Outline

Use of the Interpolating Polynomial Error Bound

Example: Tabulated Data

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- Assume that the number of decimal places to be given per entry is $d \geq 8$ and that the difference between adjacent x-values, the step size, is h.
- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

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- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in [0,1], and suppose j satisfies $x_j \leq x \leq x_{j+1}$.

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- What step size h will ensure that linear interpolation gives an absolute error of at most 10^{-6} for all x in $[0, 1]$?

Let $x_0, x_1, \ldots$ be the numbers at which f is evaluated, x be in [0,1], and suppose j satisfies $x_j \leq x \leq x_{j+1}$. The error bound theorem ▸ Theorem implies that the error in linear interpolation is

$$|f(x) - P(x)| = \left| \frac{f^{(2)}(\xi)}{2!}(x - x_j)(x - x_{j+1}) \right| = \frac{|f^{(2)}(\xi)|}{2} |(x - x_j)||(x - x_{j+1})|$$

Use of the Interpolating Polynomial Error Bound

Solution (1/3)

The step size is h, so $x_j = jh$, $x_{j+1} = (j+1)h$, and

$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!} |(x - jh)(x - (j+1)h)|.$$

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$$|f(x) - P(x)| \leq \frac{|f^{(2)}(\xi)|}{2!} |(x - jh)(x - (j+1)h)|.$$

Hence

$$\begin{aligned}|f(x) - P(x)| &\leq \frac{\max_{\xi \in [0,1]} e^{\xi}}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)| \\ &\leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |(x - jh)(x - (j+1)h)|.\end{aligned}$$

Use of the Interpolating Polynomial Error Bound

Solution (2/3)

Consider the function $g(x) = (x - jh)(x - (j + 1)h)$, for $jh \leq x \leq (j + 1)h$.

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Consider the function $g(x) = (x - jh)(x - (j+1)h)$, for $jh \leq x \leq (j+1)h$. Because

$$g'(x) = (x - (j+1)h) + (x - jh) = 2\left(x - jh - \frac{h}{2}\right),$$

Use of the Interpolating Polynomial Error Bound

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the only critical point for g is at $x = jh + \frac{h}{2}$, with

$$g\left(jh + \frac{h}{2}\right) = \left(\frac{h}{2}\right)^2 = \frac{h^2}{4}$$

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Since $g(jh) = 0$ and $g((j+1)h) = 0$, the maximum value of $|g'(x)|$ in $[jh, (j+1)h]$ must occur at the critical point.

Use of the Interpolating Polynomial Error Bound

Solution (3/3)

This implies that

$$|f(x) - P(x)| \leq \frac{e}{2} \max_{x_j \leq x \leq x_{j+1}} |g(x)| \leq \frac{e}{2} \cdot \frac{h^2}{4} = \frac{eh^2}{8}.$$

Use of the Interpolating Polynomial Error Bound

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Consequently, to ensure that the the error in linear interpolation is bounded by 10^{-6}, it is sufficient for h to be chosen so that

$$\frac{eh^2}{8} \le 10^{-6}. \quad \text{This implies that} \quad h < 1.72 \times 10^{-3}.$$

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Because $n = \frac{(1-0)}{h}$ must be an integer, a reasonable choice for the step size is $h = 0.001$.

Questions?

Reference Material

Generalized Rolle's Theorem

Suppose $f \in C[a, b]$ is n times differentiable on (a, b). If

$$f(x) = 0$$

at the $n + 1$ distinct numbers $a \leq x_0 < x_1 < \ldots < x_n \leq b$, then a number c in (x_0, x_n), and hence in (a, b), exists with

$$f^{(n)}(c) = 0$$

Return to Error Bound Theorem

The Lagrange Polynomial: Theoretical Error Bound

Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each x in $[a, b]$, a number $\xi(x)$ (generally unknown) between $x_0, x_1, \ldots, x_n$, and hence in (a, b), exists with

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n)$$

where $P(x)$ is the interpolating polynomial given by

$$P(x) = f(x_0)L_{n,0}(x) + \cdots + f(x_n)L_{n,n}(x) = \sum_{k=0}^{n} f(x_k)L_{n,k}(x)$$

Return to Second Lagrange Interpolating Polynomial Example

Return to Tabulated data example with $f(x) = e^x$

A Recursive Method

We want to construct a n^{th}-degree polynomial that interpolates $(n+1)$ points, (x_i, y_i), $i = 0, 1, \cdots, n$.

