Eksamen 2013

$H_0 = 70 \frac{km}{s\, Mpc} = (14 \cdot 10^9 \, yr)^{-1}$ $a_0 = 1$

1 Structure formation

a) The Jeans wavelength $\lambda_J = \frac{2\pi}{k_J} = c_s \sqrt{\frac{\pi}{G\rho_0}}$ is the limiting length between sizes where gravitational collapse is the dominating process, $\lambda > \lambda_J$, and where pressure and collisions dominate $\lambda < \lambda_J$. Below λ_J pressure stabilizes the perturbations against gravitational collapse and hence λ_J roughly defines the minimum size of objects like for instance galaxies. The Jeans length can also be interpreted as the length which a sound wave transverses in a collapse time explaining thus how pressure (which travels at sound speed) can't stabilize structures of sizes larger than λ_J. The Jeans mass is just the mass contained within a sphere of Jeans length diameter and is hence the largest mass (given some density) which does not collapse under its own gravity.

b) Before recombination, the baryons are charged and their pressure component is given by Thompson scattering. The sound speed of this is equal to the light speed so the Jeans length is very large, for baryonic perturbations even though the universe is matter dominated

Hence the baryons virtually do not collapse before recombination. Hoverer, the dark matter is unaffected by the electromagnetic force and has a much shorter Jeans length allowing it to collapse.
After recombination baryons are no longer charged and their Jeans length drops very substantially. Hence they collapse into the perturbations already formed by dark matter.

c) To study radiation perturbations in Ω_{rad} we start with equation (4.39):

$$\frac{d^2\Delta_{\vec{k}}}{dt^2} + 2\frac{\dot{a}}{a}\frac{d\Delta_{\vec{k}}}{dt} = \frac{32\pi G\rho_0}{3}\Delta_{\vec{k}} \quad \text{(neglecting pressure)}$$

In radiation domination $a = \left(\frac{t}{t_0}\right)^{1/2}$ so that

$$\tau - \tau_0 = \int_{\tau_0}^{\tau} d\tau = t_0^{1/2}\int_{t_0}^{t} t^{-1/2}\,dt = 2t_0\left(\left(\frac{t}{t_0}\right)^{1/2} - 1\right) = 2t_0(a-1)$$

$a = \frac{\tau-\tau_0}{2t_0} + 1$, but t_0 is an arbitrary integration constant and we can put $\tau_0 = 2t_0$ so that

$a = \frac{\tau}{\tau_0}$ and $\frac{da}{d\tau} = \frac{1}{\tau_0}$ and $H = \frac{1}{a}\frac{da}{dt} = \frac{1}{a}\frac{d\tau}{dt}\frac{da}{d\tau}$

$$= \frac{1}{a}\frac{1}{a}\frac{da}{d\tau} = \left(\frac{\tau_0}{\tau}\right)^2\frac{1}{\tau_0} = \frac{\tau_0}{\tau^2}$$

We rewrite 4.39 in terms of τ:

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{a}\frac{d}{d\tau}$$

$$\frac{1}{a}\frac{d}{d\tau}\left(\frac{1}{a}\frac{d\Delta_{\vec{k}}}{d\tau}\right) + 2\frac{1}{a^2}\frac{da}{d\tau}\frac{1}{a}\frac{d\Delta_{\vec{k}}}{d\tau} = \frac{32\pi G}{3}\rho_0\Delta_{\vec{k}} = 4H^2\Delta_{\vec{k}}$$

$$\cancel{\frac{1}{a^2}}\left[\frac{d^2\Delta_{\vec{k}}}{d\tau^2} - \frac{1}{a}\frac{da}{d\tau}\frac{d\Delta_{\vec{k}}}{d\tau} + 2\frac{1}{a}\frac{da}{d\tau}\frac{d\Delta_{\vec{k}}}{d\tau}\right] = 4\cancel{\frac{1}{a^2}}\frac{1}{a^2}\left(\frac{da}{d\tau}\right)^2\Delta_{\vec{k}}$$

