Chapter 4. Part 3

FPT Education

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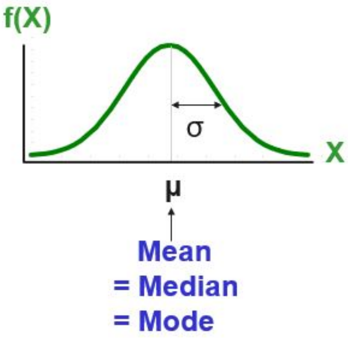
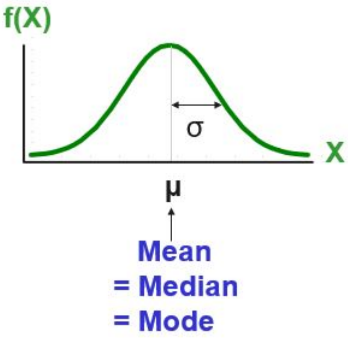
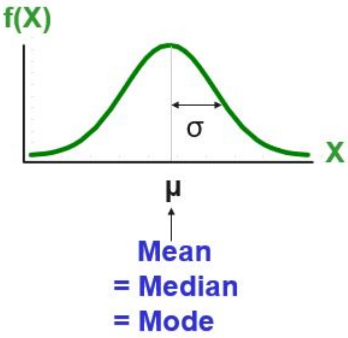
February 11, 2022

4.6 The Normal Distribution

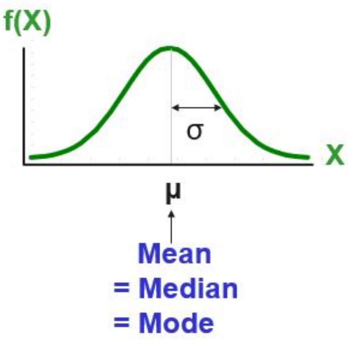




- Bell Shaped



- Bell Shaped
- Symmetrical



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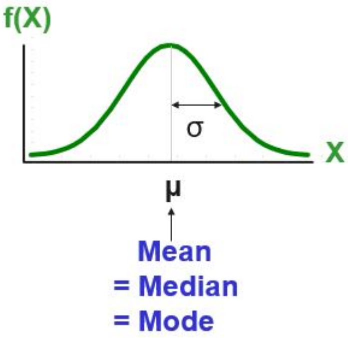

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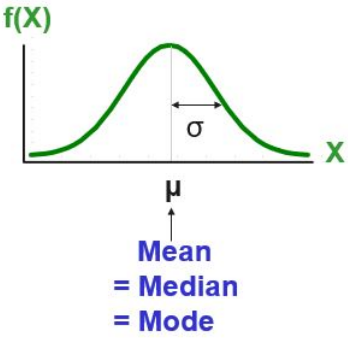

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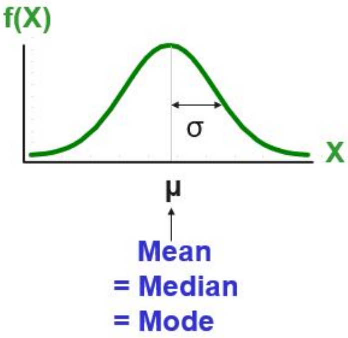

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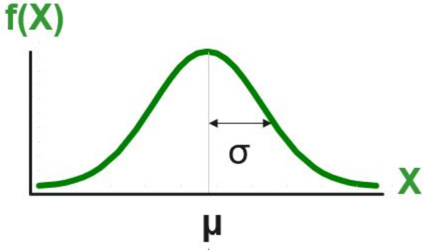
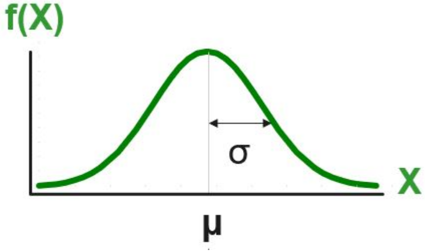


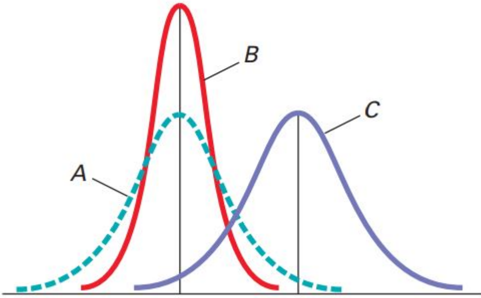
Increase μ shifts the distribution right.

