UNIVERSITY OF OSLO
FACULTY OF MATHEMATICS AND NATURAL SCIENCES

Exam in: FYS3110, Quantum mechanics

Day of exam: Dec. 7. 2012

Exam hours: 14:30-18:30 (4 hours)

This examination paper consists of 4 pages.

Permitted material: Approved calculator, D.J. Griffiths: "Introduction to Quantum Mechanics", the printed notes: "Time evolution of states in quantum mechanics", "Symmetry and degeneracy" and "WKB connection formulae", one handwritten A4-sheet(2 pages) with your own notes, and K. Rottmann: "Matematisk formelsamling"

Check that the problem set is complete before you start working. Some subproblems have more than one question.

Problem 1

1.1 Define what is meant by a hermitian operator. What are the properties of hermitian operators which make them suitable for representing physical observables. Explain.

Consider a particle with mass m in a one dimensional double-well potential, Fig. 1 left. At low energies, $E \ll V_0$, the double-well potential can be approximated by two

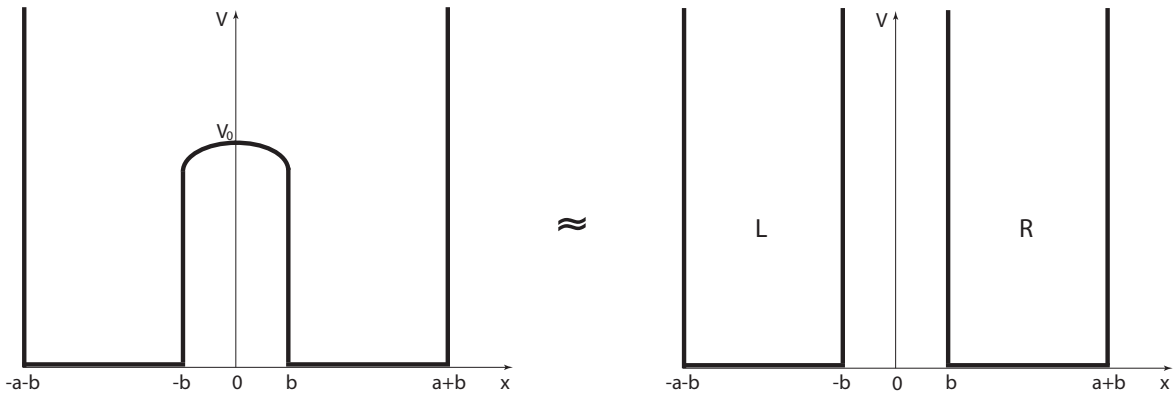

Figure 1: A double-well potential(left) approximated as two infinite-well potentials(right)

infinite potential wells, Fig. 1 right, with Hamiltonian

$$H_0 = \frac{p^2}{2m} + V(x), \qquad V(x) = \begin{cases} 0, & x \in [-a-b, -b] \cup [b, a+b] \\ \infty, & \text{otherwise.} \end{cases}$$

Label the energy eigenstates of H_0 by $|n, i\rangle$ where n labels the n'th energy eigenvalue in well i, and $i \in \{L, R\}$ labels the L(eft) and R(ight) well respectively. The energy eigenvalues of H_0 depend only on n, not on i,

$$H_0|n, i\rangle = E_n|n, i\rangle$$

where $E_n = (n\pi\hbar)^2/(2ma^2)$, $n \in \{1, 2, \ldots\}$.

1.2 What is the energy and degeneracy of the lowest energy level of H_0?

To go beyond the approximation H_0, we take into account tunneling of particles between the wells. To model this we add a term

$$H' = -\sum_{nm} \left(g_{nm}|n, L\rangle\langle m, R| + h_{nm}|n, R\rangle\langle m, L|\right)$$

to H_0.

1.3 We require that H' is hermitian. This implies a relation between g_{nm} and h_{nm}. Find this relation.

In the following let g_{nm} be real and positive.

