Problem 9.1

A particle of mass m is located in a one-dimensional potential $V(x) = \alpha|x|$ where α is a real positive constant. Use the variational method to establish an upper bound on the ground state energy of this system. Choose the class of trial wavefunctions yourself.

Problem 9.2

Consider a particle with mass m in a three-dimensional attractive Dirac delta-function potential

$$H_{3d} = \frac{\vec{p}^{\,2}}{2m} - \alpha\,\delta^3(\vec{r})$$

where $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$ and $\alpha > 0$ is a constant that characterizes the strength of the potential.

Use the variational principle with Gaussian trial wavefunctions having a real variational parameter L

$$\psi_L(\vec{r}) = Ae^{-r^2/2L^2} = Ae^{-(x^2+y^2+z^2)/2L^2}$$

to demonstrate that the ground state energy of H_{3d} is not finite. H_{3d} is therefore unphysical. You might amuse yourself (not required) with repeating this for attractive delta-function potentials in two and one dimensions. It turns out that the two-dimensional delta-function potential is also unphysical. This is in contrast to the one-dimensional attractive delta-function potential which has a finite ground state energy.

Problem 9.3

A problem of importance in astrophysics is whether or not a single proton can bind two electrons. This question can be addressed by the variational method using the trial wavefunctions

$$\psi(\vec{r}_1, \vec{r}_2) = A\left[\psi_1(r_1)\psi_2(r_2) + \psi_2(r_1)\psi_1(r_2)\right], \qquad \psi_i(r) = \sqrt{\frac{Z_i^3}{\pi a_0^3}}e^{-Z_i r/a_0}, \quad i \in 1, 2$$

where a_0 is the Bohr radius and Z_1 and Z_2 are the variational parameters. (The spin part is omitted.) Using this class of trial wavefunctions one can show that the trial energy in terms of $x = Z_1 + Z_2$ and $y = 2\sqrt{Z_1 Z_2}$ is

$$E_{tr} = \frac{E_1}{x^6 + y^6}\left(-x^8 + 2x^7 + \frac{1}{2}x^6y^2 - \frac{1}{2}x^5y^2 - \frac{1}{8}x^3y^4 + \frac{11}{8}xy^6 - \frac{1}{2}y^8\right)$$

where $E_1 = -13.6eV$. Minimize E_{tr} numerically with respect to Z_1 and Z_2 and determine if a proton and two electrons can form a bound state.