Hermite Interpolation Section 5.7

Hermite interpolation is an extension of basic polynomial interpolation that not only matches discrete information at a set of points, but also matches the slope (or rate of change) at those points.

> **Hermite Interpolation Theorem**
>
> Let **S** be a discrete data set of distinct points defined as follows:
> $$\mathbf{S} = \{(\mathbf{x_i}, \mathbf{f(x_i)}, \mathbf{f'(x_i)}) \mid \mathbf{i} = 0, 1, 2, \ldots, \mathbf{n}\}$$
> Then there exists a unique polynomial $\mathbf{H_{2n+1}(x)}$ of degree $2n + 1$ or less that interpolates the data by matching the position of each point and the slope at each at each point; that is,
> $$\mathbf{H_{2n+1}(x_i)} = \mathbf{f(x_i)} \text{ and } \mathbf{H'_{2n+1}(x_i)} = \mathbf{f'(x_i)} \ \mathbf{i} = 0, 1, 2, \ldots, \mathbf{n}.$$
> $\mathbf{H_{2n+1}(x)}$ is called the **Hermite polynomial interpolant** to the data set **S**.

Construction of the Hermite Interpolating Polynomial

There are three ways to construct the Hermite interpolant that are similar to the procedures for basic polynomial interpolation.

1. Method of undetermined coefficients; requires the solution of a linear system.
2. A modification of the Lagrange basis; the required modification is cumbersome and not very easy to use computationally.
3. A modification to the Divided Difference basis; this procedure is quite reasonable when a "modified" divided difference table construction is included with the process.

Example: Use the method of undetermined coefficients to construct the Hermite interpolant to set $\mathbf{S} = \{(1,2,1),(3,1,-1),(4,2,0)\}$.

Here we have data point (1,2) where the slope is to be m = 2, point (3,1) where the slope is to be m = -1, and point (4,2) where the slope is to be m = 0.

For set **S** the Hermite interpolant is to be of degree 5, so let

$$\mathbf{H_5(x)} = \mathbf{a_5x^5} + \mathbf{a_4x^4} + \mathbf{a_3x^3} + \mathbf{a_2x^2} + \mathbf{a_1x} + \mathbf{a_0}$$
$$\mathbf{H'_5(x)} = 5\mathbf{a_5x^4} + 4\mathbf{a_4x^3} + 3\mathbf{a_3x^2} + 2\mathbf{a_2x} + \mathbf{a_1} \quad .$$

Now use the interpolation requirements listed next to construct a linear system of 6 equations in the 6 unknown coefficients a_5, a_4, …, a_0.

Interpolation requirements:

$$\mathbf{H}_5(1) = 2, \mathbf{H}_5(3) = 1, \mathbf{H}_5(4) = 2$$
$$\mathbf{H'}_5(1) = 1, \mathbf{H'}_5(3) = -1, \mathbf{H'}_5(4) = 0$$

We obtain the following linear system to solve for the coefficients a_5, a_4, …, a_0. Here we show the augmented matrix of the system.

$$\left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 243 & 81 & 27 & 9 & 3 & 1 & 1 \\ 1024 & 256 & 64 & 16 & 4 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 & 0 & 1 \\ 405 & 108 & 27 & 6 & 1 & 0 & -1 \\ 1280 & 256 & 48 & 8 & 1 & 0 & 0 \end{array}\right]$$

The Hermite interpolant is

$$\mathbf{H}_5(\mathbf{x}) = \mathbf{-}0.4722\mathbf{x}^5 + 5.8333\mathbf{x}^4 + \mathbf{-}26.5833\mathbf{x}^3 + 54.5556\mathbf{x}^2 - 49.3333\mathbf{x} + 18.0000$$

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The divided difference basis for standard polynomial interpolation is

$$\begin{aligned} &\mathbf{D}_0(\mathbf{x}) = 1 \\ &\mathbf{D}_1(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0) \\ &\mathbf{D}_2(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_1) \\ &\vdots \\ &\mathbf{D}_\mathbf{n}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)(\mathbf{x} - \mathbf{x}_1)\cdots(\mathbf{x} - \mathbf{x}_{\mathbf{n}-1}) \end{aligned}$$

