Algorithm Analysis
Homework Assignment 4 Bonus

Dumb ways to die
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4 Vertex Double Cover

Exercise 4.1.

The LP P is
Minimize

$$\sum_{v \in V} x_v$$

Subject to

$$\forall v \in V : x_v \in \{0, 1, 2\}$$
$$\forall e = (u, v) \in E : x_u + x_v \geq 2$$

The LP relaxation P' is
Minimize

$$\sum_{v \in V} x_v$$

Subject to

$$\forall v \in V : x_v \in [0, 2]$$
$$\forall e = (u, v) \in E : x_u + x_v \geq 2$$

Exercise 4.2.

Firstly, we have to prove $\text{val}(P) \leq \text{val}(P')$.

Consider $\mathbf{x}$ as a optimal solution of P'.

Define

$$V_{01} = \{v|v \in A, 0 < x_v < 1\}$$

$$V_{12} = \{v|v \in A, 1 < x_v < 2\}$$

$$V_z = \{v|v \in A, x_v \in \mathbb{N}\}$$

$$E_f = \{e|e \in E, \sum_{v \in e} x_v \text{ is not an integer}\}$$

$$E_z = \{e|e \in E, \sum_{v \in e} x_v \in \mathbb{N}\}$$

Assume $|V_{01}|+|V_{12}|+|E_f| > 0$(Otherwise it is already a integral solution). Then we define

$$\mathbf{x}(\epsilon) = \begin{cases} x_v + \epsilon & v \in V_{01} \\ x_v - \epsilon & v \in V_{12} \\ x_v & \text{otherwise} \end{cases}$$

Now we have to prove we can find both positive and negative ϵ such that $\mathbf{x}(\epsilon)$ is feasible.

Firstly,

$$\epsilon \in [\lfloor x_v \rfloor - x_v, \lceil x_v \rceil - x_v] \quad \forall v \in V_{01}$$

$$\epsilon \in [x_v - \lceil x_v \rceil, x_v - \lfloor x_v \rfloor] \quad \forall v \in V_{12}$$

Obviously, $0 \in [\lfloor x_v \rfloor - x_v, \lceil x_v \rceil - x_v], 0 \in [x_v - \lceil x_v \rceil, x_v - \lfloor x_v \rfloor]$.

Now consider the inequality $\sum_{v \in e} x_v \geq 2$(*).

Consider edges (u, v) between V_{01} and $V_z(u \in V_{01},\ v \in V_z)$. Note that $x_u + x_v \geq 2$. For positive ϵ, $x_u(\epsilon) + x_v(\epsilon) = x_u + x_v + \epsilon > 2$.

Note that x_u is not integral, but x_v is integral. So $x_u + x_v$ is not integral, $x_u + x_v > 2$. Thus there is a negative ϵ such that $x_u + x_v + \epsilon \geq 2$.

i.e., $\epsilon \in [2 - x_u - x_v, +\infty)$, and $0 \in [2 - x_u - x_v, +\infty)$.

Consider edges (u, v) between V_{01} and $V_{12}(u \in V_{01},\ v \in V_{12})$. Note that $x_u + x_v \geq 2$, $\epsilon > 0$. $x_u(\epsilon) + x_v(\epsilon) = x_u + x_v + \epsilon - \epsilon \geq 2$. Thus (*) holds for any ϵ.

Consider edges (u, v) between V_{12} and V_{12}. $x_u(\epsilon) + x_v(\epsilon) = x_u + x_v - 2\epsilon$. Obviously (*) holds for any negative ϵ. Note that $x_u > 1$, $x_v > 1$, so $x_u + x_v > 2$. Thus there is a positive ϵ such that $x_u + x_v - 2\epsilon \geq 2$.

i.e., $\epsilon \in (-\infty, \frac{x_u+x_v-2}{2}]$, and $0 \in (-\infty, \frac{x_u+x_v-2}{2}]$.

Consider edges (u, v) between V_{12} and $V_z (u \in V_{12}, v \in V_z)$. $x_u(\epsilon)+x_v(\epsilon) = x_u+x_v-\epsilon$. Obviously (*) holds for any negative ϵ. Note that x_u is not integral, but x_v is integral. So $x_u + x_v$ is not integral, $x_u + x_v > 2$. Thus there is a positive ϵ such that $x_u + x_v - \epsilon \geq 2$.

i.e., $\epsilon \in (-\infty, x_u + x_v - 2]$, and $0 \in (-\infty, x_u + x_v - 2]$.

Now we can get ϵ_1 and $-\epsilon_2 (\epsilon_1, \epsilon_2 > 0)$ from the intersection of the intervals above. Then $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(-\epsilon_2)$ are feasible solutions of P'.

Now we can define $\mathbf{u} \in \mathbb{R}^{|V|}$.

$$\mathbf{u}_i = \begin{cases} 1 & v_i \in V_{01} \\ -1 & v_i \in V_{12} \\ 0 & \text{otherwise} \end{cases}$$

Thus $\mathbf{x}(\epsilon) = \mathbf{x} + \epsilon\mathbf{u}$. Denote $\sum_{v\in V} x_v$ as $\mathbf{C}^T\mathbf{x}$. Then $\mathbf{C}^T\mathbf{x}(\epsilon) = \mathbf{C}^T\mathbf{x} + \epsilon\mathbf{C}^T\mathbf{u}$. Note that $\mathbf{C}^T\mathbf{x}$ and $\mathbf{C}^T\mathbf{u}$ are numbers.

