Algorithm Analysis
Homework Assignment 4

Dumb ways to die
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4 Different Versions of Farkas Lemma

Exercise 4.1.

1. Prove that the Lemma 1 implies the Lemma 2.

   Proof.
   $$\begin{aligned}
   &\exists x \in \mathbb{R}^n_{\geq 0} : Ax \leq b \\
   \Leftrightarrow &\exists x, z \in \mathbb{R}^n : \begin{bmatrix} A & I_m \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = b \\
   \Leftrightarrow &\neg(\exists y \in \mathbb{R}^m : \begin{bmatrix} A & I_m \end{bmatrix}^T y \geq 0, b^T y < 0) \\
   \Leftrightarrow &\forall y \in \mathbb{R}^m : y^T \begin{bmatrix} A & I_m \end{bmatrix} \geq 0 \to y^T b \geq 0 \\
   \Leftrightarrow &\forall y \in \mathbb{R}^m : (y^T A \geq 0 \wedge y^T \geq 0) \to y^T b \geq 0 \\
   \Leftrightarrow &\forall y \in \mathbb{R}^m_{\geq 0} : y^T A \geq 0 \to y^T b \geq 0 \\
   \Leftrightarrow &\neg(\exists y \in \mathbb{R}^m_{\geq 0} : A^T y \geq 0, b^T y < 0)
   \end{aligned}$$
   □

2. Prove that the Lemma 2 implies the Lemma 3.

   Proof. Let $x = (x_1\ x_2 \cdots x_n)^T$.

Define $gez(x)$ as

$$f(x) = (\frac{1}{2}x_1 + \frac{1}{2}|x_1| \quad \frac{1}{2}x_2 + \frac{1}{2}|x_2| \ \cdots \ \frac{1}{2}x_n + \frac{1}{2}|x_n|)^T$$

Obviously, $\forall x \in \mathbb{R}^n : f(x) \in \mathbb{R}^n_{\geq 0}$. So that we has

$$\begin{aligned}
&\exists x \in \mathbb{R}^n : Ax \leq b \\
\Leftrightarrow &\exists f(x), f(-x) \in \mathbb{R}^n_{\geq 0} : \begin{bmatrix} A & -A \end{bmatrix} \begin{bmatrix} f(x) \\ f(-x) \end{bmatrix} \leq b \\
\Leftrightarrow &\neg(\exists y \in \mathbb{R}^m_{\geq 0} : \begin{bmatrix} A & -A \end{bmatrix}^T y \geq 0, b^T y < 0) \\
\Leftrightarrow &\forall y \in \mathbb{R}^m_{\geq 0} : y^T \begin{bmatrix} A & -A \end{bmatrix} \geq 0 \to b^T y \geq 0 \\
\Leftrightarrow &\forall y \in \mathbb{R}^m_{\geq 0} : y^T A \geq 0 \land y^T A \leq 0 \to y^T b \leq 0 \\
\Leftrightarrow &\forall y \in \mathbb{R}^m_{\geq 0} : y^T A = 0 \to y^T b \leq 0 \\
\Leftrightarrow &\neg(\exists y \in \mathbb{R}^m_{\geq 0} : A^T y = 0, b^T y < 0)
\end{aligned}$$

□

3. Prove that the Lemma 3 implies the Lemma 1.

Proof.

