AST3220 - final exam 14/6 - 12

1 a) $k=1$, for this the Friedmann equation reads:

$$\dot{a}^2 + kc^2 = \frac{8\pi G}{3}\rho a^2 \qquad \Omega(t) = \frac{\rho(t)}{\rho_c(t)} = \frac{\rho}{3H^2}\cdot 8\pi G$$

we put this in and divide by a:

$$\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2} = H^2\Omega(t)$$, dividing by H^2:

$$1 + \frac{kc^2}{a^2\frac{\dot{a}^2}{a^2}} = \Omega(t)$$, rearranging:

$$\Omega(t) - 1 = \frac{kc^2}{\dot{a}^2}$$, we have $k=1$ and work in units of $c=1$, so this is just

$$\Omega(t) - 1 = \frac{1}{\dot{a}^2} \quad (*)$$ as given in the text.

b) Inflation is a rapid accelerating expansion period in the early universe, meaning $\dot{a}$ is large since $\Omega(t) - 1 = \frac{1}{\dot{a}^2}$ according to $(*)$ $\Omega(t) - 1$ will go towards 0 in this epoch forcing $\Omega(t)$ closer to 1

c) The inflaton field is a field governed by quantum mechanics and hence experienses quantum fluctuations. These fluctuations mean that the field varies over space hence breaking the perfect homogeniety and isotropy of the universe. These perturbations caused by the fluctuations of the inflationary field form the seeds for structure formation in the universe.

2 a) Recombination is when electrons and nuclei combine to form neutral atoms. This neutralisation of the universe makes it transparent to photons as they will no longer be scattered continuously on free electrons, but can travel freely throughout the universe. Hence photons have been travelling pretty much straight forward since then. The photons that were at an appropriate distance from us at recombination, reach us today as the cosmic microwave background providing us with valuable information on the universes temperature, perturbation

to it and hence on the composition and physics of the early universe.

b) Matter-radiation equality:

$$\Omega_{m0}\,\cancel{(1+z_{eq})^3} = \Omega_{r0}\,(1+z_{eq})^{4}$$

$$(1+z_{eq}) = \frac{\Omega_{m0}}{\Omega_{r0}}$$

$$z_{eq} = \frac{0{,}3}{8{,}6\cdot 10^{-5}} - 1 \approx \underline{\underline{3500}}$$

This is well before recombination at $z = 1100$, so we can safely assume that the universe was matter dominated at recombination.

c) $k=0$, matter dominated universe the Friedmann equation reads:

$$H^2 = \frac{8\pi G}{3}\rho_{m0}\left(\frac{a_0}{a}\right)^3$$ at time t_0 this means

$$H_0^2 = \frac{8\pi G}{3}\rho_{m0}$$ and we can substitute this

into the Friedmann equation:

$$H^2 = H_0^2\left(\frac{a_0}{a}\right)^3, \quad H = H_0\left(\frac{a_0}{a}\right)^{\frac{3}{2}}, \quad \frac{\dot a}{a}\cdot a^{\frac{3}{2}} = H_0\, a_0^{\frac{3}{2}}$$

$$\int_0^a a^{1/2}\,da = \int_0^t H_0\, a_0^{3/2}\,dt \qquad \frac{2}{3}a^{\frac{3}{2}} = a_0^{3/2} H_0 t$$

$$a^{\frac{3}{2}} = a_0^{3/2}\,\frac{3H_0 t}{2} \Rightarrow \underline{a = a_0\left(\frac{3H_0 t}{2}\right)^{\frac{2}{3}}}$$

d) $a(z)$ is as always given by $1+z = \frac{a_0}{a}$,

$\underline{\underline{a(z) = \frac{a_0}{1+z}}}$, we solve from (c) for a and get

$$\left(\frac{a}{a_0}\right)^{\frac{3}{2}} = \frac{3H_0 t}{2} \qquad \underline{\underline{t = \frac{2}{3H_0}\left(\frac{a}{a_0}\right)^{\frac{3}{2}} = \frac{2}{3H_0}(1+z)^{-\frac{3}{2}}}}$$

$$r(t) = \int_{t}^{t_0}\frac{c\,dt}{a(t)} \qquad \frac{da}{dt} = \dot a \Rightarrow r(a) = \int_{a}^{a_0}\frac{c\,da}{\dot a\, a}, \quad \frac{dz}{da} = -\frac{a_0}{a^2} \Rightarrow r(z) = \int_0^z \frac{c\,dz}{H(z)}$$

$$H(z) = H_0 \left(\frac{a_0}{a(z)}\right)^{3/2} = H_0 (1+z)^{\frac{3}{2}} \Rightarrow$$

$$r(z) = \int_0^z \frac{c\,dz}{a_0 H_0 (1+z)^{3/2}} = \frac{c}{H_0 a_0}\left[-2(1+z)^{-\frac{1}{2}}\right]_0^z = \frac{2c}{a_0 H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$$

