TMA4220 Numerical Solution of Differential Equations by element methods

Part 1

Candidate numbers

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1 Gaussian quadrature

Evaluation of definite integrals is an integral part of every finite element code. The integrals vary in complexity and many does not even have known analytical solutions. One general way to evaluate such integrals is by the *Gaussian quadrature* approximation:

$$\int_{\hat{\Omega}} g(\zeta)\,\mathrm{d}\zeta \approx \sum_{q=1}^{N_q} \rho_q g(\zeta_q)$$

The function $g(\zeta)$ is evaluated in the N_q vector quadrature points, ζ_q, and ρ_q are the associated Gaussian weights. In all numerical quadratures, the points and associated weights are given on a reference domain, $\hat{\Omega}$. It is therefore important to have a map from this domain to the domain at hand.

1.1 1D quadrature

In 1D $\hat{\Omega} = [-1, 1]$. So for $\zeta \in [-1, 1]$ the mapping

$$x = \frac{1}{2}\left((b-a)\zeta + (b+a)\right)$$

takes any point onto a general domain [a,b] and gives the approximation

$$\int_{\Omega} g(x)\,\mathrm{d}x = \int_{\hat{\Omega}} g\left(x(\zeta)\right)\,\frac{\mathrm{d}z}{\mathrm{d}\zeta}\mathrm{d}x \approx \frac{1}{2}(b-a)\sum_{q=1}^{N_q} \rho_q g(x(\zeta_q)).$$

1.2 2D QUADRATURE

In higher dimensions a bit more care is needed. Consider triangles on the (x,y)-plane with corner points $\mathbf{p_i} = (x_i, y_i), i = 1, 2, 3$. Then define the area coordinates

$$\zeta_i = \frac{A_i}{A},\ i = 1, 2, 3,$$

as indicated in figure [FIGURE OF AREA COORDINATES]. Then area coordinates $\zeta \in [0, 1]$ map to physical coordinates $\mathbf{x}$ via

$$\mathbf{x} = \zeta_1\mathbf{p_1} + \zeta_2\mathbf{p_2} + \zeta_3\mathbf{p_3}.$$

Let the reference element be the triangle with corners in (0,0), (0,1) and (1,0) on the (ξ, η)-plane. Then the lowest order mapping corresponding in each corner is

$$\mathbf{x} = \mathbf{p_1}(1 - \xi - \eta) + \mathbf{p_2}\xi + \mathbf{p_3}\eta = B_k \begin{bmatrix} \xi \\ \eta \end{bmatrix} + \mathbf{p_1},$$

where B_k is the Jacobian matrix of the transformation. The Jacobian determinant is then

$$|J(\xi, \eta)| = \det(B_k) = \begin{vmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{vmatrix}. \tag{1}$$

Gaussian quadrature points, ζ_q, and weights, ρ_q, in area coordinates are given as triplets. The quadrature rules also introduce a scaling with the element size (here area), T_Ω. The approximation of the integral is thus

$$\int_\Omega g(\mathbf{x})\,\mathrm{d}x\mathrm{d}y = \int_{\hat{\Omega}} g(\mathbf{x})\,|J(\xi, \eta)|\mathrm{d}\xi\mathrm{d}\eta \approx T_{\hat{\Omega}}|J(\xi, \eta)| \sum_{q=1}^{N_q} \rho_q g(\mathbf{x}(\zeta_q)).$$

Note that taking g to be constant over the element gives $T_\Omega = T_{\hat{\Omega}}|J(\xi, \eta)|$. For our reference element the area is $T_{\hat{\Omega}} = 1/2$.

