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Entropy Solutions for Nonlinear Degenerate Problems

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This work is dedicated to the memory of S. N. Kruzhkov

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Abstract

We consider a class of elliptic-hyperbolic degenerate equations

$$g(u) - \Delta b(u) + \operatorname{div} \phi(u) = f$$

with Dirichlet homogeneous boundary conditions and a class of elliptic-parabolic-hyperbolic degenerate equations

$$g(u)_t - \Delta b(u) + \operatorname{div} \phi(u) = f$$

with homogeneous Dirichlet conditions and initial conditions. Existence of entropy solutions for both problems is proved for nondecreasing continuous functions g and b vanishing at zero and for a continuous vectorial function ϕ satisfying rather general conditions. Comparison and uniqueness of entropy solutions are proved for g and b continuous and nondecreasing and for ϕ continuous.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$. When $N \geqq 2$ we assume that Ω has a Lipschitz boundary Γ. We consider the initial-boundary-value problem

$$(P_E(f, g_0)(g, b, \phi)) \begin{cases} \dfrac{\partial g(u)}{\partial t} - \Delta b(u) + \operatorname{div}(\phi(u)) = f & \text{in } (0, T) \times \Omega, \\ g(u) = g_0 & \text{on } \{0\} \times \Omega, \\ b(u) = 0 & \text{on } (0, T) \times \Gamma, \end{cases}$$

and the boundary value problem

$$(P_S(f)(g, b, \phi)) \begin{cases} g(u) - \Delta b(u) + \operatorname{div}(\phi(u)) = f & \text{in } \Omega, \\ b(u) = 0 & \text{on } \Gamma, \end{cases}$$

where

$$g,\ b\ : \mathbb{R} \mapsto \mathbb{R} \text{ are continuous and nondecreasing } g(0) = b(0) = 0 \tag{H1}$$

$$\phi \in \mathscr{C}(\mathbb{R}; \mathbb{R}^N), \phi_j(0) = 0 \quad \forall\, 1 \leqq j \leqq N. \tag{H2}$$

In order to prove the existence of "good" solutions we need more assumptions related to g, b and ϕ:

$$D((g+b)^{-1}) = \mathbb{R} \tag{H3}$$

and either

$$\begin{gathered} \exists\, \phi^{(1)},\ \phi^{(2)} \in \mathscr{C}(\mathbb{R}^2; \mathbb{R}^N),\ \phi_j^{(1)}(0,0) = 0, \quad 1 \leqq j \leqq N, \\ \text{such that } \phi(s) = \phi^{(1)}(g(s), b(s)) + \phi^{(2)}(g(s), b(s))s \quad \forall\, s \in \mathbb{R}, \end{gathered} \tag{H4}$$

or

$$\begin{gathered} \exists\, \phi^{(1)},\ \phi^{(2)} \in \mathscr{C}(\mathbb{R}; \mathbb{R}^N),\ \phi_j^{(1)}(0) = 0, \quad 1 \leqq j \leqq N, \\ \text{such that } \phi(s) = \phi^{(1)}(g(s)) + \phi^{(2)}(g(s))b(s) \quad \forall\, s \in \mathbb{R}. \end{gathered} \tag{H5}$$

Remark 1. We easily check that

$$\text{(H5)} \Longrightarrow \text{(H4)} \Longrightarrow \text{(H2)}.$$

Moreover, if

$$(g+b)^{-1} \in \mathscr{C}(\mathbb{R}), \tag{H6}$$

then (H3) is fulfilled and we have

$$\text{(H2)} \Longrightarrow \text{(H4)};$$

and if

$$g^{-1} \in \mathscr{C}(\mathbb{R}) \tag{H7}$$

then (H3) and (H6) are fulfilled and we have

$$\text{(H2)} \Longrightarrow \text{(H4)} \Longrightarrow \text{(H5)};$$

The above formulations include a large class of equations. For instance

$$v - \Delta w + \operatorname{div} \psi(v, w) = f, \quad w \in \beta(v)$$

or

$$\frac{\partial v}{\partial t} - \Delta w + \operatorname{div} \psi(v, w) = f, \quad w \in \beta(v)$$

where β is a maximal monotone operator. We set $v + w = u$, $(I + \beta)^{-1} = g$ and $(I + \beta^{-1})^{-1} = b$, then $v = g(u)$, $w = b(u)$, $\phi(u) = \psi(g(u), b(u))$. Moreover

$$v - \Delta\beta(v) + e \cdot \nabla\chi = f, \quad \chi \in \gamma(v)$$

or

$$\frac{\partial v}{\partial t} - \Delta\beta(v) + e \cdot \nabla\chi = f, \quad \chi \in \gamma(v)$$

where β is a continuous and nondecreasing function, γ is a maximal monotone operator and $e \in \mathbb{R}^N$. Then we set $u = v + \chi$, $g = (I + \gamma)^{-1}$, $b = \beta \circ g$ and $\phi = e(I + \gamma^{-1})^{-1}$.

We get an interesting particular case of the above formulation when b degenerates ($b \equiv 0$). Then first-order equations arise in the above formulations:

$$\frac{\partial g(u)}{\partial t} + \operatorname{div}(\phi(u)) = f \quad \text{in } (0, T) \times \Omega,$$

$$g(u) + \operatorname{div}(\phi(u)) = f \quad \text{in } \Omega.$$

The results concerning this equation are developed in [Ca3].

It is well known that, for such equations, the above problems are ill-posed in the sense that there is no uniqueness. It is necessary to introduce Kruzhkov solutions in order to have a good theory of existence and uniqueness (see [Kr1, Kr2]). So in the general case we do not expect to have well-posed problems.

Since g and b are not strictly increasing, the above formulations include Stefan problems, filtration problems, etc. Such formulations involve a large class of problems and an important literature has been developed. Degenerate problems have been solved, when $N = 1$, by BÉNILAN & TOURÉ [BT] (see also [VH, BB, Wu]). The case $b \equiv 0$ was solved in $\mathbb{R}^N$ by KRUZHKOV [Kr1, Kr2] (see also [BK, Ol]); in a bounded domain it was studied by HIL'DEBRAND [Hi] and by BARDOS, LEROUX & NEDELEC [BLN] for smooth data. Related problems are studied in [BG, BW1, BW2, BP, Ca1, Ca2, CW, DK, GM, GP, La, Ma, Ot, WY, Yi]. In [AL] ALT & LUCKHAUS have studied systems of degenerate equations. See also [Be, Cr, CL] and the corresponding references for the semigroup approach.

As usual, in order to study existence and uniqueness of solutions for the time-dependent problem we study the steady-state problem. In Section 2 we define weak and entropy solutions and we study the existence of entropy solutions. In Section 3 we study comparison and uniqueness of entropy solutions.

In Section 4.1 we study the T-accretivity of the stationary operator and the existence and uniqueness of integral solutions of the evolution problem. In Section 4.2 we deal with the existence of entropy solutions for the time dependent problem and Section 5 is dedicated to the comparison and uniqueness of entropy solutions.

2. The Stationary Problem: Existence of Solutions

2.1. Notations and Definitions

Let γ be a maximal monotone operator; we denote by γ_ε the Yosida regularization of γ:

$$\gamma_\varepsilon(s) = \frac{s - (I + \varepsilon\gamma)^{-1}(s)}{\varepsilon};$$

we denote by γ_0 the main section of γ:

$$\gamma_0(s) = \begin{cases} \text{the element of minimal absolute value of } \gamma(s) \text{ if } \gamma(s) \neq \emptyset, \\ +\infty \quad \text{if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ -\infty \quad \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases}$$

Moreover, let $\psi \in \mathscr{C}(\mathbb{R})$ with ψ nondecreasing. Then we define $\psi \circ \gamma$ by

$$P_{(\psi\circ\gamma)}(s) = \begin{cases} \psi(\gamma(s)) & \forall s \in D(\gamma), \\ \sup_{r\in\mathbb{R}} \psi(r) & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ \inf_{r\in\mathbb{R}} \psi(r) & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset, \end{cases}$$

$$D(\psi \circ \gamma) = \{s \in \mathbb{R} / \inf_{r \in P_{(\psi\circ\gamma)}(s)} |r| < +\infty\},$$

$$(\psi \circ \gamma)(s) = P_{(\psi\circ\gamma)}(s) \quad \forall s \in D(\psi \circ \gamma), \tag{1}$$

and we define $\psi \circ \gamma_0$ by

$$(\psi \circ \gamma_0)(s) = \begin{cases} \psi(\gamma_0(s)) & \forall s \in D(\gamma), \\ \sup_{r\in\mathbb{R}} \psi(r) & \text{if } [s, +\infty) \cap D(\gamma) = \emptyset, \\ \inf_{r\in\mathbb{R}} \psi(r) & \text{if } (-\infty, s] \cap D(\gamma) = \emptyset. \end{cases} \tag{2}$$

In particular, let g and b satisfy (H1) and (H3); then we easily check that

$$D(b \circ g^{-1}) = D(g^{-1}),$$

$$b \circ g^{-1}(s) = b(g^{-1}(s)) \quad \forall s \in D(g^{-1}).$$

Throughout this paper we use the operator H (also denoted by sign^+ in the literature):

$$H(s) = \begin{cases} 1 & \text{if } s > 0, \\ [0, 1] & \text{if } s = 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Then

$$H_\varepsilon(s) = \min(s^+/\varepsilon, 1),$$

$$H_0(s) = \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \leqq 0. \end{cases}$$

Furthermore we use

$$H_{\text{Max}}(s) = \begin{cases} 1 & \text{if } s \geqq 0, \\ 0 & \text{if } s < 0. \end{cases}$$

Definition 1. Let $f \in L^1(\Omega)$. Then u is a *weak solution* of $(P_S(f)(g,b,\phi))$ if and only if

$$\begin{aligned} &u \text{ is measurable}, \quad g(u) \in L^1(\Omega),\\ &b(u) \in H_0^1(\Omega), \quad \phi(u) \in (L^2(\Omega))^N, \end{aligned}$$

$$g(u) - \Delta b(u) + \operatorname{div}(\phi(u)) = f \text{ in } \mathscr{D}'(\Omega)$$

or equivalently

$$\int_\Omega \{g(u)\xi + \nabla b(u) \cdot \nabla \xi - \phi(u) \cdot \nabla \xi\}\, dx = \int_\Omega f\xi\, dx$$

for any $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Remark 2. We easily check that if u is a weak solution of $(P_S(f)(g,b,\phi))$ then $-u$ is a weak solution of $(P_S(-f)(\tilde{g},\tilde{b},\tilde{\phi}))$ where $\tilde{g}(r) = -g(-r)$, $\tilde{b}(r) = -b(-r)$ and $\tilde{\phi}(r) = -\phi(-r)$.

Theorem 1. *Let $f \in L^q(\Omega)$ with*

$$q \begin{cases} \geqq 1 & \text{if } N = 1,\\ > 1 & \text{if } N = 2,\\ \geqq 2N/(N+2) & \text{if } N > 2, \end{cases} \tag{3}$$

and let u be a weak solution of $(P_S(f)(g,b,\phi))$. Let $\Omega \subset B(0,R)$ for some R large enough. Then there exists a constant $K = K(q,N,R)$ such that

$$\|\nabla b(u)\|_{(L^2(\Omega))^N} \leqq K\|f\|_{L^q(\Omega)}.$$

For any $p > N/2$ there exists a constant $C = C(N,p,R)$ such that

$$\|b(u)\|_{L^\infty(\Omega)} \leqq C\|f\|_{L^p(\Omega)}.$$

Moreover, if b is (strictly) increasing, then

$$\|g(u)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)}, \quad 1 \leqq r \leqq +\infty.$$

Proof. Let us define

$$b^{(k)}(s) = \max(\min(b(s), b(k)), b(-k));$$

then $b^{(k)}(u) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega \{g(u)b^{(k)}(u) + \nabla b(u) \cdot \nabla b^{(k)}(u) - \phi(u) \cdot \nabla b^{(k)}(u)\}\, dx = \int_\Omega f\, b^{(k)}(u)\, dx.$$

Obviously

$$\int_\Omega g(u)\, b^{(k)}(u)\, dx \geqq 0;$$

from [BM] we have

$$\int_\Omega \phi(u) \cdot \nabla b^{(k)}(u)\, dx = 0.$$

Then we get

$$\int_\Omega |\nabla b^{(k)}(u)|^2\, dx \leqq \int_\Omega f\, b^{(k)}(u)\, dx.$$

By taking into account the Poincaré inequality and Sobolev imbeddings, we get

$$\|\nabla b^{(k)}(u)\|_{(L^2(\Omega))^N} \leqq K \|f\|_{L^q(\Omega)},$$

and by letting $k \to +\infty$ we get

$$\|\nabla b(u)\|_{(L^2(\Omega))^N} \leqq K \|f\|_{L^q(\Omega)}.$$

In order to prove the second inequality we consider $f \in L^p(\Omega)$ (for $f \notin L^p(\Omega)$ the inequality is obvious) and we observe that for any bounded nondecreasing and Lipschitz continuous function G such that $G(0) = 0$ we have

$$\int_\Omega g(u) G(b(u))\, dx \geqq 0,$$

and from [BM]

$$\int_\Omega \phi(u) \cdot \nabla G(b(u))\, dx = 0;$$

hence

$$\int_\Omega \nabla b(u) \cdot \nabla G(b(u))\, dx \leqq \int_\Omega f G(b(u))\, dx.$$

Then, by arguing as in [GT], we find that there is a constant $C = C(N, p, R)$ such that

$$\|b(u)\|_{L^\infty(\Omega)} \leqq C \|f\|_{L^p(\Omega)}.$$

Finally, let $1 \leqq r < +\infty$ and let us assume that $f \in L^r(\Omega)$ (otherwise the last inequality of the theorem is obvious) and that b is (strictly) increasing. Let us denote $\gamma = g \circ b^{-1}$, let γ_ε be the Yosida regularization of γ, and let

$$G_\varepsilon(s) = \begin{cases} 1 & \text{for } s \geqq \varepsilon, \\ s/\varepsilon & \text{for } -\varepsilon \leqq s \leqq \varepsilon, \\ -1 & \text{for } s \leqq -\varepsilon; \end{cases}$$

$$B_\varepsilon(s) = \max(-1/\varepsilon, \min(\gamma_\varepsilon(s), 1/\varepsilon)).$$

Then $s \mapsto |B_\varepsilon(s)|^{r-1} G_\varepsilon(s)$ is a bounded nondecreasing Lipschitz continuous function vanishing at 0; hence

$$\int_\Omega \{g(u)|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u)) + \nabla b(u) \cdot \nabla[|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u))]$$
$$- \phi(u) \cdot \nabla[|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u))]\} \, dx = \int_\Omega f|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u)) \, dx.$$

Obviously we have

$$\int_\Omega \nabla b(u) \cdot \nabla[|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u))] \, dx \geqq 0,$$

$$\int_\Omega \phi(u) \cdot \nabla[|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u))] \, dx = 0,$$

and, since $|\gamma_\varepsilon(s)| \leqq |\gamma(s)|$ (see [Br]), we deduce that

$$\int_\Omega |B_\varepsilon(b(u))|^r |G_\varepsilon(b(u))| \, dx \leqq \int_\Omega \gamma(b(u))|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u)) \, dx$$
$$= \int_\Omega g(u)|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u)) \, dx \leqq \int_\Omega f|B_\varepsilon(b(u))|^{r-1} G_\varepsilon(b(u)) \, dx,$$

whence

$$\int_\Omega |B_\varepsilon(b(u))|^r |G_\varepsilon(b(u))| \, dx \leqq \|f\|_{L^r(\Omega)} \left(\int_\Omega |B_\varepsilon(b(u))|^r |G_\varepsilon(b(u))| \, dx \right)^{r-1}$$

and by letting $\varepsilon \to 0$ we get

$$\|g(u)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)}, \quad 1 \leqq r < +\infty.$$

Moreover, we have

$$\|g(u)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \leqq \|f\|_{L^\infty(\Omega)} |\Omega|^{1/r},$$

whence, by letting $r \to +\infty$ we get

$$\|g(u)\|_{L^\infty(\Omega)} \leqq \|f\|_{L^\infty(\Omega)}.$$

Remark 3. The constants K and C arising in Theorem 1 do not depend on ϕ, g and b.

Let us define

$$E = \{r \in R(b) / \ (b^{-1})_0 \text{ is discontinuous at } r\}. \tag{4}$$

Now we can prove

Lemma 1. *Let u be a weak solution of $(P_S(f)(g,b,\phi))$. Then*

$$\int_\Omega H_0(u-s)\{(\nabla b(u)+\phi(s)-\phi(u))\cdot\nabla\xi+(g(u)-f)\xi\}\,dx$$
$$= -\lim_{\varepsilon\to 0}\int_\Omega |\nabla b(u)|^2 H'_\varepsilon(b(u)-b(s))\xi\,dx \leqq 0 \quad (5)$$

for any $(s,\xi)\in\mathbb{R}\times(H_0^1(\Omega)\cap L^\infty(\Omega))$ such that $b(s)\in R(b)\setminus E$ and $\xi\geqq 0$ and for any $(s,\xi)\in\mathbb{R}\times(H^1(\Omega)\cap L^\infty(\Omega))$ such that $b(s)\in R(b)\setminus E$, $s\geqq 0$ and $\xi\geqq 0$. Moreover,

$$\int_\Omega H_0(-s-u)\{(\nabla b(u)+\phi(-s)-\phi(u))\cdot\nabla\xi+(g(u)-f)\xi\}\,dx$$
$$= \lim_{\varepsilon\to 0}\int_\Omega |\nabla b(u)|^2 H'_\varepsilon(b(-s)-b(u))\xi\,dx \geqq 0 \quad (6)$$

for any $(s,\xi)\in\mathbb{R}\times(H_0^1(\Omega)\cap L^\infty(\Omega))$ such that $b(-s)\in R(b)\setminus E$ and $\xi\geqq 0$ and for any $(s,\xi)\in\mathbb{R}\times(H^1(\Omega)\cap L^\infty(\Omega))$ such that $b(-s)\in R(b)\setminus E$, $s\geqq 0$ and $\xi\geqq 0$.

Proof. Since $(b^{-1})_0$ is a monotone function, E is a countable subset of $\mathbb{R}$; hence we have

$$\nabla b(u)=0 \quad \text{a.e. in } O=\{x\in\Omega/\ b(u(x))\in E\}. \quad (7)$$

We observe that for every s such that $b(s)\notin E$ we have

$$H_0(u-s)=H_0(b(u)-b(s)) \quad \text{for a.e. } x\in\Omega, \quad (8)$$

and we have

$$H_0(s-u)=H_0(b(s)-b(u)) \quad \text{for a.e. } x\in\Omega. \quad (9)$$

Let (s,ξ) be as in Lemma 1. Then the function $H_\varepsilon(b(u)-b(s))\xi$ belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$ and we have

$$0=\int_\Omega\{(g(u)-f)H_\varepsilon(b(u)-b(s))\xi+\nabla b(u)\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi)$$
$$-\phi(u)\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi)\}\,dx$$
$$=\int_\Omega\{(g(u)-f)H_\varepsilon(b(u)-b(s))\xi+\nabla b(u)\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi)$$
$$+(\phi(s)-\phi(u))\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi)\}\,dx,$$

whence

$$\int_\Omega \{(g(u)-f)H_\varepsilon(b(u)-b(s))\xi + \nabla b(u)\cdot\nabla\xi\, H_\varepsilon(b(u)-b(s)) + (\phi(s)-\phi(u))\cdot\nabla\xi\, H_\varepsilon(b(u)-b(s))\}\,dx = -\int_\Omega \nabla b(u)\cdot\nabla\, H_\varepsilon(b(u)-b(s))\,\xi\,dx + \int_\Omega (\phi(u)-\phi(s))\cdot\nabla H_\varepsilon(b(u)-b(s))\,\xi\,dx.$$

From the Lebesgue Theorem the integral on the left-hand side of this identity converges to

$$\int_\Omega H_0(b(u)-b(s))\{(g(u)-f)\xi + \nabla b(u)\cdot\nabla\xi + (\phi(s)-\phi(u))\cdot\nabla\xi\}\,dx.$$

Let us consider the second integral on the right-hand side:

$$I_\varepsilon = \int_\Omega (\phi(u)-\phi(s))\cdot\nabla H_\varepsilon(b(u)-b(s))\,\xi\,dx;$$

from (7) and (4) we have

$$(\phi(u)-\phi(s)\cdot\nabla H_\varepsilon(b(u)-b(s)) = (\phi((b^{-1})_0(b(u))) - \phi((b^{-1})_0(b(s)))\cdot\nabla H_\varepsilon(b(u)-b(s))$$

almost everywhere in Ω, whence

$$I_\varepsilon = \int_\Omega (\phi((b^{-1})_0(b(u))) - \phi((b^{-1})_0(b(s))))\cdot\nabla H_\varepsilon(b(u)-b(s))\,\xi\,dx = \int_\Omega \operatorname{div}\mathscr{F}_\varepsilon(b(u))\xi\,dx$$

where

$$\mathscr{F}_\varepsilon(z) = \int_0^z ((\phi((b^{-1})_0(r)) - \phi((b^{-1})_0(b(s))))H'_\varepsilon(r-b(s))dr = \frac{1}{\varepsilon}\int_{\min(z,b(s))}^{\min(z,b(s)+\varepsilon)} ((\phi((b^{-1})_0(r)) - \phi((b^{-1})_0(b(s))))\,dr.$$

Since $b(s)\notin E$, we easily check that

$$\mathscr{F}_\varepsilon(z) \overset{\varepsilon\to 0}{\to} 0 \text{ for any } z\in R(b);$$

then, since $\mathscr{F}_\varepsilon$ is bounded, we deduce from the Lebesgue Theorem that

$$I_\varepsilon = -\int_\Omega \mathscr{F}_\varepsilon(b(u))\cdot\nabla\xi\,dx \overset{\varepsilon\to 0}{\to} 0.$$

Then we get

$$\int_\Omega H_0(b(u)-b(s))\{(g(u)-f)\xi + \nabla b(u)\cdot\nabla\xi + (\phi(s)-\phi(u))\cdot\nabla\xi\}\,dx$$
$$= -\lim_{\varepsilon\to 0}\int_\Omega \nabla b(u)\cdot\nabla H_\varepsilon(b(u)-b(s))\,\xi\,dx$$
$$-\lim_{\varepsilon\to 0}\int_\Omega |\nabla b(u)|^2\,H'_\varepsilon(b(u)-b(s))\,\xi\,dx \leqq 0,$$

and thereby, from (8) we get (5).

Finally (6) is a consequence of (5) and of Remark 2.

Definition 2. Let $f\in L^1(\Omega)$. u is an *entropy solution* of $(P_S(f)(g,b,\phi))$ if and only if u is a weak solution and

$$0 \geqq \int_\Omega H_0(u-s)\{(g(u)-f)\xi + (\nabla b(u) + (\phi(s)-\phi(u)))\cdot\nabla\xi\,dx \tag{10}$$

and

$$0 \leqq \int_\Omega H_0(-s-u)\{(g(u)-f)\xi + (\nabla b(u) + (\phi(-s)-\phi(u)))\cdot\nabla\xi\,dx \tag{11}$$

for any $(s,\xi)\in\mathbb{R}\times(H_0^1(\Omega)\cap L^\infty(\Omega))$ such that $\xi\geqq 0$ and for any $(s,\xi)\in\mathbb{R}\times(H^1(\Omega)\cap L^\infty(\Omega))$ such that $s\geqq 0$ and such that $\xi\geqq 0$.

Remark 4. If u is an entropy solution of $(P_S(f)(g,b,\phi))$, then we easily check that $-u$ is an entropy solution of $(P_S(-f)(\tilde g,\tilde b,\tilde\phi))$ where $\tilde g(r)=-g(-r)$, $\tilde b(r)=-b(-r)$ and $\tilde\phi(r)=-\phi(-r)$.

Remark 5. Let $f\in L^q(\Omega)$ with

$$q\begin{cases} = 1 & \text{if } N=1,\\ > 1 & \text{if } N=2,\\ = 2N/(N+2) & \text{if } N>2.\end{cases}$$

Then, from Theorem 1, $g(u)\in L^q(\Omega)$, and from Sobolev imbeddings, $H^1(\Omega)\subset L^{q'}(\Omega)$. Hence inequalities (10) and (11) are still true for any nonnegative $\xi\in H_0^1(\Omega)$ and $\xi\in H^1(\Omega)$ respectively.

Remark 6. Let u be a weak or an entropy solution of $(P_S(f)(g,b,\phi))$ and let $\sigma\in\mathscr{C}(\mathbb{R})$, $\sigma(0)=0$, with σ strictly increasing and such that $\sigma^{-1}\in\mathscr{C}(\mathbb{R})$. Then $\tilde u=\sigma(u)$ is respectively a weak or an entropy solution of $(P_S(f)(g\circ\sigma^{-1}, b\circ\sigma^{-1},\phi\circ\sigma^{-1}))$.

Theorem 2. *Let b be a (strictly) increasing function. Then any weak solution of $(P_S(f)(g,b,\phi))$ is an entropy solution.*

Proof. When b is strictly increasing, $(b^{-1})_0 = b^{-1}$ is continuous in $R(b)$ and thus the set E is empty. Then, from Lemma 1 we deduce that a weak solution is an entropy solution.

Lemma 2. *Let $\mathscr{O}$ be a bounded open subset of $\mathbb{R}^{\mathbf{p}}$ with Lipschitz boundary $\partial\mathscr{O}$. Let $u \in L^\infty(\mathscr{O})$, $e \in \mathbb{R}^p$, $F \in \left(L^2(\mathscr{O})\right)^p$ and let G_1 and $G_2 \in L^1(\mathscr{O})$. Let*

$$\text{either } \mathscr{H}^1 = H^1(\mathscr{O}) \cap L^\infty(\mathscr{O}),$$
$$\text{or } \mathscr{H}^1 = H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O}).$$

Suppose that there are m and M with $m < M$ such that

$$\int_{\mathscr{O}} \left((u-M)^+ e \cdot \nabla\xi + F \cdot \nabla\xi + G_1\xi\right) dx \leqq 0,$$
$$\int_{\mathscr{O}} \left((u-m)^+ e \cdot \nabla\xi + F \cdot \nabla\xi + G_2\xi\right) dx \leqq 0$$

for any nonnegative $\xi \in \mathscr{H}^1$. Then

$$\int_{\mathscr{O}} \Big((u-s)^+ e \cdot \nabla\xi + F \cdot \nabla\xi + (G_1(1-\tilde{H}(u-s)) + G_2\tilde{H}(u-s)\xi\Big)\, dx \leqq 0$$

for any $s \in [m, M]$, for any nonnegative $\xi \in \mathscr{H}^1$, and for some $\tilde{H}(u-s) \in H(u-s)$.

Proof. Let us define

$$\mathscr{D} = \begin{cases} \mathscr{D}(\mathbb{R}^p) & \text{if } \mathscr{H}^1 = H^1(\mathscr{O}) \cap L^\infty(\mathscr{O}), \\ \mathscr{D}(\mathscr{O}) & \text{if } \mathscr{H}^1 = H_0^1(\mathscr{O}) \cap L^\infty(\mathscr{O}). \end{cases}$$

Let us prolong $(u-m)^+$, $(u-M)^+$, F, G_1 and G_2 by 0 outside of $\mathscr{O}$. Let ρ_n be a sequence of mollifiers and let us define

$$u_{2_n} = [(u-m)^+] * \rho_n, \quad u_{1_n} = [(u-M)^+] * \rho_n,$$
$$F_n = F * \rho_n, \quad G_{1_n} = G_1 * \rho_n, \quad G_{2_n} = G_2 * \rho_n.$$

Then

$$\int_{\mathbb{R}^p} \{(u_{2_n} - u_{1_n})\, e \cdot \nabla\zeta + (u_{1_n} e + F_n) \cdot \nabla\zeta + G_{2_n}\zeta\}\, dx \leqq 0$$

for any nonnegative $\zeta \in \mathscr{D}$, and

$$\int_{\mathbb{R}^p} \{(u_{1_n} e + F_n) \cdot \nabla\zeta + G_{1_n}\zeta\}\, dx \leqq 0$$

for any nonnegative $\zeta \in \mathscr{D}$.

Let $\xi \in \mathscr{D}$, $\xi \geqq 0$ and let κ be smooth enough and such that $0 \leqq \kappa \leqq 1$. Let us set $\zeta = \xi\kappa$ in the first inequality above and $\zeta = \xi(1-\kappa)$ in the second inequality. Then, by addition, we get

$$\int_{\mathbb{R}^p} \{(u_{2_n} - u_{1_n})\, e \cdot \nabla(\xi\kappa) + (u_{1_n} e + F_n) \cdot \nabla\xi$$
$$+ (G_{2_n}\kappa + G_{1_n}(1-\kappa))\xi\}\, dx \leqq 0;$$

and, for any $s \in (m, M)$ we have

$$\int_{\mathbb{R}^p} \{(u_{2_n} + m - s - u_{1_n})\, e \cdot \nabla(\xi\kappa) + (u_{1_n}\, e + F_n) \cdot \nabla\xi + (G_{2_n}\kappa + G_{1_n}(1-\kappa))\xi\}\, dx \leqq 0.$$

We choose $\kappa = H_\varepsilon(u_{2_n} + m - s - u_{1_n})$. Then

$$\int_{\mathbb{R}^p} (u_{2_n} + m - s - u_{1_n})\, e \cdot \nabla(\xi H_\varepsilon(u_{2_n} + m - s - u_{1_n}))\, dx$$
$$= -\int_{\mathbb{R}^p} \nabla(u_{2_n} + m - s - u_{1_n}) \cdot e\, \xi\, H_\varepsilon(u_{2_n} + m - s - u_{1_n})\, dx$$
$$\stackrel{\varepsilon\to 0}{\to} -\int_{\mathbb{R}^p} \nabla(u_{2_n} + m - s - u_{1_n})^+ \cdot e\, \xi\, dx = \int_{\mathbb{R}^p} (u_{2_n} + m - s - u_{1_n})^+\, e \cdot \nabla\xi\, dx,$$

$$\int_{\mathbb{R}^p} (G_{2_n} H_\varepsilon(u_{2_n} + m - s - u_{1_n}) + G_{1_n}(1 - H_\varepsilon(u_{2_n} + m - s - u_{1_n})))\xi\, dx$$
$$\stackrel{\varepsilon\to 0}{\to} \int_{\mathbb{R}^p} (G_{2_n} H_0(u_{2_n} + m - s - u_{1_n}) + G_{1_n}(1 - H_0(u_{2_n} + m - s - u_{1_n})))\xi\, dx.$$

Now, by letting $n \to +\infty$ we find that $u_{2_n} + m - s - u_{1_n}$ converges to $(u - m)^+ + m - s - (u - M)^+)^+ = (u-s)^+ - (u_M)^+$ in $L^r(\mathbb{R}^p)$ for any $1 \leqq r < +\infty$; $H_0(u_{2_n} + m - s - u_{1_n})$, or at least a subsequence, converges to some $\tilde{H}(u-s)$ in $L^\infty(\mathbb{R}^p)$ weak $\star$, where $\tilde{H}(u-s) \in H((u-m)^+ + m - s - (u-M)^+) = H(u-s)$. Then we get

$$\int_{\mathscr{O}} \{(u-s)^+\, e \cdot \nabla\xi + F \cdot \nabla\xi + (G_1 \tilde{H}(u-s) + G_2(1 - \tilde{H}(u-s)))\xi\}\, dx$$

for any nonnegative $\xi \in \mathscr{D}$ and, by density, for any nonnegative $\xi \in \mathscr{H}^1$.

Theorem 3. *Assume that there exist $\phi^{(1)}$ and $\phi^{(2)} \in \mathscr{C}(\mathbb{R}; \mathbb{R}^N)$ satisfying $\phi_j^{(1)}(0) = 0$ for $1 \leqq j \leqq N$ such that*

$$\phi(s) = \phi^{(1)}(b(s)) + \phi^{(2)}(b(s))s \quad \textit{for any } s \in \mathbb{R}.$$

Moreover, let $b^{-1}(0) = 0$. Then any weak solution of $(P_S(f)(g, b, \phi))$ is an entropy solution.

Proof. Let $s \in \mathbb{R}$ be such that $b(s) \notin E$. Then, from Lemma 1 we have

$$\int_\Omega H_0(u-s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\, dx \leqq 0 \tag{12}$$

for any $(s, \xi) \in \mathbb{R} \times (H^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \notin E$, $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \notin E$ and $\xi \geqq 0$.

Now let $s \in \mathbb{R}$ be such that $b(s) \in E$; let $[m, M] = b^{-1} \circ b(s)$. Since $0 \notin E$, we have two cases:

$$\text{If } 0 < r, \quad \text{then } 0 < m \leqq M \leqq +\infty.$$
$$\text{If } r < 0, \quad \text{then } -\infty \leqq m \leqq M < 0.$$

By taking into account the assumption of the theorem we have

$$\begin{aligned}\phi(s) - \phi(u) &= \phi^{(1)}(b(s)) - \phi^{(1)}(b(u)) \\ &\quad + (\phi^{(2)}(b(s)) - \phi^{(2)}(b(u)))u - (u - s)\phi^{(2)}(b(s)).\end{aligned}$$

If $M = +\infty$, then $H_0(u - M) \equiv 0$ in Ω; hence, obviously we have

$$\begin{aligned}&\int_\Omega H_0(u - M)\{(\nabla b(u) + \phi(M) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx \\ &\quad = \int_\Omega \{H_0(u - M)[(\nabla b(u) + \phi^{(1)}(b(M)) - \phi^{(1)}(b(u)) \\ &\qquad + (\phi^{(2)}(b(M)) - \phi^{(2)}(b(u)))u) \cdot \nabla\xi \\ &\qquad + (g(u) - f)\xi] - (u - M)^+\phi^{(2)}(b(s)) \cdot \nabla\xi\}\,dx = 0\end{aligned} \tag{13}$$

for any $\xi \in H^1(\Omega)$.

