UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Wednesday December 3rd 2003

Time for exam: 09.00 – 12.00

This problem set consists of 5 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

Whenever you are asked to give a numerical answer in this problem set you can use the following values: In this problem you may find the following numbers useful:

$$\begin{aligned} H_0 &= 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \\ &= h \times (9.778 \times 10^9\ \mathrm{yr})^{-1} \\ \rho_{c0} &= \frac{3H_0^2}{8\pi G} = 1.879 \times 10^{-26}\, h^2\ \mathrm{kg\,m^{-3}} \end{aligned}$$

$$
\begin{aligned}
\hbar &= 1.055 \times 10^{-34}\ \mathrm{J\,s} \\
c &= 2.998 \times 10^{8}\ \mathrm{m\,s^{-1}} \\
k_{\mathrm{B}} &= 1.381 \times 10^{-23}\ \mathrm{J\,K^{-1}}
\end{aligned}
$$

In addition you may assume $\Omega_{m0} = 0.3$ and $\Omega_{\Lambda 0} = 0.7$ (the present values of the density parameters for dust and vacuum energy) and for henholdsvis støv og kosmologisk konstant ved vår epoke), Furthermore, the present temperature of the cosmic microwave background is $T_0 = 2.73$ K, and we take the present value of the scale factor, a_0, to be 1. In the Friedmann equations we choose a system of units where $c = 1$.

a) Write dowm the equation that determines how the energy density ρ_i of a perfect fluid i varies as the universe expands. Determine how ρ_i depends on the scale factor a and redshift z in the following cases:

1. dust (with equation of state $p_m = 0$)
2. radiation (with equation of state $p_r = \rho_r/3$)
3. cosmological constant (equation of state $p_\Lambda = -\rho_\Lambda$)

b) Assume that we have three types of neutrinos which are all massless. The energy density of a gas of relativistic bosons is given by

$$\rho = \frac{\pi^2}{30} g \frac{(k_{\mathrm{B}} T)^4}{(\hbar c)^3},$$

where g is the number of internal degrees of freedom for the boson in question. The corresponding expression for fermions is

$$\rho = \frac{7}{8}\frac{\pi^2}{30} g \frac{(k_{\mathrm{B}} T)^4}{(\hbar c)^3}.$$

Make us of the fact that the present temperature of the cosmic neutrino background is Bruk at temperaturen til nøytrinobakgrunnen i dag er

$$T_{\nu 0} = \left(\frac{4}{11}\right)^{1/3} T_0,$$

and show that the present density parameter for radiation (photons plus neutrinos),$\Omega_{r0} = \rho_{r0}/(\rho_{c0}c^2)$ is given by

$$\Omega_{r0}h^2 = 4.2 \times 10^{-5}.$$

At what redshift $z_{\rm eq}$ was the energy density in the form of relativistic particles (radiation) equal to the energy density in dust?

c) Show that Friedmann's first equation in the case of a spatially flat universe with dust and radiation can be written as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{r0}}{a^4}\left(1 + \frac{a}{a_{\rm eq}}\right)$$

where $a_{\rm eq}$ is the value of the scale factor at $z_{\rm eq}$.

d) Show that the equation in c) can be rewritten as

$$H_0 dt = \frac{a da}{\sqrt{\Omega_{r0}}}\left(1 + \frac{a}{a_{\rm eq}}\right)^{-1/2}.$$

Integrate this equation and show that

$$H_0 t = \frac{4a_{\rm eq}^2}{3\sqrt{\Omega_{r0}}}\left[1 - \left(1 - \frac{a}{2a_{\rm eq}}\right)\left(1 + \frac{a}{a_{\rm eq}}\right)^{1/2}\right].$$

Useful integral:

$$\int \frac{x dx}{\sqrt{1+x}} = \frac{2}{3}(1+x)^{3/2} - 2(1+x)^{1/2} + C,$$

where C is a constant of integration. How old was the Universe at $a = a_{\rm eq}$?

e) At what redshift z_Λ is the energy density of dust equal to the vacuum energy density? If we neglect the contribution from the radiation dominated epoch and assume the Universe to be spatially flat and dominated by dust (with $\Omega_{m0} = 0.3$) up to z_Λ, how old was the Universe at z_Λ ?

Next we want to study how perturbations in the density of dust ρ_m develop in time at length scales much larger than the Jeans length $\lambda_{\rm J}$ in the different epochs. The equation that determines the time evolution of a Fourier mode of the density perturbations, $\Delta_{\mathbf{k}}(t)$, is

$$\frac{d^2\Delta_{\mathbf{k}}}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\Delta_{\mathbf{k}}}{dt} = 4\pi G\rho_m \Delta_{\mathbf{k}}(t),$$

where ρ_m is the average density of dust.

f) Explain shortly, without any calculations, how this equation is derived.

g) Show that this equation leads to the following results:

1. In a radiation-dominated universe ($\rho_m \approx 0$):

$$\Delta_{\mathbf{k}}(t) = B_1 + B_2 \ln t$$

2. In a matter-dominated universe ($\rho_m = \rho_{\rm c}$):

$$\Delta_{\mathbf{k}}(t) = C_1 t^{-1} + C_2 t^{2/3}$$

3. In a universe dominated by the cosmological constant (de Sitter universe, $\rho_m \approx 0$):

$$\Delta_{\mathbf{k}}(t) = D_1 + D_2 e^{-2H_\Lambda t},$$

where $H_\Lambda = \sqrt{8\pi G\rho_\Lambda/3}$.

