the subcategories of projective objects*

Proof. The projective objects are direct summand of object of the form $\mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0} \otimes V$ for some $V \in \mathrm{Rep}(G)$. ... □

Theorem 12.2. *There is an equivalence of abelian categories:*

$$\mathsf{HC}_\epsilon(0,0) \cong \mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}\text{-mod}^G.$$

Proof. Let $\mathcal{C} := \mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}\text{-mod}^G$

Step 1: Show that the category of projective objects in $\mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}\text{-mod}^G$ is equivalent to the the smallest two-sided cell in SB using the Abe's realization.

- Describe a collection of objects H in $\mathsf{A}^{\wedge_0} \otimes_{\mathbb{C}[\mathfrak{g}]^{\wedge_0}} \mathsf{A}^{op\wedge_0}\text{-mod}^G$ that maps to generators of SB in terms of Abe's realization. $\mathcal{R}_i$ is exact on the exotic t-structure.
- Show that tensor with these objects on the left or right preserves the categories of projective objects.

- The object Ω_0 corresponding to $\mathcal{O}_{\tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}}$ is projective and its image is contained in the smallest two-sided cell in SB (corresponding to element $\mathsf{R} \otimes_{\mathsf{R}^W} \mathsf{R}$ in $\mathsf{SB}_\hbar$). Use the description of $\mathcal{E}$ to show that any projective objects can be obtained as follows: Apply tensor product of Ω_0 with elements in H then taking direct summand. This implies Step 1.

Step 2: Both abelian categories $\mathsf{HC}_\epsilon(0,0)$ and $\mathcal{C}$ have enough projective objects. Furthermore, there is an equivalence of additive categories of projective objects $\mathcal{P}_{\mathsf{HC}_\epsilon(0,0)} \cong \mathcal{P}_{\mathcal{C}}$ via the identification with the smallest two-sided cell in SB. Hence, this equivalence extends to an equivalence of abelian categories we want. $\square$

13. Generalization

- The case of even order roots of unity: Restrict to the root lattices.
- The block of Harish-Chandra bimodules with non-integral Harish-Chandra characters.
- Enhanced version of restriction functors with equivariant structures.

14. Appendices

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