If $M < +\infty$, there exists a sequence $(s^n)_{n\in\mathbb{N}}$ such that $s^n > M$, $s^n \searrow M$, $b(s^n) \notin E$, $b(s^n) > b(M)$, $b(s^n) \searrow b(M)$ and $H_0(u - s^n) \nearrow H_0(u - M)$ almost everywhere in Ω. Then, from Lemma 1 we have

$$\begin{aligned}&\int_\Omega \{H_0(u - M)[(\nabla b(u) + \phi^{(1)}(b(M)) - \phi^{(1)}(b(u)) \\ &\qquad + (\phi^{(2)}(b(M)) - \phi^{(2)}(b(u)))u) \cdot \nabla\xi + (g(u) - f)\xi] \\ &\qquad\qquad - (u - M)^+\phi^{(2)}(b(M)) \cdot \nabla\xi\}\,dx \leqq 0\end{aligned} \tag{14}$$

for any $(M, \xi) \in \mathbb{R} \times H^1(\Omega) \cap L^\infty(\Omega)$ such that $M > 0$ and $\xi \geqq 0$ and for any $(M, \xi) \in \mathbb{R} \times H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $M < 0$ and $\xi \geqq 0$.

Moreover, if $m = -\infty$, then $H_0(u - m) \equiv 1$ in Ω. Hence we have

$$\begin{aligned}&\int_\Omega H_0(u - m)\{(\nabla b(u) + \phi(m) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx \\ &\quad = \int_\Omega \{H_{\text{Max}}(u - m)[(\nabla b(u) + \phi^{(1)}(b(m)) - \phi^{(1)}(b(u)) \\ &\qquad + (\phi^{(2)}(b(m)) - \phi^{(2)}(b(u)))u) \cdot \nabla\xi \\ &\qquad + (g(u) - f)\xi] - (u - m)^+\phi^{(2)}(b(s)) \cdot \nabla\xi\}\,dx = 0\end{aligned}$$

for any $\xi \in H_0^1(\Omega)$.

If $m > -\infty$, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n < m$, $s_n \nearrow m$, $b(s_n) \notin E$, $b(s_n) < b(m)$, $b(s_n) \nearrow b(m)$ and $H_0(u - s_n) \searrow H_{\mathrm{Max}}(u - m)$. Then, from Lemma 1 we have

$$\int_\Omega \{H_{\mathrm{Max}}(u-m)[(\nabla b(u) + \phi^{(1)}(b(m)) - \phi^{(1)}(b(u)) \\ + (\phi^{(2)}(b(m)) - \phi^{(2)}(b(u)))u) \cdot \nabla\xi + (g(u) - f)\xi\} \\ - (u-m)^+ (\phi^{(2)}(b(m)) \cdot \nabla\xi\}\, dx \leqq 0$$

for any $(m, \xi) \in \mathbb{R} \times H^1(\Omega) \cap L^\infty(\Omega)$ such that $m > 0$ and $\xi \geqq 0$; and for any $(m, \xi) \in \mathbb{R} \times H_0^1(\Omega) \cap L^\infty(\Omega)$ such that $m < 0$ and $\xi \geqq 0$.

Now, by taking into account that for any $s \in [m, M]$ we have

$$\begin{aligned} &H_0(u-M)[(\nabla b(u) + \phi^{(1)}(b(M)) - \phi^{(1)}(b(u)) \\ &\quad + (\phi^{(2)}(b(M)) - \phi^{(2)}(b(u)))u)] \\ &\quad = H_0(u-s)[(\nabla b(u) + \phi^{(1)}(b(s)) - \phi^{(1)}(b(u)) \\ &\quad\quad + (\phi^{(2)}(b(s)) - \phi^{(2)}(b(u)))u)] \\ &\quad = H_{\mathrm{Max}}(u-m)[(\nabla b(u) + \phi^{(1)}(b(m)) - \phi^{(1)}(b(u)) \\ &\quad\quad + (\phi^{(2)}(b(m)) - \phi^{(2)}(b(u)))u)]. \end{aligned} \tag{15}$$

We apply Lemma 2 with

$$\begin{aligned} F &= H_0(u-M)[(\nabla b(u) + \phi^{(1)}(b(M)) - \phi^{(1)}(b(u)) \\ &\quad + (\phi^{(2)}(b(M)) - \phi^{(2)}(b(u)))u), \\ G_1 &= H_0(u-M)(g(u) - f), \\ G_2 &= H_{\mathrm{Max}}(u-m)(g(u) - f), \end{aligned}$$

and, by taking into account (15), we get

$$0 \geqq \int_\Omega \{H_0(u-s)[\nabla b(u) + (\phi^{(1)}(b(s)) - \phi^{(1)}(b(u)) \\ + (\phi^{(2)}(b(s)) - \phi^{(2)}(b(u)))u) - (u-s)\phi^{(2)}(r)] \cdot \nabla\xi \\ + (g(u) - f)(H_0(u-M)(1 - \tilde{H}(u-s)) + H_{\max}(u-m)\tilde{H}(u-s))\xi\}\, dx$$

for some $\tilde{H}(u-s) \in H(u-s)$, whence, since $H_0(u-M)(1-\tilde{H}(u-s) = 0$ and $H_{\max}(u-m)\tilde{H}(u-s) = \tilde{H}(u-s)$, we have

$$0 \geqq \int_\Omega \{H_0(u-s)(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\tilde{H}(u-s))\xi\}\, dx$$

for any $(s, \xi) \in \mathbb{R} \times (H^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \in E$, $s > 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \in E$, $s < 0$ and $\xi \geqq 0$.

Now, for any $s \in [m, M)$ there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s < s_n < M$ and $s_n \searrow s$. Then $\tilde{H}(u - s_n) \nearrow H_0(u - s)$ and $H_0(u - s_n) \nearrow H_0(u - s)$ almost everywhere in Ω; hence we get

$$0 \geqq \int_\Omega H_0(u - s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx \tag{16}$$

for any $(s, \xi) \in \mathbb{R} \times (H^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \in E$, $s \in I(s)$, $s > 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$ such that $b(s) \in E$, $s \in I(s)$, $s < 0$ and $\xi \geqq 0$, where

$$I(s) = \left[\inf b^{-1} \circ b(s), \sup b^{-1} \circ b(s)\right).$$

Then, from (12)–(14) and (16) we deduce that

$$0 \geqq \int_\Omega H_0(u - s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx$$

for any $(s, \xi) \in \mathbb{R} \times (H^1(\Omega) \cap L^\infty(\Omega))$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$ such that $\xi \geqq 0$.

Similarly we prove that

$$0 \leqq \int_\Omega H_0(-s - u)\{(\nabla b(u) + \phi(-s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx$$

for any $(s, \xi) \in \mathbb{R} \times (H^1(\Omega) \cap L^\infty(\Omega))$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$ such that $\xi \geqq 0$, whence u is an entropy solution.

2.2. Existence for the Nondegenerate Case

In this part we study the case when b^{-1} is continuous, i.e., b is strictly increasing and $|b(s)| \overset{|s|\to+\infty}{\rightarrow} +\infty$. Under such an assumption it follows from Theorem 2 that any weak solution is an entropy solution. Moreover, if we take into account Remark 6, then u is an entropy solution of $(P_S(f)(g, b, \phi))$ if and only if $b(u)$ is an entropy solution of $(P_S(f)(g \circ b^{-1}, I, \phi \circ b^{-1}))$ (where I is the identity). Let us set

$$\gamma = g \circ b^{-1}, \quad \Phi = \phi \circ b^{-1}, \quad w = b(u).$$

Then we study

$$(P_S(f)(\gamma, I, \Phi)) \begin{cases} \gamma(w) - \Delta w + \operatorname{div}(\Phi(w)) = f & \text{in } \Omega \times (0, T), \\ w = 0 & \text{on } \Gamma, \end{cases}$$

where

$$\gamma : \mathbb{R} \mapsto \mathbb{R}, \quad \text{is continuous and nondecreasing}, \quad \gamma(0) = 0, \tag{17}$$

$$\Phi \in \mathscr{C}(\mathbb{R}; \mathbb{R}^N), \quad \Phi(0) = 0. \tag{18}$$

Theorem 4. *Let* (17) *and* (18) *hold and let* $f \in L^p(\Omega)$ *for some* $p > \frac{1}{2}N$. *Then there exists at least one entropy solution* $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *of* $(P_S(f)(\gamma, I, \Phi))$.

Proof. From Theorem 1, for any $p > \frac{1}{2}N$ there exists a constant C such that any solution w of $(P_S(f)(\gamma, I, \Phi))$ satisfies

$$\|w\|_{L^\infty(\Omega)} \leqq C\|f\|_{L^p(\Omega)},$$

where C does not depend on Φ. Let $k = C\|f\|_{L^p(\Omega)}$ and let

$$\Phi_k(s) = \Phi(\max(\min(s,k), -k)),$$
$$\gamma_k(s) = \gamma(\max(\min(s,k), -k)).$$

Then any solution of $(P_S(f)(\gamma, I, \Phi))$ is still a solution of $(P_S(f)(\gamma_k, I, \Phi_k))$ and reciprocally. Hence it is enough to prove the existence of a solution to $(P_S(f)(\gamma_k, I, \Phi_k))$.

Let us define A: $H_0^1(\Omega) \mapsto H^{-1}(\Omega)$ by

$$A(w) = \gamma_k(w) - \Delta w.$$

We easily check (see [Li]) that A maps $H_0^1(\Omega)$ one-to-one onto $H^{-1}(\Omega)$. Let us define $T : L^2(\Omega) \mapsto L^2(\Omega)$ by

for any $v \in L^2(\Omega)$ $T(v)$ is the unique solution $w \in H_0^1(\Omega)$ of
$$A(w) = -\operatorname{div}(\Phi_k(v)) + f.$$

Then T maps $L^2(\Omega)$ into a bounded subset of $H_0^1(\Omega)$:

$$\begin{aligned}
\|\nabla w\|^2_{(L^2(\Omega))^N} &\leqq \int_\Omega (\gamma_k(w)\, w + |\nabla w|^2)\, dx \\
&= \int_\Omega (\Phi_k(v) \cdot \nabla w + f w)\, dx \\
&\leqq |\Omega|^{1/2} \|\Phi_k\|_{(L^\infty(\mathbb{R}))^N} \, \|\nabla w\|_{(L^2(\Omega))^N} + \|f\|_{L^2(\Omega)} \, \|w\|_{L^2(\Omega)}.
\end{aligned}$$

Hence, from the Poincaré Theorem,

$$\|T(v)\|_{H_0^1(\Omega)} \leqq K$$

where $K = K(\Omega, k, f)$ is a constant. Since the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact, it follows from the Schauder Fixed-Point Theorem that T has at least one fixed point $w \in L^2(\Omega)$ (so that $T(w) = w$, whence $w \in H_0^1(\Omega)$). Of course, any fixed point of T is a weak (and therefore an entropy) solution of $(P_S(f)(\gamma_k, I, \Phi_k))$, and thus $(P_S(f)(\gamma, I, \Phi))$ has at least one entropy solution.

Theorem 5. *Let* (17) *and* (18) *hold, let* $\Phi \in \mathscr{C}^{0,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^N)$ *and let* $f \in L^p(\Omega)$ *for some* $p > \frac{1}{2}N$. *Then* $(P_S(f)(\gamma, I, \Phi))$ *has one and only one weak solution* w. *Moreover, let* $f_i \in L^p(\Omega)$ *for* $i = 1, 2$ *and let* w_i *be the unique weak solution of* $(P_S(f_i)(\gamma, I, \Phi))$ *for* $i = 1, 2$. *Then*

$$\|(\gamma(w_1) - \gamma(w_2))^+\|_{L^1(\Omega)} \leqq \int_\Omega (f_1 - f_2) H_0(w_1 - w_2). \tag{19}$$

Proof. Let $H_\varepsilon(s) = \min(s^+/\varepsilon, 1)$ and let w_i be a solution of $(P_S(f_i)(\gamma, I, \Phi))$ for $i = 1, 2$. Let $\xi \in \mathscr{D}(\overline{\Omega})$, $\xi \geqq 0$. Then

$$H_\varepsilon(w_1 - w_2)\xi \in H^1_0(\Omega),$$

$$\begin{aligned}\int_\Omega (f_1 - f_2) H_\varepsilon(w_1 - w_2)\xi \, dx = \int_\Omega [(\gamma(w_1) - \gamma(w_2))\, H_\varepsilon(w_1 - w_2)\, \xi \\ + (\nabla(w_1 - w_2) - \Phi(w_1) + \Phi(w_2)) \cdot \nabla(H_\varepsilon(w_1 - w_2)\, \xi)]\, dx. \end{aligned}\tag{20}$$

Obviously we have

$$\int_\Omega \nabla(w_1 - w_2) \cdot \nabla H_\varepsilon(w_1 - w_2)\, \xi \, dx \geqq 0.$$

Moreover, from Theorem 1 we have

$$\|w_i\|_{L^\infty(\Omega)} \leqq C \|f_i\|_{L^p(\Omega)}$$

for $i = 1, 2$. Thus Φ can be considered as a Lipschitz continuous function, whence

$$\begin{aligned}&\lim_{\varepsilon \to 0} \left| \int_\Omega (\Phi(w_1) - \Phi(w_2) \cdot \nabla(w_1 - w_2)\, H'_\varepsilon(w_1 - w_2)\, \xi dx \right| \\ &\leqq \lim_{\varepsilon \to 0} L \int_\Omega |\nabla(w_1 - w_2)|\; H'_\varepsilon(w_1 - w_2)\, (w_1 - w_2)\, dx = 0.\end{aligned}$$

Then, by letting $\varepsilon \to 0$ in (20), we get

$$\begin{aligned}&\int_\Omega (f_1 - f_2)\, H_0(w_1 - w_2)\, \xi \, dx \\ &\geqq \int_{\Omega \cap \{w_1 > w_2\}} ((\gamma(w_1) - \gamma(w_2)\, \xi + (\nabla(w_1 - w_2) - \Phi(w_1) + \Phi(w_2)) \cdot \nabla\xi)\, dx\end{aligned}$$

for any $\xi \in \mathscr{D}(\overline{\Omega})$, $\xi \geqq 0$.

In particular, by choosing $\xi \equiv 1$ we get (19).

By choosing $f_1 = f_2$ we get $\gamma(w_1) = \gamma(w_2)$, whence this inequality becomes

$$0 \geqq \int_{\Omega \cap \{w_1 > w_2\}} (\nabla(w_1 - w_2) - \Phi(w_1) + \Phi(w_2)) \cdot \nabla\xi \, dx$$

for any $\xi \in \mathscr{D}(\overline{\Omega})$, $\xi \geqq 0$ and therefore

$$0 = \int_{\Omega \cap \{w_1 > w_2\}} (\nabla(w_1 - w_2) - \Phi(w_1) + \Phi(w_2)) \cdot \nabla\xi \, dx$$

for any $\xi \in \mathscr{D}(\overline{\Omega})$ and, by density, for any $\xi \in H^1(\Omega)$. Then

$$0 = \int_\Omega (w_1 - w_2)^+ \, (-\Delta\xi + W \cdot \nabla\xi) \; dx$$

for any $\xi \in H^1(\Omega)$, where

$$W = \begin{cases} -(\Phi(w_1) - \Phi(w_2))/(w_1 - w_2) & \text{if } w_1 > w_2, \\ 0 & \text{if } w_1 \leqq w_2 \end{cases}$$

is a $(L^\infty(\Omega))^N$ function. Now we choose $\xi \in H_0^1(\Omega)$ such that

$$-\Delta\xi + W \cdot \nabla\xi = (w_1 - w_2)^+$$

and we deduce that

$$\int_\Omega ((w_1 - w_2)^+)^2 \, dx = 0.$$

Thus the solution of $(P_S(f)(\gamma, I, \Phi))$ is unique for any $f \in L^p(\Omega)$.

Theorem 6. *Let* (17) *and* (18) *hold, let* $\Phi \in \mathscr{C}^0(\mathbb{R}; \mathbb{R}^N)$ *and let* $f \in L^p(\Omega) \cap BV(\Omega)$ *for some* $p > \frac{1}{2}N$. *Then there exists a weak solution* w *of* $(P_S(f)(\gamma, I, \Phi))$ *such that* $\gamma(w) \in BV(\Omega)$ *and*

$$TV_\Omega(\gamma(w)) \leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)}.$$

Proof. For any $\varepsilon > 0$ let Ω_ε be a $\mathscr{C}^\infty$ open subset of $\mathbb{R}^N$ satisfying

$$B(\Omega, \varepsilon) \subset\subset \Omega_\varepsilon \subset B(\Omega, 2\varepsilon).$$

Let $\gamma_\varepsilon \in \mathscr{C}^\infty(\mathbb{R})$ and $\Phi_\varepsilon \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^N)$ satisfy

$$\gamma_\varepsilon(s) = |\Phi_\varepsilon(s)| = 0 \quad \forall\, s \in (-\varepsilon, +\varepsilon)$$

(where $|\Phi|^2 = \sum_{i=1}^N \Phi_i^2$) and be such that

$$\gamma_\varepsilon \to \gamma \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}), \quad \Phi_\varepsilon \to \Phi \text{ in } (L^\infty_{\mathrm{loc}}(\mathbb{R}))^N.$$

Let ρ_ε be a sequence of mollifiers with support in $B(0, \varepsilon)$ and let $f_\varepsilon = \overline{f} * \rho_\varepsilon \in \mathscr{D}(\Omega_\varepsilon)$, where

$$\overline{f}(x) = \begin{cases} f(x) & \text{for a.e. } x \in \Omega, \\ 0 & \text{for a.e. } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Then we consider

$$(P_S^\varepsilon(f_\varepsilon)(\gamma_\varepsilon, I, \Phi_\varepsilon)) \quad \begin{cases} \gamma_\varepsilon(w_\varepsilon) - \Delta w_\varepsilon + \operatorname{div}(\Phi_\varepsilon(w_\varepsilon)) = f_\varepsilon & \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Let w_ε be the unique weak solution of $(P_S^\varepsilon(f_\varepsilon)(\gamma_\varepsilon, I, \Phi_\varepsilon))$. Since $\partial\Omega_\varepsilon$ is smooth and since

$$\Delta w_\varepsilon = \gamma_\varepsilon(w_\varepsilon) + \operatorname{div}(\Phi_\varepsilon(w_\varepsilon)) - f_\varepsilon,$$

we easily deduce that $w_\varepsilon \in \mathscr{C}^\infty(\overline{\Omega_\varepsilon})$.

Let $\omega_\varepsilon = \{x \in \Omega_\varepsilon \,/\, w_\varepsilon(x) > \varepsilon\}$. Then ω_ε is an open subset of Ω_ε, and since $w_\varepsilon = 0$ on $\partial\Omega_\varepsilon$, we have $\omega_\varepsilon \subset\subset \Omega_\varepsilon$. Then $d(\omega_\varepsilon, \mathbb{R}^N \setminus \Omega_\varepsilon) = d_\varepsilon > 0$ and

$$\gamma_\varepsilon = |\Phi_\varepsilon(w_\varepsilon)| = 0, \quad -\Delta w_\varepsilon = f_\varepsilon \quad \text{in } \Omega_\varepsilon \setminus \omega_\varepsilon.$$

Moreover, since f_ε vanishes in $\Omega_\varepsilon \setminus B(\Omega, \varepsilon)$, we have

$$\Delta w_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon \setminus \overline{\omega_\varepsilon \cup B(\Omega, \varepsilon)},$$

whence

$$\begin{aligned} \left|\frac{\partial |\nabla w_\varepsilon|}{\partial \nu}\right| &= \left|\frac{\partial \nabla w_\varepsilon \cdot \nabla w_\varepsilon / |\nabla w_\varepsilon|}{\partial \nu}\right| \\ &= \left|\frac{\partial \nabla w_\varepsilon \cdot \nu}{\partial \nu}\right| = \left|\frac{\partial^2 w_\varepsilon}{\partial \nu^2}\right| = |\Delta w_\varepsilon| = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{aligned} \tag{21}$$

Let $H_\delta(s) = \min(s^+/\delta, 1)$ and, for any $\xi \in \mathbb{R}^N$ let

$$I_\delta(|\xi|) = \int_0^{|\xi|} H_\delta(r)\, dr.$$

Then

$$\begin{aligned} &\sum_{i=1}^N \frac{\partial}{\partial x_i} (\gamma_\varepsilon(w_\varepsilon) - \Delta w_\varepsilon + \operatorname{div}(\Phi_\varepsilon(w_\varepsilon)))\, H_\delta(|\nabla w_\varepsilon|) \frac{\partial w_\varepsilon}{\partial x_i} \Big/ |\nabla w_\varepsilon| \\ &\qquad = \sum_{i=1}^N \frac{\partial f_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|) \frac{\partial w_\varepsilon}{\partial x_i} \Big/ |\nabla w_\varepsilon|. \end{aligned}$$

Hence

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla(\gamma_\varepsilon(w_\varepsilon)) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\ &\qquad - \int_{\Omega_\varepsilon} \nabla(\Delta w_\varepsilon) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\ &\qquad + \int_{\Omega_\varepsilon} \nabla(\operatorname{div}(\Phi_\varepsilon(w_\varepsilon))) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\ &\quad = \int_{\Omega_\varepsilon} \nabla f_\varepsilon \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx. \end{aligned}$$

The first integral

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla(\gamma_\varepsilon(w_\varepsilon)) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\
&\quad = \int_{\Omega_\varepsilon} \gamma_\varepsilon'(w_\varepsilon)|\nabla w_\varepsilon| H_\delta(|\nabla w_\varepsilon|) \\
&\quad \overset{\delta\to 0}{\to} \int_{\Omega_\varepsilon} \gamma_\varepsilon'(w_\varepsilon)|\nabla w_\varepsilon|\, dx = \int_{\Omega_\varepsilon} |\nabla(\gamma_\varepsilon(w_\varepsilon))|\, dx = TV_{\Omega_\varepsilon}(\gamma_\varepsilon(w_\varepsilon)).
\end{aligned}$$

By taking into account (21) and the fact that I_δ is convex, the second integral gives

$$\begin{aligned}
&-\int_{\Omega_\varepsilon} \nabla(\Delta w_\varepsilon) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\
&\quad = -\int_{\Omega_\varepsilon} \sum_{i=1}^N \Delta \frac{\partial w_\varepsilon}{\partial x_i} \frac{\partial w_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\
&\quad = -\int_{\Omega_\varepsilon} \sum_{i=1}^N \operatorname{div}\left(\nabla \frac{\partial w_\varepsilon}{\partial x_i} \frac{\partial w_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\right) dx \\
&\qquad + \int_{\Omega_\varepsilon} \sum_{i=1}^N \nabla \frac{\partial w_\varepsilon}{\partial x_i} \cdot \nabla \left(\frac{\partial w_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\right) dx \\
&\quad = -\int_{\partial\Omega_\varepsilon} \sum_{i=1}^N \nu \cdot \nabla \frac{\partial w_\varepsilon}{\partial x_i} \frac{\partial w_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, d\sigma \\
&\qquad + \int_{\Omega_\varepsilon} \sum_{i,j,k=1}^N \frac{\partial^2 w_\varepsilon}{\partial x_i \partial x_j} \frac{\partial^2 I_\delta(|\nabla w|)}{\partial \xi_i \partial \xi_k} \frac{\partial^2 w_\varepsilon}{\partial x_k \partial x_j}\, dx \\
&\quad \geqq -\frac{1}{2} \int_{\partial\Omega_\varepsilon} \frac{\partial |\nabla w_\varepsilon|^2}{\partial \nu} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\
&\quad = -\int_{\partial\Omega_\varepsilon} \frac{\partial |\nabla w_\varepsilon|}{\partial \nu} H_\delta(|\nabla w_\varepsilon|)\, d\sigma = 0.
\end{aligned}$$

Since $\Phi_\varepsilon'(w_\varepsilon)$ is smooth and vanishes at the boundary, the third integral gives

$$\begin{aligned}
&\int_{\Omega_\varepsilon} \nabla\left(\operatorname{div}\left(\Phi_\varepsilon(w_\varepsilon)\right)\right) \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx \\
&\quad = \sum_{i=1}^N \int_{\Omega_\varepsilon} \operatorname{div}\left(\Phi_\varepsilon'(w_\varepsilon) \frac{\partial w_\varepsilon}{\partial x_i}\right) \frac{\partial w_\varepsilon}{\partial x_i} H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon|\, dx
\end{aligned}$$

$$= \int_{\Omega_\varepsilon} \left\{ \operatorname{div}\left(\Phi'_\varepsilon(w_\varepsilon)\right) |\nabla w_\varepsilon| H_\delta(|\nabla w_\varepsilon|) + \Phi'_\varepsilon(w_\varepsilon) \cdot \nabla |\nabla w_\varepsilon| H_\delta(|\nabla w_\varepsilon|) \right\} dx$$

$$= \int_{\Omega_\varepsilon} \operatorname{div}\left(\Phi'_\varepsilon(w_\varepsilon)\right) \left(|\nabla w_\varepsilon| H_\delta(|\nabla w_\varepsilon|) - \mathscr{H}_\delta(|\nabla w_\varepsilon|)\right) dx \overset{\delta \to 0}{\to} 0$$

where

$$\mathscr{H}_\delta(s) = \int_0^s H_\delta(r) dr.$$

Finally, the last integral satisfies

$$\int_{\Omega_\varepsilon} \nabla f_\varepsilon \cdot \nabla w_\varepsilon H_\delta(|\nabla w_\varepsilon|)/|\nabla w_\varepsilon| \, dx \leqq \int_{\Omega_\varepsilon} |\nabla f_\varepsilon| \, dx$$

whence we get

$$TV_{\Omega_\varepsilon}(\gamma_\varepsilon(w_\varepsilon)) \leqq TV_{\Omega_\varepsilon}(f_\varepsilon) \leqq TV_{\Omega_\varepsilon}(\overline{f}) = TV_\Omega(f) + \|f\|_{L^1(\Gamma)} \tag{22}$$

(see [Gi, GR]).

Now let R be so large that $\Omega_\varepsilon \subset B(0, R)$ for any $\varepsilon \leqq \varepsilon_0$. Then from Theorem 1 (since $p > \frac{1}{2} N$ p fulfills (3)) there exist constants $K = K(N, p, R)$ and $C = C(N, p, R)$ such that

$$\|\nabla w_\varepsilon\|_{(L^2(B(0,R)))^N} \leqq K \|f_\varepsilon\|_{L^p(B(0,R))} \leqq K \|f\|_{L^p(\Omega)} \quad \forall \varepsilon \leqq \varepsilon_0,$$

$$\|w_\varepsilon\|_{L^\infty(B(0,R))} \leqq C \|f_\varepsilon\|_{L^p(B(0,R))} \leqq C \|f\|_{L^p(\Omega)},$$

and, from (22)

$$TV_{B(0,R)}(\gamma_\varepsilon(w_\varepsilon)) \leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)} \quad \forall \varepsilon \leqq \varepsilon_0.$$

Since the canonical imbedding of $BV(B(0, R))$ into $L^1(B(0, R))$ and the canonical imbedding of $H_0^1(B(0, R))$ into $L^2(B(0, R))$ are compact, we deduce that there exists a subsequence still denoted by ε such that

$$\begin{aligned}
w_\varepsilon &\overset{\varepsilon \to 0}{\to} w \ \text{ weakly in } H_0^1(B(0, R)), \\
w_\varepsilon &\overset{\varepsilon \to 0}{\to} w \ \text{ in } L^2(B(0, R)), \\
w_\varepsilon &\overset{\varepsilon \to 0}{\to} w \ \text{ in } L^\infty(B(0, R)) \text{ weak} \star, \\
w_\varepsilon &\overset{\varepsilon \to 0}{\to} w \ \text{ a.e. in } B(0, R), \\
\gamma_\varepsilon(w_\varepsilon) &\overset{\varepsilon \to 0}{\to} \hat{\gamma} \ \text{ in } L^1(B(0, R)).
\end{aligned}$$

We easily deduce that $\hat{\gamma} = \gamma(w)$ (see [Br]) and that

$$\Phi_\varepsilon(w_\varepsilon) \overset{\varepsilon \to 0}{\to} \Phi(w) \ \text{ a.e. in } B(0, R).$$

Moreover, we easily check that w vanishes outside of Ω, whence $w \in H_0^1(\Omega)$.

Now, let $\xi \in H_0^1(\Omega)$; then

$$0 = \int_{\Omega_\varepsilon} ((\gamma_\varepsilon(w_\varepsilon) - f_\varepsilon)\xi + (\nabla w_\varepsilon - \Phi_\varepsilon(w_\varepsilon)) \cdot \nabla\xi)\; dx$$

$$\stackrel{\varepsilon\to 0}{\to} \int_{\Omega} ((\gamma(w) - f)\xi + (\nabla w - \Phi(w)) \cdot \nabla\xi)\; dx,$$

whence w is a weak solution of $(P_S(f)(\gamma, I, \Phi))$. Moreover,

$$\begin{aligned} TV_\Omega(\gamma(w)) &= TV_{B(0,R)}(\gamma(w)) \\ &\leqq \liminf_{\varepsilon\to 0} TV_{B(0,R)}(\gamma_\varepsilon(w_\varepsilon)) \leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)}, \end{aligned}$$

which achieves the proof.

2.3. Existence of Entropy Solutions for $(P_S(f)(g, b, \phi))$

Before formulating the existence theorem we prove the following technical lemma:

Lemma 3. *Let $\mathscr{O}$ be a bounded open subset of $\mathbb{R}^p$ with Lipschitz boundary $\partial\mathscr{O}$. Let v, w_1 and $w_2 \in L^2(\mathscr{O})$ be such that*

$$w_1 \geqq v \geqq w_2,$$

let $e \in \mathbb{R}^p$, let $F \in \left(L^2(\mathscr{O})\right)^p$ and let G, G_1 and $G_2 \in L^2(\mathscr{O})$ be such that

$$\int_{\mathscr{O}} \{v\, e \cdot \nabla\xi + F \cdot \nabla\xi + G\xi\}\, dx \leqq 0 \quad \forall \xi \in H_0^1(\mathscr{O}), \quad \xi \geqq 0,$$

$$\int_{\mathscr{O}} \{w_i\, e \cdot \nabla\xi + F \cdot \nabla\xi + G_i\xi\}\, dx \leqq 0 \quad \forall \xi \in H^1(\mathscr{O}), \quad \xi \geqq 0, \quad i = 1, 2.$$

Then

$$\int_{\mathscr{O}} \{v\, e \cdot \nabla\xi + F \cdot \nabla\xi + G\xi\}\, dx \leqq 0 \quad \forall \xi \in H^1(\mathscr{O}), \quad \xi \geqq 0.$$

Proof. Without loss of generality we can consider a nonnegative function $\xi \in \mathscr{D}(\mathbb{R}^p)$ such that the support of ξ is contained in a ball $B \subset \mathbb{R}^p$ for which $B \cap \partial\mathscr{O}$ is part of a Lipschitz graph. Then we can find a sequence of mollifiers $(\rho_n)_{n\in\mathbb{N}}$ satisfying

$$y \mapsto \rho_n(x - y) \in \mathscr{D}(\mathscr{O}) \quad \forall x \in B \cap \mathscr{O},$$

for n large enough.

Let us prolong all the functions arising in this lemma by 0 outside of $\mathscr{O}$ and let us define

$$\begin{aligned} v_n &= v * \rho_n, & w_{i_n} &= w_i * \rho_n & &\text{for } i = 1, 2, \\ G_n &= G * \rho_n, & G_{i_n} &= G_i * \rho_n & &\text{for } i = 1, 2, \\ F_n &= F * \rho_n. \end{aligned}$$

Finally let $(\zeta_m)_{m\in\mathbb{N}}$ be a sequence of function of $\mathscr{D}(\mathscr{O})$ satisfying $0 \leqq \zeta_m \leqq 1$ and such that $\zeta_m \to 1$ almost everywhere in $\mathscr{O}$. From the assumptions of the lemma we deduce that

$$\int_{\mathbb{R}^p} \{v_n\, e \cdot \nabla(\xi\zeta_m) + F_n \cdot \nabla(\xi\zeta_m) + G_n\xi\zeta_m\}\, dx \leqq 0,$$

$$\int_{\mathbb{R}^p} \{w_{i_n}\, e \cdot \nabla(\xi(1-\zeta_m)) + F_n \cdot \nabla(\xi(1-\zeta_m)) + G_{i_n}\xi(1-\zeta_m)\}\, dx \leqq 0$$

for $i = 1, 2$. Hence

$$\begin{aligned} \int_{\mathbb{R}^p} \{(w_{i_n} - v_n)\, e \cdot \nabla(\xi(1-\zeta_m)) + (v_n e + F_n) \cdot \nabla\xi \\ + (G_{i_n}(1-\zeta_m) + G_n\zeta_m)\xi\}\, dx \leqq 0 \end{aligned}$$

for $i = 1, 2$.