In the expressions above, B_1, B_2, C_1, C_2 og D_1, D_2 are constants of integration.

h) Give a short, physical explanation of why perturbations grow slowly/do not grow in the radiation-dominated epoch and in a de Sitter universe.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Friday December 3rd 2004

Time for exam: 09.00 – 12.00

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

In some models the universe can experience a so-called loitering phase where it spends some time without expanding significantly. We will in this problem define the existence of such a phase by the existence of a redshift $z_{\mathrm{loit}} \geq 0$ so that $H'(z_{\mathrm{loit}}) = 0$ (where $H'(z) = dH/dz$, and $H = \dot{a}/a$ is the Hubble parameter.)

a) Write down an expression for $H^2(z)$ for a universe containing non-relativistic matter (dust), curvature, and a cosmological constant. Express your answer in terms of the present-day density parameters Ω_{m0}, Ω_{k0} og $\Omega_{\Lambda 0}$.

b) Let $\Omega_{\mathrm{m}0} = \frac{1}{2}$ and $\Omega_{\Lambda 0} = 2$. Find z_{loit} and $H(z_{\mathrm{loit}})$ for this model. Make a rough, qualitative sketch of $H(z)$ for this case. In the same figure, draw $H(z)$ for a flat universe with $\Omega_{\mathrm{m}0} = \frac{1}{2}$.

c) Which of the two cases in b) gives the larger age for the universe? No numerical calculation is required, just try to give a qualitative argument.

d) Show that a loitering phase is impossible in a curved universe which contains only non-relativistic matter. (Hint: remember that we require $z_{\mathrm{loit}} \geq 0$.)

e) Show that we can have a loitering phase in a universe with curvature, non-relativistic matter, and a cosmological constant provided that it is closed and $\Omega_{\Lambda 0} \geq \frac{1}{2}\Omega_{\mathrm{m}0} + 1$.

Problem 2

In this problem you will study the growth of density perturbations in an Einstein-de Sitter universe (a flat universe with matter only, and no cosmological constant.)

a) Write down expressions for $a(t)$, $H(t) = \dot{a}/a$ and the matter density $\rho_{\mathrm{m}}(t)$ in this model for the unperturbed case.

Assume that the matter consists of two components: some form of cold, non-relativistic dark matter, and massive neutrinos. Also, assume that the neutrinos have so high thermal velocities that they do not clump, so that the only density perturbations are those in the cold dark matter. Denote the density parameter of the neutrinos by Ω_ν, and that of the cold dark matter by $\Omega_{\rm cdm}$, so that $\Omega_{\rm m} = \Omega_{\rm cdm} + \Omega_\nu$, and define $f_\nu = \Omega_\nu/\Omega_{\rm m}$.

b) Start from the equation for the time evolution of a Fourier mode Δ_k of the density perturbations, derived in the lectures, and justify that it can be written as

$$\ddot{\Delta}_k + \frac{4}{3t}\dot{\Delta}_k = \frac{2}{3}(1 - f_\nu)\frac{\Delta_k}{t^2},$$

in the situation considered in this problem.

c) Assume that this equation has a power-law solution $\Delta_k \propto t^\alpha$, and show that the growing mode solution is

$$\alpha = \frac{1}{6}\left[5\sqrt{1 - \frac{24}{25}f_\nu} - 1\right],$$

and that

$$\alpha \approx \frac{2}{3}\left(1 - \frac{3}{5}f_\nu\right),$$

for $f_\nu \ll 1$.

d) Density perturbations only start to grow after matter-radiation equality at a redshift $1 + z_{\rm eq} \approx 23900\Omega_{\rm m}h^2$. Show that the perturbations in this model by the current epoch ($a = a_0 \equiv 1$, $z = 0$) have grown by a factor

$$\frac{\Delta_k(z=0)}{\Delta_k(z=z_{\rm eq})} = (1 + z_{\rm eq})e^{-\frac{3}{5}f_\nu \ln(1+z_{\rm eq})}.$$

(Hint: write the solution for Δ_k in terms of the scale factor a.)

e) For $\Omega_{\rm m} = 1$, $h = 0.5$, compare the growth of density perturbations from $z_{\rm eq}$ to $z = 0$ in the cases $f_\nu = 0.1$ og $f_\nu = 0$.

