Solution to mid term exam 2013

1. We will say that if the peculiar velocity is small compared to the Hubble velocity if it is less than 10% ~~pace~~ of the Hubble velocity

$$v_{pec} < 0{,}1\, v_H$$
$$v_H > 10\, v_{pec}$$
$$H_0 d > 10\, v_{pec}$$
$$d > \frac{10\, v_{pec}}{H_0} = \frac{10 \cdot 300\,\frac{km}{s}}{70\,\frac{km}{s\,Mpc}} \approx \underline{43\,Mpc}$$
$$\underline{\underline{d > 50\,Mpc}}$$

(~~E~~We use only the simple hubble velocity to get a rough estimate. ~~However, we could also do a more apropriate estimate as the distance we found here was very large~~ To do better we ~~could use~~ would have to know something about the dynamics of te universe. However, we know now that the universe is accelerating, meaning that H has been smaller in the past, meaning the compansating distance is larger and $d > 50\,Mpc$ rather than 43 Mpc is propably a safe estimate).

~~(This is also an ero~~

2. Considering the Einstein deSitter model

$\Omega_{m_0} = 1$ $\quad p_m = 0$

a) We have the adiabatic expansion equation

$\dot{\rho} = -3\frac{\dot{a}}{a}(\rho + p)$ which in this case reduces to $\dot{\rho}_m = -3\frac{\dot{a}}{a}\rho_m$

$\rightarrow \frac{d\rho_m}{\rho_m} = -3\frac{da}{a}$ Integrating this we get $\rho_m = \rho_{m0}\left(\frac{a_0}{a}\right)^3$, so that

$\Omega_m = \Omega_{m0}\left(\frac{a_0}{a}\right)^3 = \left(\frac{a_0}{a}\right)^3$

The first Friedmann equation now reads

$$H^2 = H_0^2\left(\frac{a_0}{a}\right)^3 \quad \text{(assuming } H \text{ to be positive)}$$

$$H = H_0\left(\frac{a_0}{a}\right)^{\frac{3}{2}}$$

$$\frac{1}{a}\frac{da}{dt} = H_0\left(\frac{a_0}{a}\right)^{\frac{3}{2}}$$

$$\sqrt{a}\,da = H_0\,(a_0)^{\frac{3}{2}}\,dt$$

EdS has a Big Bang so we are free to integrate from $a(t=0)=0$

$$\int_0^a \sqrt{a'}\,da' = H_0 a_0^{\frac{3}{2}}\int_0^t dt$$

$$\frac{2}{3}a^{3/2} = H_0 a_0^{\frac{3}{2}} t \quad (*)$$

$$a = a_0\left(\frac{3H_0 t}{2}\right)^{\frac{2}{3}}$$

We know that at t_0 $a(t_0) = a_0$ so that

$\frac{3H_0 t_0}{2} = 1$ so $t_0 = \frac{2}{3H_0}$ and we can write

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

Going back to $(*)$ we observe that

$$t = \frac{2}{3H_0}\left(\frac{a}{a_0}\right)^{\frac{3}{2}} = t_0\left(\frac{a}{a_0}\right)^{\frac{3}{2}} = t_0(1+z)^{-\frac{3}{2}}$$

Finally the angular diameter distance d_A can be expressed as

$$d_A = \frac{a_0 r}{1+z} \quad \text{(as shown in the notes)}$$

We need to find r to find d_A. Since EdS is flat the comoving coordinat is given by:

$$r = \int_t^{t_0} \frac{c\,dt'}{a(t')} \quad \text{(see notes)}$$

Inserting for a: $r = \int_t^{t_0} \frac{t_0^{2/3} c\,dt'}{a_0 t'^{2/3}} = \frac{c t_0^{2/3}}{a_0} \int_t^{t_0} t'^{-\frac{2}{3}}\,dt'$