The modification of this basis for Hermite interpolation is shown next. Note the repeats of the linear factors and that each member is of a different degree. Also note that in the expression for $\mathbf{HD}_{2\mathbf{n}+1}(\mathbf{x})$ that the last data point's x-coordinate only appears to the first power. Hermite interpolants are of odd degree since there are an even number, 2n+2, requirements to be fulfilled. (Recall we started the indexing of points at zero.)

$$\begin{aligned} &\mathbf{HD}_0(\mathbf{x}) = 1 \\ &\mathbf{HD}_1(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0) \\ &\mathbf{HD}_2(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2 \\ &\mathbf{HD}_3(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2(\mathbf{x} - \mathbf{x}_1) \\ &\mathbf{HD}_4(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2(\mathbf{x} - \mathbf{x}_1)^2 \\ &\mathbf{HD}_5(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2(\mathbf{x} - \mathbf{x}_1)^2(\mathbf{x} - \mathbf{x}_2) \\ &\vdots \\ &\mathbf{HD}_{2\mathbf{n}}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2(\mathbf{x} - \mathbf{x}_1)^2\cdots(\mathbf{x} - \mathbf{x}_{\mathbf{n}-1})^2 \\ &\mathbf{HD}_{2\mathbf{n}+1}(\mathbf{x}) = (\mathbf{x} - \mathbf{x}_0)^2(\mathbf{x} - \mathbf{x}_1)^2\cdots(\mathbf{x} - \mathbf{x}_{\mathbf{n}-1})^2(\mathbf{x} - \mathbf{x}_\mathbf{n}) \end{aligned}$$

For the data set $\mathbf{S} = \{(1,2,1),(3,1,-1),(4,2,0)\}$ in our example $\mathbf{H_5(x)}$ is a linear combination of

$$1,(\mathbf{x}-1),(\mathbf{x}-1)^2,(\mathbf{x}-1)^2(\mathbf{x}-3),(\mathbf{x}-1)^2(\mathbf{x}-3)^2, \text{ and } (\mathbf{x}-1)^2(\mathbf{x}-3)^2(\mathbf{x}-4).$$

We need only determine the corresponding coefficients of these 6 expressions. To get the coefficients we modify the standard divided difference table construct process. We use the slopes at the data points to fill-in some entries in the 1^{st} DD column. The modified table for the case of 3 Hermite data points is displayed below. Note that we repeat the (x, y) data in the x and 0^{th} DD columns respectively.

x	0^{th} DD	1^{st} DD	2^{nd} DD	3^{rd} DD	4^{th} DD	5^{th} DD
x_0	$f(x_0)$					
	derivative value	$f'(x_0)$				
x_0	$f(x_0)$					
	slope computation	$\frac{f(x_1)-f(x_0)}{x_1-x_0}$				
x_1	$f(x_1)$					
	derivative value	$f'(x_1)$				
x_1	$f(x_1)$					
	slope computation	$\frac{f(x_2)-f(x_1)}{x_2-x_1}$				
x_2	$f(x_2)$					
	derivative value	$f'(x_1)$				
x2	$f(x_2)$					

The values in 2^{nd}DD and higher are computed in the standard way from the data in the x, 0^{th}DD and 1^{st}DD columns.

The coefficients for the Hermite divided difference basis are the entries of the table from the top diagonal just as in the standard polynomial interpolation case when the divided difference basis is used. We illustrate this in the next example.

Example: Use the modified divided difference basis to construct the Hermite interpolant to set $\mathbf{S} = \{(1,2,1),(3,1,-1),(4,2,0)\}$.

The corresponding divided difference table is

x	**0^{th}DD**	**1^{st}DD**	**2^{nd}DD**	**3^{rd}DD**	**4^{th}DD**	**5^{th}DD**
1	2					
		$\boxed{1}$				
1	2		$-3/4$			
		$-1/2$		$1/4$		
3	1		$-1/4$		$1/6$	
		$\boxed{-1}$		$3/4$		$-34/72$
3	1		2		$-5/4$	
		1		-3		
4	2		-1			
		$\boxed{0}$				
4	2					

Entries in the boxes are derivative data values.

