12

When there is only one decay channel the lifetime is inversely proportional to the spontaneous decay rate A:

$$\tau = 1/A$$

The spontaneous decay rate due to the electric dipole interaction is given by

$$A = \frac{\omega_0^3 \left(|\mathcal{P}_x|^2 + |\mathcal{P}_y|^2 + |\mathcal{P}_z|^2\right)}{3\pi\epsilon_0\hbar c^3},$$

where the electric dipole matrix elements are

$$\begin{aligned}
\mathcal{P}_x &= (-e)\langle\psi_a|x|\psi_b\rangle \\
\mathcal{P}_y &= (-e)\langle\psi_a|y|\psi_b\rangle \\
\mathcal{P}_z &= (-e)\langle\psi_a|z|\psi_b\rangle
\end{aligned}$$

where $|\psi_b\rangle$ describes the initial state and $\psi_a\rangle$ is the final state. $\hbar\omega_0$ is the (positive) energy difference between the initial and the final state. In the case considered here $|\psi_b\rangle = |210\rangle$, and the final state $|\psi_a\rangle = |100\rangle$ which follows from the fact that it is the only state with lower energy. Expressing the coordinates as $x = r\sin\theta\cos\phi$,$y = r\sin\theta\sin\phi$, $z = r\cos\theta$ we can calculate the matrix elements. These calculations involve an integral over r,θ and ϕ. Being $m = 0$ the states have no ϕ-dependence. Therefore the matix elements $\mathcal{P}_x$,$\mathcal{P}_y$ involve an integration over $\cos\phi$,$\sin\phi$ which is zero. Thus only $\mathcal{P}_z$ is non-zero Using

$$\begin{aligned}
\langle r,\theta,\phi|100\rangle &= \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \\
\langle r,\theta,\phi|210\rangle &= \frac{1}{\sqrt{32\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0}\cos\theta
\end{aligned}$$

$$\begin{aligned}
\mathcal{P}_z &= (-e)\langle 100|r\cos\theta|210\rangle \\
&= (-e)\int_0^\infty dr r^2 \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} r\cos\theta \frac{1}{\sqrt{32\pi a_0^3}} \frac{r}{a_0} e^{-r/2a_0}\cos\theta \\
&= (-e)\frac{2\pi}{\sqrt{32\pi}}\sqrt{\frac{1}{\pi}} \int_0^\infty dr \left(\frac{r}{a_0}\right)^4 e^{-3r/2a_0} \int_{-1}^1 d(\cos\theta)\cos^2\theta \\
&= (-e)\frac{1}{\sqrt{8}}\frac{a_0 2^5}{3^5}\frac{2}{3}\int_0^\infty dx x^4 e^{-x} \\
&= (-e)\frac{2^5}{\sqrt{2}}\frac{a_0}{3^6}4! \\
&= (-ea_0)\frac{2^7\sqrt{2}}{3^5}
\end{aligned}$$

Inserting into the expression for A we get

$$A = \frac{\omega_0^3}{3\pi\epsilon_0\hbar c^3} e^2 a_0^2 \frac{2^{15}}{3^{10}} = \frac{(\hbar\omega_0)^3}{\hbar^3}\left(\frac{a_0}{c}\right)^2 \frac{e^2}{4\pi\epsilon_0\hbar c}\frac{4}{3}\frac{2^{15}}{3^{10}}$$

Now the energy difference $\hbar\omega_0$ between the $n = 2$ and the $n = 1$ levels is $\hbar\omega_0 = 13.6eV\,(1 - 1/4) = (3/4)\cdot 13.6eV$. Using $\hbar = 6.582\cdot 10^{-16}eV$, $c = 3\cdot 10^8 m/s$ and $a_0 = 0.5\cdot 10^{-10}m$, $\frac{e^2}{4\pi\epsilon_0\hbar c} \approx 1/137$ we get

$$\begin{aligned}
A &= \left(\frac{(3/4)\cdot 13.6eV}{6.582\cdot 10^{-16}eVs}\right)^3 \left(\frac{0.529\cdot 10^{-10}m}{3\cdot 10^8 m/s}\right)^2 \frac{1}{137}\frac{4}{3}\frac{2^{15}}{3^{10}} \\
&= \left(\frac{13.6eV}{6.582\cdot 10^{-16}eVs}\right)^3 \left(\frac{0.529\cdot 10^{-10}m}{3\cdot 10^8 m/s}\right)^2 \frac{1}{137}\frac{2^{11}}{3^8} = 6.3\cdot 10^8 s^{-1}
\end{aligned}$$

which gives a lifetime of

$$\tau = 1/A = 1.6\cdot 10^{-9}s.$$