Tutorial 9

CS3241 Computer Graphics (AY22/23)

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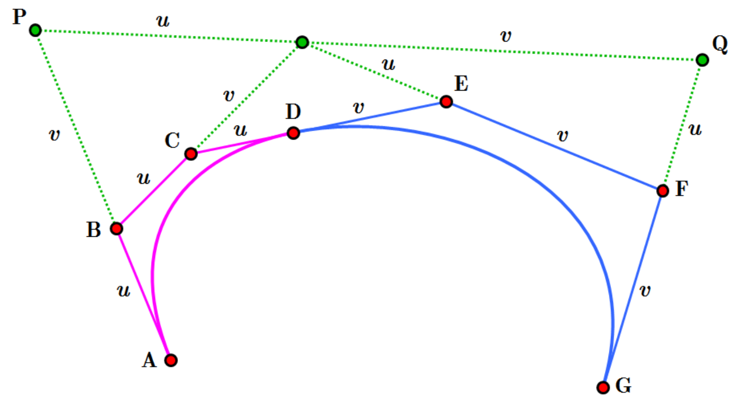
Question 1

Considering the rendering efficiency and rendering quality on a standard polygon-based rasterization renderer, what are the disadvantages of representing surfaces using polygon mesh compared to using Bézier patches?

Adaptive subdivision

De Casteljau's algorithm can be used to **generate polygons** adaptively.
Less screen space ⇒ less levels of subdivision ⇒ less vertices.
More screen space ⇒ more levels of subdivision ⇒ more vertices.



Adaptive subdivision

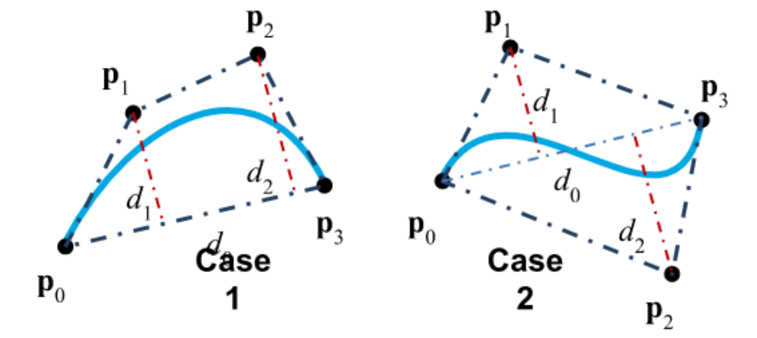
Compared to polygon mesh which has **fixed vertex/polygon count**, facing:

- poorer efficiency for small meshes which need less vertices
- poorer fidelity for large meshes which cannot achieve smooth curves

Question 2

Propose a way to measure the “flatness” of a cubic Bézier curve segment in 2D space. Can your method be extended to a cubic Bézier curve segment in 3D space?

Convex hull



Flatness estimate f

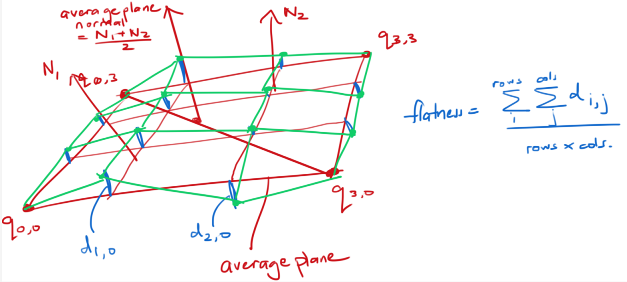
Case 1: $f = \max(d_1, d_2)$
Case 2: $f = \max(d_1, d_2)$

Q: How to determine cases 1 or 2?
A: determine if the intersection of the lines formed by p_1p_2 and p_0p_3 lies within p_0, p_3

Question 3

Propose a way to measure the “flatness” of a cubic Bézier **surface patch**.