Problem 3

The time evolution of the energy density ρ of a perfect fluid with pressure p is governed by the equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p).$$

We choose $a_0 = 1$, where a_0 is the scale factor at our present epoch t_0.

a) Find ρ as a function of a for a perfect fluid with equation of state $p = w\rho$, where w is a constant.

b) The deceleration parameter q is defined by

$$q = -\frac{\ddot{a}a}{\dot{a}^2}.$$

Determine q for a flat universe which contains a combination of non-relativistic matter and a fluid with equation of state $w = -1/3$.

c) Find q for a universe which contains non-relativistic matter only, but has spatial curvature. Compare with the result in b) and comment.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Mid-term exam for AST4220 — Cosmology I

Day of exam: Wednesday October 12th 2005

Time for exam: 14.30 – 17.30

This problem set consists of 3 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

The different parts of this problem are not necessarily related. When asked to calculate a numerical answer, you can use the value $h = 0.7$ for the Hubble constant $H_0 = 100h\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\mathrm{m0}} = 0.3$ for the present value of the density parameter of non-relativistic matter (dust). It is also useful to know that $1/H_0 = 9.778 \times 10^9\ \mathrm{yr}/h$.

a) Starting with the expression for the scale factor $a(t)$ in the Einstein-de Sitter model, show that the age of the universe at redshift z is given by

$$t(z) = t_0(1+z)^{-3/2},$$

where $t_0 = 2/3H_0$.

A quasar is observed at redshift $z = 6$. Calculate how old the universe was when the light we see now was emitted if the universe is described by

b) an Einstein-de Sitter model

c) an open model with dust only, and

d) a flat ΛCDM model.

e) And now for something completely different: Assume that the universe is spatially flat, expanding, and that the scale factor is given by $a(t) = (t/t_0)^\alpha$, where α is a constant. For which values of α does this model have a particle horizon? For which values of α does this model have an event horizon? And for which values of α is the universe expanding at an accelerating rate in this model?

f) 'The Hubble paramater increases with time in an accelerating universe.' Is this statement true or false? Justify your answer.

g) 'The quantity $a(t)H(t)$ increases with time in an accelerating universe.' Is this statement true or false? Justify your answer.

Problem 2

a) Starting with the Friedmann equations for a universe with spatial curvature, containing non-relativistic matter (dust), and a cosmological constant, show that for this model

$$H^2(z) = H_0^2[\Omega_{\rm m0}(1+z)^3 + \Omega_{\rm k0}(1+z)^2 + \Omega_{\Lambda 0}],$$

$$\frac{\ddot{a}}{a} = -\frac{H_0^2}{2}\left[\Omega_{\rm m0}(1+z)^3 - 2\Omega_{\Lambda 0}\right],$$

and that the density parameters must satisfy

$$\Omega_{\rm m0} + \Omega_{\rm k0} + \Omega_{\Lambda 0} = 1.$$

It is possible to construct models where the universe goes through a 'bounce' in the sense that the scale factor $a(t)$ is large at early times, decreases and reaches a minimum value a_*, and then increases at late times.

b) Which conditions must $\dot{a}$ and $\ddot{a}$ meet at the redshift of the 'bounce', $z_* = 1/a_* - 1$? (Hint: Which conditions must the first and the second derivative of a function meet at a local minimum?)

c) From your result in b) show that

$$\Omega_{\rm m0} \leq \frac{2}{z_*^2(z_*+3)}.$$

d) Quasars with redshifts $z = 6$ have been observed. What upper limit on $\Omega_{\rm m0}$ does this fact give for the 'bouncing universe' model? Is this a realistic model of the universe? Justify your answer.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Thursday December 1st 2005

Time for exam: 14.30 – 17.30

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

The temperature of the cosmic microwave background is to lowest order the same in all directions on the sky, and has the value $T_0 = 2.73K$. The photons have propagated freely through the universe since it became electrically neutral. We will in this problem assume that this happened when the temperature of the photons was 3000 K.

a) Show that $T \propto (1+z)$ and calculate the redshift $z_{\rm dec}$ when the Universe became neutral.

b) Write down the Friedmann equation for a spatially flat, matter-dominated universe and use it to show that the present age of the Universe is

$$t_0 = \frac{2}{3H_0},$$

where H_0 is the Hubble parameter. Calculate t_0 for $H_0 = 70\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$. Use this value of H_0 in the remaining parts of this problem.

c) Calculate the age $t_{\rm dec}$ of the Universe at $z_{\rm dec}$.

d) Calculate $d_{\rm PH}(z_{\rm dec})$, the proper distance to the particle horizon at $t_{\rm dec}$.

e) Calculate the proper distance from us out to the redshift $z_{\rm dec}$.

f) Fint the angle $\theta_{\rm PH}$ subtended by the particle horizon at p$z_{\rm dec}$ on the sky today.