ⓔ The proper distance $d_p = a(t) S_k^{-1}(r)$ and in a flat universe: $d_p = a(t)\, r(z) = \frac{a}{a_0} \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right]$

$$= \frac{1}{1+z} \frac{2c}{H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] = \frac{2}{\frac{3}{3}} c \cdot 9{,}778 \cdot 10^9 yr \cdot \frac{1}{1100}\left[1 - \frac{1}{\sqrt{1100}}\right]$$

$$= \frac{3}{1100} c \cdot 9{,}778 \cdot 10^9 yr \cdot \left[1 - \frac{1}{\sqrt{1100}}\right] = \underline{\underline{10 \text{ light years}}}$$

3. A scalar field with no potential has

$$\rho_\varphi = P_\varphi = \frac{1}{2} \dot{\varphi}^2$$

ⓐ Adiabatic expansion:

$$\dot{\rho}_\varphi = -3\frac{\dot{a}}{a}(\rho_\varphi + p_\varphi) = -3\frac{\dot{a}}{a} \cdot 2\rho_\varphi = -6\frac{\dot{a}}{a}\rho_\varphi$$

$$\int_{\rho_0}^{\rho_\varphi} \frac{d\rho_\varphi}{\rho_\varphi} = -6\int_{a_0}^{a} \frac{da}{a}$$

$$\ln\left(\frac{\rho_\varphi}{\rho_{\varphi 0}}\right) = -6 \ln\left(\frac{a}{a_0}\right)$$

$$\underline{\underline{\rho_\varphi = \rho_{\varphi 0} \left(\frac{a_0}{a}\right)^6}}$$

ⓑ Friedmann equation in this case reads:

$$H^2 = \frac{8\pi G}{3} \rho_{\varphi 0} \left(\frac{a_0}{a}\right)^6 \quad \text{at } t = t_0$$

$$H_0^2 = \frac{8\pi G}{3} \rho_{\varphi 0} \quad \text{which we can substitute into the}$$

above: $H^2 = H_0^2 \left(\frac{a_0}{a}\right)^6$, $H = H_0 \left(\frac{a_0}{a}\right)^3$

$$a^2 \frac{da}{dt} = H_0 a_0^3 \qquad \int_0^{a} a^2\, da = \int_0^t H_0 a_0^3\, dt \Rightarrow \frac{1}{3} a^3 = H_0 a_0^3 t$$

$$\underline{a = a_0 (3H_0 t)^{\frac{1}{3}}} \qquad H = H_0 \left(\frac{a_0}{a}\right)^3 = H_0 (3H_0 t)^{-1} = \underline{\underline{\frac{1}{3t}}}$$

c) 4.26 reads as:

$$\frac{d^2 \Delta_k}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\Delta_k}{dt} = \Delta_k (4\pi G \rho_{m0} - k^2 c_s^2)$$

For scales much larger than the Jeans length, the last term can be ignored yielding:

$$\frac{d^2 \Delta_k}{dt^2} + 2H \frac{d\Delta_k}{dt} = \Delta_k 4\pi G \rho_{m0}$$

This universe is assumed to be highly dominated by the scalar field φ, meaning that we can ignore ρ_{m0} yielding:

$$\frac{d^2 \Delta_k}{dt^2} + 2H \frac{d\Delta_k}{dt} = 0$$

Above we found that $H = \frac{1}{3t}$, hence we get $\frac{d^2 \Delta_{\vec{k}}}{dt^2} + \frac{2}{3t}\frac{d\Delta_k}{dt} = 0$

d) Two ways of solving this equation.

1. It's a first order equation in $\frac{d\Delta_{\vec{k}}}{dt}$, so renaming this y we get:

$$\frac{dy}{dt} = -\frac{2}{3t} y$$

It is easy to see that $y = At^{-\frac{2}{3}}$ is a solution to this and since the equation is first order, the full solution. We can integrate this to get

$$\Delta_{\vec{k}} = C t^{\frac{1}{3}} + B \qquad (\text{where } C = 3A)$$

2. Guess at solution on the form $\Delta_k = At^n$, we then get the equation

$$n(n-1) + \frac{2}{3}n = 0 \quad \text{for } n$$

$$n\left(n - \frac{1}{3}\right) = 0 \text{ yielding } n = 0 \wedge n = \frac{1}{3}, \text{ so } \underline{\underline{\Delta_{\vec{k}} = B + Ct^{\frac{1}{3}}}}$$

e) The perturbations are either constant (B term) or growing (but not very fast) ($Ct^{1/3}$ term). This means that structure formation can take place in this universe, but it will proceed more slowly than in a matter dominated universe.

~~If we instead looked at density perturbations to the scalar field itself the equation would read~~