Extending Q2



Define the "average plane" (in red) as $N \cdot p = N \cdot \frac{q_{0,0}+q_{0,3}+q_{3,0}+q3,3}{4}$.
(where N is the average of the two normals of $q_{0,0}$, $q_{0,3}$, $q_{3,0}$ and $q_{3,0}$, $q_{0,3}$, $q_{3,3}$).

> "Flatness" = average of distance between average plane and each point.

Parametric Curves and Surfaces: Polynomial and Bezier

Parametric Curves/Surfaces

Curve (2D):

$$\text{In } \mathbf{2D} \text{ space: } p(u) = \begin{bmatrix} x(u) \\ y(u) \end{bmatrix}; \quad \text{In } \mathbf{3D} \text{ space: } p(u) = \begin{bmatrix} x(u) \\ y(u) \\ z(u) \end{bmatrix}$$

where $x(u), y(u)$ can be any function of **single parameter** u.

Surface (3D):

$$p(u) = \begin{bmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{bmatrix}$$

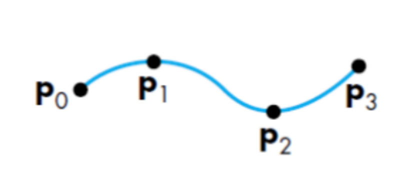
where $x(u,v), y(u,v), z(u,v)$ can be any function of **two parameters**u, v.

Parameteric Cubic Polynomial Curves

$$
\begin{aligned}
p(u) &= c_0 + c_1(u) + c_2(u^2) + c_3(u^3) \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} c_{0,x} & c_{0,y} & c_{0,z} \\ c_{1,x} & c_{1,y} & c_{1,z} \\ c_{2,x} & c_{2,y} & c_{2,z} \\ c_{3,x} & c_{3,y} & c_{3,z} \end{bmatrix} \\
&= \begin{bmatrix} x(u) & y(u) & z(u) \end{bmatrix}
\end{aligned}
$$

Cubic interpolating

Pre-determine points that **curve passes through** at $u = 0, \frac{1}{3}, \frac{2}{3}, 1$.



Our conditions are for $k \in \{0 \ldots 3\}$

$$\begin{aligned} p_k &= p\left(\tfrac{k}{3}\right) \\ &= c_0 + \tfrac{k}{3} c_1 + \left(\tfrac{k}{3}\right)^2 c_2 + \left(\tfrac{k}{3}\right)^3 c_3 \end{aligned}$$

Deriving Cubic Interpolating Curves

- We can write the equations in matric form as

$$\mathbf{p} = \mathbf{A}\mathbf{c}, \quad \text{where} \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{3} & \left(\frac{1}{3}\right)^2 & \left(\frac{1}{3}\right)^3 \\ 1 & \frac{2}{3} & \left(\frac{2}{3}\right)^2 & \left(\frac{2}{3}\right)^3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- We invert $\mathbf{A}$ to obtain the **interpolation geometry matrix**

$$\mathbf{M}_I = \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5.5 & 9 & -4.5 & 1 \\ 9 & -22.5 & 18 & -4.5 \\ -4.5 & 13.5 & -13.5 & 4.5 \end{bmatrix}$$

Note that $\mathbf{M}_I$ is the same for *any* 4 control points

- The desired coefficients are

$$\mathbf{c} = \mathbf{M}_I\mathbf{p} \qquad \text{where} \quad \mathbf{p} = \begin{bmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}$$

Cubic Bézier curves

Pre-determine points that **define the graph where**

$$\begin{aligned} p(0) &= p_0 \\ p(1) &= p_3 \\ p'(0) &= 3(p_1 - p_0) \\ p'(1) &= 3(p_3 - p_2) \end{aligned}$$

Our conditions are

$$\begin{aligned} p_0 &= c_0 && (1) \\ 3p_1 - 3p_0 &= c_1 && (2) \\ 3p_3 - 3p_2 &= c_1 + c_2 + c_3 && (3) \\ p_1 &= c_0 + c_1 + c_2 + c_3 && (4) \\ & && (5) \end{aligned}$$

Can you derive (1) and (2)?

Blending functions and geometry matrices

Relationships between p and c

$$\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

Here A contains our conditions for the curve type (Bezier/interpolating).