It follows that the Hermite interpolating polynomial is

$$\mathbf{H}_5(\mathbf{x}) = 2 + 1(\mathbf{x}-1) - \frac{3}{4}(\mathbf{x}-1)^2 + \frac{1}{4}(\mathbf{x}-1)^2(\mathbf{x}-3) + \frac{1}{6}(\mathbf{x}-1)^2(\mathbf{x}-3)^2 - \frac{34}{72}(\mathbf{x}-1)^2(\mathbf{x}-3)^2(\mathbf{x}-4)$$

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The Error In Hermite Interpolation

Theorem Let f(x) be in C^{2n+2}[a, b] and let all the x-coordinates of the data be in [a, b]. For Hermite interpolant $\mathbf{H_{2n+1}(x)}$ to the data set

$$\mathbf{S} = \{(\mathbf{x_i}, \mathbf{f(x_i)}, \mathbf{f'(x_i)}) \mid \mathbf{i} = 0, 1, 2, \ldots, \mathbf{n}\}$$

of 2n + 1 distinct points for each x in [a, b] there exist an α_x in [a, b] such that

$$\mathbf{f(x)} - \mathbf{H_{2n+1}(x)} = \frac{\prod_{k=0}^{n}(\mathbf{x}-\mathbf{x_k})^2}{(2\mathbf{n}+2)!}\mathbf{f}^{(2n+2)}(\alpha_x).$$

Example: Let $\mathbf{f(x)} = \mathbf{sin(x)}$. Construct the Hermite data set for $\mathbf{x} = 0, \pi/2$, the Hermite interpolant to the data and find a least upper bound on the error.

The data set is $\{(0,0,1),(\pi/2,1,0)\}$.

The divided difference table is ➔

x	0^{th}DD	1^{st}DD	2^{nd}DD	3^{rd}DD
0	0			
		1		
0	0		–0.2313	
		$2/\pi$		–0.1107
$\pi/2$	1		–0.4053	
		0		
$\pi/2$	1			

The Hermite interpolant is

$$\mathbf{H_3(x)} = 0 + 1(\mathbf{x}-0) - 0.2313(\mathbf{x}-0)^2 - 0.1107(\mathbf{x}-0)^2(\mathbf{x}-\pi/2).$$

The error is given by $\mathbf{E(x)} = \frac{(\mathbf{x}-0)^2(\mathbf{x}-\pi/2)^2}{4!}\mathbf{f}^{(4)}(\alpha_x)$. The fourth derivative of f(x) is sin(x) so we have

$$|\mathbf{E(x)}| = \left|\frac{(\mathbf{x}-0)^2(\mathbf{x}-\pi/2)^2}{4!}\right|\left|\mathbf{f}^{(4)}(\alpha_x)\right| \le \frac{1}{24}\max_{x\in[0,\pi/2]}\left|(\mathbf{x}-0)^2(\mathbf{x}-\pi/2)^2\right|\max_{x\in[0,\pi/2]}|\mathbf{sin(x)}|$$

$$\le \frac{1}{24}\max_{x\in[0,\pi/2]}\left|(\mathbf{x}-0)^2(\mathbf{x}-\pi/2)^2\right|$$

One way to proceed is to plot $(\mathbf{x}-0)^2(\mathbf{x}-\pi/2)^2$ over [0, π/2] and estimate its max in absolute value. Alternatively we could use calculus with max/min techniques. From the graph it is rather easy to observe that the max is at $x = \pi/4 \approx 0.7854$. It follows that

$$|\mathbf{E(x)}| \le \left(\frac{\pi}{4}\right)^4 \frac{1}{24} \approx 0.01585$$

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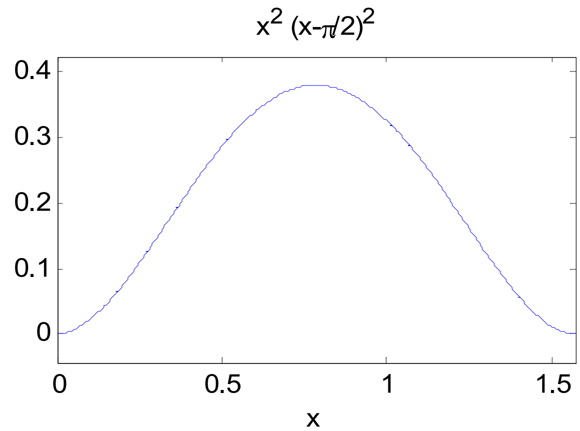

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Comment: It is possible to modify the divided difference procedure to handle mixed data; that is, a data set, which may contain points where some have derivative information and others do not.