PYTHAGOREAN TRIPLES

ODD SUMS AND BABYLONIAN TABLETS

WHAT'S A PYTHAGOREAN TRIPLE?

- Positive integers $A, B,$ and C form a "Pythagorean triple" if $A^2 + B^2 = C^2$.
- Familiar triples:
  - 3,4,5
  - 5,12,13
  - 8,15,17
  - 20,21,29
- It's convenient to call the largest member of the triple the "hypotenuse".

WHAT'S A PRIMITIVE TRIPLE?

- A Pythagorean Triple is "primitive" if the numbers have no common factors.
- Example: 5, 12, 13 is a primitive triple.
- Example: 6, 8, 10 is a triple, but not primitive

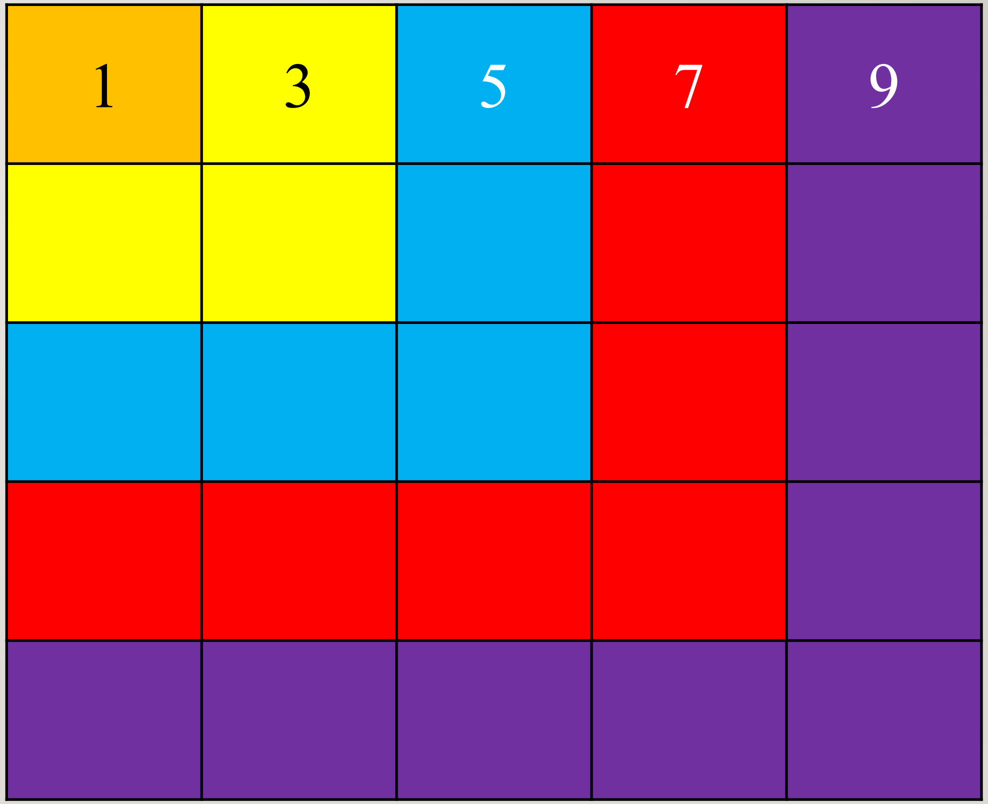
PLIMPTON 322 VS. PYTHAGORAS
BABYLONIAN CUNEIFORM TABLET, 1800 BCE

A	C
119	169
3,367	4,825
4,601	6,649
12,709	18,541
65	97
319	481
2,291	3,541
799	1,249
481	769

A	C
4,961	8,161
45	75
1,679	2,929
161	289
1,771	3,229
56	106

SUMS OF ODD NUMBERS – HOW I GOT STARTED

- N^2 is the sum of the first N odd numbers: $1 + 3 + 5 + \cdots + 2N - 1$.
- Example: $1 + 3 + 5 + 7 = 16 = 4^2$
- Example: $1 + 3 + 5 + 7 + 9 = 25 = 5^2$
- Wait! $3^2 = 9$ is the difference between 4^2 and 5^2. So $(3, 4, 5)$ is a triple!
- In general, if A is odd, then $A^2 = 2N - 1$ is odd. So A^2 is the difference between $(N-1)^2$ and N^2.



