Sample Problems from Lectures

Chapter 1

5. Boole's Inequality

If $A_1, A_2, \cdots$ is a sequence of events,

then $P(\bigcup_{i=1}^{\infty} A_i) \le \sum_{i=1}^{\infty} P(A_i)$

proof:

$B_1 = A_1$

$B_2 = A_2 \setminus A_1 = A_2 \cap A_1^c$

$B_3 = A_3 \setminus (A_1 \cup A_2) = A_3 \cap (A_1 \cup A_2)^c$

Base

$B_1 \cup B_2 = A_1 \cup (A_2 \cap A_1^c)$

$= (A_1 \cup A_2) \cap \underbrace{(A_1 \cup A_1^c)}_{S}$

$= A_1 \cup A_2$

$B_i = A_i \cap \left(\bigcup_{j=1}^{i-1} A_j\right)^c$

then we have $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ (*)

$B_1 \cap B$

$B_i \subseteq A_i \Rightarrow P(B_i) \le P(A_i)$

B_i's are p.m.e (pairwise mutually exclusive)

$B_1 \cap B_2 = \underbrace{A_1 \cap (A_2 \cap A_1^c)}_{\phi}$

$= \phi$

$$P(\bigcup_{i=1}^{\infty} A_i) \underset{(*)}{=} P(\bigcup_{i=1}^{\infty} B_i) \overset{(3)}{\Rightarrow} \underbrace{\sum_{i=1}^{\infty} P(B_i)}_{\text{p.m.e}} \le \underbrace{\sum_{i=1}^{\infty} P(A_i)}_{\text{Not p.m.e}} \quad \boxtimes$$

EX: Roll a dice twice

$S = \{(1,1), \cdots, (6,6)\}$

$P(i,j) = \frac{1}{36}$

A: sum ≤ 4

B: sum ≥ 10

$P(A \cup B) = 1/3$

$A = \{(1,1), (1,2), (2,1), (2,2), (3,1), (1,3)\}$

$P(A) = 6/36 = 1/6$

$B = \{(4,6), (5,5), (6,4), (5,6), (6,5), (6,6)\}$

$P(B) = 6/36 = 1/6$ Hence $P(A \cap B) = 1/3$

since $A \cap B = \emptyset$, A, B are p.m.e.

Let $C = \{i = j\}$ suppose we know C happened

$P(A \cup B \mid C) = \frac{4}{6} = \frac{2}{3}$

$S^* = \{\underbrace{(1,1), (2,2)}_{\text{sum} \leq 4}, \cdots, \underbrace{(5,5), (6,6)}_{\text{sum} \geq 10}\}$

7. Law of Total probability

Let $B_1, \cdots, B_n$ is a collection of m.e. and exhaustive events $\left(\bigcup_{i=1}^{n} B_i = S \right)$

Then $P(A) = \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i)$

proof $P(A) = P(A \cap S)$

$$= P\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right)$$

$$= P\left(\bigcup_{i=1}^{n} (A \cap B_i)\right)$$

$$\overset{(3)}{=} \sum_{i=1}^{n} P(A \cap B_i)$$ since $(A \cap B_1), (A \cap B_2) \cdots$ are p.m.e

$$= \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i)$$

Example Two groups of people : accident-prone and those who are not.

For those accident-prone, the chance of having accident in year $\nearrow 0.4$
For those not ... , " $\searrow 0.2$

Suppose 30% of customers are accident-prone.
What is the probability that a new policy holder will have a claim within a year?

