INVARIANT MEASURES IN $1d$

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Abstract. These are a set of revised notes on Bourgain's invariant measures result. They are based on Hiro Oh's treatment, with some exposition, corrections, and additional details of my own. In what follows, we do not track factors of 2π seriously. They are irrelevant to the discussion at hand and tend to serve only as a distraction.

1. Warmup - an Infinite Dimensional Gaussian Measure

As a warmup, let's construct the formal object

$$\mathrm{d}\rho_s = Z^{-1}\exp\left(-\frac{1}{2}\|u\|_{H^s}^2\right)\mathrm{d}u$$

as a well-defined probability measure (the measurable space on which this is defined will come along as a bonus). To do this, first recall the Littlewood-Paley projections on the torus: for any dyadic integer $N \in \mathbb{N}$, define the Littlewood-Paley projection by

$$P_{\leqslant N}u = \sum_{|n|\leqslant N} \widehat{u}_n e^{in\cdot x}$$

where $u : \mathbb{T}^d \to \mathbb{C}$ is a sufficiently regular function. Note that $\|P_{\leqslant N}u\|_{H^s}^2 = \sum_{|n|\leqslant N}\langle n\rangle^{2s}|\widehat{u}_n|^2$

Now define for each $N \in \mathbb{N}$ a measure

$$\begin{aligned}\mathrm{d}\rho_N = Z_N^{-1}\exp(-\frac{1}{2}\|u_{\leqslant N}\|_{H^s}^2)\mathrm{d}(u_{\leqslant N}) &= Z_N^{-1}\exp\left(-\frac{1}{2}\sum_{|n|\leqslant N}\langle n\rangle^{2s}|\widehat{u}_n|^2\right)\mathrm{d}\widehat{u}_n \\ &= Z_N^{-1}\prod_{|n|\leqslant N}\exp\left(-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2\right)\mathrm{d}\widehat{u}_n\end{aligned}$$

Note that the measure

$$\mathrm{d}\nu_n = \exp\left(-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2\right)\mathrm{d}\widehat{u}_n$$

is a complex Gaussian measure with mean zero and variance $2\langle n\rangle^{-2s}$. In the natural way, for each n we can consider $\widehat{u}_n$ to be the Lebesgue measure on $\mathbb{C}_{\widehat{u}_n}$ (i.e. the complex numbers with complex coordinate $\widehat{u}_n$).

With this in mind, we will now change perspective (for a not entirely obvious reason). Fix a sufficiently rich probability space $(\Omega, \mathcal{A}, \mathbb{P})$. One can think of this as being $([0,1], \mathcal{B}([0,1]), \mathrm{d}\lambda)$ where λ is the $1d$ Lebesgue measure and $\mathcal{B}$ is the Borel σ-algebra.

On the space $(\Omega, \mathcal{A}, \mathbb{P})$, let g be a standard complex Gaussian random variable (i.e. mean 0 and variance 2). We can relate this data to our current setup in the following way:

Lemma 1. $\nu_n = \left(\frac{g}{\langle n\rangle^s}\right)_\sharp \mathbb{P}$

Proof. By a standard approximation argument, it suffices to show (for any Borel subset $E \subset \mathbb{C}$)

$$\int_E \mathrm{d}\nu_n = \int_{g^{-1}(\langle n\rangle^s E)} \mathrm{d}\mathbb{P}$$

However, by definition

$$\int_{g^{-1}(\langle n\rangle^s E)} \mathrm{d}\mathbb{P} = \mathbb{P}\left[\frac{g(\omega)}{\langle n\rangle^s} \in E\right]$$
$$= \int_E (2\pi)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\langle n\rangle^{2s}|\widehat{u}_n|^2 \mathrm{d}\widehat{u}_n\right)$$

where the final line comes from the definition of a Gaussian random variable (and the rescaling we did to g). □

As a result, we can think of the measure ρ_N as being a pushforward measure: define a map $\mathcal{G}(\omega) : \Omega \to \mathbb{C}^{2N+1}$ by

$$\mathcal{G}(\omega) = \left(\frac{g_n(\omega)}{\langle n\rangle^s}\right)_{|n|\leqslant N}$$

where (g_n) are a family of iid standard complex Gaussians. Then by the exact same argument as given for ν_n, we can write

$$\rho_N = \mathcal{G}_\sharp \mathbb{P}.$$

Remark 1. Alternatively, one can see this by the identity (which is immediate from the definitions of the objects involved)

$$\rho_N = \bigotimes_{|n|\leqslant N} \nu_n$$

from which we can immediately deduce what the map $\mathcal{G}$ ought to be.