Moreover, let $\kappa \in \mathscr{D}(\mathbb{R}^p)$ satisfy $0 \leqq \kappa \leqq 1$; then we still have

$$\begin{aligned} \int_{\mathbb{R}^p} \{(w_{1_n} - v_n)\, e \cdot \nabla(\xi\kappa(1-\zeta_m)) + (v_n e + F_n) \cdot \nabla(\xi\kappa) \\ + (G_{1_n}(1-\zeta_m) + G_n\zeta_m)\xi\kappa\}\, dx \leqq 0, \end{aligned}$$

$$\begin{aligned} \int_{\mathbb{R}^p} \{(w_{2_n} - v_n)\, e \cdot \nabla(\xi(1-\kappa)(1-\zeta_m)) + (v_n e + F_n) \cdot \nabla(\xi(1-\kappa)) \\ + (G_{2_n}(1-\zeta_m) + G_n\zeta_m)\xi(1-\kappa)\}\, dx \leqq 0, \end{aligned}$$

whence

$$\begin{aligned} 0 \geqq & \int_{\mathbb{R}^p} [(w_{1_n} - v_n)\, e \cdot \nabla(\xi\kappa(1-\zeta_m)) + (w_{2_n} - v_n)\, e \cdot \nabla(\xi(1-\kappa)(1-\zeta_m))]\, dx \\ & + \int_{\mathbb{R}^p} \{(v_n e + F_n) \cdot \nabla\xi + (G_{1_n}(1-\zeta_m) + G_n\zeta_m)\xi\kappa \\ & \qquad + (G_{2_n}(1-\zeta_m) + G_n\zeta_m)\xi(1-\kappa)\}\, dx \\ = & -\int_{\mathbb{R}^p} [\nabla(w_{1_n} - v_n) \cdot e\, \xi\kappa(1-\zeta_m) + \nabla(w_{2_n} - v_n) \cdot e\, \xi(1-\kappa)(1-\zeta_m)]\, dx \\ & + \int_{\mathbb{R}^p} \{(v_n e + F_n) \cdot \nabla\xi + (G_{1_n}(1-\zeta_m) + G_n\zeta_m)\xi\kappa \\ & \qquad + (G_{2_n}(1-\zeta_m) + G_n\zeta_m)\xi(1-\kappa)\}\, dx. \end{aligned}$$

Then, by letting $m \to +\infty$ we get

$$0 \geqq -\int_{\mathbb{R}^p\setminus\mathscr{O}} [\nabla(w_{1_n} - v_n)\cdot e\,\xi\kappa + \nabla(w_{2_n} - v_n)\cdot e\,\xi(1-\kappa)]\,dx$$

$$+\int_{\mathbb{R}^p} \{(v_n e + F_n)\cdot\nabla\xi + G_n\xi\}\,dx + I_n \quad \text{where } I_n \overset{n\to\infty}{\to} 0.$$

Now, by density, this inequality is still true for $\kappa \in W^{1,1}(\mathbb{R}^p)$, $0 \leqq \kappa \leqq 1$ almost everywhere; hence we choose κ such that $\mathrm{Tr}(\kappa) = 1 - H_0(e\cdot\nu)$ in $\partial\mathscr{O}$, where ν is the inner normal unit vector at $\partial\mathscr{O}$ and Tr is the trace operator from $W^{1,1}(\mathscr{O})$ into $L^1(\partial\mathscr{O})$. Then, this inequality becomes

$$0 \geqq \int_{\mathbb{R}^p\setminus\mathscr{O}} [(w_{1_n} - v_n)\,e\cdot\nabla(\xi\kappa) + (w_{2_n} - v_n)\,e\cdot\nabla(\xi(1-\kappa))]\,dx$$

$$-\int_{\partial\mathscr{O}} [(w_{1_n} - v_n)\,e\cdot\nu\xi(1 - H_0(e\cdot\nu)) + (w_{2_n} - v_n)\,e\cdot\nu\xi H_0(e\cdot\nu)]\,d\sigma$$

$$+\int_{\mathbb{R}^p} \{(v_n e + F_n)\cdot\nabla\xi + G_n\xi\}\,dx.$$

Of course, since $w_1 \geqq v \geqq w_2$, we deduce that

$$-[(w_{1_n} - v_n)\,e\cdot\nu\xi(1 - H_0(e\cdot\nu)) + (w_{2_n} - v_n)\,e\cdot\nu\xi H_0(e\cdot\nu)] \geqq 0$$

almost everywhere in $\partial\mathscr{O}$; hence we still have

$$0 \geqq \int_{\mathbb{R}^p\setminus\mathscr{O}} [(w_{1_n} - v_n)\,e\cdot\nabla(\xi\kappa) + (w_{2_n} - v_n)\,e\cdot\nabla(\xi(1-\kappa))]\,dx$$

$$+\int_{\mathbb{R}^p} \{(v_n e + F_n)\cdot\nabla\xi + G_n\xi\}\,dx.$$

Of course, since w_{1_n}, w_{2_n} and v_n are bounded functions converging to 0 almost everywhere in $\mathbb{R}^p \setminus \mathscr{O}$, from the Lebesgue Theorem the first integral converges to 0 when $n \to +\infty$. Hence we get

$$0 \geqq \int_{\mathbb{R}^p} \{(ve + F)\cdot\nabla\xi + G\xi\}\,dx.$$

When ξ is a nonnegative function of $H^1(\mathscr{O})$, by density we can assume that $\xi \in \mathscr{D}(\mathbb{R}^p)$ and, by means of a partition of unity we can assume that the support of ξ is contained in a ball like B.

Theorem 7. *Let* (H1), (H3) *and* (H4) *hold. Let $f \in L^\infty(\Omega)$. Then there exists at least one entropy solution u of $(P_S(f)(g,b,\phi))$. Moreover $u \in L^\infty(\Omega)$.*

Proof. Step 1. Existence of a weak solution. Let $f_n \in \mathscr{D}(\overline{\Omega})$ be such that

$$f_n \overset{n\to+\infty}{\to} f \quad \text{in} L^r(\Omega) \ \ \forall\, r \in [1, +\infty),$$

$$\|f_n\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \quad \forall\, r \in [1, +\infty).$$

Let

$$b_\varepsilon = \varepsilon I + b$$

(then $b_\varepsilon^{-1} \in \mathscr{C}^{0,1}(\mathbb{R})$) and, for $i = 1, 2$, let

$$\phi_\varepsilon^{(i)} \in \mathscr{C}^0(\mathbb{R}^2; \mathbb{R}^N), \quad \phi_\varepsilon^{(i)} \overset{\varepsilon\to 0}{\to} \phi^{(i)} \text{ in } (L^\infty_{\text{loc}}(\mathbb{R}^2))^N; \tag{23}$$

let

$$\phi_\varepsilon(s) = \phi_\varepsilon^{(2)}(g(s), b(s))\, s \ + \phi_\varepsilon^{(1)}(g(s), b(s)) \quad \forall\, s \in \mathbb{R},$$
$$\gamma_\varepsilon = g \circ b_\varepsilon^{-1}, \quad \Phi_\varepsilon = \phi_\varepsilon \circ b_\varepsilon^{-1}.$$

Obviously we can choose $\phi_\varepsilon^{(i)}$ for $i = 1, 2$ such that

$$\Phi_\varepsilon \in \mathscr{C}^{0,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^N).$$

By Theorem 5 $(P_S(f_n)(\gamma_\varepsilon, I, \Phi_\varepsilon))$ has a unique weak solution $w_{\varepsilon,n}$. Let us define

$$u_{\varepsilon,n} = b_\varepsilon^{-1}(w_{\varepsilon,n}) = (g + b_\varepsilon)^{-1}(\gamma_\varepsilon(w_{\varepsilon,n}) + w_{\varepsilon,n}).$$

Then $u_{\varepsilon,n}$ is the unique solution of $(P_S(f_n)(g, b_\varepsilon, \phi_\varepsilon))$.

Since b_ε^{-1} is Lipschitz continuous, we deduce that $u_{\varepsilon,n} \in H^1_0(\Omega)$ whence $b(u_{\varepsilon,n}) = b_\varepsilon(u_{\varepsilon,n}) - \varepsilon u_{\varepsilon,n}$ still belongs to $H^1_0(\Omega)$. Then by Theorem 1 there exist C and K such that

$$\begin{aligned} \||\nabla b(u_{\varepsilon,n})|\|_{(L^2(\Omega))^N} &\leqq \||\nabla b_\varepsilon(u_{\varepsilon,n})|\|_{(L^2(\Omega))^N} \\ &\leqq C\|f_n\|_{L^p(\Omega)} \leqq C\|f\|_{L^p(\Omega)}, \end{aligned}$$
$$\|b(u_{\varepsilon,n})\|_{L^\infty(\Omega)} \leqq \|b_\varepsilon(u_{\varepsilon,n})\|_{L^\infty(\Omega)} \leqq K\|f_n\|_{L^p(\Omega)} \leqq K\|f\|_{L^p(\Omega)}$$

for some $p > \frac{1}{2}N$ and

$$\|g(u_{\varepsilon,n})\|_{L^r(\Omega)} \leqq \|f_n\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)}$$

for any $1 \leqq r \leqq +\infty$. Moreover, almost everywhere in Ω we have

$$|g(u_{\varepsilon,n}) + b(u_{\varepsilon,n})| \leqq |g(u_{\varepsilon,n}) + b_\varepsilon(u_{\varepsilon,n})| \leqq K\|f\|_{L^p(\Omega)} + \|f\|_{L^\infty(\Omega)}.$$

Then, from (H3) we deduce the existence of $C' = C'(f)$ such that

$$\|u_{\varepsilon,n}\|_{L^\infty(\Omega)} \leqq C',$$

and from (23) there exists $C'' = C''(f)$ such that

$$\|\phi_\varepsilon(u_{\varepsilon,n})\|_{L^\infty(\Omega)} \leqq C''.$$

From Theorem 6 we obtain

$$TV_\Omega(g(u_{\varepsilon,n})) \leqq TV_\Omega(f_n) + \|f_n\|_{L^1(\Gamma)}.$$

Let us denote by $\varepsilon^{(0)}$ the sequence ε; then, for any $n \in \mathbb{N}$ there exists a subsequence of $\varepsilon^{(n-1)}$ denoted by $\varepsilon^{(n)}$ such that, by taking into account that the imbedding of $BV(\Omega)$ into $L^1(\Omega)$ and the imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$ are compact, we obtain

$$\begin{aligned}
b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\rightharpoonup} w_n && \text{weakly in } H_0^1(\Omega),\\
b_{\varepsilon}(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\to} w_n && \text{in } L^2(\Omega),\\
b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\to} w_n && \text{a.e. in } \Omega,\\
b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\rightharpoonup} w_n && \text{in } L^\infty(\Omega) \text{ weak}\star,\\
g(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\to} v_n && \text{in } L^1(\Omega),\\
g(u_{\varepsilon^{(n)},n}) &\overset{\varepsilon^{(n)}\to 0}{\to} v_n && \text{a.e. in } \Omega,\\
u_{\varepsilon^{(n)},n} &\overset{\varepsilon^{(n)}\to 0}{\rightharpoonup} u_n && \text{in } L^\infty(\Omega) \text{ weak}\star.
\end{aligned}$$

Since $g(u_{\varepsilon^{(n)},n})$ is uniformly bounded in $L^\infty(\Omega)$, it still converges to v_n in $L^2(\Omega)$ and, since $u_{\varepsilon^{(n)},n}$ converges to u_n in $L^\infty(\Omega)$ weak$\star$, it still converges in $L^2(\Omega)$ weakly. Thus we deduce (see [Br]) that

$$v_n = g(u_n) \ \text{ a.e. in } \Omega.$$

Since $u_{\varepsilon^{(n)},n}$ is uniformly bounded in $L^\infty(\Omega)$, we deduce that $\varepsilon^{(n)}u_{\varepsilon^{(n)},n}$ converges to 0 in $L^\infty(\Omega)$ and $b(u_{\varepsilon^{(n)},n}) = b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) - \varepsilon^{(n)}u_{\varepsilon^{(n)},n}$ still converges to w_n in $L^2(\Omega)$, whence we deduce that

$$w_n = b(u_n).$$

Moreover, for $i = 1, 2$ we have

$$\phi^{(i)}(g(u_{\varepsilon^{(n)},n}), b(u_{\varepsilon^{(n)},n})) \overset{\varepsilon^{(n)}\to 0}{\to} \phi^{(i)}(g(u_n), b(u_n)) \quad \text{a.e. in } \Omega,$$

and, by taking into account (23),

$$\phi^{(i)}_{\varepsilon^{(n)}}(g(u_{\varepsilon^{(n)},n}), b(u_{\varepsilon^{(n)},n})) \overset{\varepsilon^{(n)}\to 0}{\to} \phi^{(i)}(g(u_n), b(u_n)) \quad \text{a.e. in } \Omega.$$

Therefore, since $g(u_{\varepsilon^{(n)},n})$ and $b(u_{\varepsilon^{(n)},n})$ are uniformly bounded in $L^\infty(\Omega)$, we deduce that $\phi^{(i)}_{\varepsilon^{(n)}}(g(u_{\varepsilon^{(n)},n}), b(u_{\varepsilon^{(n)},n}))$ are uniformly bounded in $(L^\infty(\Omega))^N$ for $i = 1, 2$: Hence, from the Lebesgue Theorem we deduce that

$$\phi^{(i)}_{\varepsilon^{(n)}}(g(u_{\varepsilon^{(n)},n}), b(u_{\varepsilon^{(n)},n})) \overset{\varepsilon^{(n)}\to 0}{\to} \phi^{(i)}(g(u_n), b(u_n)) \quad \text{in } L^r(\Omega)$$

for any $r \in [1, +\infty)$, and in particular for $r = 2$. Therefore we get

$$\phi_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) \overset{\varepsilon^{(n)}\to 0}{\rightharpoonup} \phi(u_n) \quad \text{in } L^2(\Omega) \text{ weakly.}$$

Now let $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$; then

$$0 = \int_\Omega [(g(u_{\varepsilon^{(n)},n}) - f_n)\xi + \big(\nabla b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) - \phi_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n})\big) \cdot \nabla\xi]\,dx,$$

and by letting $\varepsilon^{(n)} \to 0$ we get

$$0 = \int_\Omega [(g(u_n) - f_n)\xi + (\nabla b(u)_n - \phi(u_n))) \cdot \nabla\xi]\,dx.$$

Hence u_n is a weak solution of $(P_S(f_n)(g, b, \phi)$. Moreover, let $n \geqq m$; then $\varepsilon^{(n)}$ is a subsequence of $\varepsilon^{(m)}$. From (19) we thus obtain

$$\begin{aligned}\|g(u_{\varepsilon^{(n)},n}) - g(u_{\varepsilon^{(n)},m})\|_{L^1(\Omega)} &= \|\gamma_{\varepsilon^{(n)}}(w_{\varepsilon^{(n)},n}) - \gamma_{\varepsilon^{(n)}}(w_{\varepsilon^{(n)},m})\|_{L^1(\Omega)} \\ &\leqq \|f_n - f_m\|_{L^1(\Omega)} \quad \forall\, \varepsilon^{(n)}.\end{aligned}$$

When $\varepsilon^{(n)}$ converges to 0, we thus have

$$\|g(u_n) - g(u_m)\|_{L^1(\Omega)} \leqq \|f_n - f_m\|_{L^1(\Omega)}.$$

Hence we deduce that $g(u_n)$ is a Cauchy sequence in $L^1(\Omega)$. Moreover,

$$\begin{gathered}\|\nabla b(u_n)\|_{L^2(\Omega)} \leqq C\|f\|_{L^p(\Omega)}, \quad \|b(u_n)\|_{L^\infty(\Omega)} \leqq K\|f\|_{L^p(\Omega)}, \\ \|g(u_n)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \quad \forall\, 1 \leqq r \leqq +\infty, \\ \|u_n\|_{L^\infty(\Omega)} \leqq C', \quad \|\phi(u_n)\|_{L^\infty(\Omega)} \leqq C''.\end{gathered}$$

Then there exists a subsequence of n, still denoted by n, such that

$$\begin{aligned} b(u_n) &\overset{n\to+\infty}{\rightharpoonup} w && \text{weakly in } H_0^1(\Omega), \\ b(u_n) &\overset{n\to+\infty}{\to} w && \text{in } L^2(\Omega), \\ b(u_n) &\overset{n\to+\infty}{\to} w && \text{a.e. in } \Omega, \\ g(u_n) &\overset{n\to+\infty}{\to} v && \text{in } L^1(\Omega), \\ g(u_n) &\overset{n\to+\infty}{\to} v && \text{a.e. in } \Omega, \\ u_n &\overset{n\to+\infty}{\to} u && \text{in } L^\infty(\Omega) \text{ weak}\star. \end{aligned}$$

Arguing as above we deduce that

$$w = b(u) \text{ a.e. in } \Omega, \quad v = g(u) \text{ a.e. in } \Omega,$$

$$\phi^{(i)}(g(u_n), b(u_n)) \overset{n\to+\infty}{\to} \phi^{(i)}(g(u), b(u) \quad \text{in } L^r(\Omega)$$
$$\text{for any } r \in [1, +\infty) \text{ and for } i = 1, 2,$$

$$\phi(u_n) \overset{n\to+\infty}{\rightharpoonup} \phi(u) \quad \text{in } L^2(\Omega) \text{ weak.}$$

Then

$$0 = \int_\Omega [(g(u) - f)\xi + (\nabla b(u) - \phi(u))) \cdot \nabla\xi]\, dx$$

for any $\xi \in H_0^1(\Omega)$, and thus u is a weak solution of $(P_S(f)(g, b, \phi))$.

Step 2. u is an entropy solution of $(P_S(f)(g, b, \phi))$. Let $J \subset \mathbb{N}$ and let $\{r_i \in \mathbb{R}/ i \in J\}$ be the set where $((g + b)^{-1})_0$ is discontinuous. Then

$$(g + b)^{-1} = c + \sum_{i\in J} \mathscr{H}_i$$

where $c \in \mathscr{C}(\mathbb{R})$, $c(0) = 0$, where

$$\mathscr{H}_i(s) = \begin{cases} \lambda_i(H(s - r_i) - 1) & \text{for } r_i < 0, \\ \lambda_i H(s - r_i) & \text{for } r_i > 0, \\ \lambda_i(H(s - r_i) - \mu) & \text{for } r_i = 0, \end{cases}$$

where $0 \leqq \mu \leqq 1$, $\lambda_i > 0$ for any $i \in J$ and where

$$\sum_{\substack{i\in J \\ r_i \in K}} \lambda_i < +\infty$$

for any bounded subset K. Then

$$u_{\varepsilon^{(n)},n} = c(g(u_{\varepsilon^{(n)},n}) + b(u_{\varepsilon^{(n)},n})) + \sum_{i\in J} \chi_{i_{\varepsilon^{(n)},n}}$$

with $\chi_{i_{\varepsilon^{(n)},n}} \in \mathscr{H}_i(g(u_{\varepsilon^{(n)},n}) + b(u_{\varepsilon^{(n)},n}))$, and

$$u = c(g(u) + b(u)) + \sum_{i\in J} \chi_i$$

with $\chi_i \in \mathscr{H}_i(g(u) + b(u))$. Since

$$u = \lim_{n\to+\infty} \lim_{\varepsilon^{(n)}\to 0} u_{\varepsilon^{(n)},n} \quad \text{in } L^\infty(\Omega) \text{ weak}\star,$$

we easily check that

$$\chi_i = \lim_{n\to+\infty} \lim_{\varepsilon^{(n)}\to 0} \chi_{i_{\varepsilon^{(n)},n}} \quad \text{in } L^\infty(\Omega) \text{ weak}\star\ \forall\, i \in J.$$

Now let

$$\mathscr{A} = \{s = \max(g+b)^{-1}(r) / r \in \mathbb{R}\},$$

$$\mathscr{B} = \{s = \min(g+b)^{-1}(r) / r \in \mathbb{R}, \}$$

We deduce that, for any $s \in \mathscr{A}$,

$$\begin{aligned}
&\lim_{n\to+\infty} \lim_{\varepsilon^{(n)}\to 0} (u_{\varepsilon^{(n)},n} - s)^+ \\
&\quad = \lim_{n\to+\infty} \lim_{\varepsilon^{(n)}\to 0} \Bigg[(c(g(u_{\varepsilon^{(n)},n}) + b(u_{\varepsilon^{(n)},n})) \\
&\qquad\qquad - c(g(s)+b(s)))^+ + \sum_{\substack{i\in J \\ g(s)+b(s)<r_i}} \chi_{i_{\varepsilon^{(n)},n}} \Bigg] \\
&\quad = (c(g(u)+b(u)) - c(g(s)+b(s))^+ + \sum_{\substack{i\in J \\ g(s)+b(s)<r_i}} \chi_i = (u-s)^+ \qquad (24)
\end{aligned}$$

where the limits are taken in the $L^\infty(\Omega)$ weak$\star$ topology.

From Theorem 2 the unique weak solution of $(P_S(f_n(g, b_\varepsilon, \phi_\varepsilon)))$ is an entropy solution. Then we have

$$\begin{aligned}
0 \geqq &\int_\Omega H_0(u_{\varepsilon^{(n)},n} - s)\big[(g(u_{\varepsilon^{(n)},n}) - f_n)\xi \\
&+ (\nabla b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) + \phi_{\varepsilon^{(n)}}(s) - \phi_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n})) \cdot \nabla\xi\big]\, dx \qquad (25)
\end{aligned}$$

for any $(s, \xi) \in \mathbb{R} \times H^1(\Omega)$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

Since H_0 is bounded, there exists a subsequence of $(\varepsilon^{(n)}, n)$, still denoted by $(\varepsilon^{(n)}, n)$, such that

$$H_0(u_{\varepsilon^{(n)},n} - s) \rightharpoonup \chi_{u,s} \text{ in } L^\infty(\Omega) \text{ weak}\,\star.$$

Moreover, since

$$H_0(u_{\varepsilon^{(n)},n} - s) \in H(g(u_{\varepsilon^{(n)},n}) + b(u_{\varepsilon^{(n)},n}) - g(s) - b(s)),$$

we deduce (see [Br]) that

$$\chi_{u,s} \in H(g(u) + b(u) - g(s) - b(s)).$$

Then, for any $\xi \in H^1(\Omega)$,

$$\int_\Omega H_0(u_{\varepsilon^{(n)},n} - s)(g(u_{\varepsilon^{(n)},n}) - f_n)\xi\, dx \to \int_\Omega \chi_{u,s}(g(u) - f)\xi\, dx,$$

$$\int_\Omega H_0(u_{\varepsilon(n),n} - s)\nabla b_{\varepsilon(n)}(u_{\varepsilon(n),n}) \cdot \nabla\xi\, dx$$
$$= \int_\Omega \nabla(b_{\varepsilon(n)}(u_{\varepsilon(n),n}) - b_{\varepsilon(n)}(s))^+ \cdot \nabla\xi\, dx$$
$$\to \int_\Omega \nabla(b(u) - b(s))^+ \cdot \nabla\xi\, dx = \int_\Omega \chi_{u,s}\nabla b(u) \cdot \nabla\xi\, dx.$$

Moreover,

$$H_0(u_{\varepsilon(n),n} - s)(\phi_{\varepsilon(n)}(s) - \phi_{\varepsilon(n)}(u_{\varepsilon(n),n}))$$
$$= \phi^{(1)}_{\varepsilon(n)}(g(s), b(s)) - \phi^{(1)}_{\varepsilon(n)}(\max(g(s), g(u_{\varepsilon(n),n})), \max(b(s), b(u_{\varepsilon(n),n})))$$
$$+ u_{\varepsilon(n),n}(\phi^{(2)}_{\varepsilon(n)}(g(s), b(s))$$
$$- \phi^{(2)}_{\varepsilon(n)}(\max(g(s), g(u_{\varepsilon(n),n})), \max(b(s), b(u_{\varepsilon(n),n}))))$$
$$- (u_{\varepsilon(n),n} - s)^+\phi^{(2)}_{\varepsilon(n)}(g(s), b(s)).$$

Obviously,

$$\int_\Omega (\phi^{(1)}_{\varepsilon(n)}(g(s), b(s)) - \phi^{(1)}_{\varepsilon(n)}(\max(g(s), g(u_{\varepsilon(n),n})), \max(b(s), b(u_{\varepsilon(n),n})))$$
$$+ u_{\varepsilon(n),n}(\phi^{(2)}_{\varepsilon(n)}(g(s), b(s))$$
$$- \phi^{(2)}_{\varepsilon(n)}(\max(g(s), g(u_{\varepsilon(n),n})), \max(b(s), b(u_{\varepsilon(n),n}))))) \cdot \nabla\xi\, dx$$
$$\to \int_\Omega (\phi^{(1)}(g(s), b(s)) - \phi^{(1)}(\max(g(s), g(u)), \max(b(s), b(u)))$$
$$+ u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(\max(g(s), g(u)), \max(b(s), b(u))))) \cdot \nabla\xi\, dx$$
$$= \int_\Omega H_0(u - s)[\phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u))$$
$$+ u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi\, dx,$$

and, from (24), for $s \in \mathscr{A}$,

$$\int_\Omega (u_{\varepsilon(n),n} - s)^+\phi^{(2)}_{\varepsilon(n)}(g(s), b(s)) \cdot \nabla\xi\, dx$$
$$\to \int_\Omega (u - s)^+\phi^{(2)}(g(s), b(s)) \cdot \nabla\xi\, dx.$$

Then, for some $\chi_{u,s} \in H(g(u) + b(u) - g(s) - b(s))$ we get

$$0 \geqq \int_\Omega \{H_0(u - s)(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + \chi_{u,s}(g(u) - f)\xi\}\, dx$$

for any $(s, \xi) \in \mathscr{A} \times H^1(\Omega)$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathscr{A} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

Moreover, for any $s \in \mathscr{A}$ we can find a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n = (g+b)^{-1}(g(s_n)+b(s_n))$ (i.e., $s_n \in \mathscr{A} \cap \mathscr{B}$) such that $s_n > s$ and $s_n \searrow s$ when $n \to +\infty$. Then $\chi_{u,s_n} \in H(u - s_n) = H(g(u)+b(u)-g(s_n)-b(s_n))$, $\chi_{u,s_n} \nearrow H_0(u-s)$ and $H_0(u-s_n) \nearrow H_0(u-s)$ almost everywhere in Ω, and therefore,

$$0 \geqq \int_\Omega H_0(u-s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx \tag{26}$$

for any $(s, \xi) \in \mathscr{A} \times H^1(\Omega)$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathscr{A} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

Now, let $s \in \mathscr{B}$. Then there exist $(s_n)_{n\in\mathbb{N}}$ such that $s_n = (g+b)^{-1}(g(s_n) + b(s_n))$ (i.e., $s_n \in \mathscr{A} \cap \mathscr{B}$) such that $s_n < s$ and $s_n \nearrow s$. Hence $\chi_{u,s_n} \in H(u-s_n) = H(g(u)+b(u)-g(s_n)-b(s_n))$ and $\chi_{u,s_n} \searrow H_{\mathrm{Max}}(u-s)$ almost everywhere in Ω. Then we get

$$0 \geqq \int_\Omega H_{\mathrm{Max}}(u-s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx \tag{27}$$

for any $(s, \xi) \in \mathscr{B} \times H^1(\Omega)$ such that $s > 0$ and $\xi \geqq 0$ and for any $(s, \xi) \in \mathscr{B} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

Step 2.1. $\xi \in H_0^1(\Omega)$. Now let $i \in J$ and let $[m, M] = (g+b)^{-1}(r_i)$. Since $M \in \mathscr{A}$ and $m \in \mathscr{B}$, we have

$$0 \geqq \int_\Omega H_0(u-M)\{(\nabla b(u) + \phi(M) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx,$$

$$0 \geqq \int_\Omega H_{\mathrm{Max}}(u-m)\{(\nabla b(u) + \phi(m) - \phi(u)) \cdot \nabla\xi + (g(u) - f)\xi\}\,dx$$

for any nonnegative $\xi \in H_0^1(\Omega)$. Since we have

$$g(M) = g(s) = g(m), \quad b(M) = b(s) = b(m),$$

$$\begin{aligned} &H_0(u-M)\{\nabla b(u) + \phi^{(1)}(g(M), b(M)) - \phi^{(1)}(g(u), b(u)) \\ &\qquad + u(\phi^{(2)}(g(M), b(M)) - \phi^{(2)}(g(u), b(u)))\} \\ &\quad = H_0(u-s)\{\nabla b(u) + \phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u)) \\ &\qquad + u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))\} \\ &\quad = H_{\mathrm{Max}}(u-m)\{\nabla b(u) + \phi^{(1)}(g(m), b(m)) - \phi^{(1)}(g(u), b(u)) \\ &\qquad + u(\phi^{(2)}(g(m), b(m)) - \phi^{(2)}(g(u), b(u)))\}, \end{aligned} \tag{28}$$

for any $s \in (g+b)^{-1}(r_i)$ we deduce that

$$
\begin{aligned}
0 \geqq \int_\Omega \{ & H_0(u-s)[\nabla b(u) + \phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u)) \\
& + u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi \\
& - (u-M)^+ \phi^{(2)}(g(s), b(s)) \cdot \nabla \xi + H_0(u-M)(g(u)-f)\xi \}\, dx,
\end{aligned}
$$

$$
\begin{aligned}
0 \geqq \int_\Omega \{ & H_0(u-s)[\nabla b(u) + \phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u)) \\
& + u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi \\
& - (u-m)^+ \phi^{(2)}(g(s), b(s)) \cdot \nabla \xi + H_{\mathrm{Max}}(u-m)(g(u)-f)\xi \}\, dx
\end{aligned}
$$

for any nonnegative $\xi \in H_0^1(\Omega)$.

Then we can apply Lemma 2 with

$$
\begin{aligned}
F &= H_0(u-M)\{\nabla b(u) + \phi^{(1)}(g(M), b(M)) - \phi^{(1)}(g(u), b(u)) \\
&\quad + u(\phi^{(2)}(g(M), b(M)) - \phi^{(2)}(g(u), b(u)))\}, \\
G_1 &= H_0(u-M)(g(u)-f), \\
G_2 &= H_{\mathrm{Max}}(u-m)(g(u)-f), \\
e &= -\phi^{(2)}(g(M), b(M))
\end{aligned}
$$

to get

$$
\begin{aligned}
0 \geqq \int_\Omega \{ & H_0(u-s)[\nabla b(u) + \phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u)) \\
& + u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi - (u-s)^+ \phi^{(2)}(g(s), b(s)) \cdot \nabla \xi \\
& + [H_{\mathrm{Max}}(u-m)\tilde{H}(u-s) + H_0(u-M)(1-\tilde{H}(u-s))](g(u)-f)\xi \}\, dx
\end{aligned}
$$

for any $s \in [m, M]$, for any nonnegative $\xi \in H_0^1(\Omega)$, and for some $\tilde{H}(u-s) \in H(u-s)$.

Moreover, since

$$
\begin{aligned}
H_{\mathrm{Max}}(u-m)\tilde{H}(u-s) &= \tilde{H}(u-s), \\
H_0(u-M)(1-\tilde{H}(u-s)) &= 0,
\end{aligned}
$$

we have

$$
0 \geqq \int_\Omega \{ H_0(u-s)[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla \xi + \tilde{H}(u-s)(g(u)-f)\xi \}\, dx
$$

for any $s \in [m, M]$, for any nonnegative $\xi \in H_0^1(\Omega)$, and for some $\tilde{H}(u-s) \in H(u-s)$.

For any $s \in [m, M)$ there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \in [m, M)$, $s_n > s$, $s_n \searrow s$; then $\tilde{H}(u - s_n) \nearrow H_0(u - s)$ and $H_0(u - s_n) \nearrow H_0(u - s)$ almost everywhere in Ω. Hence we get

$$0 \geqq \int_\Omega H_0(u - s)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u) - f)\xi\}\, dx$$

for any $s \in [m, M)$ and for any nonnegative $\xi \in H_0^1(\Omega)$. Then, by taking into account (26) we get

$$0 \geqq \int_\Omega H_0(u - s)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u) - f)\xi\}\, dx \tag{29}$$

for and $s \in \mathbb{R}$ and for any nonnegative $\xi \in H_0^1(\Omega)$.

Step 2.2. $s = 0$. Now, for $s = 0$, from (25) we have

$$\begin{aligned} 0 \geqq \int_\Omega & H_0(u_{\varepsilon^{(n)},n})[(g(u_{\varepsilon^{(n)},n}) - f_n)\xi \\ & + (\nabla b_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n}) - \phi_{\varepsilon^{(n)}}(u_{\varepsilon^{(n)},n})) \cdot \nabla\xi]\, dx \end{aligned}$$

for any nonnegative $\xi \in H^1(\Omega)$.

Let us set

$$\tilde{u} = \lim_{n\to+\infty} \lim_{\varepsilon^{(n)}\to 0} u^+_{\varepsilon^{(n)},n} \quad \text{in } L^\infty(\Omega) \text{ weak}\star.$$

Then by letting $\varepsilon^{(n)} \to 0$ and then $n \to +\infty$ in the last inequality, we get

$$\begin{aligned} 0 \geqq \int_\Omega & \{H_0(u)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) \\ & + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi \\ & - \tilde{u}\phi^{(2)}(0,0) \cdot \nabla\xi + \chi_{u,0}(g(u) - f)\xi\}\, dx \end{aligned} \tag{30}$$

for some $\chi_{u,0} \in H(g(u) + b(u))$ and for any nonnegative $\xi \in H^1(\Omega)$.

Let $[m, M] = (g + b)^{-1}(0)$. Then $M \in \mathscr{A}$ and from (26) we have

$$\begin{aligned} 0 \geqq \int_\Omega & \{H_0(u - M)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) \\ & + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi \\ & - (u - M)^+\phi^{(2)}(0,0) \cdot \nabla\xi + H_0(u - M)(g(u) - f)\xi\}\, dx. \end{aligned}$$

Hence, by taking into account that

$$\begin{aligned} & H_0(u - M)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi \\ & \quad = H_0(u)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi, \end{aligned}$$

we deduce that

$$0 \geqq \int_\Omega \{H_0(u)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) \\ + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi \\ - (u-M)^+ \phi^{(2)}(0,0) \cdot \nabla \xi + H_0(u-M)(g(u) - f)\xi\}\, dx \tag{31}$$

for any nonnegative $\xi \in H^1(\Omega)$.

From (29) we deduce that

$$0 \geqq \int_\Omega \{H_0(u)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi \\ - (u)^+ \phi^{(2)}(0,0) \cdot \nabla \xi + H_0(u)(g(u) - f)\xi\}\, dx$$

for any nonnegative $\xi \in H_0^1(\Omega)$. Then, from (30), (31) and Lemma 3 we deduce that

$$0 \geqq \int_\Omega \{H_0(u)[\nabla b(u) - \phi^{(1)}(g(u), b(u)) + u(\phi^{(2)}(0,0) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla \xi \\ - (u)^+ \phi^{(2)}(0,0) \cdot \nabla \xi + H_0(u)(g(u) - f)\xi\}\, dx$$

for any nonnegative $\xi \in H^1(\Omega)$ or equivalently

$$0 \geqq \int_\Omega H_0(u)\{(\nabla b(u) - \phi(u)) \cdot \nabla \xi + (g(u) - f)\xi\}\, dx \tag{32}$$

for any nonnegative $\xi \in H^1(\Omega)$.