$$
\begin{aligned}
&\exists x \in \mathbb{R}^n_{\geq 0} : Ax = b \\
\Leftrightarrow &\exists x \in \mathbb{R}^n : I_n x \geq 0, Ax \leq b, Ax \geq b \\
\Leftrightarrow &\exists x \in \mathbb{R}^n : \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix} x = \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \\
\Leftrightarrow &\neg(\exists \bar{y} \in \mathbb{R}^{2m+n}_{\geq 0} : \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix}^T \bar{y} = 0, \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix}^T \bar{y} < 0) \\
\Leftrightarrow &\forall \bar{y} \in \mathbb{R}^{2m+n}_{\geq 0} : \bar{y}^T \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix} = 0 \to \bar{y}^T \begin{bmatrix} b \\ -b \\ 0 \end{bmatrix} \geq 0 \\
\Leftrightarrow &\forall \bar{y} \in \mathbb{R}^{2m+n}_{\geq 0} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix} (y, y' \in \mathbb{R}^m_{\geq 0}, y'' \in \mathbb{R}^n_{\geq 0}) : \\
&\quad y^T A - y'^T A - y''^T I_n = 0 \to y^T b - y'^T b \geq 0 \\
\Leftrightarrow &\forall y, y' \in \mathbb{R}^m_{\geq 0}, y'' \in \mathbb{R}^n_{\geq 0} : (y^T - y'^T)A = y''^T \to (y^T - y'^T)b \geq 0 \\
\Leftrightarrow &\forall y \in \mathbb{R}^m : y^T A \geq 0 \to y^T b \geq 0
\end{aligned}
$$

□

In conclusion, the Lemma 1 implies the Lemma 2, the Lemma 2 implies the Lemma 3 and the Lemma 3 implies the Lemma 1. Thus, the three lemmas are equivalent.

4.1 Convex Combination, vertex cover, and matching

Exercise 4.2. Let $x, y, z \in \mathbb{R}^m, \epsilon_1, \epsilon_2 > 0, S \subseteq \mathbb{R}^m$.

1. Show that x is a convex combination of $x + \epsilon_1 z$ and $x - \epsilon_2 z$.

Proof. $x + \epsilon_1 z, x - \epsilon_2 z \in \mathbb{R}^m$, and $\frac{\epsilon_1}{\epsilon_1+\epsilon_2}, \frac{\epsilon_2}{\epsilon_1+\epsilon_2} > 0$ since $\epsilon_1, \epsilon_2 > 0$

$$\frac{\epsilon_2}{\epsilon_1 + \epsilon_2} \cdot (x + \epsilon_1 z) + \frac{\epsilon_1}{\epsilon_1 + \epsilon_2} \cdot (x - \epsilon_2 z) = x$$

$$\frac{\epsilon_1}{\epsilon_1+\epsilon_2}+\frac{\epsilon_2}{\epsilon_1+\epsilon_2}=1$$

Hence x is a convex combination of $x+\epsilon_1 z$ and $x-\epsilon_2 z$. $\square$

2. Show that if x is a convex combination of y, z and y, $z \in S$, then $x \in conv(S)$.

   Proof. If S is a finite set, $conv(S)$ is the set of convex combinations of the elements in S. Since x is the convex combination of y and z. Suppose $x = \lambda y + (1-\lambda)z$, where $\lambda \in [0,1]$. Then we have

   $$x = \lambda y + (1-\lambda)z + \sum_{s_i \in S \setminus \{y,z\}} t_i s_i$$

   $$1 = \lambda + (1-\lambda) + \sum_{i \in ind(S\setminus\{y,z\})} t_i$$

   in which $t_i = 0$ for any $i \in ind(S\setminus\{y,z\})$. Thus x is the convex combination of the elements of S. Hence $x \in conv(S)$.

   If S is infinite, $conv(S)$ is the set of points that are a convex combination of finitely many points in S. $\{y,z\} \subset S$ and x is a convex combination of y, z. Thus $x \in conv(S)$. $\square$

3. Suppose P is a linear program, and $x, y, z \in \mathbb{R}^n$ are three feasible solutions. Show that if(i) x is optimal and (ii) x is a convex combination of y and z, and then y, z must both be optimal.

   Proof. Suppose linear program P is to maximize $c^T x$ (its duality can be proved in the similar way). Since x is an optimal solution, y and z are feasible solutions.

   $$c^T x \geq c^T y \text{ and } c^T x \geq c^T z$$

   x is a convex combination of y, z. $x = \lambda y + (1-\lambda)z$, where $\lambda \in [0,1]$.

   $$\Rightarrow c^T x = c^T(\lambda y + (1-\lambda)z) = \lambda c^T y + (1-\lambda)c^T z$$

   if $c^T y \neq c^T x$ or $c^T z \neq c^T x$, we will see

   $$c^T x < \lambda c^T x + (1-\lambda)c^T x = c^T x,$$

   which doesn't hold. Hence $c^T y = c^T z = c^T x$ must hold. Thus y, z are also optimal solutions. $\square$

Exercise 4.3.

