Solution to midterm exam 2014

Problem 1 - The Friedmann equations

ⓐ The second Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad (1)$$ (using units of $c=1$)

The adiabatic expansion equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p) \quad (2)$$

To eliminate the pressure term, observe that the pressure term in (1) has a prefactor of $-4\pi G$, whereas the prefactor in (2) is $-3\frac{\dot{a}}{a}$. Therefore we multiply (1) with $3\frac{\dot{a}}{a}$ and (2) by $4\pi G$ getting:

$$3\frac{\dot{a}}{a}\frac{\ddot{a}}{a} = -4\pi G\frac{\dot{a}}{a}\rho \quad - 4\pi G \cdot 3\frac{\dot{a}}{a}p \quad (1)$$

$$4\pi G\dot{\rho} = -4\pi G\left(3\frac{\dot{a}}{a}\rho\right) - 4\pi G \cdot 3\frac{\dot{a}}{a}p \quad (2)$$

Subtracting (2) from (1) we get:

$$3\frac{\dot{a}}{a}\frac{\ddot{a}}{a} - 4\pi G\dot{\rho} = -4\pi G\frac{\dot{a}}{a}\rho + 12\pi G\frac{\dot{a}}{a}\rho - 4\pi G \cdot 3\frac{\dot{a}}{a}p + 4\pi G \cdot 3\frac{\dot{a}}{a}p$$

$$3\frac{\dot{a}}{a}\frac{\ddot{a}}{a} = 4\pi G\dot{\rho} + 8\pi G\frac{\dot{a}}{a}\rho$$

$$\frac{\dot{a}}{a}\frac{\ddot{a}}{a} = \frac{4\pi G}{3}\left(\dot{\rho} + 2\frac{\dot{a}}{a}\rho\right) \quad (3)$$

b) Multiplying ~~4~~ (3) with $2a^2$ we get

$$2\dot{a}\ddot{a} - \frac{8\pi G}{3}\left(\dot{\rho}a^2 + 2\dot{a}a\rho\right) = 0 \quad (4)$$

Then we observe that $2\dot{a}\ddot{a} = \frac{d}{dt}(\dot{a}^2)$

and $\dot{\rho}a^2 + 2\dot{a}a\rho = \frac{d}{dt}(\rho a^2)$

~~So that (4)~~ So that

$$2\dot{a}\ddot{a} - \frac{8\pi G}{3}\left(\dot{\rho}a^2 + 2\dot{a}a\rho\right) =$$

$$\frac{d}{dt}\left[\dot{a}^2 - \frac{8\pi G}{3}\rho a^2\right] = 0$$

And further multiplying by $-\frac{3}{8\pi G}$:

$$\frac{d}{dt}\left[\rho a^2 - \frac{3}{8\pi G}\dot{a}^2\right] = 0 \quad (5)$$

c) (5) implies that

$$\rho a^2 - \frac{3}{8\pi G}\dot{a}^2 = k_1$$

where k is a constant w.r.t time. Moving things around a bit this gives:

$$\rho a^2 = k_1 + \frac{3}{8\pi G}\dot{a}^2$$

$$\dot{a}^2 + k_2 = \frac{8\pi G}{3}\rho a^2 \quad \text{where } k_2 = \frac{8\pi G}{3}k_1$$

If k_2 can be interpreted as the curvature of the universe, then this is the first Friedmann equation. Proof of this interpretation (via initial conditions) is necessary to get complete equivalence between (5) and the first Friedmann equation

Problem 2 - The single component Universe

Defining h by: $H_0 = 100\,km/s/Mpc \cdot h$ & $q_0 = -\frac{\ddot{a}_0 a_0}{\dot{a}_0^2}$

a) $h = 0.68$ and $q_0 = -0.53$ today. To find the equation of state $p = w\rho$ for a single component universe with this property we use that:

$$H_0^2 = \frac{8\pi G}{3}\rho_0 \quad (6) \text{ (first Friedmann equation today)}$$

$$\frac{\ddot{a}_0}{a_0} = -\frac{4\pi G}{3}\rho_0 (1+3w) \quad (7) \text{ (second Friedmann equation today)}$$