$$\frac{d^2\Delta_{\vec{k}}}{d\tau^2} + \frac{\tau_0}{\tau}\frac{1}{\tau_0}\frac{d\Delta_{\vec{k}}}{d\tau} = 4\frac{\tau_0^2}{\tau^2}\frac{1}{\tau_0^2}\Delta_{\vec{k}}$$

$$\frac{d^2\Delta_{\vec{k}}}{d\tau^2} + \frac{1}{\tau}\frac{d\Delta_{\vec{k}}}{d\tau} - 4\frac{1}{\tau^2}\Delta_{\vec{k}} = 0$$

Guessing solution on the form $\Delta_{\vec{k}} \propto \tau^n$

$$n(n-1)\tau^{n-2} + n\tau^{n-2} - 4\cancel{\tau^{n-2}} = 0$$

$$n^2 - n + n - 4 = 0 \qquad n^2 = 4 \qquad n = \pm 2.$$

The growing modes go like $\Delta_k \propto \tau^2$. Since we've neglected pressure, the Fourier equation is actually equal to the original one an all modes behave equally giving

$$\underline{\underline{\delta \propto \tau^2}}$$

d) For matter pertubations in matterdomination we use 4.26.

$$\frac{d^2\Delta_{\vec{k}}}{dt} + 2\frac{\dot{a}}{a}\frac{d\Delta_{\vec{k}}}{dt} = 4\pi G\rho_0\Delta_{\vec{k}}$$

In this case $a = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$ so that

$$\tau - \tau_0 = \int_{\tau_0}^{\tau} d\tau = t_0^{2/3}\int_{t_0}^{t} t^{-\frac{2}{3}}dt = 3t_0\left[\left(\frac{t}{t_0}\right)^{1/3} - 1\right] = 3t_0(a^{\frac{1}{2}} - 1)$$

$$a = \left(\frac{\tau - \tau_0}{3t_0} + 1\right)^2$$

Again we are free to choose τ_0 and choose $\tau_0 = 3t_0$ so that $a = \left(\frac{\tau}{\tau_0}\right)^2$

$$\frac{da}{d\tau} = 2\left(\frac{\tau}{\tau_0^2}\right)$$

We rewrite 4.26 in conformal time

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{a}\frac{d}{d\tau}$$

$$\frac{1}{a}\frac{d}{d\tau}\left(\frac{1}{a}\frac{d\Delta_{\vec{k}}}{d\tau}\right) + 2\frac{1}{a^2}\frac{da}{d\tau}\frac{1}{a}\frac{d\Delta_{\vec{k}}}{d\tau} = \frac{3}{2}\frac{8\pi G}{3}\rho_0\Delta_{\vec{k}}$$

$$\frac{1}{a^2}\left[\frac{d^2\Delta_{\vec{k}}}{d\tau^2} - \frac{1}{a}\frac{da}{d\tau}\frac{d\Delta_{\vec{k}}}{d\tau} + 2\frac{1}{a}\frac{da}{d\tau}\frac{d\Delta_{\vec{k}}}{d\tau}\right] = \frac{3}{2}H^2\Delta_{\vec{k}} = \frac{3}{2}\frac{1}{a^2}\frac{1}{a^2}\frac{da}{d\tau}\Delta_{\vec{k}}$$

$$\frac{d^2\Delta_{\vec{k}}}{d\tau^2} + \left(\frac{\tau_0}{\tau}\right)^2\frac{2\tau}{\tau_0^2}\frac{d\Delta_{\vec{k}}}{d\tau} = \frac{3}{2}\left(\frac{\tau_0}{\tau}\right)^4\frac{4\tau^2}{\tau_0^4}\Delta_{\vec{k}}$$

$$\frac{d^2\Delta_{\vec{k}}}{d\tau^2} + 2\frac{1}{\tau}\frac{d\Delta_{\vec{k}}}{d\tau} - 6\frac{1}{\tau^2}\Delta_{\vec{k}} = 0$$

