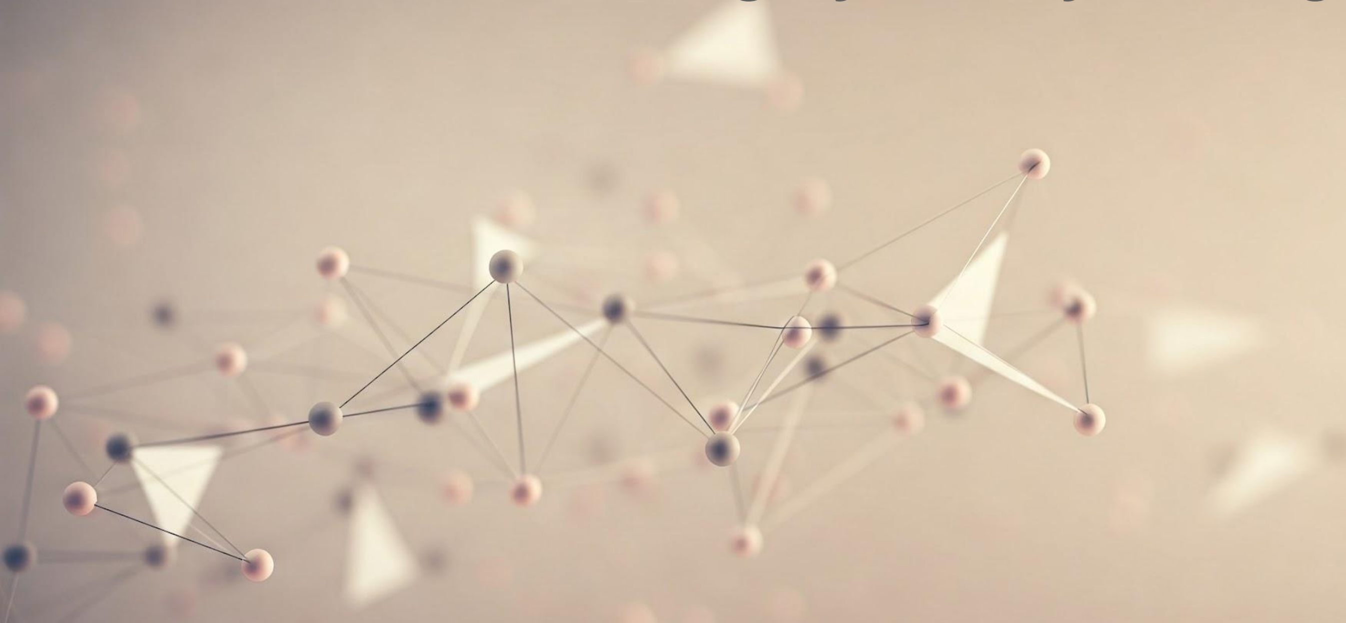
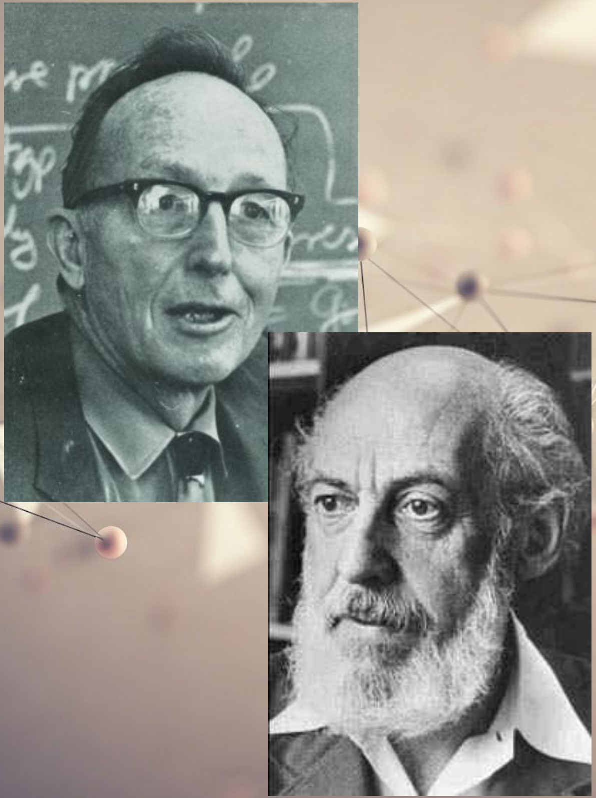
An Introduction to Category Theory: Seeing the Bigger Picture



