

## Black-Scholes Equation

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The Black-Scholes equation solves the option price using the stochastic calculus. This note is based on several lectures found on the Internet (e.g., Michael Dabkowski, University of Michigan).

### Delta Hedging

Delta hedging is a strategy that neutralizes a portfolio's sensitivity to stochastic price changes in the underlying asset by taking offsetting positions, i.e., buying or selling simultaneously the underlying stock and/or the related derivatives. Let  $P$  be a portfolio that buys one unit of call option contract (priced as  $C$ ) and sells  $\Delta$  shares of the underlying stock (priced as  $S$ ):

$$P = C - \Delta S$$

The derivative of the portfolio is:

$$dP = dC - \Delta dS$$

A delta neutral portfolio should equal to another portfolio that only earns risk free interest,  $r$ :

$$dP = rPdt$$

Using the Ito's Lemma and  $dS = \mu S dt + \sigma S dw$ , noting  $dw$  as the Wiener process:

$$dC = \frac{\partial C}{\partial t} dt + \frac{\partial C}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} dt$$

Therefore,

$$dP = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \left( \frac{\partial C}{\partial S} - \Delta \right) dS$$

If  $\Delta$  is selected such that the second parenthesis term vanishes, the portfolio becomes risk free. This leads the portfolio to be the same as the second risk free portfolio:

$$\left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) = r \left( C - \frac{\partial C}{\partial S} S \right)$$

Rearrange the terms, one has the Black-Scholes equation for call option price,  $C$ :

$$\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

and the "initial condition" (expiration condition) as shown below, where  $K$  is the strike price of the option contract at expiration,  $T$ :

$$C(S, t = T) = \max(S - K, 0)$$

## Solving the Partial Differential Equation

The solution strategy is to transform this non-homogeneous, non-standard parabolic partial differential equation to a standard parabolic partial differential equation using variable substitutions. First, one can substitute the variables  $t$  and  $S$  using the relations below so that the “expiration condition” becomes the initial condition and the logarithmic stock price is used:

$$\begin{aligned}\tau &= \frac{\sigma^2}{2}(T-t) \\ x &= \ln\left(\frac{S}{K}\right)\end{aligned}$$

The partial derivatives of the option price,  $C$ , with respect to  $t$  and  $S$  become:

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau} \frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2} \frac{\partial C}{\partial \tau}$$

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x} \frac{\partial x}{\partial S} = \frac{1}{S} \frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2} \left( \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \right)$$

Now, the partial differential equation has been transformed in terms of variables  $\tau$  (reverse the direction of time) and  $x$  (logarithmic stock price):

$$-\frac{\sigma^2}{2} \frac{\partial C}{\partial \tau} + r \frac{\partial C}{\partial x} + \frac{\sigma^2}{2} \left( \frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x} \right) = rC$$

and the initial condition has also been transformed to:

$$C(x, 0) = \max(K(e^x - 1), 0)$$

Further, the constants  $r$  and  $\sigma^2$  can be combined as a single  $\lambda$ :

$$\lambda = \frac{2r}{\sigma^2}$$

The transformed partial differential equation is much neater:

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial x^2} + (\lambda - 1) \frac{\partial C}{\partial x} - \lambda C$$

A mathematical trick is then used to eliminate the lower order terms in the equation by intentionally choosing the constants  $a$  and  $b$  in below function substitution using  $u(x, \tau)$ :

$$C(x, \tau) = K e^{ax+b\tau} u(x, \tau)$$

After carrying out all the tedious differentiations, the partial differential equation of  $u(x, \tau)$  is:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\lambda - 1 + 2a) \frac{\partial u}{\partial x} + [a(\lambda - 1) - \lambda + a^2 - b] u$$

