Suggested solution to March 2015 midterm exam in AST3220

1. $\frac{1}{a} = 1+z$

a) Adiabatic expansion equation:

$$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$$, must show that this is equivalent to $\dot{\rho} = \frac{3\dot{z}}{1+z}(\rho+p)$

$$a = \frac{1}{1+z} \qquad \dot{a} = -\frac{\dot{z}}{(1+z)^2}$$

$$\frac{\dot{a}}{a} = -\frac{\dot{z}}{(1+z)^2} \cdot \left(\frac{1}{1+z}\right)^{-1} = -\frac{\dot{z}}{(1+z)}$$

$$\dot{\rho} = -3\left(-\frac{\dot{z}}{1+z}\right)(\rho+p) = 3\frac{\dot{z}}{1+z}(\rho+p) \quad (i)$$

Q.E.D.

b) Dust: $p = 0$

Cosmological constant: $p = -\rho$

Using (i) to find $\rho(z)$ directly:

Dust: $\dot{\rho}_m = 3\frac{\dot{z}}{1+z}\rho_m$

$$\frac{\dot{\rho}_m}{\rho_m} = 3\frac{\dot{z}}{1+z}$$

$$\int_{\rho_{m0}}^{\rho_m}\frac{d\rho_m}{\rho_m} = 3\int_0^z\frac{dz}{1+z}$$

$$\ln\left(\frac{\rho_m}{\rho_{m0}}\right) = 3\ln(1+z)$$

$$\Rightarrow \rho_m = \rho_{m0}(1+z)^3$$

Cosmological constant: $\dot{\rho}_\Lambda = 3\frac{\dot{z}}{1+z}(\rho_\Lambda - \rho_\Lambda) = 0$

$d\rho_\Lambda = 0 \quad \Rightarrow \quad \underline{\underline{\rho_\Lambda = \rho_0}}$

c) 1. Friedmann equation:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) \qquad \rho_{c0} = \frac{3H_0^2}{8\pi G}$$

$$\frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}(\rho_m + \rho_\Lambda) \qquad \Omega_{m0} = \frac{\rho_{m0}}{\rho_{c0}}$$

$$\frac{H^2}{H_0^2} = \frac{1}{\rho_{c0}}\left(\rho_{m0}(1+z)^3 + \rho_{\Lambda 0}\right) \qquad \Omega_{\Lambda 0} = \frac{\rho_{\Lambda 0}}{\rho_{c0}}$$

$$H^2 = H_0^2\left[\frac{\rho_{m0}}{\rho_{c0}}(1+z)^3 + \frac{\rho_{\Lambda 0}}{\rho_{c0}}\right]$$

$$\underline{\underline{H^2 = H_0^2\left[\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}\right]}}$$

d) Universe is flat, so

$$r = \int_{t}^{t_0} \frac{c\,dt'}{a(t')}$$

where t is the time in the past when a lightray originating from a point at coordinate distance r from us reaching us now must have been emitted.

Changing coordinates from t to a:

$$\frac{da}{dt} = \dot{a} \Rightarrow$$

$$r = \int_{a}^{1} \frac{c\,da}{a\,\dot{a}} = \int_{a}^{1} \frac{c\,da}{a^2 H(a)}$$

Changing coordinates from a to z:

$$\frac{dz}{da} = -\frac{1}{a^2} \Rightarrow \quad r = \int_z^0 \frac{c\,dz'}{a^2 H(z')}\left(-a^2\right) = \int_0^z \frac{c\,dz'}{H(z')}$$

$$E(z) \equiv \frac{H(z)}{H_0} \Rightarrow r = \int_0^z \frac{c\,dz'}{H_0 E(z')} = \frac{c}{H_0}\int_0^z \frac{dz'}{E(z')}$$

2. a) In $R_H = t$ - cosmology we have:

$$R_H = \frac{1}{H} = t \Rightarrow \quad H(t) = \frac{1}{t} \quad (ii)$$

In power law cosmology we have:

$$a = \left(\frac{t}{t_0}\right)^n \qquad \dot{a} = n\left(\frac{t}{t_0}\right)^n \cdot \frac{1}{t}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{n}{t} \quad (iii)$$

b) When $n = 1$ i (iii) we see that (iii) is equivalent to (ii), that is $R_H = t$ - cosmology is equivalent to a power law cosmology for $n=1$. However, this is only true in the limit for $H(t)$ letting n go to this value. From the definition $a = \frac{t}{t_0}$ of power law cosmology for $n=1$, $H(t) = 0$, therefore the $R_H = t$ - cosmology needs its own definition.

c) $E(z) \equiv \frac{H(z)}{H_0}$, for power law cosmologies (and the special case $R_H = t$-cosmology for $n=1$)

$$E(z) = \left(\frac{n}{t}\right)\cdot\left(\frac{n}{t_0}\right)^{-1} = \frac{t_0}{t}$$ regardless of the value for n.

