

Stuff that may well go into the project report

Trygve Bærland

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Abstract

Funker dette?

1 Basics

Let $\Omega \subset \mathbb{R}^d$ be an open, simply connected domain. Then the filtration, or general porous medium equation is

$$\partial_t u = \Delta \Phi(u) + f, \quad (1)$$

where $u, f : \Omega \times (0, \infty) \rightarrow \mathbb{R}$, and $\phi : \mathbb{R} \rightarrow \mathbb{R}$. More specific assumptions on the nonlinearity ϕ and the forcing term f will follow.

1.1 Problem statement

I will consider on the following Dirichlet problem:

$$\begin{cases} \partial_t u = \Delta \Phi(u) + f, & \text{in } Q_T \\ u(x, 0) = u_0(x), & \text{in } \Omega \\ u(x, t) = 0, & \text{on } \partial\Omega \times (0, T) \end{cases} \quad (2)$$

1.2 Weak formulation

Definition 1 (Weak solutions). A locally integrable function u defined in Q_T is said to be a **weak solution** of 2 if

- i) $u \in L^1(Q_T)$ and $\Phi(u) \in L^1(0, T; W_0^{1,1}(\Omega))^1$;
- ii) u satisfies

$$\iint_{Q_T} \{\nabla \Phi(u) \cdot \nabla \eta - u \partial_t \eta\} dx dt = \int_{\Omega} u_0(x) \eta(x, 0) dx + \iint_{Q_T} f \eta dx dt \quad (3)$$

for any $\eta \in C^1(\overline{Q_T})$ that vanishes on the boundary and has compact support.

2 Some estimates

Now we need to make some assumptions on Φ and f :

- i) The function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, strictly increasing and we assume $\Phi(0) = 0$ without any loss of generality.

For now the assumption that $\phi' > 0$ is kind of strong, but I hope to generalise (or find analogues for) most of the estimates to the case $\Phi \geq 0$. In that case (1) changes from nondegenerate parabolic to degenerate parabolic. But hopes are high. In the following proof of uniqueness of weak solutions I'll try out assuming that $\Phi' \geq 0$.

¹ $f : \Omega \rightarrow \mathbb{R}$ is an element of $W^{k,p}(\Omega)$ if all weak derivatives with $|\alpha| \leq k$ exists and are in $L^p(\Omega)$. $W_0^{k,p}$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

3 Uniqueness

Theorem 1 (Uniqueness). *Assuming $u \in L^2(Q_T)$ and that $\Phi \in L^2(0, T : H_0^1(\Omega))$, (2) has at most one weak solution.*

Proof. This is a pretty neat proof using a smart choice of η . So let's assume u_1 and u_2 are weak solutions. Then

$$\iint_{Q_T} \nabla(\Phi(u_2) - \Phi(u_1)) \cdot \nabla \eta - (u_2 - u_1) \partial_t \eta \, dx dt = 0.$$

Now the inspired move² is choosing

$$\eta(x, t) = \begin{cases} \int_t^T (\Phi(u_2) - \Phi(u_1)) \, ds, & \text{if } 0 < t < T \\ 0 & \text{if } t \geq T. \end{cases}$$

It is then easily verified that

$$\begin{cases} \partial_t \eta &= -(\Phi(u_2) - \Phi(u_1)) \\ \nabla \eta &= \int_t^T \nabla(\Phi(u_2) - \Phi(u_1)) \, ds, \end{cases}$$

which put into the integral equality above yields

$$\begin{aligned} & \iint_{Q_T} (\Phi(u_2) - \Phi(u_1))(u_2 - u_1) \, dx dt \\ & + \iint_{Q_T} \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \left(\int_t^T \nabla(\Phi(u_2(s)) - \Phi(u_1(s))) \, ds \right) dt dx = 0. \end{aligned} \quad (4)$$

Let's take these in order:

- i) In the first integral we notice that $u_2 > u_1 \Rightarrow \Phi(u_2) \geq \Phi(u_1)$ and vice versa, so this integral is nonnegative.
- ii) For the second integral it is helpful to first only consider integration of the temporal dimensions, leaving us with

$$\begin{aligned} & \int_0^T \int_t^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s))) \, ds dt \\ & = \int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s))) \, dt ds. \end{aligned}$$

Due to the symmetry about the line $s = t$, we have

$$\begin{aligned} & \int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s))) \, dt ds \\ & = \frac{1}{2} \int_0^T \int_0^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s))) \, dt ds \\ & = \frac{1}{2} \left(\int_0^T \nabla(\Phi(u_2) - \Phi(u_1)) \, dt \right)^2, \end{aligned}$$

which is also nonnegative, and therefore the second integral in (4) must also be nonnegative.

From this we may conclude that both integral terms in (4) are zero, which in turn implies that $(\Phi(u_2) - \Phi(u_1))(u_2 - u_1) = 0$ a.e. in Q_T . Wherever $\Phi'(u) > 0$ we have $u_2 = u_1$, but more care is needed when u_1 and u_2 take values where $\Phi'(u) = 0$. So suppose $u_1 \neq u_2$ on a set $E \subset Q_T$ of positive measure. That means $\nabla(\Phi(u_2) - \Phi(u_1)) = 0$ a.e. in E , and so we can deduce that

$$\iint_{Q_T} (u_2 - u_1) \partial_t \eta \, dx dt = 0$$

for all test functions η with support in E . This gives us that $u_2 = u_1$ a.e. in E as well, so $u_2 = u_1$ a.e. in all of Q_T . Whoop-dee-doo! \square

²Of course, this isn't my proof so that sort of horn-tooting is in good taste.