However we usually define our points p instead of c, and we want to derive c instead. We can do so by

$$c_k = A^{-1}C = M_I p_k \text{ for interpolating, or } M_B p_k \text{ for Bezier}$$

Blending functions

$$
\begin{aligned}
p(u) &= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} 1 & u & u^2 & u^3 \end{bmatrix} M_I \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix} \\
&= \begin{bmatrix} b_0 & b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}
\end{aligned}
$$

Our $b(u) = \begin{bmatrix} b_0(u) & b_1(u) & b_2(u) & b_3(u) \end{bmatrix}^T$.

Question 4

Given 4 control points p_0, p_1, p_2, p_3 for a cubic Bézier curve segment $p(u)$, find the 4 control points q_0, q_1, q_2, q_3 for the cubic interpolating curve segment $q(u)$ such that $q(u) = p(u)$.

Cubic Bézier to Cubic interpolating

The corresponding $q_k = q(\frac{k}{3}) = p(\frac{k}{3})$ (since $q(u) = p(u)$).

$$
\begin{aligned}
q(0) &= \quad p_0 \\
q(\frac{1}{3}) &= \quad p(\tfrac{1}{3}) \\
q(\frac{2}{3}) &= \quad p(\tfrac{2}{3}) \\
q(1) &= \quad p_1
\end{aligned}
$$

Directly compute q from p.

Question 5

Given 4 control points q_0, q_1, q_2, q_3 for the cubic interpolating curve segment $q(u)$, find the 4 control points p_0, p_1, p_2, p_3 for a cubic Bézier curve segment $p(u)$ such that $p(u) = q(u)$.

Cubic interpolating to Cubic bezier

Let's look at our constraints:

$$
\begin{aligned}
p(0) &= p_0 \\
p(1) &= p_3 \\
p'(0) &= 3(p_1 - p_0) \\
p'(1) &= 3(p_3 - p_2)
\end{aligned}
$$

Cubic interpolating to Cubic bezier

Now we use the fact that $p(u) = q(u)$:

$$\begin{aligned} p_0 = p(0) &= q(0) = q_0 \\ p_3 = p(1) &= q(1) = q_3 \\ p'(0) &= q'(0) = 3(p_1 - p_0) \\ p'(1) &= q'(1) = 3(p_3 - p_2) \end{aligned}$$

For p_1, p_2 we obtain them via:

$$q'(0) = 3(p_1 - p_0) \Rightarrow \frac{q'(0)}{3} + p_0 = p_1$$

and

$$q'(1) = 3(p_3 - p_2) \Rightarrow p_3 - \frac{q'(1)}{3} = p_2$$

Question 6

Given two cubic Bezier curve segments, $p(u)$ and $q(u)$, that are to be joined together, where $p(1) = q(0)$. The control points of $p(u)$ are p_0, p_1, p_2, p_3 and the control points of $q(u)$ are q_0, q_1, q_2, q_3. How should the control points of $q(u)$ be positioned so that there is C^1 continuity at the join point of $p(u)$ and $q(u)$?

C^1 continuity

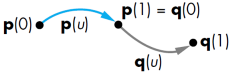
1. $p(1) = p(0)$
2. First derivative of p at end = first derivative of q at start: $p'(1) = q'(0)$.

$$\begin{aligned} p(1) &= p_3 = q_0 = q(0) \\ p'(1) &= 3(p_3 - p_2) = 3(q_1 - q_0) = q'(0) \end{aligned}$$

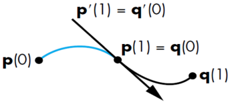
Continuity

Geometric and Parametric Continuity

- Consider two curve segments, $\mathbf{p}(u)$ and $\mathbf{q}(u)$
- If $\mathbf{p}(1) = \mathbf{q}(0)$, we say there is C^0 **parametric continuity** at the join point



- If $\mathbf{p}'(1) = \mathbf{q}'(0)$, we say there is C^1 **parametric continuity** at the join point