Oppgave 2

In this problem we will use a set of units where $\hbar = c = 1$. Observations of the present state of the Universe reval that it is currently in an accelerated phase of expansion. This can be explained by introducing a positive cosmological constant, $\Lambda > 0$, but there are alternatives. We will consider one alternative in this problem: a homogeneous scalar field $\phi(t)$ (NOTE: This is not the field that drove inflation, and we will not consider the inflationary epoch in this problem.) Assume that the field follows the equation of state $p_\phi = w\rho_\phi$, that it is the only contribution to the mass-energy density of the Universe, and that the Universe is spatially flat. In the lectures we have shown that for such a universe model,

$$\begin{aligned} \rho_\phi(a) &= \frac{\rho_\phi^0}{a^{3(1+w)}} \\ a(t) = &= \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} . \end{aligned}$$

a) Find expressions for $H(t)$ and $\rho_\phi(t)$. What condition must w satisfy if we want accelerated expansion?

b) The energy density and pressure of the scalar field are given by, respectively,

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi) \\ p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi) \end{aligned}$$

If we assume the equation of state given in the introduction to be correct, show that ϕ can be written in terms of a as

$$\phi(a) = \phi_0 + \sqrt{\frac{3(1+w)}{8\pi G}} \ln a$$

and that

$$V(\phi) = \frac{1}{2}(1-w)\rho_\phi^0 \exp\left[-\sqrt{24\pi G(1+w)}(\phi - \phi_0)\right],$$

where ϕ_0 is the value of the scalar field for $a = a(t_0) = 1$.

c) The potential energy for this particular scalar field model is often written as

$$V(\phi) = V_0 e^{-\lambda\phi\sqrt{8\pi G}},$$

where λ is a positive constant. Find the condition λ must satisfy if we want both accelerated expansion and $p_\phi + \rho_\phi > 0$.

Problem 3

Consider a matter-dominated universe with $\Omega_{\mathrm{m}0} < 1$.

a) Explain why we can neclect the curvature term in the first Friedmann equation early in the matter-dominated period. Show that the Hubble parameter can be written

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2 \Omega_{\mathrm{m}0}}{a^3}.$$

b) Determine $a(t)$.

c) We will now consider how density perturbations grow on scales much larger than the Jeans length in this model. Show that we get the same growing mode, $\Delta_{\mathbf{k}}(t) \propto t^{2/3}$, as in the Einstein-de Sitter model.

d) If we want to distinguish between an open universe and the Einstein-de Sitter model by observing the growth of perturbations, should we use observations made at high redshifts? Justify your conclusion.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Thursday January 12th 2006

Time for exam: 09.00 – 12.00

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

In this problem we wish to study the time evolution of the (energy) density in the Universe with the help of the equation

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p).$$

For simplicity we will use a set of units where $\hbar = 1 = c$.

a) Show that for a perfect fluid/gas with equation of state $p = w\rho$, where w is a constant, we have

$$\rho = \rho_0 a^{-3(1+w)},$$

where ρ_0 is the density at$a = a_0 = 1$.

b) Explain why the density in a universe with dust and a cosmological constant is given by

$$\rho(a) = \rho_{c0}[(1 - \Omega_{m0}) + \Omega_{m0} a^{-3}],$$

where the symbols have their usual meaning.

c) A so-called Chaplygin gas has the equation of state

$$p = -\frac{A}{\rho^\alpha},$$

where A and α are constants and $A > 0$. Show that in a universe that contains this gas only the density will vary with the scale factor as

$$\rho(a) = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{1/(1+\alpha)},$$

where $B = \rho_0^{\alpha+1} - A$.

d) Determine how ρ for the Chaplygin gas behaves in the limits $a \gg 1$ and $a \ll 1$. In particular, show that $p \approx -\rho$ for $a \gg 1$.

e) Show that we for the Chaplygin gas can write

$$\rho(a) = \rho_* \left[(1 - \Omega_m^*) + \Omega_m^* a^{-3(1+\alpha)}\right]^{1/(1+\alpha)},$$

where $\rho_* = (A + B)^{1/(1+\alpha)}$ and $\Omega_m^* = B/(A + B)$.

f) What happens when $\alpha = 0$?

Problem 2

Models for the inflationary epoch in the very early universe make use of a homogeneous scalar field $\phi = \phi(t)$ with energy density and pressure given by

$$\begin{aligned}
\rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi), \\
p_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi),
\end{aligned}$$

where $V(\phi)$ is the scalar field potential. We will again use a system of units where $\hbar = 1 = c$, and we also introduce the so-called reduced Planck mass by defining $M_{\rm Pl} = (8\pi G)^{-1/2}$. We will also assume that $\dot{\phi} > 0$.

a) Show that in a universe dominated by this scalar field the dynamics of the universe and the scalar field are determined by the equations

$$\begin{aligned}
H^2 &= \frac{1}{3M_{\rm Pl}^2}\left[V(\phi) + \frac{1}{2}\dot{\phi}^2\right], \\
\ddot{\phi} + 3H\dot{\phi} &= -\frac{dV}{d\phi}.
\end{aligned}$$

b) Use the equations in a) to show that

$$\dot{\phi} = -2M_{\rm Pl}^2 H'(\phi),$$

where $H'(\phi) \equiv dH/d\phi$.