Now, we want to try to take the limit of the ρ_N in a suitable sense. Formally, this is trying to define the limit of

$$Z_N^{-1} \exp\left(-\frac{1}{2}\|u_{\leqslant N}\|_{H^s}^2\right) d(u_{\leqslant N})$$

as $N \to \infty$. As a start, we should see when

$$u_{\leqslant N} = \sum_{|n|\leqslant N} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}$$

converges in $H^\sigma(\mathbb{T}^d)$. Directly computing, we see

$$\mathbb{E}\left[\|u_{\leqslant N} - u_{\leqslant M}\|_{H^\sigma}^2\right] = \mathbb{E}\left[\sum_{N<|n|\leqslant M} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2s-2\sigma}}\right]$$
$$= 2 \sum_{N<|n|\leqslant M} \langle n\rangle^{2\sigma-2s}.$$

One can clearly see that the latter sum converges if and only if $2\sigma - 2s < -d$, which is equivalent to $\sigma < s - \frac{d}{2}$. In this case, we define

$$\rho_s := \left(\sum_{n\in\mathbb{Z}^d} \frac{g_n(\omega)}{\langle n\rangle^s} e^{inx}\right)_\sharp \mathbb{P}$$

which is a measure with $\mathrm{supp}(\rho_s) \subset H^{s-\frac{d}{2}-}(\mathbb{T}^d)$.

Remark 2. One might wonder why we defined ρ_N using a sequence, whereas ρ_s is defined with a random Fourier series. However, this is purely artificial; by Plancherel's theorem, we can define

$$\rho_s = \left(\frac{g_n(\omega)}{\langle n \rangle^s} \right)_\sharp \mathbb{P}$$

and the two definitions give coincident measures. The point is that any apparent difference in definitions is just a red herring, which is hopefully apparent upon looking at the two seemingly different definitions of ρ_N given initially.

Remark 3. (This one is actually important, so read it).
The construction given above is an example of an 'abstract Weiner space'. The general data of an abstract Weiner space are as follows: Let H be an infinite dimensional (separable) Hilbert space. Consider the formal 'measure'

$$\mathrm{d}\rho = Z^{-1} \exp\left(-\frac{1}{2} \|u\|_H^2 \right) \mathrm{d}u$$

One can show that ths is not a measure because it is not countably additive (when $\dim(H) = \infty$). A possible fix is to enlarge H by embedding it in a Banach space B:

$$i \colon H \hookrightarrow B$$

where the embedding i is continuous and dense. We then say the triple (H, B, ρ) is an abstract Weiner space if

$$\int_B e^{i\langle u, \phi \rangle} \mathrm{d}\rho(u) = e^{-\frac{1}{2}\|\phi\|_H^2}$$

for all $\phi \in B^* \subset H$. That is, $\langle u, \phi \rangle$ is a standard Gaussian random variable for every ϕ.

In our case, the space H is $H^s(\mathbb{T}^d)$, and the space B is the (also Hilbert) space $H^{s-\frac{d}{2}-}(\mathbb{T}^d)$.

The first thing we'll prove is an exponential tail bound for ρ_s, which intuitively should be reasonable, as ρ_s is a Gaussian measure.

Lemma 2. *Let $\sigma < s - \frac{d}{2}$. Then there exists $c > 0$ so that*

$$\rho_s(\|u\|_{H^\sigma} > K) \lesssim \exp\left(-2cK^2\right) \quad \forall K > 0$$

Proof. Since $t \mapsto t^2$ and $t \mapsto \exp(t)$ are increasing functions, we have

$$\rho_s(\|u\|_{H^\sigma} > K) = \rho_s(e^{c\|u\|_{H^\sigma}^2} > e^{cK^2})$$

By Chebyshev's inequality, the latter measure can be bounded by

$$\rho_s(e^{c\|u\|_{H^\sigma}^s} > e^{cK^2}) \leqslant e^{-2cK^2} \int_{H^\sigma} e^{2c\|u\|_{H^\sigma}^2} \mathrm{d}\rho_s(u).$$