Step 2.3. $s \geqq 0$ and $\xi \in H^1(\Omega)$. Let $r_i \geqq 0$, $i \in J$, let $[m, M] = (g+b)^{-1}(r_i) \cap [0, +\infty)$. From (26) we have

$$0 \geqq \int_\Omega H_0(u-M)\{[\nabla b(u) + \phi^{(1)}(g(M), b(M)) - \phi^{(1)}(g(u), b(u)) \\ + u(\phi^{(2)}(g(M), b(M)) - \phi^{(2)}(g(u), b(u))) \\ - (u-M)\phi^{(2)}(g(M), b(M))] \cdot \nabla \xi + (g(u) - f)\xi\}\, dx$$

for any nonnegative $\xi \in H^1(\Omega)$. From (27) and (32) we have

$$0 \geqq \int_\Omega K(u-m)\{[\nabla b(u) + \phi^{(1)}(g(m), b(m)) - \phi^{(1)}(g(u), b(u)) \\ + u(\phi^{(2)}(g(m), b(m)) - \phi^{(2)}(g(u), b(u))) \\ - (u-m)\phi^{(2)}(g(m), b(m))] \cdot \nabla \xi + (g(u) - f)\xi\}\, dx$$

for any nonnegative $\xi \in H^1(\Omega)$, where

$$K(u-m) = \begin{cases} H_0(u-m) & \text{for } m = 0, \\ H_{\mathrm{Max}}(u-m) & \text{for } m > 0. \end{cases}$$

Then, by taking into account (28), we can apply Lemma 2 with

$$\begin{aligned} F &= H_0(u-M)[\nabla b(u) + \phi^{(1)}(g(M), b(M)) - \phi^{(1)}(g(u), b(u)) \\ &\quad + u(\phi^{(2)}(g(M), b(M)) - \phi^{(2)}(g(u), b(u)))], \\ G_1 &= H_0(u-M)(g(u)-f), \\ G_2 &= K(u-m)(g(u)-f), \\ e &= -\phi^{(2)}(g(s), b(s)). \end{aligned}$$

to get

$$\begin{aligned} 0 \geqq \int_\Omega &\{H_0(u-s)[\nabla b(u) + \phi^{(1)}(g(s), b(s)) - \phi^{(1)}(g(u), b(u)) \\ &+ u(\phi^{(2)}(g(s), b(s)) - \phi^{(2)}(g(u), b(u)))] \cdot \nabla\xi - (u-s)^+ \phi^{(2)}(g(s), b(s)) \cdot \nabla\xi \\ &+ [K(u-m)\tilde{H}(u-s) + H_0(u-M)(1-\tilde{H}(u-s))](g(u)-f)\xi\}\,dx \end{aligned}$$

for any nonnegative $\xi \in \mathscr{D}(\mathbb{R}^N)$.

We easily check that

$$\tilde{H}(u-s)K(u-m) = \tilde{H}(u-s) \quad \text{for } m > 0 \text{ and for } 0 = m < s,$$
$$H_0(u-M)(1-\tilde{H}(u-s)) = 0,$$
$$(u-M)^+ + ((u-m)^+ + m - s - (u-M)^+)^+ = (u-s)^+,$$

whence

$$0 \geqq \int_\Omega \{H_0(u-s)[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + \tilde{H}(u-s)(g(u)-f)\xi\}\,dx$$

for any $s \in [m, M)$, $s > 0$, for any nonnegative $\xi \in \mathscr{D}(\mathbb{R}^N)$ and, by density, for any nonnegative $\xi \in H^1(\Omega)$.

For any $s \in [m, M)$ there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \in [m, M)$, $s_n > s$, $s_n \searrow s$; then $\tilde{H}u - s_n) \nearrow H_0(u-s)$ and $H_0(u-s_n) \nearrow H_0(u-s)$ almost everywhere in Ω. Hence we get

$$0 \geqq \int_\Omega H_0(u-s)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u)-f)\xi\}\,dx$$

for any $s \in [m, M)$, $s \geqq 0$, and for any nonnegative $\xi \in H^1(\Omega)$. Then, by taking into account (26) and (32) we get

$$0 \geqq \int_\Omega H_0(u-s)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u)-f)\xi\}\,dx$$

for any nonnegative $s \in \mathbb{R}$ and for any nonnegative $\xi \in H^1(\Omega)$.

Then we have proved that

$$0 \geqq \int_\Omega H_0(u-s)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u)-f)\xi\}\,dx$$

for any $(s,\xi) \in \mathbb{R} \times H^1(\Omega)$ such that $s \geqq 0$ and $\xi \geqq 0$ and for any $(s,\xi) \in \mathbb{R} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

By taking into account Remark 4 and the fact that $u_{\varepsilon(n),n}$ is an entropy solution, and by arguing as above we still prove

$$0 \leqq \int_\Omega H_0(s-u)\{[\nabla b(u) + \phi(s) - \phi(u)] \cdot \nabla\xi + (g(u) - f)\xi\}\, dx$$

for any $(s,\xi) \in \mathbb{R} \times H^1(\Omega)$ such that $s \leqq 0$ and $\xi \geqq 0$; and for any $(s,\xi) \in \mathbb{R} \times H_0^1(\Omega)$ such that $\xi \geqq 0$.

3. Comparison and Uniqueness

Theorem 8. *Let* (H1) *and* (H2) *hold, let f_1 and $f_2 \in L^1(\Omega)$ and let u_i be an entropy solution of $(P_S(f_i)(g,b,\phi))$ for $i = 1, 2$. Then there exists $\kappa \in H(u_1 - u_2)$ such that*

$$\int_\Omega \kappa(f_1 - f_2)\xi\, dx \geqq \int_\Omega \{(g(u_1) - g(u_2))^+\xi + \nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi \\ - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla\xi\}\, dx \tag{33}$$

for any nonnegative $\xi \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let x and y be two different generic variables in $\mathbb{R}^N$ and let

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_N}\right),$$
$$\nabla_y = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_N}\right).$$

We consider that f_1 and u_1 depend only on x and that f_2 and u_2 depend only on y. Let E be defined as in (4). Then

$$\nabla_x b(u_1) = 0 \quad \text{a.e. in } \mathscr{O}_1 = \{x \in \Omega /\ b(u_1(x)) \in E\}, \tag{34}$$

$$\nabla_y b(u_2) = 0 \quad \text{a.e. in } \mathscr{O}_2 = \{y \in \Omega /\ b(u_2(y)) \in E\}. \tag{35}$$

From (8) and (9) we have

$$H_0(u_1 - u_2) = H_0(b(u_1) - b(u_2)) \\ \text{a.e. in } ((\Omega \setminus \mathscr{O}_1) \times \Omega) \cup (\Omega \times (\Omega \setminus \mathscr{O}_2)). \tag{36}$$

We consider a nonnegative function $\xi \in \mathscr{D}(\Omega \times \Omega)$. We replace u by u_1 and s by u_2 in (10) and we integrate over $\mathscr{O}_2$. We replace u by u_1 and s by u_2 in (5) of Lemma 1 and we integrate over $\Omega \setminus \mathscr{O}_2$. Then we get

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_1)-f_1)\xi + \nabla_x b(u_1)\cdot\nabla_x\xi - (\phi(u_1)-\phi(u_2))\cdot\nabla_x\xi\}\,dx\,dy \leqq -\lim_{\varepsilon\to 0}\int_{\Omega\times(\Omega\setminus\mathscr{O}_2)} |\nabla_x b(u_1)|^2\; H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dx\,dy \leqq 0,$$

and by taking into account (34),

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_1)-f_1)\xi + \nabla_x b(u_1)\cdot\nabla_x\xi - (\phi(u_1)-\phi(u_2))\cdot\nabla_x\xi\}\,dx\,dy \leqq -\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} |\nabla_x b(u_1)|^2\; H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dxdy \leqq 0. \quad (37)$$

Moreover, for $\xi\in\mathscr{D}(\Omega\times\Omega)$ we have

$$\int_\Omega \nabla_x b(u_1)\cdot\nabla_y(H_\varepsilon(b(u_1)-b(u_2))\xi)dy = 0$$

for almost every $x\in\Omega$, whence, by taking into account (34)–(36) we deduce that

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\nabla_x b(u_1)\cdot\nabla_y\xi\,dx\,dy = \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \nabla_x b(u_1)\cdot\nabla_y b(u_2)\,H'_\varepsilon(b(u_1)-b(u_2))\xi\,dx\,dy. \quad (38)$$

Then, from (37) and (38) we get

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_1)-f_1)\xi + \nabla_x b(u_1)\cdot(\nabla_x\xi+\nabla_y\xi) - (\phi(u_1)-\phi(u_2))\cdot\nabla_x\xi\}\,dx\,dy \leqq -\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \Big(|\nabla_x b(u_1)|^2 - \nabla_x b(u_1)\cdot\nabla_y b(u_2)\Big) H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dx\,dy \quad (39)$$

for any nonnegative $\xi\in\mathscr{D}(\Omega\times\Omega)$.

Now we replace u by u_2 and $-s$ by u_1 in (11) and we integrate over $\mathscr{O}_1$. We replace u by u_2 and $-s$ by u_1 in (6) and we integrate over $\Omega\setminus\mathscr{O}_1$. Then we get

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_2)-f_2)\xi + \nabla_y b(u_2)\cdot\nabla_y\xi - (\phi(u_2)-\phi(u_1))\cdot\nabla\xi\}\,dx\,dy \geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times\Omega} \left|\nabla_y b(u_2)\right|^2 H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dx\,dy \geqq 0,$$

and by taking into account (35),

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_2)-f_2)\xi+\nabla_y b(u_2)\cdot\nabla_y\xi \\ -(\phi(u_2)-\phi(u_1))\cdot\nabla\xi\}\,dx\,dy \\ \geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \left|\nabla_y b(u_2)\right|^2 H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dx\,dy \geqq 0 \quad (40)$$

for any nonnegative $\xi\in\mathscr{D}(\Omega\times\Omega)$. Moreover, for $\xi\in\mathscr{D}(\Omega\times\Omega)$ we have

$$\int_\Omega \nabla_y b(u_2)\cdot\nabla_x(H_\varepsilon(b(u_1)-b(u_2))\xi)\,dx=0$$

for almost every $y\in\Omega$, whence, taking into account (34)–(36), we deduce that

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\nabla_y b(u_2)\cdot\nabla_x\xi\,dx\,dy \\ =-\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \nabla_y b(u_2)\cdot\nabla_x b(u_1)\,H'_\varepsilon(b(u_1)-b(u_2))\xi\,dx\,dy. \quad (41)$$

Now, from (40) and (41) we get

$$\int_{\Omega\times\Omega} H_0(u_1-u_2)\{(g(u_2)-f_2)\xi+\nabla_y b(u_2)\cdot(\nabla_y\xi+\nabla_x\xi) \\ -(\phi(u_2)-\phi(u_1))\cdot\nabla_y\xi\}\,dx\,dy \\ \geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \left(\left|\nabla_y b(u_2)\right|^2-\nabla_y b(u_2)\cdot\nabla_x b(u_1)\right) \\ H'_\varepsilon(b(u_1)-b(u_2))\,\xi\,dx\,dy. \quad (42)$$

By forming the difference between (39) and (42) we get

$$\int_{\Omega\times\Omega}\{(g(u_1)-g(u_2))^+\xi+(\nabla_x+\nabla_y)(b(u_1)-b(u_2))^+\cdot(\nabla_x+\nabla_y)\xi \\ -H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot(\nabla_x+\nabla_y)\xi\}\,dx\,dy \\ \leqq \int_{\Omega\times\Omega} H_0(u_1-u_2)(f_1-f_2)\xi\,dx\,dy \\ -\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)} \left|\nabla_x b(u_1)-\nabla_y b(u_2)\right|^2 H'_\varepsilon(b(u_1)-b(u_2))\xi\,dx\,dy \\ \leqq \int_{\Omega\times\Omega} H_0(u_1-u_2)(f_1-f_2)\xi\,dx\,dy \quad (43)$$

for any nonnegative $\xi\in\mathscr{D}(\Omega\times\Omega)$.

Now let $\zeta\in\mathscr{D}(\Omega)$, $\zeta\geqq 0$ and let ρ_n be a sequence of mollifiers with n so large that ξ_n belongs to $\mathscr{D}(\Omega\times\Omega)$, where

$$\xi_n(x,y)=\zeta\left(\frac{x+y}{2}\right)\rho_n\left(\frac{x-y}{2}\right).$$

From (43) we have

$$\int_{\Omega\times\Omega}\{(g(u_1)-g(u_2))^+\xi_n+(\nabla_x+\nabla_y)(b(u_1)-b(u_2))^+\cdot(\nabla_x+\nabla_y)\xi_n - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot(\nabla_x+\nabla_y)\xi_n\}\,dx\,dy \leqq \int_{\Omega\times\Omega}H_0(u_1-u_2)(f_1-f_2)\xi_n\,dx\,dy. \quad (44)$$

We set

$$(X,Y)=T(x,y)=\left(\frac{x+y}{2},\frac{x-y}{2}\right),\quad \nabla=\left(\frac{\partial}{\partial X_1},\frac{\partial}{\partial X_2},\cdots,\frac{\partial}{\partial X_N}\right),$$

whence, from (44) we deduce that

$$\int_{T(\Omega\times\Omega)}\{(g(u_1)-g(u_2))^+\zeta+\nabla(b(u_1)-b(u_2))^+\cdot\nabla\zeta - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\zeta\}\rho_n\,dXdY \leqq \int_{T(\Omega\times\Omega)}H_0(u_1-u_2)(f_1-f_2)\zeta\rho_n\,dX\,dY$$

where $\nabla=(\frac{\partial}{\partial X_1},\frac{\partial}{\partial X_2},\cdots,\frac{\partial}{\partial X_N})$. Then, by letting $n\to+\infty$ we get (33).

Moreover, the inequality (33) is still true for any nonnegative ξ belonging to $H^1(\Omega)\cap L^\infty(\Omega)$:

Theorem 9. *Let* (H1) *and* (H2) *hold, let f_1 and $f_2\in L^1(\Omega)$ and let u_i be an entropy solution of $(P_S(f_i)(g,b,\phi))$ for $i=1,2$. Then there exists $\kappa\in H(u_1-u_2)$ such that*

$$\int_\Omega\kappa(f_1-f_2)\xi\,dx \geqq \int_\Omega\left\{(g(u_1)-g(u_2))^+\xi+\nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\xi\right\}dx \quad (45)$$

for any nonnegative $\xi\in H^1(\Omega)\cap L^\infty(\Omega)$.

Proof. We consider two different variables x and y of $\mathbb{R}^N$ and we assume that $u_1=u_1(x)$, $f_1=f_1(x)$, $u_2=u_2(y)$ and $f_2=f_2(y)$. We use the same notation as that in the proof of Theorem 8.

From (5) and from (10) we deduce that inequality (37) is still true for any nonnegative $\zeta\in\mathscr{D}(\mathbb{R}^N\times\mathbb{R}^N)$ when replacing u_2 by any measurable function $v=v(y)$ satisfying

$$x\mapsto H_\varepsilon(u_1-v)\zeta\in H_0^1(\Omega)\quad\text{for a.e. } y\in\Omega$$

and $\mathscr{O}_2$ by $\{y \in \Omega / b(v(y)) \in E\}$. In particular, let $\zeta \in \mathscr{D}(\overline{\Omega} \times \Omega)$, $\zeta \geqq 0$, and let us replace u_2 by u_2^+ and $\mathscr{O}_2$ by $\mathscr{O}_2^+ = \{y \in \Omega / b(u_2^+(y)) \in E\}$. Then we get

$$\begin{aligned}\int_{\Omega\times\Omega} & H_0(u_1^+ - u_2^+)\{(g(u_1^+) - f_1)\zeta + \nabla_x b(u_1^+)\cdot\nabla_x\zeta \\ & \qquad - (\phi(u_1^+) - \phi(u_2^+))\cdot\nabla_x\zeta\}\,dx\,dy \\ & \leqq -\lim_{\varepsilon\to 0}\int_{(\Omega\backslash\mathscr{O}_1^+)\times(\Omega\backslash\mathscr{O}_2^+)} \left|\nabla_x b(u_1^+)\right|^2 H_\varepsilon'(b(u_1^+) - b(u_2^+))\,\zeta\,dx\,dy, \quad (46)\end{aligned}$$

where $\mathscr{O}_1^+ = \{x \in \Omega / b(u_1^+(x)) \in E\}$. Moreover, for such function $\zeta \in \mathscr{D}(\overline{\Omega} \times \Omega)$ we have

$$\int_\Omega \nabla_x b(u_1^+)\cdot\nabla_y(H_\varepsilon(b(u_1^+) - b(u_2^+))\zeta)\,dy = 0.$$

Then, by taking into account (34)–(36) we get

$$\begin{aligned}&\int_{\Omega\times\Omega} H_0(u_1^+ - u_2^+)\nabla_x b(u_1^+)\cdot\nabla_y\zeta\,dx\,dy \\ &= \int_{\Omega\times\Omega} H_0(b(u_1^+) - b(u_2^+))\nabla_x b(u_1^+)\cdot\nabla_y\zeta\,dx\,dy \qquad (47)\\ &= \lim_{\varepsilon\to 0}\int_{(\Omega\backslash\mathscr{O}_1^+)\times(\Omega\backslash\mathscr{O}_2^+)} \nabla_x b(u_1^+)\cdot\nabla_y b(u_2^+) H_\varepsilon'(b(u_1^+) - b(u_2^+))\zeta\,dx\,dy.\end{aligned}$$

Then, from (46) and (47) we get

$$\begin{aligned}\int_{\Omega\times\Omega} & H_0(u_1^+ - u_2^+)\{(g(u_1^+) - f_1)\zeta + \nabla_x b(u_1^+)\cdot(\nabla_x\zeta + \nabla_y\zeta) \\ & \qquad - (\phi(u_1^+) - \phi(u_2^+))\cdot\nabla_x\zeta\}\,dxdy \\ & \leqq -\lim_{\varepsilon\to 0}\int_{(\Omega\backslash\mathscr{O}_1^+)\times(\Omega\backslash\mathscr{O}_2^+)} \left(\left|\nabla_x b(u_1^+)\right|^2 - \nabla_x b(u_1^+)\cdot\nabla_y b(u_2^+)\right) \\ & \qquad\qquad H_\varepsilon'(b(u_1^+) - b(u_2^+))\,\zeta\,dxdy. \quad (48)\end{aligned}$$

Now from (6) and from (11) we deduce that, for any nonnegative $\zeta \in \mathscr{D}(\overline{\Omega} \times \Omega)$, (40) is still true when u_1 is replaced by u_1^+ and $\mathscr{O}_1$ by $\mathscr{O}_1^+$. Then we get

$$\begin{aligned}\int_{\Omega\times\Omega} & H_0(u_1^+ - u_2)\{(g(u_2) - f_2)\zeta + \nabla_y b(u_2)\cdot\nabla_y\zeta \\ & \qquad - (\phi(u_2) - \phi(u_1^+))\cdot\nabla_y\zeta\}\,dx\,dy \\ & \geqq \lim_{\varepsilon\to 0}\int_{(\Omega\backslash\mathscr{O}_1^+)\times(\Omega\backslash\mathscr{O}_2)} \left|\nabla_y b(u_2)\right|^2 H_\varepsilon'(b(u_1^+) - b(u_2))\,\zeta\,dxdy\end{aligned}$$

for any nonnegative $\zeta \in \mathscr{D}(\overline{\Omega} \times \Omega)$. And, since

$$H_0(u_1^+ - u_2) = H_0(u_1^+ - u_2^+)(1 - H_0(u_2^-)) + H_0(u_2^-),$$

we get

$$
\begin{aligned}
&\int_{\Omega\times\Omega} H_0(u_1^+ - u_2^+)\{(g(u_2^+) - (1 - H_0(u_2^-) f_2)\zeta + \nabla_y b(u_2^+)\cdot\nabla_y\zeta \\
&\quad - (\phi(u_2^+) - \phi(u_1^+))\cdot\nabla_y\zeta\}\,dx\,dy \\
&\quad + \int_{\Omega\times\Omega} H_0(u_2^-)\{g(u_2) - f_2)\zeta + \big(\nabla_y b(u_2) - \phi(u_2)\big)\cdot\nabla_y\zeta\}\,dx\,dy \\
&\geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1^+)\times(\Omega\setminus\mathscr{O}_2^+)} \left|\nabla_y b(u_2^+)\right|^2 H_\varepsilon'(b(u_1^+) - b(u_2^+))\,\zeta\,dx\,dy \\
&\quad + \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1^+)\times(\Omega\setminus\mathscr{O}_2)} \left|\nabla_y b(-u_2^-)\right|^2 H_\varepsilon'(b(u_1^+) - b(-u_2^-))\,\zeta\,dx\,dy \\
&\geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1^+)\times(\Omega\setminus\mathscr{O}_2^+)} \left|\nabla_y b(u_2^+)\right|^2 H_\varepsilon'(b(u_1^+) - b(u_2^+))\,\zeta\,dx\,dy \geqq 0
\end{aligned}
\tag{49}
$$

for any nonnegative $\zeta \in \mathscr{D}(\overline{\Omega}\times\Omega)$.

Moreover, for $\zeta \in \mathscr{D}(\overline{\Omega}\times\Omega)$ we have

$$
\int_\Omega \nabla_y b(u_2^+)\cdot\nabla_x(H_\varepsilon(b(u_1^+) - b(u_2^+))\zeta)\,dx = 0
$$

for almost every $y \in \Omega$. Then, by taking into account (34)–(36) we deduce that

$$
\begin{aligned}
&\int_{\Omega\times\Omega} H_0(u_1^+ - u_2^+)\nabla_y b(u_2^+)\cdot\nabla_x\zeta\,dxdy \\
&\qquad = -\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1^+)\times(\Omega\setminus\mathscr{O}_2^+)} \nabla_y b(u_2^+)\cdot\nabla_x b(u_1^+) \\
&\qquad\qquad H_\varepsilon'(b(u_1^+) - b(u_2^+))\zeta\,dxdy.
\end{aligned}
\tag{50}
$$

Now, from (49) and (50) we get

$$
\begin{aligned}
&\int_{\Omega\times\Omega} H_0(u_1^+ - u_2^+)\{(g(u_2^+) - (1 - H_0(u_2^-)) f_2)\zeta \\
&\quad + \nabla_y b(u_2^+)\cdot(\nabla_y\zeta + \nabla_x\zeta) - (\phi(u_2^+) - \phi(u_1^+))\cdot\nabla_y\zeta\}\,dx\,dy \\
&\quad + \int_{\Omega\times\Omega} H_0(u_2^-)\{(g(u_2) - f_2)\zeta + \nabla_y b(u_2)\cdot\nabla_y\zeta - \phi(u_2)\cdot\nabla_y\zeta\}\,dx\,dy \\
&\geqq \lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1^+)\times(\Omega\setminus\mathscr{O}_2^+)} \Big(\left|\nabla_y b(u_2^+)\right|^2 - \nabla_y b(u_2^+)\cdot\nabla_x b(u_1^+)\Big) \\
&\qquad\qquad H_\varepsilon'(b(u_1^+) - b(u_2^+))\,\zeta\,dx\,dy.
\end{aligned}
\tag{51}
$$

From (48) and (51) we get

$$\int_{\Omega\times\Omega}\{(g(u_1^+)-g(u_2^+))^+\zeta - H_0(u_1^+-u_2^+)(f_1-(1-H_0(u_2^-))f_2)\zeta$$
$$+(\nabla_x+\nabla_y)(b(u_1^+)-b(u_2^+))^+\cdot(\nabla_x+\nabla_y)\zeta$$
$$-H_0(u_1^+-u_2^+)(\phi(u_1^+)-\phi(u_2^+))\cdot(\nabla_x+\nabla_y)\zeta\}\,dxdy$$
$$-\int_{\Omega\times\Omega}H_0(u_2^-)\{(g(u_2)-f_2)\zeta+\big(\nabla_y b(u_2)-\phi(u_2)\big)\cdot\nabla_y\zeta\}\,dx\,dy$$
$$\leqq -\lim_{\varepsilon\to 0}\int_{(\Omega\setminus\mathscr{O}_1)\times(\Omega\setminus\mathscr{O}_2)}\big|\nabla_x b(u_1^+)-\nabla_y b(u_2^+)\big|^2$$
$$H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\,dx\,dy \tag{52}$$

for any nonnegative $\zeta\in\mathscr{D}(\overline{\Omega}\times\Omega)$.

Now let $\xi\in\mathscr{D}(\mathbb{R}^N)$, $\xi\geqq 0$ such that $\operatorname{Supp}(\xi)\subset B$ where B is a ball satisfying the property:

$$\begin{gathered}\text{either } B\cap\partial\Omega=\emptyset\\ \text{or } B\subset\subset B' \text{ and } B'\cap\partial\Omega \text{ is a part of the graph}\\ \text{of a Lipschitz continuous function.}\end{gathered}\tag{53}$$

Then there exists a sequence of mollifiers ρ_n defined in $\mathbb{R}^N$ such that, for n large enough,

$$y\mapsto\rho_n\left(\frac{x-y}{2}\right)\in\mathscr{D}(\Omega)\quad\forall x\in B,\tag{54}$$

$$\chi_n(y)=\int_\Omega\rho_n\left(\frac{x-y}{2}\right)dx\quad\text{is an increasing sequence for } y\in B,\tag{55}$$

$$\chi_n(y)=1\quad\text{for any } y\in B \text{ such that } d(y,\mathbb{R}^N\setminus\Omega)>c/n\tag{56}$$

where c is a positive constant depending on B. Then we can define the nonnegative function

$$\zeta_n(x,y)=\xi(y)\rho_n\left(\frac{x-y}{2}\right)\in\mathscr{D}(\overline{\Omega}\times\Omega).$$

By substituting ζ_n into (52), we obtain

$$\int_{\Omega\times\Omega}\{(g(u_1^+)-g(u_2^+))^+\xi - H_0(u_1^+-u_2^+)(f_1-(1-H_0(u_2^-))f_2)\xi$$
$$+(\nabla_x+\nabla_y)(b(u_1^+)-b(u_2^+))^+\cdot(\nabla_x+\nabla_y)\xi$$
$$-H_0(u_1^+-u_2^+)(\phi(u_1^+)-\phi(u_2^+))\cdot(\nabla_x+\nabla_y)\xi\}\,\rho_n\,dx\,dy$$
$$\leqq\int_{\Omega\times\Omega}H_0(u_2^-)\{(g(u_2)-f_2)(\xi\rho_n)+\big(\nabla_y b(u_2)-\phi(u_2)\big)\cdot\nabla_y(\xi\rho_n)\}\,dx\,dy$$

and therefore

$$\begin{aligned}
&\int_{\Omega\times\Omega}\{(g(u_1^+)-g(u_2^+))^+\xi - H_0(u_1^+-u_2^+)(f_1-(1-H_0(u_2^-))f_2)\xi \\
&\qquad + (\nabla_x+\nabla_y)(b(u_1^+)-b(u_2^+))^+\cdot(\nabla_x+\nabla_y)\xi \\
&\qquad - H_0(u_1^+-u_2^+)(\phi(u_1^+)-\phi(u_2^+))\cdot(\nabla_x+\nabla_y)\xi\}\,\rho_n\,dx\,dy \\
&\quad \leqq \int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_n)+\big(\nabla_y b(u_2)-\phi(u_2)\big)\cdot\nabla_y(\xi\chi_n)\}dy \qquad (57)
\end{aligned}$$

where $\xi=\xi(y)$. When $n\to+\infty$, the integral at the left side of (57) converges to

$$\begin{aligned}
\int_\Omega\{((g(u_1^+)-g(u_2^+))^+ - \kappa_+ H_0(u_1^+)(f_1-(H_0(u_2^-))f_2))\xi \\
+\nabla(b(u_1^+)-b(u_2^+))^+\cdot\nabla\xi \\
-H_0(u_1^+-u_2^+)(\phi(u_1^+)-\phi(u_2^+))\cdot\nabla\xi\}\,dx \qquad (58)
\end{aligned}$$

where $\kappa_+\in H(u_1^+-u_2^+)$ for almost every $x\in\Omega$, and any function in the integrand of this integral is now considered as a function of the variable x.

The integral on the right of (57) is nonnegative: In fact, u_2 is an entropy solution and therefore, from (11) we deduce that $\mathscr{M}:\mathscr{D}(\overline{\Omega})\mapsto\mathbb{R}$ defined by

$$\varphi\mapsto\mathscr{M}(\varphi)=\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)\varphi+(\nabla b(u_2)-\phi(u_2))\cdot\nabla\varphi\}dx$$

is monotone. In particular, since $\xi\chi_n$ is an increasing sequence satisfying

$$0\leqq\xi\chi_n\leqq\xi,$$

we deduce that $\mathscr{M}(\xi\chi_n)$ is a bounded and increasing sequence and, therefore it converges. Then, from (57) and (58) we deduce that

$$\begin{aligned}
&\int_\Omega\{((g(u_1^+)-g(u_2^+))^+-\kappa_+(f_1-(1-H_0(u_2^-))f_2))\xi \\
&\qquad +\nabla(b(u_1^+)-b(u_2^+))^+\cdot\nabla\xi - H_0(u_1^+-u_2^+)(\phi(u_1^+)-\phi(u_2^+))\cdot\nabla\xi\}\,dx \qquad (59) \\
&\quad\leqq \lim_{n\to+\infty}\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_n)+(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi\chi_n)\}\,dx
\end{aligned}$$

for any nonnegative $\xi\in\mathscr{D}(B)$.

Now, in view of Remark 4, inequality (59) is still true when u_1 is replaced by $-u_2$, u_2 by $-u_1$, f_1 by $-f_2$, f_2 by $-f_1$, g by $\tilde{g}$, b by $\tilde{b}$ and ϕ by $\tilde{\phi}$. Then we have

$$\begin{aligned}&\int_\Omega \{((g(-u_1^-) - g(-u_2^-))^+ - \kappa_- H_0(u_2^-)((1 - H_0(u_1^+))f_1 - f_2))\xi \\ &+\nabla(b(-u_1^-)-b(-u_2^-))^+ \cdot \nabla\xi - H_0(-u_1^- + u_2^-)(\phi(-u_1^-) - \phi(-u_2^-)) \cdot \nabla\xi\}\,dx \\ &\leqq - \lim_{n\to+\infty} \int_\Omega H_0(u_1^+)\{(g(u_1^+) - f_1)(\xi\chi_n) + \big(\nabla b(u_1^+) - \phi(u_1^+)\big) \cdot \nabla(\xi\chi_n)\}\,dx\end{aligned} \tag{60}$$

where $\kappa_- \in H(-u_1^- + u_2^-)$ for almost every $x \in \Omega$.

Let $\kappa_B = \kappa_-(1 - H_0(u_1^+))H_0(u_2^-) + \kappa_+ H_0(u_1^+)$. Then we easily check that

$$\begin{aligned}\kappa_B &= \kappa_-(1 - H_0(u_1^+))H_0(u_2^-) + \kappa_+ H_0(u_1^+) \\ &= \kappa_+(1 - H_0(u_2^-))H_0(u_1^+) + \kappa_- H_0(u_2^-) \in H(u_1 - u_2).\end{aligned}$$

Then (59) and (60) lead to

$$\begin{aligned}&\int_\Omega \{((g(u_1) - g(u_2))^+ - \kappa_B(f_1 - f_2))\xi \\ &\quad+\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla\xi\}\,dx \\ &\leqq \lim_{n\to+\infty} \int_\Omega H_0(u_2^-)\{(g(u_2) - f_2)(\xi\chi_n) + (\nabla b(u_2) - \phi(u_2)) \cdot \nabla(\xi\chi_n)\}\,dx \\ &\quad - \lim_{n\to+\infty} \int_\Omega H_0(u_1^+)\{(g(u_1^+) - f_1)(\xi\chi_n) \\ &\qquad + \big(\nabla b(u_1^+) - \phi(u_1^+)\big) \cdot \nabla(\xi\chi_n)\}dx\end{aligned} \tag{61}$$

for any nonnegative $\xi \in \mathscr{D}(B)$.

Let ξ be a nonnegative function of $\mathscr{D}(B)$. Then $\xi\chi_{n'} \in \mathscr{D}(\Omega)$ for n' large enough, and from inequality (33) we have

$$\begin{aligned}&\int_\Omega \{((g(u_1) - g(u_2))^+ - \kappa(f_1 - f_2))\xi\chi_{n'} \\ &\quad+\nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi\chi_{n'}) - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla(\xi\chi_{n'})\}\,dx \leqq 0\end{aligned}$$

and therefore

$$\begin{aligned}&\int_\Omega \{((g(u_1) - g(u_2))^+ - \kappa(f_1 - f_2))\xi \\ &\quad + \nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla\xi\}\,dx \\ &\qquad \leqq \int_\Omega \{((g(u_1) - g(u_2))^+ - \kappa(f_1 - f_2))\xi(1 - \chi_{n'}) \\ &\qquad\quad + \nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi(1 - \chi_{n'})) \\ &\qquad\quad - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla(\xi(1 - \chi_{n'}))\}\,dx\end{aligned} \tag{62}$$

and, since $\xi(1-\chi_{n'})$ is a nonnegative function of $\mathscr{D}(B)$, from (61) we have

$$\begin{aligned}\int_\Omega \{((g(u_1)-g(u_2))^+ - \kappa_B(f_1-f_2))\xi(1-\chi_{n'}) \\ + \nabla(b(u_1)-b(u_2))^+ \cdot \nabla(\xi(1-\chi_{n'})) \\ - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2)) \cdot \nabla(\xi(1-\chi_{n'}))\}\,dx\end{aligned} \tag{63}$$

$$\begin{aligned}\leqq \lim_{n\to+\infty}\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi(1-\chi_{n'})\chi_n) \\ + (\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\}dx \\ - \lim_{n\to+\infty}\int_\Omega H_0(u_1^+)\{(g(u_1^+)-f_1)(\xi(1-\chi_{n'})\chi_n) \\ + \left(\nabla b(u_1^+)-\phi(u_1^+)\right)\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\}dx.\end{aligned}$$

Now, since $\mathrm{Supp}(\xi\chi_{n'}) \subset \Omega\cap B$, from (56) we deduce that for n large enough we have

$$\xi\chi_{n'}\chi_n = \xi\chi_{n'}.$$

Then we have

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi(1-\chi_{n'})\chi_n) \\ + (\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\}dx \\ - \lim_{n\to+\infty}\int_\Omega H_0(u_1^+)\{(g(u_1)-f_1)(\xi(1-\chi_{n'})\chi_n) \\ + (\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\}dx\end{aligned} \tag{64}$$

$$\begin{aligned}= \lim_{n\to+\infty}\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_n) + (\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi\chi_n)\}dx \\ - \lim_{n\to+\infty}\int_\Omega H_0(u_1^+)\{(g(u_1)-f_1)(\xi\chi_n) + (\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi\chi_n)\}dx \\ - \int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_{n'}) + (\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi\chi_{n'})\}dx \\ + \int_\Omega H_0(u_1^+)\{(g(u_1)-f_1)(\xi\chi_{n'}) + (\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi\chi_{n'})\}dx.\end{aligned}$$

From (62)–(64) we deduce that

$$\begin{aligned}\int_\Omega \{((g(u_1)-g(u_2))^+ - \kappa(f_1-f_2))\xi \\ + \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\xi\}\,dx\end{aligned}$$

$$\leqq \lim_{n\to+\infty}\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_n)+(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi\chi_n)\}dx$$

$$-\lim_{n\to+\infty}\int_\Omega H_0(u_1^+)\{(g(u_1)-f_1)(\xi\chi_n)+(\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi\chi_n)\}dx$$

$$-\int_\Omega H_0(u_2^-)\{(g(u_2)-f_2)(\xi\chi_{n'})+(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi\chi_{n'})\}dx$$

$$+\int_\Omega H_0(u_1^+)\{(g(u_1)-f_1)(\xi\chi_{n'})+(\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi\chi_{n'})\}dx$$

$$-\int_\Omega(\kappa-\kappa_B)(f_1-f_2)\xi(1-\chi_{n'})\,dx.$$

We check easily that the right side of this inequality converges to 0 when $n'\to+\infty$, and therefore

$$\int_\Omega\{((g(u_1)-g(u_2))^+-\kappa(f_1-f_2))\xi + \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\xi\}\,dx\leqq 0 \quad (65)$$

for any nonnegative $\xi\in\mathscr{D}(B)$ (where the function κ is the same as that in Theorem 8).