Thus if $\mathbf{C}^T\mathbf{x}(\epsilon_1) > \mathbf{C}^T\mathbf{x}$, then $\mathbf{C}^T\mathbf{x} > \mathbf{C}^T\mathbf{x}(\epsilon_2)$. There comes a contradiction. i.e. $\mathbf{C}^T\mathbf{x}(\epsilon)$ is linear, and we have feasible ϵ both positive and negative. So both $\mathbf{x}(\epsilon_1)$ and $\mathbf{x}(\epsilon_2)$ are both optimal solutions.

We can do the same steps for $\mathbf{x}(\epsilon_1)$.

According to the analyse above, we can choose ϵ_1 to let $|V_{01}|+|V_{12}|+|E_f|$ decrease by at least 1.

So the algorithm will stop in finite time. Then $|V_{01}| + |V_{12}| + |E_f| = 0$, we get a integral solution $\mathbf{x}'$. Note that $\mathbf{C}^T\mathbf{x} = \mathbf{C}^T\mathbf{x}'$.

Thus for every optimal solution of P', there is a integral solution with the same size of vertex double cover. Therefore, $\text{val}(P) \leq \text{val}(P')$.

Secondly, we have to prove $\text{val}(P) \geq \text{val}(P')$.

Obviously, if $\mathbf{x}$ is a optimal solution of P, then it must be a solution of P'. So $\text{val}(P) \geq \text{val}(P')$.

Therefore, $\text{val}(P) = \text{val}(P')$.

Exercise 4.3.

Algorithm Firstly, we construct a new bipartite graph $G^b = (U^b \cup W^b, E^b)$ from original graph $G = (V, E)$ with U^b and W^b are both of size $|V|$.

In our new graph G^b, edge $e^b = \{u_i^b, w_j^b\} \in E^b$ iff $\{v_i, v_j\} \in E$.[1]

Secondly we run a classic algorithm to find the minimum *vertex cover* problem in bipartite graph and the result is the same with the *vertex double cover* problem.

Time Complexity The step of constructing the new bipartite graph are just "splitting" vertices, which costs only linear time.

The classic algorithm for *vertex cover* problem in bipartite graph is a polynomial time algorithm. Thus the composite algorithm costs only polynomial time.

Correctness. We prove the correctness of the algorithm above with linear programming. We are going to prove the LP relaxation P^b of the *vertex cover* in bipartite graph G^b have the same optimal value.

P' has been stated in **Exercise 4.1**. We stated the LP relaxation P^b of vertex cover in G^b here

$$\begin{aligned}
\min \quad & \sum_{u_i^b \in U^b} x_{u_i^b} + \sum_{w_j^b \in W^b} x_{w_j^b} \\
s.t. \quad & x_{u_i^b} + x_{w_j^b} \geq 1 \quad \forall \{u_i^b, w_j^b\} \in E^b \\
& x_{u_i^b} \geq 0 \quad \forall u_i^b \in U^b \\
& x_{w_j^b} \geq 0 \quad \forall w_j^b \in W^b
\end{aligned}$$

To begin with, we construct P'' from P' that replace x_{v_i} with $x_{u_i''} \geq 0$ and $x_{w_i''} \geq 0$:

$$x_{v_i} = x_{u_i''} + x_{w_i''}$$

This does not affect the optimal value:

$$\text{val}(P') = \text{val}(P'')$$

[1] https://en.wikipedia.org/wiki/Bipartite_double_cover

State P'' here:

$$\begin{aligned} \min \quad & \sum_{v_i \in V} x_{u_i''} + x_{w_i''} \\ s.t. \quad & x_{u_i''} + x_{w_i''} + x_{u_j''} + x_{w_j''} \geq 2 \quad \forall \{v_i, v_j\} \in E \\ & x_{u_i''} \geq 0 \quad \forall v_i \in V \\ & x_{w_i''} \geq 0 \quad \forall v_i \in V \end{aligned}$$

Comparing P^b with P''. If we have a feasible solution x^b of P^b, we make $x'' = x^b$. It is obviously that x'' is a feasible solution of P''. Then

$$\mathrm{val}(P'') \leq \mathrm{val}(P^b)$$

If we have a feasible solution x'' of P'', then we make

$$x_{u_i^b} = x_{w_i^b} = \frac{x_{u_i''} + x_{w_i''}}{2}$$

Compare the first line of constrains, we have

$$x_{u_i^b} + x_{w_j^b} = \frac{x_{u_i''} + x_{w_i''} + x_{u_j''} + x_{w_j''}}{2} \geq \frac{2}{2} = 1$$

And the target function

$$\sum_{u_i^b \in U^b} x_{u_i^b} + \sum_{w_j^b \in W^b} x_{w_j^b} = \sum_{u_i^b \in U^b} \frac{x_{u_i''} + x_{w_i''}}{2} + \sum_{w_j^b \in W^b} \frac{x_{u_j''} + x_{w_j''}}{2} = \sum_{v_i \in V} x_{u_i''} + x_{w_i''} = \sum_{v_i \in V} x_{v_i}$$

remains the same.

Thus

$$\mathrm{val}(P'') \geq \mathrm{val}(P^b)$$

Then we reach the goal

$$\mathrm{val}(P'') = \mathrm{val}(P^b)$$

$\square$