Geometric Brownian Motion
05/04/2025

Brownian motion is a stochastic process, specifically a random walk or a Wiener process $\{w_t\}$, which can be regarded as a stochastic time series and $\Delta w_t = w_{t+\Delta t} - w_t$ with the following characteristics:

1) $\Delta w_t = \epsilon\sqrt{\Delta t}$, where ϵ is a standard normal random variable, *i.e.*, $\epsilon \sim N(0,1)$
2) Δw_t is independent of w_j for all j < t

The first condition reflects $\Delta w_t \sim N\left(0, \Delta t\right)$ and the second is the Markov property.

A generalized Wiener process has a non-zero drift μ and non-unity volatility σ as shown below:

$$\Delta x_t = \mu \Delta t + \sigma \Delta w_t$$

where μ and σ are constants. Further as $\Delta w_t \sim N(0, \Delta t)$,

$$\Delta x_t \sim N(\mu\Delta t, \sigma^2 \Delta t)$$

Ito's Process

An Ito's process allows μ and σ in the Wiener process to be functions of x_t, *i.e.*,

$$\Delta x_t = \mu(x_t, t)\Delta t + \sigma(x_t, t)\Delta w_t$$

However, w_t is not continuous and non-differentiable that makes analytical study of the Ito's process difficult. Kiyoshi Ito (1951) worked out a differentiation "chain rule" applicable to functions that contain the Wiener process. Let $F(x,t)$ be a differentiable function of x and t, and x is a generalized Wiener process, the conventional Taylor expansion of ΔF is:

$$\Delta F = \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 F}{\partial x \partial t}\Delta x \Delta t + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(\Delta t)^2 + \cdots$$

Now, let us introduce the Wiener process into x:

$$\Delta x = \mu \Delta t + \sigma \Delta w_t$$

$$(\Delta x)^2 = \mu^2(\Delta t)^2 + \sigma^2\epsilon^2\Delta t + 2\mu\sigma\epsilon(\Delta t)^{3/2} = \sigma^2\epsilon^2\Delta t + O((\Delta t)^{3/2})$$

$$E\left[\epsilon\right] = 0$$

$$E\left[\epsilon^2\right] = Var\left[\epsilon\right] = 1$$

$$E\left[(\Delta x)^2\right] = E\left[\sigma^2\epsilon^2\Delta t\right] = \sigma^2\Delta t$$

Therefore, as $\Delta t \to dt$, $\Delta x \to \mu dt + \sigma dw_t$ and $\left(\Delta x\right)^2 \to \sigma^2 dt$. Substituting these results into the ΔF Taylor expansion and ignore the higher order terms (*e.g.*, $dxdt\ and\ (dt)^2 etc.$):

$$
\begin{aligned}
dF &= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 dt \\
&= \left(\frac{\partial F}{\partial x}\mu + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2\right)dt + \frac{\partial F}{\partial x}\sigma dw_t
\end{aligned}
$$

This is the Ito's Lemma for stochastic calculus that expresses the differentiation of a general function that contains the Wiener process.

Geometric Brownian Motion

Financial asset prices follow a special Ito's process where the drift and volatility are proportional to the asset price, P; specifically:

$$
\begin{aligned}
&\mu(x_t, t) = \mu P \\
&\sigma(x_t, t) = \sigma P
\end{aligned}
$$

$$dP_t = \mu P dt + \sigma P dw_t$$

This is a geometric Brownian motion. The term “geometric” suggests that the change in P is a product of P as opposed to the term “arithmetic” that suggests a sum. Apply Ito's lemma to $F = ln(P_t)$:

$$
\begin{aligned}
&\frac{\partial F}{\partial P} = \frac{1}{P} \\
&\frac{\partial F}{\partial t} = 0 \\
&\frac{1}{2}\frac{\partial^2 F}{\partial P^2} = \frac{1}{2}\frac{-1}{P^2}
\end{aligned}
$$

The differentiation of $F = ln(P_t)$ becomes:

$$
\begin{aligned}
dF = dln(P) &= \left(\frac{1}{P}\mu P + \frac{1}{2}\frac{-1}{P^2}\sigma^2 P^2\right)dt + \frac{1}{P}\sigma P dw_t \\
&= \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dw_t
\end{aligned}
$$

It shows that the logarithmic asset price follows a generalized Wiener process with a drift $\mu - \sigma^2/2$ and a volatility σ . In fact, the logarithmic asset price has a normal distribution, or the asset price follows a lognormal distribution. Studies in the literature have repeatedly demonstrated this important finding with the early one (1959) attributed to M. F. Osborne at the Naval Research Lab.

$$\ln(\mathrm{P}_t) - ln(P_0) \sim N\left[\left(\mu - \tfrac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

In other words, ln (P) follows a straight line versus t with the slope of $\mu - \sigma^2/2$. The plots on the pages below show the logarithmic prices of several stock indices and their least square straight lines.