It then remains to show that the integral is bounded. By Plancherel, we can rewrite the integral as

$$\int_{H^\sigma} \exp\left(2C \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2\sigma} |\widehat{u}_n|^2 \right) \mathrm{d}\rho_s(u)$$

Now we recall that $\rho_s = \left(\frac{g_n(\omega)}{\langle n \rangle^s} \right)_\sharp \mathbb{P}$. Making the change of measure amounts to replacing (for each n) $\widehat{u}_n \mapsto \langle n \rangle^{-s} g_n(\omega)$. Hence we need to evaluate

$$\prod_{n \in \mathbb{Z}^d} \int_\Omega \exp\left(2c\langle n \rangle^{2\sigma - 2s} |g_n(\omega)|^2\right) \mathrm{d}\mathbb{P} = \prod_{n \in \mathbb{Z}^d} \mathbb{E}\left[2c\langle n \rangle^{2\sigma-2s} |g_n(\omega)|^2\right]$$

However, since $g_n(\omega)$ is a standard complex Gaussian, we rewrite

$$\prod_{n\in\mathbb{Z}^d} \mathbb{E}\left[2c\langle n\rangle^{2\sigma-2s}|g_n(\omega)|^2\right] = \prod_{n\in Z^d} \int_{\mathbb{C}} \exp(2c\langle n\rangle^{2\sigma-2s}|z|^2)\exp\left(-\frac{1}{2}|z|^2\right)\mathrm{d}z$$

$$= \prod_{n\in\mathbb{Z}^d} \frac{1}{1-4c\langle n\rangle^{2\sigma-2s}}$$

$$= \prod_{n\in\mathbb{Z}^d}\left(1+\frac{4c\langle n\rangle^{2\sigma-2s}}{1-4c\langle n\rangle^{2\sigma-2s}}\right)$$

By the criterion for the convergence of a product of nonnegative terms $\prod_n(1+a_n)$, we see that convergence of the product is equivalent to being able to sum $\langle n\rangle^{2\sigma-2s}$, which as above is equivalent to $\sigma < s - \frac{d}{2}$. □

2. Invariant Measures

In this section, we will construct an invariant measure associated to

$$i\partial_t u + \partial_x^2 u = \pm|u|^{p-1}u \tag{2.1}$$

where $u = u(t,x)\colon \mathbb{R}\times\mathbb{T} \to \mathbb{C}$. One can obtain (2.1) as the Fréchet derivative of the following Hamiltonian:

$$H[u] := \frac{1}{2}\int_{\mathbb{T}} |\partial_x u|^2\,\mathrm{d}x \pm \frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\,\mathrm{d}x \tag{2.2}$$

The measure will be given by

$$\mathrm{d}\mu := Z^{-1}\exp\left(-\frac{1}{2}M[u]-H[u]\right)\mathrm{d}u$$

By using the notation above, we can rewrite μ as

$$\mathrm{d}\mu := Z^{-1}\exp\left(\mp\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right)\mathrm{d}\rho_1(u)$$

where $\operatorname{supp}(\rho_1)\subset H^{\frac{1}{2}-}(\mathbb{T})$, using the numerology from above. If we want the measure μ to be a probability measure, we should at least ask that $\exp\left(\mp\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right)$ be integrable with respect to ρ_1. We separate this into the defocusing and focusing cases. In the defocusing case, we can exploit Sobolev embedding:

$$\|u\|_{L^{p+1}}^{p+1} \lesssim \|u\|_{H^{\frac{1}{2}-}}^{p+1}.$$

Since $\operatorname{supp}(\rho_1)\subset H^{\frac{1}{2}-}$, this quantity is finite ρ_1-a.s. This gives that

$$0 < \exp\left(-\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) \leqslant 1$$

and thus μ is a perfectly sensible probability measure on $H^{\frac{1}{2}-}$.

