Chapter 4. Continuous Random Variables

FPT Education
FPT UNIVERSITY

February 7, 2022

1. 4.1 Continuous Random Variables

2. 4.2 Probability distributions and probability density functions

3. 4.3 Cumulative distribution function

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. The model provides for any precision in length measurements. The number of possible values of X is uncountably infinite.

4.1 Continuous Random Variables

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. The model provides for any precision in length measurements. The number of possible values of X is uncountably infinite.

Example

The height of a student at FPT University, weight of a cat, the mass of an object in a classroom...

A continuous random variable is a random variable with an interval (either finite or infinite) of real numbers for its range. The model provides for any precision in length measurements. The number of possible values of X is uncountably infinite.

Example

The height of a student at FPT University, weight of a cat, the mass of an object in a classroom...

4.2 Probability distributions and probability density functions

FPT Education
FPT UNIVERSITY

Definition

For a continuous random variable X, a probability density function is a function such that

- $f(x) \geq 0$
- $\int\limits_{-\infty}^{\infty} f(x)dx = 1$

4.2 Probability distributions and probability density functions

FPT Education
FPT UNIVERSITY

Definition

For a continuous random variable X, a probability density function is a function such that

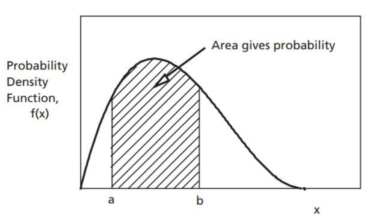
- $f(x) \geq 0$
- $\int\limits_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = \int_a^b f(x)dx$, this is the area under the curve $y = f(x)$ from $x = a$ to b.

4.2 Probability distributions and probability density functions

Definition

For a continuous random variable X, a probability density function is a function such that

- $f(x) \geq 0$
- $\int\limits_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = \int_a^b f(x)dx$, this is the area under the curve $y = f(x)$ from $x = a$ to b.

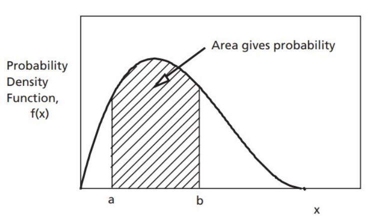


FPT Education FPT UNIVERSITY

Definition

For a continuous random variable X, a probability density function is a function such that

- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$
- $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b) = \int_a^b f(x)dx$, this is the area under the curve $y = f(x)$ from $x = a$ to b.



Example 1

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \leq X < 5),\ \ b)\ P(X < 8),\ \ c)\ P(X \geq 6),\ \ d)\ P(X \geq 0)?$$

Example 1

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \leq X < 5),\ b)\ P(X < 8),\ c)\ P(X \geq 6),\ d)\ P(X \geq 0)?$$

Answer:

a) $P(1 \leq X < 5) = P(3 < X < 5) = \int_3^5 e^{-(x-3)} dx = -e^{-(x-3)}\Big|_3^5 = 1 - e^{-2}$.

Example 1

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \leq X < 5),\ \ b)\ P(X < 8),\ \ c)\ P(X \geq 6),\ \ d)\ P(X \geq 0)?$$

Answer:

a) $P(1 \leq X < 5) = P(3 < X < 5) = \int_3^5 e^{-(x-3)}dx = -e^{-(x-3)}\Big|_3^5 = 1 - e^{-2}$.

b) $P(X < 8) = P(3 < X < 8) = \int_3^8 e^{-(x-3)}dx = -e^{-(x-3)}\Big|_3^8 = 1 - e^{-5}$.

Example 1

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \le X < 5),\ b)\ P(X < 8),\ c)\ P(X \ge 6),\ d)\ P(X \ge 0)?$$

Answer:

a) $P(1 \le X < 5) = P(3 < X < 5) = \int_3^5 e^{-(x-3)}dx = -e^{-(x-3)}\Big|_3^5 = 1 - e^{-2}$.

b) $P(X < 8) = P(3 < X < 8) = \int_3^8 e^{-(x-3)}dx = -e^{-(x-3)}\Big|_3^8 = 1 - e^{-5}$.

c) $P(X \ge 6) = \int_6^\infty e^{-(x-3)}dx = -e^{-(x-3)}\Big|_6^\infty = e^{-3}$.

