Problem 10.1a)

Since $\omega \sim \sqrt{k}$, the new energies are trivially

$$\tilde{E} = (n + \frac{1}{2})\hbar\omega\sqrt{1+\epsilon}. \tag{1}$$

Expanding $\sqrt{1+\epsilon} = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + ...$ we thus have

$$\tilde{E} = E_n \left(1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + ...\right). \tag{2}$$

Problem 10.1b)

Next, find the first order correction from the perturbation expression

$$E_n^1 = \langle n|H'|n\rangle \tag{3}$$

From $H = \frac{p^2}{2m} + \frac{1}{2}kx^2 \to \frac{p^2}{2m} + \frac{1}{2}(1+\epsilon)kx^2$ we read off the perturbation

$$H' = \epsilon \cdot \frac{1}{2}kx^2 = \epsilon V \tag{4}$$

where V is the potential term of the harmonic oscillator Hamiltonian. Thus,

$$E_n^1 = \epsilon\langle n|V|n\rangle \tag{5}$$

But we know (recall problems 2.11, 2.12, and the virial theorem discussed in problem 3.31) that for the harmonic oscillator, $\langle n|T|n\rangle = \langle n|V|n\rangle = E_n/2$. This gives us

$$E_n^1 = \epsilon\frac{1}{2}E_n \tag{6}$$

which is the same as the order ϵ correction in Eq.(2)

Problem 10.2a)

The unperturbed system has two eigenstates, $|\uparrow\rangle_z \equiv |1\rangle$ and $|\downarrow\rangle_z \equiv |2\rangle$, with eigenvalues given by

$$\begin{aligned} H_0\,|1\rangle &= -\frac{B}{\hbar}S^z\,|1\rangle = -\frac{B}{\hbar}\frac{\hbar}{2}\,|1\rangle = -\frac{B}{2}\,|1\rangle & (7)\\ H_0\,|2\rangle &= +\frac{B}{2}\,|2\rangle\,. & (8)\end{aligned}$$

In order to compute the first order corrections to these, we need to recall how S^x acts on the eigenstates of S^z:

$$\begin{aligned} S^x\,|1\rangle &= \frac{\hbar}{2}\,|2\rangle & (9)\\ S^x\,|2\rangle &= \frac{\hbar}{2}\,|1\rangle\,. & (10)\end{aligned}$$

Thus,

$$E_1^{(1)} = \langle 1|\,H'\,|1\rangle = -\frac{g}{2}\,\langle 1|\,S^x\,|1\rangle = -\frac{g}{2}\,\langle 1|2\rangle = 0 \tag{11}$$

and similarly, $E_2^{(2)} = 0$. It can be seen from the Hamiltonian that B and g must have the same physical dimensions, and we can write the Hamiltonian in the suggestive form

$$H = B\left(-\frac{S^z}{\hbar} - \frac{g}{B}\frac{S^x}{\hbar}\right) \tag{12}$$

where the expression inside the brackets is dimensionless. The dimensionless parameter characterizing the perturbative expansion is thus g/B.

Problem 10.2b)

The second order correction to the n'th eigenenergy is in general

$$E_n^{(2)} = \sum_{m\neq n} \frac{|\langle m| H' |n\rangle|^2}{E_n^{(0)} - E_m^{(0)}}. \tag{13}$$

But here we only have two states, so the sums vanish, and we get

$$\begin{aligned} E_1^{(2)} &= \frac{|\langle 2| H' |1\rangle|^2}{E_1^{(0)} - E_2^{(0)}} && (14)\\ &= \frac{|\langle 2| (-gS^x/\hbar) |1\rangle|^2}{-B} && (15)\\ &= -\left(\frac{g}{\hbar B}\frac{\hbar}{2}\right)^2 |\langle 2|2\rangle|^2 && (16)\\ &= -\left(\frac{g}{B}\right)^2 \frac{B}{4}. && (17) \end{aligned}$$

A completely analogous calculation gives

$$E_2^{(2)} = +\left(\frac{g}{B}\right)^2 \frac{B}{4}. \tag{18}$$

Problem 10.2c)

The general expression for first order correction to the eigen*states* is

$$|n\rangle^{(1)} = \sum_{m\neq n} \frac{\langle m| H' |n\rangle}{E_n^{(0)} - E_m^{(0)}} |m\rangle \tag{19}$$

$$\tag{20}$$

Again, in our case the sum reduces to a single term for each eigenstate,

$$|1\rangle^{(1)} = \frac{\langle 2| H' |1\rangle}{E_1^{(0)} - E_2^{(0)}} |2\rangle = \frac{-g/\hbar \langle 2| S^x |1\rangle}{-B} |2\rangle = \frac{g}{2B} |2\rangle. \tag{21}$$

Similarly,

$$|2\rangle^{(1)} = -\frac{g}{2B} |1\rangle. \tag{22}$$

Problem 10.3a)

The unperturbed ground state is $|0\rangle \otimes |0\rangle$ i.e. $n_x = n_y = 0$, so that the ground state energy is $\hbar\omega$. The first excited state is two-fold degenerate: Either $n_x = 1, n_y = 0$ or vice versa, corresponding to the states $|1\rangle \otimes |0\rangle$ and $|0\rangle \otimes |1\rangle$, respectively. They both have energy $2\hbar\omega$.