$$= \frac{3 c t_0^{2/3}}{a_0} \Big[t'^{1/3} \Big]_t^{t_0} = \frac{3 c t_0}{a_0} \left(1 - \left(\frac{t}{t_0}\right)^{1/3} \right)$$

$$d_A(z) = \frac{a_0 r}{1+z} = \frac{3 a_0 c t_0}{(1+z)\,a_0} \left[1 - \left((1+z)^{-\frac{3}{2}}\right)^{\frac{1}{3}} \right]$$

$$= \underline{\frac{3 c t_0}{(1+z)} \left[1 - \frac{1}{\sqrt{1+z}} \right]}$$

$$= \underline{\underline{\frac{2c}{(1+z) H_0} \left[1 - \frac{1}{\sqrt{1+z}} \right]}}$$

b) The particle horizon is ~~situated at~~ given by the distance to light sent out at the Big Bang $a = t = 0$

For this we have $z = \frac{a_0}{a} - 1 \rightarrow \infty$

So $d_A^{PH} = \lim_{z \to \infty} \frac{2c}{(1+z) H_0} \left[1 - \frac{1}{\sqrt{1+z}} \right]$

$= \underline{0}$

c) Universets alder idag er

$$t_0 = \frac{2}{3H_0} = \frac{2}{3} \cdot 14 \cdot 10^9 \text{ år} = \underline{\underline{9{,}3 \cdot 10^9 \text{ år}}}$$

$$t(z=1090) = \frac{2}{3} \cdot 14 \cdot 10^9 \text{ år } (1+z)^{-\frac{3}{2}} =$$

$$\frac{2}{3} \cdot 14 \cdot 10^9 \text{ år } \cancel{(1+z)} (1091)^{-\frac{3}{2}} = \underline{\underline{2{,}6 \cdot 10^5 \text{ år}}}$$

d) $$d_A(z=1090) = \frac{2c}{(1+z)H_0}\left[1 - \frac{1}{\sqrt{1+z}}\right] = \frac{2c}{(1091)H_0}\left[1 - \frac{1}{\sqrt{(1091)}}\right]$$

$$= \underline{\underline{7{,}6 \text{ MpC}}}$$

This is much less than the distance we found ~~to~~ for the Hubble flow to be negligable. However, two things must be considered when comparing these values: We saw that the angular diameter distance was 0 to the particle horizon. This means that this distance measure peaks somwhere between now und then. This peak is located when $\frac{d\, d_A(z)}{dz} = 0$

So $$\frac{2c}{H_0}\left(-\frac{1}{(1+z)^2} + \frac{3}{2(1+z)^{5/2}}\right) = \frac{2c}{H_0(1+z)^2}\left(\frac{3}{2\sqrt{(1+z)}} - 1\right) = 0$$

So when $\frac{3}{2\sqrt{1+z}} = 1$ $\quad \frac{1}{\sqrt{1+z}} = \frac{2}{3}$

$$\sqrt{1+z} = \frac{3}{2} \qquad (1+z) = \frac{9}{4} \qquad \underline{\underline{z = \frac{5}{4}}}$$

so way earlier than this, this means peculiar velocities are probably washed away well before $z = 1090$.

In addition we note that really Hubble flow measurements are made with lots of objects in different directions. Since their average v_{pec} ~~is~~ towards us should be as large as their average v_{pec} away from us, all these velocities should cancel out and we will be left with the Hubble flow.

e) As we noted above d_A has a maximum at $\frac{d\, d_A(z)}{dz} = 0$ which we found to be at ~~$z = \frac{5}{4}$~~

$$\underline{\underline{z = \frac{5}{4}}}$$

3. The concordance model $\Omega_{m0} = 0{,}3$, $\Omega_\Lambda = 0{,}7$

a) The Friedmann equation in this case is

$$H^2 = H_0^2 \left(\Omega_{m0} \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda \right)$$

$$H = H_0 \sqrt{\Omega_{m0} \left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda}$$

$$\int_0^a \frac{da}{a\sqrt{\Omega_{m0}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda}} = \int_0^t H_0\, dt$$

(where again we have used that we know this model has a Big Bang.)