SUMS OF ODD NUMBERS – HOW I GOT STARTED

- For odd A, $A^2 = 2N - 1$ is odd, and it's the difference between $(N-1)^2$ and N^2.
- So $N = \frac{A^2+1}{2}$ and $N - 1 = \frac{A^2-1}{2}$ and $\left(A, \frac{A^2-1}{2}, \frac{A^2+1}{2}\right)$ is a triple.
- Example: $\left(11, \frac{121-1}{2}, \frac{121+1}{2}\right) = (11, 60, 61)$ is a triple. Note $60 + 61 = 121$.
- Example: $\left(25, \frac{625-1}{2}, \frac{625+1}{2}\right) = (25, 312, 313)$ is a triple.

SUMS OF ODD NUMBERS – EXTENDING…

- $8^2 = 64 = 31 + 33$, which are the 16th and 17th odd numbers, so 8^2 is the difference between 15^2 and 17^2. So $(8,15,17)$ is a triple.

- Fiddling and generalizing, we get the form $(\mathrm{A}, \mathrm{B}, \mathrm{C}) = \left(n, \frac{n^2-g^2}{2g}, \frac{n^2+g^2}{2g}\right)$, about which more later.

BASICS: COMMON FACTORS

- **If two numbers of a triple have a common factor, then all three share that factor.**
- If $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ is a triple and $A = da, B = db$, then $C^2 = (da)^2 + (db)^2 = d^2(a^2 + b^2)$

BASICS: EVEN AND ODD

- **Claim: A primitive triple must have exactly one even number.**
- If none of the numbers is even, then we have $odd^2 + odd^2 = odd^2$ which can't happen.
- If more than one number is even, they are all even and the triple is not primitive.

CAN A PRIMITIVE HYPOTENUSE BE EVEN?

- **Claim: The hypotenuse (largest) number of a primitive triple must be odd.**
- Only one number of a primitive triple can be even.
- If C is even and A and B are odd, then C^2 must be divisible by 4.
- Write odd $A = 2a + 1$ and $B = 2b + 1$.
- But then $C^2 = A^2 + B^2 = (4a^2 + 4a + 1) + (4b^2 + 4b + 1)$, which is not divisible by 4.

EUCLID'S FORMULA – "GENERATING PAIRS"

- **For any two positive integers $\boldsymbol{m > n}$, $(\mathbf{A}, \mathbf{B}, \mathbf{C}) = (\boldsymbol{m^2 - n^2},\ \ \boldsymbol{2mn},\ \ \boldsymbol{m^2 + n^2})$ is a triple.**
- Simple verification:

$$\mathrm{A}^2 = (m^2 - n^2)^2 = m^4 - 2m^2n^2 + n^4$$

$$B^2 = (2mn)^2 = 4m^2n^2$$

$$A^2 + B^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 + 2m^2n^2 + n^4 = (m^2 + n^2)^2 = C^2$$

EUCLID'S FORMULA – "GENERATING PAIRS"

- Example: $a = 2, b = 1$, generates $(2^2 - 1^2, 2 \cdot 2 \cdot 1, 2^2 + 1^2) = (3, 4, 5)$
- Example: $a = 4, b = 1$, generates $(15, 8, 17)$.
- Example $a = 5, b = 4$, generates $(9, 40, 41)$.

EUCLID'S FORMULA $(m^2 - n^2, 2mn, m^2 + n^2)$

- Advantages
  - Easy to generate triples.
  - Handy for many proofs regarding triples.

- Disadvantages
  - Can't generate all triples. Example: (9, 12, 15) has no generating pair.
  - Hard to generate triples with a given odd leg. (Possible with adjusted form.)

WHEN IS $(m^2 - n^2, 2mn, m^2 + n^2)$ PRIMITIVE?

- **Claim: The triple is primitive as long as a and b are coprime and one of them is even.**
- Suppose $(A, B, C) = (m - n^2, 2mn, m^2 + n^2)$ with m, n coprime and one even. Can A, B, and C have a common factor $d > 1$?
- Since B is even, A and C must be odd, $d \neq 2$.
- If d divides A and C, d must divide $A + C = 2m^2$ and $C - A = 2n^2$. Since $d \neq 2$, it must divide m^2 and n^2. But m and n are coprime, so this will not happen.