Decrease μ shifts the distribution left.

Changing σ increases or decreases the spread.

Normal Distribution Shape





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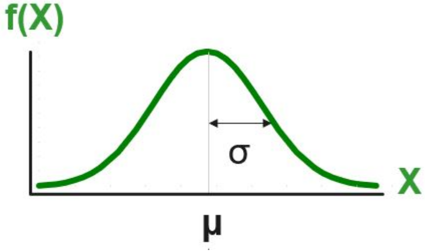

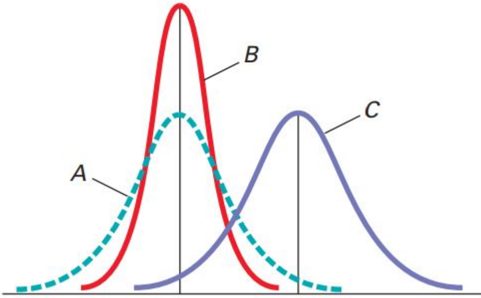

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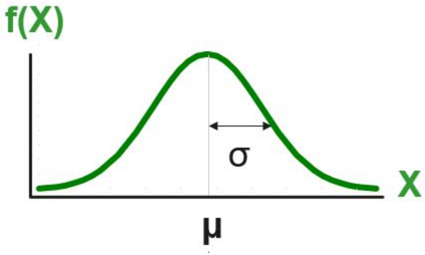

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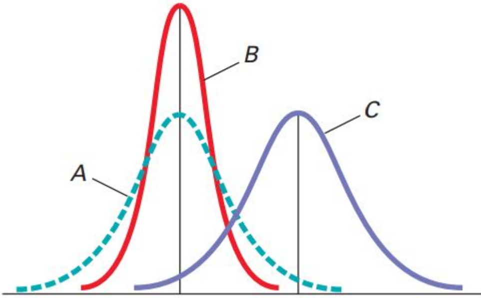

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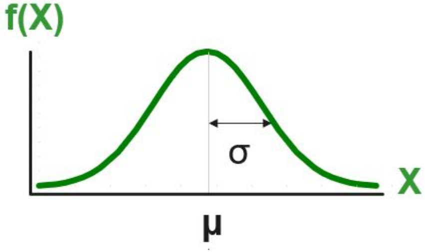

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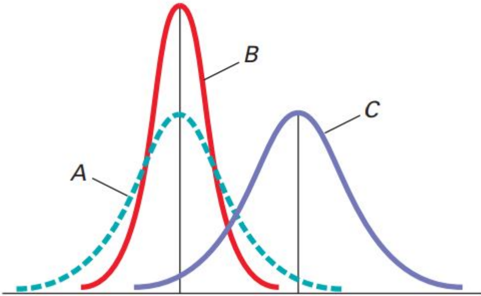

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The normal probability density function of a normal distributed variable X with mean μ and variance σ^2:

$$\boxed{f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(X-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma} e^{-0.5\left(\frac{X-\mu}{\sigma}\right)^2}}$$

where

- $e =$ the mathematical constant approximated by 2.71828
- $\pi =$ the mathematical constant approximated by 3.14159
- $\mu =$ the population mean
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- $X =$ any value of the continuous variable

The Normal Distribution Density Function

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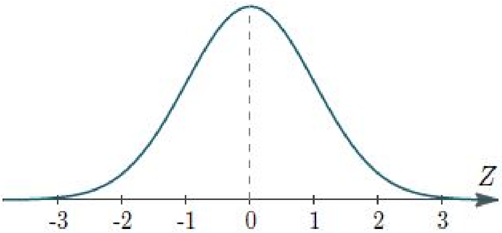
A normal distribution with $\mu = 0$ and $\sigma = 1$ is called standardized normal distribution, denoted by Z:

$$\boxed{f(Z) = \frac{1}{\sqrt{2\pi}} e^{-0.5Z^2}}$$



The Standardized Normal

Suppose that X has normal distribution with μ, σ^2. Set

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Then Z is the standardized normal random variable.

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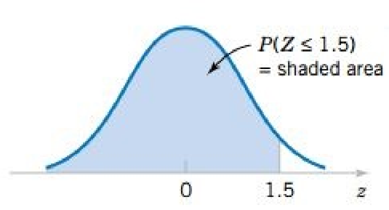
Definition

The cumulative standardized distribution of Z is

$$\Phi(a) = \boxed{P(Z \le a) = P(Z < a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-0.5z^2} dz}$$

given in Appendix Table *III* (page 708).