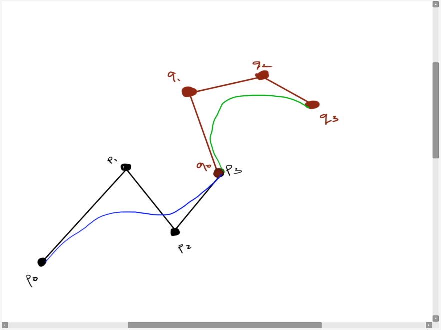


- If $\mathbf{p}'(1) = \alpha\mathbf{q}'(0)$, for some positive number α, we say there is G^1 **geometric continuity** at the join point
- We can extend the idea to higher derivatives and talk about C^n and G^n continuity

Question 7

Sketch two 2D curves that have C^0 continuity but not C^1 continuity.

Question 8

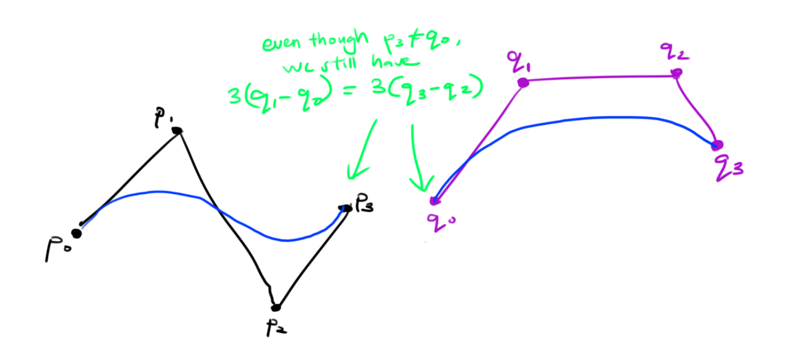


C^0 as both curves are joined $p_3 = q_0$.
Not C^1 as $p'(1) \neq q'(0)$.

Question 8

Sketch two 2D curves that have C^1 continuity but not C^0 continuity. Is it even possible to have such a situation?

Question 8



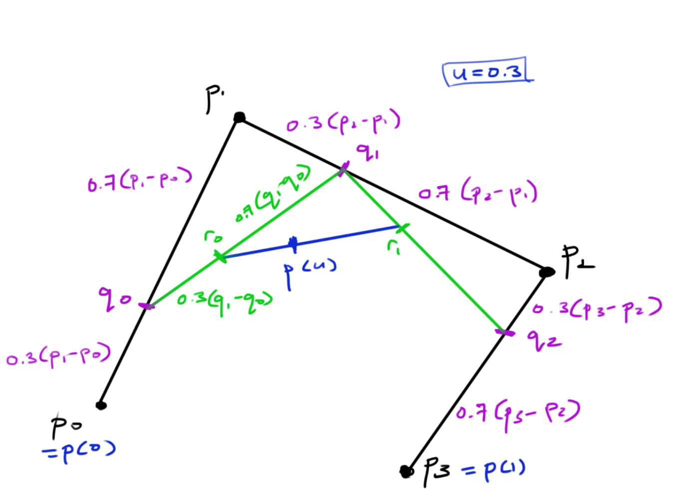
C^1 as $p'(1) = q'(0)$.

Not C^0 as $p(1) \neq q(0)$.

Question 9

Given 4 control points p_0, p_1, p_2, p_3 for a cubic Bézier curve segment $p(u)$, and any $0 \leq u \leq 1$ show that the De Casteljau algorithm produces the point $p(u)$.

De Casteljau's Algorithm



Pseudocode

```
vec3 recursive_decasteljau(vector<vec3> points, float u) {
    // points.size() >= 1.
    if (points.size() == 1) {
        return points[0];
    }

    vector<vec3> interpolated_points(points.size() - 1);
    for (int i = 0; i < points.size() - 1; i++) {
        interpolated_points[i] = interpolate(points[i], points[i+1], u);
    }

    return recursive_decasteljau(interpolated_points, u);
}
```

Attendance taking

Thanks! Get the slides here after the tutorial.

https://trxe.github.io/cs3241-notes