ALTERNATE GENERATING PAIRS

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- **For odd m and n with $m > n$,** $(A, B, C) = \left(mn, \frac{m^2-n^2}{2}, \frac{m^2+n^2}{2}\right)$ **is a triple.**
- The triple is primitive if m and n are coprime.

HOW MANY PRIMITIVE TRIPLES ARE THERE?

- Using $m = 1$ and n any even number will generate a unique primitive triple
$$(\boldsymbol{m^2 - n^2, 2mn, m^2 + n^2})$$
- **So there are infinitely many primitive triples.**

(n, g) GENERATION $\quad (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \left(n, \frac{n^2-g^2}{2g}, \frac{n^2+g^2}{2g}\right)$

- Note: $n = A$ and $g = C - B$.
- (A, B, C) will be a triple as long as all three are integers, which happens when…
  - n and g have the same parity (both even or both odd).
  - If g is odd, then g must divide n^2.
  - If g is even, then $2g$ must divide n^2.

(n, g) GENERATION $\quad (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \left(n, \frac{n^2-g^2}{2g}, \frac{n^2+g^2}{2g}\right)$

- **Claim: This form generates all triples.**
- Let (A, B, C) be a triple. Let $n = A$ and $g = C - B \Leftrightarrow C = B + g$.
- Since $A^2 + B^2 = C^2$, $\;n^2 + B^2 = (B^2 + 2Bg + g^2)$

$$n^2 = 2Bg + g^2$$

$$B = \frac{n^2 - g^2}{2g}$$

$$C = B + g = \frac{n^2 + g^2}{2g}$$

(n, g) GENERATION $\quad (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \left(n, \frac{n^2-g^2}{2g}, \frac{n^2+g^2}{2g}\right)$

- Advantages:
  - Generates **all** triples.
  - Generates all triples with a given leg easily.
- Disadvantages:
  - Test for primitive is a little ugly.

(n, g) GENERATION $\quad (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \left(n, \frac{n^2 - g^2}{2g}, \frac{n^2 + g^2}{2g}\right)$

- **When does (n, g) generate a primitive triple? (Don't ask!)**
- Criteria for a triple must be met
  - n and g have the same parity. If g is odd, g divides n^2. If g is even, $2g$ divides n^2.
- In addition:
  - If g is even, then the exponent of 2 in g must be one less than twice the exponent of 2 in n.
  - For any odd prime p in g, the exponent of p in g must be exactly twice the exponent of p in n.

CAN EUCLID GENERATE ALL PRIMITIVE TRIPLES?

- **Claim: Euclid's form $(\boldsymbol{a^2-b^2, 2ab, a^2+b^2})$ can generate all primitive triples.**
- Let (A, B, C) be a primitive triple with A the even leg. Let $n = A$ and $g = C - B$.
- Write $(A, B, C) = \left(n, \frac{n^2-g^2}{2g}, \frac{n^2+g^2}{2g}\right)$
- C and B are odd, so $g = 2\mathrm{d}$ is even. So $2g = 4d$ divides $n^2 = 4dk$.
- Then $B = \frac{4dk-4d^2}{4d} = k - d$. Prime p can't divide k and d or p divides A and B.
- But $n^2 = 4dk$, so $d = b^2$ and $k = a^2$ are perfect squares. Voila! $(2ab, a^2 - b^2, a^2 + b^2)$

SQUARE DIFFERENCES

- **Claim: In any primitive triple, one leg must differ from the hypotenuse by a perfect square, and the other from the hypotenuse by twice a perfect square.**

- Let $(A, B, C) = (a^2 - b^2, 2ab, a^2 + b^2)$ be a primitive triple.
- Then $C - A = 2b^2$ and $C - B = a^2 - 2ab + b^2 = (a - b)^2$.
- Example: $(20,21,29)$ is a primitive triple. $29 - 20 = 9 = 3^2$ and $29 - 21 = 8 = 2 \cdot 2^2$