$$q_0 = -\frac{\ddot{a}_0}{\dot{a}_0}\frac{a_0}{\dot{a}_0^2} = -\frac{(7)}{(6)} = \frac{\frac{4\pi G}{3}\rho_0(1+3w)}{\frac{8\pi G}{3}\rho_0} = \frac{(1+3w)}{2}$$

$$2q_0 = (1+3w) \qquad 3w = 2q_0 - 1$$

$$w = \frac{-2q_0 - 1}{3} = \frac{-2\cdot 0{,}53 - 1}{3} = \cancel{\ \ } \underline{\underline{-0{,}69}}$$

b) Luminosity distance:

$$d_L = a_0 r(1+z) = a_0(1+z) r$$

$$r = \int_t^{t_0} \frac{c\,dt'}{a(t')} =$$

We need to find $a(t)$ and $t(z)$ in this universe.

Adiabatic expansion equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}\rho(1+w)$$

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

$$\underline{\rho = \rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}} \quad (8)$$

First Friedmann equation using (8):

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}$$

$$H = \sqrt{\frac{8\pi G}{3}\rho_0 \left(\frac{a_0}{a}\right)^{3(1+w)}}$$

$$\frac{da}{a} = \sqrt{\frac{8\pi G}{3}\rho_0}\left(\frac{a_0}{a}\right)^{\frac{3(1+w)}{2}} dt$$

$$\int_{a_0}^{a} a^{\frac{1+3w}{2}}\, da = \int_{t_0}^{t} \sqrt{\frac{8\pi G}{3}\rho_0}\,\cancel{(\ )}\, a_0^{\frac{3(1+w)}{2}}\, dt$$

$$\frac{2}{3(1+w)}\left[a^{\frac{3(1+w)}{2}} - a_0^{\frac{3(1+w)}{2}}\right] = \sqrt{\frac{8\pi G}{3}\rho_0}\, a_0^{\frac{3(1+w)}{2}}\left[t - t_0\right]$$

$$t = t_0 - \frac{\cancel{2}}{3(1+w)}\sqrt{\frac{3}{8\pi G\rho_0}} + \cancel{a}\left(\frac{a}{a_0}\right)^{\frac{3(1+w)}{2}} \cdot \frac{2}{3(1+w)}\sqrt{\frac{8\pi G}{3}\rho_0}$$

In our case: $\frac{3(1+w)}{2} = \frac{3(1-0{,}69)}{2} = 0{,}46 > 0$

So we can go to $a \to 0$ safely, meaning this universe has a Big Bang. It is natural to ~~call that age~~ define that time ~~age~~ as $t=0$, so the age of this Universe is $t_0 = \sqrt{\frac{3}{8\pi G\rho_0}}\,\frac{2}{3(1+w)} = \frac{2}{3(1+w)H_0}$

(as $H_0 = \sqrt{\frac{8\pi G}{3}\rho_0}$) this means that:

$$\cancel{a}\quad t = t_0\left(\frac{a}{a_0}\right)^{\frac{3(1+w)}{2}} = \underline{t_0(1+z)^{-\frac{3(1+w)}{2}}}$$

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}$$

$$\Rightarrow \quad r = \int_t^{t_0} \frac{c\,dt'}{a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}} = \frac{c}{a_0}\, t_0^{\frac{2}{3(1+w)}}\left[\frac{1}{1-\frac{2}{3(1+w)}}\right]\left[t^{\left(1-\frac{2}{3(1+w)}\right)}\right]_t^{t_0}$$

$$= \frac{c}{a_0}\, t_0\left[\frac{1}{1-\frac{2}{3(1+w)}}\right]\left[1 - \left(\frac{t}{t_0}\right)^{\left(1-\frac{2}{3(1+w)}\right)}\right]$$

$$r = \frac{c}{a_0} t_0 \frac{3(1+w)}{1+3w} \left[1 - \left((1+z)^{-\frac{3(1+w)}{2}} \right)^{\frac{1+3w}{3(1+w)}} \right]$$

$$= \frac{c}{a_0} t_0 \frac{3(1+w)}{1+3w} \left[1 - (1+z)^{-\frac{1+3w}{2}} \right]$$

$$d_L = c\, t_0 \frac{3(1+w)}{1+3w} \left[1 - (1+z)^{-\frac{1+3w}{2}} \right] (1+z)$$

