Algorithms and Complexity
Homework Assignment 2

Dumb Ways to Die

2015-10-25

2 Network Flows, Matchings, and Paths

2.1 Flows

Exercise 2.1. [Flow Conservation Properties]

1. *Proof.* Obviously,
$$\sum_{u,v\in S} f(u,v) = \sum_{u,v\in S} f(v,u) \tag{1}$$
holds for any $S \subseteq U$.
Because An *s-t-flow* f satisfies skew symmetry, which indicates
$$f(u,v) + f(v,u) = 0$$
Thus, we have
$$\sum_{u,v\in S} f(u,v) + \sum_{u,v\in S} f(v,u) = \sum_{u,v\in S} (f(u,v) + f(v,u)) = 0 \tag{2}$$
According to (1) and (2), we prove that
$$\sum_{u,v\in S} f(u,v) = 0 \tag{3}$$
holds for any $S \subseteq V$. □

2. *Proof.* Let $S = V$, because of (3) we have
$$\sum_{u,v\in V} f(u,v) = \sum_{u\in V\setminus\{s,t\}} \sum_{v\in V} f(u,v) + \sum_{v\in V} f(s,v) + \sum_{v\in V} f(t,v) = 0$$
The flow conservation shows
$$\sum_{v\in V} f(u,v) = 0, \forall u \in V\setminus\{s,t\}$$
Thus
$$\sum_{u,v\in V} f(u,v) = 0 + \sum_{v\in V} f(s,v) + \sum_{v\in V} f(t,v) = 0$$

Besides, $val(f) = \sum_{v\in V} f(s,v)$ and $f(t,v) = -f(v,t)$, which indicates

$$val(f) = \sum_{v\in V} f(s,v) = 0 - \sum_{v\in V} f(t,v) = \sum_{v\in V} f(v,t)$$

□

3. *Proof.* Because of the flow conservation law

$$\sum_{v\in V} f(u,v) = 0, \forall u \in V\backslash\{s,t\}$$

We have

$$val(f) = \sum_{v\in V} f(s,v) + \sum_{u\in S\backslash\{s\}} \sum_{v\in V} f(u,v) = \sum_{u\in S}\sum_{v\in V} f(u,v)$$

We can resolve the sum into:

$$\sum_{u\in S}\sum_{v\in V} f(u,v) = \sum_{u\in S}\sum_{v\in V\backslash S} f(u,v) + \sum_{u\in S}\sum_{v\in S} f(u,v)$$

Because of the equation(3), we know that $\sum_{u,v\in S} f(u,v) = 0$. Thus

$$val(f) = \sum_{u\in S}\sum_{v\in V\backslash S} f(u,v) + \sum_{u,v\in S} f(u,v) = \sum_{u\in S}\sum_{v\in V\backslash S} f(u,v)$$

So the following holds.

$$f(S, V\backslash S) = val(f) \tag{4}$$

□

Exercise 2.2. [Flow is Smaller than Cut]

Proof. f is a flow in G and S be a cut. By (4) we have

$$val(f) = f(S, V\backslash S) = \sum_{u\in S}\sum_{v\in V\backslash S} f(u,v)$$

And capacity constrains shows

$$f(u,v) \le c(u,v), \forall u,v \in V$$

Then we have

$$\sum_{u\in S}\sum_{v\in V\backslash S} f(u,v) \le \sum_{u\in S}\sum_{v\in V\backslash S} c(u,v)$$

That is

$$val(f) = f(S, V\backslash S) \le c(S, V\backslash S)$$

□

Exercise 2.3.

Proof. If there is no flow from s to t of value k, then there must be at least a cut S, $V\backslash S$ of value $c(S) < k$.

If $r \in S$ then S, V is a cut that separates r and t. Since there is a flow from r to t of value k and flow is smaller than cut, there comes a contradiction.

If $r \in V\backslash S$ then S, V is a cut that separates s and r. Since there is a flow from s to r of value k and flow is smaller than cut, there comes a contradiction. □

2.2 Matchings

Exercise 2.4. [Matchings in Regular Bipartite Graphs]

Proof. For simplicity assume that $|V_1| \leq |V_2|$. If $|V_1| > |V_2|$, we can swap $|V_1|$ and $|V_2|$.