I have to proof something first. If $\mathbf{a}$ is a convex combination of $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{b}$ is a convex combination of $\mathbf{d}$ and $\mathbf{e}$, $\mathbf{c}$ is a convex combination of $\mathbf{f}$ and $\mathbf{g}$, then $\mathbf{a}$ is a convex combination of $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$.

We konw that

$$\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$$
$$\mathbf{b} = \alpha_1\mathbf{d} + \beta_1\mathbf{e}$$
$$\mathbf{c} = \alpha_2\mathbf{f} + \beta_2\mathbf{g}$$

Thus

$$\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c} = \alpha(\alpha_1\mathbf{d} + \beta_1\mathbf{e}) + \beta(\alpha_2\mathbf{f} + \beta_2\mathbf{g}) = \alpha\alpha_1\mathbf{d} + \alpha\beta_1\mathbf{e} + \beta\alpha_2\mathbf{f} + \beta\beta_2\mathbf{g}$$

$$\alpha\alpha_1 + \alpha\beta_1 + \beta\alpha_2 + \beta\beta_2 = \alpha(\alpha_1 + \beta_1) + \beta(\alpha_2 + \beta_2) = \alpha + \beta = 1$$

Thus $\mathbf{a}$ is a convex combination of $\mathbf{d}$, $\mathbf{e}$, $\mathbf{f}$ and $\mathbf{g}$. So convex combination is "transtive".

1. Firstly, we have to prove that $S(P) \subseteq \mathrm{conv}(S_{\mathrm{int}}(P))$.

   Consider a $\mathbf{x} \in S(P)$.

   G is a bipartite graph, so we can denote nodes on the left side as A, nodes on the right side as B.

   Define

   $$A_f = \{v | v \in A, x_v \text{ is not an integer}\}$$
   $$A_z = \{v | v \in A, x_v \in \mathbb{N}\}$$
   $$B_f = \{v | v \in B, x_v \text{ is not an integer}\}$$
   $$B_z = \{v | v \in B, x_v \in \mathbb{N}\}$$
   $$E_f = \{e | e \in E, \sum_{v \in e} x_v \text{ is not an integer}\}$$
   $$E_z = \{e | e \in E, \sum_{v \in e} x_v \in \mathbb{N}\}$$

   Assume $|A_f| + |B_f| + |E_f| > 0$(Otherwise it is already a integral solution).

If $|A_f| + |B_f| = 0$, then obviously $|E_f| = 0$. But $|A_f| + |B_f| + |E_f| > 0$. So $|A_f| + |B_f| > 0$.

Then we define

$$\mathbf{x}(\epsilon) = \begin{cases} x_v + \epsilon & v \in A_f \\ x_v - \epsilon & v \in B_f \\ x_v & \text{otherwise} \end{cases}$$

Note that $\sum_{v \in e} x_v \geq 1(*)$ for all $e \in E$. Now we have to prove we can find a positive ϵ and another negative ϵ such that $\mathbf{x}(\epsilon)$ is feasible. Firstly,

$$\epsilon \in [\lfloor x_v \rfloor - x_v, \lceil x_v \rceil - x_v] \quad \forall v \in A_f$$
$$\epsilon \in [x_v - \lceil x_v \rceil, x_v - \lfloor x_v \rfloor] \quad \forall v \in B_f$$

Obviously, $0 \in [\lfloor x_v \rfloor - x_v, \lceil x_v \rceil - x_v], 0 \in [x_v - \lceil x_v \rceil, x_v - \lfloor x_v \rfloor]$.

For $\mathbf{x}(\epsilon)$, consider the edges(u, v) between A_f and $B_z(u \in A_f,\ v \in B_z)$. Note that $x_u(\epsilon) + x_v(\epsilon) = x_u(\epsilon) + x_v$. Obviously (*) holds for any positive ϵ.

