# Quadratic Model Algorithm

The polynomial model has a formula like below ($let\ s_t = ln(S_t)$):

$$s_t = \theta_0 1 + \theta_1 t + \theta_2 t^2 + \cdots + \theta_n t^n + \epsilon_t \tag{1}$$

where $t$ is the number of trading days, $\theta$ are model parameters, and $\epsilon$ are the error terms. For prices collected over m trading days, the data would fit below equation.

$$\begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} 1 & 1 & \dots & 1^n \\ 1 & 2 & \dots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & m & \dots & m^n \end{pmatrix} \begin{pmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_n \end{pmatrix} + \begin{pmatrix} \epsilon_0 \\ \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} \tag{2}$$

Or,

$$\overrightarrow{s} = \mathbf{K}\overrightarrow{\theta} + \overrightarrow{\epsilon} \tag{3}$$

The objective for the model determination is to select a vector of $\overrightarrow{\theta}$ so that the magnitude of the error term, or the square of it, is minimized.

$$\min \|\overrightarrow{\epsilon}\|^2 = \min \|\mathbf{K}\overrightarrow{\theta} - \overrightarrow{s}\|^2 = \min \left[ \sum_{i=1}^{m} \left( \sum_{j=0}^{n} K_{ij}\theta_j - s_i \right)^2 \right] \tag{4}$$

The partial derivative of $\|\overrightarrow{\epsilon}\|^2$ with respect to $\theta_k$ is

$$\nabla\left(\|\overrightarrow{\epsilon}\|^2\right)_k = \sum_{i=1}^{m} \left( \sum_{j=0}^{n} K_{ij}\theta_j - s_i \right) 2K_{ik} = \sum_{i=1}^{m} 2\left(\mathbf{K}^T\right)_{ki} \left(\mathbf{K}\overrightarrow{\theta} - \overrightarrow{s}\right)_i \tag{5}$$

$$\nabla\left(\|\overrightarrow{\epsilon}\|^2\right)_k = \left(2\mathbf{K}^T\left(\mathbf{K}\overrightarrow{\theta} - \overrightarrow{s}\right)\right)_k \tag{6}$$

To minimize $\|\overrightarrow{\epsilon}\|^2$, the first derivative of it must be zero.

$$\nabla\left(\|\overrightarrow{\epsilon}\|^2\right) = 2\mathbf{K}^T\left(\mathbf{K}\overrightarrow{\theta} - \overrightarrow{s}\right) = 0 \tag{7}$$

As the result, the model parameters are solved using the equation below

$$\overrightarrow{\theta} = \left(\mathbf{K}^T\mathbf{K}\right)^{-1}\mathbf{K}^T\overrightarrow{s} \tag{8}$$

Specifically,

$$\mathbf{K}^T\mathbf{K} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & m \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \dots & m^n \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1^n \\ 1 & 2 & \dots & 2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & m & \dots & m^n \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^{m} 1 & \sum_{i=1}^{m} i & \dots & \sum_{i=1}^{m} i^n \\ \sum_{i=1}^{m} i & \sum_{i=1}^{m} i^2 & \dots & \sum_{i=1}^{m} i^{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{m} i^n & \sum_{i=1}^{m} i^{n+1} & \dots & \sum_{i=1}^{m} i^{n+n} \end{pmatrix} \tag{9}$$

$$\mathbf{K}^T\overrightarrow{s} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 2 & \dots & m \\ \vdots & \vdots & \ddots & \vdots \\ 1^n & 2^n & \dots & m^n \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{m} 1 s_i \\ \sum_{i=1}^{m} i s_i \\ \vdots \\ \sum_{i=1}^{m} i^n s_i \end{pmatrix} \tag{10}$$

Further, the summation terms in $\mathbf{K}^T\mathbf{K}$ may be evaluated using Faulhaber's formula.

$$\sum_{i=1}^{n} i^p = \frac{1}{p+1} \sum_{r=0}^{p} \begin{pmatrix} p+1 \\ r \end{pmatrix} B_r n^{p-r+1} \tag{11}$$

where $B_r$ are the Bernoulli numbers.

# 1 Constant Model (n=0) :

$$\mathbf{K}^T\mathbf{K} = m \tag{12}$$

$$\mathbf{K}^T\overrightarrow{s} = \sum_{i=1}^{m} s_i \tag{13}$$

$$\overrightarrow{\theta} = \left(\mathbf{K}^T\mathbf{K}\right)^{-1}\mathbf{K}^T\overrightarrow{s} = \frac{1}{m}\sum_{i=1}^{m} s_i \tag{14}$$

The zero-th constant model predicts the average logarithmic price as the future prices.