Consider a network $G = (V', E', c)$ with

$$V' = V_1 \cup V_2 \cup \{s, t\}$$

and

$$E' = E \cup \{(s, u) | u \in V_1\} \cup \{(v, t) | t \in V_2\}$$

and

$$c(u, v) = \begin{cases} +\infty & u \in V_1, v \in V_2, (u, v) \in V \\ 1 & u = s, v \in V_1 \\ 1 & u \in V_2, v = t \\ 0 & otherwise \end{cases}$$

Let

$$f(u, v) = \begin{cases} \frac{1}{d_1} & u \in V_1, v \in V_2, (u, v) \in E \\ -\frac{1}{d_1} & u \in V_2, v \in V_1, (v, u) \in E \\ 1 & u = s, v \in V_1 \\ -1 & u \in V_1, v = s \\ \frac{d_2}{d_1} & u \in V_2, v = t \\ -\frac{d_2}{d_1} & u = t, v \in V_2 \\ 0 & otherwise \end{cases}$$

$|V_1| d_1 = |V_2| d_2$, so $d_1 \geq d_2$. Thus $\frac{d_2}{d_1} \leq 1$.

Obviously,

$$\forall u, v \in V' \quad f(u, v) = f(v, u), f(u, v) \leq c(u, v)$$

Additionally, We can find that

$$\sum_{v \in V'} f(u, v) = 1 - d_1 \times \frac{1}{d_1} = 0$$

and

$$\sum_{v \in V'} f(u, v) = \frac{d_2}{d_1} - d_2 \times \frac{1}{d_1} = 0$$

for all $u \in V_1$.

So f is a flow and $val(f) = |V_1|$.

Let $S = \{s\}$, thus $S, V' \backslash S$ is a cut.

$$c(S, V' \backslash S) = |V_1| = val(f)$$

Therefore, f is a maximum flow.

Lemma 1. *If each edge in a flow network has integral capacity, then there exists an integral maximal flow.*

From the lemma above we can know there is an integral maximal flow of value $|V_1|$.

From **Exercise 2.1.2** we can get

$$\sum_{u \in V'} f(u,v) = 0 \quad \forall v \in V_1$$

thus

$$\sum_{u \in V', u \neq s} f(u,v) = -f(s,v) = -1 \quad \forall v \in V_1$$

Thus

$$\sum_{u \in V', u \neq s} f(v,u) = 1 \quad \forall v \in V_1$$

and

$$f(v,u) \geq 0 \quad \forall u,v \in V_1,\ u \neq s$$

So for all $v \in V_1$, there is a $m_v \in V_2$ such that

$$f(v, m_v) = 1$$

and m_v is unique for any v in V_1.

Similarly, for all $v \in V_2$, there is a $m_v \in V_1$ such that

$$f(m_v, v) = 1$$

and m_v is unique for any v in V_2.

Let

$$M = \{(v, m_v) | \forall v \in V_1\}$$

Obviously, $M \subseteq E$. And every $v \in V$ is incident to exactly one edge of M.

So M is a *matching* in G of size $|V_1|$.

Finally, there must be a *matching* in G of size $\min(|V_1|, |V_2|)$.

□

Note that the conclusion might not be true when $d_1 = 0$ or $d_2 = 0$.

Counterexample: $G = (V = \{1,2\},\ E = \phi)$, then we make $V_1 = \{1\}$, $d_1 = 0$, $V_2 = \{2\}$ and $d_2 = 0$. We can see that $\min(|V_1|,\ |V_2|) = 1$ but the maximum matching is 0.

Exercise 2.5. [Matchings in H_n]

Proof. Consider $i \leq n/2$ and the graph $H_n[L_i \cup L_{i+1}]$. For any u in L_i, there are exactly i "1" in u. So we can choose one "0", replace it with one "1". Thus we can get a new node u', and u, u' are connected.

Thus $deg(u) = n-i \quad \forall u \in L_i$. Similarly $deg(u) = n-(i+1) \quad \forall u \in L_{i+1}$. From **Exercise 2.4** we can know that there is a matching of size $min(|L_i|, |L_{i+1}|) = |L_i| = \binom{n}{i}$. □

2.3 Vertex Disjoint Paths

Exercise 2.6.

1. The transformation from G to G' has some functions following:

   (a) Two injections from V to V':

   $$f_1 : V \longrightarrow V'$$
   $$v \longmapsto v_{in}$$

   $$f_2 : V \longrightarrow V'$$
   $$v \longmapsto v_{out}$$

   and $f_1(V) \bigcap f_2(V) = \emptyset$ and $f_1(V) \bigcup f_2(V) = V'$ are satisfied.