1.3 3D GUADRATURE

The extention into 3 dimensions and for tetrahedral elements is straight forward. Let

$$\mathbf{x} = \zeta_1\mathbf{p_1} + \zeta_2\mathbf{p_2} + \zeta_3\mathbf{p_3} + \zeta_4\mathbf{p_4}$$

for $\mathbf{p_i} = (x_i, y_i, z_i)$ and the reference tetrahedral defined by the corner points $(0, 0, 0)$,$(0, 1, 0)$,$(0, 0, 1)$ and $(1, 0, 0)$ in the (ξ, η, ψ)-coordinate system. Then the mapping is

$$\mathbf{x} = B_k \begin{bmatrix} \xi \\ \eta \\ \psi \end{bmatrix} + \mathbf{p_1}.$$

where the columns of B_k are given by $\mathbf{p_i} - \mathbf{p_1}$ for $i \neq 1$ as before. The integral over the element can be approximated by

$$\int_\Omega g(\mathbf{x})\,\mathrm{d}x\mathrm{d}y\mathrm{d}z \approx T_{\hat{\Omega}}|J(\xi,\eta,\psi)|\sum_{q=1}^{N_q}\rho_q g(\mathbf{x}(\zeta_q)).$$

$T_{\hat{\Omega}} = 1/6$ is the volume of the reference element.

2 Poisson equation in 2 dimensions

Consider the two-dimensional Poisson problem

$$\begin{aligned}\nabla^2 u(x,y) &= -f(x,y)\\ u(x,y)|_{r=1} &= 0,\end{aligned} \tag{2}$$

on the domain $\Omega = \{(x,y) : x^2 + y^2 \leq 1\}$, and with f given in polar coordinates as

$$f(r,\theta) = -8\pi\cos(2\pi r^2) + 16\pi^2 r^2 \sin(2\pi^2 r^2).$$

2.1 Analytical solution

An analytical solution to problem (2) is

$$u(x,y) = \sin\left(2\pi(x^2+y^2)\right) \tag{3}$$

since

$$\nabla^2 u(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u(x,y) = \frac{\partial}{\partial x}4\pi x\cos\left(2\pi(x^2+y^2)\right) + \frac{\partial}{\partial y}4\pi y\cos\left(2\pi(x^2+y^2)\right)$$

$$= 8\pi\cos\left(2\pi(x^2+y^2)\right) - 16\pi(x^2+y^2)\sin\left(2\pi(x^2+y^2)\right) = -f(x,y)$$

2.2 Weak formulation

To reach the weak formulation of the problem multiply with a test function v and integrate over the domain. Using Green's theorem one get

$$\int_\Omega (\nabla^2 u)v = \int_{\partial\Omega}(\partial_n u)v - \int_\Omega \nabla u\cdot\nabla v = \int_\Omega fv.$$

The weak formulation of problem (2) is then: Find $u \in H^1_{\partial\Omega}(\Omega)$ such that

$$a(u,v) = l(v), \forall v \in H^1_{\partial\Omega}$$

where

$$\begin{aligned} H^1(\Omega) &= \{u \in L^2(\Omega) : u_x, u_{xx} \in L^2(\Omega)\} \\ H^1_{\partial\Omega}(\Omega) &= \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\} \\ a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x\mathrm{d}y \\ l(v) &= \int_\Omega fv \, \mathrm{d}x\mathrm{d}y. \end{aligned} \tag{4}$$

Now $a(u,v)$ is a bilinear functional since $a(\lambda u, v_1 + v_2) = \lambda\left(a(u,v_1) + a(u,v_2)\right)$ and

$$a(w,w) = \int_\Omega (\nabla w)^2 \geq 0$$

with equalilty only for $w \equiv 0$. $\nabla w = 0$ gives constant w, but since $w = 0$ on the boundary w must be zero everywhere. In the following let $X = H^1_{\partial\Omega}(\Omega)$.