Finally, let ξ be a nonnegative function of $\mathscr{D}(\overline{\Omega})$, let $\Omega_0\subset\subset\Omega$ be such that

$$\Omega\subset\Omega_0\cup(\cup_{j=1}^k B_j)$$

where the B_j are balls satisfying (53), let $(\varphi_j)_{j=0}^k$ be a partition of unity related to the above covering of Ω, and let $\xi_j=\xi\varphi_j$ for $0\leqq j\leqq k$. Then, by applying (33) for $j=0$ and (65) for $1\leqq j\leqq k$ we have

$$\int_\Omega\{((g(u_1)-g(u_2))^+-\kappa(f_1-f_2))\xi_j + \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi_j - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\xi_j\}\,dx\leqq 0$$

for $0\leqq j\leqq k$ and therefore

$$\int_\Omega\{((g(u_1)-g(u_2))^+-\kappa(f_1-f_2))\xi + \nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi - H_0(u_1-u_2)(\phi(u_1)-\phi(u_2))\cdot\nabla\xi\}\,dx\leqq 0$$

for any nonnegative $\xi\in\mathscr{D}(\overline{\Omega})$, which achieves the proof.

As a consequence of (45) we have

Corollary 1. *Let* (H1) *and* (H2) *hold, let f_1 and $f_2 \in L^1(\Omega)$ and let u_i be an entropy solution of $(P_S(f_i)(g, b, \phi))$ for $i = 1, 2$. Then*

$$\int_\Omega (g(u_1) - g(u_2))^+ \, dx \leqq \int_\Omega (f_1 - f_2)\kappa \, dx \tag{66}$$

for some $\kappa \in H(u_1 - u_2)$, and

$$\|g(u_1) - g(u_2)\|_{L^1(\Omega)} \leqq \|f_1 - f_2\|_{L^1(\Omega)}. \tag{67}$$

Moreover, if $f_1 \leqq f_2$ almost everywhere in Ω, then

$$g(u_1) \leqq g(u_2) \quad \text{a.e. in } \Omega, \tag{68}$$

and if $f_1 = f_2$ almost everywhere in Ω, then

$$g(u_1) = g(u_2) \quad \text{a.e. in } \Omega. \tag{69}$$

Proof. Let $\xi = 1$ in the inequality (45). Then we get (66). From (66) we deduce (67)–(69).

Corollary 2. *Let* (H1), (H3) *and* (H4) *hold. Let $f \in L^p(\Omega)$ for some $1 \leqq p \leqq +\infty$ and let u be an entropy solution of $(P_S(f)(g, b, \phi))$. Then $g(u) \in L^p(\Omega)$ and*

$$\|g(u)\|_{L^p(\Omega)} \leqq \|f\|_{L^p(\Omega)}.$$

Moreover, let $f \in BV(\Omega)$. Then $g(u) \in BV(\Omega)$ and

$$TV_\Omega(g(u)) + \|g(u)\|_{L^1(\Gamma)} \leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)}.$$

Proof. For the time being let $f \in L^\infty(\Omega) \cap BV(\Omega)$. Let

$$b_\varepsilon = \varepsilon I + b,$$

$$\phi_\varepsilon \in \mathscr{C}^{0,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^N), \quad \phi_\varepsilon \overset{\varepsilon \to 0}{\to} \phi \text{ in } (L^\infty_{\text{loc}}(\mathbb{R}))^N,$$

and let u_ε be the unique solution of $(P_S(f)(g, b_\varepsilon, \phi_\varepsilon))$. From Theorem 1 we know that

$$\|g(u_\varepsilon)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \quad \text{for any } 1 \leqq r \leqq +\infty$$

and from Theorem 6 we know that

$$TV_\Omega(g(u_\varepsilon)) \leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)}.$$

Arguing as in the proof of Theorem 7 we deduce that, for a subsequence of ε still denoted by ε, we have

$$u_\varepsilon \overset{\varepsilon \to 0}{\longrightarrow} u \quad \text{in } L^q(\Omega) \quad \text{for any } 1 \leqq q < +\infty,$$

$$g(u_\varepsilon) \overset{\varepsilon \to 0}{\longrightarrow} g(u) \quad \text{in } L^1(\Omega)$$

where u is an entropy solution of $(P_S(f)(g, b, \phi))$. Moreover, since $g(u_\varepsilon)$ is uniformly bounded in $L^\infty(\Omega)$, we deduce that

$$g(u_\varepsilon) \stackrel{\varepsilon\to 0}{\longrightarrow} g(u) \quad \text{in } L^q(\Omega) \text{ for any } 1 \leqq q < +\infty \text{ and in } L^\infty(\Omega) \text{ weak}\star.$$

Hence we deduce that

$$\|g(u)\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \qquad \text{for any } 1 \leqq r \leqq +\infty.$$

Moreover, let us consider an open subset Ω' of $\mathbb{R}^N$ such that $\Omega \subset\subset \Omega'$; we prolong by 0 the functions in $\Omega' \setminus \Omega$. Then, by taking into account that $g(u_\varepsilon) = 0$ on $\partial\Omega$, we have

$$\begin{aligned} TV_{\Omega'}(g(u)) = TV_\Omega(g(u)) + \|g(u)\|_{L^1(\Gamma)} &\leqq \liminf_{\varepsilon\to 0} TV_{\Omega'}(g(u_\varepsilon)) \\ &= \liminf_{\varepsilon\to 0} TV_\Omega(g(u_\varepsilon)) \\ &\leqq TV_\Omega(f) + \|f\|_{L^1(\Gamma)} = TV_{\Omega'}(f). \end{aligned}$$

Then, since for any entropy solution v of $(P_S(f)(g, b, \phi))$ we have $g(v) = g(u)$, the corollary is proved for a function $f \in L^\infty(\Omega) \cap BV(\Omega)$.

Now let us assume either that $f \in L^p(\Omega)$ for some $p \geqq 1$ or that $f \in BV(\Omega)$. We still denote by f the prolongation of f by 0 outside of Ω. Let ρ_δ be a sequence of mollifiers and let f_δ be the restriction to Ω of $f * \rho_\delta$ ($f_\delta = f * \rho_\delta$ in Ω and $f_\delta = 0$ outside of Ω). Let u_δ be an entropy solution of $(P_S(f_\delta)(g, b, \phi))$. Then we have

$$\|g(u_\delta)\|_{L^r(\Omega)} \leqq \|f_\delta\|_{L^r(\Omega)} \quad \text{for any } 1 \leqq r \leqq +\infty,$$

$$\begin{aligned} TV_{\Omega'}(g(u_\delta)) &= TV_\Omega(g(u_\delta)) + \|g(u_\delta)\|_{L^1(\Gamma)} \\ &\leqq TV_\Omega(f_\delta) + \|f_\delta\|_{L^1(\Gamma)} = TV_{\Omega'}(f_\delta). \end{aligned}$$

Moreover, from Corollary 1 we have

$$\|g(u) - g(u_\delta)\|_{L^1(\Omega)} \leqq \|f - f_\delta\|_{L^1(\Omega)} \stackrel{\delta\to 0}{\longrightarrow} 0.$$

Then we have

$$\|g(u))\|_{L^1(\Omega)} \leqq \|f\|_{L^1(\Omega)}$$

(this inequality can be deduce either from the preceding inequality or from Corollary 1).

If $f \in L^p(\Omega)$ for some $p \in (1, +\infty]$, then we have

$$\|g(u_\delta)\|_{L^r(\Omega)} \leqq \|f_\delta\|_{L^r(\Omega)} \leqq \|f\|_{L^r(\Omega)} \quad \text{for any } 1 \leqq r \leqq p, \tag{70}$$

and we deduce that

$$g(u_\delta) \stackrel{\delta\to 0}{\rightharpoonup} g(u) \text{ in } \begin{cases} L^p(\Omega) \text{ weak for } p < +\infty, \\ L^\infty(\Omega) \text{ weak}\star \text{ for } p = +\infty, \end{cases}$$

$$\|g(u)\|_{L^p(\Omega)} \leqq \liminf_{\delta\to 0} \|g(u_\delta)\|_{L^p(\Omega)} \leqq \|f\|_{L^p(\Omega)}.$$

If $f \in BV(\Omega)$, then we deduce that $g(u) \in BV(\Omega)$ and that

$$\begin{aligned} TV_{\Omega'}(g(u)) &= TV_\Omega(g(u)) + \|g(u)\|_{L^1(\Gamma)} \leqq \liminf_{\delta\to 0} TV_{\Omega'}(g(u_\delta)) \\ &\leqq \limsup_{\delta\to 0} TV_{\Omega'}(f_\delta) \leqq \lim_{\delta\to 0} TV_{\Omega'}(f * \rho_\delta) \\ &= TV_{\Omega'}(f) = TV_\Omega(f) + \|f\|_{L^1(\Gamma)}. \end{aligned}$$

Since for any entropy solution v of $(P_S(f)(g,b,\phi))$ we have $g(v) = g(u)$, the proof is achieved.

Corollary 3. *Let* (H1) *and* (H2) *hold, let* f_1 *and* $f_2 \in L^1(\Omega)$ *and let* u_i *be an entropy solution of* $(P_S(f_i)(g,b,\phi))$ *for* $i = 1, 2$. *Assume that for some* $1 \leqq i_0 \leqq N$,

$$g(s) = g(s') \implies \phi_{i_0}(s) = \phi_{i_0}(s') \quad \forall\, s,\ s' \in \mathbb{R}. \tag{A1}$$

If $f_1 \leqq f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) \leqq b(u_2) \quad \textit{a.e. in } \Omega,$$

and if $f_1 = f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) = b(u_2) \quad \textit{a.e. in } \Omega.$$

Proof. From Corollary 1 we have

$$g(u_1) \leqq g(u_2) \quad \text{for a.e. } x \in \Omega$$

and then

$$g(u_1) = g(u_2) \quad \text{for a.e. } x \in \Omega \text{ s.t } u_1(x) > u_2(x),$$

whence, from (A1), we deduce that

$$\phi_{i_0}(u_1) = \phi_{i_0}(u_2) \quad \text{for a.e. } x \in \Omega \text{ such that } u_1(x) > u_2(x).$$

Then from (45) we have

$$0 \geqq \int_\Omega \kappa(f_1 - f_2)\,\xi\,dx \geqq \int_\Omega -(b(u_1) - b(u_2))^+ \xi''\,dx$$

for some $\kappa \in H(u_1 - u_2)$ and for any nonnegative $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $\xi = \xi(x_{i_0})$ (where $\xi'' = d^2\xi/dx^2$). Therefore we deduce that

$$0 = \int_\Omega (b(u_1) - b(u_2))^+ \xi''\,dx$$

for any $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$ such that $\xi = \xi(x_{i_0})$. By choosing ξ such that $\xi'' > 0$ we easily deduce that $b(u_1) \leqq b(u_2)$.

Corollary 4. *Let* (H1) *and* (H2) *hold, let* f_1 *and* $f_2 \in L^1(\Omega)$ *and let* u_i *be an entropy solution of* $(P_S(f_i)(g, b, \phi))$ *for* $i = 1, 2$. *Assume that*

$$\text{for some } 1 \leqq i_0 \leqq N \text{ either } \phi_{i_0} \text{ or } -\phi_{i_0} \text{ is not decreasing.} \tag{A2}$$

If $f_1 \leqq f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) \leqq b(u_2) \quad \text{a.e. in } \Omega, \tag{71}$$

and if $f_1 = f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) = b(u_2) \quad \text{a.e. in } \Omega.$$

Proof. We assume that $f_1 \leqq f_2$. Then, from Corollary 1 we have $g(u_1) \leqq g(u_2)$. Inequality (45) leads to

$$\begin{aligned} 0 &\geqq \int_\Omega \kappa(f_1 - f_2)\xi\, dx \\ &\geqq \int_\Omega \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla\xi\}\, dx \end{aligned}$$

for any nonnegative $\xi \in H^1(\Omega)$, whence we deduce that

$$\begin{aligned} 0 &= \int_\Omega \kappa(f_1 - f_2)\xi\, dx \\ &= \int_\Omega \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi - H_0(u_1 - u_2)(\phi(u_1) - \phi(u_2)) \cdot \nabla\xi\}\, dx \end{aligned} \tag{72}$$

for any $\xi \in H^1(\Omega)$.

In particular, let $\xi(x) = e^{\lambda x_{i_0}}$. Then, since $b(u_1) - b(u_2)$ vanishes at the boundary of Ω, we have

$$\begin{aligned} 0 = \int_\Omega \{&-(b(u_1) - b(u_2))^+\lambda^2 \\ &- H_0(u_1 - u_2)(\phi_{i_0}(u_1) - \phi_{i_0}(u_2))\lambda\} e^{\lambda x_{i_0}}\, dx. \end{aligned} \tag{73}$$

By choosing $\lambda > 0$ when ϕ_{i_0} is nondecreasing and $\lambda < 0$ when ϕ_{i_0} is nonincreasing, we deduce that $(b(u_1) - b(u_2))^+ = 0$.

Corollary 5. *Let* (H1) *and* (H2) *hold, let* f_1 *and* $f_2 \in L^1(\Omega)$ *and let* u_i *be an entropy solution of* $(P_S(f_i)(g, b, \phi))$ *for* $i = 1, 2$. *Assume that for some* $1 \leqq i_0 \leqq N$ *there exists* C *such that*

$$|\phi_{i_0}(s) - \phi_{i_0}(r)| \leqq C|b(s) + g(s) - b(r) - g(r)| \quad \forall s,\ r \in \mathbb{R}. \tag{A3}$$

If $f_1 \leqq f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) \leqq b(u_2) \quad \text{a.e. in } \Omega,$$

and if $f_1 = f_2$ *almost everywhere in* Ω, *then*

$$b(u_1) = b(u_2) \quad \text{a.e. in } \Omega.$$

Proof. Arguing as above, we still have (73), whence we deduce

$$0 = \int_\Omega (b(u_1) - b(u_2))^+ e^{\lambda x_{i_0}} (\lambda^2 + W\lambda)\, dx \tag{74}$$

where

$$W = \begin{cases} \dfrac{\phi(u_1) - \phi(u_2)}{b(u_1) - b(u_2)} & \text{if } b(u_1) > b(u_2), \\ 0 & \text{otherwise.} \end{cases}$$

From $(A3)$ we deduce that W is essentially bounded (because where $b(u_1) > b(u_2)$ we have $u_1 > u_2$ and, therefore, $g(u_1) = g(u_2)$ whence $b(u_1) - b(u_2) = b(u_1) + g(u_1) - b(u_2) - g(u_2)$), and that $|W| \leqq C$ almost everywhere in Ω. From (74) we deduce that

$$0 \geqq \int_\Omega (b(u_1) - b(u_2))^+ e^{\lambda x_{i_0}} (\lambda^2 - C\lambda)\, dx$$

for any positive λ. In particular, by choosing $\lambda > C$ we deduce that $(b(u_1) - b(u_2))^+ = 0$.

Corollary 6. *Let* (H1), (H2) *and* (H3) *hold and let* $f \in L^1(\Omega)$. *Let either* (A1) *or* (A2) *or* (A3) *hold. If* $g + b$ *is (strictly) increasing, then there exists no more than one entropy solution of* $(P_S(f)(g, b, \phi))$. *Moreover, if* b *is (strictly) increasing, then there exists no more than one weak solution of* $(P_S(f)(g, b, \phi))$.

Remark 7. When ϕ satisfies neither (A2) nor (A3), in general $b(u)$ is not unique. For instance, let $N = 1$, $g \equiv 0$, $b = I$, $\phi(s) = \sqrt{\min(s, 1-s)}$ for $1 \geqq s \geqq 0$ and $\phi(s) = 0$ otherwise. Let

$$-u'' + (\phi(u))' = \chi(4, 4 + \sqrt{2}) \quad \text{in } (0, 4 + 2\sqrt{2}\,\mathrm{Log}(2)),$$
$$u(0) = u(4 + 2\sqrt{2}\,\mathrm{Log}(2)) = 0.$$

Then for any $\lambda \in [0, 4 - 2\sqrt{2}]$ the function u_λ defined by

$$u_\lambda(x) = \left[\frac{(x-\lambda)^+}{2}\right]^2 \quad \text{for } 0 \leqq (x-\lambda)^+ \leqq \sqrt{2},$$

$$u_\lambda(x) = 1 - \left[\frac{2\sqrt{2} - (x-\lambda)^+}{2}\right]^2 \quad \text{for } \sqrt{2} \leqq (x-\lambda)^+ \leqq 2\sqrt{2},$$

$$u_\lambda(x) = 1 \quad \text{for } 2\sqrt{2} + \lambda \leqq x \leqq 4,$$

$$u_\lambda(x) = 1 - \left(\frac{x}{2} - 2\right)^2 \quad \text{for } 4 \leqq x \leqq 4 + \sqrt{2},$$

$$u_\lambda(x) = \mathscr{F}^{-1}(x + 2\sqrt{2}\mathrm{Log}(\sqrt{2}/2) - 4) \quad \text{for } 4 + \sqrt{2} \leqq x \leqq 4 + 2\sqrt{2}\,\mathrm{Log}(2),$$

where

$$\mathscr{F}(s) = 2\sqrt{s} + 2\sqrt{2}\,\mathrm{Log}(\sqrt{2} - \sqrt{s}) \quad \text{for } 0 \leqq s \leqq \tfrac{1}{2},$$

is a weak solution of this problem and therefore, since $b = I$, it is still an entropy solution.

Corollary 7. *Let* (H1) *and* (H2) *hold, let f_1 and $f_2 \in L^1(\Omega)$ and let u_i be entropy solutions of $(P_S(f_i)(g, b, \phi))$ for $i = 1, 2$ with $f_1 \leqq f_2$. Then*

$$(f_1 - f_2)(u_1 - u_2)^+ = 0 \quad \textit{a.e. in } \Omega.$$

In particular, if

$$f_1 < f_2 \quad \textit{a.e. in } \Omega,$$

then

$$u_1 \leqq u_2 \quad \textit{a.e. in } \Omega.$$

Proof. Under the assumptions of this corollary the identity (72) is still true, in particular,

$$0 = \int_\Omega \kappa(f_1 - f_2)\xi \, dx$$

for some $\kappa \in H(u_1 - u_2)$ and for any $\xi \in H^1(\Omega)$ (and therefore for any $\xi \in L^2(\Omega)$). Then we deduce that

$$(f_1 - f_2)\kappa = 0 \quad \text{a.e. in } \Omega,$$

whence we get

$$(f_1 - f_2)(u_1 - u_2)^+ = 0 \quad \text{a.e. in } \Omega.$$

Corollary 8. *Under the assumptions of Theorem 7, $(P_S(f)(g, b, \phi))$ has a minimal entropy solution $u_{\min}$ and a maximal entropy solution $u_{\max}$ such that for any entropy solution u of $(P_S(f)(g, b, \phi))$ we have*

$$u_{\min} \leqq u \leqq u_{\max}$$

almost everywhere in Ω.

Proof. Let u_{n^+} be an entropy solution of $(P_S(f + 1/n)(g, b, \phi))$ and let u_{n^-} be an entropy solution of $(P_S(f - 1/n)(g, b, \phi))$. Then for any entropy solution u of $(P_S(f)(g, b, \phi))$, from Corollary 7 we have

$$u_{m^-} \leqq u_{n^-} \leqq u \leqq u_{n^+} \leqq u_{m^+}$$

for any $n, m \in \mathbb{N}$ such that $n > m$. Let

$$u_{\min} = \lim_{n \to +\infty} u_{n^-}, \quad u_{\max} = \lim_{n \to +\infty} u_{n^+}.$$

Then, arguing as in the proof of Theorem 7 we easily check that $u_{\min}$ and $u_{\max}$ are entropy solutions of $(P_S(f)(g, b, \phi))$ satisfying

$$u_{\min} \leqq u \leqq u_{\max}$$

for any entropy solution u of $(P_S(f)(g, b, \phi))$.

4. The Evolution Problem: Existence of Entropy Solutions

4.1. Existence and Uniqueness of Integral Solutions

Let us define the operator $A_{(g,b,\phi)} \subset L^\infty(\Omega) \times L^\infty(\Omega) \subset L^1(\Omega) \times L^1(\Omega)$ by $(v, f) \in A_{(g,b,\phi)}$ if and only if there exists $u \in L^\infty(\Omega)$ such that

$$v = g(u) \quad \text{and } u \text{ is an entropy solution of } (P_S(f+v)(g,b,\phi)).$$

Theorem 10. *Let* (H1), (H3) *and* (H4) *hold. Then* $A_{(g,b,\phi)}$ *satisfies:*
i) $A_{(g,b,\phi)}$ *is a T-accretive operator in* $L^1(\Omega)$,

$$\forall (v_i, f_i) \in A_{(g,b,\phi)}, \quad \int_\Omega \kappa(f_1 - f_2)\, dx \geqq 0 \quad \textit{for some } \kappa \in H(v_1 - v_2).$$

ii) *For any* $\lambda > 0$, $R(I + \lambda A_{(g,b,\phi)}) = L^\infty(\Omega)$.
iii) $\overline{D(A_{(g,b,\phi)})} = \{v \in L^1(\Omega)/\, v(x) \in \overline{R(g)}$ *for a.e.* $x \in \Omega\}$.
iv) *For any* $\lambda > 0$, *let* $J^\lambda_{(g,b,\phi)} = (I + \lambda A_{(g,b,\phi)})^{-1}$. *Then*

$$J^\lambda_{(g,b,\phi)}(L^\infty(\Omega) \cap BV(\Omega)) \subset L^\infty(\Omega) \cap BV(\Omega),$$

$$TV_\Omega(J^\lambda_{(g,b,\phi)}(f)) \leqq TV_{\mathbb{R}^N}(J^\lambda_{(g,b,\phi)}(f)) \leqq TV_{\mathbb{R}^N}(f)$$

where $f = J^\lambda_{(g,b,\phi)}(f) = 0$ *almost everywhere in* $\mathbb{R}^N \setminus \Omega$.
v) *Let* $g^{-1} \in \mathscr{C}(\mathbb{R})$. *Then* $A_{(g,b,\phi)}$ *is single-valued.*

Proof. i) Let $(v_i, f_i) \in A_{(g,b,\phi)}$. Then there exists $u_i \in L^\infty(\Omega)$ such that $v_i = g(u_i)$ and u_i is an entropy solution of $(P_S(f+v_i)(g,b,\phi))$. Therefore, if we choose $\xi \equiv 1$ in Theorem 9, we deduce that

$$\int_\Omega \kappa(f_1 - f_2)\, dx \geqq 0.$$

ii) is a direct consequence of Theorem 7.

To prove iii) let $f \in L^\infty(\Omega)$ be such that $f \pm \varepsilon \in R(g)$ and let u_h be an entropy solution of $(P_S(f)(g, hb, h\phi))$. Then $g(u_h) \in D(A_{(g,b,\phi)}$ and

$$\|g(u_h)\|_{L^q(\Omega)} \leqq \|f\|_{L^q(\Omega)} \quad \text{for } 1 \leqq q \leqq +\infty,$$

in particular, for $q = +\infty$. Therefore there exists a constant $C(f)$ such that

$$\|u_h\|_{L^\infty(\Omega)} \leqq C(f),$$

whence we deduce that

$$h(\Delta b(u_h) - \operatorname{div}(\phi(u_h))) \overset{h\to 0}{\longrightarrow} 0 \quad \text{in } \mathscr{D}'(\Omega),$$

and therefore

$$g(u_h) \overset{h\to 0}{\longrightarrow} f \quad \text{in } \mathscr{D}'(\Omega).$$

Moreover, since $g(u_h)$ is bounded in $L^2(\Omega)$, we deduce that

$$g(u_h) \overset{h\to 0}{\rightharpoonup} f \quad \text{weakly in } L^2(\Omega),$$

whence

$$\liminf_{h\to 0} \|g(u_h)\|_{L^2(\Omega)} \geqq \|f\|_{L^2(\Omega)}$$

and therefore

$$g(u_h) \overset{h\to 0}{\longrightarrow} f \quad \text{in } L^2(\Omega), \text{ and therefore in } L^1(\Omega).$$

Now we conclude by observing that the set of functions $f \in L^\infty(\Omega)$ such that $f \pm \varepsilon \in R(g)$ for some $\varepsilon > 0$ is dense in

$$\{v \in L^1(\Omega)/\, v(x) \in \overline{R(g)} \text{ for a.e. } x \in \Omega\}.$$

iv) can be deduced from Corollary 2.

v) can be deduced from Corollary 3.

From the theory of semi-groups in Banach spaces (see BÉNILAN [Be] and CRANDALL & LIGGETT [CL]) we have

Theorem 11. *Let us assume* (H1), (H3) *and* (H4) *hold. Then, for any* $f \in L^1(Q)$ *(where* $Q = (0,T) \times \Omega$*) and for any* $v_0 \in \overline{D(A_{(g,b,\phi)})}$ *there exists a unique integral solution* v *of*

$$(P_E(f, v_0)(A_{(g,b,\phi)})) \quad \begin{cases} \dfrac{dv}{dt} + A_{(g,b,\phi)}(v) \ni f, \\ v(0) = v_0, \end{cases}$$

and $v \in \mathscr{C}([0,T]; L^1(\Omega))$.

Moreover, let v_i *be the integral solution of* $(P_E(f_i, v_{i_0})(A_{(g,b,\phi)}))$ *for* $i = 1, 2$. *Then*

$$\int_Q \kappa\{(v_1 - v_2)\zeta' + (f_1 - f_2)\zeta\}\, dx\, dt \geqq 0$$

for any nonnegative $\zeta \in \mathscr{D}(0,T)$*, and for some* $\kappa \in H(v_1 - v_2)$ *almost everywhere in* Q*. In particular,* $v_1 \leqq v_2$ *for* $v_{1_0} \leqq v_{2_0}$ *and* $f_1 \leqq f_2$.

Proof. See, for instance, [Be] Propositions 1.28 and 2.2.

4.2. Existence of Entropy Solutions of $(P_E(f, g_0)(g, b, \phi))$

For any continuous and nondecreasing or nonincreasing function ψ we define (by taking into account (1) and (2)) the proper lower semi-continuous and convex or upper semi-continuous and concave function

$$B_\psi(s) = \begin{cases} \displaystyle\int_0^s \psi(b \circ (g^{-1})_0(r))\, dr & \text{for } s \in \overline{(\psi \circ b) \circ g^{-1}}, \\ +\infty & \text{otherwise,} \end{cases}$$

and we have $(\psi \circ b) \circ g^{-1} \subset \partial B_\psi$.

According to [BW1, AL] we define a weak solution of $(P_E(f, g_0)(g, b, \phi))$ by

Definition 3. Let $f \in L^2((0,T); H^{-1}(\Omega))$ and let $g_0 \in L^1(\Omega)$. Then a *weak solution* of $(P_E(f, g_0)(g, b, \phi))$ is a measurable function u satisfying

$$g(u) \in L^1(Q), \quad \frac{\partial g(u)}{\partial t} \in L^2((0,T); H^{-1}(\Omega)), \tag{75}$$

$$b(u) \in L^2((0,T); H_0^1(\Omega))), \quad \phi(u) \in (L^2(Q))^N, \tag{76}$$

$$\frac{\partial g(u)}{\partial t} - \Delta b(u) + \operatorname{div} \phi(u) = f \text{ in } \mathscr{D}'(Q), \tag{77}$$

$$g(u(0,x) = g_0(x) \quad \text{a.e. in } \Omega. \tag{78}$$

The condition that $g(u(0,x) = g_0(x)$ almost everywhere in Ω must be understood in the sense that

$$\int_0^T \left\langle \frac{\partial g(u)}{\partial t}, \xi \right\rangle dt = -\int_Q g(u) \frac{\partial \xi}{\partial t}\, dx\, dt - \int_\Omega g_0 \xi(0)\, dx$$

for any $\xi \in L^2((0,T); H_0^1(\Omega)) \cap W^{1,1}((0,T); L^\infty(\Omega))$ such that $\xi(T) = 0$, where $\langle \cdot, \cdot \rangle$ represents the duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Of course a necessary condition to get a weak solution should be

$$g_0(x) \in R(g) \quad \text{for a.e. } x \in \Omega.$$

Then there exists a measurable function u_0 such that $g_0 = g(u_0)$ almost everywhere in Ω. Moreover, in [BW1] and [AL] the existence of a weak solution is related to an energy estimate on the initial data. In our case such an energy estimate is

$$B_I(g_0) \in L^1(\Omega).$$

In general, the definition of weak solution does not suffice enough to get uniqueness. So we have to define entropy solutions:

Definition 4. Let $f \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$ and let $g_0 \in L^1(\Omega)$. An *entropy solution* of $(P_E(f, g_0)(g, b, \phi))$ is a weak solution u satisfying

$$\int_Q H_0(u-s)\{\nabla b(u) \cdot \nabla \xi - (\phi(u) - \phi(s)) \cdot \nabla \xi \\ - (g(u) - g(s))\xi_t - f\xi\}\, dx\, dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx \leqq 0,$$

$$\int_Q H_0(-s-u)\{\nabla b(u) \cdot \nabla \xi - (\phi(u) - \phi(-s)) \cdot \nabla \xi \\ - (g(u) - g(-s))\xi_t - f\xi\}\, dx\, dt + \int_\Omega (g_0 - g(-s))^- \xi(0)\, dx \geqq 0$$

for any $(s, \xi) \in \mathbb{R} \times (L^2((0,T); H^1(\Omega)) \cap W^{1,1}((0,T); L^\infty(\Omega)))$ such that $s \geqq 0$, $\xi \geqq 0$ and $\xi(T) = 0$ and for any $(s, \xi) \in \mathbb{R} \times (L^2((0,T); H_0^1(\Omega)) \cap W^{1,1}((0,T); L^\infty(\Omega)))$ such that $\xi \geqq 0$ and $\xi(T) = 0$.

Remark 8. If u is an entropy solution of $(P_E(f, g_0)(g, b, \phi))$, then we easily check that $-u$ is an entropy solution of $(P_E(-f, -g_0)(\tilde{g}, \tilde{b}, \tilde{\phi}))$ where $\tilde{g}(r) = -g(-r)$, $\tilde{b}(r) = -b(-r)$ and $\tilde{\phi}(r) = -\phi(-r)$.

In order to prove the existence of weak (entropy) solutions we need an energy estimate similar to the one set in Lemma 1.5 of [AL]. However, to prove the uniqueness of the entropy solution we need a more general estimate:

Lemma 4. *Let $\psi \in \mathscr{C}^{0,1}(\mathbb{R})$, let ψ be monotone, let $g_0 \in L^1(\Omega)$, let $g_0 \in R(g)$ almost everywhere in Ω such that $B_\psi(g_0) \in L^1(\Omega)$, let u be a measurable function such that $g(u)$ satisfies* (75) *and* (78) *and such that $b(u)$ satisfies* (76). *Then*

$$B_\psi(g(u)) \in L^\infty((0,T); L^1(\Omega))$$

and, for almost every $t \in [0,T]$,

$$\int_\Omega B_\psi(g(u(t)))\xi(t)\, dx - \int_\Omega B_\psi(g_0)\xi(0)\, dx \\ = \int_0^t \langle g(u)_t, \psi(b(u))\xi \rangle\, dt + \int_0^t \int_\Omega B_\psi(g(u))\xi_t\, dx\, dt \quad (79)$$

for any $\xi \in \mathscr{C}^{0,1}(\overline{Q})$ such that $\psi(b(u))\xi \in L^2((0,T); H_0^1(\Omega))$.

Proof. For the time being we assume that ψ is nondecreasing and that $\psi(0) = 0$. Then B_ψ is a nonnegative convex function. We define a bounded approximation of ψ:

$$\psi_n(s) = \begin{cases} \psi(n) & \text{for } s > n, \\ \psi(s) & \text{for } -n \leqq s \leqq n, \\ \psi(-n) & \text{for } s < -n. \end{cases}$$

Then B_{ψ_n} is Lipschitz continuous, $B_{\psi_n}(g_0) \in L^1(\Omega)$ (since $0 \leqq B_{\psi_n}(g_0) \leqq B_\psi(g_0)$) and, for almost every $t \in (0, T)$ we have

$$|B_{\psi_n}(g(u(t,x)))| \leqq |B_{\psi_n}(g_0(x))| + C(n)|g(u(t,x)) - g_0(x)|;$$

hence $B_{\psi_n}(g(u)) \in L^1(Q)$.

For $\tau > 0$ and for almost every $t > 0$ we have

$$B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u(t-\tau))) \leqq (g(u(t)) - g(u(t-\tau)))\psi_n(b(u(t))$$

almost everywhere in Ω, and for almost every $t > \tau$ we have

$$B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u(t-\tau))) \geqq (g(u(t)) - g(u(t-\tau)))\psi_n(b(u(t-\tau)))$$

almost everywhere in Ω, where $g(u(t)) = g_0$ for $-\tau < t < 0$.

