Chapter 4. Continuous Random Variables

FPT Education
FPT UNIVERSITY

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- The mean or expected value of X:

$$\mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx.$$

- The variance of X:

$$V(X) = \sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2.$$

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The cumulative distribution function of X given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 0.25x & \text{if } 0 \le x < 4 \\ 1 & \text{if } x \ge 4. \end{cases}$$

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$$\mu = \int_0^4 xf(x)dx = \int_0^4 0.25xdx = 0.125x^2\Big|_0^4 = 2$$

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Definition

A continuous random variable X with probability density function

$$f(x) = \frac{1}{b-a}, \text{ for } a \leq x \leq b$$

is a continuous uniform random variable.

Theorem

If X is a continuous uniform random variable over $[a, b]$, then

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(a) $\mu = \frac{1+10}{2} = 5.5,\ \sigma^2 = \frac{(10-1)^2}{12} = 6.75,\ \sigma = \sqrt{6.75} \approx 2.598.$

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