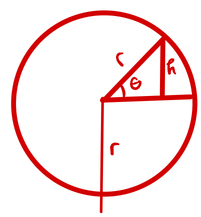
Math 199 CD2: Midterm 3 review

October 12, 2021

1 Related Rates

1. A Ferris wheel with a radius of 15 m is rotating at a rate of one revolution every two minutes. Exactly how fast (in m/min) is a rider rising when his seat is 27 m above ground level? $\frac{d\theta}{dt} = \frac{2\pi}{2\text{ min}} = \pi$ rad/min

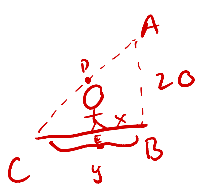
$\frac{h}{r} = \sin\theta \Rightarrow \frac{h}{15} = \sin\theta \Rightarrow \frac{1}{15}\frac{dh}{dt} = \cos\theta \frac{d\theta}{dt}$



At height 27 m, $h = 12$, $\cos\theta = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{9}{15} = \frac{3}{5}$

$\Rightarrow \frac{dh}{dt} = 15\cos\theta\frac{d\theta}{dt} = 15 \cdot \frac{3}{5} \cdot \pi = 9\pi$ (m/min)

2. A 5-foot girl is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamp) moving?

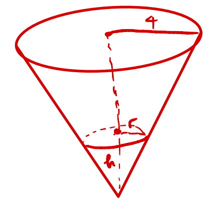


$\frac{AB}{DE} = \frac{y}{y-x} = \frac{20}{5} = 4 \Rightarrow 3y = 4x \Rightarrow 3\frac{dy}{dt} = 4\frac{dx}{dt}$, $\frac{dx}{dt} = 6$ (feet/sec)

$\Rightarrow \frac{dy}{dt} = \frac{4 \cdot 6}{3} = 8$ feet/sec

3. A small funnel in the shape of a cone is being emptied of fluid at the rate of 12cubic centimeters per second. The height of the funnel is 20 centimeters and the radius of the top is 4 centimeters. How fast is the fluid level dropping when the level stands 5 centimeters above the vertex of the cone?

$\frac{r}{4} = \frac{h}{20} \Rightarrow r = \frac{h}{5} \Rightarrow \frac{dr}{dt} = \frac{1}{5}\frac{dh}{dt}$ | $\frac{dV}{dt} = 12$,



$V = \frac{1}{3}\pi r^2 h$, when $h = 5 \Rightarrow r = 1$

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(2r\frac{dr}{dt}\cdot h + r^2\frac{dh}{dt}\right)$$

$$12 = \frac{1}{3}\pi\left(2\cdot 1\cdot\frac{1}{5}\frac{dh}{dt}\cdot 5 + \frac{dh}{dt}\right)$$

$$\frac{36}{\pi} = 3\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{12}{\pi}$$

4. A particle moves on the hyperbola $x^2 - 18y^2 = 9$ in such a way that its y-coordinate increases at a constant rate of 9 units per second. How fast is its x-coordinate changing when $x = 9$?

$2x\frac{dx}{dt} - 36y\frac{dy}{dt} = 0$ at $x = 9 \Rightarrow y = \pm 2$.

$$\Rightarrow 18\frac{dx}{dt} - 36(\pm 2)\,9 = 0 \Rightarrow \frac{dx}{dt} = \frac{(\pm)36.2.9}{18}$$

$$= \pm 36.$$

2 Log/Exp differentiation

1. Find the derivative of the following function:

(a) $y = x^x$

$\log y = x\log x$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \log x + 1 \Rightarrow \frac{dy}{dx} = x^x\log x + x^x$

(b) $y = (4x+1)^{5x}$

$\log y = 5x\,(4x+1)$

$\frac{1}{y}\frac{dy}{dx} = 5(4x+1) + 5x.4$

$= 40x+5$

$\Rightarrow \frac{dy}{dx} = (40x+5)(4x+1)^{5x}$

(c) $y = x^{x^2+4}$

$\log y = (x^2+4)\log x$

$\frac{1}{y}\frac{dy}{dx} = 2x\log x + \frac{(x^2+4)}{x}$

$\Rightarrow \frac{dy}{dx} = 2x\log x \cdot x^{x^2+4} + (x^2+4)x^{x^2+3}$

3 Rolle's Theorem, MVT, IVT

Please state all 3 theorem before you even attempt

Consider the polynomial $f(x) = 5x^3 - 2x^2 + 3x - 4$. Prove that $f(x)$ has a zero between 0 and 1 that is the only zero of $f(x)$.

$f(0) = -4 < 0 \overset{IVT}{\Rightarrow} f(x) = 0$ for some $x \in (0,1)$

$f(1) = 2 > 0$

$f'(x) = 15x^2 - 4x + 3 > 0 \Rightarrow f(x)$ is $\uparrow$

$\Rightarrow$ Can take on the value 0 at most 1.

4 Exponential Growth or Decay

1. A certain chemical decomposes exponentially. Assume that 200grams becomes 50 grams in 1hour. How much will remain after 3 hours?

   let y be # of grams at time t $\Rightarrow$ $y = y_0 e^{kt}$ $\Rightarrow$ $50 = 200 \cdot e^{k}$

   y_0 be # of grams at time $t=0$

   $\Rightarrow e^{k} = 1/4$

   $\Rightarrow k = \ln(1/4) = -\ln(4)$

   $\Rightarrow y = 200\, e^{-\ln 4 (t)}$ $\Rightarrow y(3) = 200 \cdot e^{-\ln(4) \cdot 3}$ $= \frac{200}{64} = 3.125$.

2. If the world population in 1980 was 4.5 billion and if it is growing exponentially with a growth constant $K = 0.04 \ln 2$, find the population in the year 2030.

   y = population in (m.)

   $y = 4.5\, e^{0.04 \ln(2)\, t}$

   $\Rightarrow y(50) = 4.5\text{ m.} \left(e^{\ln(2)}\right)^{0.04 \times 50}$

   $= 4.5\text{m} \cdot 2^{0.04 \times 50}$ $= 18$ m (people)

3. Fruit flies are being bred in an enclosure that can hold a maximum of 640 flies. If the flies multiply exponentially, with a growth constant $K = 0.05$ and where time is measured in days, how long will it take an initial population of 20 to fill the enclosure?

$y(t) = 20e^{0.05t}$ y = population

$\Rightarrow 640 = 20 \cdot e^{0.05t}$

$\Rightarrow 32 = e^{0.05t}$ $\Rightarrow \log 32 = 0.05t$

$\Rightarrow t = \frac{\log 32}{0.05} \approx 69.31$

4. A bacterial culture, growing exponentially, increases from 100to 400grams in 10 hours. How much was present after 3 hours?

$y(t) = 100e^{kt}$

$\Rightarrow 400 = 100e^{k \cdot 10}$ $\Rightarrow 4 = e^{10k}$ $\Rightarrow \log 4 = 10k$

$\Rightarrow k = \frac{\log 4}{10}$

$\Rightarrow y(3) = 100 \cdot e^{\frac{\log(4)}{10} \cdot 3}$

≈ 151.56. y = amount.