Problem 1

Setting $\rho = 0$, $k \neq 0$

a) Friedmann equation reads:

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = 0$$

$$\dot{a}^2 = -kc^2$$

We only want real solutions so k must be -1 (i.e. we have an open universe)

$$\rightarrow \quad \dot{a}^2 = c^2$$

$$\dot{a} = \pm c$$

$$da = \pm c\,dt$$

$$\int_{a_0}^{a} da = \pm \int_0^t c\,dt$$

$$a = a_0 \pm ct$$ where a_0 is the value $a_0 = a_0(t=0)$.

If we assume a big bang model $a(t=0) = 0$ we get $a = \pm ct$ where only the plus solution is viable, that is the universe is open and forever increasing in size linearly. If $a(t=0) \neq 0$ we can have this increasing model, but with big bang at some time t_{BB} where $a_0 + ct_{BB} = 0$, or we could have a big crunch model with t_{BC} given by $a_0 - ct_{BC} = 0$ i.e. linearly decreasing universe size.

1 b)

Proper distance: $d_p(t) = a(t)\, S^{-1}(r)$

where $S^{-1}(r) = \sinh^{-1}(r)$ for our open universe case

In this case we also have comoving coordinate

$$r = \sinh\left[\int_{t_e}^{t_0} \frac{c\,dt'}{a(t')}\right] = \sinh\left[\int_{t_e}^{t_0} \frac{c\,dt'}{\pm ct' + a_0}\right]$$

$$= \sinh\left[\pm \ln[\pm ct_0 \pm a_0] \mp \ln[\pm ct_e \pm a_0]\right]$$

$$= \cancel{\sinh} = \begin{cases} \ln\left[\frac{ct_0 + a_0}{ct_e + a_0}\right] & \text{for } + \\ \ln\left[\frac{ct_e + a_0}{ct_0 + a_0}\right] & \text{for } - \end{cases} = \pm \ln\left[\frac{ct_0 + a_0}{ct_e + a_0}\right]$$

redshift $1+z = \frac{a_0}{a}$, so that

$$\underline{d_p(t) = \pm \frac{a_0}{1+z} \ln(1+z)}$$

2 Flat universe $k=0$, containing matter Ω_M and cosmological constant Ω_Λ.

Friedmann equation: $\frac{H^2}{H_0^2} = \Omega_M + \Omega_\Lambda = \Omega_{M,0}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda$

a) When the expansion stops $H = 0$

$$\rightarrow \Omega_{M,0}\left(\frac{a_0}{a_{max}}\right)^3 + \Omega_\Lambda = 0$$

$$-\frac{\Omega_\Lambda}{\Omega_{M,0}} = \left(\frac{a_0}{a_{max}}\right)^3$$

$$\frac{a_{max}}{a_0} = \left(-\frac{\Omega_\Lambda}{\Omega_{M,0}}\right)^{-\frac{1}{3}}$$

$$\cancel{a_{max} = (}$$

We usually define $a_0 = 1$, so that:

$$\underline{a_{max} = \left(-\frac{\Omega_\Lambda}{\Omega_{M,0}}\right)^{-\frac{1}{3}}}$$

(which is what we wanted.)

2b) We want an expression for the age of the universe at crunching time.

$$\left(\frac{\dot a}{a}\right)^2 = H_0 \left(\Omega_{M,0}\, a^{-3} + \Omega_\Lambda\right)$$

$$\dot a^2 = H_0^2 \left(\Omega_{M,0}\, a^{-1} \quad \Omega_\Lambda a^2\right)$$

$$\frac{da}{H_0\sqrt{(\Omega_{M,0}\, a^{-1} - \Omega_\Lambda a^2)}} = dt$$

It starts from zero at $a=0$, however we cannot integrate past a_{max}. Luckily the contraction phase is exactly similar to the expansion phase in time span so that

$$\underline{\underline{t_{crunch} = 2\int_0^{a_{max}} \frac{da}{H_0\sqrt{\Omega_{M,0}\, a^{-1} - \Omega_\Lambda a^2}}}}$$

3 Decceleration parameter: (I set $a_0 = 1$)

$q = -\frac{\ddot a\, a}{\dot a^2}$ Universe is flat and matter dominated so that

$$\frac{\dot a^2}{a^2} = H_0\, \Omega_{M,0}\, a^{-3}$$

$$\dot a^2 = H_0\, \Omega_{M,0}\, a^{-1} \qquad 2\ddot a \dot a = -H_0\, \Omega_{M,0}\, a^{-2} \dot a$$

So that $\ddot a = -\frac{1}{2} H_0\, \Omega_{M,0}\, a^{-2}$ and hence

$$q = -\frac{\ddot a\, a}{\dot a^2} = \frac{-\left(-\frac{1}{2}\cancel{H_0}\,\cancel{\Omega_{M,0}}\, a^{-2}\right) a}{\cancel{H_0}\,\cancel{\Omega_{M,0}}\, a^{-1}} = \underline{\frac{1}{2}}$$ and

is independent of the density $\Omega_M = \Omega_{M,0}\, a^{-3}$

4. One-component universe with fluid of density ρ and pressure p $\quad p = w\rho \quad (c=1) \quad w \neq -1$

a) We use the adiabatic fluid equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho(1+w)$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -3(1+w)\int_{a_0}^{a}\frac{da}{a}$$

$$\ln\rho - \ln\rho_0 = -3(1+w)[\ln a - \ln a_0]$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = \ln\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

$$\underline{\rho = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)} = \rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}}$$