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \leq X < 5),\ b)\ P(X < 8),\ c)\ P(X \geq 6),\ d)\ P(X \geq 0)?$$

Answer:

a) $P(1 \leq X < 5) = P(3 < X < 5) = \int_3^5 e^{-(x-3)} dx = -e^{-(x-3)}\Big|_3^5 = 1 - e^{-2}$.

b) $P(X < 8) = P(3 < X < 8) = \int_3^8 e^{-(x-3)} dx = -e^{-(x-3)}\Big|_3^8 = 1 - e^{-5}$.

c) $P(X \geq 6) = \int_6^\infty e^{-(x-3)} dx = -e^{-(x-3)}\Big|_6^\infty = e^{-3}$.

d) $P(X \geq 0) = P(X > 3) = 1$.

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-3)}$ for $x > 3$. Determine:

$$a)\ P(1 \leq X < 5),\ b)\ P(X < 8),\ c)\ P(X \geq 6),\ d)\ P(X \geq 0)?$$

Answer:

a) $P(1 \leq X < 5) = P(3 < X < 5) = \int_3^5 e^{-(x-3)} dx = -e^{-(x-3)}\Big|_3^5 = 1 - e^{-2}$.

b) $P(X < 8) = P(3 < X < 8) = \int_3^8 e^{-(x-3)} dx = -e^{-(x-3)}\Big|_3^8 = 1 - e^{-5}$.

c) $P(X \geq 6) = \int_6^\infty e^{-(x-3)} dx = -e^{-(x-3)}\Big|_6^\infty = e^{-3}$.

d) $P(X \geq 0) = P(X > 3) = 1$.

Example 2

The probability density function of the length of a metal rod is $f(x) = cx^2$ for $2 \leq x < 3$ meters.

a) What is the value of c?   b) Find $P(X < 2.5 \text{ or } X \geq 2.8)$?

Example 2

The probability density function of the length of a metal rod is $f(x) = cx^2$ for $2 \leq x < 3$ meters.
a) What is the value of c? b) Find $P(X < 2.5 \text{ or } X \geq 2.8)$?

Answer:

a) We must have

$$\int_2^3 cx^2 dx = \frac{cx^3}{3}\Big|_2^3 = 9c - \frac{8c}{3} = \frac{19c}{3} = 1.$$

Thus, $c = 3/19$.

Example 2

The probability density function of the length of a metal rod is $f(x) = cx^2$ for $2 \leq x < 3$ meters.
a) What is the value of c? b) Find $P(X < 2.5 \text{ or } X \geq 2.8)$?

Answer:

a) We must have

$$\int_2^3 cx^2 dx = \frac{cx^3}{3}\Big|_2^3 = 9c - \frac{8c}{3} = \frac{19c}{3} = 1.$$

Thus, $c = 3/19$.

b) We have

$$P(X < 2.5 \text{ or } X \geq 2.8) = P(2 < X < 2.5) + P(2.8 \leq X < 3)$$
$$= \int_2^{2.5} \frac{3}{19}x^2 dx + \int_{2.8}^3 \frac{3}{19}x^2 dx \approx 0.667$$

Example 2

The probability density function of the length of a metal rod is $f(x) = cx^2$ for $2 \leq x < 3$ meters.
a) What is the value of c? b) Find $P(X < 2.5 \text{ or } X \geq 2.8)$?

Answer:

a) We must have

$$\int_2^3 cx^2 dx = \frac{cx^3}{3}\Big|_2^3 = 9c - \frac{8c}{3} = \frac{19c}{3} = 1.$$

Thus, $c = 3/19$.

b) We have

$$P(X < 2.5 \text{ or } X \geq 2.8) = P(2 < X < 2.5) + P(2.8 \leq X < 3)$$
$$= \int_2^{2.5} \frac{3}{19}x^2 dx + \int_{2.8}^{3} \frac{3}{19}x^2 dx \approx 0.667$$

4.3 Cumulative distribution function

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = P(X < x) = \int_{-\infty}^{x} f(t)dt,$$

for any $-\infty < x < \infty$.

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = P(X < x) = \int_{-\infty}^{x} f(t)dt,$$

for any $-\infty < x < \infty$.