Selecting the appropriate  $a$  and  $b$  such that all the lower order terms vanish:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where  $a = -(\lambda - 1)/2$  and  $b = -(\lambda + 1)^2/4$ . The initial condition of  $u(x, \tau)$  (at  $\tau = 0$ ) becomes:

$$u(x, 0) = \max(e^{ax+b\tau}(e^x - 1), 0) = \max\left(e^{(1-a)x} - e^{-ax}, 0\right), \text{ or}$$

$$u(x, 0) = \max\left(e^{(\lambda+1)x/2} - e^{(\lambda-1)x/2}, 0\right)$$

### Applying the Initial Condition

The solution of the standard parabolic partial differential equation with initial condition  $u(x, 0)$  is given by the following convolution:

$$u(x, \tau) = (\Gamma * u(x, 0)) = \int_{-\infty}^{\infty} \Gamma(x - z) u(z, 0) dz$$

where the kernel (filter) of the convolution is:

$$\Gamma(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-x^2/4\tau}$$

Since  $u(z, 0)$  has a non-zero value only if  $z > 0$ , note that  $z$  or  $x = \ln(S/K)$ ; otherwise  $u(z, 0) = 0$  as the payoff at the contract expiration. Therefore, the non-zero integration for  $z$  is from 0 to  $\infty$ . Further, let  $y = (z - x)/\sqrt{2\tau}$ , the solution becomes:

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} u(x + \sqrt{2\tau}y, 0) e^{-y^2/2} dy$$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \left[ e^{(\lambda+1)(x+\sqrt{2\tau}y)} - e^{(\lambda-1)(x+\sqrt{2\tau}y)} \right] e^{-y^2/2} dy$$

The next step is to rewrite the integrals as a form of the cumulative standard normal distribution function given the familiar term  $\exp(-y^2/2)dy$  in the integrals. Adding and subtracting  $(\lambda + 1)^2\tau/2$  to the first exponential term, and adding and subtracting  $(\lambda - 1)^2\tau/2$  to the second exponential term, the solution expands to:

$$u(x, \tau) = \frac{e^{(\lambda+1)x/2 + (\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-(\lambda+1)\sqrt{\tau/2})^2/2} dy$$

$$- \frac{e^{(\lambda-1)x/2 + (\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{-(y-(\lambda-1)\sqrt{\tau/2})^2/2} dy$$

Let  $-v_1 = y - (\lambda + 1)\sqrt{\tau/2}$  and  $-v_2 = y - (\lambda - 1)\sqrt{\tau/2}$ . Please note the negative sign of these two new variables so that the upper and lower limits of the integrals are flipped:

$$u(x, \tau) = \frac{e^{(\lambda+1)x/2 + (\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau} + (\lambda+1)\sqrt{\tau/2}} e^{-v_1^2/2} dv_1$$

$$- \frac{e^{(\lambda-1)x/2 + (\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau} + (\lambda-1)\sqrt{\tau/2}} e^{-v_2^2/2} dv_2$$

Now, both integrals are the cumulative standard normal distribution function with the “z-scores” at their upper integral limits. Therefore,

$$u(x, \tau) = e^{(\lambda+1)x/2 + (\lambda+1)^2\tau/4} N\left(\frac{x + (\lambda+1)\tau}{\sqrt{2\tau}}\right) - e^{(\lambda-1)x/2 + (\lambda-1)^2\tau/4} N\left(\frac{x + (\lambda-1)\tau}{\sqrt{2\tau}}\right)$$

### Back Substitution

Now, use  $u(x, \tau) = C(x, \tau)/Ke^{ax+b\tau}$  where  $a = -(\lambda - 1)/2$  and  $b = -(\lambda + 1)^2/4$ :

$$C(x, \tau) = Ke^x N\left(\frac{x + \lambda\tau + \tau}{\sqrt{2\tau}}\right) - Ke^{-\lambda\tau} N\left(\frac{x + \lambda\tau - \tau}{\sqrt{2\tau}}\right)$$

Note  $\tau = \sigma^2(T - t)/2$  and  $\lambda\tau = r(T - t)$ ; therefore

$$C(x, t) = Ke^x N\left(\frac{x + r(T - t) + \sigma^2(T - t)/2}{\sqrt{\sigma^2(T - t)}}\right) - Ke^{-r(T-t)} N\left(\frac{x + r(T - t) - \sigma^2(T - t)/2}{\sqrt{\sigma^2(T - t)}}\right)$$

Finally, substitute  $e^x = S/K$ ,  $x = \ln(S/K)$ , and reorganize the terms:

$$C(S, t) = SN\left(\frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right) - Ke^{-r(T-t)} N\left(\frac{\ln(S/K) + (r - \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}\right)$$

where  $t \in [0, T]$ .