   (b) A injection from E to E'

   $$e : \quad E \longrightarrow E'$$
   $$< u, v > \longmapsto < f_2(u), f_1(v) >$$

   And for any $v \in V$, there is an *extra edge* $< f_1(v), f_2(v) >$ in E'.

   (c) The function c' is defined as

   $$c'(< u', v' >) = \begin{cases} c(v) & < u', v' > \text{ is an extra edge which is referred in (b).} \\ \infty & < u', v' > \text{ is not an extra edge.} \end{cases}$$

   where $u', v' \in V'$.

2. For example, there is a graph with vertex capacities.

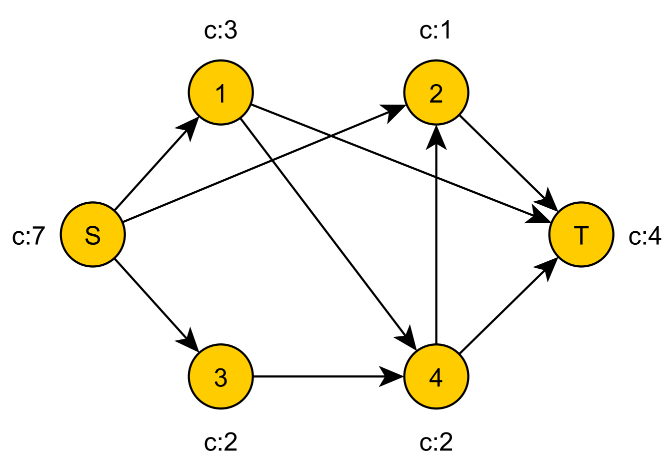

Figure 1: A graph G with vertex capacities

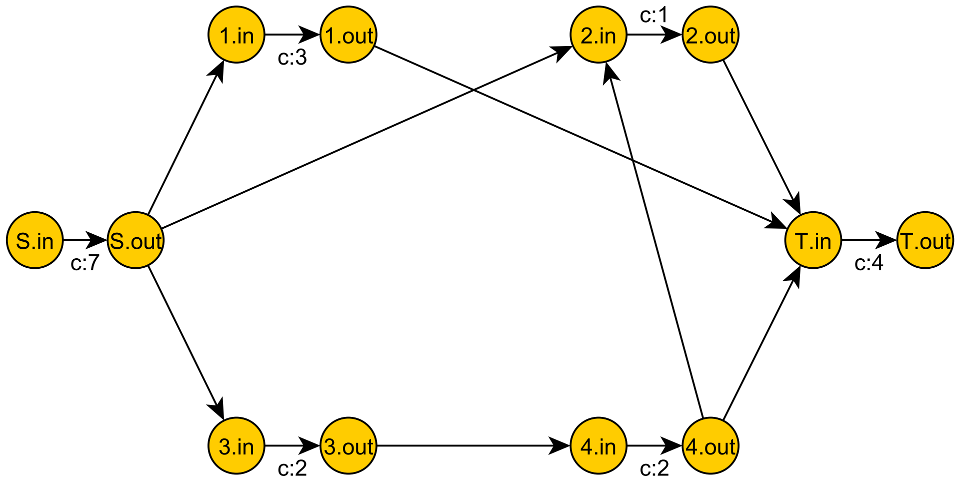

Figure 2: G' which is transformed from G

The mapping from V to V' can be showed in following picture.

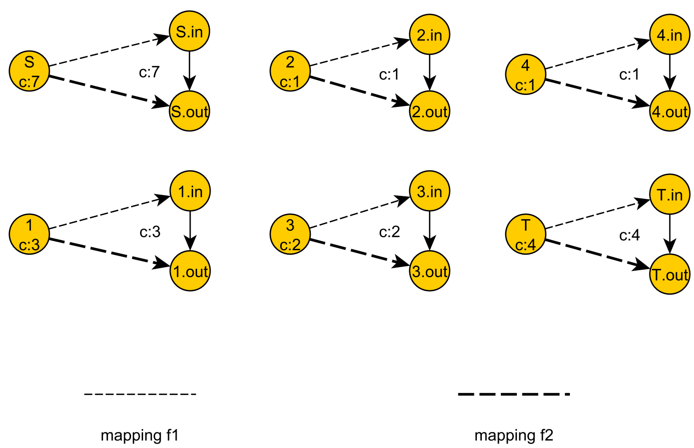

Figure 3: The mapping between V and V'

The maximum flow in G(from s to t) is equal to the maximum flow in G' (from $S.1$ to $T.1$) which equals to 6.