Example

$$\Phi(1.5) = 0.9332, \quad \Phi(Z \le 0) = 0.5, \quad \Phi(Z < -0.5) = 0.3085$$

Example

Compute $P(0.5 < Z \leq 1.2)$.

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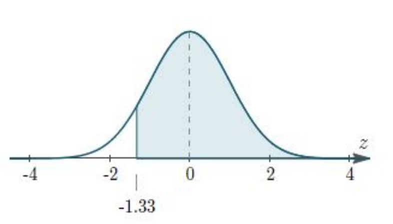
$$P(0.5 < Z \leq 1.2) = \Phi(1.2) - \Phi(0.5) = 0.885 - 0.691 = 0.194$$

Example

Compute $P(Z > -1.33)$.



$$P(Z > -1.33) = 1 - P(Z \leq -1.33) = 1 - \Phi(-1.33) = 1 - 0.092 = 0.908$$

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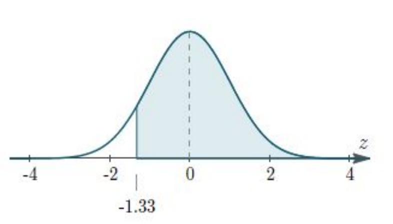

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Suppose X has a normal distribution with mean μ and standard deviation σ.

Normalize $\boxed{Z = \dfrac{X - \mu}{\sigma}}$.

$$\boxed{P(X < b) = P\left(Z < \frac{b - \mu}{\sigma}\right)}$$

$$\boxed{P(X > a) = P\left(Z > \frac{a - \mu}{\sigma}\right)}$$

$$\boxed{P(a < X < b) = P\left(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma}\right)}$$

Example

Suppose X is normal with mean 12 and standard deviation 2. Find $P(X > 13.2)$.

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We have

$$P(X > 13.2) = P(Z > \frac{13.2 - 12}{2}) = P(Z > 0.6)$$
$$= 1 - P(Z < 0.6) = 1 - \Phi(0.6) = 1 - 0.726 = 0.274$$

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Suppose X is normal with mean 12 and standard deviation 2. Find $P(X > 13.2)$.

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$$\begin{aligned} P(X > 13.2) &= P(Z > \frac{13.2 - 12}{2}) = P(Z > 0.6) \\ &= 1 - P(Z < 0.6) = 1 - \Phi(0.6) = 1 - 0.726 = 0.274 \end{aligned}$$

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Answer: Let X be the amount of beer the machine will pour into the next bottle. Then X is normal with mean $\mu = 16$ and standard deviation $\sigma = \sqrt{2.69} = 1.64$. We want to find the probability $P(15.5 < X < 16)$.

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4.8 Exponential Distribution

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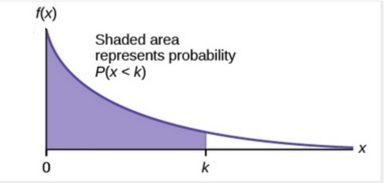
Cumulative Exponential Distribution

Suppose X is exponential with parameter λ. For $k > 0$

$$\boxed{P(X \leq k) = \int_0^k \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^k = 1 - e^{-\lambda k}}$$



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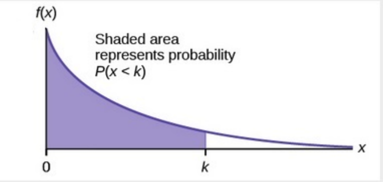

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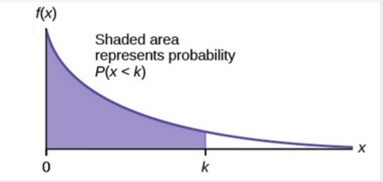

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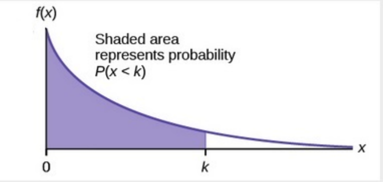

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Answer: Let X the time between customer arrivals. Then X has an exponential distribution with parameter $\lambda = \frac{1}{9}$. We want to compute $P(X > 15)$. We have

$$P(X > 15) = \int_{15}^{\infty} \frac{1}{9} e^{-x/9} dx = -e^{-\frac{1}{9}x}\Big|_{15}^{\infty} = e^{-15/9} = 0.189$$

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