AST 4220 final exam h11

1 a) Puzzles of the Big Bang model:

i) The Flatness problem: We observe the curvature parameter today:

$$\underbrace{\Omega}_{\text{total density parameter}}(t) - 1 = \frac{k c^2}{a^2 H^2}$$

k — Curvature parameter, c — light speed, H — Hubble parameter, a — scale factor

to be very small. Evolving the universe back in time with the Big Bang model from this we get a value at Planck time of $\Omega(t_{PL}) - 1 \lesssim 10^{-61}$, so the universe must have been extreemly flat at this point. In principle, however, $\Omega(t_{PL}) - 1$ should be relatively arbitrary so the extremly flat universe seems highly unlikely.

ii) Entropy problem: A related way of stating this is by looking at this in the radiation dominated epoch. Then we have:

$$\boxed{H^2 \simeq \rho_R = \frac{T^4}{M_{PL}^2} \qquad \Omega - 1 = \frac{k M_{PL}^2}{a^4 T^4} = \frac{k M_{PL}^2}{\underbrace{S^{2/3}}_{\text{Entropy}} T^2}}$$

If the expansion of the universe has been adiabatic, which the Big Bang model predicts, the entropy S_U of the universe is constant so

$$|\Omega - 1|_{t=t_{PL}} = \frac{m_{PL}^2}{T_{PL}^2} \frac{1}{S_U^{2/3}} = \frac{1}{S^{2/3}} = 10^{-60}$$

This means that the universe started in a state of immense entropy, which is also intuitively unlikely.

iii) Horizon problem: When we observe the CMB ~~bac~~ it turns out that it is extremly isotropic and homogeneous with only 10^{-5}K temperature fluctuations that seem to be correlated throughout the whole universe. However, if we calculate the particle horizon size, that is the size of causally connected regions, at the time of recombination, (which is essentially what we see when we look at the CMB) it is in fact very small compared to the size of the particle horizon today (i.e. to the full size of the observed CMB today). It is therefore truly puzzeling that the whole CMB is so highly correlated.

iv) Relic problem: As the universe cools down ~~the state of the un~~ it goes through several phase changes from the quark-gluon-plasma through the different decoupling scales. This is quite analogous to the phase transition in H_2O from water to ice. In the case of water to ice, we observe surfaces between different domains in the ice where the ice has formed simultanously and hence formed geometrically differentoriented crystal structures. The same is excpected in the cooling of the universe and the boundary points and surfaces should be marked with magnetic monopoles or other topological objects all denoted relics. None of these have been observed although the Big Bang models predict that they should be dense enough to have been detected.

1 b) Inflation is generically a theory in which the universe undergoes an epoch of highly accelerated expansion. This expansion is not adiabatic and can hence blow up the entropy and at the same time flattening it out solving the flatness and entropy problems. If the relics are formed before inflation or early in the inflationary epoch, the relics will be scattered so widely by the inflationary expansion that they are much too sparse today for us to have any hope of ever detecting them. The horizon problem will also be solved because the seemingly disconnected regions were in causal contact before inflation, when the whole observable universe was only a tiny part of the preinflationary particle horizon.

1 c) Inflation sets up the initial conditions for the perturbations to the homogeneity and isotropy of the universe. We can use quantities calculated in the specific inflationary scenario to calculate our way into the structure distributions today.

1 d) At the end of inflation we naively expect the universe to be very cold and the matter and radiation content of it to be completely inflated away just like the curvature of the universe. However, the universe we observe today clearly didn't evolve from such a state, so we need the universe to reheat

creating matter and radiation in order for the universe to follow the observed radiation- and matterdominated evolution. ~~Above~~

2) ~~a~~ The deuterium bottleneck: As the universe cools down and neutral hydrogen forms at recombination it becomes energetically favourable for the remaining neutrons to combine with the protons to form ^4He (in fact iron would be the most favoured element however there are even tighter bottlenecks that can only be surpassed in stars for that to form). To achieve this the particles must undergo a chain of nuclear reactions starting with formation of deuterium. However, the single reaction of a proton and a neutron to form deuterium

$$p + n \rightarrow d + \gamma$$

is not very energetically favoured and the photon energy needed to reverse the reaction is so low that there will be an abundance of photons of this energy ~~for~~ ready to break any deuterium particles apart. Only when the average photon energy is far below this ~~poin~~ energy, will deuterium be able to form stably and after that all further reactions to ^4He will be fast and proceed stably, hence the stable formation of deuterium is the bottleneck for the Helium formation.

2 b) The freeze-out of a particle species is when the production and ~~annihilati~~ destruction of this particle type falls out of equilibrium with the rest of the universe, that is the reaction rate is slower than the expansion rate of the universe. The particle species is then no longer in thermal equilibrium with the surrounding universe so if the universe is boosted with energy from a total annihilation of a particle species, say, the frozen out particle species will not be heated by this.