c) Use the result in b) to show that the first Friedmann equation can be written as

$$[H'(\phi)]^2 - \frac{3}{2M_{\mathrm{Pl}}^2} H^2(\phi) = -\frac{1}{2M_{\mathrm{Pl}}^4} V(\phi). \tag{1}$$

An important property of the solutions of the equations governing the inflationary phase is that they have a so-called attractor. In practice, this means that regardless of the inital value of ϕ, the Hubble parameter $H(\phi)$ will quickly end up on the same curve in the ϕ–H-planet. You will now demonstrate that this is the case by considering linear, homogeneous perturbatins around solutions of equation (1): take $H(\phi) = H_0(\phi) + \delta H(\phi)$, where $H_0(\phi)$ is a solution of (1).

d) Show that we to first order in the perturbation δH have

$$H_0' \delta H' = \frac{3}{2M_{\mathrm{Pl}}^2} H_0 \delta H,$$

where $'$ again denotes differentiation with respect to ϕ.

e) Show that the equation in d) has the general solution

$$\delta H(\phi) = \delta H(\phi_{\mathrm{i}}) \exp\left(\frac{3}{2M_{\mathrm{Pl}}^2} \int_{\phi_{\mathrm{i}}}^{\phi} \frac{H_0(\phi)}{H_0'(\phi)} d\phi \right),$$

where ϕ_{i} is the inital value of the scalar field ϕ. Explain why this result shows that the perturbation δH quickly dies out.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Wednesday October 11th 2006

Time for exam: 14.30 – 17.30

This problem set consists of 3 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

We will consider flat universe models with dust and vacuum energy (a cosmological constant).

a) Calculate the comoving radial coordinate r_{PH} of the particle horizon in the Einstein-de Sitter model at the present epoch, and to the event horizon r_{EH} in a de Sitter universe at the present epoch ($t = t_0$).

b) Show that the event horizon in the de Sitter universe corresponds to a redshift of $z = 1$.

c) An unnamed teacher of a course on cosmology wants to get as far away as possible before the student evaluations of his course are in. He lives in an Einstein-de Sitter universe, and decides to go for a trip to the particle horizon. Assuming that he starts his journey at $t = t_0$ and travels at the speed of light, determine the cosmic time t_f when he arrives at his destination.

d) If the teacher lived in a de Sitter universe and wanted to go to the event horizon, at what cosmic time will he get there if he starts today and travels at the speed of light.

e) Show that the event horizon at our epoch, $t = t_0$, in models with both dust and vacuum energy has comoving radial coordinate

$$r_{\rm EH} = \frac{c}{H_0}\int_{-1}^{0} \frac{dz}{\sqrt{\Omega_{\rm m0}(1+z)^3 + 1 - \Omega_{\rm m0}}},$$

and that the particle horizon at our epoch has comoving radial coordinate

$$r_{\rm PH} = \frac{c}{H_0}\int_{0}^{\infty} \frac{dz}{\sqrt{\Omega_{\rm m0}(1+z)^3 + 1 - \Omega_{\rm m0}}}.$$

f) Show that models of the type in e) with $0 < \Omega_{\rm m0} < 1$ have both a particle horizon and a event horizon. That is, that both integrals above have finite values. (Hint: you don't have to calculate the integrals explicitly, only show that they converge).

Problem 2

In this problem we will use Newtonian mechanics to look at the motion of a test particle of mass m in an expanding universe. Assume that the universe containts dust only, and that we can neglect spatial curvature. We place ourselves at the origin and assume that the particle starts its motion at a distance R_0 from us at the time $t = t_0$ with zero velocity with respect to us. We wish to determine the particle's distance from us as a function of cosmic time, $R = R(t)$. Only gravitational forces affect the motion of the particle.

a) Show that the equation of motion of the particle is

$$\ddot{R} = -\frac{\Omega_{\mathrm{m0}} H_0^2}{2a^3} R,$$

where the symbols have their usual meaning.

b) Find $R(t)$ if the universe is described by an Einstein-de Sitter model.

c) How does the particle move initially? How does it move for $t \gg t_0$? Compare this to a particle that starts in the same position, but follows the expansion of the universe.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Monday December 11th 2006

Time for exam: 15.30 – 18.30

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

The following problem is found in physics textbook for the upper secondary school: A quasar is observed at redshift $z = 3.78$. In special relativity the relationship between z and the speed at which the quasar is moving away from us is given by

$$1 + z = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

a) Calculate the quasar's speed.

b) Use Hubble's law $v = H_0 d$ to calculate the distance d to the quasar. Let $H_0 = 16$ kilometer per second per million lightyears.