Because x_u is not an integer and x_v is an integer, so $x_u + x_v > 1$. Thus (*) holds for $\epsilon = max_{u \in A_f}(1 - x_u - x_v)$ which is less than 0.

i.e., $\epsilon \in [1 - x_u - x_v, +\infty)(u \in A_f,\ v \in B_z)$, and $0 \in [1 - x_u - x_v, +\infty)$.

Consider the edges(u, v) between A_f and $B_f(u \in A_f,\ v \in B_f)$. We know that $x_u + x_v \geq 1$, so $(x_u + \epsilon) + (x_v - \epsilon) = x_u + x_v \geq 1$. So (*) holds for any ϵ.

Consider the edges(u, v) between A_z and $B_f(u \in A_z,\ v \in B_f)$. Similarly, (*) holds for any negative ϵ. Because x_u is not an integer and x_v is an integer, so $x_u + x_v > 1$. Thus (*) holds for $\epsilon = min_{v \in B_f}(x_u + x_v - 1)$, which is greater than 0.

i.e., $\epsilon \in [-\infty, x_u + x_v - 1)(u \in A_z,\ v \in B_f)$, and $0 \in [-\infty, x_u + x_v - 1)$.

Now we can get ϵ_1 and $-\epsilon_2(\epsilon_1, \epsilon_2 > 0)$ from the intersection of the intervals above. Then $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(-\epsilon_2)$ are feasible solutions of P.

And from the analyse above we konw that we can choose ϵ_1 and $-\epsilon_2$ to make $|A_f| + |B_f| + |E_f|$ of the new solution $\mathbf{x}(\epsilon)$ to decrease by at least 1. Because at least one node in A_f or B_f will be an element of A_z or B_z in the new solution, or at least one edge in E_f will be an element of E_z in the new solution.

Now we can define $\mathbf{u} \in \mathbb{R}^{|V|}$.

$$u_i = \begin{cases} 1 & v_i \in A_f \\ -1 & v_i \in B_f \\ 0 & \text{otherwise} \end{cases}$$

So $\mathbf{x}(\epsilon_1) = \mathbf{x} + \epsilon_1\mathbf{u}$, $\mathbf{x}(-\epsilon_2) = \mathbf{x} - \epsilon_2\mathbf{u}$. From **Exercise 4.2.** we know $\mathbf{x}$ is a convex combination of $\mathbf{x} + \epsilon_1\mathbf{u}$ and $\mathbf{x} - \epsilon_2\mathbf{u}$, so $\mathbf{x}$ is a convex combination of $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(-\epsilon_2)$.

Note that $|A_f| + |B_f| > 0$, thus $\mathbf{u} \neq \mathbf{0}$.

We can do it recursively. Note that we can choose ϵ to make $|A_f| + |B_f| + |E_f|$ decrease in every iteration. So the algorithm will stop in finite steps. Then $|A_f| + |B_f| + |E_f| = 0$, i.e., feasible integral solutions.

We have already proofed that convex combination is "transtive", so $\mathbf{x}$ is a convex combination of feasible integral solutions. Thus, $\mathbf{x} \subseteq \mathrm{conv}(S_{\mathrm{int}}(P))$.

Secondly, we have to prove that $S(P) \supseteq \mathrm{conv}(S_{\mathrm{int}}(P))$.

For every $\mathbf{x} \in \mathrm{conv}(S_{\mathrm{int}}(P))$, we know

$$\mathbf{x} = \sum_{i=1}^{n} t_i \mathbf{x}^i$$

where $\mathbf{x}^i \in S_{\mathrm{int}}(P) \subseteq S(P)$ for all i and $\sum_{i=1}^{n} t_i = 1$. Additionly, $\sum_{v\in e} x_v^i \geq 1 \quad \forall e \in E$. So

$$\sum_{v\in e} x_v = \sum_{v\in e}\sum_{i=1}^{n} t_i x_v^i = \sum_{i=1}^{n} t_i \sum_{v\in e} x_v^i \geq \sum_{i=1}^{n} t_i = 1$$

for all $e \in E$.

And it is obviously that $x_v \geq 0 \quad \forall v \in V$.