In the focusing case, things are more complicated. Let $p > 1$. Then since $\mathbb{T}$ is a compact manifold, by Hölder's inequality,

$$\|u\|_{L^2}^{p+1} \lesssim \|u\|_{L^{p+1}}^{p+1}$$

However, we can explicitly write out what the L^2-norm is for u in the support of the measure ρ_1. Indeed, we have

$$\begin{aligned}\|u\|_{L^2}^{p+1} &= \left(\sum_{n\in\mathbb{Z}} |\widehat{u}_n|^2\right)^{\frac{p+1}{2}} \\ &= \left(\sum_{n\in\mathbb{Z}} \left|\frac{g_n(\omega)}{\langle n\rangle}\right|^2\right)^{\frac{p+1}{2}}\end{aligned}$$

Since $2 < p+1$, we have the inclusion $\ell^2(\mathbb{Z}) \hookrightarrow \ell^{p+1}(\mathbb{Z})$. Hence

$$\left(\sum_{n\in\mathbb{Z}} \left|\frac{g_n(\omega)}{\langle n\rangle}\right|^2\right)^{\frac{p+1}{2}} \geqslant \left\|\frac{g_n(\omega)}{\langle n\rangle}\right\|_{\ell^{p+1}}$$

In particular, we have the lower bound

$$\mathbb{E}_{\rho_1}\left[\exp\left(\frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\right)\right] \geqslant \prod_{n\in\mathbb{Z}} \mathbb{E}\left[\exp\left(\langle n\rangle^{-p-1}|g_n|^{p+1}\right)\right]$$

However, since $p+1 > 2$, the expectation on the right-hand side is infinite, which follows from direct computation of the expected value. This shows us that we cannot expect to construct the measure μ as it currently stands.

A new tack is to somehow introduce a cutoff and hope for better behavior as a result. Since the mass $\|u\|_{L^2}$ is conserved over the evolution of the nonlinear Schrödinger equation, we introduce a mass cutoff and instead consider the measure defined by

$$\mathrm{d}\mu := Z^{-1}\mathbb{1}_{\{\|u\|_{L^2}\leqslant K\}} \exp\left(\frac{1}{p+1}\int |u|^{p+1}\,\mathrm{d}x\right)\mathrm{d}\rho_1(u)$$

Proposition 1.

(1) If $1 < p < 5$, we have

$$\mathcal{R}(u) = \mathcal{R}_K(u) := \mathbb{1}_{\{\|u\|_{L^2}\leqslant K\}} \exp\left(\frac{1}{p+1}\int_{\mathbb{T}} |u|^{p+1}\,\mathrm{d}x\right) \in L^r(d\rho_1) \tag{2.3}$$

for every K and $1 \leqslant r < \infty$.

(2) If $p = 5$, (2.3) holds by choosing K sufficiently small.

Remark 4. It can be shown that in the second case the upper bound is $K < \|Q\|_{L^2}$ where Q is the $1d$ NLS ground state. Further, note that $p = 5$ corresponds to $1d$ mass-critical NLS, which is known to exhibit finite-time blowup.

To prove this, we will need a so-called 'large deviation estimate' for iid Gaussian random variables. The remainder of the proof is the harmonic analysis argument given in Bourgain's original paper from 1994.

Lemma 3 (Large deviations for sums of Gaussians)**.** *Let $(g_n)_n$ be a family of iid standard real-valued Gaussian random variables on a sufficiently rich probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Then*

$$\mathbb{P}\left[\left(\sum_{n=1}^{M} |g_n|^2\right)^{\frac{1}{2}} \geqslant R\right] \lesssim \exp\left(-\frac{1}{4}R^2\right)$$

where $R \geqslant 3M^{\frac{1}{2}}$ and $M \geqslant 1$.

Proof. Since $x \mapsto tx$ and $x \mapsto \exp(x)$ are strictly increasing (as long as $t > 0$), by Chebyshev's inequality we have

$$\begin{aligned}\mathbb{P}\left[\left(\sum_{n=1}^{M}|g_n|^2\right)^{\frac{1}{2}} \geqslant R\right] &\leqslant \exp\left(-2tR^2\right)\mathbb{E}\left[\exp\left(2t\sum_{n=1}^{M}|g_n|^2\right)\right] \\ &\leqslant \exp\left(-2tR^2\right)\prod_{n=1}^{M}\mathbb{E}[2tg_n^2]\end{aligned}$$

for any $0 < t < \frac{1}{4}$. By direct computation, the expected value is

$$\mathbb{E}[2tg_n^2] = (1-4t)^{-\frac{1}{2}}$$

so that we may write (up to factors of 2π)

$$\exp(-2tR^2)\prod_{n=1}^{M}\mathbb{E}[2tg_n^2] = \exp(-2tR^2)(1-4t)^{-\frac{M}{2}}.$$