We consider a nonnegative $\xi \in \mathscr{C}^{0,1}(\mathbb{R}^N)$, and we multiply the last two inequalities respectively by $\xi(t)$ and by $\xi(t-\tau)$:

$$\begin{aligned}
&(B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u(t-\tau))))\xi(t)\\
&= B_{\psi_n}(g(u(t)))\xi(t) - B_{\psi_n}(g(u(t-\tau)))\xi(t-\tau)\\
&\quad + B_{\psi_n}(g(u(t-\tau)))(\xi(t-\tau) - \xi(t))\\
&\leqq (g(u(t)) - g(u(t-\tau)))\psi_n(b(u(t))\xi(t),
\end{aligned} \tag{80}$$

$$\begin{aligned}
&(B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u(t-\tau))))\xi(t-\tau)\\
&= B_{\psi_n}(g(u(t)))\xi(t) - B_{\psi_n}(g(u(t-\tau)))\xi(t-\tau)\\
&\quad + B_{\psi_n}(g(u(t)))(\xi(t-\tau) - \xi(t))\\
&\geqq (g(u(t)) - g(u(t-\tau)))\psi_n(b(u(t-\tau))\xi(t-\tau).
\end{aligned} \tag{81}$$

We divide (80) by τ and we integrate the quotient over $(0, t_0) \times \Omega$:

$$\begin{aligned}
&\frac{1}{\tau}\int_{t_0-\tau}^{t_0}\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt - \frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g_0)\xi(t-\tau)\,dx\,dt\\
&\quad + \frac{1}{\tau}\int_0^{t_0}\int_\Omega B_{\psi_n}(g(u(t-\tau)))(\xi(t-\tau) - \xi(t))\,dx\,dt\\
&\qquad \leqq \frac{1}{\tau}\int_0^{t_0}\int_\Omega (g(u(t)) - g(u(t-\tau)))\psi_n(b(u(t))\xi(t)\,dx\,dt.
\end{aligned}$$

By letting $\tau \to 0$, for almost every $t_0 \in (0, T)$ we have

$$\begin{aligned}
&\int_\Omega B_{\psi_n}(g(u(t_0)))\xi(t_0)\,dx - \int_\Omega B_{\psi_n}(g_0)\xi(0)\,dx\\
&\quad - \int_0^{t_0}\int_\Omega B_{\psi_n}(g(u))\xi_t\,dx\,dt \leqq \int_0^{t_0}\langle g(u)_t, \psi_n(b(u))\xi\rangle\,dt.
\end{aligned} \tag{82}$$

Hence (we set $\xi = 1$ and) we have

$$B_{\psi_n}(g(u)) \in L^\infty((0,T); L^1(\Omega)).$$

Now we divide (81) by τ and we integrate the quotient over $(\tau, t_0) \times \Omega$:

$$\begin{aligned}
&\frac{1}{\tau}\int_{t_0-\tau}^{t_0}\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt - \frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt\\
&+\frac{1}{\tau}\int_\tau^{t_0}\int_\Omega b_{\psi_n}(g(u(t)))(\xi(t-\tau)-\xi(t))\,dx\,dt\\
&\qquad\geqq \frac{1}{\tau}\int_\tau^{t_0}\int_\Omega (g(u(t))-g(u(t-\tau)))\psi_n(b(u(t-\tau))\xi(t-\tau)\,dx\,dt.
\end{aligned}$$

By letting $\tau \to 0$, for almost every $t_0 \in (0,T)$ we have

$$\begin{aligned}
&\int_\Omega B_{\psi_n}(g(u(t_0)))\xi(t_0)\,dx - \limsup_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt\\
&\qquad - \int_0^{t_0}\int_\Omega B_{\psi_n}(g(u))\xi_t\,dx\,dt \geqq \int_0^{t_0}\langle g(u)_t, \psi_n(b(u))\xi\rangle\,dt. \qquad (83)
\end{aligned}$$

From inequalities (82) and (83) we deduce that

$$\limsup_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt \leqq \int_\Omega B_{\psi_n}(g_0)\xi(0)\,dx.$$

Let us prove that

$$\liminf_{\tau\to 0}\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt \geqq \int_\Omega B_{\psi_n}(g_0)\xi(0)\,dx.$$

Since $g_0 \in L^1(\Omega)$ and $g_0 \in R(g)$ a.e. in Ω, we have $g_0 \in \overline{D(A_{(g,b,\phi)})}$ and therefore there exists a sequence $g_\varepsilon \in D(A_{(g,b,\phi)})$ such that

$$\|g_\varepsilon - g_0\|_{L^1(\Omega)} \overset{\varepsilon\to 0}{\to} 0$$

and there exists a sequence of measurable functions u_ε such that $g_\varepsilon = g(u_\varepsilon)$ and such that $b(u_\varepsilon) \in H_0^1(\Omega)$. Then

$$\begin{aligned}
&\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\xi(t)\,dx\,dt - \frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g_0)\xi(0)\,dx\,dt\\
&\quad= \frac{1}{\tau}\int_0^\tau\int_\Omega (B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u_\varepsilon)))\xi(0)\,dx\,dt\\
&\qquad+\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))(\xi(t)-\xi(0))\,dx\,dt\\
&\qquad+\int_\Omega (B_{\psi_n}(g(u_\varepsilon)) - B_{\psi_n}(g_0))\xi(0)\,dx.
\end{aligned}$$

We have

$$\frac{1}{\tau}\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))(\xi(t)-\xi(0))\,dx\,dt$$
$$=\int_0^\tau\int_\Omega B_{\psi_n}(g(u(t)))\frac{(\xi(t)-\xi(0))}{\tau}\,dx\,dt \overset{\tau\to 0}{\longrightarrow} 0,$$

$$\int_\Omega (B_{\psi_n}(g(u_\varepsilon)) - B_{\psi_n}(g_0))\xi(0)\,dx \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Moreover,

$$\frac{1}{\tau}\int_0^\tau\int_\Omega (B_{\psi_n}(g(u(t))) - B_{\psi_n}(g(u_\varepsilon)))\xi(0)\,dx\,dt$$
$$\geqq \frac{1}{\tau}\int_0^\tau\int_\Omega (g(u(t)-g(u_\varepsilon))\psi_n(b(u_\varepsilon))\xi(0)\,dx\,dt$$
$$=\frac{1}{\tau}\int_0^\tau\int_\Omega (g(u(t)-g(u_0))\psi_n(b(u_\varepsilon))\xi(0)\,dx\,dt$$
$$+\int_\Omega (g(u_0-g(u_\varepsilon))\psi_n(b(u_\varepsilon))\xi(0)\,dx$$
$$=-\frac{1}{\tau}\int_0^\tau\left\langle\frac{\partial g(u(t)}{\partial t},\psi_n(b(u_\varepsilon))\xi(0)\right\rangle(t-\tau)\,dt$$
$$+\int_\Omega (g(u_0-g(u_\varepsilon))\psi_n(b(u_\varepsilon))\xi(0)\,dx.$$

The right side of this inequality converges to 0 when $\tau\to 0$ and $\varepsilon\to 0$:

$$\frac{1}{\tau}\int_0^\tau\left\langle\frac{\partial g(u(t)}{\partial t},\psi_n(b(u_\varepsilon))\xi(0)\right\rangle(t-\tau)\,dt \overset{\tau\to 0}{\longrightarrow} 0,$$

$$\int_\Omega (g(u_0-g(u_\varepsilon))\psi_n(b(u_\varepsilon))\xi(0)\,dx \overset{\varepsilon\to 0}{\longrightarrow} 0,$$

whence the identity (79) is proved for ψ_n and for a nonnegative Lipschitz continuous function ξ and then of course, for any Lipschitz continuous function ξ (since ξ^+ and ξ^- are still Lipschitz continuous).

Since $B_{\psi_n}(0)=B_\psi(0)=0$ and since $|\psi_n|\leqq|\psi|$, we deduce that

$$B_{\psi_n}\nearrow B_\psi \quad \text{when } n\to+\infty.$$

Then, if we consider $\xi\equiv 1$ in the above identity, we have

$$\int_\Omega B_{\psi_n}(g(u(t_0)))\,dx = \int_\Omega B_{\psi_n}(g_0)\,dx + \int_0^{t_0}\langle g(u)_t,\psi_n(b(u))\rangle\,dt.$$

We easily check that

$$\|\psi_n(b(u))\|_{H_0^1(\Omega)} \leqq \|\psi(b(u))\|_{H_0^1(\Omega)}$$

$$\psi_n(b(u)) \overset{n\to+\infty}{\to} \psi(b(u)) \quad \text{a.e. in } Q$$

and we deduce that

$$\psi_n(b(u)) \overset{n\to+\infty}{\rightharpoonup} \psi(b(u)) \quad \text{weakly in } H_0^1(\Omega) \text{ for a.e. } t \in (0,T),$$

$$\langle g(u)_t, \psi_n(b(u))\rangle \overset{n\to+\infty}{\to} \langle g(u)_t, \psi(b(u))\rangle \quad \text{a.e. in } (0,T),$$

$$\begin{aligned} |\langle g(u)_t, \psi_n(b(u))\rangle| &\leqq \|g(u)_t\|_{H^{-1}(\Omega)} \|\psi_n(b(u))\|_{H_0^1(\Omega)} \\ &\leqq \|g(u)_t\|_{H^{-1}(\Omega)} \|\psi(b(u))\|_{H_0^1(\Omega)} \quad \text{for a.e. } t \in (0,T), \end{aligned}$$

which is a function of $L^1(0,T)$. From the Lebesgue theorem we deduce that

$$\int_0^{t_0} \langle g(u)_t, \psi_n(b(u))\rangle\, dt \overset{n\to+\infty}{\to} \int_0^{t_0} \langle g(u)_t, \psi(b(u))\rangle\, dt \quad \text{for any } t_0 \in [0,T].$$

Moreover,

$$\int_\Omega B_{\psi_n}(g_0)\, dx \nearrow \int_\Omega B_\psi(g_0)\, dx \quad \text{when } n \to +\infty.$$

Then we have

$$\begin{aligned} \int_\Omega B_{\psi_n}(g(u(t_0)))\, dx &= \int_\Omega B_{\psi_n}(g_0)\, dx + \int_0^{t_0} \langle g(u)_t, \psi_n(b(u))\rangle\, dt \\ &\nearrow \int_\Omega B_\psi(g_0)\, dx + \int_0^{t_0} \langle g(u)_t, \psi(b(u))\rangle\, dt, \end{aligned}$$

$$\begin{aligned} &\int_\Omega B_{\psi_n}(g(u(t_0)))\, dx \\ &\qquad \leqq \int_\Omega B_\psi(g_0)\, dx + \int_0^{t_0} \|g(u)_t\|_{H^{-1}(\Omega)} \|\psi(b(u))\|_{H_0^1(\Omega)}\, dt. \end{aligned}$$

Then, from the Monotone Convergence Theorem we deduce that the function $B_\psi(g(u(t_0)))$ belongs to $L^1(\Omega)$ for almost every $t_0 \in (0,T)$, that

$$\int_\Omega B_{\psi_n}(g(u(t_0)))\, dx \overset{n\to+\infty}{\longrightarrow} \int_\Omega B_\psi(g(u(t_0)))\, dx,$$

that

$$\|B_\psi(g(u(t_0))) - B_{\psi n}(g(u(t_0)))\|_{L^1(\Omega)} \overset{n\to+\infty}{\longrightarrow} 0$$

since $B_{\psi_n} \leqq B_\psi$, and, for a.e. $t_0 \in (0,T)$, that

$$\int_\Omega B_\psi(g(u(t_0)))\,dx = \int_\Omega B_\psi(g_0)\,dx + \int_0^{t_0} \langle g(u)_t, \psi(b(u))\rangle\,dt.$$

In particular, we deduce that $B_\psi(g(u)) \in L^\infty((0,T); L^1(\Omega))$. Since (79) is true for ψ_n and for any nonnegative Lipschitz continuous function ξ, by letting $n \to +\infty$ we get (79) for ψ and for any nonnegative Lipschitz continuous function ξ, and therefore for any Lipschitz continuous function ξ.

When ψ is a nonincreasing function, we replace ψ by $-\psi$ to get (79).

When $\psi(0) \neq 0$, we consider the function $\tilde\psi = \psi - \psi(0)$. Obviously we have $B_\psi(g(u)) = B_{\tilde\psi}(g(u)) + \psi(0)g(u)$. In this case, $\xi \in L^2((0,T); H_0^1(\Omega))$ (since $\psi(b(u))\xi \in L^2((0,T); H_0^1(\Omega))$). Then we have

$$\begin{aligned}\int_0^t \langle g(u)_t, \psi(b(u))\xi\rangle dt &= \int_0^t \langle g(u)_t, \tilde\psi(b(u))\xi\rangle dt + \int_0^t \langle g(u)_t, \psi(0)\xi\rangle dt\\ &= \int_\Omega B_{\tilde\psi}(g(u(t)))\xi(t)\,dx - \int_\Omega B_{\tilde\psi}(g_0)\xi(0)\,dx\\ &\quad + \psi(0)\int_\Omega g(u(t))\xi(t)\,dx - \psi(0)\int_\Omega g_0\xi(0)\,dx\\ &\quad - \int_0^t\int_\Omega B_{\tilde\psi}(g(u))\xi_t\,dx\,dt - \int_0^t\int_\Omega g(u)\xi_t\,dx\,dt.\end{aligned}$$

Remark 9. Equation (79) is still true for any Lipschitz continuous function ψ since we can write $\psi = \psi_1 + \psi_2$ where ψ_1 is nondecrasing and ψ_2 is nonincreasing.

The set E is defined as in (4).

Lemma 5. *Let* (H1) *and* (H2) *hold. Let $f \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$, let $g_0 \in L^1(\Omega)$, and let $g_0 \in R(g)$ almost everywhere in Ω. Let u be a weak solution of $(P_E(f, g_0)(g, b, \phi))$. Then*

$$\begin{aligned}\int_Q H_0(u-s)\{(\nabla b(u) + \phi(s) - \phi(u))\cdot\nabla\xi\\ + (g(s) - g(u))\xi_t - f\xi\}\,dx\,dt - \int_\Omega (g_0 - g(s))^+\xi(0)\,dx\\ = -\lim_{\varepsilon\to 0}\int_Q |\nabla b(u)|^2 H_\varepsilon'(b(u) - b(s))\,dx\,dt \quad (84)\end{aligned}$$

for any $(s,\xi) \in \mathbb{R} \times \mathscr{D}([0,T) \times \overline{\Omega})$ such that $b(s) \notin E$, $s \geqq 0$ and $\xi \geqq 0$ and for any $(s,\xi) \in \mathbb{R} \times \mathscr{D}([0,T) \times \Omega)$ such that $b(s) \notin E$ and $\xi \geqq 0$. Moreover,

$$\int_Q H_0(s-u)\{(\nabla b(u)+\phi(s)-\phi(u))\cdot\nabla\xi + (g(s)-g(u))\xi_t - f\xi\}\,dx\,dt + \int_\Omega (g(s)-g_0)^+\xi(0)\,dx = \lim_{\varepsilon\to 0}\int_Q |\nabla b(u)|^2 H'_\varepsilon(b(s)-b(u))\,dx\,dt \quad (85)$$

for any $(s,\xi)\in\mathbb{R}\times\mathscr{D}([0,T)\times\overline{\Omega})$ *such that* $b(s)\notin E$, $s\leqq 0$ *and* $\xi\geqq 0$ *and for any* $(s,\xi)\in\mathbb{R}\times\mathscr{D}([0,T)\times\Omega)$ *such that* $b(s)\notin E$ *and* $\xi\geqq 0$.

Proof. We observe that for s such that $b(s)\notin E$ we have

$$H_0(u-s)=H_0(b(u)-b(s))\quad \text{a.e. in } Q.$$

Let (s,ξ) be as in the lemma. Then $H_\varepsilon(b(u)-b(s))\xi\in L^2((0,T);H_0^1(\Omega))$. We denote by $\psi_\varepsilon(z)=H_\varepsilon(z-b(s))$. Since ψ_ε is bounded, we have $B_{\psi_\varepsilon}(g_0)\in L^1(\Omega)$ and, from Lemma 4 we have $B_{\psi_\varepsilon}(g(u))\in L^\infty((0,T);L^1(\Omega))$ and

$$\int_Q B_{\psi_\varepsilon}(g(u))\xi_t\,dx\,dt+\int_\Omega B_{\psi_\varepsilon}(g_0)\xi(0)\,dx = -\int_0^T\langle g(u)_t, H_\varepsilon(b(u)-b(s))\xi\rangle\,dt.$$

Moreover, since u is a weak solution and since $H_\varepsilon(b(u)-b(s))\xi$ belongs to $L^2((0,T);H_0^1(\Omega))$, we have

$$-\int_0^T\langle g(u)_t, H_\varepsilon(b(u)-b(s))\xi\rangle\,dt$$

$$=\int_Q\{(\nabla b(u)+\phi(s)-\phi(u))\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi)-fH_\varepsilon(b(u)-b(s))\xi\}\,dx\,dt,$$

and therefore

$$\int_Q B_{\psi_\varepsilon}(g(u))\xi_t\,dx\,dt+\int_\Omega B_{\psi_\varepsilon}(g_0)\xi(0)\,dx = \int_Q\{(\nabla b(u)+\phi(s)-\phi(u))\cdot\nabla(H_\varepsilon(b(u)-b(s))\xi) - fH_\varepsilon(b(u)-b(s))\xi\}\,dx\,dt. \quad (86)$$

The function $B_{\psi_\varepsilon}(g(u))$ was defined by

$$B_{\psi_\varepsilon}(g(u))=\int_0^{g(u)} H_\varepsilon(b\circ(g^{-1})_0(r)-b(s))\,dr,$$

and therefore

$$B_{\psi_\varepsilon}(g(u))=\int_{g(s)}^{g(u)} H_\varepsilon(b\circ(g^{-1})_0(r)-b(s))\,dr \xrightarrow{\varepsilon\to 0} (g(u)-g(s))^+$$

almost everywhere in Q because g is continuous and because, since $b(s) \notin E$, we have $b \circ (g^{-1})_0(r) - b(s) > 0$ for any $r > g(s)$, whence $H_\varepsilon(b \circ (g^{-1})_0(r) - b(s))$ converges to 1 when $\varepsilon \to 0$ for any $r > g(s)$.

Similarly we have

$$\lim_{\varepsilon \to 0} B_{\psi_\varepsilon}(g_0) = (g_0 - g(s))^+ \quad \text{a.e. in } \Omega.$$

Moreover, $|B_{\psi_\varepsilon}(g(u))| \leqq |g(u)|$ and $|B_{\psi_\varepsilon}(g_0)| \leqq |g_0|$. Then, by applying the Lebesgue Theorem we deduce that

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_Q B_{\psi_\varepsilon}(g(u))\xi_t \, dx\, dt + \int_\Omega B_{\psi_\varepsilon}(g_0)\xi(0)\, dx \\ = \int_Q (g(u) - g(s))^+ \xi_t \, dx\, dt + \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx.\end{aligned}$$

By arguing as in the proof of Lemma 1, for s such that $b(s) \notin E$, we have

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_Q &\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla(H_\varepsilon(b(u) - b(s))\xi) \\ &\quad - f H_\varepsilon(b(u) - b(s))\xi\}\, dx\, dt \\ &= \int_Q H_0(u - s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi) - f\xi\}\, dx\, dt \\ &\quad + \lim_{\varepsilon \to 0} \int_Q \nabla b(u) \cdot \nabla H_\varepsilon(b(u) - b(s))\, \xi \, dx\, dt \\ &= \int_Q H_0(u - s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi - f\xi\}\, dx\, dt \\ &\quad + \lim_{\varepsilon \to 0} \int_Q |\nabla b(u)|^2 H_\varepsilon'(b(u) - b(s))\, \xi \, dx\, dt.\end{aligned}$$

Therefore, by letting $\varepsilon \to 0$ in (86) we get

$$\begin{aligned}\int_Q (g(u) - g(s))^+ \xi_t \, dx\, dt + \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx \\ = + \int_Q H_0(u - s)\{(\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi - f\xi\}\, dx\, dt \\ + \lim_{\varepsilon \to 0} \int_Q |\nabla b(u)|^2 H_\varepsilon'(b(u) - b(s))\, \xi \, dx\, dt.\end{aligned}$$

Then (85) is a consequence of (84) and of Remark 8.

Then we have

Corollary 9. *Let* (H1) *and* (H2) *hold. Let* $b^{-1} \in \mathscr{C}(\mathbb{R})$. *Then any weak solution of* $(P_E(f, g_0)(g, b, \phi))$ *is an entropy solution.*

Proof. In this case the set E is empty. Hence identities (84) and (85) hold for any $s \in \mathbb{R}$.

To complete this section we prove

Theorem 12. *Let* (H1), (H3) *and* (H5) *hold. Let $g_0 \in L^\infty(\Omega)$, let $g_0 \in R(g)$, let $f \in L^\infty(Q)$ and let v be the unique integral solution of $(P_E(f, g_0)(A_{(g,b,\phi)}))$. Then there exists an entropy solution u of problem $(P_E(f, g_0)(g, b, \phi))$ such that $g(u) = v$.*

Proof. For any $M \in \mathbb{N}$ we define $\tau = T/M$. For $i = 0, 1, \dots, M$ we define $t_i = i \times \tau$. Let $f_1, f_2, \dots, f_M \in L^\infty(\Omega)$ be such that

$$\|f_i\|_{L^\infty(\Omega)} \leqq \|f\|_{L^\infty(Q)},$$

$$\sum_{i=1}^{M} \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_{L^1(\Omega)}\, dt \overset{M\to+\infty}{\longrightarrow} 0.$$

For $i = 1, \dots, M$ let v_i be the unique solution of

$$\tau f_i + v_{i-1} \in (I + \tau A_{(g,b,\phi)})v_i,$$

i.e., there exists a measurable function u_i such that $v_i = g(u_i)$ and u_i is an entropy solution of $(P_S(\tau f_i + v_{i-1})(g, \tau b, \tau\phi))$ for $1 \leqq i \leqq M$, where $v_0 = g_0$. Moreover, v_i satisfies

$$\|v_i\|_{L^\infty(\Omega)} \leqq \|v_{i-1}\|_{L^\infty(\Omega)} + \tau\|f_i\|_{L^\infty(\Omega)}, \quad 1 \leqq i \leqq M,$$

and therefore

$$\begin{aligned}\|v_i\|_{L^\infty(\Omega)} &\leqq \|v_0\|_{L^\infty(\Omega)} + \tau \sum_{j=1}^{i} \|f_j\|_{L^\infty(\Omega)}\\ &= \|v_0\|_{L^\infty(\Omega)} + \int_0^{t_i} \|f^\tau\|_{L^\infty(\Omega)}\, dt\end{aligned}$$

where

$$f^\tau(t) = f_i \quad \text{for } t_{i-1} < t \leqq t_i \text{ and } 1 \leqq i \leqq M.$$

Similarly we define $v^\tau = g(u^\tau)$ where

$$u^\tau(t) = u_i \quad \text{for } t_{i-1} < t \leqq t_i \text{ and } 1 \leqq i \leqq M.$$

Since v is an integrable solution, we know (see [Be]) that

$$\|v(t) - v^\tau\|_{L^\infty((0,T);L^1(\Omega))} \overset{M\to+\infty}{\longrightarrow} 0,$$

and then, since $v^\tau = g(u^\tau)$ is uniformly bounded in $L^\infty(Q)$,

$$\begin{aligned}\|g(u^\tau)\|_{L^\infty(Q)} &\leqq \|g_0\|_{L^\infty(\Omega)} + \int_0^T \|f^\tau\|_{L^\infty(\Omega)}\, dt\\ &\leqq \|g_0\|_{L^\infty(\Omega)} + T\|f\|_{L^\infty(Q)}.\end{aligned}$$

From the Lebesgue Theorem we deduce that

$$\|v(t) - v^\tau\|_{L^p(Q)} \overset{M\to+\infty}{\longrightarrow} 0 \quad \text{for } 1 \leqq p < +\infty.$$

Moreover, since $\phi^{(i)}$ is continuous for $i = 1, 2$, it follows that $\phi^{(i)}(g(u^\tau))$ is bounded in $L^\infty(Q)$ and since $g(u^\tau)$ converges to v in $L^\infty((0, T); L^1(\Omega))$, we deduce the existence of a subsequence of τ, still denoted by τ such that

$$g(u^\tau) \overset{\tau\to 0}{\longrightarrow} v \quad \text{a.e. in } Q.$$

Hence, from the Lebesgue Theorem, we obtain

$$\phi^{(i)}(g(u^\tau)) \overset{\tau\to 0}{\longrightarrow} \phi^{(i)}(v), \quad \text{in } L^p(Q) \text{ for } 1 \leqq p < +\infty, \text{ for } i = 1, 2.$$

Now, since u_i is an entropy (and therefore a weak) solution of problem $(P_S(\tau f_i + g(u_{i-1}))(g, \tau b, \tau\phi))$ for $1 \leqq i \leqq M$, we have

$$\int_\Omega \{(g(u_i) - g(u_{i-1}) - \tau f_i)\xi + \tau(\nabla b(u_i) - \phi(u_i)) \cdot \nabla\xi\}\, dx = 0$$

for any $\xi \in H_0^1(\Omega)$. In particular, for $\xi = b(u_i)$ we have

$$\int_\Omega \{(g(u_i) - g(u_{i-1}))b(u_i) + \tau|\nabla b(u_i)|^2\}\, dx = \tau \int_\Omega f_i b(u_i)\, dx.$$

Since B_I is convex we deduce that

$$\int_\Omega \{(B_I(g(u_i)) - B_I(g(u_{i-1}))) + \tau|\nabla b(u_i)|^2\}\, dx \leqq \tau \int_\Omega f_i b(u_i)\, dx,$$

where $B_I(g(u_i)) \in L^\infty(\Omega)$. Then we deduce that

$$\int_\Omega \{(B_I(g(u_M)) - B_I(g(u_0)))\, dx + \tau \sum_{i=1}^M \int_\Omega |\nabla b(u_i)|^2\}\, dx$$
$$\leqq \tau \sum_{i=1}^M \int_\Omega f_i b(u_i)\, dx.$$

Hence

$$\|b(u^\tau)\|_{L^2((0,T);H_0^1(\Omega))} \leqq C$$

and therefore there exists a subsequence of τ, still denoted by τ, such that

$$b(u^\tau) \overset{\tau\to 0}{\rightharpoonup} w \quad \text{weakly in } L^2((0, T); H_0^1(\Omega)).$$

Since $g(u^\tau)$ converges in $L^2(Q)$ and $b(u^\tau)$ converges weakly in $L^2(Q)$ and since $b \circ g^{-1}$ is a maximal monotone operator (in $L^2(Q)$), we deduce that

$$w \in b \circ g^{-1}(v),$$

whence there exists $\tilde{u} \in g^{-1}(v)$ such that $w = b(\tilde{u})$. Then we set

$$u = ((b+g)^{-1})_0(v+w) = ((b+g)^{-1})_0(b(\tilde{u}) + g(\tilde{u})).$$

Obviously, u is a measurable function and we have $v = g(u)$ and $w = b(u)$. Moreover, since $\phi^{(2)}(g(u^\tau))$ converges to $\phi^{(2)}(v) = \phi^{(2)}(g(u))$ in $(L^p(Q))^N$ for any $p \in [1, +\infty)$, we deduce that

$$\phi^{(2)}(g(u^\tau))b(u^\tau) \rightharpoonup \phi^{(2)}(g(u))b(u) \quad \text{weakly in } (L^p(Q))^N \text{ for } 1 \leqq p < 2.$$

Since $\phi^{(2)}(g(u^\tau))$ is uniformly bounded in $(L^\infty(Q))^N$, we easily deduce that

$$\phi^{(2)}(g(u^\tau))b(u^\tau) \rightharpoonup \phi^{(2)}(g(u))b(u) \quad \text{weakly in } (L^2(Q))^N.$$

Now let $\xi \in \mathscr{D}(\overline{Q})$ be such $\xi = 0$ on $((0,T) \times \partial\Omega) \cup (\{T\} \times \Omega)$. Then

$$\begin{aligned}
&\int_Q f^\tau(t)\xi(t)\,dx\,dt \\
&= \int_Q \Bigg\{ \frac{g(u^\tau(t)) - g(u^\tau(t-\tau))}{\tau}\xi(t) \\
&\qquad + (\nabla b(u^\tau(t)) - \phi(u^\tau(t))) \cdot \nabla\xi(t) \Bigg\}\,dx\,dt \\
&= \int_Q \Bigg\{ g(u^\tau(t)\frac{\xi(t) - \xi(t+\tau)}{\tau} + (\nabla b(u^\tau(t)) - \phi(u^\tau(t))) \cdot \nabla\xi(t) \Bigg\}\,dx\,dt \\
&\quad - \frac{1}{\tau}\int_0^\tau \int_\Omega g_0\xi(t)\,dx\,dt
\end{aligned}$$

where $g(u^\tau(t) = g_0$ for $t \leqq 0$. By letting $\tau \to 0$ we get

$$\int_Q f\xi\,dx\,dt = \int_Q \{-g(u)\xi_t + (\nabla b(u) - \phi(u)) \cdot \nabla\xi\}\,dx\,dt - \int_\Omega g_0\xi(0)\,dx.$$

Hence we easily deduce that $g(u)_t \in L^2((0,T); H^{-1}(\Omega))$ and that u is a weak solution of $(P_E(f, g_0)(g, b, \phi))$. Moreover, we deduce that

$$\frac{g(u^\tau(\cdot)) - g(u^\tau(\cdot - \tau))}{\tau} \overset{\tau \to 0}{\rightharpoonup} g(u(\cdot))_t \quad \text{weakly in } L^2((0,T); H^{-1}(\Omega)).$$

Then

$$\begin{aligned}
&\int_Q f^\tau(b(u^\tau) - b(u))\,dx\,dt \\
&\qquad = \int_Q \Bigg\{ \frac{g(u^\tau(t)) - g(u^\tau(t-\tau))}{\tau}(b(u^\tau(t)) - b(u(t))) \\
&\qquad\qquad + |\nabla(b(u^\tau(t)) - b(u(t)))|^2\}\,dx\,dt \Bigg\} \\
&\qquad\quad + \int_Q (\nabla b(u(t)) - \phi(u^\tau(t))) \cdot \nabla(b(u^\tau(t)) - b(u(t)))\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&\geqq \frac{1}{\tau}\int_Q (B_I(g(u^\tau(t))) - B_I(g(u^\tau(t-\tau))))\,dx\,dt\\
&\quad - \int_Q \frac{g(u^\tau(t)) - g(u^\tau(t-\tau))}{\tau} b(u(t))\,dx\,dt\\
&\quad + \int_Q |\nabla(b(u^\tau(t)) - b(u(t)))|^2\,dx\,dt\\
&\quad + \int_Q \nabla(b(u(t)) \cdot \nabla(b(u^\tau(t)) - b(u(t)))\,dx\,dt\\
&\quad - \int_Q \phi(u^\tau(t)) \cdot \nabla(b(u^\tau(t)) - b(u(t)))\,dx\,dt.
\end{aligned}
$$

If we consider that $b(u^\tau) \rightharpoonup b(u)$ weakly in $L^2((0,T), H_0^1(\Omega))$ and that

$$
\begin{aligned}
&\int_Q \phi(u^\tau(t)) \cdot \nabla(b(u^\tau(t)) - b(u(t)))\,dx\,dt\\
&\qquad = -\int_Q \phi(u^\tau(t)) \cdot \nabla b(u(t))\,dx\,dt\\
&\qquad \to -\int_Q \phi(u(t)) \cdot \nabla b(u(t))\,dx\,dt = 0
\end{aligned}
$$

(since $g(u) \in g \circ (b)^{-1}(b(u))$), by letting $\tau \to 0$ we get

$$
\begin{aligned}
0 &\geqq \int_\Omega (B_I(g(u(T))) - B_I(g_0))\,dx - \int_0^T \langle g(u)_t, b(u)\rangle\,dt\\
&\quad + \limsup_{\tau\to 0}\int_Q |\nabla(b(u^\tau(t)) - b(u(t)))|^2\,dx\,dt\\
&= \limsup_{\tau\to 0}\int_Q |\nabla(b(u^\tau(t)) - b(u(t)))|^2\,dx\,dt,
\end{aligned}
$$

whence we have

$$
\|b(u^\tau) - b(u)\|_{L^2((0,T);H_0^1(\Omega))} \overset{\tau\to 0}{\longrightarrow} 0.
$$

Hence

$$
\begin{aligned}
\phi(u^\tau) &= \phi^{(2)}(g(u^\tau))b(u^\tau) + \phi^{(1)}(g(u^\tau))\\
&\to \phi^{(2)}(g(u))b(u) + \phi^{(1)}(g(u)) = \phi(u)
\end{aligned}
$$

in $(L^p(Q))^N$ for any $1 \leqq p < 2$, and

$$
\begin{aligned}
\phi(u^\tau) &= \phi^{(2)}(g(u^\tau))b(u^\tau) + \phi^{(1)}(g(u^\tau))\\
&\rightharpoonup \phi^{(2)}(g(u))b(u) + \phi^{(1)}(g(u)) = \phi(u)
\end{aligned}
$$

weakly in $(L^2(Q))^N$.

Now, since u_i is an entropy solution of $(P_S(\tau f_i + g(u_{i-1}))(g, b, \phi))$, we have

$$0 \geqq \int_\Omega H_0(u_i - s)\{(g(u_i) - g(u_{i-1}) - \tau f_i)\xi + \tau(\nabla b(u_i) + \phi(s) - \phi(u_i)) \cdot \nabla\xi\}\, dx$$

for any $(s, \xi) \in \mathbb{R} \times H_0^1(\Omega)$ such that $\xi \geqq 0$ and for any $(s, \xi) \in \mathbb{R} \times H^1(\Omega)$ such that $s \geqq 0$ and $\xi \geqq 0$. Therefore, we have

$$0 \geqq \int_Q H_0(u^\tau(t) - s)\left\{\left(\frac{g(u^\tau(t)) - g(u^\tau(t-\tau))}{\tau} - f^\tau(t)\right)\xi(t) + (\nabla b(u(t)) + \phi(s) - \phi(u^\tau(t))) \cdot \nabla\xi(t)\right\} dx\, dt$$

for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$; and for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $s \geqq 0$.

