Chapter 3: Expectation & Moment Generating Function

1. Expectation
2. Expectations for a function of X
3. $E[ag(x) + bh(x)] = a \cdot E[g(x)] + b \cdot E[h(x)]$
4. Special Expectaions
   (1) Variance
   (2) $K^{th} - momentdf$
   (3) $K^{th} - moment$ (about the mean)
   (4) $K^{th} - factorial \ \ moment$
5. $Var(aX + b) = a^2 Var(X)$
6. Markov's Inequality
7. Chebyshev's Inequality
8. Degenerate Distribution
9. Variance Stabilizing Tranformation
10. ----- NO EXAM

1. Expectation

- Discrete R.V. with p.m.f $f(x)$ and support set A, then

$$E(X) = \sum_{x \in A} x \cdot f(x)$$

  provided the sum converges absolutely, i.e.

$$E[\ |X|\ ] = \sum_{x \in A} |x| \ \cdot f(x) < \infty$$

  if $E[|X|] = \infty$, then $E(X)$ doesn't exist.
- Continous R.V. with p.d.f. $f(x)$ and support set A, then

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\ dx$$

  provided the sum converges absolutely, i.e.

$$E[\ |X|\ ] = \int_{-\infty}^{\infty} |x| \ \cdot f(x) < \infty$$

2. Expectations of a function of X

If X is discrete,

$$E[h(X)] = \sum_{x \in A} h(x) \cdot f(x) \qquad \text{(i.e. weighted average)}$$

- Provided that $E(\mid h(x) \mid) < \infty$, if X is continous,

$$E(\mid h(x) \mid) = \int_{-\infty}^{\infty} h(x)\, f(x)\; dx$$

3. $E[ag(x) + bh(x)] = a \cdot E[g(x)] + b \cdot E[h(x)]$

Provided that $E(\mid h(x) \mid) < \infty$, if X is continous

4. Special Expectations

(1) Variance

$$\sigma^2 = Var(X) = E(X^2) - [E(X)]^2$$

(2) k^{th} moment

$$E(X^k)$$

(3) k^{th} moment (about the mean)

$$E\left[(X - \mu)^k\right]$$

(4) k^{th} factorial moment

$$E\left[X^{(k)}\right] = E\left[X \cdot (X-1) \cdots (X-k+1)\right]$$

5. $Var(aX + b) = a^2 Var(X)$

6. Markov's Inequality

$$P\left(|X| \geq c\right) \leq \frac{E|X|^k}{C^k} \qquad (\forall\ k, c > 0)$$

7. Chebyshev's Inequality

Suppose X is a r.v. w/ finite mean μ and finite variance σ^2. Then, $\forall k > 0$,

$$P\left(|X - \mu| \geq k\sigma\right) \leq \frac{E\left(X-\mu\right)^2}{\left(k\sigma\right)^2} = \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{\sigma^2}$$

8. Degenerate Distribution

If $\mu = E(X), \sigma^2 = Var(X) = 0$, then we have

$$P(X = \mu) = 1$$

9. Variance Stabilizing Transformation

- R.V. X
- $E(X) = \theta$
- $Var(X) = \sigma^2(\theta)$

We aim to find $Y = g(X) \quad \text{s.t.} \quad Var(g(X)) \quad$ is a constant.

$$
\begin{gathered}
Y = \alpha + \beta z + \epsilon \\
E(\epsilon) = 0 \quad \Rightarrow \quad E(Y) = \alpha + \beta z \\
Var(\epsilon) = \sigma^2 \Rightarrow Var(Y) = \sigma^2
\end{gathered}
$$

$$
\begin{gathered}
Y = g(x) \approx g(\theta) + g'(\theta)(X - \theta) \qquad \text{(form: } Y = \alpha + \beta z + \epsilon \\
E(Y) \approx g(\theta) + g'(\theta)(\theta - \theta) = g(\theta) \\
Var(Y) \approx \left[g'(\theta)\right]^2 \cdot Var(X - \theta) = \left[g'(\theta)\right]^2 \sigma^2(\theta) = \left[g'(\theta) \cdot \sigma(\theta)\right]^2
\end{gathered}
$$

If we want $Var(Y)$ to be a constant, we just need $g'(\theta) \cdot \sigma(\theta) \approx$ constant.
That is, $g'(\theta) = \frac{k}{\sigma(\theta)}$ where k is a conveniently chosen constant.