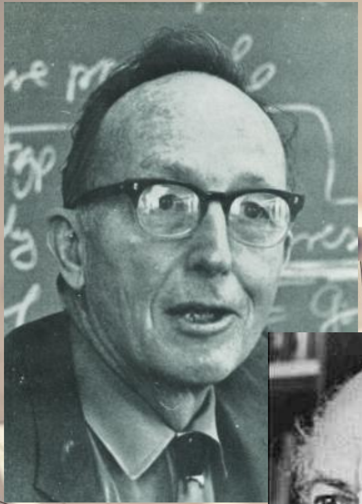
Understanding Structures and Relationships Across Diverse Fields

Unifying Math: Goals & Origins



- Category Theory began in the mid-1940s by Eilenberg and Mac Lane.
- Working to unify and formalize concepts in **algebraic topology** and **homological algebra**
- Mac Lane: “[we] had to discover the notion of a **natural transformation**. That in turn forced us to look at **functors**, which in turn made us look at **categories**."

Unifying Math: Goals & Origins



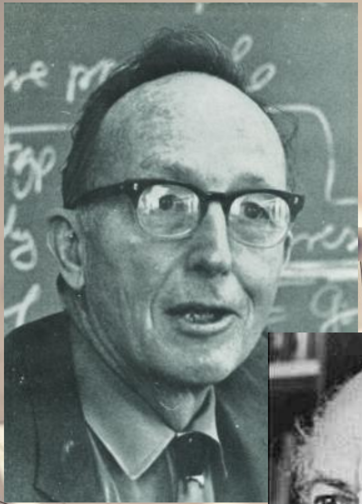
GENERAL THEORY OF NATURAL EQUIVALENCES

BY

SAMUEL EILENBERG AND SAUNDERS MACLANE

It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a *functor* and of a natural transformation (the latter is defined in the next chapter). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Hom" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves.

Unifying Math: Goals & Origins



GENERAL THEORY OF NATURAL EQUIVALENCES

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The coboundary homomorphism

$$\delta_q(K, G): C_q(K, G) \rightarrow C_{q+1}(K, G)$$

is defined by setting, for each cochain $f \in C_q(K, G)$,

$$(\delta_q f)(c^{q+1}) = f(\partial^{q+1} c^{q+1}).$$

This leads to a natural transformation of functors

$$\delta_q: C_q \rightarrow C_{q+1}.$$

Elements of a Category

- A category has abstract objects (represented by upper case letters)
- Morphisms are arrows showing relationships between objects.
- Each arrow has a source, the starting object and a target. These are typically called the domain and codomain of the morphism (arrow)

.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[loop left, "1_A"] & B \arrow[d, "g"] \arrow[loop right, "1_B"] \\
& C \arrow[loop right, "1_C"]
\end{tikzcd}
$$

Rules of a Category

The composition of $f : a \to b$ and $g : b \to c$ is written as $g \circ f$ or gf,[b] governed by two axioms:

1. Associativity: If $f : a \to b$, $g : b \to c$, and $h : c \to d$ then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

2. Identity: For every object x, there exists a morphism $1_x : x \to x$ (also denoted as id_x) called the *identity morphism for* x, such that for every morphism $f : a \to b$, we have