However, $1+z = \frac{1}{a} = \left(\frac{t_0}{t}\right)^n$, so $E(z) = (1+z)^{\frac{1}{n}}$

d) ~~r(z)~~ Since the universe is still assumed flat, our derivation of $r(z)$ in 1 d) is still valid, i.e.:

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz}{E(z)}$$

$$= \frac{c}{H_0} \int_0^z \frac{dz}{(1+z)^{1/n}}$$

If $n \neq 1$: $r(z) = \frac{c}{H_0} \left[\frac{1}{1-\frac{1}{n}} (1+z)^{1-\frac{1}{n}} \right]_0^z$

$$= \frac{c}{H_0} \frac{(1+z)^{1-\frac{1}{n}} - 1}{1 - \frac{1}{n}} \quad \text{for } n \neq 1$$

If $n = 1$: $r(z) = \frac{c}{H_0} \left[\ln(1+z) \right]_0^z = \frac{c}{H_0} \ln(1+z)$

So with $c = 1$:

$$r(z) = \begin{cases} \frac{(1+z)^{1-\frac{1}{n}} - 1}{1 - \frac{1}{n}}, & n \neq 1 \\ \ln(1+z), & n = 1 \end{cases}$$

3. $m = M + 5 \log_{10}(H_0 d_L(z))$

a) We know that

$$d_L(z) = (1+z)\, r(z) \quad (\text{with } a_0 = 1)$$

Hence $m = M + 5 \log_{10}(H_0 (1+z)\, r(z))$

$$= M + 5 \left[\log_{10}((1+z)) + \log_{10}(H_0\, r(z)) \right]$$

$$= M + 5 \left[\log_{10}(H_0(1+z)) + \log_{10}(r(z)) \right]$$

$$m = M + 5\log_{10}(H_0(1+z)) + 5\log_{10}(r(z))$$

If we observe a supernova at redshift z the first two terms are equal for all universe models. Only the last term, which only depends on $r(z)$, is model dependent. Hence, specifying $r(z)$ is enough to compare the models using observations.

b) Finding $r(z)$ for ΛCDM in the small z limit:

$$r(z) = \frac{1}{H_0}\int_0^z \frac{dz'}{E(z')}$$

$$= \frac{1}{H_0}\int_0^z \frac{dz'}{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}}$$

$$\approx \frac{1}{H_0}\int_0^z \frac{dz'}{\Omega_{m0} + 3\Omega_{m0} z + \Omega_{\Lambda 0}}$$

At $t = t_0$ we have $\frac{H^2}{H_0^2} = 1 = \Omega_{m0}\cdot 1^3 + \Omega_{\Lambda 0}$

so that $\Omega_{\Lambda 0} = 1 - \Omega_{m0}$

$$\Rightarrow \quad r(z) \approx \frac{1}{H_0}\int_0^z \frac{dz'}{1 + 3\Omega_{m0} z}$$

$$\approx \frac{1}{H_0}\int_0^z dz'(1 - 3\Omega_{m0} z)$$

$$\approx \frac{1}{H_0}\left[z - \frac{3}{2}\Omega_{m0} z^2\right]_0^z$$

$$\approx \frac{z}{H_0}$$

c) For z small & in power law cosmology:

$$r(z) = \frac{1}{H_0} \begin{cases} \dfrac{(1+z)^{1-\frac{1}{n}} - 1}{1-\frac{1}{n}} & n \neq 1 \\ \ln(1+z) & \end{cases}$$

Take the to cases and consider each separately:

$n \neq 1$: $$r(z) = \frac{1}{H_0}\,\frac{(1+z)^{1-\frac{1}{n}} - 1}{1-\frac{1}{n}}$$

$$\approx \frac{1}{H_0}\,\frac{1 + (1-\frac{1}{n})z - 1}{1-\frac{1}{n}} = \frac{1}{H_0}\,\frac{(1-\frac{1}{n})z}{1-\frac{1}{n}} = \underline{\frac{z}{H_0}}$$