Setting for convenience $a_0 = 1$

$$\int_0^a \frac{da}{\sqrt{\frac{\Omega_{m0}}{a} + \Omega_\Lambda a^2}} = \int_0^a \frac{\sqrt{a}\, da}{\sqrt{\Omega_{m0} + (1-\Omega_{m0})\, a^3}} = \int_0^a \frac{\sqrt{a}\, da}{\sqrt{1-\Omega_{m0}}\sqrt{b^2 + a^3}}$$

where $b^2 = \frac{\Omega_{m0}}{(1-\Omega_{m0})}$

$$\Rightarrow t = \frac{1}{H_0\sqrt{1-\Omega_{m0}}} \int_0^a \frac{\sqrt{a}\, da}{\sqrt{b^2 + a^3}}$$

Now we use the substitution $a^{3/2} = b\sinh\psi$

here $a = (b\sinh\psi)^{\frac{2}{3}}$

$$\frac{da}{d\psi} = \frac{2}{3}(b\sinh\psi)^{-1/3}\, b\cosh\psi$$
$$= \frac{2}{3}\frac{1}{\sqrt{a}}\, b\cosh\psi$$

So that $\sqrt{a}\,da = \frac{2}{3} b\cosh\psi\, d\psi$

Inserting this in the integral we get:

$$t = \frac{1}{H_0\sqrt{1-\Omega_{m0}}}\,\frac{2}{3}\int_0^{\psi}\frac{b\cosh\psi\, d\psi}{\sqrt{b^2+b^2\sinh^2\psi}} = \frac{2}{3H_0\sqrt{1-\Omega_{m0}}}\int_0^{\psi}\frac{b\cosh\psi\, d\psi}{b\cosh\psi}$$

$$= \frac{2}{3H_0\sqrt{1-\Omega_{m0}}}\int_0^{\psi} d\psi = \frac{2}{3H_0\sqrt{1-\Omega_{m0}}}\,\psi$$

$$\cancel{= \frac{2}{3H_0\sqrt{1-\Omega_{m0}}}}\quad \psi = \sinh^{-1}\left(\frac{a^{3/2}}{b}\right) = \sinh^{-1}\left(\frac{(1+z)^{-3/2}}{b}\right)$$

$$t_0 = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\sinh^{-1}\left(\frac{(1+z=0)^{-3/2}}{b}\right) = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\sinh^{-1}\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}}\right)$$

$$= \frac{2\cdot 14\cdot 10^9 \text{ år}}{3\sqrt{0{,}7}}\ln\left(\sqrt{\frac{0{,}7}{0{,}3}}+\sqrt{\frac{0{,}7}{0{,}3}+1}\right)$$

$$= \underline{\underline{13{,}4\cdot 10^9 \text{ år}}}$$

This is considerably older than what we found in 2c) and more in accordance with the universe we observe today. ~~We also~~

$$t(z=1091) = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \sinh^{-1}\left(\frac{(1+z)^{-3/2}}{\sqrt{\Omega_{m0}}}\sqrt{\Omega_\Lambda}\right)$$

$$= \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}}(1+z)^{-3/2} + \sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}\frac{1}{(1+z)^3}+1}\right)$$

$$= \frac{2 \cdot 14 \cdot 10^9\,yr}{3\sqrt{0,7}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}}(1+z)^{-\frac{3}{2}} + \sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}\frac{1}{(1+z)^3}+1}\right)$$

$= \underline{\underline{4{,}7 \cdot 10^5\,yr}}$ which is also considerably higher than what we fount in 2c).
In fact it is even higher than the scientific consenus for our universe for this redshift. However, this number is dominated by the dynamics of the very early universe where radiation also plays an important part. The lack of radiaton in this model accounts for the to high value of the age of the universe at this time.