3 a) Radiation dominated universe before t_d with $p = \rho/3$. ~~The~~ The equation for adiabatic expansion yields:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) = -4\frac{\dot{a}}{a}\rho$$

$$\frac{d\rho}{\rho} = -4\frac{da}{a} \qquad \int_{\rho}^{\rho_d} \frac{d\rho}{\rho} = -\int_{a}^{a_d} 4\frac{da}{a}$$

$$-4\ln\left(\frac{a_d}{a}\right) = \ln\left(\frac{\rho_d}{\rho}\right) \rightarrow \underline{\rho = \rho_d\left(\frac{a_d}{a}\right)^4}$$

$$\left(\frac{a_d}{a}\right)^{-4} = \frac{\rho_d}{\rho}$$

3 b) $k = 0$, 1. Friedmann equation:

$$\dot{a}^2 = \frac{8\pi G}{3}\rho(a)a^2 = \frac{8\pi G}{3}\rho_d\frac{a_d^4}{a^2}$$

$$a\dot{a} = a_d^2\left(\frac{8\pi G}{3}\rho_d\right)^{1/2}$$

$$\int_0^a a\,da = \int_0^t a_d^2\left(\frac{8\pi G}{3}\rho_d\right)^{1/2}$$

$$\frac{1}{2}a^2 = a_d^2\left(\frac{8\pi G}{3}\rho_d\right)^{1/2} t \qquad \underline{a = a_d\left(\frac{32\pi G}{3}\rho_d\right)^{1/4} t^{1/2}}$$

3 ⓐ $\rho(t) = \rho_j \left(\frac{a_j}{a}\right)^4 = \rho_j \left(\frac{a_j}{a_j \left(\frac{32\pi G}{3}\rho_j\right)^{1/4} t^{1/2}}\right)^4$

$= \rho_j \frac{3}{32\pi G \rho_j} t^{-2} = \underline{\underline{\frac{3}{32\pi G} t^{-2}}}$

3 ⓓ $\rho = N \alpha T^4$ ← temperature

\# radiative species (above N); Stefan-Boltzmann constant (below α)

$\frac{3}{32\pi G} t^{-2} = N\alpha T^4 \qquad t^2 = \frac{3}{32\pi G N\alpha} T^{-4}$

$\underline{\underline{t(T) = \left(\frac{3}{32\pi G N\alpha}\right)^{1/2} T^{-2}}}$

3 ⓔ Nucleosynthesis ends at $T \approx 10^9\,K$
$t(T=10^9 K) \propto N^{-1/2}$, so when N increases t decreases, t decreases with N.

3 ⓕ As $t(T=10^9K)$ increases nucleosynthesis ends later. Before nucleosynthesis the neutron to proton ratio $\frac{\rho_n}{\rho_p}$ decreases exponentially with time, but freezes out at $t(10^9 K)$, so as $t(10^9 K)$ increases the neutron to proton ratio decreases.

3 ⓖ At the end of nucleosynthesis basically all the neutrons remaining in the universe end up in helium, so for every two neutrons there is one Helium atom now (disregarding stellar Helium formation) so:

$$\underline{\underline{\frac{\rho_n}{\rho_p} = \frac{2\rho_{He}}{\rho_p} \quad \text{or} \quad \frac{\rho_{He}}{\rho_p} = \frac{1}{2}\frac{\rho_n}{\rho_p}}}$$

4 Static universe $\left(\frac{da}{dt}=0\right)$, filled with gas of density ρ, pressure p and velocity $\vec{v}$ and gravitational acceleration obeying the mass equation: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \rho \vec{v} = 0$ (1) and momentum equation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{v}\right) = -\vec{\nabla}p + \rho\vec{F} \quad (2)$$

a) Assuming linear perturbations around a static background: $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, $\vec{v} = \delta\vec{v}$, $\vec{F} = \delta\vec{F}$, $\nabla\cdot\delta\vec{F} = -4\pi G\delta\rho$, $c_s^2 \equiv \frac{dp}{d\rho}$

Taking $\frac{\partial}{\partial t}$ of (1) yields ~~: to f~~

$$\frac{\partial^2(\bar{\rho}+\delta\rho)}{\partial t^2} + \frac{\partial}{\partial t}\vec{\nabla}(\bar{\rho}+\delta\rho)\delta\vec{v} = 0$$

However the zeroth order equation is assumed to hold so $\frac{\partial^2 \bar{\rho}}{\partial t^2} + 0 = 0$ and the combination $\frac{\partial}{\partial t}\vec{\nabla}\delta\rho\delta\vec{v}$ is second order so to first order we get:

$$\frac{\partial^2 \delta\rho}{\partial t^2} + \frac{\partial}{\partial t}\vec{\nabla}\bar{\rho}\delta\vec{v} = 0 \quad \frac{\partial}{\partial t}(1)$$