Now let us observe that for such functions ξ we have

$$\begin{aligned}
&\int_Q H_0(u^\tau(t) - s)\frac{g(u^\tau(t)) - g(u^\tau(t-\tau))}{\tau}\xi(t)\, dx\, dt \\
&\quad = \int_Q \frac{H_0(u^\tau(t) - s)(g(u^\tau(t)) - g(s))}{\tau}\xi(t)\, dx\, dt \\
&\qquad - \int_Q \frac{H_0(u^\tau(t) - s)(g(u^\tau(t-\tau)) - g(s))}{\tau}\xi(t)\, dx\, dt \\
&\quad \geqq \int_Q \frac{H_0(u^\tau(t) - s)(g(u^\tau(t)) - g(s))}{\tau}\xi(t)\, dx\, dt \\
&\qquad - \int_Q \frac{H_0(u^\tau(t-\tau) - s)(g(u^\tau(t-\tau)) - g(s))}{\tau}\xi(t)\, dx\, dt \\
&\quad = \int_Q H_0(u^\tau(t) - s)(g(u^\tau(t)) - g(s))\frac{\xi(t) - \xi(t+\tau)}{\tau}\, dx\, dt \\
&\qquad - \frac{1}{\tau}\int_0^\tau \int_\Omega H_0(u_0 - s)(g_0 - g(s))\xi(t)\, dx\, dt.
\end{aligned}$$

Then we deduce that

$$\begin{aligned}
0 \geqq \int_Q H_0(u^\tau(t) - s)\Big\{&(g(u^\tau(t)) - g(s))\frac{\xi(t) - \xi(t+\tau)}{\tau} - f^\tau(t))\xi(t) \\
&+ (\nabla b(u(t)) + \phi(s) - \phi(u^\tau(t))) \cdot \nabla\xi(t)\Big\}\, dx\, dt \\
&- \frac{1}{\tau}\int_0^\tau \int_\Omega H_0(u_0 - s)(g_0 - g(s))\xi(t)\, dx\, dt
\end{aligned}$$

for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$ and for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $s \geqq 0$.

Since H_0 is bounded, there exists a subsequence of τ still denoted by τ such that

$$H_0(u^\tau - s) \overset{\tau\to 0}{\rightharpoonup} \chi_{u,s} \quad \text{in } L^\infty(Q) \text{ weak} \star.$$

We easily check that

$$H_0(u^\tau - s) \in H(g(u^\tau) + b(u^\tau) - g(s) - b(s)).$$

Then, since $g(u^\tau) + b(u^\tau)$ converges to $g(u) + b(u)$ in $L^2(Q)$, we deduce that

$$\chi_{u,s} \in H(g(u) + b(u) - g(s) - b(s)).$$

By letting $\tau \to 0$ we get

$$\begin{aligned} 0 \geqq \int_Q &\chi_{u,s}\{(g(s) - g(u))\xi_t - f\xi + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi\}\, dx\, dt \\ &- \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx \end{aligned}$$

for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$ and for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $s \geqq 0$.

Since $\chi_{u,s} \in H(g(u) + b(u) - g(s) - b(s))$, we still get

$$\begin{aligned} 0 \geqq \int_Q &\{-\chi_{u,s} f\xi + H_0(u - s)((g(s) - g(u))\xi_t \\ &+ (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi)\}\, dx\, dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx \end{aligned} \tag{87}$$

for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$; and for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $s \geqq 0$.

Now let $[s^{(m)}, s^{(M)}] = (g + b)^{-1}(g(s) + b(s))$ and let $s_n \searrow s^{(M)}$. Then $\chi_{u,s_n} \nearrow H_0(u - s^{(M)})$ almost everywhere in Q. From the last inequality and from the Lebesgue Theorem we deduce that

$$\begin{aligned} 0 \geqq \int_Q &H_0(u - s^{(M)})\{(g(s^{(M)}) - g(u))\xi_t - f\xi \\ &+ (\nabla b(u) + \phi(s^{(M)}) - \phi(u)) \cdot \nabla\xi\}\, dx\, dt - \int_\Omega (g_0 - g(s^{(M)}))^+ \xi(0)\, dx \end{aligned}$$

for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$ and such that $\xi = 0$ on $(0, T) \times \partial\Omega$.

Since $g(s) = g(s^{(M)}) = g(s^{(m)})$ and $b(s) = b(s^{(M)}) = b(s^{(m)})$ for any $s \in (g+b)^{-1}(g(s)+b(s))$, from assumption (H5) we have $\phi(s) = \phi(s^{(M)}) = \phi(s^{(m)})$. Hence we deduce that

$$0 \geqq \int_Q \{-H_0(u - s^{(M)}) f\xi + H_0(u-s)((g(s) - g(u))\xi_t - f\xi + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi)\}\,dx\,dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\,dx \tag{88}$$

for any $s \in (g+b)^{-1}(g(s)+b(s))$ and for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$ and such that $\xi = 0$ on $(0,T) \times \partial\Omega$. Now let $s_n < s^{(m)}$, $s_n \nearrow s^{(m)}$. Then $\chi_{u,s_n)} \searrow H_{\max}(u - s^{(m)})$ and $H_0(u - s_n) \searrow H_{\max}(u - s^{(m)})$ almost everywhere in Q. From (87) applied to s_n and by letting $n \to +\infty$ we deduce that

$$0 \geqq \int_Q H_{\mathrm{Max}}(u - s^{(m)})\{(g(s^{(m)}) - g(u))\xi_t - f\xi + (\nabla b(u) + \phi(s^{(m)}) - \phi(u)) \cdot \nabla\xi\}\,dx\,dt - \int_\Omega (g_0 - g(s^{(m)}))^+ \xi(0)\,dx,$$

whence we get

$$0 \geqq \int_Q \{-H_{\mathrm{Max}}(u - s^{(m)}) f\xi + H_0(u-s)[(g(s) - g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi]\}\,dx\,dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\,dx \tag{89}$$

for any $s \in (g+b)^{-1}(g(s)+b(s))$ and for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$ and $\xi = 0$ on $(0,T) \times \partial\Omega$.

Now let κ be smooth enough, $0 \leqq \kappa \leqq 1$ and let $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0,T) \times \partial\Omega$. We substitute $\xi\kappa$ into (89) and we substitute $\xi(1-\kappa)$ into (88), whence we get

$$\begin{aligned}0 \geqq \int_Q \{&(H_0(u - s^{(M)}) - H_{\mathrm{Max}}(u - s^{(m)}))\kappa f\xi - H_0(u - s^{(M)}) f\xi \\ &+ H_0(u-s)((g(s) - g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi)\}\,dx\,dt \\ &- \int_\Omega (g_0 - g(s))^+ \xi(0)\,dx\end{aligned}$$

for any $s \in (g+b)^{-1}(g(s)+b(s))$, for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$ and $\xi = 0$ on $(0,T) \times \partial\Omega$, and for any $0 \leqq \kappa \leqq 1$, with κ smooth enough. In particular, by density, it is enough to have $\kappa \in L^\infty(Q)$; then we choose

$$\kappa = \frac{H_0(u - s^{(M)}) - H_0(u-s)}{H_0(u - s^{(M)}) - H_0(u - s^{(m)})}$$

and the preceding inequality becomes

$$0 \geqq \int_Q H_0(u-s)\{-f\xi + (g(s)-g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla \xi\}\, dx\, dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx \quad (90)$$

for any $s \in (g+b)^{-1}(g(s)+b(s))$, and hence for any $s \in \mathbb{R}$, and for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$.

Finally let $\xi \in \mathscr{D}(\overline{Q})$ with $\xi \geqq 0$, $\xi(T) = 0$, let $s \geqq 0$ and let $\zeta_n \in \mathscr{D}(\Omega)$, with $0 \leqq \zeta_n \leqq 1$, $\zeta_n(x) \to 1$ for any $x \in \Omega$. We substitute $\xi\zeta_n$ into (90) and $\xi(1-\zeta_n)$ into (87) to get

$$0 \geqq \int_Q \{-(H_0(u-s)\zeta_n + \chi_{u,s}(1-\zeta_n)) f\xi + H_0(u-s)((g(s)-g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi)\}\, dx\, dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx.$$

By letting $n \to +\infty$, we get

$$0 \geqq \int_Q H_0(u-s)\{-f\xi + (g(s)-g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi\}\, dx\, dt - \int_\Omega (g_0 - g(s))^+ \xi(0)\, dx$$

for any nonnegative $s \in \mathbb{R}$ and for any nonnegative $\xi \in \mathscr{D}(\overline{Q})$ such that $\xi(T) = 0$.

Arguing as above we prove

$$0 \leqq \int_Q H_0(s-u)\{-f\xi + (g(s)-g(u))\xi_t + (\nabla b(u) + \phi(s) - \phi(u)) \cdot \nabla\xi\}\, dx\, dt + \int_\Omega (g(s) - g_0)^+ \xi(0)\, dx$$

for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $\xi = 0$ on $(0, T) \times \partial\Omega$ and for any $(s, \xi) \in \mathbb{R} \times \mathscr{D}(\overline{Q})$ such that $\xi \geqq 0$, $\xi(T) = 0$ and $s \leqq 0$.

5. Comparison and Uniqueness of Entropy Solutions of $(P_E(f, g_0)(g, b, \phi))$

Theorem 13. *Let* (H1) *and* (H2) *hold. Let* $g_{i_0} \in L^1(\Omega)$, *let* $g_{i_0} \in R(g)$ ($g_{i_0} = g(u_{i_0})$), *let* $f_i \in L^2((0, T); H^{-1}(\Omega)) \cap L^1(Q)$ *and let* u_i *be an entropy solution* *of* $(P_E(f_i, g_{i_0})(g, b, \phi))$ *for* $i = 1, 2$. *Then*

$$\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\phi(u_2) - \phi(u_1)) \cdot \nabla\xi$$
$$- (g(u_1) - g(u_2))^+ \xi_t\}\, dx dt - \int_\Omega (g_{1_0} - g_{2_0})^+ \xi(0)\, dx$$
$$\leqq \int_Q \kappa(f_1 - f_2)\xi\, dx\, dt \tag{91}$$

for some $\kappa \in H(u_1 - u_2)$ *and for any nonnegative* $\xi \in \mathscr{D}([0, T) \times \Omega)$.

Proof. Let us consider two different pairs of variables (s, y) and (t, x) in Q. We assume that $u_1 = u_1(s, y)$, $f_1 = f_1(s, y)$, $g_{1_0} = g_{1_0}(y)$ and that $u_2 = u_2(t, x)$, $f_2 = f_2(t, x)$, $g_{2_0} = g_{2_0}(x)$. Let ζ be a smooth nonnegative function such that

$$(s, y) \mapsto \zeta(t, x, s, y) \in \mathscr{D}([0, T) \times \Omega) \quad \text{for any } (t, x) \in Q, \tag{92}$$

$$(t, x) \mapsto \zeta(t, x, s, y) \in \mathscr{D}([0, T) \times \Omega) \quad \text{for any } (s, y) \in Q. \tag{93}$$

Let us define

$$Q_1 = \{(s, y) \in Q / u_1(s, y) \in E\}, \quad Q_2 = \{(t, x) \in Q / u_2(t, x) \in E\}.$$

We deduce that

$$\nabla_y u_1 = 0 \quad \text{a.e. in } Q_1, \tag{94}$$

$$\nabla_x u_2 = 0 \quad \text{a.e. in } Q_2. \tag{95}$$

Moreover, we easily check that

$$\begin{gathered} H_0(u_1 - u_2) = H_0(b(u_1) - b(u_2)) \\ \text{a.e. in } (Q \setminus Q_1) \times Q \cup Q \times (Q \setminus Q_2). \end{gathered} \tag{96}$$

By taking into account the definition of entropy solution and Lemma 5 we deduce that

$$\int_{Q\times Q} H_0(u_1 - u_2)\{(\nabla_y b(u_1) + \phi(u_2) - \phi(u_1)) \cdot \nabla_y \zeta$$
$$- (g(u_1) - g(u_2))\zeta_s - f_1\zeta\}\, dy\, ds\, dx\, dt - \int_{(\{0\}\times\Omega)\times Q} (g_{1_0} - g_2)^+ \zeta\, dy\, dx\, dt$$
$$\leqq - \lim_{\varepsilon\to 0} \int_{Q\times(Q\setminus Q_2)} |\nabla_y b(u_1)|^2 H'_\varepsilon(b(u_1) - b(u_2))\zeta\, dy\, ds\, dx\, dt$$
$$= - \lim_{\varepsilon\to 0} \int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_y b(u_1)|^2 H'_\varepsilon(b(u_1) - b(u_2))\zeta\, dy\, ds\, dx\, dt. \tag{97}$$

Similarly, from the definition of entropy solution and from Lemma 5 we deduce that

$$\int_{Q\times Q} H_0(u_1-u_2)\{(\nabla_x b(u_2)+\phi(u_1)-\phi(u_2))\cdot\nabla_x\zeta$$

$$-(g(u_2)-g(u_1))\zeta_t - f_2\zeta\}\,dy\,ds\,dx\,dt + \int_{(Q\times\{0\}\times\Omega)} (g_1-g_{2_0})^+\zeta\,dy\,ds\,dx$$

$$\geqq \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times Q} |\nabla_x b(u_2)|^2 H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt$$

$$= \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_x b(u_2)|^2 H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt. \tag{98}$$

Now, from (93) we have

$$\int_Q \nabla_y b(u_1)\cdot\nabla_x(H_\varepsilon(b(u_1)-b(u_2))\zeta)\,dx\,dt = 0 \quad \text{for a.e. } (s,y)\in Q,$$

and from (92) we have

$$\int_Q \nabla_x b(u_2)\cdot\nabla_y(H_\varepsilon(b(u_1)-b(u_2))\zeta)\,dy\,ds = 0 \quad \text{for a.e. } (t,x)\in Q.$$

Then we deduce that

$$\int_{Q\times Q} H_0(b(u_1)-b(u_2))\nabla_y b(u_1)\cdot\nabla_x\zeta\,dy\,ds\,dx\,dt$$
$$= +\lim_{\varepsilon\to 0}\int_{Q\times Q} \nabla_y b(u_1)\cdot\nabla_x b(u_2)\,H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt,$$

$$\int_{Q\times Q} H_0(b(u_1)-b(u_2))\nabla_x b(u_2)\cdot\nabla_y\zeta\,dy\,ds\,dx\,dt$$
$$= -\lim_{\varepsilon\to 0}\int_{Q\times Q} \nabla_x b(u_2)\cdot\nabla_y b(u_1)\,H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt.$$

Therefore, by taking into account (94)–(96) we get

$$\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} \nabla_y b(u_1)\cdot\nabla_x b(u_2)\,H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt$$
$$= \int_{Q\times Q} H_0(u_1-u_2)\nabla_y b(u_1)\cdot\nabla_x\zeta\,dy\,ds\,dx\,dt, \tag{99}$$

$$-\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} \nabla_x b(u_2)\cdot\nabla_y b(u_1)\,H_\varepsilon'(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt$$
$$= \int_{Q\times Q} H_0(u_1-u_2)\nabla_x b(u_2)\cdot\nabla_y\zeta\,dy\,ds\,dx\,dt. \tag{100}$$

By subtracting (98) and (100) from the sum of (97) and (99), we deduce that

$$\begin{aligned}
&\int_{Q\times Q} H_0(u_1-u_2)\{(\nabla_y b(u_1)-\nabla_x(b(u_2))\cdot(\nabla_y\zeta+\nabla_x\zeta)\\
&+(\phi(u_2)-\phi(u_1))\cdot(\nabla_y\zeta+\nabla_x\zeta)+(g(u_2)-g(u_1)(\zeta_s+\zeta_t)\\
&+(f_2-f_1)\zeta\}\,dy\,ds\,dx\,dt\\
&-\int_{(\{0\}\times\Omega)\times Q}(g_{1_0}-g(u_2))^+\zeta\,dy\,dx\,dt-\int_{Q\times(\{0\}\times\Omega)}(g(u_1)-g_{2_0})^+\zeta\,dy\,ds\,dx\\
&\leqq -\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}|\nabla_y b(u_1)-\nabla_x b(u_2)|^2\\
&\qquad\qquad H'_\varepsilon(b(u_1)-b(u_2))\zeta\,dy\,ds\,dx\,dt\leqq 0 \qquad (101)
\end{aligned}$$

for any nonnegative function ζ satisfying (92) and (93).

Now let $\xi\in\mathscr{D}([0,T)\times\Omega)$ be such that $\xi\geqq 0$, let ϱ_l be a classical sequence of mollifiers in $\mathbb{R}$ and let ρ_n be a classical sequence of mollifiers in $\mathbb{R}^N$. We define

$$\zeta^{(n,l)}(t,x,s,y)=\xi\left(\frac{t+s}{2},\frac{x+y}{2}\right)\rho_n\left(\frac{x-y}{2}\right)\varrho_l\left(\frac{t-s}{2}\right).$$

Then $\zeta^{(n,l)}$ are nonnegative functions satisfying (92) and (93) for n and l large enough.

From (101), for n and l large enough, we get

$$\begin{aligned}
0\geqq&\int_{Q\times Q} H_0(u_1-u_2)\{(\nabla_y b(u_1)-\nabla_x(b(u_2))\cdot(\nabla_y\xi+\nabla_x\xi)\\
&+(\phi(u_2)-\phi(u_1))\cdot(\nabla_y\xi+\nabla_x\xi)+(g(u_2)-g(u_1)(\xi_s+\xi_t)\\
&+(f_2-f_1)\xi\}\rho_n\varrho_l\,dy\,ds\,dx\,dt \qquad (102)\\
&-\int_{(\{0\}\times\Omega)\times Q}(g_{1_0}-g(u_2))^+\xi\rho_n\varrho_l\,dy\,dx\,dt\\
&-\int_{Q\times(\{0\}\times\Omega)}(g(u_1)-g_{2_0})^+\xi\rho_n\varrho_l\,dy\,ds\,dx.
\end{aligned}$$

Let us define

$$\varphi^{(l)}(t,x,y)=\int_t^T\xi\left(\frac{r}{2},\frac{x+y}{2}\right)\varrho_l\left(\frac{r}{2}\right)dr=\int_{\min(t,1/l)}^{1/l}\xi\left(\frac{r}{2},\frac{x+y}{2}\right)\varrho_l\left(\frac{r}{2}\right)dr$$

since $\mathrm{Supp}(\varrho_l) \subset (-1/l, +1/l)$. Since u_2 is an entropy solution, we have

$$-\int_{(\{0\}\times\Omega)\times Q} (g_{1_0} - g(u_2))^+ \xi \rho_n \varrho_l \, dy\, dx\, dt$$

$$= \int_{(\{0\}\times\Omega)\times Q} (g_{1_0} - g(u_2))^+ \varphi_t^{(l)} \rho_n dy\, dx\, dt$$

$$\geqq - \int_{(\{0\}\times\Omega)\times Q} H_0(u_{1_0} - u_2)\{(\nabla_x b(u_2) + \phi(u_{1_0}) - \phi(u_2)) \cdot \nabla_x(\rho_n \varphi^{(l)})$$

$$- f_2 \rho_n \varphi^{(l)}\}\, dy\, dx\, dt - \int_{(\{0\}\times\Omega)\times(\{0\}\times\Omega)} (g_{1_0} - g_{2_0})^+ \rho_n \varphi^{(l)}\, dy\, dx,$$

and, since $\varphi^{(l)}$ vanishes for $t \geqq 1/l$, we get

$$-\int_{(\{0\}\times\Omega)\times Q} (g_{1_0} - g(u_2))^+ \xi \rho_n \varrho_l \, dy\, dx\, dt$$

$$\geqq - \int_{(\{0\}\times\Omega)\times((0,1/l)\times\Omega)} H_0(u_{1_0} - u_2)\{(\nabla_x b(u_2)$$

$$+ \phi(u_{1_0}) - \phi(u_2)) \cdot \nabla_x(\rho_n \varphi^{(l)}) - f_2 \rho_n \varphi^{(l)}\}\, dy\, dx\, dt$$

$$- \int_{(\{0\}\times\Omega)\times(\{0\}\times\Omega)} (g_{1_0} - g_{2_0})^+ \rho_n \varphi^{(l)}\, dy\, dx.$$

We easily check that the first integral on the right side of this inequality converges to 0 when $l \to +\infty$. Moreover, without loss of generality, we can assume that $\varrho_l(s) = \varrho_l(-s)$ for any $s \in \mathbb{R}$; then

$$\varphi^{(l)}(0, x, y) \overset{l\to+\infty}{\longrightarrow} \xi\left(0, \frac{x+y}{2}\right) \lim_{l\to+\infty} \int_0^T \varrho_l(r)\, dr = \frac{1}{2}\xi\left(0, \frac{x+y}{2}\right)$$

for any $(x, y) \in \Omega \times \Omega$. Since $\varphi^{(l)}(0, x, y)$ is uniformly bounded in $L^\infty(\Omega) \times L^\infty(\Omega)$, we deduce that the second integral on the right side of the above inequality converges to

$$\frac{1}{2} \int_{(\{0\}\times\Omega)\times(\{0\}\times\Omega)} (g_{1_0} - g_{2_0})^+ \rho_n \xi\, dy\, dx.$$

Then we conclude that

$$- \limsup_{n\to+\infty} \limsup_{l\to+\infty} \int_{(\{0\}\times\Omega)\times Q} (g_{1_0} - g(u_2))^+ \xi \rho_n \varrho_l \, dy\, dx\, dt$$

$$\geqq - \lim_{n\to+\infty} \frac{1}{2} \int_{(\{0\}\times\Omega)\times(\{0\}\times\Omega)} (g_{1_0} - g_{2_0})^+ \xi \rho_n\, dy\, dx$$

$$= -\frac{1}{2} \int_{\{0\}\times\Omega} (g_{1_0} - g_{2_0})^+ \xi\, dx \tag{103}$$

where, in the last integral, all the functions depend on x and t.

Similarly, by considering the function

$$\tilde{\varphi}^{(l)}(s,x,y) = \int_s^T \xi\left(\frac{r}{2}, \frac{x+y}{2}\right)\varrho_l\left(-\frac{r}{2}\right) dr$$
$$= \int_{\min(s,1/l)}^{1/l} \xi\left(\frac{r}{2}, \frac{x+y}{2}\right)\varrho_l\left(-\frac{r}{2}\right) dr$$

and the fact that u_1 is an entropy solution, we deduce that

$$-\limsup_{n\to+\infty}\limsup_{l\to+\infty}\int_{Q\times(\{0\}\times\Omega)} (g(u_1)-g_{2_0})^+\xi\rho_n\varrho_l\,dy\,dx\,ds$$
$$\geqq -\lim_{n\to+\infty}\frac{1}{2}\int_{(\{0\}\times\Omega)\times(\{0\}\times\Omega)} (g_{1_0}-g_{2_0})^+\xi\rho_n\,dy\,dx$$
$$= -\frac{1}{2}\int_{\{0\}\times\Omega}(g_{1_0}-g_{2_0})^+\xi\,dx \tag{104}$$

where, in the last integral, all the functions depend on x and t.

Finally we have

$$\lim_{n,l\to+\infty}\int_{Q\times Q} H_0(u_1-u_2)\{(\nabla_y b(u_1)-\nabla_x(b(u_2))\cdot(\nabla_y\xi+\nabla_x\xi)$$
$$+(\phi(u_2)-\phi(u_1))\cdot(\nabla_y\xi+\nabla_x\xi)+(g(u_2)-g(u_1)(\xi_s+\xi_t)$$
$$+\kappa(f_2-f_1)\xi\}\rho_n\varrho_l\,dy\,ds\,dx\,dt$$
$$=\int_Q\{\nabla(b(u_1)-b(u_2))^+\nabla\xi+H_0(u_1-u_2)(\phi(u_2)-\phi(u_1))\cdot\nabla\xi$$
$$-(g(u_1)-g(u_2))^+\xi_t+(f_2-f_1)\xi\,dx\,dt \tag{105}$$

where $\kappa\in H(u_1-u_2)$.

Then, from (102)–(105) we get the inequality (91).

Theorem 14. *Let* (H1) *and* (H2) *hold. Let $g_{i_0}\in L^1(\Omega)$, let $g_{i_0}\in R(g)$ ($g_{i_0}=g(u_{i_0})$), let $f_i\in L^2((0,T);H^{-1}(\Omega))\cap L^1(Q)$ and let u_i be an entropy solution of $(P_E(f_i,g_{i_0})(g,b,\phi))$ for $i=1,2$. Then*

$$\int_Q\{\nabla(b(u_1)-b(u_2))^+\cdot\nabla\xi+H_0(u_1-u_2)(\phi(u_2)-\phi(u_1))\cdot\nabla\xi$$
$$-(g(u_1)-g(u_2))^+\xi_t\}\,dxdt-\int_\Omega(g_{1_0}-g_{2_0})^+\xi(0)\,dx$$
$$\leqq\int_Q\kappa(f_1-f_2)\xi\,dx\,dt \tag{106}$$

for some $\kappa\in H(u_1-u_2)$ and for any nonnegative $\xi\in\mathscr{D}([0,T)\times\overline{\Omega})$.

Proof. As in the proof of the previous theorem we consider two different pairs of variables (s, y) and (t, x) in Q and we assume that $u_1 = u_1(s, y)$, $f_1 = f_1(s, y)$, $g_{1_0} = g_{1_0}(y)$ and that $u_2 = u_2(t, x)$, $f_2 = f_2(t, x)$, $g_{2_0} = g_{2_0}(x)$. Let Q_1 and Q_2 be defined as in the proof of the previous theorem.

Let $\zeta = \zeta(t, x, s, y)$ be a nonnegative and smooth function in $\mathbb{R}^{2N+2}$ such that

$$\begin{aligned} &(s, y) \mapsto \zeta(t, x, s, y) \in \mathscr{D}((0, T) \times \overline{\Omega}) \quad \text{for any } (t, x) \in Q, \\ &(t, x) \mapsto \zeta(t, x, s, y) \in \mathscr{D}([0, T) \times \Omega) \quad \text{for any } (s, y) \in Q. \end{aligned} \tag{107}$$

Then, from the definition of entropy solution, from Lemma 5 and from (94) we have

$$\begin{aligned} 0 &\geqq - \lim_{\varepsilon\to 0} \int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_y b(u_1)|^2 H_\varepsilon'(b(u_1) - b(u_2^+(t, x)))\zeta \, dy\, ds\, dx\, dt \\ &= - \lim_{\varepsilon\to 0} \int_{Q\times(Q\setminus Q_2)} |\nabla_y b(u_1)|^2 H_\varepsilon'(b(u_1) - b(u_2^+(t, x)))\zeta \, dy\, ds\, dx\, dt \\ &\geqq \int_{Q\times Q} H_0(u_1 - u_2^+(t, x))\{(\nabla_y b(u_1) + \phi(u_2^+(t, x)) - \phi(u_1)) \cdot \nabla_y \zeta \\ &\quad + (g(u_2^+(t, x)) - g(u_1))\zeta_s - f_1\zeta\} \, dy\, ds\, dx\, dt. \end{aligned}$$

Since $(t, x) \mapsto H_\varepsilon(b(u_1(s, y)) - b(u_2^+(t, x)))\zeta(t, x, s, y) \in L^2((0, T); H_0^1(\Omega))$ for almost every $(s, y) \in Q$, we have

$$0 = \int_{Q\times Q} \nabla_y b(u_1) \cdot \nabla_x (H_\varepsilon(b(u_1) - b(u_2^+))\zeta) \, dy\, ds\, dx\, dt$$

and therefore, by taking into account (94)–(96),

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} \nabla_y b(u_1) \cdot \nabla_x b(u_2^+)\, H_\varepsilon'(b(u_1) - b(u_2^+))\zeta \, dy\, ds\, dx\, dt \\ &\quad = \lim_{\varepsilon\to 0} \int_{Q\times Q} \nabla_y b(u_1) \cdot \nabla_x b(u_2^+)\, H_\varepsilon'(b(u_1) - b(u_2^+))\zeta \, dy\, ds\, dx\, dt \\ &\quad = \int_{Q\times Q} H_0(b(u_1) - b(u_2^+))\nabla_y b(u_1) \cdot \nabla_x \zeta \, dy\, ds\, dx\, dt \\ &\quad = \int_{(Q\setminus Q_1)\times Q} H_0(b(u_1) - b(u_2^+))\nabla_y b(u_1) \cdot \nabla_x \zeta \, dy\, ds\, dx\, dt \\ &\quad = \int_{(Q\setminus Q_1)\times Q} H_0(u_1 - u_2^+)\nabla_y b(u_1) \cdot \nabla_x \zeta \, dy\, ds\, dx\, dt \\ &\quad = \int_{Q\times Q} H_0(u_1 - u_2^+)\nabla_y b(u_1) \cdot \nabla_x \zeta \, dy\, ds\, dx\, dt. \end{aligned}$$

Then we deduce that

$$\begin{aligned}\int_{Q\times Q} & H_0(u_1-u_2^+)\{\nabla_y b(u_1)\cdot(\nabla_y\zeta+\nabla_x\zeta)+(\phi(u_2^+)-\phi(u_1))\cdot\nabla_y\zeta \\ & +(g(u_2^+)-g(u_1))\zeta_s-f_1\zeta\}\,dy\,ds\,dx\,dt \\ & \leqq -\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}(|\nabla_y b(u_1)|^2-\nabla_y b(u_1)\cdot\nabla_x b(u_2^+)) \\ & \qquad H_\varepsilon'(b(u_1)-b(u_2^+))\zeta\,dy\,ds\,dx\,dt.\end{aligned}$$

Since we are integrating where u_1 is positive and where g_{1_0} is positive, this inequality can be written as

$$\begin{aligned}\int_{Q\times Q} & H_0(u_1^+-u_2^+)\{\nabla_y b(u_1^+)\cdot(\nabla_y\zeta+\nabla_x\zeta)+(\phi(u_2^+)-\phi(u_1^+))\cdot\nabla_y\zeta \\ & +(g(u_2^+)-g(u_1^+))\zeta_s-f_1\zeta\}\,dy\,ds\,dx\,dt \\ & \leqq -\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}(|\nabla_y b(u_1^+)|^2-\nabla_y b(u_1^+) \\ & \quad \cdot\nabla_x b(u_2^+))H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\,dy\,ds\,dx\,dt. \qquad (108)\end{aligned}$$

Now, from the definition of entropy solution, from Lemma 5 and from (95), we have

$$\begin{aligned}0 & \leqq \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}|\nabla_x b(u_2)|^2 H_\varepsilon'(b(u_1^+)-b(u_2))\zeta\,dy\,dsdx\,dt \\ & = \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times Q}|\nabla_x b(u_2)|^2 H_\varepsilon'(b(u_1^+)-b(u_2))\zeta\,dy\,dsdx\,dt \\ & \leqq \int_{Q\times Q} H_0(u_1^+-u_2)\{(\nabla_x b(u_2)+\phi(u_1^+)-\phi(u_2))\cdot\nabla_x\zeta \\ & \quad +(g(u_1^+)-g(u_2))\zeta_t-f_2\zeta\}\,dy\,ds\,dx\,dt \\ & \quad +\int_{Q\times(\{0\}\times\Omega)}(g(u_1^+)-g(u_{2_0}))^+\zeta\,dy\,ds\,dx,\end{aligned}$$

and therefore

$$
\begin{aligned}
0 &\leqq \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_x b(u_2^+)|^2 H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\, dy\, ds dx\, dt \\
&\quad + \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_x b(-u_2^-)|^2 H_\varepsilon'(b(u_1^+)-b(u_2^-))\zeta\, dy\, ds\, dx\, dt \\
&\leqq \int_{Q\times Q} H_0(u_1^+-u_2^+)\{(\nabla_x b(u_2^+)+\phi(u_1^+)-\phi(u_2^+))\cdot\nabla_x\zeta \\
&\quad + (g(u_1^+)-g(u_2^+))\zeta_t - (1-H_0(u_2^-))f_2\zeta\}\, dy\, ds\, dx\, dt \\
&\quad + \int_{Q\times(\{0\}\times\Omega)} (g(u_1^+)-g(u_{2_0}^+))^+\zeta\, dy\, ds\, dx \\
&\quad + \int_{Q\times Q} H_0(u_2^-)\{(\nabla_x b(u_2)-\phi(u_2))\cdot\nabla_x\zeta - g(u_2)\zeta_t - f_2\zeta\}\, dy\, ds\, dx\, dt \\
&\quad + \int_{Q\times(\{0\}\times\Omega)} g_{2_0}^-\zeta\, dy\, ds\, dx.
\end{aligned}
$$

Since the two integrals on the left side of this inequality are nonnegative, we still have

$$
\begin{aligned}
0 &\leqq \lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} |\nabla_x b(u_2^+)|^2 H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\, dy\, ds dx\, dt \\
&\leqq \int_{Q\times Q} H_0(u_1^+-u_2^+)\{(\nabla_x b(u_2^+)+\phi(u_1^+)-\phi(u_2^+))\cdot\nabla_x\zeta \\
&\quad + (g(u_1^+)-g(u_2^+))\zeta_t - (1-H_0(u_2^-))f_2\zeta\}\, dy\, ds\, dx\, dt \\
&\quad + \int_{Q\times(\{0\}\times\Omega)} (g(u_1^+)-g(u_{2_0}^+))^+\zeta\, dy\, ds\, dx \\
&\quad + \int_{Q\times Q} H_0(u_2^-)\{(\nabla_x b(u_2)-\phi(u_2))\cdot\nabla_x\zeta - g(u_2)\zeta_t - f_2\zeta\}\, dy\, ds\, dx\, dt \\
&\quad + \int_{Q\times(\{0\}\times\Omega)} g_{2_0}^-\zeta\, dy\, ds\, dx.
\end{aligned}
$$

Moreover, since for almost every $(t,x)\in Q$ the function

$$
(s,y)\mapsto H_\varepsilon(b(u_1^+(s,y))-b(u_2^+(t,x)))\zeta(t,x,s,y)\in L^2((0,T);H_0^1(\Omega)),
$$

we have

$$
\int_{Q\times Q}\nabla_x b(u_2^+)\cdot\nabla_y(H_\varepsilon(b(u_1^+)-b(u_2^+))\zeta)\, dy\, ds\, dx\, dt = 0,
$$

and therefore, by taking into account (94)–(96)

$$-\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)} \nabla_x b(u_2^+)\cdot\nabla_y b(u_1^+)H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\,dy\,ds\,dx\,dt$$
$$=\int_{Q\times Q} H_0(u_1^+-u_2^+)\nabla_x b(u_2^+)\cdot\nabla_y\zeta\,dy\,ds\,dx\,dt.$$

From the above inequalities we deduce that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}(|\nabla_x b(u_2^+)|^2-\nabla_x b(u_2^+)\cdot\nabla_y b(u_1^+))\\
&\qquad\qquad H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\,dy\,dsdx\,dt\\
&\leqq\int_{Q\times Q} H_0(u_1^+-u_2^+)\{\nabla_x b(u_2^+)\cdot(\nabla_y\zeta+\nabla_x\zeta)+(\phi(u_1^+)-\phi(u_2^+))\cdot\nabla_x\zeta\\
&\quad+(g(u_1^+)-g(u_2^+))\zeta_t-(1-H_0(u_2^-))f_2\zeta\}\,dy\,ds\,dx\,dt\\
&\quad+\int_{Q\times(\{0\}\times\Omega)}(g(u_1^+)-g(u_2^+))^+\,dy\,ds\,dx\\
&\quad+\int_{Q\times Q} H_0(u_2^-)\{(\nabla_x b(u_2)-\phi(u_2))\cdot\nabla_x\zeta-g(u_2)\zeta_t-f_2\zeta\}\,dy\,ds\,dx\,dt\\
&\quad+\int_{Q\times(\{0\}\times\Omega)} g_{2_0}^-\zeta\,dy\,ds\,dx.
\end{aligned}$$

Then, from (108) and this inequality we get

$$\begin{aligned}
0&\geqq-\lim_{\varepsilon\to 0}\int_{(Q\setminus Q_1)\times(Q\setminus Q_2)}|\nabla_y b(u_1^+)-\nabla_x b(u_2^+)|^2\\
&\qquad\qquad H_\varepsilon'(b(u_1^+)-b(u_2^+))\zeta\,dy\,ds\,dx\,dt\\
&\geqq\int_{Q\times Q} H_0(u_1^+-u_2^+)\{(\nabla_y b(u_1^+)-\nabla_x b(u_2^+))\cdot(\nabla_y\zeta+\nabla_x\zeta)\\
&\quad+(\phi(u_2^+)-\phi(u_1^+))\cdot(\nabla_y\zeta+\nabla_x\zeta)\\
&\quad+(g(u_2^+)-g(u_1^+))(\zeta_s+\zeta_t)-(f_1-(1-H_0(u_2^-))f_2)\zeta\}\,dy\,ds\,dx\,dt\\
&\quad-\int_{Q\times(\{0\}\times\Omega)}(g(u_1^+)-g_{2_0}^+)^+\zeta dy\,ds\,dx\\
&\quad-\int_{Q\times Q} H_0(u_2^-)\{(\nabla_x b(u_2)-\phi(u_2))\cdot\nabla_x\zeta-g(u_2)\zeta_t-f_2\zeta\}\,dy\,ds\,dx\,dt\\
&\quad-\int_{Q\times(\{0\}\times\Omega)} g_{2_0}^-\zeta\,dy\,ds\,dx. \qquad (109)
\end{aligned}$$

Now let $\xi \in \mathscr{D}([0,T) \times \mathbb{R}^N)$, $\xi \geqq 0$, be such that

$$\operatorname{Supp}(\xi) \cap [0,T) \times \mathbb{R}^N \subset [0,T) \times B$$

where B is a ball for which

$$\begin{gathered} \text{either} B \cap \partial\Omega = \emptyset \quad \text{or} \quad B \subset\subset B' \text{ and } B' \cap \partial\Omega \text{ is a part} \\ \text{of the graph of a Lipschitz continuous function.} \end{gathered} \tag{110}$$

Then there exists a sequence of mollifiers ϱ_l, defined on $\mathbb{R}$, with Supp$(\varrho_l)\subset$ $(-2/l, 0)$ and there exists a sequence of mollifers ρ_n defined on $\mathbb{R}^N$ such that, for n large enough,

$$\begin{gathered} x \mapsto \rho_n(x-y) \in \mathscr{D}(\Omega) \quad \forall\, y \in B, \\ \chi_n(x) = \int_\Omega \rho_n(x-y)\,dy \quad \text{is an increasing sequence for } x \in B, \\ \chi_n(x) = 1 \quad \text{for any } x \in B \text{ such that } d(x, \mathbb{R}^N \setminus \Omega) > c/n \end{gathered}$$

where c is a positive constant depending on B. Then, for n and l large enough, the function

$$\zeta^{(n,l)}(t,x,s,y) = \xi(t,x)\rho_n(x-y)\varrho_l(t-s)$$

satisfies (107) and the function

$$\begin{aligned} \xi^{(n)}(t,x) &= \int_Q \zeta^{(n,l)}(t,x,s,y)\,dy\,ds \\ &= \xi(t,x)\int_\Omega \rho_n(x-y)\,dy \int_0^T \varrho_l(t-s)\,ds = \xi\chi_n \end{aligned}$$

satisfies

$$\begin{gathered} \xi^{(n)} \in \mathscr{D}([0,T) \times \Omega), \quad \xi^{(n)} \leqq \xi^{(n')} \quad \forall\, n' \geqq n, \\ \xi^{(n)}(t,x) = \xi(t,x) \quad \forall\, x \text{ such that } d(x, \mathbb{R}^N \setminus \Omega) > c/n, \end{gathered}$$

where c is a positive constant depending on B. Obviously $\xi^{(n)} \leqq \xi$, and $\xi^{(n)}$ converges to ξ in $L^r(Q)$ for any $1 \leqq r < +\infty$.