Sol^n $A = \{\text{have a claim}\}$
$B_1 = \{\text{accident-prone}\}$
$B_2 = \{\text{not accident-prone}\}$

For accident-prone, chance of accident
$\Rightarrow P(A|B_1)$
" not "
$\Rightarrow P(A|B_2)$

$B_1 \cap B_2 = \emptyset$ (mutually exclusive)
$B_1 \cup B_2 = S \Rightarrow B_1, B_2$ is a partition. (exhaustive)

(Also Exhaustive)

By Law of Total probability,

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2)$$
$$= 0.4 \times 0.3 + 0.2 \times 0.7$$
$$= 0.26 \quad \blacksquare$$

$P(A) = \sum P(A|B_i)P(B_i)$
where $\bigcup_{i=1}^{n} B_i = S$ ① p.m.e ② exh

Example John & Michelle communicate thru email. They agree they will reply on the same day. Due to unstable connection, $\frac{1}{n}$ emails cannot reach the destination on the same day.

Now, J sent an email to M., but didn't receive a reply on the same day. what is the prob that M received J's email?

Sol^n $A = \{M \text{ received J's email}\}$

$B = \{J \text{ received M's reply on the same day}\}$

$B^c = \{$ " didn't receive " $\}$

we should find $P(A|B^c)$.

$$P(A|B^c) = \frac{P(B^c|A)P(A)}{P(B^c|A)P(A) + P(B^c|A^c)P(A^c)} = \frac{\frac{n-1}{n^2}}{\frac{1}{n}\cdot\frac{n-1}{n} + 1\cdot\frac{1}{n}}$$

$\cdot\ P(A) = \frac{n-1}{n}$ $\quad \cdot\ P(B^c|A) = \frac{1}{n}$ $\quad \cdot\ P(B^c|A^c) = 1$

$P(B^c|A^c)$ = Given M didn't receive J's email, what is the probability that J didn't receive M's reply?

⇒ Since M didn't receive any email from J, there is no chance M will reply. Hence, M absolutely doesn't reply Hence, 1.

$$\underset{\left(\substack{\text{Mult } n^2 \text{ on}\\ \text{both sides}}\right)}{=} \frac{n-1}{n-1+n} = \frac{n-1}{2n-1}$$

Example : toss a fair coin twice.

$$S = \{HH, HT, TH, TT\}$$

$$P_i = 1/4 \text{ for } i = 1, \cdots, 4$$

$A_1 = \{H \text{ on } 1^{st} \text{ toss}\}$, $A_2 = \{H \text{ on } 2^{nd} \text{ toss}\}$
$B = \{\text{exactly } 1H \text{ and } 1T\}$

	A_1	A_2	B
HH	✓	✓	
HT	✓		✓
TH		✓	✓
TT			

$$P(A_1) = P(A_2) = P(B) = \frac{1}{2}$$

$$\left.\begin{aligned} P(A_1 \cap A_2) &= \frac{1}{4} = P(A_1)P(A_2) \\ P(A_1 \cap B) &= \frac{1}{4} = P(A_1)P(B) \\ P(A_2 \cap B) &= \frac{1}{4} = P(A_2)P(B) \end{aligned}\right\}$$

A_1, A_2, B are pair-wise independent (refer to #9)

$P(A_1 \cap A_2 \cap B) = 0$ } Not Mutually Exclusive (refer to # 10)

Chapter 2

Example Toss a coin 3 times.
$X = \#$ of heads

$X(w) : S \longrightarrow \mathbb{R}$

S	$\mathbb{R}$
TTT	0
HHH	3
HHT	2
HTT	1
HTH	2
THH	2
THT	1
TTH	1

- $p(X=0) = 1/8$
- $p(X=1) = 3/8$
- $p(X=2) = 3/8$
- $p(X=3) = 1/8$

(p.m.f: probability mass function; aka frequency function)

$P(\{w, X(w) = 3\})$
$= P(\{HHH\})$
$= 1/8$

p.s.f of x

x	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$2/8$	$1/8$

Ex Toss a fair coin 3 times. $X = \#$ of heads

$X : S \to \mathbb{R}$

HHH	3
HHT	2
HTH	2
HTT	1
THH	2
THT	1
TTH	1
TTT	0

x	0	1	2	3
$p(X=x)$	1/8	3/8	3/8	1/8

Example Show $f(x) = \dfrac{\mu^x e^{-\mu}}{x!}$ for 1) $x = 0, 1, \cdots$ 2) $\mu > 0$

is a p.m.f.

