Tutorial 3

CS3241 Computer Graphics (AY22/23)

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Question 1a

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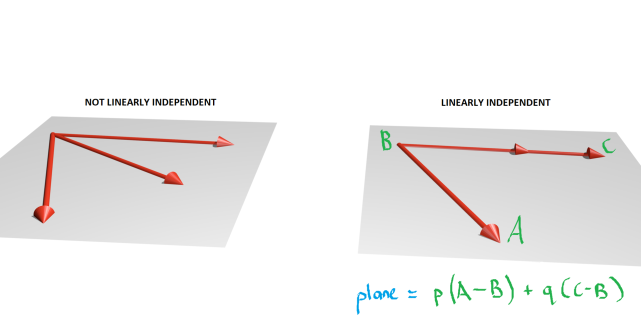
Given three points A, B, and C in 3D space, write an expression for the **normal vector** of the plane that contains the three points.

Normal vector



The **linear combination** of any two **linearly independent** vectors forms a plane.

Question 1b

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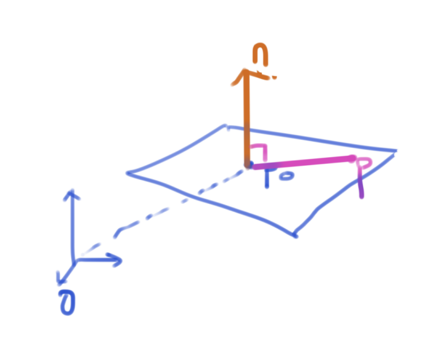
Given a point $R = [r_x \; r_y \; r_z]^T$ on a plane Π and a normal vector $n = [n_x \; n_y \; n_z]^T$ of the plane, write the implicit-form equation of the plane in the form $ax + by + cz + d = 0$.

Implicit form



$$
\begin{aligned}
n \cdot ([x\, y\, z]^T - R) &= 0 && (1)\\
n_x x + n_y y + n_z z - n \cdot R &= 0 && (2)
\end{aligned}
$$

Question 1c

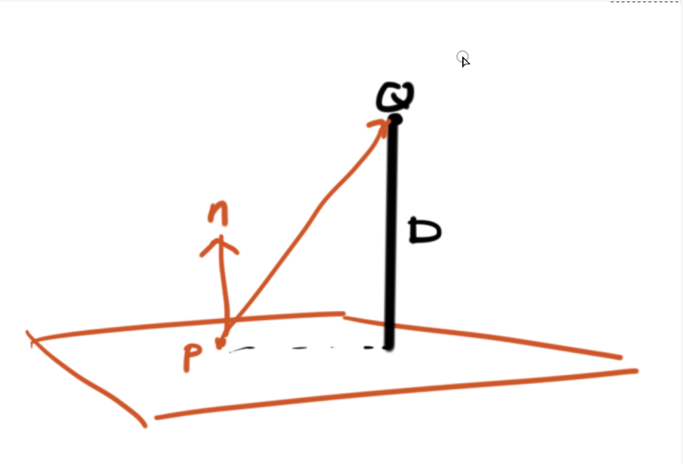
What is the perpendicular distance of the point Q from the plane $ax + by + cz + d = 0$?

Perpendicular distance of point to plane

Let Q $= [q_x \; q_y \; q_z]^T$. Then the distance D from point to plane is

$$D = \left| \frac{aq_x + bq_y + cq_z + d}{\sqrt{a^2 + b^2 + c^2}} \right| \tag{3}$$

Proof?



Outline of proof

$$
\begin{aligned}
D &= \|PQ\| \cos\theta \\
\cos\theta &= \widehat{n} \cdot \widehat{PQ} \\
&= \frac{n \cdot PQ}{\|n\|\|PQ\|} \\
\Rightarrow D &= \|PQ\| \frac{n \cdot PQ}{\|n\|\|PQ\|} \\
&= \frac{n \cdot (OQ - OP)}{\|n\|}
\end{aligned}
$$

Question 2a

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What does the homogenous coordinates $[6, 4, 2, 0.5]^T$ represent?

Question 2a

What does the homogenous coordinates $[6, 4, 2, 0.5]^T$ represent?

$$\begin{aligned} & \begin{bmatrix} x & y & z & w \end{bmatrix} \\ = \quad & \begin{bmatrix} x/w & y/w & z/w & 1 \end{bmatrix} \end{aligned}$$

Question 2b

Why does OpenGL (and many other graphics APIs) use homogeneous coordinates to represent points?

Why homogenous coordinates?

1. Different representations for points and vectors.
2. 4×4 Matrix multiplication
3. Perspective projection with matrix mult. and *perspective division*.

Point vector distinguishing

For **vector**; $w = 0$ e.g. $\begin{bmatrix} x & y & z & 0 \end{bmatrix}$.
For **point**; $w = 1$ e.g. $\begin{bmatrix} x & y & z & 1 \end{bmatrix}$.