Ansatz $\Delta_{\vec{k}} \propto \tau^n$:

$$n(n-1)\tau^{n-2} + 2n\tau^{n-2} - 6\tau^{n-2} = 0$$

$$n^2 - n + 2n - 6 = 0$$

$$n^2 + n - 6 = 0$$

$$n = \frac{-1 \pm \sqrt{1+24}}{2} = \frac{-1 \pm 5}{2} \quad n = -3 \wedge n = 2$$

~~Only the grown~~ So $\Delta_k (= \delta) = D_1 \tau^{-3} + D_2 \tau^2$

Only the growing mode is of interest so

$\underline{\underline{\Delta_k \propto \tau^2}}$ for matter perturbations in matterdomination as well.

2 Primordial nucleosynthesis

a) There are no stable nuclei of 7 or 8 nucleons hence to create carbon (or a related 12 nucleon atom like N or O) one needs three helium atoms to come close enough with a high enough energy to overcome the electromagnetic repulsive force between them. The reaction to form such states is

$$^{4}He + {}^{4}He + {}^{4}He \rightarrow {}^{12}C + \gamma$$

The conditions in the early universe after the formation of deuterium and subsequent helium formation is neither hot nor dense enough for this reaction to proceed with any observable (even with trace yields) outcome.

b) The binding energy of the helium atom is 13.6eV However, since the early universe is very dense with photons, even when the expectation energy for photons has dropped beneath 13.6eV, there are still plenty of high energy photons energetic enough to ionize any neutral hydrogen that forms. Only when the temperature drops beneath 0,5 eV are such photons rare enough for neutral hydrogen to form stably.

3 Inflation

① The Universe seems very flat, however curvature goes like a^{-2} while matter and radiation goes as a^{-3} or a^{-4} so as the universe expands throug matter and radiation domination, an initially small curvature component would soon outgrow the two other components unless it was initially extremely small which seems unlikely. <u>This is the flatness problem.</u>

The CMB is only has fluctuations of 10^{-5} degrees and fluctuations are correlated throughout the whole sky. However, evolving back to the ~~[illegible]~~ time of decoupling, only what looks like about 2 degrees today was actually in causal contact at this time, so why is the CMB so correlated. This is the horizon problem.

In the early universe according to current particle physics the universe underwent ~~the~~ the strong phase transition. Just like when ice settles on a pond, starting differently in different places, creating unaligned crystal structures that meet up making domain walls corners and cracks, ~~the~~ the same kind of domain walls etc. would result from the strong phase transition. These would have quite ~~[illegible]~~ large physical consequences, ~~[illegible]~~ for instance they would manifest as monopoles or other observable relics and according to Big Bang theory should be very common in the Universe. So common in fact that it seems extremely weird that we haven't seen any yet. This is the relic or monopole problem.

b) The inflationary ~~[illegible]~~ Universe is one where at very early times, just after the strong phase transition, the Universe is dominated by an inflaton field φ with energy density $\rho_\varphi = \frac{1}{2\hbar c^3}\dot{\varphi}^2 + V(\varphi)$ and pressure

$p_\varphi = \frac{1}{2\hbar c^3}\dot{\varphi}^2 - V(\varphi)$ and subsequent Friedmann equation: $H^2 = \frac{8\pi G}{3c^2}\rho_\varphi c^2$

it also obeys the equation of motion:

$$\ddot{\varphi} + 3H\dot{\varphi} + \hbar c^3 V'(\varphi) = 0$$

The field is assumed to cause ~~an~~ very strong accelerated expansion which is insured by the potential being substantially larger than the kinetic term. In a very short time the inflationary field has blown the universe up to about e^{60} times its original size.

c) The slow-roll approximation is the extreme accelerating limit where very near deSitter conditions are reached. This is insured by $\varepsilon = \frac{E_{Pl}^3}{16\pi}\left(\frac{V'}{V}\right)^2 \ll 1$ and $|\eta| = \left|\frac{E_{Pl}}{8\pi}\frac{V''}{V}\right| \ll 1$.