Using the restriction on M and R, we choose $t = \frac{1}{4}(1-\frac{M}{R^2}) < \frac{1}{4}$. Plugging this in, we obtain

$$\begin{aligned}\exp(-2tR^2)(1-4t)^{-\frac{M}{2}} &\leqslant \left(\frac{R^2}{M}\right)^{-\frac{M}{2}}\exp(-\frac{1}{2}R^2+\frac{1}{2}M) \\ &\leqslant e^{\frac{M}{2}\log\frac{R^2}{M}+\left(\frac{1}{18}-\frac{1}{2}\right)R^2} \\ &\leqslant e^{-\frac{1}{4}R^2}\end{aligned}$$

which uses the fact that $\log(x) \leqslant \frac{x}{4}$ for $x \geqslant 9$. □

Proof of Proposition 1. To be totally explicit, we want to understand

$$\left\|\mathbb{1}_{\{\|u\|_{L^2}\leqslant K\}}\exp\left(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right)\right\|_{L^r(\rho_1)}$$

By the layer cake decomposition, this is equivalent to

$$\begin{aligned}&r\int_0^\infty \lambda^{r-1}\rho_1\left[\mathbb{1}_{\{\|u\|_{L^2}\leqslant K\}}\exp\left(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) > \lambda\right]\mathrm{d}\lambda \\ &\qquad = r\int_0^\infty \lambda^{r-1}\rho_1\left[\exp\left(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) > \lambda, \|u\|_{L^2}\leqslant K\right]\mathrm{d}\lambda\end{aligned}$$

We will get this into a more tractable form by changing variables. By using the fact that ρ_1 is a probability measure, we can bound the latter integral by

$$\begin{aligned}&r\int_0^\infty \lambda^{r-1}\rho_1\left[\exp\left(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) > \lambda, \|u\|_{L^2}\leqslant K\right]\mathrm{d}\lambda \\ &\qquad \leqslant C_0 + r\int_{\lambda_0}^\infty \lambda^{r-1}\rho_1\left[\exp\left(\frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}\right) > \lambda, \|u\|_{L^2}\leqslant K\right]\mathrm{d}\lambda\end{aligned}$$

where $C_0 \sim \lambda_0^r$ and $\lambda_0 > 2$ (say). We only need enough room to take logarithms. Indeed, by rearranging inside the measure above, we can rewrite the integral as

$$r\int_{\lambda_0}^\infty \lambda^{r-1}\rho_1\left[\|u\|_{L^{p+1}} > (\log(\lambda^{p+1}))^{\frac{1}{p+1}}, \|u\|_{L^2}\leqslant K\right]\mathrm{d}\lambda$$

Then define $t = (\log(\lambda^{p+1}))^{\frac{1}{p+1}}$. If we follow this change of variables through, we can rewrite the integral as

$$r \int_{\widetilde{\lambda_0}}^{\infty} t^p \exp\left(\frac{r}{p+1} t^{p+1}\right) \rho_1\left[\|u\|_{L^{p+1}} > t, \|u\|_{L^2} \leqslant K\right] \mathrm{d}t \tag{2.4}$$

where $\widetilde{\lambda_0}$ is the image of λ_0 under the change of variables. In particular, this representation of the L^r norm tells us that we should endeavor to obtain good estimates for $\rho_1[\|u\|_{L^{p+1}} > t, \|u\|_{L^2} \leqslant K]$.

