Stuff that may well go into the project report

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Abstract

Funker dette?

1 Basics

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded, and simply connected domain. Then the filtration, or general porous medium equation is

$$\partial_t u = \Delta\Phi(u) + f, \tag{1}$$

where $u, f : \Omega \times (0,\infty) \to \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$. For now we assume that Φ is continuous and nondecreasing.

1.1 Problem statement

I will consider on the following Dirichlet problem:

$$\begin{cases} \partial_t u = \Delta\Phi(u) + f, & \text{in } Q_T \\ u(x,0) = u_0(x), & \text{in } \Omega \\ u(x,t) = 0, & \text{on } \partial\Omega \times (0,T) \end{cases} \tag{2}$$

1.2 Weak formulation

Definition 1 (Weak solutions)**.** *A locally integrale function u defined in Q_T is said to be a* ***weak solution*** *of 2 if*

i) $u \in L^1(Q_T)$ *and* $\Phi(u) \in L^1(0,T : W_0^{1,1}(\Omega))$[1]*;*

ii) *u satisfies*

$$\iint_{Q_T} \{\nabla\Phi(u) \cdot \nabla\eta - u\partial_t\eta\} dxdt = \int_\Omega u_0(x)\eta(x,0)dx + \iint_{Q_T} f\eta dxdt \tag{3}$$

for any $\eta \in C^1(\overline{Q}_T)$ *that vanishes on* $\partial\Omega[0,T)$ *and for* $t = T$.

2 Some estimates

Now we need to make some assumptions on Φ and f:

i) The function $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous, strictly increasing and we assume $\Phi(0) = 0$ without any loss of generality.

For now the assumption that $\Phi' > 0$ is kind of strong, but I hope to generalise (or find analogues for) most of the estimates to the case $\Phi \geq 0$. In that case (1) changes from nodegenerate parabolic to degenerate parabolic. But hopes are high. In the following proof of uniqueness of weak solutions I'll try out assuming that $\Phi' \geq 0$.

[1] $f : \Omega \to \mathbb{R}$ is an element of $W^{k,p}(\Omega)$ if all weak derivatives with $|\alpha| \leq k$ exists and are in $L^p(\Omega)$. $W_0^{k,p}$ is the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

3 Uniqueness

Theorem 1 (Uniqueness)**.** *Assuming in addition that $u \in L^2(Q_T)$ and $\Phi(u) \in L^2(0,T : H_0^1(\Omega))$, (2) has at most one weak solution.*

Proof. This is a pretty neat proof using a smart choice of η. So let's assume u_1 and u_2 are weak solutions. Then

$$\iint_{Q_T} \nabla(\Phi(u_2) - \Phi(u_1)) \cdot \nabla\eta - (u_2 - u_1)\partial_t \eta \, dxdt = 0.$$

Now the inspired move[2] is choosing

$$\eta(x,t) = \begin{cases} \int_t^T (\Phi(u_2) - \Phi(u_1))ds, & \text{if } 0 < t < T \\ 0 & \text{if } t \geq T. \end{cases}$$

It is then easily verified that

$$\begin{cases} \partial_t \eta &= -(\Phi(u_2) - \Phi(u_1)) \\ \nabla\eta &= \int_t^T \nabla(\Phi(u_2) - \Phi(u_1))ds, \end{cases}$$

which put into the integral equality above yields

$$\begin{aligned} &\iint_{Q_T} (\Phi(u_2) - \Phi(u_1))(u_2 - u_1)dxdt \\ &+ \iint_{Q_T} \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \left(\int_t^T \nabla(\Phi(u_2(s)) - \Phi(u_1(s))ds \right) dtdx = 0. \end{aligned} \tag{4}$$

Let's take these in order:

i) In the first integral we notice that $u_2 > u_1 \Rightarrow \Phi(u_2) \geq \Phi(u_1)$ and vice versa, so this integral is nonnegative.

ii) For the second integral it is helpful to first only consider integration of the temporal dimensions, leaving us with

$$\begin{aligned} &\int_0^T \int_t^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dsdt \\ &= \int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds. \end{aligned}$$

Due to the symmetry about the line $s = t$, we have

$$\begin{aligned} &\int_0^T \int_0^s \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds \\ &= \frac{1}{2} \int_0^T \int_0^T \nabla(\Phi(u_2(t)) - \Phi(u_1(t))) \cdot \nabla(\Phi(u_2(s)) - \Phi(u_1(s)))dtds \\ &= \frac{1}{2} \left(\int_0^T \nabla(\Phi(u_2) - \Phi(u_1))dt \right)^2, \end{aligned}$$

which is also nonnegative, and therefore the second integral in (4) must also be nonnegative.

From this we may conclude that both integral terms in (4) are zero, which in turn implies that $(\Phi(u_2) - \Phi(u_1))(u_2 - u_1) = 0$ a.e. in Q_T. Wherever $\Phi'(u) > 0$ we have $u_2 = u_1$, but more care is needed when u_1 and u_1 take values where $\Phi'(u) = 0$. So suppose $u_1 \neq u_2$ on a set $E \subset Q_T$ of positive measure. That means $\nabla(\Phi(u_2) - \Phi(u_1)) = 0$ a.e. in E, and so we can deduce that

$$\iint_{Q_T} (u_2 - u_1)\partial_t \eta \, dxdt = 0$$

for all test functions η with support in E. This gives us that $u_2 = u_1$ a.e. in E as well, so $u_2 = u_1$ a.e. in all of Q_T. Whoop-dee-doo! □

[2]Of course, this isn't my proof so that sort of horn-tooting is in good taste.