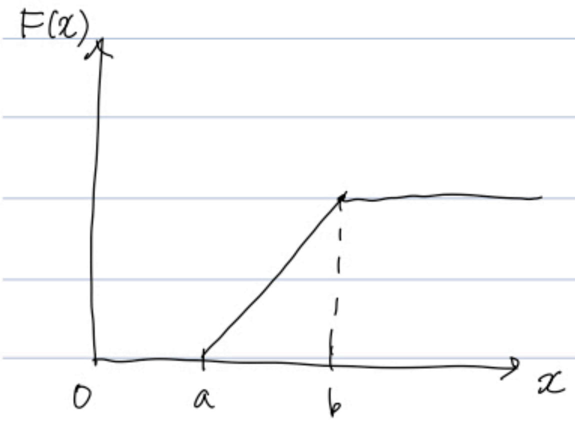
Sol^n (1) $f(x) > 0$

(2) $\displaystyle\sum_{x \in 0}^{\infty} f(x) = \sum_{x=0}^{\infty} \frac{\mu^x e^{-\mu}}{x!} = e^{-\mu} \boxed{\sum_{x=0}^{\infty} \frac{\mu^x}{x!}} = e^{-\mu} \cdot e^{\mu} = 1$

$\Downarrow$

$\left(\displaystyle\sum_{x=0}^{\infty} \frac{K^x}{x!} = \frac{K^0}{0!} + \frac{K^1}{1!} + \frac{K^2}{2!} + \cdots = e^K \right)$

Example $F(x) = \begin{cases} 0 & x \le a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$



For $a < x < b$, $F(x) = \frac{1}{b-a} x - \frac{a}{b-a}$

Hence,

$$f(x) = F'(x) = \begin{cases} 0 & x < a \\ \frac{1}{b-a} & a < x < b \\ 0 & x > b \end{cases}$$

$F(x)$ is not differentiable at $x = a$ or $x = b$.

Hence, the support set of X is (a, b)

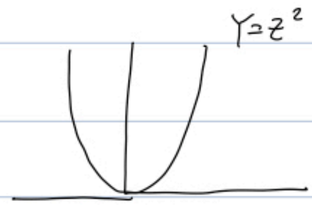
$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$$

Ex: $f(x) = \frac{\theta}{x^{\theta+1}}$ for $x \ge 1$

For what values of θ is the function a p.d.f?

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_1^{\infty} \frac{\theta}{x^{\theta+1}}\,dx = -\frac{1}{x^{\theta}}\Big|_1^{\infty} = -\lim_{a\to\infty} a^{-\theta} + 1 = 1 \text{ for } \theta > 0$$

Ex : If $Z \sim N(0,1)$, find the p.d.f of $Y = Z^2$



Sol^n : Let $\underbrace{G(y)}_{\text{c.d.f}} = P(Y \le y) = P(Z^2 \le y)$

$$= P(-\sqrt{y} \le Z \le \sqrt{y})$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}}_{\text{even function}} dz$$

How can we know the function is even? If you put sqrt(y) and -sqrt(y) into z, both result e^(-y/2).

$$= 2\int_0^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$(\sqrt{y})' = (y^{1/2})' = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$

p.d.f

$$g(y) = G'(y) = \frac{2}{\sqrt{2\pi}} e^{-\frac{(\sqrt{y})^2}{2}} (\sqrt{y})'$$

$$= \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}} \quad \text{for } y > 0$$

which is a p.d.f of $\chi^2(1)$.