Pretend that you are a gullible student and answer question a) and b).

c) What serious error are the authors of this textbook guilty of? If you have plenty of time, you are welcome to suggest a suitable punishment (but no extra credit will be given for this.)

d) Assume that the Universe is described by an Einstein-de Sitter model and calculate the proper distance to the quasar. Use the same value for H_0 as in b) and compare the result to the distance computed in b).

Problem 2

a) Explain briefly what we mean by the flatness problem.

We will consider inflation driven by a scalar field with the potential

$$V(\phi) = \lambda \phi^p,$$

where the constant λ has the appropriate units so that $V(\phi)$ is an energy density.

b) Calculate the slow-roll parameters ϵ and η.

c) Let inflation end when $\epsilon = 1$. Determine the value of ϕ, $\phi_{\rm end}$ when this happens.

d) We need roughly 60 e-foldings for inflation to solve the flatness problem. Use the slow-roll expression for the number of e-foldings to determine the value ϕ_i where the field has to start in order to produce 60 e-foldings in this model.

Problem 3

The first Friedmann equation for a universe with radiation and spatial curvature can be written as

$$H^2 = H_0^2\left(\frac{\Omega_{\rm r0}}{a^4} + \frac{1-\Omega_{\rm r0}}{a^2}\right).$$

We will in this problem let $\Omega_{\rm r0}$ and H_0 denote the values of these quantities at the time $t = 10^{-35}$ s. We take $H_0 = 5 \times 10^{34}\ {\rm s}^{-1}$ and $\Omega_{\rm r0} = 0.99$.

a) Show that the curvature term in the Friedmann equation is larger than the radiation term for

$$a > a_{\rm m} = \sqrt{\frac{\Omega_{\rm r0}}{1-\Omega_{\rm r0}}}.$$

b) Show that the Friedmann equation can be written as

$$\frac{1}{a}\frac{da}{dt} = H_0\frac{\sqrt{\Omega_{\rm r0}}}{a^2}\sqrt{1+\frac{a^2}{a_{\rm m}^2}}.$$

c) Integrate this equation and show that

$$a(t) = a_{\mathrm{m}} \sqrt{\left(1 + \frac{H_0\sqrt{\Omega_{\mathrm{r}0}}}{a_{\mathrm{m}}^2} t\right)^2 - 1}.$$

Hint: the substitution $x = a^2/a_{\mathrm{m}}^2$ might be useful.

d) How old was the Universe at $a = a_{\mathrm{m}}$?

e) How will a vary with t far into the curvature-dominated epoch? How will $\Omega_{\mathrm{r}}(t) = \rho_{\mathrm{r}}(t)/\rho_{\mathrm{c}}(t)$ vary with t in the same epoch?

f) Assume that teh radiation is converted to dust far into the curvature-dominated epoch. Explain why it is hard to get structure formation going based on this dust.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Exam in AST4220 — Cosmology I

Day of exam: Tuesday January 11th 2007

Time for exam: 09.00 – 12.00

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

Light emitted from an object at time t_{e} and received by an observer at time t has a cosmological redshift

$$1 + z = \frac{a(t)}{a(t_{\mathrm{e}})}.$$

In this problem we will calculate how this redshift changes with the time of observation t.

a) Show that

$$\frac{dz}{dt} = \frac{a(t)}{a(t_{\mathrm{e}})} \frac{\dot{a(t)}}{a(t)} - \frac{a(t)}{a(t_{\mathrm{e}})} \frac{\dot{a}(t_{\mathrm{e}})}{a(t_{\mathrm{e}})} \frac{dt_{\mathrm{e}}}{dt}.$$

b) The comoving radial coordinate of the object is given by

$$r = \int_{t_{\mathrm{e}}}^{t} \frac{cdt'}{a(t')}.$$

Use the fact that $dr/dt = 0$, assume the Universe to be described by an Einstein-de Sitter model and show that

$$\frac{dt_{\mathrm{e}}}{dt} = \frac{1}{1+z}.$$

This result can be shown to be valid in general.

c) Use the results in a) and b) to show that

$$\frac{dz}{dt} = (1+z)H(t) - H(t_{\mathrm{e}}).$$

d) For the Einstein-de Sitter model, what is the value of dz/dt at the present epoch ($t = t_0$) for an object with $z = 4$?

e) For the same object, calculate dz/dt at $t = t_0$ if the Universe is described by the de Sitter model. Give a qualitative explanation of this result and the result in d).

Problem 2

The theme of this problem is inflation, but the questions are not necessarily connected. Use units where $\hbar = c = 1$.

a) Describe briefly the horizon problem.

b) Is inflation possible if $V(\phi) = 0$?

c) Is inflation possible if the dynamics of the scalar field is such that $\dot{\phi}^2 = 2V(\phi)$ always?

d) For $V(\phi) = V_0 e^{-\phi/E_{\mathrm{Pl}}}$ calculate the slow-roll parameters ϵ and η. Will inflation end in this model?

