Nonlinear Optimization

Trust-region methods

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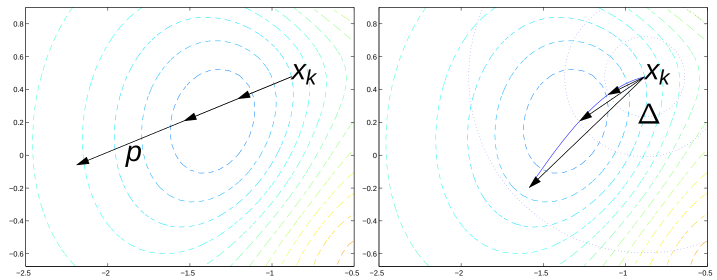
Line search and trust-region

- Line search and trust-region and two examples of *global strategies* that modify a (usually) locally convergent algorithm, e.g. Newton, to become globally convergent.
- At every iteration k, both global strategies enforce the descent condition
$$f(x_{k+1}) < f(x_k)$$
by controlling the length and direction of the step.

- Line search strategies choose the direction first, followed by the distance.
- Trust-region strategies choose the maximum distance first, followed by the direction.



The trust-region model

- Trust-region methods use the quadratic model
$$m_k(p) = f_k + p^T g_k + \frac{1}{2} p^T B_k p,$$
$f_k = f(x_k)$, $g_k = \nabla f(x_k)$.
- Newton-type trust-region methods have $B_k = \nabla^2 f(x_k)$.
- The model is "trusted" within a limited region around the current point x_k defined by
$$\|p\| \leq \Delta_k.$$
This will limit the length of the step from x_k to x_{k+1}.
- The value of Δ_k will be adjusted up if the model is found to be in "good" agreement with the objective function, and down if the model is a "poor" approximation.

The trust-region subproblem

- At iteration k of a trust-region method, the following subproblem must be solved:

$$\begin{aligned}\min_p m_k(p) =& f_k + p^T g_k + \frac{1}{2}p^T B_k p,\\ \text{s.t. } & \|p\| \leq \Delta_k\end{aligned}$$

- It can be shown that the solution p^* of this constrained problem is the solution of the linear equation system

$$(B_k + \lambda I)p^* = -g_k$$

  for some $\lambda \geq 0$ such that the matrix $(B_k + \lambda I)$ is positive semidefinite.
- Furthermore,

$$\lambda(\Delta_k - \|p^*\|) = 0.$$

- Note that if $B_k = \nabla^2 f(x_k)$ is positive definite and Δ_k big enough, the solution of the trust-region subproblem is the solution of

$$\nabla^2 f(x_k)p = -\nabla f(x_k),$$

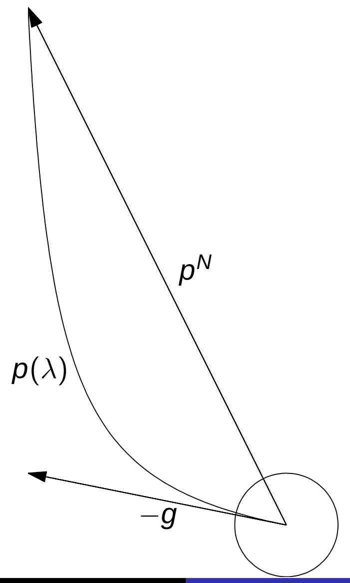
  i.e. p is a Newton-direction.
- Otherwise,

$$\Delta_k \geq \|p_k\| = \|(\nabla^2 f(x_k) + \lambda I)^{-1}\nabla f(x_k)\|,$$

  so if $\Delta_k \to 0$, then $\lambda \to \infty$ and

$$p_k \to -\frac{1}{\lambda}\nabla f(x_k).$$

- When λ varies between 0 and ∞, the corresponding search direction $p_k(\lambda)$ will vary between the Newton direction and a multiple of the negative gradient.



The reduction ratio

- To enable adaption of the trust-region size Δ_k, the reduction ratio

$$\rho_k = \frac{f(x_k) - f(x_k + p_k)}{m_k(0) - m_k(p_k)} = \frac{\text{actual reduction}}{\text{predicted reduction}}$$

  is defined.
- If the reduction ratio is large, e.g. $\rho_k > \frac{3}{4}$, the trust-region size is increased in the next iteration.
- If the reduction ratio is small, e.g. $\rho_k < \frac{1}{4}$, the trust-region size is decreased in the next iteration.
- Furthermore, a step p_k will only be accepted if the reduction ratio is not too small.