BASICS: MODULO ARITHMETIC

- For our purposes, $a \cong b\ mod\ c$ means that a and b have the same remainder when divided by c.
- Properties:
  - $a\ mod\ c + \mathrm{b\ mod\ c} \cong (a + b)\ mod\ c.$ (Addition)
  - $(a\ mod\ c)(b\ mod\ c) \cong (ab)\ mod\ c$ (Multiplication)
- Example: $23 \cong 2\ mod\ 7$ and $45 \cong 3\ mod\ 7$, so $68 \cong 5\ mod\ 7.$
- Example: $17 \cong 3\ mod\ 7$ and $25 \cong 4\ mod\ 7$, so $425 \cong 12 \cong 5\ mod\ 7.$

DIVISIBILITY BY 3

- **Claim: In a primitive triple (A, B, C), exactly one number is divisible by 3. This will not be the hypotenuse.**
- Note: If $n \cong 1 \; mod \; 3$ or $n \cong 2 \; mod \; 3$, in either case $n^2 \cong 1 \; mod \; 3$.

			Remarks
			Not primitive
			Not possible

DIVISIBILITY BY 4

- **Claim: In a primitive triple, exactly one number is divisible by 4. This will not be the hypotenuse.**
- This one's easy: If $(A, B, C) = (a^2 - b^2, 2ab, a^2 + b^2)$ is primitive, either a or b must be even, and so $2ab$ is divisible by 4.

DIVISIBILITY BY 5

- Claim: In a primitive triple, exactly one number will be divisible by 5. This may be the hypotenuse.
- First, consider $n^2 \ mod \ 5$. Note that 2 and 3 can't occur.

DIVISIBILITY BY 5

- Claim: In a primitive triple, exactly one number will be divisible by 5. This may be the hypotenuse.

			Remarks
			Not primitive
			Not possible
			Not possible

DIVISIBILITY BY 3, 4, 5

- Since all non-primitive triples are multiples of primitive triples, all triples must have **at least** one number each divisible by 3, 4, and 5.
- Can one of the legs be divisible by more than one of 3, 4, and 5? Yes!
  - $(5, 12\ 13)$ and $12 = 3 \times 4$
  - $(8, 15, 17)$ and $15 = 3 \times 5$
  - $(20, 21, 29)$ and $20 = 4 \times 5$
  - $(11, 60, 61)$ and $60 = 3 \times 4 \times 5$

Solving Plimpton 322 $(a^2 - b^2, 2ab, a^2 + b^2)$

- Since $C - A = 2b^2, b = \sqrt{\frac{C-A}{2}}$. And $a^2 = A + b^2$, so $a = \sqrt{A + b^2}$
- Given $\mathrm{A} = a^2 - b^2 = 119$ and $\mathrm{C} = a^2 + b^2 = 169$, find $2ab$.
- $C - A = 2b^2 = 50$, so $b^2 = 25$ and $b = 5$.
- $a^2 = A + b^2 = 119 + 25 = 144$, so $a = 12$. So $B = 2ab = 120$.

SOLVING PLIMPTON 322

$$(a^2 - b^2, 2ab, a^2 + b^2)$$

- Since $C - A = 2b^2, b = \sqrt{\frac{C-A}{2}}$. And $a^2 = A + b^2$, so $a = \sqrt{A + b^2}$
- How about $A = 4{,}601$ and $C = 6{,}649$?
- Remember how to do square roots by hand????
- $2b^2 = 6{,}649 - 4{,}601 = 2{,}048 \Rightarrow b^2 = 1{,}024 \Rightarrow b = 32$
- $a^2 = 4{,}601 + 1{,}024 = 5{,}625 \Rightarrow a = 75$, so $B = 2ab = 4800$.

