Black-Scholes Equation
12/16/2025

The Black-Scholes equation solves the option price using stochastic calculus. This note is based on the lectures found on the Internet (*e.g.,* Michael Dabkowski, University of Michigan).

Delta Hedging

Delta hedging is a strategy that neutralizes a portfolio's sensitivity to stochastic price movements of the underlying asset by taking offsetting positions, i.e., buying or selling the underlying stock (or the related derivatives).   Let P be a portfolio that buys one unit of call option contract (C) and sells Δ shares of the underlying stock (S):

$$P = C - \Delta S$$

The derivative of the portfolio is:

$$dP = dC - \Delta dS$$

If the portfolio is delta neutral, it should be equal to another portfolio that earns risk free interest, r:

$$dP = rPdt$$

Using Ito's Lemma and $dS = \mu S dt + \sigma S dw$, noting dw as the Wiener's process:

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}dt$$

Therefore,

$$dP = \left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right)dt + \left(\frac{\partial C}{\partial S} - \Delta\right)dS$$

If one selects Δ such that the second parenthesis vanishes, the portfolio becomes risk free.   This leads the portfolio to be the same as the second risk free portfolio:

$$\left(\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}\right) = r\left(C - \frac{\partial C}{\partial S}S\right)$$

Rearrange the terms, one has the Black-Scholes equation for call options:

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} = rC$$

and the "initial condition" (terminal condition) is as below, where K is the strike price of the option contract:

$$C(S, t = T) = max(S - K, 0)$$

Solving the Partial Differential Equation

The solution strategy is to transform this non-homogeneous, non-standard parabolic partial differential equation to a standard parabolic partial differential equation through variable substitutions. First, one can substitute the variables t and S using the following relations so that the "terminal condition" becomes the initial condition and the equation uses the logarithmic stock price:

$$\tau = \frac{\sigma^2}{2}(T-t)$$
$$x = ln\left(\frac{S}{K}\right)$$

The partial derivative of the option price, C, with respect to t and S become:

$$\frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial C}{\partial \tau}$$

$$\frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial t} = \frac{1}{S}\frac{\partial C}{\partial x}$$

$$\frac{\partial^2 C}{\partial S^2} = \frac{1}{S^2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right)$$

Now, one has transformed the partial differential equation in variables τ (reverse the direction of time) and x (logarithmic stock price):

$$-\frac{\sigma^2}{2}\frac{\partial C}{\partial \tau} + r\frac{\partial C}{\partial x} + \frac{\sigma^2}{2}\left(\frac{\partial^2 C}{\partial x^2} - \frac{\partial C}{\partial x}\right) = rC$$

and the initial condition becomes:

$$C(x,0) = max\left(K(e^x - 1), 0\right)$$

Further, the constants r and σ^2 can be simplified as:

$$\lambda = \frac{2r}{\sigma^2}$$

The partial differential equation is much neater:

$$\frac{\partial C}{\partial \tau} = \frac{\partial^2 C}{\partial x^2} + (\lambda - 1)\frac{\partial C}{\partial x} - \lambda C$$

A mathematical trick is used to eliminate the lower differentiation terms in the above equation by selecting the constants a and b in the substitution form below:

$$C(x,\tau) = Ke^{ax+b\tau}u(x,\tau)$$

After carrying out the differentiations, the partial differential equation becomes:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (\lambda - 1 + 2a)\frac{\partial u}{\partial x} + \left[a(\lambda - 1) - \lambda + a^2 - b\right] u$$

Selecting the appropriate a and b so that all the lower order differentiation terms vanish:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

where $a = -(\lambda - 1)/2$ and $b = -(\lambda + 1)^2/4$. The initial condition ($\tau = 0$) now becomes:

$u(x, 0) = max\left(e^{ax+b\tau}(e^x - 1), 0\right) = max\left(e^{(1-a)x} - e^{-ax}, 0\right)$, or

$$u(x, 0) = max\left(e^{(\lambda+1)x/2} - e^{(\lambda-1)x/2}, 0\right)$$

<u>Applying the Initial Condition</u>

The solution for the standard parabolic partial differential equation with initial condition $u(x, 0)$ is given by the following convolution:

$$u(x, \tau) = (\Gamma * u(x, 0)) = \int_{-\infty}^{\infty} \Gamma(x - z)u(z, 0)dz$$

where the kernel (filter) for the convolution is:

$$\Gamma(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} e^{-x^2/4\tau}$$

Since $u(z, 0)$ has a non-zero value only if $z > 0$ (note z or $x = ln(S/K)$); otherwise $u(z, 0) = 0$ as the payoff at the contract expiration. Therefore, the integration range for z must be from 0 to ∞. Further, let $y = (z - x)/\sqrt{2\tau}$, the solution becomes:

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} u(x + \sqrt{2\tau}y, 0)e^{-y^2/2}dy$$

$$u(x, \tau) = \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \left[e^{(\lambda+1)(x+\sqrt{2\tau}y)} - e^{(\lambda-1)(x+\sqrt{2\tau}y)}\right] e^{-y^2/2}dy$$

Adding and subtracting $(\lambda + 1)^2\tau/2$ to the first exponential term, and adding and subtracting $(\lambda - 1)^2\tau/2$ to the second exponential term, the solution becomes:

$$u(x,\tau) = \frac{e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(y-(\lambda+1)\sqrt{\tau/2})^2/2} dy$$

$$- \frac{e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} e^{(y-(\lambda-1)\sqrt{\tau/2})^2/2} dy$$

Let $-v_1 = y - (\lambda+1)\sqrt{\tau/2}$ and $-v_2 = y - (\lambda-1)\sqrt{\tau/2}$. Please note the negative sign of these two new variables so that the range of the integrals are flipped:

$$u(x,\tau) = \frac{e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau}+(\lambda+1)\sqrt{\tau/2}} e^{-v_1^2/2} dv_1$$

$$- \frac{e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4}}{\sqrt{2\pi}} \int_{-\infty}^{x/\sqrt{2\tau}+(\lambda-1)\sqrt{\tau/2}} e^{-v_2^2/2} dv_2$$

Both integrals are the cumulative standard normal distribution function with the “z-scores” at their upper integral limits. Therefore,

$$u(x,\tau) = e^{(\lambda+1)x/2+(\lambda+1)^2\tau/4} N\left(\frac{x+(\lambda+1)\tau}{\sqrt{2\tau}}\right) - e^{(\lambda-1)x/2+(\lambda-1)^2\tau/4} N\left(\frac{x+(\lambda-1)\tau}{\sqrt{2\tau}}\right)$$

Back Substitution

Now, use $u(x,\tau) = C(x,\tau)/Ke^{ax+b\tau}$ where $a = -(\lambda-1)/2$ and $b = -(\lambda+1)^2/4$:

$$C(x,\tau) = Ke^{x} N\left(\frac{x+\lambda\tau+\tau}{\sqrt{2\tau}}\right) - Ke^{-\lambda\tau} N\left(\frac{x+\lambda\tau-\tau}{\sqrt{2\tau}}\right)$$

Note $\tau = \sigma^2(T-t)/2$ and $\lambda\tau = r(T-t)$; therefore

$$C(S,t) = Ke^{x} N\left(\frac{x+r(T-t)+\sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}\right) - Ke^{-r(T-t)} N\left(\frac{x+r(T-t)-\sigma^2(T-t)/2}{\sqrt{\sigma^2(T-t)}}\right)$$

Finally, substitute $e^x = S/K$, $x = ln(S/K)$, and reorganize the terms:

$$C(S,t) = SN\left(\frac{ln(S/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)} N\left(\frac{ln(S/K)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

where $t \in [0,T)$.