Taking $\vec{\nabla}\cdot(2)$ we get:

$$\vec{\nabla}\cdot(\bar{\rho}+\delta\rho)\left(\frac{\partial\delta\vec{v}}{\partial t} + \delta\vec{v}\vec{\nabla}\delta\vec{v}\right) = -\vec{\nabla}^2(\bar{p}+\delta p) + \vec{\nabla}\cdot(\bar{\rho}+\delta\rho)\delta\vec{F}$$

again we use the zeroth order $0 = -\vec{\nabla}\bar{p}$ and take away all second (or higher) order terms getting:

$$\vec{\nabla}\cdot\bar{\rho}\frac{\partial\delta\vec{v}}{\partial t} = -\vec{\nabla}^2\delta p + \vec{\nabla}\cdot\bar{\rho}\delta\vec{F} \quad \vec{\nabla}(2)$$

Subtracting $\vec{\nabla}\cdot(2)$ from $\frac{\partial}{\partial t}(1)$ we get:

$$\frac{\partial^2\delta\rho}{\partial t^2} + \frac{\partial}{\partial t}\vec{\nabla}\bar{\rho}\delta\vec{v} - \vec{\nabla}\cdot\bar{\rho}\frac{\partial\delta\vec{v}}{\partial t} = \vec{\nabla}^2\delta p - \vec{\nabla}\cdot\bar{\rho}\delta\vec{F}$$

Since $\frac{\partial\bar{\rho}}{\partial t}=0$ the second and third terms cancel →

$$\boxed{\frac{\partial^2\delta\rho}{\partial t^2} - \vec{\nabla}^2\frac{dp}{d\rho}\delta\rho + \vec{\nabla}\cdot\bar{\rho}\delta\vec{F} = 0}$$

$$\boxed{\frac{\partial^2\delta\rho}{\partial t^2} - c_s^2\vec{\nabla}^2\delta\rho - 4\pi G\bar{\rho}\delta\rho = 0}$$

because $\vec{\nabla}\bar{\rho}=0$ since $\vec{\nabla}\bar{p}=0$ →

4 b) If we move to Fourier space $\delta\rho = \delta\tilde{\rho}\, e^{i\vec{k}\cdot\vec{x}}$, we get:

$$\frac{\partial^2 \delta\tilde{\rho}}{\partial t^2} + c_s^2 \vec{k}^2 \delta\tilde{\rho} - 4\pi G\bar{\rho}\,\delta\tilde{\rho} = 0$$

$$\frac{\partial^2 \delta\tilde{\rho}}{\partial t^2} = -\left(c_s^2 \vec{k}^2 - 4\pi G\bar{\rho}\right)\delta\tilde{\rho}$$

This is of course the harmonic oscillator with solutions (for $c_s^2 k^2 - 4\pi G\bar{\rho} > 0$)

$$\delta\tilde{\rho} = \hat{A}\, e^{-i\omega t} \quad \text{where } \omega = \omega(\vec{k}) = \pm\sqrt{c_s^2 \vec{k}^2 - 4\pi G\bar{\rho}}$$

~~so~~ ω and $\hat{A}$ will depend on k, so that the full solution is

$$\underline{\delta\rho = A(\vec{k})\, e^{i(\vec{k}\cdot\vec{x} - \omega(\vec{k})t)}}$$

4 c) As shown in 4 b) $\omega(\vec{k}) = \sqrt{c_s^2 \vec{k}^2 - 4\pi G\bar{\rho}}$ (for the positive frequency wave and $\omega_2(\vec{k}) = -\omega(\vec{k})$ also solution) so

$$\underline{\omega(\vec{k}) = c_s\sqrt{\vec{k}^2 - \frac{4\pi G\bar{\rho}}{c_s^2}} = c_s\sqrt{\vec{k}^2 - k_J^2}}$$

4 d) c_s is the sound speed in the gas. k_J is the Jeans length and seperates the two regimes of solutions. For $k > k_J$ the term in the square root for ω is positive and the solution describes acoustic oscillations. If $k < k_J$ it is negative and the solution describes exponential growth of the density perturbations eventually driving the universe away from the static state and making the linear approximation unvalid.

4 e) $\lambda = \frac{2\pi}{k}$ the perturbations grow if $k < k_J$ so for $\underline{\lambda > \frac{2\pi}{k_J} = \lambda_J}$, when $\lambda < \lambda_J$ the perturbations

exhibit acoustic oscillations.

4f) A standing wave starts at $t=0$ with maximum amplitude so $A(k)\, e^{i(\vec{k}_1 \cdot x \mp \omega(k_1) \cdot 0)}$ ~~has~~ is maximum amplitude. The next maximum is reached when $\omega(k_1) t = 2\pi$ and this happens at decoupling time t_d so

$$\omega(k_1)\, t_d = 2\pi$$

$$c_s \sqrt{k_1^2 - k_J^2}\; t_d = 2\pi$$

$$k_1^2 - k_J^2 = \frac{4\pi^2}{c_s^2 t_d^2}$$

$$k_1 = \sqrt{k_J^2 + \frac{4\pi^2}{c_s^2 t_D^2}}$$

$$\lambda_1 = \frac{2\pi}{k_1} = \frac{2\pi}{\sqrt{k_J^2 + \frac{4\pi^2}{c_s^2 t_d^2}}}$$