Redshift $\quad 1+z = \frac{a_0}{a} \rightarrow \underline{\rho = \rho_0(1+z)^{3(1+w)}}$

b) Assume $k=0$: $\quad \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^{3(1+w)}$

$$\rho_c = \frac{3H^2}{8\pi G} \qquad \frac{\dot{a}^2}{a^2} = H_0^2\frac{\rho_0}{\rho_{c0}}\left(\frac{a_0}{a}\right)^{3(1+w)}$$

But at t_0 we get $H_0 = H_0\frac{\rho_0}{\rho_{c0}} \rightarrow \rho_0 = \rho_{c0}$

so: $$\frac{\dot{a}^2}{a^2} = H_0^2\left(\frac{a_0}{a}\right)^{3(1+w)}$$

$$\dot{a} = H_0 a_0^{\frac{3(1+w)}{2}} a^{-\frac{3(1+w)}{2}+1}$$

$$da\, a^{\frac{3(1+w)}{2}-1} = H_0 a_0^{\frac{3(1+w)}{2}} dt$$

$$\int_{a_0}^{a} da\, a^{\frac{3(1+w)}{2}-1} = \int_{t_0}^{t} H_0 a_0^{\frac{3(1+w)}{2}} dt$$

4 b) cont.

$$\frac{2}{3(1+w)}\left[a^{\frac{3(1+w)}{2}} - a_0^{\frac{3(1+w)}{2}}\right] = H_0\, a_0^{\frac{3(1+w)}{2}}\,(t - t_0)$$

Assuming a big bang where $a=0$ at $t=0$, we get that $\frac{2}{3(1+w)}\left(-a_0^{\frac{3(1+w)}{2}}\right) = -H_0 t_0\, a^{\frac{3(1+w)}{2}}$

So that $H_0 t_0 = \frac{2}{3(1+w)}$ and

$$a^{\frac{3(1+w)}{2}} = a_0^{\frac{3(1+w)}{2}}\,\frac{t}{t_0}$$

$$\underline{\underline{a = a_0 \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}}}$$

4 c) Cold dark matter, i.e. dust, is pressureless so $w=0$, Using the result from b) we get:

$$\underline{\underline{a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}}}$$

d) Radiation has pressure given by $p = \frac{1}{3}\rho$, so:

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{4}} = \underline{\underline{a_0\left(\frac{t}{t_0}\right)^{\frac{1}{2}}}}$$

e) The present age of the universe t_0, we found an expression for in b), recall that

$$H_0 t_0 = \frac{2}{3(1+w)}, \text{ so } \underline{t_0 = \frac{2}{3(1+w)H_0}}$$

$H_0 = h \cdot 100\,\mathrm{km s^{-1} Mpc^{-1}}$ so if $h = 2/3$ and $w > -1$

$t_0 \lesssim 15 \cdot 10^9$ years & then $(1+w) < \frac{2}{3H_0 t}$

$$w < \frac{2}{3\,\cancel{H_0}\,\frac{2}{3}\,100\,\mathrm{kms^{-1}Mpc^{-1}} \cdot 15 \cdot 10^9\,\mathrm{years}} - 1 = -1 + \frac{1}{100\,\mathrm{kms^{-1}Mpc^{-1}} \cdot 15 \cdot 10^9\,\mathrm{years}}$$

$$100 \ km \ s^{-1} \ Mpc^{-1} = \frac{100 \ km}{30{,}857 \cdot 10^{18} \ km \ s} =$$

$$= \frac{100 \cdot 3600 \cdot 24 \cdot 365}{30{,}857 \cdot 10^{18} \ year} = 1{,}022 \cdot 10^{-10} \ year^{-1}$$

$$\rightarrow w < \frac{1}{1{,}022 \cdot 10^{-10} \cdot 15 \cdot 10^{9}} - 1 = 0{,}651 - 1$$

$$= -0{,}348 \quad \text{so} \quad \underline{w < -0{,}35}$$

If $w < -1$

we get $H_0 t_0 = \frac{2}{3(1+w)}$ a negative number, which does not make sense in our universe as H_0 here is positive (expanding universe). However we could consider such a universe, we then get

$$t_0 = \frac{2}{3H_0(1+w)} > 15 \cdot 10^{9} \ years$$

Since $(1+w)$ is negative multiplying changes sign so

$$(1+w) > \frac{2}{3H_0 \cdot 15 \cdot 10^{9} \ years}$$

$$w > \frac{2}{3 H_0 \cdot 15 \cdot 10^{9} \ years} - 1 = -0{,}35$$

which is an absurd statement given that we assumed $w < -1$. Hence the allowed values for w are:

$$\underline{-0{,}35 > w > -1}$$