TMA4220 Numerical Solution of Differential Equations by element methods

Part 2: Vibration analysis

Candidate numbers

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Abstract

Introduction

With the displacement of spatial points in x_1- and x_2-direction represented as

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

the strain on each point is

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \end{bmatrix}$$

as many notes on linear elasticity will let you know.[1]

The connection between the strain ϵ and the stress σ is

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{12} \end{bmatrix}$$

$$\boldsymbol{\sigma} = C\boldsymbol{\epsilon}$$

where E is the Young's modulus and ν is the Poisson's ratio. Young's modulus characterises the solid's stiffness, i.e. large E means that you need a large force to deform

[1] Note however that this is the form usually called engineer strain.

the solid. Poisson's ratio is the ratio between the strains in x_1- and x_2-direction when submitting the solid to a stress in only one of the directions. To clarify: Consider a stress in only the x_1-direction direction[2], then $\nu > 0$ will say that the solid contracts in the x_2-direction and elongates in the x_1-direction. Worth noting is that $\nu < 0$ is possible, and some materials actually have this property. Weird. During the course of this analysis we will let E and ν completely describe a solid's properties. No stress or strain due to difference in temperature.

The equation, or how we ended up with an element method

In continuum mechanics

$$\rho \ddot{\boldsymbol{u}} = \nabla \sigma(\boldsymbol{u}) \tag{1}$$

describes the spatial displacement due to internal stresses alone. It is the extension of Hook's law and Newton's second law into a continuous medium. So rather than total mass we look at mass density, here denoted ρ.

To eventually arrive at variational formulation of 1 we should spend some time with the right hand side of this equation to better understand it. For that reason we will take a slight detour and start our journey by considering the equation

$$\nabla \sigma(\boldsymbol{u}) = -\boldsymbol{f} \tag{2}$$

$$\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right] \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} = -[f_1, f_2]$$

working on some domain Ω, together with some boundary conditions, which in its general form looks a little something like

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{g}, \quad \text{on } \partial\Omega_D \\ \sigma(\boldsymbol{u}) \cdot \hat{\boldsymbol{n}} &= \boldsymbol{h}, \quad \text{on } \partial\Omega_N \end{aligned}$$

As is usual procedure when trying to derive a variational formulation, we multiply 2 with some test function $\boldsymbol{v} \in V$, where the function space V will be determined from what seems useful during the derivation. Then we integrate over the domain to get

$$\int_\Omega (\nabla \sigma(\boldsymbol{u})) \cdot \boldsymbol{v} d\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} d\Omega.$$

We aim at applying Green's formula to the left hand side, and since Green's formula states that

$$\int_\Omega \nabla \cdot \boldsymbol{a} d\Omega = \int_{\partial\Omega} \boldsymbol{a} \cdot \hat{\boldsymbol{n}} dS$$

we should come up with a good candidate for $\boldsymbol{a}$. A good guess seems to be $\boldsymbol{a} = \sigma(\boldsymbol{u})\boldsymbol{v}$.

[2] σ_{11} being the only stress different from zero.