Problem 3

The Friedmann equation for a universe with dust, radiation and spatial curvature can be written as

$$H^2 = H_0^2 \left(\frac{\Omega_{\mathrm{m0}}}{a^3} + \frac{\Omega_{\mathrm{r0}}}{a^4} + \frac{\Omega_{\mathrm{k0}}}{a^2} \right)$$

where $\Omega_{\mathrm{k0}} = 1 - \Omega_{\mathrm{m0}} - \Omega_{\mathrm{r0}}$.

a) Show that

$$a(t) = H_0 t,$$

is a solution of this equation in the case $\Omega_{\mathrm{m0}} = \Omega_{\mathrm{r0}} = 0$.

b) Show that the Friedmann equation can be written as

$$\frac{H^2}{H_0^2} = \frac{\Omega_{\mathrm{r0}}}{a^4} \left[\left(\alpha a + \frac{\beta}{2\alpha} \right)^2 + 1 - \frac{\beta^2}{4\alpha^2} \right]$$

where $\alpha^2 = \Omega_{\mathrm{k0}}/\Omega_{\mathrm{r0}}$ and $\beta = \Omega_{\mathrm{m0}}/\Omega_{\mathrm{r0}}$.

c) Show that in the special case $\beta = 2\alpha$ the equation in b) has the implicit solution

$$\frac{a}{\alpha} - \frac{\ln(\alpha a + 1)}{\alpha^2} = \sqrt{\Omega_{\mathrm{r}0}} H_0 t.$$

d) Finn approximate expressions for $a(t)$ in the limits $a \gg 1$ and $\alpha a \ll 1$. Explain why these results were to be expected. In the case $\alpha a \ll 1$ you might find the result $\ln(1 + x) \approx x - \frac{1}{2}x^2$ for $x \ll 1$ useful.

UNIVERSITY OF OSLO
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Mid-term exam in AST4220 — Cosmology I

Day of exam: Wednesday October 10th 2007

Time for exam: 14.30 – 17.30

This problem set consists of 3 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Oppgave 1

The deceleration parameter q is defined by

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2}.$$

a) Calculate q for an Einstein-de Sitte runiverse.

b) Calculate q as a function of redshift z in a spatially flat universe ($k = 0$) containing dust and a cosmological constant. Calculate the value q_0 at $z = 0$ if $\Omega_{\mathrm{m}0} = 0.3$.

c) Show that
$$\frac{d}{dt}\frac{1}{H} = 1 + q.$$

d) Show that
$$\frac{d}{dt}\frac{1}{H} = \frac{1+z}{H}\frac{dH}{dz}.$$
Hint: Use the chain rule, first to replace d/dt by d/da, then to replace d/da by d/dz.

e) Use c) and d) to show
$$H(z) = H_0 \exp\left[\int_0^z \frac{1+q(z')}{1+z'}dz'\right].$$

f) Find an expression for $H(z)$ when $q = q(z=0) \equiv q_0$ =konstant.

The luminosity distance out to a redshift z in a spatially flat universe is given by
$$d_{\mathrm{L}} = c(1+z)\int_0^z \frac{dz'}{H(z')}. \tag{1}$$

g) Calculate d_L as a function of z if $H(z)$ is given by the result in f).

h) The supernova Sn1997AP has $z = 0.83$ and $d_{\mathrm{L}} = 1.16c/H_0$. Compare this observation with the result in g) in the cases q_0 where q_0 has the values found in a) and b) respectively.

i) Carry out a Taylor expansion about $z = 0$ of the integrand in the expression (1) for d_{L} in order to show that
$$d_{\mathrm{L}} \approx \frac{cz}{H_0}\left[1 + \frac{1}{2}(1-q_0)z\right].$$
Hint: the results in c)and d) are useful when Taylor expanding $1/H$.

j) Assume that luminosity distances can be determined observationally with an accuracy of ten percent. Use the result in i) to estimate how high redshifts we must go to in order to distinguish between the Einstein-de Sitter model and the model in b).

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Exam in AST4220 — Cosmology I

Day of exam: Wednesday December 12th 2007

Time for exam: 14.30 – 17.30

This problem set consists of 4 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

In this problem we are going to have a look at the inflationary epoch in the very early universe. The scalar field that drove this phase has potential energy

$$V(\phi) = \lambda|\phi|,$$

where λ is a positive constant. Assume that the scalar field is positive initially.

a) Find the slow-roll parameters ϵ and η, and find the value of the field, ϕ_{end}, for which inflation stops.

b) Find the initial value of the field necessary for producing 60 e-foldings of inflation.

c) Calculate the ratio r of the amplitudes of gravitational waves and density perturbations produced by quantum fluctuations in this model for inflation.