Problem 10.3b)

The first-order correction to the ground state is

$$E_0^{(1)} = \langle GS| \hat{H}' |GS\rangle = \langle 0| \otimes \langle 0| g\hat{x}\hat{y} |0\rangle \otimes |0\rangle \tag{23}$$

One may solve this problem by working with creation and annihilation operators (dropping hats throughout the problem), and start by recalling the relation

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger). \tag{24}$$

Introducing corresponding operators for y,

$$y = \sqrt{\frac{\hbar}{2m\omega}}(b + b^\dagger) \tag{25}$$

we have

$$E_0^{(1)} = \frac{g\hbar}{2m\omega} \langle 0| (a + a^\dagger) |0\rangle \langle 0| (b + b^\dagger) |0\rangle = 0. \tag{26}$$

So, no change in the ground state energy.

Problem 10.3c)

Now for the perturbative correction to the first excited state. Not having selected the "good combinations" of the two degenerate states, we do a brute force degenerate perturbation theory calculation,

$$E_\pm^{(1)} = \frac{1}{2}\left[W_{aa} + W_{bb} \pm \sqrt{(W_{aa} - W_{bb})^2 + 4|W_{ab}|^2}\right] \tag{27}$$

where we use the notation of Griffiths, i.e.

$$W_{ij} = \left\langle \psi_i^{(0)} \middle| H' \middle| \psi_j^{(0)} \right\rangle \tag{28}$$

with $\psi_a^{(0)} = |1\rangle \otimes |0\rangle$, $\psi_b^{(0)} = |0\rangle \otimes |1\rangle$, and

$$H' = \frac{g\hbar}{2m\omega}(a + a^\dagger)(b + b^\dagger). \tag{29}$$

It is easily seen that

$$W_{aa} = \frac{g\hbar}{2m\omega} \langle 1| (a + a^\dagger) |1\rangle \langle 0| (b + b^\dagger) |0\rangle = 0, \tag{30}$$

and similarly for W_{bb}. Thus we are left with

$$E_\pm^{(1)} = \pm |W_{ab}| \tag{31}$$

where

$$W_{ab} = \frac{g\hbar}{2m\omega} \langle 1| (a + a^\dagger) |0\rangle \langle 0| (b + b^\dagger) |1\rangle = \frac{g\hbar}{2m\omega}. \tag{32}$$

Thus

$$E_\pm^{(1)} = \pm \frac{g\hbar}{2m\omega}. \tag{33}$$

Problem 10.3d)

Eigenstates of R with energy $2\hbar\omega$ must be linear combinations of the degenerate eigenstates $|1\rangle\otimes|0\rangle$ and $|0\rangle\otimes|1\rangle$. By inspection, the (normalized) linear combinations that are eigenstates of R, are

$$|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle\otimes|0\rangle \pm |0\rangle\otimes|1\rangle\right) \tag{34}$$

with eigenvalues ± 1.

Problem 10.3e)

R commutes with the full Hamiltonian H, since both the 2D harmonic oscillator Hamiltonian and the perturbation $H' = gxy$ are invariant under interchange of x and y. So we have found the "good" states and can use them in doing ordinary (non-degenerate) perturbation theory:

$$\begin{aligned}
E_\pm^{(1)} &= \langle\psi_\pm|\,H'\,|\psi_\pm\rangle && (35)\\
&= \frac{g\hbar}{2m\omega}\,\langle\psi_\pm|\,(a+a^\dagger)(b+b^\dagger)\,|\psi_\pm\rangle && (36)\\
&= \frac{g\hbar}{4m\omega}\left[(\langle 1|\otimes\langle 0| \pm \langle 0|\otimes\langle 1|)(a+a^\dagger)(b+b^\dagger)(|1\rangle\otimes|0\rangle \pm |0\rangle\otimes|1\rangle)\right] && (37)\\
&= \pm\frac{g\hbar}{2m\omega}. && (38)
\end{aligned}$$

(Note that the last step of the calculation is essentially just the expression we computed when finding W_{ab}). Thus, as expected, we reproduce the result of **10.3.c)**.