In this case simplified versions of the equations can be used, namely

$$\boxed{\begin{aligned} H^2 &\simeq \frac{8\pi G}{3c^2}V(\varphi) \\ 3H\dot{\varphi} &\simeq -\hbar c^3 V'(\varphi) \end{aligned}}$$

making exact solutions accessible for many more potential types.

d) The inflationary Universe solves the flatness problem because $\Omega_k \propto a^{-2}$ and when the universe expands 60 efolds all traces of this will go away. The horizons (or areas in causal contact) are also blown up to sizes that are much larger than our observable Universe today, and the relics from the strong phase transition are scattered ~~so~~ wide that the chance for us to have detected one is virtually zero.

4. Modified Chaplygin gas

Equation of state: $p = A\rho - \frac{B}{\rho^\alpha}$

ⓐ Adiabatic expansion: $\dot{\rho} = -3H(\rho + p) = -3H\left((A+1)\rho - \frac{B}{\rho^\alpha}\right)$

ⓑ Solving the adiabatic expansion equation:

$$\frac{d\rho}{dt} = -3\frac{1}{a}\frac{da}{dt}\left((A+1)\rho - \frac{B}{\rho^\alpha}\right) = -\frac{3}{a}\rho^{-\alpha}\left((A+1)\rho^{1+\alpha} - B\right)$$

$$\frac{\rho^\alpha d\rho}{(A+1)\rho^{1+\alpha} - B} = -3\frac{da}{a}$$

$$\frac{1}{B}\int\left(\frac{\rho^\alpha d\rho}{1 - \frac{(A+1)}{B}\rho^{1+\alpha}}\right) = 3\int\frac{da}{a}$$ Changing variables

$$u = \frac{A+1}{B}\rho^{1+\alpha} \qquad \frac{du}{d\rho} = \frac{(1+\alpha)(A+1)}{B}\rho^\alpha$$

$$\frac{\rho^\alpha d\rho}{B} = \frac{du}{(1+\alpha)(A+1)}$$

$$\frac{1}{(1+\alpha)(1+A)}\int\frac{du}{1-u} = 3\int\frac{da}{a}$$

$$-\frac{1}{(1+\alpha)(1+A)}\ln(1-u) = 3\ln\left(\frac{a}{a_0}\right) + C_1$$

$$1 - u = C_2\left(\frac{a_0}{a}\right)^{3(1+\alpha)(1+A)}$$

$$u = -C_2\left(\frac{a_0}{a}\right)^{3(1+\alpha)(1+A)} + 1$$

$$\frac{A+1}{B}\rho^{(1+\alpha)} = -C_2\left(\frac{a_0}{a}\right)^{3(1+\alpha)(1+A)} + 1$$

$$\rho = \left[\frac{B}{A+1} + C_3\left(\frac{a_0}{a}\right)^{3(1+\alpha)(1+A)}\right]^{\frac{1}{1+\alpha}}$$

Where C_1, C_2 and C_3 are all constants, but not the same one.

E) When $B \to 0$. We see from the equation of state that we have a normal ideal gas equation with $w = A$. This also corresponds nicely in the results for equation (7) which now reduces to:

$$\rho = C_4 \left(\frac{a_0}{a}\right)^{3(1+A)}$$ which is the usual expression.

F) When $A = 0$ and $\alpha = -1$ we again get a normal ideal gas equation of state in the original equation. However, in this case the expression (7) becomes singular and we can not use it to define $\rho(a)$. Instead we must begin the derivation from the new equation of state or look at a reasonable way to define the limit of the equation. However, in this case $w = -B$.