$$1_b \circ f = f = f \circ 1_a$$

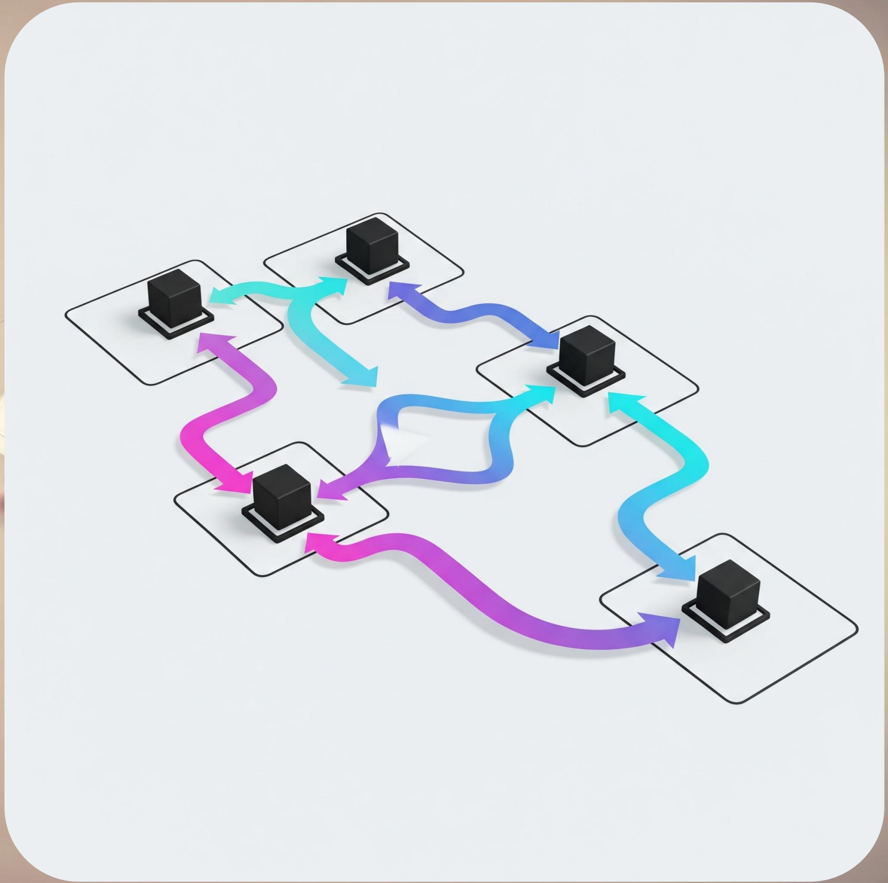
From the axioms, it can be proved that there is exactly one identity morphism for every object.

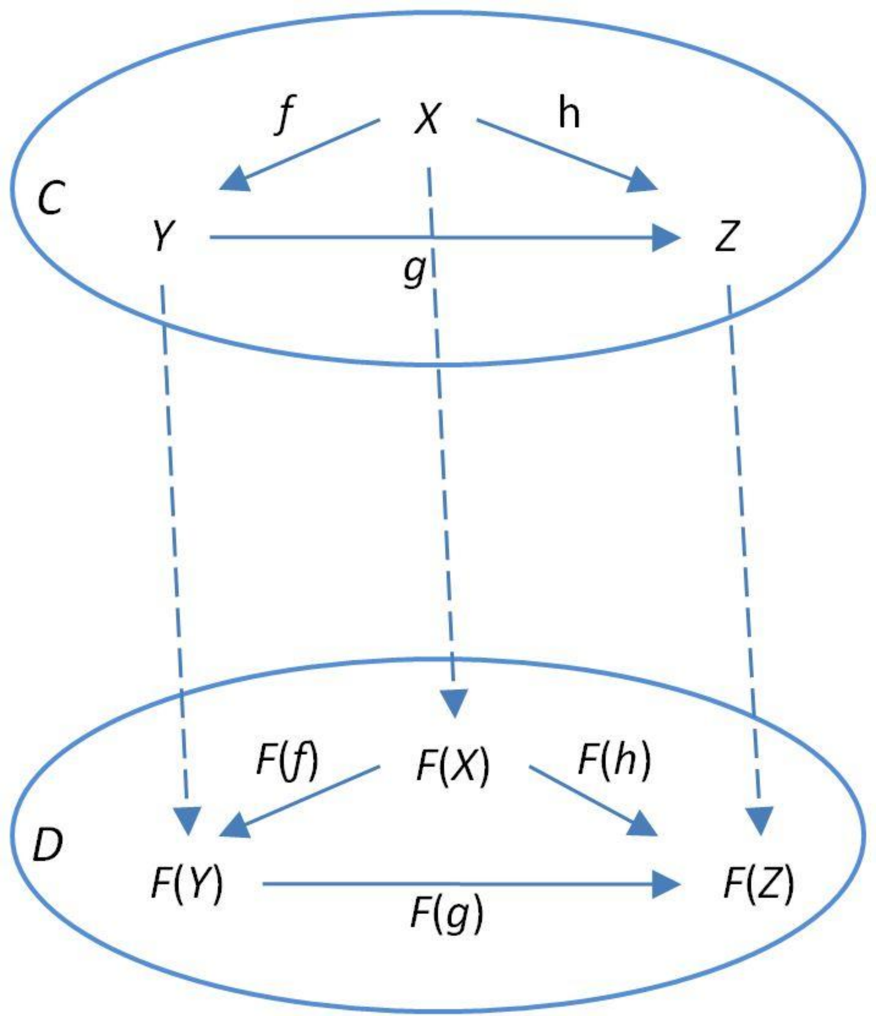
Simple Category Examples

Category	Objects	Morphisms
Sets	Sets	Functions
Groups	Groups	Group Homomorphisms
Topological Spaces	Topological Spaces	Continuous Functions
Posets	Partially Ordered Sets	Order-Preserving Functions
Monoids	One Object (the Monoid itself)	Monoid Elements

Shift to Categorical Thinking

- Focus shifts from internal details to relationships (Yoneda lemma)
- Objects as 'black boxes'; interactions matter most.
- Morphisms show object interactions, central to understanding.
- Category theory is arrow-centric, set theory is element-centric.

