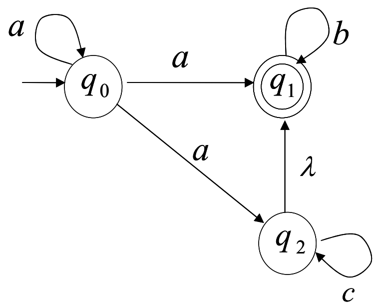
REGULAR LANGUAGES (14 POINTS)

Problem 1 (6 points) For the following NFA (Sudkamp)



a) Write the transition function and extended transition function (with λ-transitions)

b) Convert the NFA to DFA

Solution 1

a) The transition function for M

δ	a	b	c	λ
q_0	$\{q_0, q_1, q_2\}$	$\varnothing$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\{q_1\}$	$\varnothing$	$\varnothing$
q_2	$\varnothing$	$\varnothing$	$\{q_2\}$	$\{q_1\}$

The extended transition function with λ-transitions

t	a	b	c
q_0	$\{q_0, q_1, q_2\}$	$\varnothing$	$\varnothing$
q_1	$\varnothing$	$\{q_1\}$	$\varnothing$
q_2	$\varnothing$	$\{q_1\}$	$\{q_1, q_2\}$

b) Conversion of the NFA to DFA

We find the corresponding DFA by subset construction algorithm:

$\{q_0\} \xrightarrow{a} \{q_0, q_1, q_2\}$

$\{q_0\} \xrightarrow{b} \{\varnothing\}$

$\{q_0\} \xrightarrow{c} \{\varnothing\}$

$\{q_0, q_1, q_2\} \xrightarrow{a} \{q_0, q_1, q_2\}$

$\{q_0, q_1, q_2\} \xrightarrow{b} \{q_1\}$

$\{q_0, q_1, q_2\} \xrightarrow{c} \{q_1, q_2\}$

$\{q_1\} \xrightarrow{a} \{\varnothing\}$

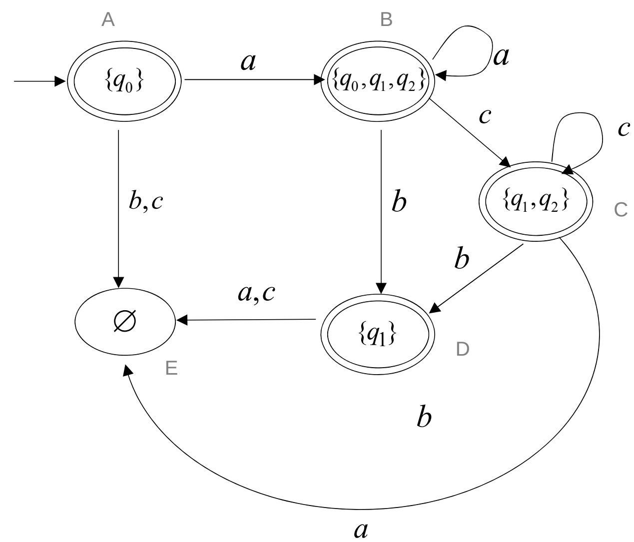
$\{q_1\} \xrightarrow{b} \{q_1\}$

$\{q_1\} \xrightarrow{c} \{\varnothing\}$

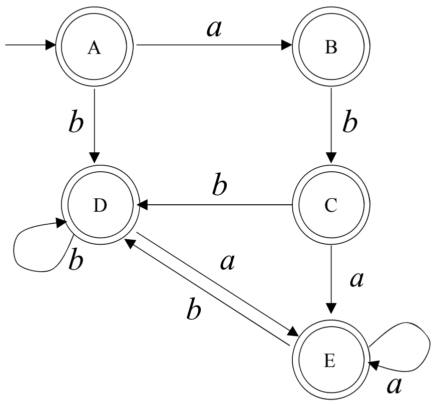
$\{q_1, q_2\} \xrightarrow{a} \{\varnothing\}$

$\{q_1, q_2\} \xrightarrow{b} \{q_1\}$

$\{q_1, q_2\} \xrightarrow{c} \{q_1, q_2\}$



Problem 2 (4 points) Minimize the following automaton (Sudkamp)



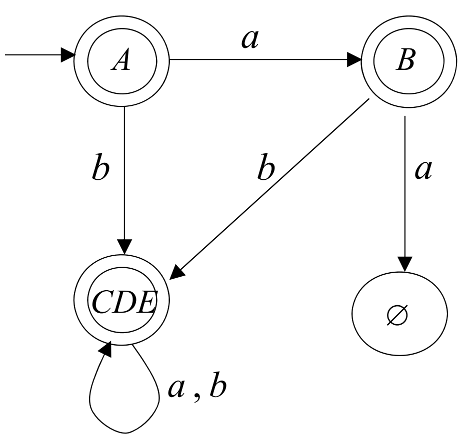
Solution 2

From the transition table below C, D, E are indistinguishable:

	a	b
A	B	D
B	∅	C
C	E	D
D	E	D
E	E	D

The minimal DFA is:

	a	b
A	B	CDE
B	∅	CDE
CDE	CDE	CDE
∅	∅	∅



Problem 3 (4 points) Prove by using Pumping lemma that the following language is not regular (Linz):

$$L = \{a^n\ b^l \mid n \text{ is an integral multiple of } l\}$$

Solution 3

We use the Pumping lemma as follows.

