Chapter 2: Random Variables

Summary

1. Definition

A random vairalbe is a function from S to R. $X : S \to R$ in which $P(X \leq x)$ always exists for $\forall x \in R$.

2. Cumulative Distribution Function (c.d.f.)

The c.d.f. or a r.v. is defined by $F(x) = P(X \leq x), x \in R$.
$F(x) = \mathbb{R}[0, 1]$.

3. Properties of c.d.f.

(1) F is non-decreasing

5. Probability Mass Function (p.m.f.)

If x is discrete, the p.m.f. of x is

$$f(x) = P(X = x) = F(x) - \lim_{\epsilon \to 0^+} F(X - \epsilon) = F(x) - \lim_{a \to x^-} F(a)$$

6. Properties of p.m.f.

f is a p.m.f. for some discrete r.v. if

$$(1) \quad f(x) = 0$$

$$(2) \quad \sum_{x \in A} f(x) = 1$$

$$\text{Note: } \sum_{x \in A} f(x) = \sum_{\forall x_i \in A} P(X = x_i) = P(\cup_{i=1}^{\infty} \{X = x_i\} = P(S) = 1$$

7. Continous Random Variables

If $F(x)$ is a continous function $\forall x \in \mathbb{R}$ and F is differentiable except possibly at countable many points, then X is a continous r.v.

8.Probability Density Function(p.d.f.)

If X is a continuous r.v. w/ c.d.f. $P(x)$, then the p.d.f. of X is

$$F'(x) = \frac{d}{dx} F(x).$$

If $F(x)$ is differentiable at x, and otherwise we define $f(x) = 0$.
$A = \{x : f(x) > 0\}$ is the support set of X.

9. Properties of p.d.f.

f is the p.d.f. for some continous random variables X iff

$$\begin{aligned}
&(1) \quad f(x) \geq 0 \\
&(2) \quad \int_{-\infty}^{\infty} f(x)dx = 1 \\
&(3) \quad f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} && \text{(if the limit exists)} \\
&(4) \quad F(x) = \int_{-\infty}^{\infty} f(t)dt && (\text{for } x \in \mathbb{R}) \\
&(5) \quad P(a < x \leq b) \\
&\quad = P(X \leq b) - P(X \leq a) \\
&\quad = \int_{-\infty}^{b} f(t)dt - \int_{-\infty}^{a} f(t)dt \\
&\quad = \int_{a}^{b} f(t)dt \\
&(6) \quad P(X = b) = 0 = \int_{b}^{b} f(t)dt \\
&(7) \quad P(a < X \leq b) = P(a \leq X \leq b) = P(a \leq X < b) = P(a < X < B) \\
&\quad \text{(this is not true for discrete r.v.)}
\end{aligned}$$

10.Distribution of functions of a r.v. (c.d.f. technique)

Suppose X is a continous r.v. with p.d.f. f and c.d.f. F , and we wish to find the p.d.f. of the r.v. $Y = h(x)$

11. One-to-One Transformations fo a continuous r.v.

Suppose X is a continuous r.v. with p.d.f. f and $A = x : f(x) > 0$ and $Y = h(x)$ where h is one-to-one. Let y be the p.d.f. of Y, then

$$g(y) = f(h^{-1}(y)) \cdot \left| \frac{d}{dh} h^{-1}(y) \right| \quad y \in B \text{ where } B = \{y : g(y) \geq 0\}$$

12. One-to-One transformation of a discrete r.v.

Suppose X is a discrete r.v. w/ p.m.f. f and $A = \{x \mid f(x) > 0\}$ and $Y = h(X)$ where h is one-to-one. Then, the p.m.f of Y is

$$g(y) = P(Y = y) = f(h^{-1}(y)), \quad y \in B \text{ where } B = \{y : g(y) \geq 0\}$$

13. Probability Integral Transformation

If X is a continuous r.v. w/ c.d.f. F and F is strictly increasing, then $Y = F(X) \sim uniform(0, 1)$

We want to generate $y_1, y_2, \cdots, y_n$ from F.
We first generate $x_1, x_2, \cdots, x_n$ from $Uniform(0, 1)$, then
Let $y_i = F^{-1}(x_i)$, then $y_1, y_2, \cdots, y_n \sim F$

Example

If $Z \sim N(0, 1)$, find the p.d.f. of $Y = Z^2$

Proof:

Let $F'(z) = f(z) = e^{-z^2/2}$

$$
\begin{aligned}
G(y) &= P(Y \leq y) \\
&= P(Z^2 \leq y) \\
&= P(-\sqrt{y} \leq z \leq \sqrt{y}) \\
&= \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz && \text{(even function)} \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{y}} e^{-z^2/2} dz \\
&= \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{y}} f(z) dz
\end{aligned}
$$

$$
\begin{aligned}
g(y) &= G'(y) \\
&= \frac{d}{dy} \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{y}} f(z) dz \\
&= \frac{2}{\sqrt{2\pi}} \frac{d}{dy} \left[F(\sqrt{y}) - F(0) \right] \\
&= \frac{2}{\sqrt{2\pi}} \left[f(\sqrt{y}) \cdot \frac{d}{dy} (\sqrt{y}) - f(0) \cdot \frac{d}{dy} 0 \right] && \text{(chain rule)} \\
&= \frac{2}{\sqrt{2\pi}} \left[e^{-(\sqrt{y})^2/2} \cdot \frac{1}{2\sqrt{y}} - 0 \right] \\
&= \frac{1}{\sqrt{2\pi y}} \cdot e^{-y/2} \qquad for \qquad y > 0
\end{aligned}
$$

Since $G(y)$ is c.d.f, $g(y)$ is p.d.f.

$$
Y \sim \chi(0, 1)
$$