Let $p_i(x)$ the i^{th}-degree polynomial that interpolates (x_k, y_k), $k = 0, 1, \cdots, i$. We have

$$p_0(x) = y_0$$

$$p_1(x) = y_0 + \frac{x - x_0}{x_1 - x_0}(y_1 - y_0)$$

Suppose we have already obtain $P_k(x)$, $k \le i$.
We can construct $P_{i+1}(x)$ as follows

$$P_{i+1}(x) = P_i(x) + \frac{\pi_i(x)}{\pi_i(x_{i+1})}\left(y_{i+1} - P_i(x_{i+1})\right)$$

where

$$\pi_i(x) = \prod_{k=0}^{i}(x - x_k)$$

It can be easily verified that

(i) $P_{i+1}(x_k) = y_k$, $k = 0, 1, \cdots, i+1$

(ii) $P_{i+1}(x)$ is an $(i+1)^{st}$-degree polynomial.

(iii) It is easy, by induction, to program.

```
INPUT  n, {(x_i, y_i)}_{i=0}^{n}, x
P = y_0

For i = 0, 1, ..., n-1 repeat

    Π = 1

    For j = 0, 1, ..., i

        Π = Π × (x - x_j) / (x_{i+1} - x_j)

    End

    P = P + Π × ( y_{i+1} - P_i(x_{i+1}) )

End

OUTPUT  P

STOP
```

```
INPUT  n, {(x_i, y_i)}_{i=0}^{n}

For i=0, 1, ..., n repeat
    P_0(x_i) = y_0
End

For i=0, 1, ..., n-1 repeat
    For j = i+1, ..., n repeat
        Π = 1
        For k=0, ..., i repeat
            Π = Π × (x_j - x_k)/(x_{i+1} - x_k)
        End
        P_{i+1}(x_j) = P_i(x_j) + Π × (y_{i+1} - P_i(x_{i+1}))
    End
End
OUTPUT  P_i(x_{i+1}), i=0, 1, ..., n-1
STOP
```

$P_j(x_i)$ \ x_i	x_0	x_1	x_2	x_3	$\cdots$	x_{n-1}	x_n
P_0		$P_0(x_1)$	$P_0(x_2)$	$P_0(x_3)$	$\cdots$	$P_0(x_{n-1})$	$P_0(x_n)$
P_1			$P_1(x_2)$	$P_1(x_3)$	$\cdots$	$P_1(x_{n-1})$	$P_1(x_n)$
P_2				$P_2(x_3)$	$\cdots$	$P_2(x_{n-1})$	$P_2(x_n)$
					$\ddots$	$\vdots$	$\vdots$
P_{n-2}						$P_{n-2}(x_{n-1})$	$P_{n-2}(x_n)$
P_{n-1}							$P_{n-1}(x_n)$

Interpolation & Polynomial Approximation

Hermite Interpolation I

Numerical Analysis (9th Edition)

R L Burden & J D Faires

Beamer Presentation Slides
prepared by
John Carroll
Dublin City University

Outline

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Hermite Interpolation: Osculating Polynomials

Osculating Polynomials

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Osculating Polynomials

- Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials.

Hermite Interpolation: Osculating Polynomials

Osculating Polynomials

- Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials.
- Suppose that we are given $n+1$ distinct numbers $x_0, x_1, \ldots, x_n$ in $[a, b]$ and nonnegative integers $m_0, m_1, \ldots, m_n$, and $m = \max\{m_0, m_1, \ldots, m_n\}$.

Hermite Interpolation: Osculating Polynomials

Osculating Polynomials

- Osculating polynomials generalize both the Taylor polynomials and the Lagrange polynomials.
- Suppose that we are given $n+1$ distinct numbers $x_0, x_1, \ldots, x_n$ in $[a, b]$ and nonnegative integers $m_0, m_1, \ldots, m_n$, and $m = \max\{m_0, m_1, \ldots, m_n\}$.
- The osculating polynomial approximating a function $f \in C^m[a, b]$ at x_i, for each $i = 0, \ldots, n$, is the polynomial of least degree that has the same values as the function f and all its derivatives of order less than or equal to m_i at each x_i.