Assume (by contradiction) that L is regular. From Pumping lemma,
$\exists$ m such that $\forall\ w \in L, |w| \geq m, \exists\ x, y, z \ni |xy| \leq m, |y| = k > 0,$
$w = xyz$ and $xy^iz \in L\ \forall\ i \geq 0.$

Let $w = xyz = a^m b^m \in L$ $\qquad (m = 1 * m)$

Since $|xy| \leq m,\ xy \subset a^m$

$\therefore\ y = a^k$

Consider $xy^2z \in L$

$a^{(m+k)} b^m \in L$

$\therefore\ \exists\ n \ni (m+k) = n * m$ $\qquad$ (by definition of L)

but $\quad k < m$

$\therefore\ 1m = m < m+k < m+m = 2m$

$\therefore\ !\exists\ n \ni (m+k) = n * m$

which is a contradiction

$\therefore$ L is not regular

CONTEXT FREE LANGUAGES (14 POINTS)

Problem 4 (4 points) Is the following language context-free? Justify your answer. If it is CFL, give a grammar. Otherwise prove that the language is not CF using the pumping lemma. (Chen)

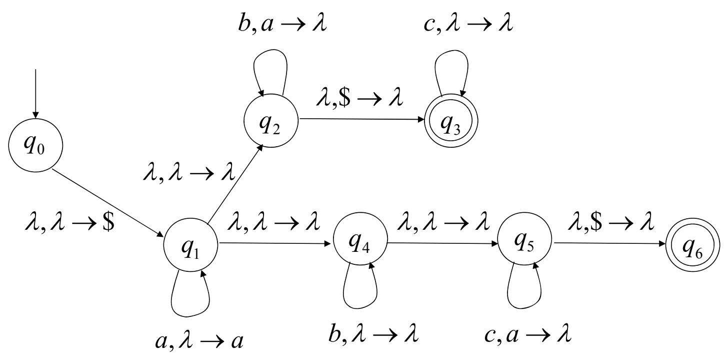
$$L = \{a^n\ b^m \mid n \neq m\}$$

Solution 4

Yes, it has the CFL with the following grammar:

n>m	m>n	n>m | m>n
$S \to AS_1$	$S \to S_1B$	$S \to AS_1 \mid S_1B$
$S_1 \to aS_1b \mid \lambda$	$S_1 \to aS_1b \mid \lambda$	$S_1 \to aS_1b \mid \lambda$
$A \to aA \mid a$	$B \to bB \mid b$	$A \to aA \mid a$
		$B \to bB \mid b.$

Problem 5 (6 points) The PDA is given by the following graph: (Sipser)



a) What language is accepted by the automaton?

b) Test run on the strings aabbc and aabcc.

Solution 5

a) The language is: $L = \{a^i\, b^j\, c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$

b) Test run on the strings aabbc and aabcc.

State	*String*	*Stack*
q_0	*aabbc*	λ
q_1	*aabbc*	$\$$
q_1	*abbc*	$a\$$
q_1	*bbc*	$aa\$$
q_2	*bbc*	$aa\$$
q_2	*bc*	$a\$$
q_2	*c*	$\$$
q_3	*c*	λ
q_3	λ	λ
	Accepted!	

State	*String*	*Stack*
q_0	*aabcc*	λ
q_1	*aabcc*	$\$$
q_1	*abcc*	$a\$$
q_1	*bcc*	$aa\$$
q_4	*bcc*	$aa\$$
q_5	*cc*	$aa\$$
q_5	*c*	$a\$$
q_5	λ	$\$$
q_6	λ	λ
	Accepted!	

Problem 6 (4 points) Which of the following languages is context-free? Justify! If context-free construct a CFG or a PDA, if not use Pumping lemma.

a) L= $\{ a^m b^n c^p d^r \mid m+n+p=r; \quad m,n,p,r \geq 0 \}$ with the alphabet $\Sigma = \{a, b, c, d\}$.

b) L= $\{ a^{k^2} b^k \mid k \geq 0 \}$ with the alphabet $\Sigma = \{a, b\}$.

Solution 6

a) Context-free. Grammar:

$$S \rightarrow aSd \mid A$$
$$A \rightarrow bAd \mid B$$
$$B \rightarrow cBd \mid \lambda$$

b) Not context-free. Pumping lemma, see Lecture 10, p.123.