Recall that $t_0 = \frac{2}{3(1+w)H_0}$

$$d_L = c \frac{2}{3(1+w)H_0} \frac{3(1+w)}{1+3w} \left[1 - (1+z)^{-\frac{1+3w}{2}} \right] (1+z)$$

$$= c \frac{2}{H_0} \frac{1}{1+3w} \left[1 - (1+z)^{-\frac{1+3w}{2}} \right] (1+z)$$

Inserting the numbers $(z=2{,}5)$ for our single component universe:

$$d_L = 3 \cdot 10^5 \frac{km}{s} \cdot \frac{2}{100 \frac{km}{s\, Mpc} \cdot 0{,}68} \cdot \frac{1}{1+3\cdot(-0{,}69)} \left[1 - (3.5)^{-\frac{(1-3\cdot 0{,}69)}{2}} \right] \cdot 3{,}5$$

$$= \underline{\underline{28\ Gpc}}$$

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c) By doing calculations like we did above, we can predict the luminosity distance to redshift relation by observing the supernovas and ~~b~~ comparing to this, we can ~~predict the d~~ dissprove the single component Universe. As the universe close to us ~~is~~ has low redshifts and hence smaller effects of the particulars of the universe evolution, the supernovae that are more distant will be more important in this endeavour.
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Problem 3 - A ~~single~~ two-dimensional universe

2D adiabatic expansion

$$\dot{\rho} = -2H(\rho + P)$$

ⓐ ρ_m is pressureless matter, so $P_m = 0$

$$\dot{\rho}_m = -2H\rho_m$$

$$\frac{d\rho_m}{\rho_m} = -2\frac{da}{a}$$

$$\underline{\rho_m = \rho_{m_0}\left(\frac{a_0}{a}\right)^2}$$

This value makes a lot of sense as pressureless matter is just spread out with the expansion of the universe diluting with the square of the scale factor, so diluting with the area.

ⓑ A cosmological constant fluid would have $\dot{\rho}_\Lambda = 0 = -2H(\rho_\Lambda + P_\Lambda)$ so $\rho_\Lambda = -P_\Lambda$ so the equation of state parameter $w_\Lambda = -1$ just like in three dimensions

ⓒ Assuming that the first Friedmann equation ~~takese~~ takes the form

$H^2 = D_2\rho$ where D_2 are some dimensionally specific positive constant and ρ is the sum total of the energy densities, we find that

$$D_2\dot{\rho} = 2\dot{H}H \qquad \dot{\rho} = -2H(\rho + P)(9) \qquad \dot{\rho} = \frac{2}{D_2}H\dot{H}$$

$$\dot{H} = \frac{d}{dt}\left(\frac{\dot{a}}{a}\right) = \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \qquad \dot{\rho} = \frac{2}{D_2}H\left(\frac{\ddot{a}}{a} - H^2\right)(10)$$

Equating (9) and 10

$$-2H(\rho + P) = \frac{2}{D_2} H\left(\frac{\ddot{a}}{a} - H^2\right)$$

$$-D_2(\rho + P) = \frac{\ddot{a}}{a} - H^2 = \frac{\ddot{a}}{a} - D_2\rho$$

$$\Rightarrow \underline{\underline{\frac{\ddot{a}}{a} = -D_2 P}} \quad \left(\text{if } D_2 \text{ was } \frac{8\pi G}{3} \quad \frac{\ddot{a}}{a} = -\frac{8\pi G}{3} P\right)$$

d) To get accelerated expansion in this case we need $\ddot{a} > 0$, so $\frac{\ddot{a}}{a} > 0$, assuming D_2 is positive (negative D_2 is not a correct case) this means $P < 0$

$P = w\rho$, and ρ is always positive so $\underline{\underline{w < 0 \text{ would give accelerated expansion}}}$ in this two-dimensional case.