TREES OF PRIMITIVE TRIPLES (F.J.M. BARNING)

-
- If $\begin{bmatrix} A \\ B \\ C \end{bmatrix}$ is a primitive triple then so are

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}, \text{ and } \begin{bmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

- In other words,

$$(A-2B+2C, 2A-B+2C, 2A-2B+3C)$$
$$(A+2B+2C, 2A+B+2C, 2A+2B+3C)$$
$$(-A+2B+2C, -2A+B+2C, -2A+2B+3C)$$

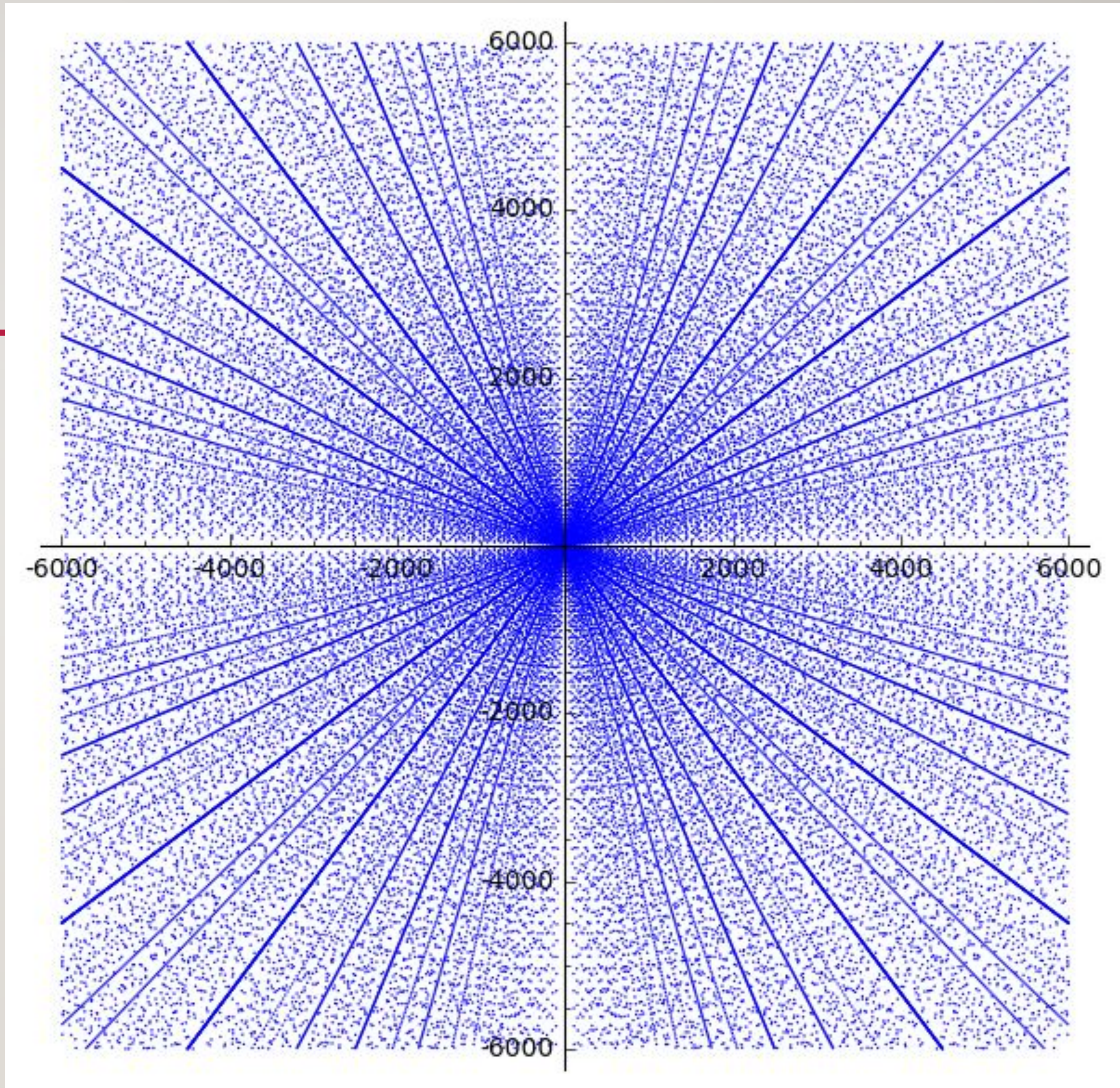
TREES OF PRIMITIVE TRIPLES (F.J.M. BARNING)

- Generates all and only primitive triples, forming a ternary tree.
- (3, 4, 5)
  - (5, 12, 13)
    - (7, 24, 25)
    - (55, 48, 73)
    - (45, 28, 53)