$\Rightarrow Y \sim \chi^2(1)$

Note $g(x) = \int_0^x f(t)\,dt$

$g'(x) = f(x)$

$g(x) = \int_0^{h(x)} f(t)\,dt$ chain rule

$g'(x) = f(h(x)) \cdot h'(x)$

...is method generalizes:

If f is a continuous function and g and h are differentiable functions, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(s)\,ds = \frac{d}{dx}\left[F(h(x)) - F(g(x))\right]$$
$$= F'(h(x))\,h'(x) - F'(g(x))\,g'(x)$$
$$= f(h(x))\,h'(x) - f(g(x))\,g'(x).$$

II. One-to-One Transformation of a continuous r.v.

Suppose X is a cont r.v. with p.d.f f and $A = \{x : f(x) > 0\}$ and $Y = h(X)$ where h is one-to-one. Let g be the p.d.f of Y, then

$$g(y) = f(h^{-1}(y)) \left|\frac{d}{dh} h^{-1}(y)\right|, \quad y \in B \text{ where } B = \{y : g(y) \ge 0\}$$

Case (1) h is increasing for some x \in A
Case (2) h is decreasing for some x \in A

proof since h is one-to-one, it is either monotonic increasing or decreasing.

(1) If $h \uparrow$ for $x \in A$, then $h^{-1}(y)$ is also $\uparrow$ for $y \in B$

and $\frac{d}{dy} h^{-1}(y) > 0$.

Think about mapping
f(x)=x^2
as x inc, f(x) inc
As f^{-1}(x) inc, x inc

Inverse of Strictly Monotone Function

Theorem

Let f be a real function which is defined on $I \subseteq \mathbb{R}$.

Let f be strictly monotone on I.

Let the image of f be J.

Then f always has an inverse function f^{-1} and:

- if f is strictly increasing then so is f^{-1}
- if f is strictly decreasing then so is f^{-1}.

$$G(y) = P(Y \le y) = P(h(x) \le y) = P(X \le h^{-1}(y))$$

$$= F(h^{-1}(y))$$

Put h^{-1} on both sides

$$g(y) = G'(y) = \frac{d}{dy} F(h^{-1}(y))$$

$$= f(h^{-1}(y)) \cdot \frac{d}{dy} h^{-1}(y) \quad \text{(by chain rule)}$$

$$= f(h^{-1}(y)) \left| \frac{d}{dy} h^{-1}(y) \right|$$

(2) If $h \downarrow$ for $x \in A$, then $h^{-1}(y)$ is also $\downarrow$ for all $y \in B$

and $\frac{d}{dy} h^{-1}(y) < 0$

$$G(y) = P(Y \le y) = P(h(x) \le y) = P(X \ge h^{-1}(y)) = 1 - F(h^{-1}(y))$$

$$g(y) = G'(y) = \underset{(-)}{-} f(h^{-1}(y)) \underbrace{\frac{d}{dy} h^{-1}(y)}_{(-)} = f(h^{-1}(y)) \underbrace{\left| \frac{d}{dy} h^{-1}(y) \right|}_{(+)}$$

Ex: find the p.d.f of y $Y = \ln(X)$ if X is the cont r.v. w/ p.d.f.

$f(x) = \frac{\theta}{x^{\theta+1}}, \quad x \ge 1, \theta > 0$ (note: pareto distribution)

- $Y = \ln(X)$ is one-to-one

Sol$^{\underline{n}}$ $Y = h(X) = \ln(X)$

$X = h^{-1}(Y) = e^{Y}$

$g(y) = \frac{\theta}{(e^{y})^{\theta+1}} |e^{y}| = \frac{\theta}{(e^{y})^{\theta}} = \theta \cdot e^{-y\theta}, \quad y \ge 0$

12. one-to-one transformation of a discrete r.v.