Then $\zeta^{(n,l)}$ satisfies the inequality (109):

$$0 \geqq \int_{Q\times Q} H_0(u_1^+ - u_2^+)\{(\nabla_y b(u_1^+) - \nabla_x b(u_2^+)) \cdot (\nabla_y \zeta^{(n,l)} + \nabla_x \zeta^{(n,l)})$$
$$+ (\phi(u_2^+) - \phi(u_1^+)) \cdot (\nabla_y \zeta^{(n,l)} + \nabla_x \zeta^{(n,l)})$$
$$+ (g(u_2^+) - g(u_1^+))(\zeta_s^{(n,l)} + \zeta_t^{(n,l)} - (f_1 - (1 - H_0(u_2^-)) f_2)\zeta^{(n,l)}\}\, dy\, ds\, dx\, dt$$
$$- \int_{Q\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \zeta^{(n,l)} dy\, ds\, dx$$
$$- \int_{Q\times Q} H_0(u_2^-)\{(\nabla_x b(u_2) - \phi(u_2)) \cdot \nabla_x \zeta^{(n,l)} - g(u_2)\zeta_t^{(n,l)} - f_2 \zeta^{(n,l)}\}\, dy\, ds\, dx\, dt$$
$$- \int_{Q\times(\{0\}\times\Omega)} g_{2_0}^- \zeta^{(n,l)} dy\, ds\, dx.$$

Therefore

$$0 \geqq \int_{Q\times Q} H_0(u_1^+ - u_2^+)\{(\nabla_y b(u_1^+) - \nabla_x b(u_2^+)) \cdot \nabla_x \xi + (\phi(u_2^+) - \phi(u_1^+)) \cdot \nabla_x \xi$$
$$+ (g(u_2^+) - g(u_1^+))\xi_t - (f_1 - (1 - H_0(u_2^-)) f_2)\xi\}\rho_n \varrho_l\, dy\, ds\, dx\, dt$$
$$- \int_{Q\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \xi \rho_n \varrho_l dy\, ds\, dx$$
$$- \int_Q H_0(u_2^-)\{(\nabla_x b(u_2) - \phi(u_2)) \cdot \nabla_x \xi^{(n)} - g(u_2)\xi_t^{(n)} - f_2 \xi^{(n)}\}\, dx\, dt$$
$$- \int_{\{0\}\times\Omega} g_{2_0}^- \xi^{(n)} dx.$$

Passage to the limit. First integral. When $l, n \to +\infty$ we have

$$\int_{Q\times Q} H_0(u_1^+ - u_2^+)\{(\nabla_y b(u_1^+) - \nabla_x b(u_2^+)) \cdot \nabla_x \xi + (\phi(u_2^+) - \phi(u_1^+)) \cdot \nabla_x \xi$$
$$+ (g(u_2^+) - g(u_1^+))\xi_t - (f_1 - (1 - H_0(u_2^-)) f_2)\xi\}\rho_n \varrho_l\, dy\, ds\, dx\, dt$$
$$= \int_{Q\times Q} H_0(u_1^+ - u_2^+)\{(\nabla_y b(u_1^+) - \nabla_x b(u_2^+)) \cdot \nabla_x \xi + (\phi(u_2^+) - \phi(u_1^+)) \cdot \nabla_x \xi$$
$$+ (g(u_2^+) - g(u_1^+))\xi_t - (f_1 - (1 - H_0(u_2^-)) f_2) H_0(u_1^+)\xi\}\rho_n \varrho_l\, dy\, ds\, dx\, dt$$
$$\longrightarrow \int_Q \kappa_+\{\nabla(b(u_1^+) - b(u_2^+)) + \phi(u_2^+) - \phi(u_1^+)) \cdot \nabla \xi$$
$$+ (g(u_2^+) - g(u_1^+))\xi_t - (f_1 - (1 - H_0(u_2^-)) f_2) H_0(u_1^+)\xi\}\, dx\, dt$$

where $\kappa_+(t,x) \in H(u_1^+(t,x) - u_2^+(t,x))$ and, in the last integral, where every function depends on (t,x).

Second integral. Let

$$\varphi^{(n,l)}(t,x,s,y) = \xi(t,x)\rho_n(x-y)\int_s^T \varrho_l(t-r)dr.$$

Then, since u_1 is an entropy solution, we have

$$\begin{aligned}
&\int_{Q\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \xi\rho_n\varrho_l dy\, ds\, dx \\
&\quad = \int_{((0,2/l)\times\Omega)\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \xi\rho_n\varrho_l dy\, ds\, dx \\
&\quad = -\int_{((0,2/l)\times\Omega)\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \varphi_s^{(n,l)} dy\, ds\, dx \\
&\quad \leqq -\int_{((0,2/l)\times\Omega)\times(\{0\}\times\Omega)} H_0(u_1 - u_{2_0})\{(\nabla_y b(u_1) \\
&\qquad + \phi(u_{2_0}) - \phi(u_1))\cdot\nabla_y\varphi^{(n,l)} - f_1\varphi^{(n,l)}\}\, dy\, ds\, dx \\
&\qquad + \int_{(\{0\}\times\Omega)\times\{0\}\times\Omega)} (g_{1_0} - g_{2_0}^+)^+ \varphi^{(n,l)} dy\, dx \\
&\quad = -\int_{((0,2/l)\times\Omega)\times(\{0\}\times\Omega)} H_0(u_1 - u_{2_0})\{(\nabla_y b(u_1) \\
&\qquad + \phi(u_{2_0}) - \phi(u_1))\cdot\nabla_y\rho_n\, \xi\int_s^T \varrho_l(-r)dr \\
&\qquad - f_1\rho_n\xi\int_s^T \varrho_l(-r)dr\}\, dy\, ds\, dx + \int_{(\{0\}\times\Omega)\times\{0\}\times\Omega)} (g_{1_0} - g_{2_0}^+)^+ \rho_n\xi\, dy\, dx,
\end{aligned}$$

because

$$\int_0^T \varrho_l(-r)dr = 1.$$

Obviously, the first integral on the right side of this inequality converges to 0 when $l \to +\infty$. Then we have

$$\begin{aligned}
&\limsup_{n\to+\infty}\limsup_{l\to+\infty}\int_{Q\times(\{0\}\times\Omega)} (g(u_1^+) - g_{2_0}^+)^+ \xi\rho_n\varrho_l\, dy\, ds\, dx \\
&\quad \leqq \lim_{n\to+\infty}\int_{(\{0\}\times\Omega)\times\{0\}\times\Omega)} (g_{1_0} - g_{2_0}^+)^+ \rho_n\xi\, dy\, dx = \int_{\{0\}\times\Omega} (g_{1_0} - g_{2_0}^+)^+ \xi\, dx
\end{aligned}$$

where, in the last integral, every function depends on x.

Third and fourth integrals. The integrands depend only on (t, x) and therefore we replace the operator∇_x by the operator ∇. Since u_2 is an entropy solution,

$$\varsigma \mapsto \mathscr{L}(\varsigma) = \int_{\{0\}\times\Omega} g_{2_0}^- \varsigma \, dx$$
$$+ \int_Q H_0(u_2^-)\{(\nabla b(u_2) - \phi(u_2)) \cdot \nabla\varsigma - g(u_2)\varsigma_t - f_2\varsigma\} \, dx \, dt$$

is a monotone increasing operator. In particular, since

$$0 \leqq \xi^{(n)} \leqq \xi^{(n')} \leqq \xi \quad \text{for any } n' \text{ such that } n \leqq n',$$

we deduce that $\mathscr{L}(\xi^{(n)})$ is an increasing sequence satisfying

$$0 \leqq \mathscr{L}(\xi^{(n)}) \leqq \mathscr{L}(\xi)$$

and therefore it converges when $n \to +\infty$. Then

$$\begin{aligned}
&\int_Q \{\nabla(b(u_1^+) - b(u_2^+))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\phi(u_2^+) - \phi(u_1^+)) \cdot \nabla\xi \\
&\qquad + (g(u_2^+) - g(u_1^+))^+\xi_t - \kappa_+(f_1 - (1 - H_0(u_2^-))f_2)H_0(u_1^+)\xi\} \, dx \, dt \\
&\qquad - \int_{\{0\}\times\Omega} (g_{1_0} - g_{2_0}^+)^+ \xi \, dx \\
&\quad \leqq \lim_{n\to+\infty} \left(\int_Q H_0(u_2^-)\{(\nabla b(u_2) - \phi(u_2)) \cdot \nabla\xi^{(n)} \right. \\
&\qquad \left. - g(u_2)\xi_t^{(n)} - f_2\xi^{(n)}\} \, dx \, dt + \int_{\{0\}\times\Omega} g_{2_0}^- \xi^{(n)} \, dx \right)
\end{aligned} \tag{111}$$

for any nonnegative $\xi \in \mathscr{D}([0, T) \times B)$.

Now, by considering Remark 8, by replacing u_1 by $-u_2$, u_2 by $-u_1$, g_{1_0} by $-g_{2_0}$, g_{2_0} by $-g_{1_0}$, f_1 by $-f_2$, f_2 by $-f_1$, g by $\tilde{g}$, b by $\tilde{b}$ and ϕ by $\tilde{\phi}$, and by arguing as above, we deduce the existence of a function $\kappa_- \in H(u_2^- - u_1^-)$ such that

$$\begin{aligned}
&\int_Q \{\nabla(b(-u_1^-) - b(-u_2^-))^+ \cdot \nabla\xi + H_0(u_2^- - u_1^-)(\phi(-u_2^-) - \phi(-u_1^-)) \cdot \nabla\xi \\
&\qquad + (g(-u_2^-) - g(-u_1^-))^+\xi_t - \kappa_-(f_1(1 - H_0(u_1^+)) - f_2)H_0(u_2^-)\xi\} \, dx \, dt \\
&\qquad - \int_{\{0\}\times\Omega} (-g_{1_0}^- + g_{2_0}^-))^+ \xi \, dx \\
&\quad \leqq - \lim_{n\to+\infty} \left(\int_Q H_0(u_1^+)\{(\nabla b(u_1) - \phi(u_1)) \cdot \nabla\xi^{(n)} \right. \\
&\qquad \left. - g(u_1)\xi_{(n,l)_t} - f_1\xi^{(n)}\} \, dx \, dt - \int_{\{0\}\times\Omega} g_{1_0}^+ \xi^{(n)} \, dx \right)
\end{aligned} \tag{112}$$

for any nonnegative $\xi \in \mathscr{D}([0, T) \times B)$.

Let

$$\kappa_B = \kappa_-(1 - H_0(u_1^+))H_0(u_2^-) + \kappa_+ H_0(u_1^+).$$

Then we easily check that

$$\begin{aligned}\kappa_B &= \kappa_-(1 - H_0(u_1^+))H_0(u_2^-) + \kappa_+ H_0(u_1^+)\\ &= \kappa_+(1 - H_0(u_2^-))H_0(u_1^+) + \kappa_- H_0(u_2^-) \in H(u_1 - u_2).\end{aligned}$$

From (111) and (112) we deduce that

$$\begin{aligned}&\int_Q \{(\nabla(b(u_1) - b(u_2))^+ + H_0(u_1 - u_2)(\phi(u_2) - \phi(u_1))) \cdot \nabla\xi\\ &\qquad + (g(u_2) - g(u_1))^+\xi_t - \kappa_B(f_1 - f_2)\xi\}\,dx\,dt\\ &\qquad - \int_{\{0\}\times\Omega} (g_{1_0} - g_{2_0})^+\xi\,dx\\ &\leqq \lim_{n\to+\infty}\left(\int_Q H_0(u_2^-)\{(\nabla b(u_2) - \phi(u_2)) \cdot \nabla\xi^{(n)}\right.\\ &\qquad \left. - g(u_2)\xi_t^{(n)} - f_2\xi^{(n)}\}\,dx\,dt + \int_{\{0\}\times\Omega} g_{2_0}^-\xi^{(n)}dx\right)\\ &\qquad - \lim_{n\to+\infty}\left(\int_Q H_0(u_1^+)\{(\nabla b(u_1) - \phi(u_1)) \cdot \nabla\xi^{(n)}\right.\\ &\qquad \left. - g(u_1)\xi_{(n,l)_t} - f_1\xi^{(n)}\}\,dx\,dt - \int_{\{0\}\times\Omega} g_{1_0}^+\xi^{(n)}dx\right)\end{aligned}$$

for any nonnegative $\xi \in \mathscr{D}([0, T) \times B)$.

Now let $\xi \in \mathscr{D}([0, T) \times B)$, $\xi \geqq 0$. Then $\xi^{(n')} = \xi\chi_{n'} \in \mathscr{D}([0, T) \times \Omega)$ and by applying Theorem 13 we have

$$\begin{aligned}&\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla(\xi\chi_{n'}) + H_0(u_1 - u_2)(\phi(u_2) - \phi(u_1)) \cdot \nabla(\xi\chi_{n'})\\ &\qquad - (g(u_1) - g(u_2))^+\xi_t\,\chi_{n'} - \kappa(f_1 - f_2)\xi\chi_{n'}\}\,dxdt\\ &\qquad - \int_\Omega (g_{1_0} - g_{2_0})^+\xi(0)\,\chi_{n'}\,dx \leqq 0.\end{aligned}$$

Therefore

$$\int_Q \{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla\xi$$

$$+(g(u_2)-g(u_1))^+\xi_t - \kappa(f_1-f_2)\xi\}\,dx\,dt$$

$$-\int_\Omega (g_{1_0}-g_{2_0})^+\xi(0)\,dx$$

$$\leqq \int_Q \{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla(\xi(1-\chi_{n'}))$$

$$+(g(u_2)-g(u_1))^+\xi_t\,(1-\chi_{n'}) - \kappa(f_1-f_2)\xi(1-\chi_{n'})\}\,dx\,dt$$

$$-\int_\Omega (g_{1_0}-g_{2_0})^+\xi(0)\,(1-\chi_{n'})\,dx$$

$$= \int_Q \{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla(\xi(1-\chi_{n'}))$$

$$+(g(u_2)-g(u_1))^+\xi_t\,(1-\chi_{n'}) - \kappa_B(f_1-f_2)(\xi(1-\chi_{n'}))\}\,dx\,dt$$

$$+\int_Q (\kappa_B-\kappa)(f_1-f_2)\xi(1-\chi_{n'})\,dx\,dt$$

$$-\int_\Omega (g_{1_0}-g_{2_0})^+\xi(0)\,(1-\chi_{n'}))\,dx$$

$$\leqq \int_Q (\kappa_B-\kappa)(f_1-f_2)\xi(1-\chi_{n'})\,dx\,dt$$

$$+\lim_{n\to+\infty}\left(\int_Q H_0(u_2^-)\{(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\right.$$

$$-g(u_2)\xi_t\,(1-\chi_{n'})\chi_n - f_2(\xi(1-\chi_{n'})\chi_n)\}\,dx\,dt$$

$$\left.+\int_\Omega g_{2_0}^-\xi(0)\,(1-\chi_{n'})\chi_n\,dx\right)$$

$$-\lim_{n\to+\infty}\left(\int_Q H_0(u_1^+)\{(\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\right.$$

$$-g(u_1)\xi_t\,(1-\chi_{n'})\chi_n - f_1\xi(1-\chi_{n'})\chi_n\}\,dx\,dt$$

$$\left.-\int_\Omega g_{1_0}^+\xi(0)\,(1-\chi_{n'})\chi_n\,dx\right)$$

Obviously we have

$$\lim_{n'\to+\infty}\int_Q (\kappa_B-\kappa)(f_1-f_2)\xi(1-\chi_{n'})\,dx\,dt = 0.$$

Moreover, for any n' there exists n'_0 such that $\chi_n = 1$ in $\mathrm{Supp}(\chi_{n'})$ for $n > n'_0$. Therefore

$$\begin{aligned}
&\lim_{n'\to+\infty}\lim_{n\to+\infty}\Big(\int_Q H_0(u_2^-)\{(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\\
&\quad - g(u_2)\xi_t\,(1-\chi_{n'})\chi_n - f_2\xi(1-\chi_{n'})\chi_n\}\,dx\,dt\\
&\quad + \int_\Omega g_{2_0}^-\xi(0)\,(1-\chi_{n'})\chi_n\,dx\Big)\\
&\quad - \lim_{n'\to+\infty}\lim_{n\to+\infty}\Big(\int_Q H_0(u_1^+)\{(\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi(1-\chi_{n'})\chi_n)\\
&\quad - g(u_1)\xi_t\,(1-\chi_{n'})\chi_n - f_1\xi(1-\chi_{n'})\chi_n\}\,dx\,dt\\
&\quad - \int_\Omega g_{1_0}^+\xi(0)\,(1-\chi_{n'})\chi_n\,dx\Big)\\
&= \lim_{n'\to+\infty}\lim_{n\to+\infty}\Big(\int_Q H_0(u_2^-)\{(\nabla b(u_2)-\phi(u_2))\cdot\nabla(\xi(\chi_n-\chi_{n'}))\\
&\quad - g(u_2)\xi_t(\chi_n-\chi_{n'}) - f_2\xi(\chi_n-\chi_{n'})\}\,dx\,dt\\
&\quad + \int_\Omega g_{2_0}^-\xi(0)(\chi_n-\chi_{n'})\,dx\Big)\\
&\quad - \lim_{n'\to+\infty}\lim_{n\to+\infty}\Big(\int_Q H_0(u_1^+)\{(\nabla b(u_1)-\phi(u_1))\cdot\nabla(\xi(\chi_n-\chi_{n'}))\\
&\quad - g(u_1)\xi_t\,(\chi_n-\chi_n') - f_1\xi(\chi_n-\chi_{n'})\}\,dx\,dt\\
&\quad - \int_{\{0\}\times\Omega} g_{1_0}^+\xi(0)(\chi_n-\chi_{n'})\,dx\Big) = 0
\end{aligned}$$

and therefore

$$\begin{aligned}
&\int_Q\{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla\xi\\
&\quad + (g(u_2)-g(u_1))^+\xi_t - \kappa(f_1-f_2)\xi\}\,dx\,dt - \int_\Omega (g_{1_0}-g_{2_0})^+\xi dx \leqq 0
\end{aligned}$$

where κ *is the same function as that arising in Theorem 13*.

Now let $\Omega_0 \subset\subset \Omega$ be such that $\Omega_0 \cup (\cup_{i=1}^k B_i)$ is a covering of Ω where B_i, for $1 \leqq i \leqq k$, are balls satisfying (110). Let $(\varphi_i)_{i=0}^k$ be a partition of unity related to the above covering ($\varphi_0 \in \mathscr{D}(\Omega_0)$, $\varphi_i \in \mathscr{D}(B_i)$, for $1 \leqq i \leqq k$). Let

$\xi \in \mathscr{D}([0,T)\times\overline{\Omega})$, $\xi \geqq 0$, and let $\xi_i = \xi\varphi_i$. Then, for $0 \leqq i \leqq k$ we have

$$\int_Q \{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla\xi_i$$
$$+ (g(u_2)-g(u_1))^+\xi_{i_t} - \kappa(f_1-f_2)\xi_i\}\,dx\,dt - \int_\Omega (g_{1_0}-g_{2_0})^+\xi_i dx \leqq 0.$$

Since $\xi = \sum_{i=0}^k \xi_i$, we therefore have

$$\int_Q \{(\nabla(b(u_1)-b(u_2))^+ + H_0(u_1-u_2)(\phi(u_2)-\phi(u_1)))\cdot\nabla\xi$$
$$+ (g(u_2)-g(u_1))^+\xi_t - \kappa(f_1-f_2)\xi\}\,dx\,dt - \int_\Omega (g_{1_0}-g_{2_0})^+\xi dx \leqq 0$$

for any nonnegative $\xi \in \mathscr{D}([0,T)\times\overline{\Omega})$.

Corollary 10. *Let* (H1) *and* (H2) *hold. Let* $g_{i_0} \in L^1(\Omega)$, *let* $g_{i_0} \in R(g)$ ($g_{i_0} = g(u_{i_0})$), *let* $f_i \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$ *and let* u_i *be an entropy solution of* $(P_E(f_i, g_{i_0})(g,b,\phi))$ *for* $i = 1, 2$. *Then*

$$\int_\Omega (g(u_1(t)) - g(u_2(t)))^+\,dx$$
$$\leqq \int_\Omega (g_{1_0}-g_{2_0})^+\,dx + \int_0^t\int_\Omega \kappa(f_1-f_2)\,dx\,dt \tag{113}$$

for some $\kappa \in H(u_1-u_2)$, *and therefore*

$$\|g(u_1(t)) - g(u_2(t))\|_{L^1(\Omega)}$$
$$\leqq \|g_{1_0}-g_{2_0}\|_{L^1(\Omega)} + \int_0^t \|f_1-f_2\|_{L^1(\Omega)}dt. \tag{114}$$

In particular, if $g_{1_0} \leqq g_{2_0}$ *almost everywhere in* Ω *and* $f_1 \leqq f_2$ *almost everywhere in* Q, *then*

$$g(u_1) \leqq g(u_2) \quad \text{a.e. in } Q. \tag{115}$$

Moreover, if $f_1 = f_2$ *and* $g_{1_0} = g_{2_0}$, *then*

$$g(u_1) = g(u_2). \tag{116}$$

Proof. Let $\xi \in \mathscr{D}([0,T)\times\overline{\Omega})$ be such that $\xi \geqq 0$ and $\xi = \xi(t)$ ($\nabla\xi \equiv 0$). From (106) we have

$$-\int_0^T\left(\int_\Omega (g(u_1)-g(u_2))^+\,dx\right)\xi_t\,dt - \int_\Omega (g_{1_0}-g_{2_0})^+\xi(0)\,dx$$
$$\leqq \int_0^T\left(\int_\Omega \kappa(f_1-f_2)\,dx\right)\xi\,dt$$

and therefore

$$-\int_0^T \left(\int_\Omega \left((g(u_1)-g(u_2))^+ - (g_{1_0}-g_{2_0})^+\right) dx\right) \xi_t \, dt$$
$$\leqq \int_0^T \left(\int_\Omega \kappa(f_1-f_2)\, dx\right) \xi \, dt. \qquad (117)$$

Let us introduce the functions

$$G(t) = \begin{cases} \displaystyle\int_\Omega \left((g(u_1(t))-g(u_2(t)))^+ - (g_{1_0}-g_{2_0})^+\right) dx & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0), \end{cases}$$

$$F(t) = \begin{cases} \displaystyle\int_\Omega \kappa(t)(f_1(t)-f_2(t))\, dx & \text{for } t \in (0,T), \\ 0 & \text{for } t \in (-T,0). \end{cases}$$

From (117) we deduce that

$$\frac{dG}{dt} \leqq F \quad \text{in } \mathscr{D}'(-T,+T),$$

and therefore, since G and F vanish for $t<0$, we deduce that

$$G(t) \leqq \int_0^t F(s)ds$$

whence we easily deduce (113). From (113) we easily deduce (114)–(116).

Corollary 11. *Let* (H1) *and* (H2) *hold. Assume that there is an* i_0 *with* $1 \leqq i_0 \leqq N$ *such that*

$$g(s) = g(s') \Longrightarrow \phi_{i_0}(s) = \phi_{i_0}(s') \quad \forall s,\ s' \in \mathbb{R}.$$

Let $g_{i_0} \in L^1(\Omega)$*, let* $g_{i_0} \in R(g)$ $(g_{i_0} = g(u_{i_0}))$*, and let* $f_i \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$ *with*

$$g_{1_0} \leqq g_{2_0} \quad \text{a.e. in } \Omega, \qquad f_1 \leqq f_2 \quad \text{a.e. in } Q.$$

Let u_i *be an entropy solution of* $(P_E(f_i, g_{i_0})(g,b,\phi))$ *for* $i=1,2$*. Then*

$$b(u_1) \leqq b(u_2) \quad \text{a.e. in } Q.$$

Moreover, if $f_1 = f_2$ *and* $g_{1_0} = g_{2_0}$*, then*

$$b(u_1) = b(u_2).$$

Proof. From Corollary 10, from Theorem 14 and from the fact that $f_1 \leqq f_2$ we deduce that

$$\int_Q \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\phi(u_2) - \phi(u_1)) \cdot \nabla\xi\}\, dx\, dt$$
$$\leqq -\int_Q \kappa(f_2 - f_1)\xi\, dx\, dt \leqq 0$$

for any nonnegative $\xi \in \mathscr{D}([0,T) \times \overline{\Omega})$. Then we get

$$\int_\Omega \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi + H_0(u_1 - u_2)(\phi(u_2) - \phi(u_1)) \cdot \nabla\xi\}\, dx \leqq 0 \tag{118}$$

for almost every $t \in (0,T)$ and for any nonnegative $\xi \in \mathscr{D}(\overline{\Omega})$. By arguing as in the proof of Corollary 3 we deduce that $b(u_1) \leqq b(u_2)$ almost everywhere in Q.

Corollary 12. *Let* (H1) *and* (H2) *hold. Assume that there is an* i_0 *with* $1 \leqq i_0 \leqq N$ *such that*

$$\phi_{i_0} \quad \textit{is monotone (not decreasing or not increasing)}.$$

Let $g_{i_0} \in L^1(\Omega)$*, let* $g_{i_0} \in R(g)$ $(g_{i_0} = g(u_{i_0}))$*, and let* $f_i \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$ *with*

$$g_{1_0} \leqq g_{2_0} \quad \textit{a.e. in } \Omega, \qquad f_1 \leqq f_2 \quad \textit{a.e. in } Q.$$

Let u_i *be an entropy solution of* $(P_E(f_i, g_{i_0})(g, b, \phi))$ *for* $i = 1, 2$*. Then*

$$b(u_1) \leqq b(u_2) \quad \textit{a.e. in } Q\ .$$

Moreover, if $f_1 = f_2$ *and* $g_{1_0} = g_{2_0}$*, then*

$$b(u_1) = b(u_2).$$

Proof. We still have (118). Then, by arguing as in the proof of Corollary 4 we deduce that $b(u_1) \leqq b(u_2)$ almost everywhere in Q.

Corollary 13. *Let* (H1) *and* (H2) *hold. Assume that there is an* i_0 *with* $1 \leqq i_0 \leqq N$ *such that there exists a C for which*

$$|\phi_{i_0}(s) - \phi_{i_0}(r)| \leqq C|g(s) + b(s) - g(r) - b(r)| \quad \forall\, s,\ r \in \mathbb{R}.$$

Let $g_{i_0} \in L^1(\Omega)$*, let* $g_{i_0} \in R(g)$ $(g_{i_0} = g(u_{i_0}))$*, let* $f_i \in L^2((0,T); H^{-1}(\Omega)) \cap L^1(Q)$ *with*

$$g_{1_0} \leqq g_{2_0} \quad \textit{a.e. in } \Omega, \qquad f_1 \leqq f_2 \quad \textit{a.e. in } Q.$$

Let u_i *be an entropy solution of* $(P_E(f_i, g_{i_0})(g, b, \phi))$ *for* $i = 1, 2$*. Then*

$$b(u_1) \leqq b(u_2) \quad \textit{a.e. in } Q.$$

Moreover, if $g_{1_0} = g_{2_0}$ *and* $f_1 = f_2$*, then*

$$b(u_1) = b(u_2).$$

Proof. We still have (118). Then, by arguing as in the proof of Corollary 5 we deduce that $b(u_1) \leqq b(u_2)$ almost everywhere in Q.

Corollary 14. *Under the assumptions of Theorem 12 there exists a unique pair $(g(u), b(u))$ such that u is an entropy solution of $(P_E(f, g_0)(g, b, \phi))$.*

Proof. Let u_1 and u_2 be entropy solutions of $(P_E(f, g_0)(g, b, \phi))$. Since (H1) and (H2) are fulfilled, we deduce that $g(u_1) = g(u_2)$. From (106) and from (H5), we then have

$$\int_0^T \int_\Omega \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi \\ - \phi^{(2)}(g(u_1))(b(u_1) - b(u_2))^+ \cdot \nabla\xi\}\, dx\, dt \leqq 0$$

for any nonnegative $\xi \in \mathscr{D}([0, T) \times \overline{\Omega})$. Hence we deduce that

$$\int_0^T \int_\Omega \{\nabla(b(u_1) - b(u_2))^+ \cdot \nabla\xi \\ - \phi^{(2)}(g(u_1))(b(u_1) - b(u_2))^+ \cdot \nabla\xi\}\, dx\, dt = 0$$

for any $\xi \in \mathscr{D}([0, T) \times \overline{\Omega})$. Then we have

$$0 = \int_\Omega (b(u_1) - b(u_2))^+\{-\Delta\xi - \phi^{(2)}(g(u_1)) \cdot \nabla\xi\}\, dx$$

for almost every $t \in (0, T)$ and for any $\xi \in \mathscr{D}(\overline{\Omega})$.

We take into account that $g(u_1) \in L^\infty(Q)$ and we choose $\xi(x) = e^{\lambda x_i}$ for some i with $1 \leqq i \leqq N$ and some

$$\lambda > \|\phi_i^{(2)}(g(u_1))\|_{L^\infty(Q)}.$$

Then we have

$$-\Delta\xi - \phi^{(2)}(g(u_1)) \cdot \nabla\xi < 0,$$

whence we deduce that

$$(b(u_1) - b(u_2))^+ = 0.$$

Of course, the inequality (106) allows us to study a more general evolution problem. For instance, let $a \in \mathscr{C}(\mathbb{R})$ be a nondecreasing function vanishing at zero. We set

$$(P_E(f, g_0)(g, b, \phi, a)) \begin{cases} \dfrac{\partial g(u)}{\partial t} - \Delta b(u) + \operatorname{div}(\phi(u)) + a(u) = f & \text{in } Q, \\ b(u) = 0 & \text{on } (0, T) \times \partial\Omega, \\ g(u) = g_0 & \text{on } \{0\} \times \Omega. \end{cases}$$

A measurable function u is an entropy solution of $(P_E(f, g_0)(g, b, \phi, a))$ if and only if u is an entropy solution of $(P_E(f - a(u), g_0)(g, b, \phi))$. Then inequality (106) is still true when the f_i are replaced by $f_i - a(u_i)$ for $i = 1, 2$. Then the above corollaries still hold.

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