- (3, 4, 5)
  - (21, 20, 29)
    - (39, 80, 89)
    - (119, 120, 169)
    - (77, 36, 85)
  - (15, 8, 17)
    - (33, 56, 65)
    - (65, 72, 97)
    - (35, 12, 37)

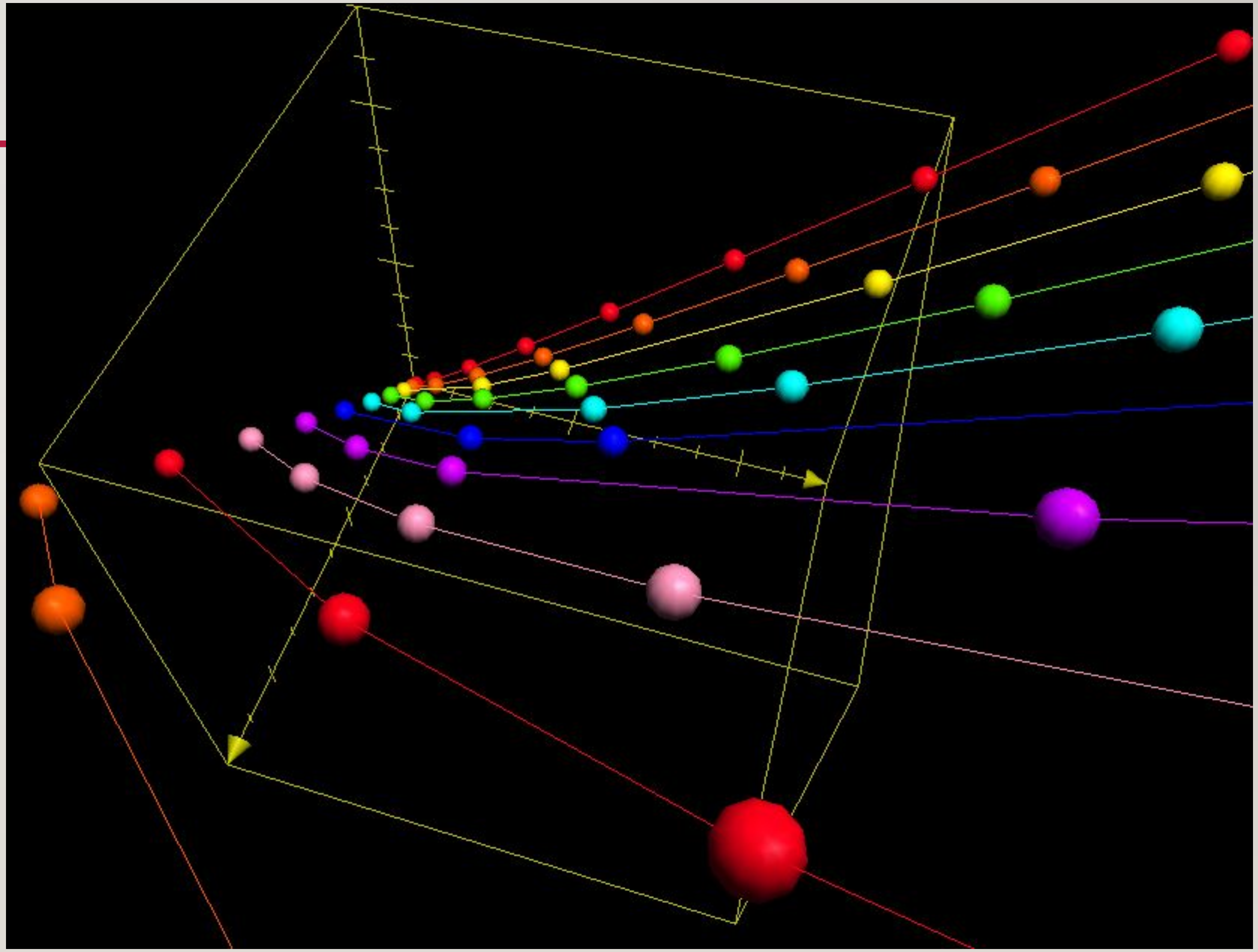
PRETTY PICTURES

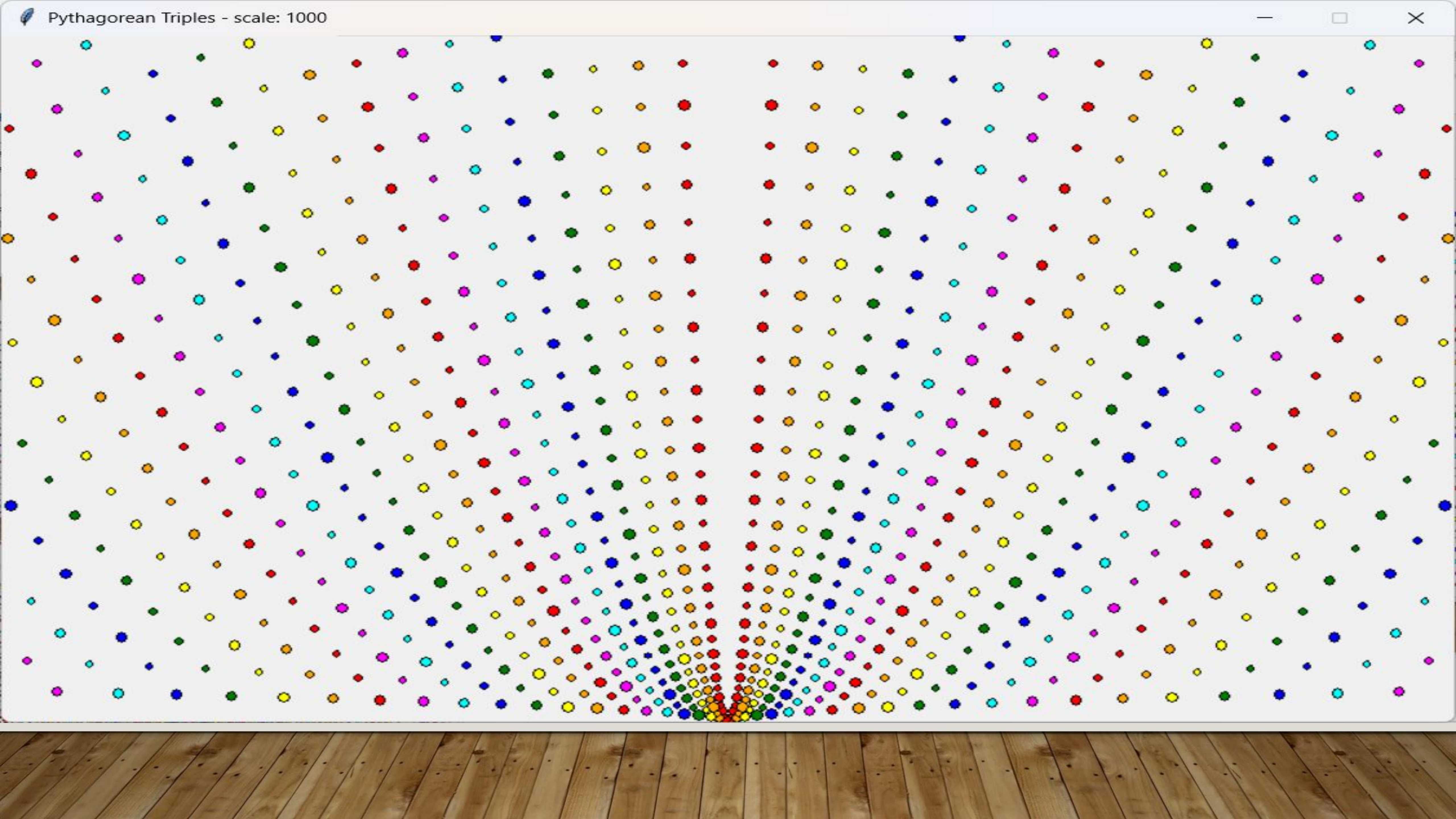
- From Wikipedia,…
- Legs of triples up to 6,000.