So $\mathbf{x}$ is a feasible solution of P. Thus $\mathbf{x} \in S(P)$.

Thus $S(P) = \mathrm{conv}(S_{\mathrm{int}}(P))$.

2. Firstly, we have to prove that $S(Q) \subseteq \mathrm{conv}(S_{\mathrm{int}}(Q))$. Consider a $\mathbf{x} \in S(Q)$.

Define

$$E_f = \{e|e \in E, x_e \text{ is not an integer}\}$$

$$E_z = \{e|e \in E, x_e \in \mathbb{N}\}$$

$$V_f = \{v|v \in V, \sum_{e \in E: v \in e} x_e \text{ is not an integer}\}$$

$$V_z = \{v|v \in V, \sum_{e \in E: v \in e} x_e \in \mathbb{N}\}$$

Assume $|V_f| + |E_f| > 0$(Otherwise it is already a integral solution).

If $|E_f| = 0$, then obviously $|V_f| = 0$. But $|V_f| + |E_f| > 0$. So $|E_f| > 0$.

While $|E_f| \neq 0$, we can choose one edge from E_f and dfs from this edge, and we only go through the edges in E_f. We should not go through a same edge twice. If some edges form a cycle, we have to stop and **only** choose edges in this cycle. Then we can get a sequence $e_1, e_2 \cdots e_m$(They form a cycle or not). They are all in E_f.

Denote the vertices in the sequence as $a_0, a_1 \cdots a_m$. Then we can define

$$\mathbf{x}(\epsilon) = \begin{cases} x_e + \epsilon & e = e_i \text{ and i is odd} \\ x_e - \epsilon & e = e_i \text{ and i is even} \\ 0 & \text{otherwise} \end{cases}$$

Similarly, we have to find a positive ϵ and a negative ϵ, such that $\sum_{e \in E: v \in e} x_e(\epsilon) \leq 1 \quad \forall v \in V$(denote it as (*)) and $x_e(\epsilon) \geq 0 \quad \forall e \in E$.

Case 1: $e_1, e_2 \cdots e_m$ form a cycle

Note that in this case $a_0 = a_m$.

Obviously for every vertex $a_i (0 < i < m)$, $a_i = e_i \cap e_{i+1}$, one of x_{e_i} and $x_{e_{i+1}}$ must increase by ϵ, and the other must decrease by ϵ. So

$$\sum_{e \in E: a_i \in e} x_e(\epsilon) = (\sum_{e \in E: a_i \in e} x_e) + \epsilon - \epsilon = \sum_{e \in E: a_i \in e} x_e \leq 1$$

Note that G is a bipartite graph, so m must be even. Thus for $a_0 (= a_m)$

$$\sum_{e \in E: a_0 \in e} x_e(\epsilon) = (\sum_{e \in E: a_0 \in e} x_e) + \epsilon - \epsilon = \sum_{e \in E: a_0 \in e} x_e \leq 1$$

also holds.

Thus this inequality holds for any ϵ.

But

$$\epsilon \in [\lfloor x_e \rfloor - x_e, \lceil x_e \rceil - x_e] \quad \forall e = e_i \text{ and i is odd}$$
$$\epsilon \in [x_e - \lceil x_e \rceil, x_e - \lfloor x_e \rfloor] \quad \forall e = e_i \text{ and i is even}$$

And

$$0 \in [\lfloor x_e \rfloor - x_e, \lceil x_e \rceil - x_e] \quad \forall e = e_i \text{ and i is odd}$$
$$0 \in [x_e - \lceil x_e \rceil, x_e - \lfloor x_e \rfloor] \quad \forall e = e_i \text{ and i is even}$$

Note that in the new solution, we can choose ϵ to make $|E_f| + |V_f|$ decrease by at least one. Because one edge in E_f will be in E_z in the new solution.

