Midterm exam 2012

1 ⓐ Starting from the first Friedmann equation : $H^2 + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$ and dividing by H^2 : $\frac{H^2}{H^2} + \frac{kc^2}{a^2H^2} = \frac{8\pi G}{3H^2}\rho$

$\Omega = \frac{\rho}{\rho_{c0}}$ were $\rho_{c0} = \frac{3H_0^2}{8\pi G}$ we get

$$1 + \frac{kc^2}{a^2H^2} = \Omega \quad \text{at } t=0$$

the quantity $\left(\frac{c^2}{a^2H^2}\right) = \left(\frac{c}{aH}\right)^2$ must be positive so the term $\frac{kc^2}{a^2H^2}$ has the same sign as k so if $k=+1$, then

$$\Omega = 1 + \frac{c^2}{a^2H^2} > 1$$

if $k=0$ then $\Omega = 1$ and if $k=-1$ then $\Omega = 1 - \frac{c^2}{a^2H^2} < 1$

ⓑ $\rho = \rho_0\left(\frac{a_0}{a}\right)^3$ for matter dominated universe.

In matter domination $\rho \approx \rho_m$. The adiabatic expansion equation states that $\dot\rho_i = -3H(1+w_i)\rho_i$ were $w_i = \frac{p_i}{\rho_i}$ is the eqution of state parameter for universe component i. Matter is pressureless so $w_m = 0$ hence

$$\dot\rho_m = -3H\rho_m$$

$$\frac{1}{\rho_m}\frac{d\rho_m}{dt} = -3\frac{1}{a}\frac{da}{dt} \Rightarrow \int_{\rho_{m0}}^{\rho_m}\frac{d\rho_m}{\rho_m} = -3\int_{a_0}^{a}\frac{da}{a}$$

$$\ln\left(\frac{\rho_m}{\rho_{m0}}\right) = -3\ln\left(\frac{a}{a_0}\right)$$

$$\rho_m = \rho_{m0}\left(\frac{a_0}{a}\right)^3$$

and since $\rho \simeq \rho_m$ in matter domination

$\rho = \rho_0 \left(\frac{a_0}{a}\right)^3$ for such a universe.

c) $q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2} = -\frac{\ddot{a}_0}{a_0} \cdot \frac{1}{H_0^2}$ which is minus the second Friedmann equation divided by the first one: 2nd Friedmann

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad p = 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \quad \text{(for matter dominated universe)}$$

1st Friedmann: $H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$

$$q_0 = \frac{\frac{4\pi G}{3}\rho_0}{\frac{8\pi G}{3}\rho_0 - \frac{kc^2}{a_0^2}} = \frac{\frac{4\pi G}{3H_0^2}\rho_0}{\frac{8\pi G}{3H_0^2} - \frac{kc^2}{a_0^2 H_0^2}} = \frac{\frac{1}{2}\Omega_{m0}}{\Omega_{m0} - \frac{kc^2}{a_0^2 H_0^2}}$$

but $\Omega_{m0} - \frac{kc^2}{a_0^2 H_0^2} = \frac{H_0^2}{H_0^2} = 1$ so

$$\underline{q_0 = \frac{1}{2}\Omega_{m0}}$$

d) First Friedmann with cosmological constant:

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho_m a^2 + \frac{\Lambda}{3}a^2 \quad (c = 1)$$

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho_{m0}\left(\frac{a_0}{a}\right)^3 a^2 + \frac{\Lambda}{3}a^2$$

$$\dot{a}^2 + k = \frac{8\pi G}{3}\frac{\rho_{m0} a_0^3}{a} + \frac{\Lambda}{3}a^2$$

is on the form $\dot{a}^2 + k = \frac{C}{a} + Da^2$

where e) $C = \frac{8\pi G}{3}\rho_{m0} a_0^3 = \underline{H_0^2 \Omega_{m0} a_0^3}$ and

f) $\underline{D = \frac{\Lambda}{3}}$.