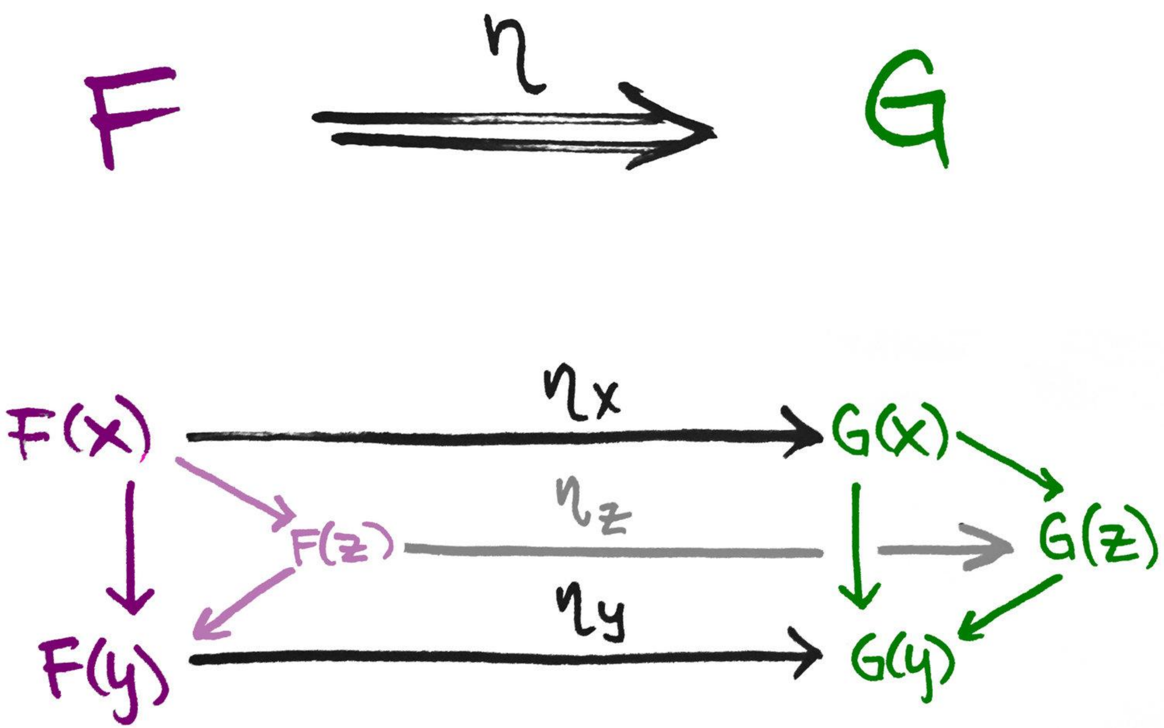




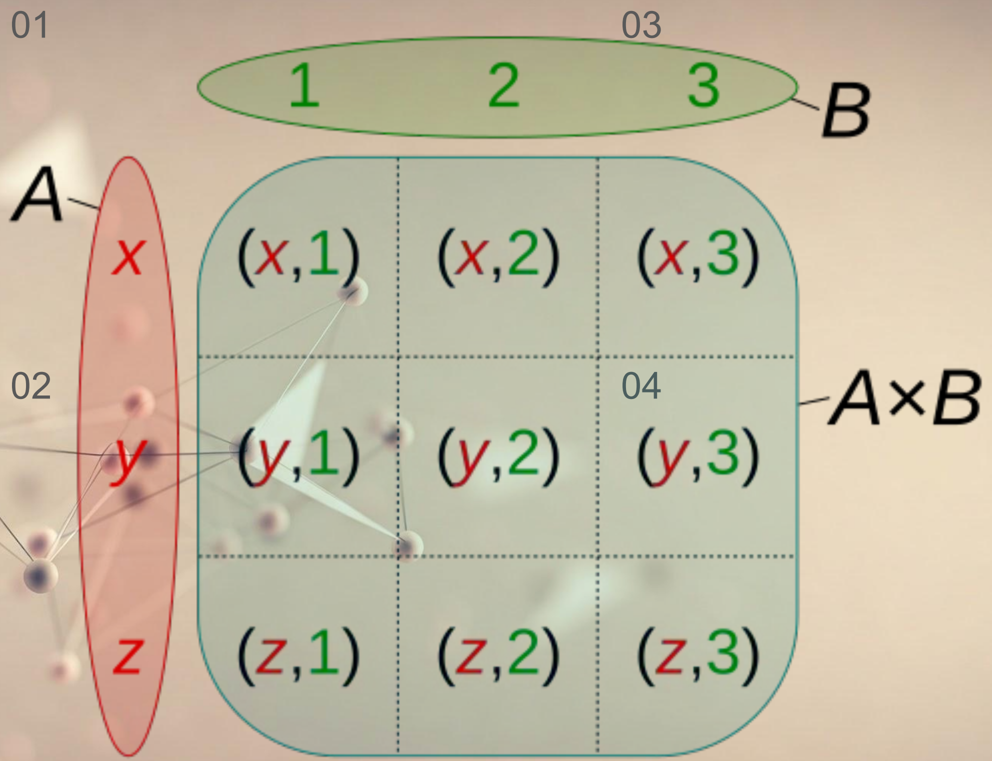
Functors: Bridges Between Categories

- Functors serve as bridges or translators between categories.
- Functor F maps objects in C to objects in D.
- Each morphism in C maps to a morphism in D.
- Functors preserve the relationships between objects and morphisms.
- The functor F maps the identity morphisms in C to identity morphisms in D. $F(1_X) = 1_{F(x)}$

Connecting Functors Naturally (Natural Transformations)



Cartesian Product



Category Product

Reads: The product of A and B, is an object P and two projections such that for any object X with arrows to A and B, there is a unique f from X to P such that diagram commutes.

This is an example of a universal property/arrow

$$
\begin{tikzcd}
 & X \arrow[dl, "f_1"'] \arrow[d, dashed, "f"] \arrow[dr, "f_2"] & \\
A & P \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B
\end{tikzcd}
$$

Universal Property (Arrow)

A **universal morphism from X to F** is a unique pair $(A, u : X \to F(A))$ in $\mathcal{D}$ which has the following property, commonly referred to as a **universal property**:

For any morphism of the form $f : X \to F(A')$ in $\mathcal{D}$, there exists a *unique* morphism $h : A \to A'$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[dr, "f"'] & F(A) \arrow[d, dashed, "F(h)"] & A \arrow[d, dashed, "h"] \\
 & F(A') & A'
\end{tikzcd}$$

Geometric Langlands Conjecture

Yale Daily News