$n = 1$ $$r(z) = \frac{1}{H_0}\ln(z+1) = \frac{1}{H_0}\left((1+z) - 1\right) + \text{Higher order}$$

$$\text{in } \left.(1+z) - 1 = z\right] \text{ so}$$

$$\underline{r(z) \simeq \frac{1}{H_0}(1+z) - 1 = \frac{z}{H_0}}$$

d) As the small z limit for ΛCDM, power law cosmology and $R_H = t$ - cosmology are all $r(z) = \frac{z}{H_0}$, we can not hope to distinguish between them from observing ~~see~~ supernovae at very low redshifts.

Solutiouns to problems 3 b) - d) in the postponed version of the exam:

b) Assume $z \gg 1$: For ΛCDM:

$$r(z) = \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3 + \Omega_{\Lambda 0}}}$$

$$= \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}}}$$

Assume that z is so large that the value of the integral is dominated by the behaviour at hight redshift:

$$r(z) \approx \frac{1}{H_0}\int_0^z \frac{dz}{\sqrt{\Omega_{m0}(1+z)^3}}$$

$$= \frac{1}{H_0\sqrt{\Omega_{m0}}}\int_0^z (1+z)^{-\frac{3}{2}}\, dz$$

$$= \frac{1}{H_0\sqrt{\Omega_{m0}}}\left[-2(1+z)^{-\frac{1}{2}}\right]_0^z$$

$$= \frac{2}{H_0\sqrt{\Omega_{m0}}}\left[1 - (1+z)^{-\frac{1}{2}}\right]$$

$$\approx \frac{2}{H_0\sqrt{\Omega_{m0}}}$$

c) Power-law cosmology in the $z \gg 1$ limit

$n \neq 1$: $r(z) = \frac{1}{H_0}\frac{(1+z)^{1-\frac{1}{n}} - 1}{1-\frac{1}{n}}$

Assuming n is positive (otherwise we would have a negative Hubble constant.

If $n < 1$ we have $r(z) = \frac{1}{H_0}\left(\frac{1}{\frac{1}{n}-1} - \frac{(1+z)^{\frac{1}{n}-1}}{1-\frac{1}{n}}\right)$

$$\approx \frac{1}{H_0}\frac{1}{\frac{1}{n}-1}$$

If $\frac{2}{\sqrt{\Omega_{m0}}} = \frac{1}{\frac{1}{n}-1}$ this could compare well to with ΛCDM, however, this will yield an age for the Universe

$$t_0 = \frac{n}{H_0}$$

$$\frac{1}{n} - 1 = \frac{\sqrt{\Omega_{m0}}}{2}$$

$$n = \frac{1}{\frac{\sqrt{\Omega_{m0}}}{2} + 1}$$

which might be to small.

~~For Also in these mo~~

More importantly $\ddot{a} = n(n-1)\left(\frac{t}{t_0}\right)^n t^{-2}$ is negative, so these models can not explain our current expanding Universe.

If $n > 1$ $\quad r(z) = \frac{1}{H_0} \frac{(1+z)^{1-\frac{1}{n}} - 1}{\left(1-\frac{1}{n}\right)}$

$$\approx \frac{1}{H_0} \frac{(1+z)^{1-\frac{1}{n}}}{\left(1-\frac{1}{n}\right)}$$

which rather than converging blows up for $z \to \infty$, signalling that there is no particle horizon, ~~and hence no Big Bang for these models~~.

If $n = 1$ $r(z) = \frac{1}{H_0} \ln(1+z)$, the solution again blows up for large z.

d) The power law cosmologies which can give accelerated expansion $(n \geq 1)$, blow up in the large z limit, signalling that these universes have no particle horizon. ~~This is very~~ Howere. for $n \geq 1$ we can define a s.t.

$a = 0$ for $t < 0$.

Thus these models have a Big Bang.
This combination signals very different early Universe ~~solutions~~ evolutions for these models compared to ΛCDM. Since we know that ΛCDM works well for the early universe (up to inflation) ~~relation to~~ we conclude that the power law cosmologies are not valid there

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This exam set corresponds to deriving the equations in the section "II Models" of the paper "Robust model comparison disfavors power law cosmology" by Daniel L. Schafer, which you can find on:

arxiv.org/abs/1502.05416

After completing the exam, you might be able to understand the main message of said article :)