RECURSIVELY ENUMERABLE LANGUAGES (12 POINTS)

Problem 7 (4 points) Is the function g(x) primitive recursive? If yes, define it as a primitive recursive function (you can use basic primitive recursive functions from the course). (Salling)

x	0	1	2	3	4	5	6	7	8	9	10	11	12	1
g(x)	1	1	3	2	5	3	7	4	9	5	11	6	13	7

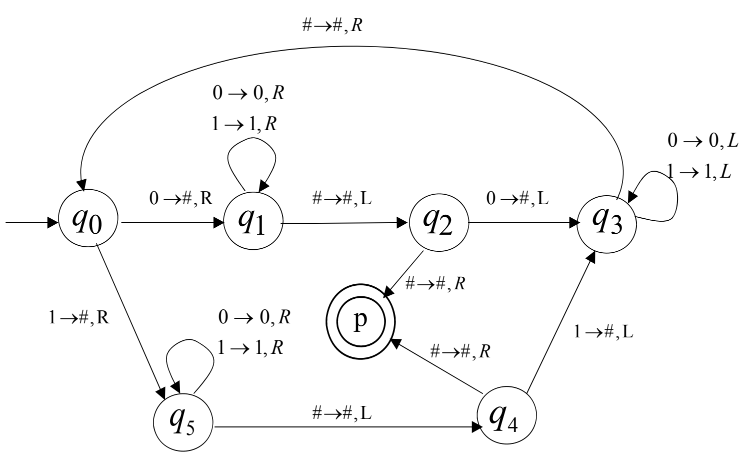
Solution 7

Function $g(x)$ is primitive recursive. From the function table we see that for even x it returns $x+1$ and for odd x it gives $(x+1)/2$. Thus, we can define g as follows:

$g(0) = 1$
$g(x) = if\ (even(x), f(x), h(x))$
$f(x) = succ\ (x)$
$h(x)= quotient\ (succ(x),2)$

Problem 8 (4 points)

How does the following TM work? Trace the execution of this machine on three input strings (Krasnogor).



Solution 8:

This is a Turing Machine that recognizes palindromes over the alphabet {0,1}.
For example fM(1) = yes, fM(10101)= yes, fM(1101011)= yes, but fM(1101)= no

The actions carried out by the machine are:

1. Search the first symbol
2. Remember the first symbol and go to the end of string
3. Delete the last symbol if it is the same as the one remembered
4. If there is only one symbol, the string is accepted.

Test run

Processing string ”0”

$\#q_0 0\# \rightarrow \#\#q_1\# \rightarrow \#q_2\#\# \rightarrow \#\#p\# \rightarrow$ halts and accepts

Processing string ”10”

$\#q_0 10\# \rightarrow \#\#q_5 0\# \rightarrow \#\#0q_5\# \rightarrow \#\#q_4 0\# \rightarrow$ halts and fails

Processing string ”10101”

$\#q_0 10101\# \rightarrow \#\#q_5 0101\# \rightarrow \#\#0q_5 101 \rightarrow \#\#01q_5 01\# \rightarrow \#\#010q_5 1\# \rightarrow$

$\#\#0101q_5\# \rightarrow \#\#010q_4 1\# \rightarrow \#\#01q_3 0\#\# \rightarrow \#\#0q_3 10\#\# \rightarrow \#\#q_3 010\#\# \rightarrow$

$\#q_3\#010\#\# \rightarrow \#\#q_0 010\#\# \rightarrow \#\#\#q_1 10\#\# \rightarrow \#\#\#1q_1 0\#\# \rightarrow \#\#\#10q_1\#\# \rightarrow$

$\#\#\#1q_2 0\#\# \rightarrow \#\#\#q_3 1\#\#\# \rightarrow \#\#q_3\#1\#\#\# \rightarrow \#\#\#q_0 1\#\#\# \rightarrow \#\#\#\#q_5\#\#\# \rightarrow$

$\#\#\#q_4\#\#\#\# \rightarrow \#\#\#\#p\#\#\# \rightarrow$ halts and accepts

Problem 9 (4 points) Is the language L decidable? Justify your answer! (Hopcroft)

a) L = { <M> | L(M) is infinite and M is an arbitrary DFA }.

b) L = { <M> | L(M) is infinite and M is an arbitrary TM }

Solution 9:

a) DECIDABLE. It is enough to check if M contains any loops, which can be done in a finitely meny steps.

b) UNDECIDABLE. Follows from Rice’s theorem: Any nontrivial property of the language recognized by a Turing machine is undecidable. The property of being infinite Turing recognizable language is nontrivial as not all Turing recognizable languages have that property.

References

Sudkamp, Languages and Machines, Addison Wesley 1998

Sipser Michael, Introduction to the Theory of Computation, PWS 1997

Linz Peter, An Introduction to Formal Languages and Automata, Jones & Bartlett, 2006

Salling: Formella språk, automater och beräkningar 2001

Hopcroft, Motwani, Ullman, Introduction to Automata Theory, Languages, and Computation, Addison Wesley 2001