Stock Indeces	μ	σ	$\mu - \sigma^2/2$	Slope ($\ln P_t$ vs t)
SPY (2005-2025)	0.105	0.193	0.0864	0.0995
QQQ (2000-2025)	0.128	0.273	0.0907	0.1075
TQQQ (2010-2025)	0.512	0.617	0.3217	0.3441

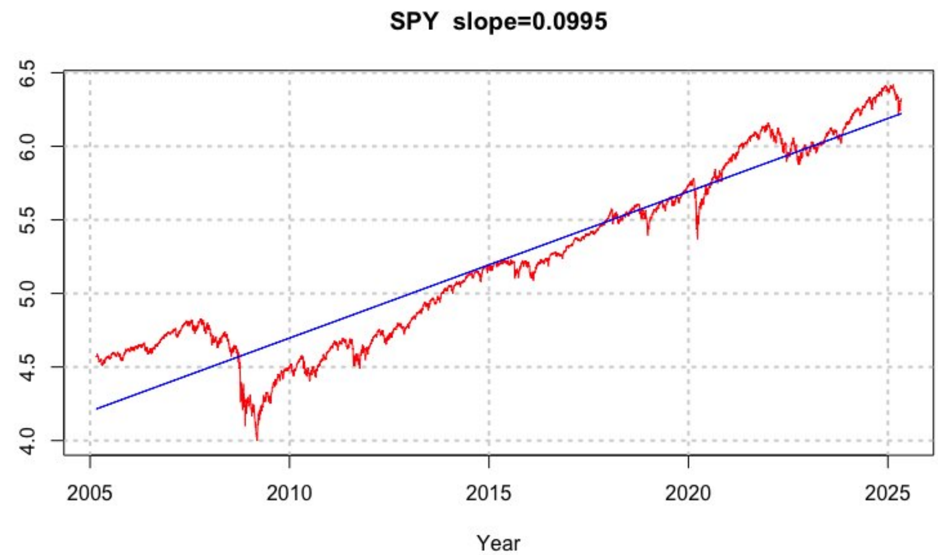

Figure 1. Logarithmic prices of SPY and the straight line fit from 2005 to 2025

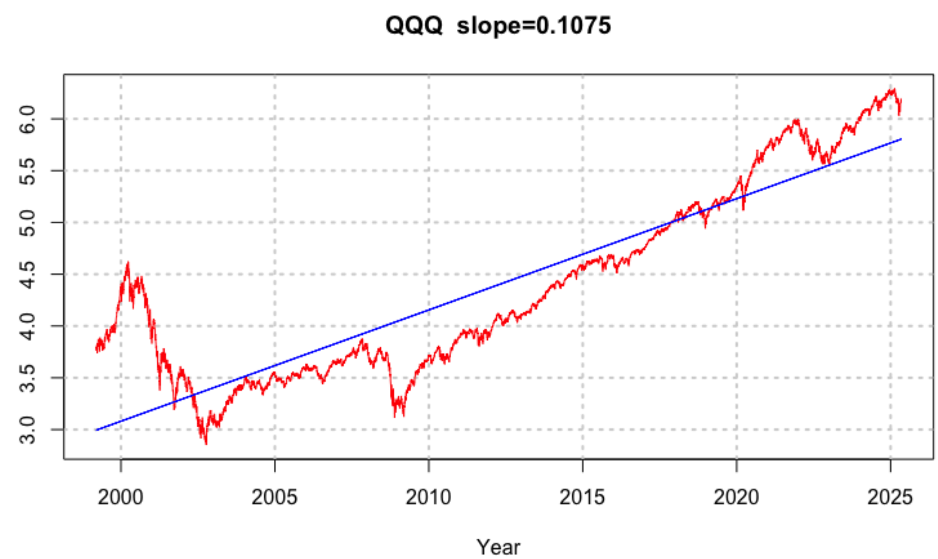

Figure 2. Logarithmic prices of QQQ and the straight line fit from 2000 to 2025

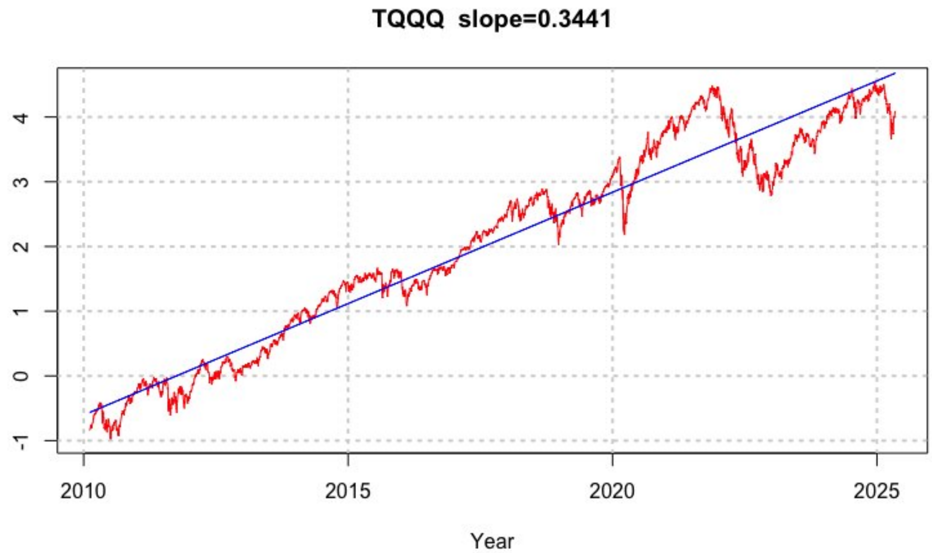

Figure 3. Logarithmic prices of TQQQ and the straight line fit from 2010 to 2025