Case 2: $e_1, e_2 \cdots e_m$ do not form a cycle

Similarly for every vertex $a_i (0 < i < m)$, $a_i \in e_i$, $a_i \in e_{i+1}$, one of x_{e_i} and $x_{e_{i+1}}$ must increase by ϵ, and the other must decrease by ϵ. So for $0 < i < m$

$$\sum_{e \in E: a_i \in e} x_e(\epsilon) = (\sum_{e \in E: a_i \in e} x_e) + \epsilon - \epsilon = \sum_{e \in E: a_i \in e} x_e \le 1$$

For a_0, note that $\forall e \in E$ if $e_1 \neq e$ and $a_0 \in e$ then x_e is integral. If not, the sequence can be extended from a_0.

But x_{e_1} is not integral. Thus $\sum_{e \in E: a_0 \in e} x_e$ is not integral. So $0 < \sum_{e \in E: a_0 \in e} x_e < 1$.

$$\sum_{e \in E: a_0 \in e} x_e(\epsilon) = (\sum_{e \in E: a_0 \in e} x_e) + \epsilon$$

Thus there will be a feasible positive ϵ and a feasible negative ϵ for (*). Similarly for a_m.

But

$$\epsilon \in [\lfloor x_e \rfloor - x_e, \lceil x_e \rceil - x_e] \quad \forall e = e_i \text{ and i is odd}$$
$$\epsilon \in [x_e - \lceil x_e \rceil, x_e - \lfloor x_e \rfloor] \quad \forall e = e_i \text{ and i is even}$$

And

$$0 \in [\lfloor x_e \rfloor - x_e, \lceil x_e \rceil - x_e] \quad \forall e = e_i \text{ and i is odd}$$
$$0 \in [x_e - \lceil x_e \rceil, x_e - \lfloor x_e \rfloor] \quad \forall e = e_i \text{ and i is even}$$

So in the new solution, we can choose ϵ to make $|E_f| + |V_f|$ decrease by at least one. Because one edge in E_f will be in E_z in the new solution, or one vertex in V_f will be in V_z in the new solution.

Now we can get ϵ_1 and $-\epsilon_2 (\epsilon_1, \epsilon_2 > 0)$ from the intersection of the intervals above. Then $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(-\epsilon_2)$ are feasible solutions of Q.

Now we can define $\mathbf{u} \in \mathbb{R}^{|E|}$.

$$u_i = \begin{cases} 1 & e = e_i \text{ and i is odd} \\ -1 & e = e_i \text{ and i is even} \\ 0 & \text{otherwise} \end{cases}$$

Note that $|E_f| > 0$, thus $\mathbf{u} \neq \mathbf{0}$.

So $\mathbf{x}(\epsilon_1) = \mathbf{x} + \epsilon_1 \mathbf{u}$, $\mathbf{x}(-\epsilon_2) = \mathbf{x} - \epsilon_2 \mathbf{u}$. From **Exercise 4.2.** we know $\mathbf{x}$ is a convex combination of $\mathbf{x} + \epsilon_1 \mathbf{u}$ and $\mathbf{x} - \epsilon_2 \mathbf{u}$, so $\mathbf{x}$ is a convex combination of $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(-\epsilon_2)$.

We can do it recursively. Note that we can choose ϵ to make $|E_f| + |V_f|$ decrease in every iteration. So the algorithm will stop in finite steps. Then $|E_f| + |V_f| = 0$, i.e., feasible integral solutions.

We have already proofed that convex combination is "transtive", so $\mathbf{x}$ is a convex combination of feasible integral solutions.

Thus, $\mathbf{x} \subseteq \text{conv}(S_{\text{int}}(Q))$.

Secondly, we have to prove that $S(Q) \supseteq \text{conv}(S_{\text{int}}(Q))$.

For every $\mathbf{x} \in \text{conv}(S_{\text{int}}(Q))$, we know

$$\mathbf{x} = \sum_{i=1}^{n} t_i \mathbf{x}^i$$

where $\mathbf{x}^i \in S_{\text{int}}(Q) \subseteq S(Q)$ for all i and $\sum_{i=1}^{n} t_i = 1$. Additionly, $\sum_{e \in E : v \in e} x_e^i \leq 1$. So

$$\sum_{e \in E : v \in e} x_e = \sum_{e \in E : v \in e} \sum_{i=1}^{n} t_i x_e^i = \sum_{i=1}^{n} t_i \sum_{e \in E : v \in e} x_e^i \leq \sum_{i=1}^{n} t_i = 1$$

for all $v \in V$.