In this week's lecture, you will learn that to transform **normal vectors** instead of points, we use the matrix

$$M_n = (M_t^{-1})^T$$

where M_t is the upper left 3×3 submatrix.

Question 3

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Which of the followings is the matrix that rotates objects about the fixed 3D point $[2\ 3\ 4]^T$, where the rotation axis is parallel to and in the same direction as the x-axis, and the rotation angle is θ?

A. $\mathbf{T}(-2, -3, -4) \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T}(2, 3, 4)$

B. $\mathbf{T}(-2, -3, -4) \cdot \mathbf{R}_x(-\theta) \cdot \mathbf{T}(2, 3, 4)$

C. $\mathbf{T}(2, 3, 4) \cdot \mathbf{R}_x(\theta) \cdot \mathbf{T}(-2, -3, -4)$

D. $\mathbf{T}(2, 3, 4) \cdot \mathbf{R}_x(-\theta) \cdot \mathbf{T}(-2, -3, -4)$

E. $\mathbf{T}(-2, -3, -4) \cdot \mathbf{R}_x(\theta)$

Things to note about transformation

Order of Transformations

- Conceptually, the transformation on the right is applied first

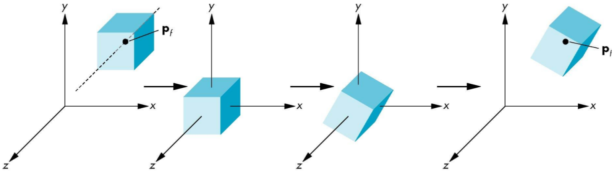
$$\mathbf{p}' = \mathbf{ABCp} = \mathbf{A}(\mathbf{B}(\mathbf{Cp}))$$

Things to note about transformation

Rotation About a Fixed Point Other than the Origin

1. Move fixed point to origin
2. Rotate
3. Move fixed point back

$$\mathbf{M} = \mathbf{T}(\mathbf{p}_f)\,\mathbf{R}(\theta)\,\mathbf{T}(-\mathbf{p}_f)$$



Ans: $T(2, 3, 4) \cdot R(\theta) \cdot T(-2, -3, -4)$.

Question 4

What is the inverse matrix of

$$M = \begin{bmatrix} s_1 r_{11} & s_1 r_{12} & s_1 r_{13} & t_1 \\ s_2 r_{22} & s_2 r_{22} & s_2 r_{23} & t_2 \\ s_3 r_{33} & s_3 r_{32} & s_3 r_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 1: Decompose

You can tell there is a S, R, and T matrix multiplied together in some order to give M.

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} & 0 \\ r_{22} & r_{22} & r_{23} & 0 \\ r_{33} & r_{32} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} s_{11} & 0 & 0 & 0 \\ 0 & s_{22} & 0 & 0 \\ 0 & 0 & s_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, T = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2: Order

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$M = ?$

Step 2: Order

$M = TSR.$

Thus $M^{-1} = R^{-1}S^{-1}T^{-1}.$

Step 3: Substitute and Compute

$$\mathbf{M}^{-1} = \mathbf{R}^{-1} \cdot \mathbf{S}^{-1} \cdot \mathbf{T}^{-1} = \begin{bmatrix} r_{11} & r_{21} & r_{31} & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/s_1 & 0 & 0 & 0 \\ 0 & 1/s_2 & 0 & 0 \\ 0 & 0 & 1/s_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -t_1 \\ 0 & 1 & 0 & -t_2 \\ 0 & 0 & 1 & -t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} r_{11}/s_1 & r_{21}/s_2 & r_{31}/s_3 & -(r_{11}t_1/s_1)-(r_{21}t_2/s_2)-(r_{31}t_3/s_3) \\ r_{12}/s_1 & r_{22}/s_2 & r_{32}/s_3 & -(r_{12}t_1/s_1)-(r_{22}t_2/s_2)-(r_{32}t_3/s_3) \\ r_{13}/s_1 & r_{23}/s_2 & r_{33}/s_3 & -(r_{13}t_1/s_1)-(r_{23}t_2/s_2)-(r_{33}t_3/s_3) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Question 5

Let $M_1 = TR$.
Let $M_2 = RT$.
Is $M_1 = M_2$? Prove it.

Proof and counterexample

In general $TR \neq RT$.

i.e. $M_1 \neq M_2$. Counterexample:

$$T(1,0,0)R_z(\pi/2)P(1,0,0) \neq R_z(\pi/2)T(1,0,0)P(1,0,0)$$

$$P(1,1,0) \neq P(0,2,0)$$

(Note: $P(x,y,z)$ refers to the point (x,y,z).)

Question 6

The computation of $\sin\theta$ and $\cos\theta$ is considered relatively slow for some real-time rendering applications.