Hermite Interpolation: Osculating Polynomials

Osculating Polynomials (Cont'd)

The degree of this osculating polynomial is at most

$$M = \sum_{i=0}^{n} m_i + n$$

Hermite Interpolation: Osculating Polynomials

Osculating Polynomials (Cont'd)

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$$M = \sum_{i=0}^{n} m_i + n$$

because the number of conditions to be satisfied is $\sum_{i=0}^{n} m_i + (n+1)$, and a polynomial of degree M has $M+1$ coefficients that can be used to satisfy these conditions.

Hermite Interpolation: Osculating Polynomials

Definition: Osculating Polynomial

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- Suppose that $f \in C^m[a, b]$, where $m = \max_{0 \leq i \leq n} m_i$.
- The osculating polynomial approximating f is the polynomial $P(x)$ of least degree such that

$$\frac{d^k P(x_i)}{dx^k} = \frac{d^k f(x_i)}{dx^k}$$

for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

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for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

The osculating polynomial approximating f is the m_0th Taylor polynomial for f at x_0 when $n = 0$ and the nth Lagrange polynomial interpolating f on $x_0, x_1, \ldots, x_n$ when $m_i = 0$ for each i.

Hermite Polynomials

Osculating Polynomials

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for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

Hermite Polynomials

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for each $i = 0, 1, \ldots, n$ and $k = 0, 1, \ldots, m_i$.

Hermite Polynomials

- The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the Hermite polynomials.

Hermite Polynomials

Osculating Polynomials

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Hermite Polynomials

- The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the Hermite polynomials.
- For a given function f, these polynomials agree with f at $x_0, x_1, \ldots, x_n$.

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Hermite Polynomials

- The case when $m_i = 1$, for each $i = 0, 1, \ldots, n$, gives the Hermite polynomials.
- For a given function f, these polynomials agree with f at $x_0, x_1, \ldots, x_n$.
- In addition, since their first derivatives agree with those of f, they have the same “shape” as the function at $(x_i, f(x_i))$ in the sense that the tangent lines to the polynomial and the function agree.

Outline

Precise Form of the Hermite Polynomials

Theorem

If $f \in C^1[a,b]$ and $x_0, \ldots, x_n \in [a,b]$ are distinct, the unique polynomial of least degree agreeing with f and f' at $x_0, \ldots, x_n$ is the Hermite polynomial of degree at most $2n+1$ given by

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

where, for $L_{n,j}(x)$ denoting the jth Lagrange coefficient polynomial of degree n, we have

$$H_{n,j}(x) = [1 - 2(x - x_j)L'_{n,j}(x_j)]L^2_{n,j}(x) \quad \text{and} \quad \hat{H}_{n,j}(x) = (x - x_j)L^2_{n,j}(x)$$

Continued on the next slide . . .

Precise Form of the Hermite Polynomials

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\hat{H}_{n,j}(x)$$

Theorem (Cont'd)

Moreover, if $f \in C^{2n+2}[a, b]$, then

$$f(x) = H_{2n+1}(x) + \frac{(x - x_0)^2 \ldots (x - x_n)^2}{(2n+2)!} f^{(2n+2)}(\xi(x))$$

for some (generally unknown) $\xi(x)$ in the interval (a, b).

