SCHRIFTELIJK TENTAMEN
COSMOLOGY
2^{nd} term 2005/2006

NOTE: THIS EXAM HAS 5 QUESTIONS, ONE QUESTION PER PAGE

Question 1.

The cosmological principle says that the Universe is isotropic and homogeneous.

a) Einstein's general theory of relativity is a metric theory of gravity. Give the expression of the Einstein equation, and provide a physical explanation for the significance of this equation. Emphasize the essential difference between Newtonian gravity and Relativistic gravity with respect to behaviour of spacetime in the general relativistic world !!!

b) What is the crucial significance of the cosmological principle in the context of relativistic cosmology. Ie., why is the cosmological principle necessary ? Describe the possible geometries allowed for a medium obeying the cosmological principle.

c) Translate the cosmological principle into the metric expression for the spacetime of an evolving universe obeying the cosmological principle and Weyl's principle, the Robertson-Walker metric,
   - write out the metric form, $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$, for this spacetime
   - explain the meaning of each of the characteristic quantities. Focus in particular on those concerning the radius of curvature of spacetime, the related definition of the "scale factor" a_{exp} of the medium, and the factors expressing its curvature.

d) Indicate how you can derive the Friedman-Robertson-Walker-Lemaitre equations from the Einstein equation, and give the expressions for these equations. Indicate the meaning for each factor in the FRW equations (density ρ, etc.).

e) Discuss and indicate the crucial differences between Newtonian cosmology and relativistic cosmology as encapsulated in the FRW equations.

Question 2.

A fundamental concept in cosmology, crucial for the analysis of cosmological observations, is *redshift*.

a) One of the fundamental concepts in cosmology is the redshift z. Derive the relation between redshift z and the expansion factor a of the Universe on the basis of the RW metric.

b) When you observe an object at redshift z the intrinsic timescale Δt of such an object has changed. Give an expression for this '*cosmological time dilation*. How would you be able to test the cosmological origin of such a time dilation ?

c) Imagine you observe a quasar at a cosmological redshift z_{cosm}. The quasar is moving a peculiar velocity v_{pec} with respect to the cosmological background, hence causing a Doppler redshift z_{Dopp}. In the center of the quasar there is a supermassive black hole. While climbing out of its deep potential well ϕ, radiation gets gravitationally redshifted to z_{grav}. Upon finally observing the radiation with a telescope on planet Earth you observe it with a total redshift z_{tot}. Give the expression for the total redshift z_{tot}.

d) In the metric we usually work in terms of the comoving distance r, while an observational cosmologist usually thinks in terms of the redshift z. How is the redshift z related to the expansion factor a_{exp} ? How can you translate comoving distance r in terms of the "redshift" z of an object (hint: I am asking for an integral expression, in which one also finds the Hubble expansion rate $H(z) \equiv \dot{a}/a$).

e) Often we wish to know the number of objects within a redshift interval $[z, z + dz]$. If the (comoving) space density N_0 of a particular class of object does not change, derive from the RW metric how many objects within a redshift internval dz you will find. (Hint: express in terms of present-day density N_0, the radius of curvature R_c and curvature term $S_k(r/R_c)$).

Question 3.

Key cosmological factors are the cosmological density parameter $\Omega(t)$, the Hubble parameter $H(t)$, the cosmological constant Λ abd the curvature parameter k.

a) For a FRW Universe:
   - define the meaning of the critical density ρ_{crit}.
   - give the expression for the critical density.
   - give an estimate of the value of the present-day critical density of the universe, in $[g/cm^3]$ and in $[M_\odot/Mpc^3]$.

b) Subsequently, provide the definition of the cosmological density parameter $\Omega(t)$.

c) Provide the definition for the Hubble parameter $H(t)$. What are the usual units in which we express $H(t)$? What are the present-day estimates for H_0 ?

d) Give the relation between the cosmological constant Λ and its contribution in terms of Ω, Ω_Λ.

e) Derive the expression of the curvature parameter k in terms of Ω and H.

Question 4.