Suppose X is a discrete r.v. w/ p.m.f f and $A = \{x \mid f(x) > 0\}$ and $Y = h(X)$ where h is one-to-one. Then, the p.m.f of Y

$g(y) = P(Y = y) = f(h^{-1}(y)), \quad y \in B$

<u>proof</u> $g(y) = P(Y = y) = P(h(X) = y)$

$= P(X = h^{-1}(y))$

$= f(h^{-1}(y)), \quad y \in B$

Example Let $X \sim NB(r, p)$ (NB: negative binomial)

number of successes: r; chance of success: p

# of trials required to obtain "r" successes in repeated independent Bernoulli trials.

$$f(x) = \underbrace{P(X=x)}_{} = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \cdots$$ (Formula ①)

Let $Y = X - r$: # of failures to obtain "r" successes

$$g(y) = P(Y=y) = P(\underbrace{X-r}_{Y=X-r}=y) = \underbrace{P(X=y+r)}_{\text{(Formula ①)}} = \binom{y+r-1}{r-1} p^r (1-p)^y, \quad y=0,1,2,\cdots$$

Chapter 3

e.g. $f(x) = \dfrac{\theta}{x^{\theta+1}}, \quad x \geq 1, \ \theta > 0$

Find $E(X)$. For what θ values does $E(X)$ exist? → use $E(X)$

$$E(X) = \int_1^\infty x \left(\frac{\theta}{x^{\theta+1}}\right) dx$$

Rule $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + C$

$$= \theta \int_1^\infty \frac{1}{x^\theta} dx = \theta \cdot \left[\frac{x^{1-\theta}}{1-\theta}\right]_1^\infty$$

$$= \frac{\theta}{1-\theta}\left(\lim_{a\to\infty} a^{1-\theta} - 1\right)$$

want $1-\theta < 0$

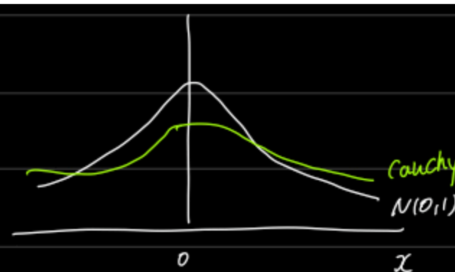
$1 < \theta$

$\therefore \theta > 1$

$\therefore$ when $\theta > 1$, $E(X) = -\left(\dfrac{\theta}{1-\theta}\right) = \dfrac{\theta}{\theta - 1}$

Ex Cauchy dist^r

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$



① $f(x) \geq 0$ for $\forall x \in \mathbb{R}$

② $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{1}{\pi}\arctan(x)\Big|_{-\infty}^{\infty}$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

$$E|X| = \int_{-\infty}^{\infty} |x| \frac{1}{\pi} \frac{1}{1+x^2}\,dx$$

$$= 2\int_0^{\infty} \frac{1}{\pi} \cdot \frac{x}{1+x^2}\,dx$$

$$= \frac{2}{\pi} \cdot \frac{\ln(1+x^2)}{2}\Big|_0^{\infty}$$

$$= \infty$$

$$\int \frac{1}{u}\,du = \ln(u)$$

$$\int \frac{x}{1+x^2}\,dx = \int \frac{1}{1+u}\frac{du}{2} = \frac{1}{2}\int \frac{1}{1+u}\,du$$

$u = x^2$, $\frac{du}{dx} = 2x$, $x\,dx = \frac{du}{2}$

$$= \frac{1}{2}\ln(1+u)$$

$$= \frac{1}{2}\ln(1+x^2)$$

Hence, $E|X|$ does not exist.