b) Matter and vacuum energy equality happens when

$$\rho_m = \cancel{\rho_{c0}}\,\Omega_{m0}(1+z_{eq})^3 = \cancel{\rho_{c0}}\,\Omega_\Lambda = \rho_\Lambda$$

$$(1+z_{eq})^3 = \frac{\Omega_\Lambda}{\Omega_{m0}}$$

$$1+z_{eq} = \left(\frac{\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3}$$

$$z_{eq} = \left(\frac{\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3} - 1$$

$$\underline{\underline{z_{eq} = \left(\frac{0,7}{0,3}\right)^{1/3} - 1 = 0,33}}$$

$$t_{eq}(z=z_{eq}) = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}}(1+z_{eq})^{-\frac{3}{2}} + \sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}(1+z_{eq})^{-3}+1}\right)$$

Using ~~t~~hat $(1+z_{eq})^{-3} = \frac{\Omega_{m0}}{\Omega_\Lambda}$ so that

$$t(z=z_{eq}) = \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}}\sqrt{\frac{\Omega_{m0}}{\Omega_\Lambda}} + \sqrt{\frac{\Omega_\Lambda}{\Omega_{m0}}\frac{\Omega_{m0}}{\Omega_\Lambda}+1}\right)$$

$$= \frac{2}{3H_0\sqrt{\Omega_\Lambda}} \ln\left(1+\sqrt{2}\right)$$

$$= \frac{2\cdot 14\cdot 10^9\,\text{yr}}{3\sqrt{0{,}7}} \ln\left(1+\sqrt{2}\right)$$

$$t_{eq} = \underline{\underline{9{,}8\cdot 10^9\,\text{yr}}}$$

c) We define the infliction point to be when $\ddot{a}=0$. We know that $p_m=0$ and $p_\Lambda = -\rho_\Lambda$, so the second Friedmann (Raychaudri) equation becomes:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho_m + \rho_\Lambda - 3\rho_\Lambda\right)$$

$$= -\frac{4\pi G}{3}\left(\rho_m - 2\rho_\Lambda\right)$$

$$= -\frac{4\pi G}{3}\rho_{c0}\left(\Omega_{m0}(1+z)^3 - 2\Omega_\Lambda\right)$$

This means the inflection point is ~~found~~ given by:

$$\Omega_{m0}(1+z_{infl})^3 = 2\,\Omega_\Lambda$$

$$(1+z_{infl})^3 = \frac{2\,\Omega_\Lambda}{\Omega_{m0}}$$

$$(1+z_{infl}) = \left(\frac{2\,\Omega_\Lambda}{\Omega_{m0}}\right)^{1/3}$$

$$z_{infl} = \left(\frac{2\cdot 0{,}7}{0{,}3}\right)^{1/3} - 1 = \underline{\underline{0{,}67}}$$

~~$t_{infl} = \frac{2}{3}$~~ To find t_{infl} we use that

$$(1+z_{infl})^{-3} = \frac{\Omega_{m0}}{2\,\Omega_\Lambda}$$ so that

$$t_{infl} = \frac{2}{3 H_0 \sqrt{\Omega_\Lambda}} \ln\left(\sqrt{\frac{\cancel{\Omega_\Lambda}}{\cancel{\Omega_m}}}\sqrt{\frac{\cancel{\Omega_{m0}}}{2\,\cancel{\Omega_\Lambda}}} + \sqrt{\frac{\cancel{\Omega_\Lambda}}{\cancel{\Omega_{m0}}}\frac{\cancel{\Omega_{m0}}}{2\cancel{\Omega_\Lambda}} + 1}\right)$$

$$= \frac{2\cdot 14\cdot 10^9\,\text{år}}{3\,\sqrt{0{,}7}} \ln\left(\frac{1}{\sqrt{2}} + \sqrt{\frac{3}{2}}\right)$$

$$= \underline{\underline{7{,}3\cdot 10^9\,\text{år}}}$$