However, when the angle θ is very small, we can make use of the small-angle approximation: $\sin\theta \approx \theta$, and $\cos\theta \approx 1 - \theta_{small}/2$, for better speed.

Write the 4x4 matrix for rotation about the z-axis by a very small rotation angle θ_{small}, which is given in radians.

Substitute in

$$\mathbf{R}_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

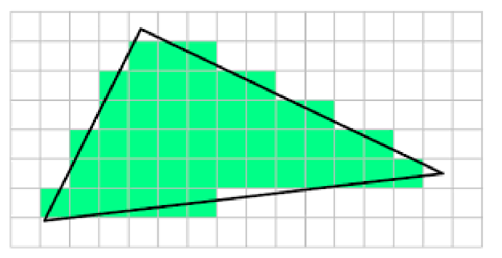
$$\cos\theta \approx 1 - \theta_{small}/2$$

$$\sin\theta \approx \theta_{small}$$

Question 7

In OpenGL, what is the purpose of transforming vertices to window space?

For the rasterizer!



Question 8

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What are the matrices applied to each vertex $v_1 \ldots v_6$?

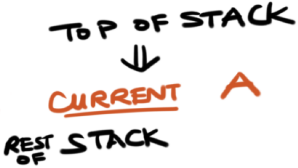
Question 8

For the following application program code fragment, write down the transformations that are actually applied to each vertex vi.

```
glMatrixMode( GL_MODELVIEW );
glLoadMatrixd( A );
glPushMatrix();
   glBegin( GL_POINTS ); glVertex3dv( v1 ); glEnd();
   glMultMatrixd( B );
   glPushMatrix();
      glMultMatrixd( C );
      glBegin( GL_POINTS ); glVertex3dv( v2 ); glEnd();
   glPopMatrix();
   glBegin( GL_POINTS ); glVertex3dv( v3 ); glEnd();
   glPushMatrix();
      glMultMatrixd( D );
      glPushMatrix();
         glMultMatrixd( E );
         glBegin( GL_POINTS ); glVertex3dv( v4 ); glEnd();
      glPopMatrix();
      glBegin( GL_POINTS ); glVertex3dv( v5 ); glEnd();
   glPopMatrix();
glPopMatrix();
glMultMatrixd( F );
glPushMatrix();
   glMultMatrixd( G );
glPopMatrix();
glBegin( GL_POINTS ); glVertex3dv( v6 ); glEnd();
```

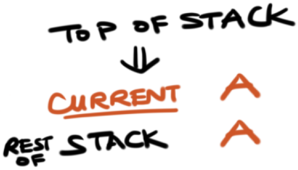
Understanding OpenGL's matrix stacks

```
glLoadMatrixd( A );
```



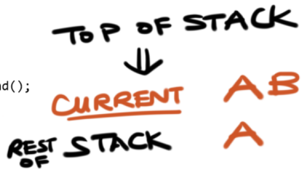
Understanding OpenGL's matrix stacks

```
glLoadMatrixd( A );
glPushMatrix();
```



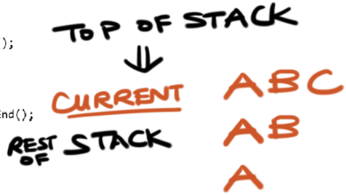
Understanding OpenGL's matrix stacks

```
LoadMatrixd( A );
PushMatrix();
 glBegin( GL_POINTS ); glVertex3dv( v1 ); glEnd();
 glMultMatrixd( B );
```



Understanding OpenGL's matrix stacks

```
glLoadMatrixd( A );
glPushMatrix();
   glBegin( GL_POINTS ); glVertex3dv( v1 ); glEnd();
   glMultMatrixd( B );
   glPushMatrix();
      glMultMatrixd( C );
      glBegin( GL_POINTS ); glVertex3dv( v2 ); glEnd();
```



Understanding OpenGL's matrix stacks

```
glLoadMatrixd( A );
glPushMatrix();
   glBegin( GL_POINTS ); glVertex3dv( v1 ); glEnd();
   glMultMatrixd( B );
   glPushMatrix();
      glMultMatrixd( C );
      glBegin( GL_POINTS ); glVertex3dv( v2 ); glEnd();
   glPopMatrix();
```

TOP OF STACK
⇓
CURRENT AB

REST OF STACK A

Note that

1. `glMultMatrix` **post-multiplies** the current matrix
2. `glPushMatrix` makes a copy of the topmost matrix of the stack, and makes it the current matrix
3. `glPopMatrix` deletes the top-most matrix and replaces it with the next one on the stack.
4. `glLoadIdentity` replaces the **current matrix** with I.

Attendance taking

Thanks! Get the slides here after the tutorial.

https://trxe.github.io/cs3241-notes

Feel free to ask me questions about Assignment 1.