**Lemma 1** (Convolution Estimates)**.** *Let $\alpha, \beta \in \mathbb{R}$ obeying $\alpha \geqslant \beta \geqslant 0$ and $\alpha + \beta > 1$. Then the following inequalities hold:*

$$\int_{\mathbb{R}} \langle x - k_1 \rangle^{-\alpha} \langle x - k_2 \rangle^{-\beta} \, \mathrm{d}x \lesssim \langle k_1 - k_2 \rangle^{-\beta} \phi_\alpha (k_1 - k_2) \tag{1}$$

*and*

$$\sum_{n \in \mathbb{Z}} \langle n - k_1 \rangle^{-\alpha} \langle n - k_2 \rangle^{-\beta} \lesssim \langle k_1 - k_2 \rangle^{-\beta} \phi_\alpha (k_1 - k_2) \tag{2}$$

*where the function $\phi_\alpha(k)$ is defined by the following relationship:*

$$\phi_\alpha(k) := \sum_{|n| \leqslant |k|} \langle n \rangle^{-\alpha} \sim \begin{cases} 1, & \alpha > 1 \\ \log(1 + \langle k \rangle), & \alpha = 1 \\ \langle k \rangle^{1-\beta}, & \alpha < 1 \end{cases} \tag{3}$$

*Proof.* Before we begin, we remark that our proof is agnostic to the notion of sum or integral. A similar argument to the one given here will furnish the proof for the other case.

To make the problem simpler, note that by making a clever $u$-substitution, we may rewrite the integral appearing in (1) as an honest-to-goodness convolution:

$$\int_{\mathbb{R}} \langle x - k_1 \rangle^{-\alpha} \langle x - k_2 \rangle^{-\beta} \, \mathrm{d}x = [\langle \cdot \rangle^{-\alpha} * \langle \cdot \rangle^{-\beta}](k_1 - k_2) \tag{4}$$

In particular, it suffices to establish the estimates claimed for the convolution integral

$$\int \langle x - y \rangle^{-\alpha} \langle y \rangle^{-\beta} \, \mathrm{d}y \tag{5}$$

which can be written in the following form:

$$\int_{\{y+z=x\}} \langle y \rangle^{-\alpha} \langle z \rangle^{-\beta} \mathrm{d}y \mathrm{d}z \tag{6}$$

We divide the set $A = \{(y, z) \in \mathbb{R}^2 \mid y + z = x\}$ into 4 smaller pieces:

$$A_1 = \{(y, z) \in A \mid |y| \geqslant 2|x|\} \tag{7}$$

$$A_2 = \{(y, z) \in A \mid |y| \leqslant \frac{1}{2}|x|\} \tag{8}$$

$$A_3 = \{(y, z) \in A \mid |z| \leqslant \frac{1}{2}|x|\} \tag{9}$$

$$A_4 = A \backslash \left( \bigcup_{i=1}^{3} A_i \right) \tag{10}$$

We then estimate each piece separately.

For $A_1$, use the triangle inequality to write

$$|z| = |x-y| \geqslant |y| - |x| \geqslant \frac{1}{2}|y|$$

where the final inequality follows from membership in $A_1$. Then we have

$$\begin{aligned}\int_{A_1} \langle y\rangle^{-\alpha}\langle z\rangle^{-\beta}\mathrm{d}y\mathrm{d}z &\leqslant \int_{A_1} \langle y\rangle^{-\alpha}\left(\frac{1}{2}\langle y\rangle\right)^{-\beta}\mathrm{d}y \\ &\lesssim \int_{|y|\geqslant 2|x|} \langle y\rangle^{-\alpha-\beta}.\end{aligned}$$

We then analyze this by cases; if $\alpha > 1$, we estimate the integral above by $\langle x\rangle^{1-\alpha-\beta}$; we can then bound the $\langle x\rangle^{1-\alpha}$ by 1 by the condition on $\alpha$, leaving the desired $\langle x\rangle^{-\beta}$. If $\alpha = 1$, we pull out $\sup_{|y|\geqslant 2|x|}\langle y\rangle^{-\beta}$ and again integrate $\langle x\rangle^{-1}$ to obtain the desired logarithmic behavior. Finally, if $\alpha < 1$, we again integrate to get $\langle x\rangle^{1-\alpha-\beta}$, which using the condition on $\alpha$, we crudely control by $\langle x\rangle^{1-\beta}\langle x\rangle^{-\beta}$.

Next, note that the sets $A_2$ and $A_3$ are symmetric in $y$ and $z$. Thus, it suffices to show the desired estimates for $A_2$ without loss of generality. On this set, we use the triangle inequality to write

$$|z| = |x-y| \geqslant |x| - |y| \geqslant \frac{1}{2}|x|$$

which tells us that

$$\int_{A_2} \langle y\rangle^{-\alpha}\langle z\rangle^{-\beta} \lesssim \langle x\rangle^{-\beta}\int_{|y|\leqslant\frac{1}{2}|x|}\langle y\rangle^{-\alpha}$$

which, upon performing the same analysis as above, leads to the same set of estimates.

Finally, on the set $A_4$, we have $|y|, |z| \geqslant \frac{1}{2}|x|$. This permits us to bound the integral over $A_4$ in the following way:

$$\int_{A_4} \langle y\rangle^{-\alpha}\langle z\rangle^{-\beta} \lesssim \langle x\rangle^{-\beta}\int_{|y|\geqslant\frac{1}{2}|x|}\langle y\rangle^{-\alpha}$$

which is amenable to the same analysis as above. $\square$