Precise Form of the Hermite Polynomials

Proof (1/4)

First recall that

$$L_{n,j}(x_i) = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

Precise Form of the Hermite Polynomials

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Hence when $i \neq j$,

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Hence when $i \neq j$,

$$H_{n,j}(x_i) = 0 \qquad \text{and} \qquad \hat{H}_{n,j}(x_i) = 0$$

whereas, for each i,

$$\begin{aligned} H_{n,i}(x_i) &= [1 - 2(x_i - x_i)L'_{n,i}(x_i)] \cdot 1 = 1 \\ \text{and} \quad \hat{H}_{n,i}(x_i) &= (x_i - x_i) \cdot 1^2 = 0 \end{aligned}$$

Precise Form of the Hermite Polynomials

Proof (2/4)

- As a consequence

$$H_{2n+1}(x_i) = \sum_{\substack{j=0 \\ j\neq i}}^{n} f(x_j)\cdot 0 + f(x_i)\cdot 1 + \sum_{j=0}^{n} f'(x_j)\cdot 0 = f(x_i)$$

so H_{2n+1} agrees with f at $x_0, x_1, \ldots, x_n$.

Precise Form of the Hermite Polynomials

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- As a consequence

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  so H_{2n+1} agrees with f at $x_0, x_1, \ldots, x_n$.
- To show the agreement of H'_{2n+1} with f' at the nodes, first note that $L_{n,j}(x)$ is a factor of $H'_{n,j}(x)$, so $H'_{n,j}(x_i) = 0$ when $i \neq j$.

Precise Form of the Hermite Polynomials

Proof (3/4)

In addition, when $i = j$ we have $L_{n,i}(x_i) = 1$, so

$$\begin{aligned} H'_{n,i}(x_i) &= -2L'_{n,i}(x_i) \cdot L^2_{n,i}(x_i) + [1 - 2(x_i - x_i)L'_{n,i}(x_i)]2L_{n,i}(x_i)L'_{n,i}(x_i) \\ &= -2L'_{n,i}(x_i) + 2L'_{n,i}(x_i) = 0 \end{aligned}$$

Hence, $H'_{n,j}(x_i) = 0$ for all i and j.

Precise Form of the Hermite Polynomials

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Hence, $H'_{n,j}(x_i) = 0$ for all i and j.
Finally,

$$\begin{aligned} \hat{H}'_{n,j}(x_i) &= L^2_{n,j}(x_i) + (x_i - x_j)2L_{n,j}(x_i)L'_{n,j}(x_i) \\ &= L_{n,j}(x_i)[L_{n,j}(x_i) + 2(x_i - x_j)L'_{n,j}(x_i)] \end{aligned}$$

so $\hat{H}'_{n,j}(x_i) = 0$ if $i \neq j$ and $\hat{H}'_{n,i}(x_i) = 1$.

Precise Form of the Hermite Polynomials

Proof (4/4)

Combining these facts, we have

$$H'_{2n+1}(x_i) = \sum_{j=0}^{n} f(x_j) \cdot 0 + \sum_{\substack{j=0 \\ j\neq i}}^{n} f'(x_j) \cdot 0 + f'(x_i) \cdot 1 = f'(x_i)$$

Precise Form of the Hermite Polynomials

Proof (4/4)

Combining these facts, we have

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Therefore, H_{2n+1} agrees with f and H'_{2n+1} with f' at $x_0, x_1, \ldots, x_n$.

Outline

Constructing the Hermite Polynomial

Example: Constructing $H_5(x)$

Use the Hermite polynomial that agrees with the data listed in the following table to find an approximation to $f(1.5)$.

k	x_k	$f(x_k)$	$f'(x_k)$
0	1.3	0.6200860	−0.5220232
1	1.6	0.4554022	−0.5698959
2	1.9	0.2818186	−0.5811571

Constructing the Hermite Polynomial $H_5(x)$

Solution (1/5)

We first compute the Lagrange polynomials and their derivatives.

Constructing the Hermite Polynomial $H_5(x)$

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We first compute the Lagrange polynomials and their derivatives. This gives

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}$$

Constructing the Hermite Polynomial $H_5(x)$

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$$L'_{2,0}(x) = \frac{100}{9}x - \frac{175}{9}$$

Constructing the Hermite Polynomial $H_5(x)$

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$$
\begin{aligned}
L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \\
L'_{2,0}(x) &= \frac{100}{9}x - \frac{175}{9} \\
L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (1/5)

We first compute the Lagrange polynomials and their derivatives. This gives

$$
\begin{aligned}
L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9} \\
L'_{2,0}(x) &= \frac{100}{9}x - \frac{175}{9} \\
L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9} \\
L'_{2,1}(x) &= \frac{-200}{9}x + \frac{320}{9}
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (2/5)

and, finally

$$L_{2,2} = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (2/5)