NEW HAVEN, CONNECTICUT • **TUESDAY, MAY 27, 2025** • VOL. CXLVII

Math's "Rosetta Stone:" Yale professor proves decades-old mathematical conjecture

Yale professor Sam Raskin led a team to prove the geometric Langlands conjecture, solving a major part of one of math's most sweeping paradigms.

HARI VISWANATHAN | 2:20 AM, NOV 14, 2024

CONTRIBUTING REPORTER

MOST READ

Baala Shakya

Geometric Langlands Conjecture

building on the vision of Beilinson and Drinfeld, it posits a canonical equivalence between two complex derived categories (or stable ∞-categories): the category of $\mathcal{D}$-modules on the moduli stack of G-bundles, $DMod(Bun_G)$, and the category of Ind-coherent sheaves with nilpotent singular support on the moduli stack of LG-local systems, $IndCoh_{Nilp}(Loc_{LG})$. This equivalence is expected to intertwine

Lawvere’s Fixed Point Theorem

See
https://en.wikipedia.org/wiki/Lawvere%27s_fixed-point_theorem

Conquering Category Theory Hurdles

- Category Theory is highly abstract initially.
- 'Element-free' thinking can be difficult to grasp.
- Provides a powerful structural perspective overall.
- Offers a precise language for abstraction.
- Furnishes robust tools for modeling relationships.

Beyond Basics: Learning & Questions

- Good online resources:
  https://math.jhu.edu/~eriehl/161/context.pdf

  https://math.mit.edu/~hrm/palestine/maclane-categories.pdf

  https://www.math3ma.com/categories/category-theory

- Any questions are welcome now.