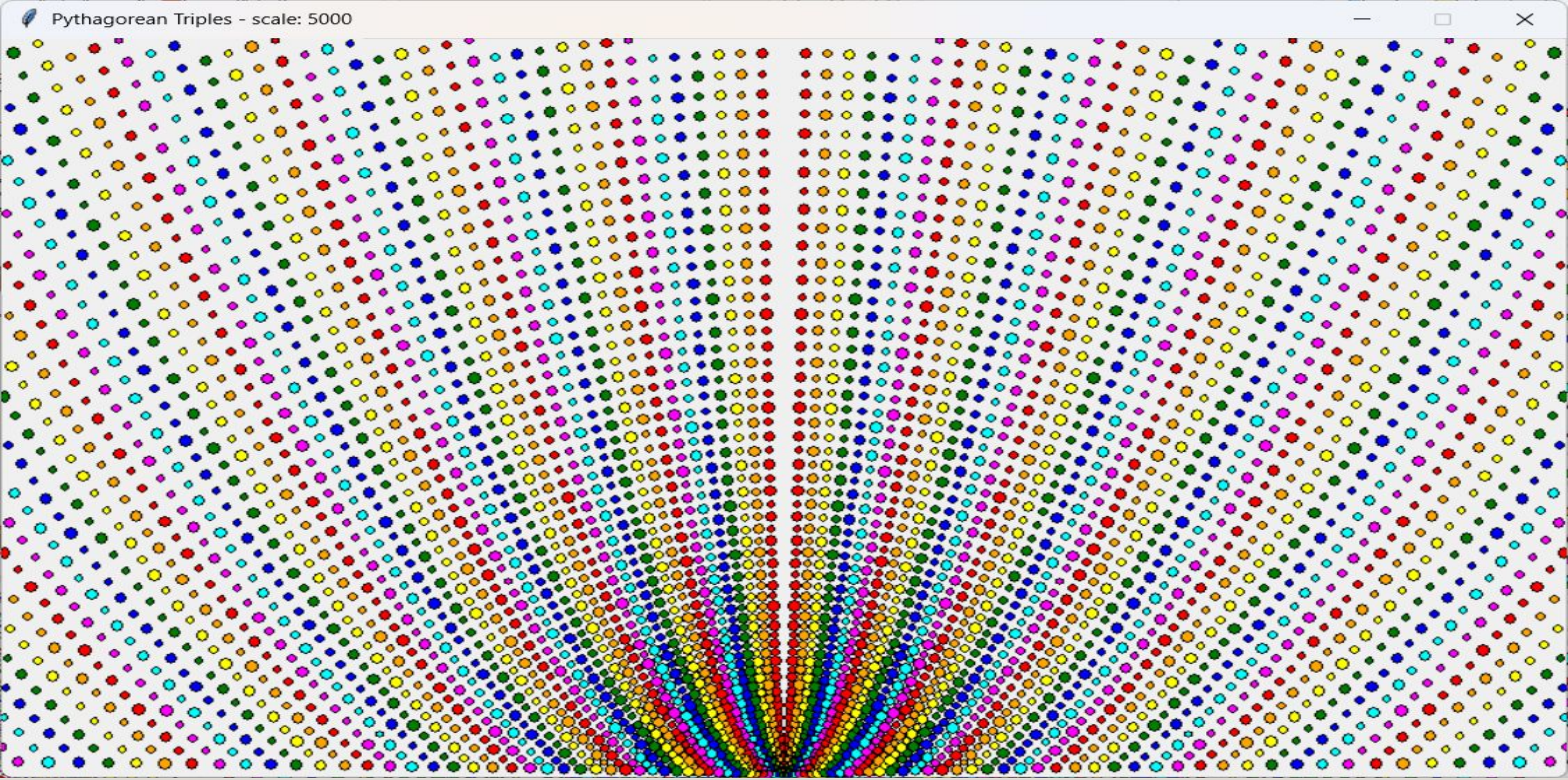
PRETTY PICTURES

- From Wikipedia,…
- Plot of Euclid triples on the cone
$$z^2 = x^2 + y^2$$
- "A constant m or n traces out part of a parabola on the cone."
- (cf. Geogebra version)





Pythagorean Triples - scale: 1000


Pythagorean Triples - scale: 5000


Pythagorean Triples - scale: 5000 symmetric