To this end, we will find a 'scale of concentration'. Let $M_0 \in 2^{\mathbb{Z}}$ to be determined, and for notational brevity write $\alpha = p + 1$. By Bernstein's inequality,

$$\begin{aligned} \|P_{\leqslant M_0} u\|_{L^\alpha} &\leqslant M_0^{\frac{1}{2}-\frac{1}{\alpha}} \|P_{\leqslant M_0} u\|_{L^2} \\ &\leqslant M_0^{\frac{1}{2}-\frac{1}{\alpha}} K \end{aligned}$$

Now for any $\lambda > 1$, choose M_0 so that $K M_0^{\frac{1}{2}-\frac{1}{\alpha}} \sim \frac{1}{2}\lambda$. Having made this choice, set $M_j := 2^j M_0$. Also note that

$$\|P_{\leqslant M_0} u\|_{L^\alpha} \leqslant \frac{1}{2}\lambda.$$

Next, choose a sequence $(\sigma_j)_{j\geqslant 1}$ so that $\sum_{j=1}^{\infty} \sigma_j \sim \frac{1}{2}$. For example, one can choose $\sigma_j \sim M_0 M_j^{-1}$. Then by our choice of M_0, we know that

$$\begin{aligned} \rho_1\left[\|u\|_{L^\alpha} > \lambda, \|u\|_{L^2} \leqslant K\right] &\leqslant \sum_{j\geqslant 1} \rho_1\left[\|P_{M_j} u\|_{L^\alpha} > \sigma_j \lambda\right] \\ &\leqslant \sum_{j\geqslant 1} \rho_1\left[\|P_{M_j} u\|_{L^2} \gtrsim M_j^{\frac{1}{\alpha}-\frac{1}{2}} \sigma_j \lambda\right] \\ &\leqslant \sum_{j\geqslant 1} \mathbb{P}\left[\left\|\left(\langle n\rangle^{-1} g_n(\omega)\right)_{|n|\sim M_j}\right\| \gtrsim M_j^{\frac{1}{\alpha}-\frac{1}{2}} \sigma_j \lambda\right] \\ &\leqslant \sum_{j\geqslant 1} \mathbb{P}\left[\left(\sum_{|n|\sim M_j} |g_n(\omega)|^2\right)^{\frac{1}{2}} \gtrsim M_j^{\frac{1}{2}+\frac{1}{\alpha}} \sigma_j \lambda\right]. \end{aligned}$$

Next, we will apply Lemma 3, since the sum is over a finite number of Gaussian random variables. This gives us the bound

$$\begin{aligned} \sum_{j\geqslant 1} \mathbb{P}\left[\left(\sum_{|n|\sim M_j} |g_n(\omega)|^2\right)^{\frac{1}{2}} \gtrsim M_j^{\frac{1}{2}+\frac{1}{\alpha}} \sigma_j \lambda\right] &\lesssim \sum_{j=1}^{\infty} \exp\left(-\frac{1}{4}\lambda^2 \sigma_j^2 M_j^{1+\frac{2}{\alpha}}\right) \\ &\lesssim \exp\left(c\lambda^2 M_1^{1+\frac{2}{\alpha}}\right) \\ &\lesssim \exp\left(-c\lambda^2 \lambda^{\left(1+\frac{2}{\alpha}\right)\left(\frac{2\alpha}{\alpha-2}\right)}\right) \\ &\lesssim \exp(-c\lambda^{\frac{4\alpha}{\alpha-2}} K^{-\frac{2\alpha+4}{\alpha-2}}), \end{aligned}$$

where in the first inequality we applied Lemma 3, in the second we used the fact that a series of decaying exponentials is as large as its first term, and the remaining two steps are just plugging in the definition of M_1 and doing some algebra.

Note that $\frac{4\alpha}{\alpha-2} > \alpha$ if and only if $\alpha = p+1 < 6 \iff p < 5$, as desired. If we remember where we started, this shows us that

$$\rho_1\left[\|u\|_{L^{p+1}>t, \|u\|_{L^2}\leqslant K}\right] \lesssim \exp\left(-ct^{\frac{4(p+1)}{p-1}}K^{-\frac{2p+6}{p-1}}\right)$$

so that plugging into the integral (2.4), we have

$$(2.4) \lesssim \int_{\widetilde{\lambda_0}}^{\infty} t^p \exp\left(\frac{r}{p+1}t^{p+1} - ct^{\frac{4(p+1)}{p-1}}K^{-\frac{2p+6}{p-1}}\right)\, \mathrm{d}t.$$

By the numerological computations done above, we see that this integral is finite **for any** $r \in [1, \infty)$ and any $0 < K < \infty$ as long as $1 < p < 5$. If $p = 5$, then the powers of t in the exponential match exactly. In this case, choosing K sufficiently small guarantees finiteness of the integral (again, for any $r \in [1, \infty)$. □

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