$$E(X) = \frac{1}{\pi}\int_{-\infty}^{\infty} x \cdot \frac{1}{1+x^2}\,dx$$

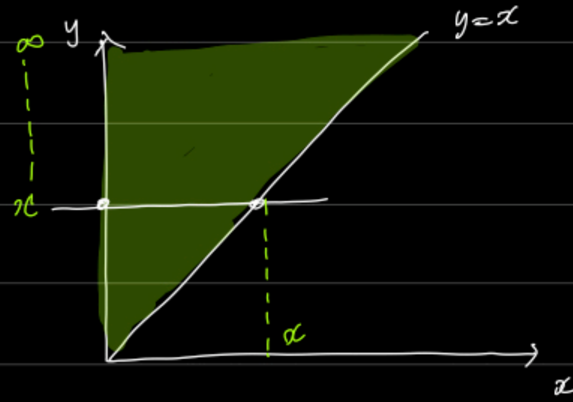
$$= \frac{1}{\pi} \cdot \frac{\ln(1+x^2)}{2}\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\pi}\left[\underbrace{\lim_{a\to\infty}\frac{\ln(1+a^2)}{2}}_{\infty} - \underbrace{\lim_{b\to-\infty}\frac{\ln(1+b^2)}{2}}_{\infty}\right]$$

Hence,

$E(X)$ d.n.e. ☒

Example Suppose X is a non-negative continuous r.v. w/ c.d.f $F(x)$. Show



$$E(X) = \int_0^\infty [1-F(x)]\,dx$$

$$= \int_0^\infty P(X > x)\,dx$$

$$= \int_0^\infty \underbrace{\int_x^\infty f_X(y)\,dy}_{y}\,dx$$

if confused ? ⇒ check ① w3 supplement, ② change of order – math insight

$$= \int_0^\infty \int_0^y f(y)\,dx\,dy$$

$$= \underbrace{\int_0^\infty y f_X(y)\,dy = E(X)}_{\text{By definition of E(X)}}$$

3. $E[a g(X) + b h(X)] = a \cdot E[g(X)] + b \cdot E[h(X)]$

proof If X is cont,

$$E[a g(X) + b h(X)] = \int_{-\infty}^{\infty} [a g(x) + b h(x)] f(x)\,dx$$

$$= a \int_{-\infty}^{\infty} g(x) f(x)\,dx + b \int_{-\infty}^{\infty} h(x) f(x)\,dx$$

$$= a \cdot E[g(x)] + b \cdot E[h(x)]$$

6. Markov's Inequality

$$p(|X| \geq C) \leq \frac{E|X|^K}{C^K} \quad \text{for all } K, C > 0$$

<u>proof</u> Assume X is cont.

$$\frac{E|X|^K}{C^K} = \int_{-\infty}^{\infty} \left|\frac{X}{C}\right|^K f(x)\,dx$$

$$= \int_{\left|\frac{X}{C}\right| \geq 1} \left|\frac{X}{C}\right|^K f(x)\,dx + \underbrace{\int_{\left|\frac{X}{C}\right| < 1} \left|\frac{X}{C}\right| f(x)\,dx}_{\geq 0}$$

$$\geq \int_{\left|\frac{X}{C}\right| \geq 1} \left|\frac{X}{C}\right|^K f(x)\,dx$$

$$\geq \int_{\left|\frac{X}{C}\right| \geq 1} f(x)\,dx = p\left(\left|\frac{X}{C}\right| \geq 1\right) = p(|X| \geq C)$$

7. Chebyshev's Inequality

Suppose X is a r.v. w/ finite mean μ and finite variance σ^2. Then, for $\forall K > 0$,

$$p(|X-\mu| \geq K\sigma) < \frac{1}{K^2}$$

(useful when the distribution is unknown)

<u>proof</u> by Markov's Inequality,

$$p(|X-\mu| \geq K\sigma) \leq \frac{E(X-\mu)^2}{(K\sigma)^2} = \frac{\sigma^2}{K^2\sigma^2} = \frac{1}{K^2}$$

In conjunction with Markov's Inequality

7. Chebyshev's Inequality

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In conjunction with Markov's Inequality

8. Degenerate Distribution

If $\mu = E(X)$, $\sigma^2 = Var(X) = 0$, then we have $p(X=\mu) = 1$.

