Stuff that may well go into the project report

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Abstract

Funker dette?

1 Basics

Let $\Omega \subset \mathbb{R}^d$ be an open, simply connected domain. Then the filtration-, or general porous medium equation is

$$\partial_t u = \Delta\Phi(u) + f, \tag{1}$$

where $u, f : \Omega \times (0,\infty) \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$. More specific assumptions on the nonlinearity Φ and the forcing term f will follow.

1.1 Problem statement

I will consider on the following Dirichlet problem:

$$\begin{cases} \partial_t u = \Delta\Phi(u) + f, & \text{in } Q_T \\ u(x,0) = u_0(x), & \text{in } \Omega \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2}$$

Here $Q_T = \Omega \times (0,T)$, where T may very well be infinite.

1.2 Weak formulation

Definition 1 (Weak solutions)**.** *A locally integrable function u defined in Q_T is said to be a **weak solution** of 2 if*

i) *$u \in L^1(Q_T)$ and $\Phi(u) \in L^1(0,T : W_0^{1,1}(\Omega))$*[1]*;*

ii) *u satisfies*

$$\iint_{Q_T} \{\nabla\Phi(u)\cdot\nabla\eta - u\partial_t\eta\}dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint_{Q_T} f\eta dxdt \tag{3}$$

for any $\eta \in C^1(\overline{Q}_T)$ that vanishes on the boundary and has compact support.

2 Some estimates

Now we need to make some assumptions on Φ and f:

i) The function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and we assume $\Phi(0) = 0$ without any loss of generality.

[1] $f : \Omega \to \mathbb{R}$ is an element of $W^{k,p}(\Omega)$ if all weak derivatives with $|\alpha| \le k$ exists and are in $L^p(\Omega)$. $W_0^{k,p}$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

For now the assumption that $\Phi'(u) > 0$ is kind of strong, but I hope to generalise (or find analogues for) most of the estimates to the case $\Phi'(u) \geq 0$. In that case (1) changes from nodegenerate parabolic to degenerate parabolic. But hopes are high. In the following proof of uniqueness of weak solutions I'll try out assuming that $\Phi'(u) \geq 0$.

3 Uniqueness

Theorem 1 (Uniqueness)**.** *Assuming $u \in L^2(Q_T)$ and that $\Phi \in L^2(0,T : H_0^1(\Omega))$, (2) has at most one weak solution.*

Proof. This is a pretty neat proof using a smart choice of η. So let's assume u_1 and u_2 are weak solutions. Then

$$\iint_{Q_T} \nabla(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\eta - (u_2 - u_1)\partial_t\eta dxdt = 0.$$

Now the inspired move[2] is choosing

$$\eta(x,t) = \begin{cases} \int_t^T (\Phi(u_2) - \Phi(u_1))ds, & \text{if } 0 < t < T \\ 0 & \text{if } t \geq T. \end{cases}$$

It is then easily verified that

$$\begin{cases} \partial_t\eta & = -(\Phi(u_2) - \Phi(u_1)) \\ \nabla\eta & = \int_t^T \nabla(\Phi(u_2) - \Phi(u_1))ds, \end{cases}$$

which put into the integral equality above yields

$$\begin{aligned} &\iint_{Q_T} (\Phi(u_2) - \Phi(u_1))(u_2 - u_1)dxdt \\ &+ \iint_{Q_T} \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \left(\int_t^T \nabla(\Phi(u_2(s)) - \Phi(u_1(s))ds \right) dtdx = 0. \end{aligned} \tag{4}$$

Let's take these in order:

i) In the first integral we notice that $u_2 > u_1 \Rightarrow \Phi(u_2) \geq \Phi(u_1)$ and vice versa, so this integral is nonnegative.

ii) For the second integral it is helpful to first only consider integration of the temporal dimensions, leaving us with

$$\begin{aligned} &\int_0^T \int_t^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dsdt \\ &= \int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds. \end{aligned}$$

Due to the symmetry about the line $s = t$, we have

$$\begin{aligned} &\int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds \\ &= \frac{1}{2} \int_0^T \int_0^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds \\ &= \frac{1}{2} \left(\int_0^T \nabla(\Phi(u_2) - \Phi(u_1))dt \right)^2, \end{aligned}$$

which is also nonnegative, and therefore the second integral in (4) must also be nonnegative.

[2] Of course, this isn't my proof so that sort of horn-tooting is in good taste.

From this we may conclude that both integral terms in (4) are zero, which in turn implies that $(\Phi(u_2) - \Phi(u_1))(u_2 - u_1) = 0$ a.e. in Q_T. Wherever $\Phi'(u) > 0$ we have $u_2 = u_1$, but more care is needed when u_1 and u_2 take values where $\Phi'(u) = 0$. So suppose $u_1 \neq u_2$ on a set $E \subset Q_T$ of positive measure. That means $\nabla(\Phi(u_2) - \Phi(u_1)) = 0$ a.e. in E, and so we can deduce that

$$\iint_{Q_T} (u_2 - u_1)\partial_t \eta dxdt = 0$$

for all test functions η with support in E. This gives us that $u_2 = u_1$ a.e. in E as well, so $u_2 = u_1$. Whoop-dee-doo! $\square$