Theorem

Suppose that $f(x)$ and $F(x)$ are the probability density and the cumulative distribution functions of a continuous random variable X respectively. Then

$$f(x) = F'(x) = \frac{dF(x)}{dx}.$$

4.3 Cumulative distribution function

The cumulative distribution function of a continuous random variable X is

$$F(x) = P(X \leq x) = P(X < x) = \int_{-\infty}^{x} f(t)dt,$$

for any $-\infty < x < \infty$.

Theorem

Suppose that $f(x)$ and $F(x)$ are the probability density and the cumulative distribution functions of a continuous random variable X respectively. Then

$$f(x) = F'(x) = \frac{dF(x)}{dx}.$$

Example 1

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

Find:

$$a)\ P(X < 2.8), \quad b)\ P(X < 0), \quad c)\ P(X > 1.5)?$$

Example 1

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \le x < 3 \\ 1 & \text{if } x \ge 3 \end{cases}$$

Find:

$$a)\ P(X < 2.8),\quad b)\ P(X < 0),\quad c)\ P(X > 1.5)?$$

Answer:

a) $P(X < 2.8) = F(2.8) = 0.5 * 2.8 - 0.5 = 0.9.$

Example 1

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

Find:

$$a)\ P(X < 2.8), \quad b)\ P(X < 0), \quad c)\ P(X > 1.5)?$$

Answer:

a) $P(X < 2.8) = F(2.8) = 0.5 * 2.8 - 0.5 = 0.9.$

b) $P(X < 0) = F(0) = 0.$

Example 1

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

Find:

$$a)\ P(X < 2.8), \quad b)\ P(X < 0), \quad c)\ P(X > 1.5)?$$

Answer:

a) $P(X < 2.8) = F(2.8) = 0.5 * 2.8 - 0.5 = 0.9.$
b) $P(X < 0) = F(0) = 0.$
c)

$$P(X > 1.5) = 1 - P(X \leq 1.5) = 1 - F(1.5)$$

$$= 1 - (0.5 * 1.5 - 0.5) = 0.75.$$

Example 1

Suppose the cumulative distribution function of the random variable X is

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ 0.5x - 0.5 & \text{if } 1 \leq x < 3 \\ 1 & \text{if } x \geq 3 \end{cases}$$

Find:

$$a)\ P(X < 2.8), \quad b)\ P(X < 0), \quad c)\ P(X > 1.5)?$$

Answer:

a) $P(X < 2.8) = F(2.8) = 0.5 * 2.8 - 0.5 = 0.9.$

b) $P(X < 0) = F(0) = 0.$

c)

$$P(X > 1.5) = 1 - P(X \leq 1.5) = 1 - F(1.5)$$
$$= 1 - (0.5 * 1.5 - 0.5) = 0.75.$$

Example 2

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-2)}$ for $x > 2$. Find the cumulative distribution function of X.

Example 2

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-2)}$ for $x > 2$. Find the cumulative distribution function of X.

Answer: For $x > 2$, we have

$$\begin{aligned} F(x) &= \int_2^x f(t)dt = \int_2^x e^{-(t-2)}dt = -e^{-(t-2)}\Big|_2^x \\ &= 1 - e^{-(x-2)}. \end{aligned}$$

Example 2

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-2)}$ for $x > 2$. Find the cumulative distribution function of X.

Answer: For $x > 2$, we have

$$\begin{aligned} F(x) &= \int_2^x f(t)dt = \int_2^x e^{-(t-2)}dt = -e^{-(t-2)}\Big|_2^x \\ &= 1 - e^{-(x-2)}. \end{aligned}$$

Since $f(x) = 0$ when $x \leq 2$, hence $F(x) = 0$ for $x \leq 2$. Thus,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 1 - e^{-(x-2)} & \text{if } x > 2. \end{cases}$$

Example 2

Suppose that the probability density of a continuous random variable X is $f(x) = e^{-(x-2)}$ for $x > 2$. Find the cumulative distribution function of X.

Answer: For $x > 2$, we have

$$\begin{aligned} F(x) =& \int_2^x f(t)dt = \int_2^x e^{-(t-2)}dt = -e^{-(t-2)}\Big|_2^x \\ =& 1 - e^{-(x-2)}. \end{aligned}$$

Since $f(x) = 0$ when $x \leq 2$, hence $F(x) = 0$ for $x \leq 2$. Thus,

$$F(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 1 - e^{-(x-2)} & \text{if } x > 2. \end{cases}$$