Problem 2

The equation for the time evolution of a Fourier mode Δ_k of the density perturbations for wave numbers k that are much smaller than the Jeans wave number is

$$\ddot{\Delta}_k + 2H\dot{\Delta}_k = 4\pi G\rho_0\Delta_k.$$

For perturbations $\delta\phi_k$ in the gravitational potential we have

$$\frac{k^2}{a^2}\delta\phi_k = -4\pi G\rho_0\Delta_k.$$

a) Show that $\delta\phi_k$ is independent of time in an Einstein-de Sitter universe.

Consider a spatially flat model of the universe, filled with a fluid with equation of state $p = w\rho c^2$, where $w > -1$ is a constant. Perturbations in this model obey the equations above.

b) Show that the growing mode of density perturbations in this fluid is given by $\Delta_k \propto t^n$, where

$$n = \frac{3w - 1 + \sqrt{25 - 6w + 9w^2}}{6(1+w)}.$$

c) Specialize to the case $w = -2/3$ and show that $\delta\phi_k$ will depend on time. Suggest an observational test based on the cosmic microwave background that can distinguish between the EdS model and an accelerating model.

Problem 3

The angular diameter distance to an object at redshift z is given by

$$d_{\mathrm{A}} = \frac{a_0 r(z)}{1+z}$$

where

$$r(z) = \frac{c}{a_0\sqrt{|\Omega_{\mathrm{k}0}|}H_0}\mathcal{S}_k\left(\sqrt{|\Omega_{\mathrm{k}0}|}\int_0^z \frac{H_0 dz'}{H(z')}\right),$$

with $\mathcal{S}_k(x) = \sin x$ for $k = +1$, $\mathcal{S}_k(x) = x$ for $k = 0$ and $\mathcal{S}_k(x) = \sinh(x)$ for $k = -1$. We will in this problem consider universe models where only one component contributes to the energy density and pressure. This component follows the equation of state

$$p = w\rho c^2.$$

a) Write down expressions for $H(z)$ for the Einstein-de Sitter model and the de Sitter model.

b) Show that

$$d_{\mathrm{A}}(z) = \frac{c}{H_0}\frac{z}{1+z}$$

for the de Sitter model, and that

$$d_{\mathrm{A}}(z) = \frac{2c}{H_0}\left[\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right]$$

for the Einstein-de Sitter model.

c) In one of the models, d_A has a maximum for a finite, positive value of z. Which model, for what value of z, and what is the maximum value? Draw a rough sketch of how $d_{\mathrm{A}}(z)$ varies with z for both models.

d) Write down an expression for $H(z)$ for a universe model with $k = 0$ and $w = -1/3$, and for en empty universe model with $k = -1$. Calculate $d_A(z)$ for both models and explain how we can distinguish between them observationally.

UNIVERSITY OF OSLO
Faculty of Mathematics and Natural Sciences

Mid-term exam for AST4220 — Cosmology I

Day of exam: Tuesday October 7th 2008

Time for exam: 15.00 – 18.00

This problem set consists of 3 pages.

Attachments: None

Allowed aids: All non-communicative aids.

Make sure that the problem set is complete before you start answering the questions.

Problem 1

We will in this problem consider a model where the universe is spatially flat, and where the density is composed of radiation and a cosmological constant. To simplify calculations you may take the present value of the scale factor, a_0, to be equal to 1. The density parameters for the radiation and the cosmological constant are denoted by, respectively, Ω_{r0} and $\Omega_{\Lambda 0}$.

a) Determine the value of the scale factor, a_{eq}, when the density of the radiation is equal to the density contributed by the cosmological constant. Write your answer in terms of the density parameters.

b) Write down the first Friedmann equation for this model.

c) Based on b), show that

$$t = \frac{1}{H_0\sqrt{\Omega_{r0}}} \int_0^a \frac{a' da'}{\sqrt{1 + (a'/a_{eq})^4}}.$$

d) If $1/H_0 = 9.52 \times 10^9$ years and $\Omega_{r0} = 10^{-4}$, how old was the universe at $a = a_{eq}$? Hint: use the substitution $x = (a/a_{eq})^2$ to transform the integral into one you have seen before.

e) How old was the universe when the expansion started to accelerate?

Problem 2

In this problem we will assume that the universe is described by the Einstein-de Sitter model.

a) Write down the expression for the scale factor as a function of time. Use this expression to show that the age of the universe at redshift z is given by

$$t(z) = t_0(1 + z)^{-3/2},$$

where t_0 is the present age of the Universe.

b) Assume that we at the present epoch observe two objects, one with $z = z_1 = 3$ and one with $z = z_2 = 8$. When, in units of t_0, was the light emitted by these objects?

c) Determine the comoving radial coordinates of the two objects.

d) The light we receive now from the object at $z = 8$ was emitted at the time $t_{\rm e}$. Determine the comoving radial coordinate where the light moving towards us is at an arbitrary later time t, $r = r(t)$.

e) Imagine that there is an observer situated at the object with $z = 3$. What redshift did she observe for the light emitted from the object we observe today at redshift $z = 8$?

THE END