# 2 Linear Model (n=1) :

$$\mathbf{K}^T\mathbf{K} = \begin{pmatrix} m & \sum\limits_{i=1}^{m} i \\ \sum\limits_{i=1}^{m} i & \sum\limits_{i=1}^{m} i^2 \end{pmatrix} \tag{15}$$

where

$$\sum_{i=1}^{m} i = \frac{1}{2}\left(m^2 + m\right) \tag{16}$$

$$\sum_{i=1}^{m} i^2 = \frac{1}{3}\left(m^3 + \frac{3}{2}m^2 + \frac{1}{2}m\right) \tag{17}$$

$$\mathbf{K}^T\overrightarrow{s} = \begin{pmatrix} \sum\limits_{i=1}^{m} s_i \\ \sum\limits_{i=1}^{m} i s_i \end{pmatrix} \tag{18}$$

One year model (m = 252):

$$\overrightarrow{\theta} = \begin{pmatrix} 0.016 \cdot \sum\limits_{i=1}^{m} s_i - 9.4859 \cdot 10^{-5} \sum\limits_{i=1}^{m} i s_i \\ -9.4859 \cdot 10^{-5} \cdot \sum\limits_{i=1}^{m} s_i + 7.4987 \cdot 10^{-7} \sum\limits_{i=1}^{m} i s_i \end{pmatrix} \tag{19}$$

Ten year model (m = 2520):

$$\overrightarrow{\theta} = \begin{pmatrix} 0.0016 \cdot \sum\limits_{i=1}^{m} s_i - 9.452 \cdot 10^{-7} \sum\limits_{i=1}^{m} i s_i \\ -9.452 \cdot 10^{-7} \cdot \sum\limits_{i=1}^{m} s_i + 7.4986 \cdot 10^{-10} \sum\limits_{i=1}^{m} i s_i \end{pmatrix} \tag{20}$$

# 3 Quadratic Model (n=2) :

$$\mathbf{K}^T\mathbf{K} = \begin{pmatrix} m & \sum_{i=1}^{m} i & \sum_{i=1}^{m} i^2 \\ \sum_{i=1}^{m} i & \sum_{i=1}^{m} i^2 & \sum_{i=1}^{m} i^3 \\ \sum_{i=1}^{m} i^2 & \sum_{i=1}^{m} i^3 & \sum_{i=1}^{m} i^4 \end{pmatrix} \tag{21}$$

where

$$\sum_{i=1}^{m} i^3 = \frac{1}{4}\left(m^4 + 2m^3 + m^2\right) \tag{22}$$

$$\sum_{i=1}^{m} i^4 = \frac{1}{5}\left(m^5 + \frac{5}{2}m^4 + \frac{5}{3}m^3 - \frac{5}{30}m\right) \tag{23}$$

$$\mathbf{K}^T\overrightarrow{s} = \begin{pmatrix} \sum_{i=1}^{m} s_i \\ \sum_{i=1}^{m} i s_i \\ \sum_{i=1}^{m} i^2 s_i \end{pmatrix} \tag{24}$$

One year model (m = 252):

$$\overrightarrow{\theta} = \begin{pmatrix} 0.0289 \sum_{i=1}^{m} s_i - 0.0004 \sum_{i=1}^{m} i s_i + 1.3322 \cdot 10^{-6} \sum_{i=1}^{m} i^2 s_i \\ -0.0004 \sum_{i=1}^{m} s_i + 8.9733 \cdot 10^{-6} \sum_{i=1}^{m} i s_i - 3.3334 \cdot 10^{-8} \sum_{i=1}^{m} i^2 s_i \\ 1.3322 \cdot 10^{-6} \sum_{i=1}^{m} s_i - 3.3334 \cdot 10^{-8} \sum_{i=1}^{m} i s_i + 1.3493 \cdot 10^{-10} \sum_{i=1}^{m} i^2 s_i \end{pmatrix} \tag{25}$$

Ten year model (m = 2520):

$$\overrightarrow{\theta} = \begin{pmatrix} 0.0035 \sum_{i=1}^{m} s_i - 5.4828 \cdot 10^{-6} \sum_{i=1}^{m} i s_i + 1.8048 \cdot 10^{-9} \sum_{i=1}^{m} i^2 s_i \\ -5.4828 \cdot 10^{-6} \sum_{i=1}^{m} s_i + 1.1608 \cdot 10^{-8} \sum_{i=1}^{m} i s_i - 4.3177 \cdot 10^{-12} \sum_{i=1}^{m} i^2 s_i \\ 1.8048 \cdot 10^{-9} \sum_{i=1}^{m} s_i - 4.3177 \cdot 10^{-12} \sum_{i=1}^{m} i s_i + 1.7168 \cdot 10^{-15} \sum_{i=1}^{m} i^2 s_i \end{pmatrix} \tag{26}$$

# 4 Conclusion

Although generic algorithms for a polynomial model may not be expressed in simple programming codes, models of specific time periods (e.g., 1 year, 5 years, or 10 years) with linear, quadratic, or higher degrees can be implemented as concise programming codes. If one adopts array programming techniques, the summation terms can further be composed using short reduce() statements.