2.3 Galerkin projection

Instead of searching for a solution in the entire space X consider a smaller space $X_h \subset X$. Let Ω be discretized into M triangles such that $\Omega = \cup_{k=1}^{M} K_k$. Each triangle K_k is defined by its three corner nodes $\mathbf{p_i}$ as in section 1.2. Each node has a corresponding basis function ϕ_i. The space X_h is then defined by

$$X_h = \{v \in X : v|_{K_k} \in \mathbb{P}_1(K_k), 1 \leq k \leq M\}$$

for which the basis functions $\{\phi_i\}_{i=1}^n$ satisfy

$$X_h = \operatorname{span}\{\phi_i\}_{i=1}^n \qquad \phi_j(\mathbf{p_i}) = \delta_{ij}$$

and δ_{ij} is the Kronecker delta. For solutions $u_h \in X_h$ it is then possible to write $u_h = \sum_{i=1}^n u_h^i \phi_i(x,y)$. This reduces the problem to finding $u_h \in X_h$ such that $a(u_h, v) = l(v) \forall v \in X_h$. Since v also can be expressed by the basis functions

$$\begin{gathered} a(u_h, v) = l(v) \\ \int_\Omega \sum_{i=1}^n u_h^i \nabla\phi_i \sum_{j=1}^n v^j \nabla\phi_j = \int_\Omega f \sum_{j=1}^n v^j \phi_j. \end{gathered}$$

The coefficients in v cancel and the weak formulation for the Poisson problem on the triangulation is equivalent to finding $\mathbf{u}$ such that

$$\mathbf{Au} = \mathbf{f}$$

with

$$\begin{aligned} \mathbf{A} &= [A_{ij}] = [a(\phi_i, \phi_j)] \\ \mathbf{u} &= [u_h^i] \\ \mathbf{f} &= [f_i] = [l(\phi_i)]. \end{aligned}$$

Taking the definitions of $a(\cdot,\cdot)$ and $l(\cdot)$ in (4).

2.4 Impementation

3 plots. Different grids. N=10,100,1000?

2.5 Stiffness matrix

As shown earlier the elements of the stiffness matrix is given by

$$A_{ij} = a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \nabla\phi_j \mathrm{d}x\mathrm{d}y.$$

Consider one element of the triangulation K_k with corner points $\mathbf{p}_\alpha = (x_\alpha, y_\alpha)$, $\alpha = 1, 2, 3$. Since the basis functions are linear they are uniquely determined by three coefficients, i.e

$$\phi_\alpha(x, y) = a_\alpha + b_\alpha x + c_\alpha y.$$

The Kronecker delta-property of the basis functions then gives that (a_1, b_1, c_1) is governed by the equations $\phi_1(x_1, y_1) = 1$, $\phi_1(x_2, y_2) = 0$ and $\phi_1(x_3, y_3) = 0$, or written in matrix form

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \tag{5}$$

Similarly one can find (a_2, b_2, c_2) and (a_3, b_3, c_3). Note that $\nabla\phi_\alpha = (b_\alpha, c_\alpha)$, and hence the gradients are independent of x and y and can be moved outside the integral. The elements of the local stiffness matrix is thus given by

$$A^K_{\alpha\beta} = \nabla\phi_\alpha \nabla\phi_\beta \int_K \mathrm{d}x\mathrm{d}y = \nabla\phi_\alpha \nabla\phi_\beta T_K.$$

This reduces the problem to solving (5) for the gradients and using equation (1) from section 1.2 to find T_K.

To build the whole matrix A loop over each element and pick out the corner points. Then for each combination of basis functions (9 per element) add the scaled contribution to the full matrix.

2.6 Loading vector

2.7 Boundary conditions

2.8 Verification

Compare analytical and FE solution

3 Neumann boundary conditions

$$\begin{aligned} \nabla^2 u(x,y) &= -f(x,y) \\ u(x,y)|_{r=1} &= 0, \\ \left.\frac{\partial u(x,y)}{\partial n}\right|_{\partial\Omega_N} &= g(x,y) \end{aligned} \tag{6}$$

$$g(r,\theta) = 4\pi r \cos(2\pi r^2). \tag{7}$$

$\partial\Omega_D = \{x^2 + y^2 = 1, y < 0\}$ and $\partial\Omega_N = \{x^2 + y^2 = 1, y > 0\}$

3.1 Boundary condition

3.2 Variational formulation

3.3 Gauss quadrature

3.4 Implementation

4 Problems in 3 dimensions

4.1 The Poisson problem in 3D

4.2 Volume visualization

Plot the ball, triscatteredInterp, isosurface

4.3 Neumann boundary condition