The evolution of the Universe is first and foremost determined by the development of the expansion factor $a(t)$, ie. the solution of the FRW equations. The expansion of the Universe is determined by its contents, the components of the Universe. Interestingly, the nature of their contribution is determined via their microphysical properties, ie. via their equation of state $p = p(\rho_{comp}) = w\rho_{comp}$.

a) How can you determine the evolution of the energy density ρ_{comp} ? Derive this relation (of energy conservation) from the FRW equations (ie. derive $\dot{\rho}$).).

b) What is the physical significance of this equation ? That is, what does it imply for the expansion of the Universe ?

c) Describe the various components of the Universe (matter, radiation, dark energy, curvature). Broadly speaking there are four different contributions, one of them geometric. For the non-geometric ones discuss their nature, identify the various ingredients of each contribution, and provide the expressions for their equation of state. What is the effect of dark energy on the acceleration $\ddot{a}$ of the Universe, and what does this imply for the value of w (referring to the FRW acceleration equation).

d) On the basis of the energy equation derive the evolution of the energy density ρ_{comp} as a function of expansion factor $a(t)$ for each component. For dark energy, give the solution in terms of general equation of state parameter w.

e) Write down the evolution of the total evolution of $\rho = \Omega H^2$ in terms of the present day values Ω_0 and H_0 of the various components and expansion factor $a(t)$.

f) As each component has a different evolution with expansion factor $a(t)$ we can identify different epochs in the evolution of the Universe. Discuss the different epochs and transitions for two options: 1) a non-Lambda Universe with radiation, matter and general curvature, 2) a Universe with radiation, matter and cosmological constant. If $\Omega_{r,0}$ is the present-day radiation density parameter, $\Omega_{m,0}$ the matter density and $\Omega_{v,0}$ the dark energy density parameter, derive the expression for the redshift of the transitions.

g) Write down the expression for the evolution of the Hubble parameter $H(t)$, in terms of the Ω contributions and expansion factor $a(t)$. Subsequently, derive the integral expression for the cosmological time $t(a)$ as function for a(t).

h) Derive the time evolution $a(t)$ for an Einstein-de Sitter Universe (matter-only, Ω_m=1), for a critical radiation only-Universe ($\Omega_{r,0} = 1$), for an empty Universe (ie. the free expansion phase for an underdense Universe) and for de Sitter expansion (ie. a Universe dominated by a cosmological constant).

Question 5.

Standard FRW cosmology has three important problems or "fine-tuning" issues. One of them is the monopole problem, created approximately one within the horizon at GUT transition and which should therefore crowd the present-day Universe. They do not ! Inflationary cosmology, stating that the Universe underwent a rapid de Sitter expansion at a very early phase, tries to explain why not.

The two additional problems of standard cosmology are the "flatness" problem and the "horizon" problem. Here we will elaborate on these:

a) For the specific case of a curved matter-dominated Universe, work out the evolution of Ω as a function of expansion factor a (or redshift z) for various possible Universes. What will be the value of Ω at $a(t) = 0.001$ (recombination epoch) if the present $\Omega_0 = 0.3$. And at $a(t) \approx 0.0001$ (around matter-radiation equivalence). In general, what do you expect Ω to be for $a(t) \to 0$. Include a sketch of the Ω evolution. Explain now what the flatness problem is.

b) What is the definition for the (particle) horizon in a FRW Universe. Give the expression for the horizon distance $d_{Hor}(t)$. How does the horizon distance evolve with time t in a radiation-dominated Universe (assume $\Omega_r = 1$), and in a matter-dominated Universe (assume $\Omega_m = 1$). How can it be that the horizon seems to grow faster than the velocity of light ? Subsequently, express these horizon distances in terms of the Hubble parameter H(t).

c) Work out the typical angular diameter θ of the horizon at recombination ($z_{dec} \approx 1000$).
   - You may use the following expression for the angular diameter distance D_a in a matter-dominated Universe for an object at redshift z:

$$d_A = \frac{2c}{H_0 \Omega_0^2} \frac{1}{(1+z)^2} \left\{ \Omega_0 z + (\Omega_0 - 2)\sqrt{1 + \Omega_0 z} - 1) \right\} \approx \frac{2c}{H_0 \Omega_0} \frac{1}{z} \text{ (for } z \gg 1)$$

   - if you are left with an equation including the Hubble parameter H(t), use the expression of the evolution of H(z) as function of redshift z (from FRW equation), approximating $(1+z) \approx z$ for $z \gg 1$ and using

$$\Omega_0 H_0^2 = \frac{8\pi G}{3} \rho_0$$

.

d) Given the impressive isotropy of the Cosmic Microwave Background, how does the derived value of $d_{Hor}(z_{dec})$ imply the "horizon" problem ?

e) Inflation is supposed to solve this situation. Explain how inflation does this. Specify the de Sitter expansion history during inflation, and how this may solve the three major Bib Bang problems.

f) Which physical mechanism is responsible for inflation. Provide a description of the inflation potential, relate this to the various stages of inflation.

SUCCES !!!!

BEDANKT VOOR JULLIE AANDACHT EN INTERESSE DIT KWARTAIR !!!!

Rien & Erwin