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\begin{aligned}
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L'_{2,2}(x) &= \frac{100}{9}x - \frac{145}{9}
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$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (3/5)

The polynomials $H_{2,j}(x)$ and $\hat{H}_{2,j}(x)$ are then

$$H_{2,0}(x) = [1 - 2(x - 1.3)(-5)] \left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (3/5)

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\begin{aligned}
H_{2,0}(x) &= [1-2(x-1.3)(-5)]\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2 \\
&= (10x-12)\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2
\end{aligned}
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Constructing the Hermite Polynomial $H_5(x)$

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H_{2,1}(x) &= 1\cdot\left(\frac{-100}{9}x^2+\frac{320}{9}x-\frac{247}{9}\right)^2
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Constructing the Hermite Polynomial $H_5(x)$

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&= (10x-12)\left(\frac{50}{9}x^2-\frac{175}{9}x+\frac{152}{9}\right)^2 \\
H_{2,1}(x) &= 1\cdot\left(\frac{-100}{9}x^2+\frac{320}{9}x-\frac{247}{9}\right)^2 \\
H_{2,2}(x) &= 10(2-x)\left(\frac{50}{9}x^2-\frac{145}{9}x+\frac{104}{9}\right)^2
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (4/5)

$$\hat{H}_{2,0}(x) \quad = \quad (x - 1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (4/5)

$$\hat{H}_{2,0}(x) = (x - 1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2$$

$$\hat{H}_{2,1}(x) = (x - 1.6)\left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (4/5)

$$
\begin{aligned}
\hat{H}_{2,0}(x) &= (x-1.3)\left(\frac{50}{9}x^2 - \frac{175}{9}x + \frac{152}{9}\right)^2 \\
\hat{H}_{2,1}(x) &= (x-1.6)\left(\frac{-100}{9}x^2 + \frac{320}{9}x - \frac{247}{9}\right)^2 \\
\hat{H}_{2,2}(x) &= (x-1.9)\left(\frac{50}{9}x^2 - \frac{145}{9}x + \frac{104}{9}\right)^2
\end{aligned}
$$

Constructing the Hermite Polynomial $H_5(x)$

Solution (5/5): and finally . . .

Constructing the Hermite Polynomial $H_5(x)$

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$$
\begin{aligned}
H_5(x) = {} & 0.6200860 H_{2,0}(x) + 0.4554022 H_{2,1}(x) + 0.2818186 H_{2,2}(x) \\
& - 0.5220232 \hat{H}_{2,0}(x) - 0.5698959 \hat{H}_{2,1}(x) \\
& - 0.5811571 \hat{H}_{2,2}(x)
\end{aligned}
$$

so that

$$
\begin{aligned}
H_5(1.5) = {} & 0.6200860 \left(\frac{4}{27}\right) + 0.4554022 \left(\frac{64}{81}\right) + 0.2818186 \left(\frac{5}{81}\right) \\
& - 0.5220232 \left(\frac{4}{405}\right) - 0.5698959 \left(\frac{-32}{405}\right) \\
& - 0.5811571 \left(\frac{-2}{405}\right) = 0.5118277
\end{aligned}
$$

a result that is accurate to the places listed.

Constructing the Hermite Polynomial

Observation

Although the theorem provides a complete description of the Hermite polynomials, it is clear from this example that the need to determine and evaluate the Lagrange polynomials and their derivatives makes the procedure tedious even for small values of n.

Constructing the Hermite Polynomial

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Although the theorem provides a complete description of the Hermite polynomials, it is clear from this example that the need to determine and evaluate the Lagrange polynomials and their derivatives makes the procedure tedious even for small values of n.

Remedy

We will turn to an alternative method for generating Hermite approximations that has as its basis the Newton interpolatory divided-difference formula at $x_0, x_1, \ldots, x_n$, that is,

$$P_n(x) = f[x_0] + \sum_{k=1}^{n} f[x_0, x_1, \ldots, x_k](x - x_0) \cdots (x - x_{k-1})$$

Questions?