The trust-region algorithm

- Specify starting approximation x_0, maximum step length $\hat{\Delta}$, initial trust-region size $\Delta_0 \in (0, \hat{\Delta})$ and acceptance constant $\eta \in [0, \frac{1}{4})$.
- For $k = 0, 1, \ldots$ until x_k is optimal
  - Solve

$$\min_p m_k(p) = f_k + p^T g_k + \frac{1}{2} p^T B_k p,$$
$$\text{s.t. } \|p\| \leq \Delta_k$$

    approximately for a trial step p_k.
  - Calculate the reduction ratio

$$\rho_k = \frac{f(x_k) - f(x_k + p_k)}{m_k(0) - m_k(p_k)}$$

    for p_k.

- - Update the current point

$$x_{k+1} = \begin{cases} x_k + p_k & \text{if } \rho_k > \eta, \\ x_k & \text{otherwise.} \end{cases}$$

  - Update the trust-region radius

$$\Delta_{k+1} = \begin{cases} \frac{1}{4}\Delta_k & \text{if } \rho_k < \frac{1}{4}, \\ \min(2\Delta_k, \hat{\Delta}) & \text{if } \rho_k > \frac{3}{4} \text{ and } \|p_k\| = \Delta_k, \\ \Delta_k & \text{otherwise.} \end{cases}$$

The Levenberg-Marquardt algorithm

- The first trust-region algorithm was developed for least squares problems by Levenberg (1944) and Marquardt (1963).
- The original algorithm uses the approximation $B_k = J_k^T J_k$ and solves

$$(B_k + \lambda_k I)p = -g_k$$

  for different values of λ_k.
- The original algorithm adapts by modifying the λ value, i.e. if the reduction produced by p is good enough, $\lambda_{k+1} = \frac{1}{10}\lambda_k$, otherwise $\lambda_{k+1} = 10\lambda_k$ and the step is rejected.

- The Levenberg-Marquardt algorithm was put into the trust-region framework (Δ-parameterized) in the early 80-ies (Moré, 1981).
- The Δ version of Levenberg-Marquardts has a number of advantages over the λ version:
  - λ is nontrivially related to the problem. Δ is related to the size of x. E.g. $\Delta_0 = \|x_0\|$ is often a reasonable choice.
  - The transition to $\lambda = 0$ is handled transparently.
  - The λ algorithm need to re-solve

$$(B_k + \lambda_k I)p = -g_k$$

    when a step is rejected and λ is reduced. The Δ algorithm has ways to avoid that.
- However, many popular implementation of Levenberg-Marquardt still use the original, λ-parameterized, formulation.

The Dogleg algorithm

- The trust-region subproblem

$$\min_p m_k(p) = f_k + p^T g_k + \frac{1}{2} p^T B_k p,$$
$$\text{s.t. } \|p\| \leq \Delta_k$$

is a hard problem.

- If the unconstrained solution

$$p^B = -B_k^{-1} g_k$$

is too long, $\|p^B\| > \Delta_k$, we have to find a λ such that

$$\|p_k(\lambda)\| = \|(B_k + \lambda I)^{-1} g_k\| = \Delta_k.$$

- This is a non-linear equation in λ.

Trust-region methods
Least Squares; Levenberg-Marquardt
The Dogleg algorithm

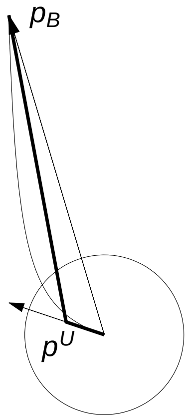
- The dogleg algorithm solves this problem by approximating the function $p_k(\lambda)$ with a piecewise linear polygon $\tilde{p}(\tau)$ and solving $\|\tilde{p}(\tau)\| = \Delta_k$.
- The polygon $\|\tilde{p}(\tau)\|$ is defined as

$$\tilde{p}(\tau) = \begin{cases} \tau p^U, & 0 \leq \tau \leq 1, \\ p^U + (\tau - 1)(p^B - p^U), & 1 \leq \tau \leq 2. \end{cases}.$$

- The point p^U is the *Cauchy point*, i.e. the minimizer of m along the steepest descent direction

$$p^U = -\frac{g^T g}{g^T B g} g.$$

- The dogleg algorithm works only if B_k is positive definite, e.g. for least squares problems.



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