UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Mid-term exam for AST4220 — Cosmology I

Day of exam: Tuesday October 12th 2010

Time for exam: 15.00 – 18.00

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

Taking that the Robertson-Walker metric is given by

$$ds^2 = c^2dt^2 - R^2(t)[dr^2 + S_k^2(r)d\psi^2]$$

where $S_k(r) = \sin r$ for $k = +1$, r for $k = 0$ and $\sinh r$ for $k = -1$.

a) Show that the R-W metric can also be written in the form

$$ds^2 = c^2dt^2 - R^2(t)[dr^2/(1 - kr^2) + r^2d\psi^2]$$

b) For a $k = -1$ Friedmann cosmology ($\Lambda = 0$), with $\rho = p = 0$, show that the R-W metric line element becomes:

$$ds^2 = c^2 dt^2 - c^2 t^2 [dr^2 + sinh^2 r d\psi^2]$$

Problem 2

a) Show that the general relativistic relation between recession velocity and cosmological redshift is

$$v_{rec}(t, z) = \frac{c}{R_0} \dot{R}(t) \int_0^z \frac{dz'}{H(z')}$$

where $H(z')$ is the Hubble constant at redshift z'.

b) Consider a galaxy at redshift $z = z_G$. At what rate is the galaxy receding from us now? At what rate was the object receding when it emitted the light we now observe?

c) From the special relativistic Doppler formula

$$1 + z = \sqrt{\frac{1 + v/c}{1 - v/c}}$$

show that the special relativistic relation between peculiar velocity and redshift is

$$v_{pec}(z) = c \frac{(1 + z)^2 - 1}{(1 + z)^2 + 1}$$

d) Show that both relativistic relations are approximately

$$v \approx cz$$

at small distance.

e) Because the general and special relativistic relations have the same low-redshift approximation one may be tempted to use the special relativistic form to interpret cosmological redshifts. Why would that be wrong?

Problem 3

Show that the dominant form of mass-energy at early times must scale as $\rho \propto R^{-\alpha}$, with $\alpha > 2$ for a particle horizon to exist.

Problem 4

Consider a non-flat Universe with a mixture of pressureless matter and radiation.

a) Express the total energy density in terms of the value of the scale factor at matter-radiation equality a_{eq} (when $\rho_m = \rho_r = \rho_{eq}/2$).

b) Write down the Fridemann and the Raychaudhuri equations in terms of the conformal time η. The conformal time is defined as $d\eta = a\,dt$.

c) Use the above equations to obtain a first order differential equation for $y(a)$, where $y = \frac{da}{d\eta}$.

d) Show that, for $k = -1$ the solution is

$$a(\eta) = a_m(\eta_* \sinh \eta + \cosh \eta - 1)$$

and for $k = 1$ one has

$$a(\eta) = a_m(\eta_* \sin \eta + 1 - \cos \eta)$$

where, and a_m and η_* are opportune constants.