And it is obviously that $x_e \geq 0 \quad \forall e \in E$.

So $\mathbf{x}$ is a feasible solution of Q. Thus $\mathbf{x} \in S(Q)$.

Thus $S(Q) = \text{conv}(S_{\text{int}}(Q))$.

3. Firstly, we have to prove that $\text{OPT}(P) \subseteq \text{conv}(\text{OPT}_{\text{int}}(P))$.

   Consider a $\mathbf{x} \in \text{OPT}(P)$.

   $\text{OPT}(P) \subseteq S(P)$, so from **Point 1** above we know $\mathbf{x}$ is a convex combination of feasible integral solutions. Now we have to prove $\mathbf{x}$ is a convex combination of optimal integral solutions.

   Note that we have to minimize $\sum_{v \in V} x_v$. For simplicity, denote it as $\mathbf{C}^T\mathbf{x}$. But for $\mathbf{x}(\epsilon)$, $\sum_{v \in V} x(\epsilon)_v = \mathbf{C}^T\mathbf{x}(\epsilon) = \mathbf{C}^T\mathbf{x} + \epsilon\mathbf{C}^T\mathbf{u}$. ($\mathbf{u}$ has been defined in **Point 1**). Note that $\mathbf{C}^T\mathbf{x}$ and $\mathbf{C}^T\mathbf{u}$ are numbers. So if $\mathbf{C}^T\mathbf{x}(\epsilon) < \mathbf{C}^T\mathbf{x}$, then there exist a ϵ' with different sign(in **Point 1** we have already proved that it exists) such that $\mathbf{C}^T\mathbf{x} < \mathbf{C}^T\mathbf{x}(\epsilon')$. But $\mathbf{x}$ is optimal. There comes a contradiction. i.e. $\mathbf{C}^T\mathbf{x}(\epsilon)$ is linear, and we have feasible ϵ both positive and negative. So if $\mathbf{x}$ is optimal and $\mathbf{x}(\epsilon)$ is feasible, then $\mathbf{x}(\epsilon)$ is optimal.

   Similarly, we can do the steps recursively, and because the convex combination is "transtive", $\mathbf{x}$ is a convex combination of optimal integral solutions.

   Thus, $\mathbf{x} \subseteq \text{conv}(\text{OPT}_{\text{int}}(P))$.

   Secondly, we have to prove that $\text{OPT}(P) \supseteq \text{conv}(\text{OPT}_{\text{int}}(P))$.

   For every $\mathbf{x} \in \text{conv}(S_{\text{int}}(P))$, we know

   $$\mathbf{x} = \sum_{i=1}^{n} t_i \mathbf{x}^{\mathbf{i}}$$

   where $\mathbf{x}^{\mathbf{i}} \in \text{OPT}_{\text{int}}(P)$ for all i and $\sum_{i=1}^{n} t_i = 1$.

   $\text{OPT}_{\text{int}}(P) \subseteq S_{\text{int}}(P)$. We have already known $\text{conv}(S_{\text{int}}(P)) = S(P)$. So for every $\mathbf{x} \in \text{conv}(\text{OPT}_{\text{int}}(P)) \subseteq \text{conv}(S_{\text{int}}(P))$, $\mathbf{x}$ is a feasible solution of P.

   Now we have to prove that $\mathbf{x}$ is optimal. Let us denote the optimal value as op. So $\sum_{v \in V} x_v^i = op$.

$$\sum_{v \in V} x_v = \sum_{v \in V} \sum_{i=1}^{n} t_i x_v^i = \sum_{i=1}^{n} t_i \sum_{v \in V} x_v^i = \sum_{i=1}^{n} t_i op = op \sum_{i=1}^{n} t_i = op$$

Therefore, $\mathbf{x}$ is also optimal. So $\mathbf{x} \in \mathrm{OPT}(P)$.

Thus $\mathrm{OPT}(P) = \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(P))$.