1.4 Use the variational method with variational states $A_k\left(|1, L\rangle + k|1, R\rangle\right)$ where k is a real variational parameter (A_k is a normalization constant) to find a lowest upper bound on the ground state energy of $H_0 + H'$.

The double-well potential is symmetric under the operation P: $x \to -x$ (parity).

1.5 Write down explicit expressions for the position representation $\psi_{n,i}(x)$ of the states $|n, i\rangle$ and work out how they behave under the operation P. (Hint: The energy eigenfunctions of a mass m particle in an infinite potential well of width a centered on $a/2$

are $\psi_n(x) = \sqrt{\frac{2}{a}}\sin(n\pi x/a)$ for $x \in [0, a]$. Outside this interval $\psi_n(x) = 0$.)

1.6 Find the first order perturbative energy corrections to the lowest energy level of H_0 using H' as a perturbation.

1.7 Three spin-1/2 non-interacting identical fermions each of mass m are located in the double-well potential. Use the results from problem 1.6 to find the lowest possible total energy of these three particles. What is the total spin of the three particles in this lowest energy state?

Problem 2

The hyperfine interaction in Hydrogen is an interaction between the proton spin and the electron spin. For the spin states with $n = 1$, $l = 0$ the interaction in the spin space of the proton and the electron takes the form

$$H_{hf} = \frac{\kappa}{\hbar^2}\,\vec{S}_p \cdot \vec{S}_e$$

where κ is a positive constant and $\vec{S}_p$ and $\vec{S}_e$ are the spin-1/2 operators of the proton and electron respectively. This interaction splits the ground state energy level of Hydrogen into two very closely spaced energy levels with energy difference $\Delta E = 5.88 \cdot 10^{-6}$eV.

2.1 Use H_{hf} to determine the energy splitting of the $n = 1$, $l = 0$ hyperfine levels in terms of κ. What is the degeneracy of each energy level after the splitting?

The transition between two $n = 1, l = 0$ hyperfine levels is a "forbidden" transition. The leading decay mechanism is not the electric but the *magnetic* dipole interaction

$$H_B = -B_0 \cos(\omega t)\hat{k} \cdot (\vec{\mu}_p + \vec{\mu}_e).$$

Here the proton magnetic moment $\vec{\mu}_p = g_p \frac{e}{2m_p}\vec{S}_p$ where $g_p \approx 5.59$, m_p is the proton mass and the electron magnetic moment is $\vec{\mu}_e = -g_e \frac{e}{2m}\vec{S}_e$ where $g_e \approx 2.00$ and m is the electron mass. B_0, $\hat{k}$ and ω are the amplitude, polarization and frequency of a time-varying external magnetic field.

2.2 Generalize the electric dipole formula for the lifetime τ due to *spontaneous* decay

$$\tau = 1/A, \quad A = \frac{\omega_0^3}{3\pi\epsilon_0\hbar c^3}\left(|\mathcal{P}_x|^2 + |\mathcal{P}_y|^2 + |\mathcal{P}_z|^2|\right), \quad \vec{\mathcal{P}} = (-e)\langle\psi_f|\vec{r}|\psi_i\rangle$$

via the electric dipole interaction

$$H_E = -E_0\cos(\omega t)\hat{k}\cdot(-e\vec{r})$$

to a similar lifetime formula valid when the decay is governed by the magnetic dipole interaction H_B. In order to do this use the fact that the energy density of the vacuum $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$ is equally distributed on the electric and magnetic parts such that $u = \epsilon_0 E^2 = \frac{1}{\mu_0}B^2$.

Use your result to calculate the lifetime of one of the hyperfine triplet states, give your answer in units of years. You may use the relations and the (approximate) numerical values listed below.

$\epsilon_0\mu_0 = 1/c^2,$	$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} = 1/137,$	$mc^2\alpha^2 = 2\cdot 13.6eV,$
$m_p/m = 1836,$	$c = 3.0\cdot 10^8 m/s,$	$\hbar = 6.582\cdot 10^{-16}eVs.$

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