2. a) $\dot{\rho}_{DE} = -3\frac{\dot{a}}{a}(1+w_{DE})\rho_{DE}$

$$\int_{\rho_{DE0}}^{\rho_{DE}} \frac{d\rho_{DE}}{\rho_{DE}} = -3(1+w_{DE})\int_{a_0}^{a} \frac{da}{a}$$

$$\ln\left(\rho_{DE}/\rho_{DE0}\right) = -3(1+w_{DE})\ln\left(\frac{a}{a_0}\right)$$

$$\rho_{DE} = \rho_{DE0}\left(\frac{a_0}{a}\right)^{3(1+w_{DE})} \qquad 1+z = \left(\frac{a_0}{a}\right)$$

$$= \rho_{DE0}(1+z)^{3(1+w_{DE})}$$

b) Dark energy and matter equality occurs when

$$\rho_m = \rho_{DE}$$

~~We know that~~ From the above and the results of problem 1 we know that this means that

$$\rho_{m0}(1+z_{eq})^3 = \rho_{DE0}(1+z_{eq})^{3(1+w_{DE})} \quad \text{dividing by } \rho_{c0}$$

$$\Omega_{m0}(1+z_{eq})^3 = \Omega_{DE0}(1+z_{eq})^{3(1+w_{DE})}$$

But the first Friedmann equation for this universe states that $\frac{H^2}{H_0^2} = \Omega_{m0}(1+z)^3 + \Omega_{DE}(1+z)^{3(1+w_{DE})}$

and for $z=0$ this becomes:

$$1 = \Omega_{m0} + \Omega_{DE} \Rightarrow \Omega_{DE} = 1-\Omega_{m0}$$

$$\Rightarrow \Omega_{m0} = (1-\Omega_{m0})(1+z_{eq})^{3w_{DE}}$$

$$(1+z_{eq})^{3w_{DE}} = \frac{\Omega_{m0}}{1-\Omega_{m0}}$$

$$\boxed{z_{eq} = \left(\frac{\Omega_{m0}}{1-\Omega_{m0}}\right)^{\frac{1}{3w_{DE}}} - 1}$$

b)

c) With the formula

$$\mu = 5 \log_{10}\left(\frac{d_L}{10pc}\right) - M$$

and observations of supernovae at different redshifts z with observed magnitudes μ, we can extract information about their comoving distances by inverting the formula:

$$\frac{\mu + M_0}{5} = \log_{10}\left(\frac{d_L}{10pc}\right)$$

$$d_L = 10pc \cdot 10^{\left(\frac{\mu + M_0}{5}\right)}$$

$$d_L = r\,(1+z)$$

$$\underline{\underline{r = \frac{1}{1+z}\, 10^{\left(1 + \frac{\mu + M_0}{5}\right)}\, pc}}$$

d) In a flat universe

$$r = \int_t^{t_0} \frac{c\,dt'}{a(t')} \qquad \frac{da}{dt} = \dot{a} \Rightarrow dt = \frac{da}{\dot{a}}$$

$$= \int_a^{a_0} \frac{c\,da'}{\dot{a}\,a'} \qquad z = \frac{1}{a} - 1 \qquad \frac{dz}{da} = -\frac{1}{a^2} \Rightarrow \frac{da}{a} = -a\,dz$$

$$= -\int_z^0 \frac{c\,a\,dz}{\dot{a}} = \underline{\int_0^z \frac{c\,dz}{H}}$$

, but this means that

$$\frac{dr}{dz} = \frac{c}{H(z)} \Rightarrow \underline{\underline{H(z) = \frac{c}{\frac{dr}{dz}}}}$$

e)

$$q = -\frac{\ddot{a}\,a}{\dot{a}^2} = -\frac{\ddot{a}}{a}\frac{1}{H^2} = -\left(\dot{H} + H^2\right)\frac{1}{H^2}$$