4. Firstly, we have to prove that $\mathrm{OPT}(Q) \subseteq \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(Q))$.

   Consider a $\mathbf{x} \in \mathrm{OPT}(Q)$.

   $\mathrm{OPT}(Q) \subseteq S(Q)$, so from **Point 2** above we know $\mathbf{x}$ is a convex combination of feasible integral solutions. Now we have to prove $\mathbf{x}$ is a convex combination of optimal integral solutions.

   Similarly, $\mathbf{C}^T\mathbf{x}(\epsilon)$ is linear, and we have feasible ϵ both positive and negative. So if $\mathbf{x}(\epsilon)$ is feasible, then it is optimal.

   Similarly, we can do the steps recursively, and because the convex combination is "transtive", $\mathbf{x}$ is a convex combination of optimal integral solutions.

   Thus, $\mathbf{x} \subseteq \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(Q))$.

   Secondly, we have to prove that $\mathrm{OPT}(Q) \supseteq \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(Q))$.

   For every $\mathbf{x} \in \mathrm{conv}(S_{\mathrm{int}}(Q))$, we know

$$\mathbf{x} = \sum_{i=1}^{n} t_i \mathbf{x^i}$$

   where $\mathbf{x^i} \in \mathrm{OPT}_{\mathrm{int}}(Q)$ for all i and $\sum_{i=1}^{n} t_i = 1$.

   $\mathrm{OPT}_{\mathrm{int}}(Q) \subseteq S_{\mathrm{int}}(Q)$. We have already known $\mathrm{conv}(S_{\mathrm{int}}(Q)) = S(Q)$. So for every $\mathbf{x} \in \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(Q)) \subseteq \mathrm{conv}(S_{\mathrm{int}}(Q))$, $\mathbf{x}$ is a feasible solution of Q.

   Now we have to prove that $\mathbf{x}$ is optimal. Let us denote the optimal value as op. So $\sum_{e \in E} x_e^i = op$.

$$\sum_{e \in E} x_e = \sum_{e \in E} \sum_{i=1}^{n} t_i x_e^i = \sum_{i=1}^{n} t_i \sum_{e \in E} x_e^i = \sum_{i=1}^{n} t_i op = op \sum_{i=1}^{n} t_i = op$$

Therefore, $\mathbf{x}$ is also optimal. So $\mathbf{x} \in \mathrm{OPT}(Q)$.

Thus $\mathrm{OPT}(Q) = \mathrm{conv}(\mathrm{OPT}_{\mathrm{int}}(Q))$.

Exercise 4.4.

Let $G = (U \cup V, E)$ where $Us = \{u_1, u_2, \ldots, u_n\}$, $V = s\{v_1, v_2, \ldots, v_n\}$ and $E = \{(u_i, v_j) | u_i \in U, v_j \in V\}$.

Obviously this is a bipartite graph. Consider its matching LP problem Q:

$$\begin{array}{ll} \max & \displaystyle\sum_{(u_i,v_j)\in E} y_{(u_i,v_j)} \\ s.t. & \sum_{v_j\in V} y_{(u_i,v_j)} \leq 1 \quad \forall u_i \in U \\ & \sum_{u_i\in U} y_{(u_i,v_j)} \leq 1 \quad \forall v_j \in V \\ & \qquad\quad y_{(u_i,v_j)} \geq 0 \quad \forall (u_i, v_j) \in E \end{array}$$

The numbers $y_{(u_i,v_j)}$ construct a matrix in $\mathbb{R}^{n\times n}$ and since the optimal solution of this matching problem is n (choose all $y_{(u_i,v_i)} = 1$ and others set to zero), then every doubly stochastic matrix is an optimal solution.

The optimal solution of *Integer Linear Program of the Matching Problem* is also doubly stochastic matrix with integer matrix elements. From the constriction we know that the elements can only be 0 and 1. Every row and column can only have one 1. These matrices are *permutation matrices*.

From **Exercise 4.3, Point 4**, we know that every optimal solution of Q is the convex combination of the integer problem Q_{int}, then every doubly stochastic matrix is a convex combination of permutation matrices.