3. (a) Make a graph with vertex capacities $G' = (V, E, c)$ where $c : V \longrightarrow R$ is defined as

$$c(v) = \begin{cases} k & v \in \{s, t\} \\ 1 & v \in V \setminus \{s, t\} \end{cases}$$

.

   (b) Then, if there is a flow from s to t of value k, k disjoint paths from S to T will be existed. As we have known in subproblem 2, we can transform G' to G'' and use *Dinic* which can compute the maximum flow in $\mathrm{O}(|V|^2|E|)$ to solve this problem.

(c) After this transformation, we have $|V'| = 2|V|$, $|E'| = |E| + |V|$. So the running time will be

$$\mathrm{O}(|V'|^2|E'|) = \mathrm{O}(4|V|^2|E| + 4|V|^3) = \mathrm{O}(|V|^2|E|)$$

.[1]

Exercise 2.7.

1. Produce the graph G with the following steps:
   (a) For any $v \in L_i$, there is a directed edge from s to v.
   (b) For any $u \in L_j, v \in L_{j+1} (j \in \{i, i+1, ..., n-i-1\})$, there is a directed edge from u to v if and only if $\{u, v\} \in H_n$.
   (c) For any $v \in L_{n-i}$, there is a directed edge from v to t.
   (d) Set the capacities of s and t as $\binom{n}{i}$, set the capacities of other vertices as 1.
   (e) Define $\{s\}$ as L_{i-1} and $\{t\}$ as L_{n-i+1} to simplify the statement.
2. The problem can be transformed to whether there is a flow from s to t of value $\binom{n}{i}$.
3. Show that there must be a flow from s to t of value $\binom{n}{i}$.
   (a) Push a flow of value 1 from s to each vertex in L_i. Obviously, the total flow of s is equal to $\binom{n}{i}$ which is no more the capacity of s.
   (b) Each vertex in L_i has $n-i$ directed edges to the vertics in L_{i+1} and pushes the flow it received totally to them averagely. So that each vertex in L_{i+1} will get a flow of value $1 \cdot \frac{i+1}{n-i} = \frac{i+1}{n-i}$ which will no more than 1 when $2i \leq n$.
   (c) For each level $k(i+1 \leq k \leq n-i)$ and $k \in N^*$, the vertices in L_k will receive flows from the level $k-1$ and push all it received to the level $k+1$ averagely.
   (d) So for each level $k = i+1, i+2, ..., n-i$, the flow of single vertex in L_k can be calculate by the formula following:

$$\begin{aligned} f &= \prod_{j=i+1}^{k} \frac{j}{n-j+1} \\ &= \frac{\prod\limits_{a=i+1}^{k} a}{\prod\limits_{b=n-k+1}^{n-i} b} \\ &= \frac{(i+1)(i+2)\cdots k}{(n-k+1)(n-k+2)\cdots(n-i)} \\ &= \prod_{j=0}^{k-i-1} \frac{i+1+j}{n+1-k+j} \end{aligned}$$

[1] In general, $|E|$ will be greater than $|V|$.

Obviously, for any $j \in \{0, 1, 2, \cdots, k-i-1\}$ where $i+1 \leq k \leq n-i$, $\frac{i+1+j}{n+1-k+j} \leq 1$ is satisfied. Because of this, f will be never greater than 1. So for any vertex $u \in G$ has

$$\sum_{v \in V, f(u,v) > 0} \leq c(u)$$

.

(e) Each vertex v in L_{n-i} will receive flows of value 1 in total from L_{n-i-1} and it will push them to t. So t will get $\binom{n}{i}$ flows. Thus there exists a flow from s to t of value $\binom{n}{i}$.

4. Because the capacities in G are all integers, there must be an integral maximum flow in G.

5. Find the integral maximum flow.

6. The capacity of each vertex in $V \setminus \{s, t\}$ is exactly 1, so the flow of each edge will be exactly 0 or 1 in G with an integral maximum flow. *Delete edges whose flows are equal to* 0. And then delete s, t, the edges from s to L_i and the edges from L_{n-i} to t as well.

7. After that G will remain $\binom{n}{i}$ paths that satisfy the requirements in the problem.