proof $\{X \neq \mu\} = \bigcup_{i=1}^{\infty} \{|X-\mu| > \frac{1}{i}\}$

$$= \bigcup_{i=1}^{\infty} \left\{ X > \mu + \frac{1}{i} \text{ or } X < \mu - \frac{1}{i} \right\}$$

$$p(X \neq \mu) = p\left(\bigcup_{i=1}^{\infty} |X-\mu| > \frac{1}{i}\right)$$

(1) $\leq \sum_{i=1}^{\infty} p\left(|X-\mu| > \frac{1}{i}\right)$ By Boole's Inequality (for sum)

(2) $\leq \sum_{i=1}^{\infty} \frac{E(X-\mu)^2}{(1/i)^2}$ By Markov's Inequality, k=2

$p(|X-\mu| \geq k\sigma) < \frac{1}{k^2}$

$k\sigma = \frac{1}{i}$ $k^2 = \frac{1}{i^2\sigma^2}$

Refer to "Degenerate Distribution-Textbook" in Notability/STAT330

$\frac{1}{k^2} = i^2\sigma^2$

$$= \sum_{i=1}^{\infty} i^2 \cdot \sigma^2$$

$$= 0$$

$\Rightarrow p(X \neq \mu) = 0$

$\Rightarrow p(X = \mu) = 1$

Example If $X \sim exp(\theta)$, then show that $Y = g(X) = \ln(X)$ has approx constant variance.

$E(X) = \int x f(x)\, dx$

$f(x) = \frac{1}{\theta} \cdot e^{-x/\theta}$, $x > 0$, $\theta > 0$

$E(X) = \int^{\infty} \ldots$

$$E(X) = \int_0^\infty x \cdot \frac{1}{\theta} e^{-\frac{x}{\theta}} dx$$

$$= \int_0^\infty y e^{-y} d\, y\theta \qquad y = \frac{x}{\theta}$$

$$= \theta \int_0^\infty y e^{-y} dy$$

$$= \theta \cdot \Gamma(2) \qquad \alpha = 2$$

$$= \theta$$

Gamma Function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$

(1) $\Gamma(1) = 1$

(2) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(3) $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$

(4) $\Gamma(n) = (n-1)!$ for $n = 1, 2, \cdots$

$$E(X^2) = \int_0^\infty x^2 \frac{1}{\theta} \cdot e^{-\frac{x}{\theta}} dx$$

$$= \int_0^\infty y^2 \theta \cdot e^{-y} d\, y\theta \qquad y = \frac{x}{\theta}$$

$$= \theta^2 \int_0^\infty y^2 e^{-y} dy$$

$$= \theta^2 \Gamma(3)$$

$$= 2\theta^2 \qquad \Gamma(3) = 2! = 2$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= 2\theta^2 - \theta^2$$

$$= \theta^2$$

$$Y = g(X) = \ln(X) \qquad \text{then } Var(Y) \approx \left[\frac{1}{\theta} \cdot \theta\right]^2 = 1$$

$$g'(X) = \frac{1}{X} \qquad Var(Y) \approx [g'(\theta) \cdot \sigma(\theta)]^2$$

$$g'(\theta) = \frac{1}{\theta}$$

Example If $X \sim poi(\theta)$, $Y = g(X) = \sqrt{X}$ has approx constant variance.

- $E(X) = Var(X) = \theta$ (since $X \sim poi(\theta)$)
- $Y = g(X) = \sqrt{X}$ • $g'(x) = \frac{1}{2\sqrt{x}}$
- $g'(\theta) = \frac{1}{2}\theta^{-1/2} = \frac{1}{2\sqrt{\theta}}$

$$Var(Y) \approx \left(\frac{1}{2\sqrt{\theta}} \cdot \sqrt{\theta}\right)^2 = 1/4$$

```
E(X) = \theta
Var(X) = \theta
\sigma(\theta) =
\sqrt{Var(x)} =
\sqrt{\theta}
```