$$\left(\dot{H} = \frac{d\,\frac{\dot{a}}{a}}{dt} = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = \frac{\ddot{a}}{a} - H^2 \Rightarrow \frac{\ddot{a}}{a} = \dot{H} + H^2\right)$$

$$q = -\left(1 + \frac{1}{H^2}\frac{d}{dt}\frac{c}{\frac{dr}{dz}}\right) = -\left(1 - \frac{c}{H^2}\frac{1}{\left(\frac{dr}{dz}\right)^2}\frac{dz}{da}\frac{da}{dt}\frac{d^2r}{dz^2}\right)$$

$$q = -\left(1 - \frac{c}{\frac{dr}{dz} r} \frac{1}{H} \cdot \frac{1}{H \frac{dr}{dz}} \left(-\frac{1}{a^2}\right) \dot{a} \frac{d^2 r}{dz^2}\right)$$

$$= -\left(1 + \cancel{H} \cdot \frac{1}{\cancel{H}} \cdot \frac{1}{\frac{dr}{dz}} \cdot \frac{1}{\cancel{H}} \cdot \cancel{H} \cdot \frac{1}{a} \frac{d^2 r}{dz^2}\right)$$

$$= -\left(1 + \frac{(1+z)}{\frac{dr}{dz}} \frac{d^2 r}{dz^2}\right)$$

$$= -\left(1 + (1+z) \frac{d}{dz} \ln\left(\frac{dr}{dz}\right)\right)$$

f) $$q = -\frac{\ddot{a}}{a} \frac{1}{H^2} = \frac{\frac{4\pi G}{3} \sum_i (1+3w_i) \rho_i}{\frac{8\pi G}{3} \sum_i \rho_i}$$

$$= \frac{1}{2} \left(\frac{\Omega_{m0} (1+z)^3 + (1+3w_{DE})(1-\Omega_{m0})(1+z)^{3(1+w_{DE})}}{\Omega_{m0} (1+z)^3 + (1-\Omega_{m0})(1+z)^{3(1+w_{DE})}} \right)$$

$$= \frac{1}{2} \left(\frac{\Omega_{m0} + (1+3w_{DE})(1-\Omega_{m0})(1+z)^{3w_{DE}}}{\Omega_{m0} + (1-\Omega_{m0})(1+z)^{3w_{DE}}} \right)$$

$$= \frac{1}{2} \left(\frac{1 + (1+3w_{DE}) \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}{1 + \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}} \right)$$

$$= \frac{1}{2} \left(1 + \frac{3w_{DE} \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}{1 + \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}} \right)$$

g) $$-\left(1 + (1+z) \frac{d}{dz} \ln\left(\frac{dr}{dz}\right)\right) = \frac{1}{2} \left(1 + \frac{3w_{DE} \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}{1 + \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}\right)$$

$$-2\left(1 + (1+z) \frac{d}{dz} \ln\left(\frac{dr}{dz}\right)\right) = 1 + \frac{3w_{DE} \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}{1 + \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}$$

$$-\left(3 + 2(1+z) \frac{d}{dz} \ln\left(\frac{dr}{dz}\right)\right) = \frac{3w_{DE} \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}{1 + \left(\frac{1+z}{1+z_{eq}}\right)^{3w_{DE}}}$$

$z = z_{eq}$: $$\cancel{w_{DE}} \frac{3w_{DE}(z_{eq})}{2} = -3 - 2(1+z_{eq}) \left.\frac{d}{dz}\right|_{z=z_{eq}} \ln\left(\left.\frac{dr}{dz}\right|_{z=z_{eq}}\right)$$

$$w_{DE}(z_{eq}) = -\frac{1}{2} - \frac{4}{3}(1+z_{eq}) \left.\frac{d}{dz}\right|_{z=z_{eq}} \ln\left(\left.\frac{d}{dz}\right|_{z=z_{eq}} r\right)$$