UNIVERSITY OF OSLO
FACULTY OF MATHEMATICS AND NATURAL SCIENCES

Exam in: FYS3110, Quantum mechanics

Day of exam: Dec. 6. 2010

Exam hours: 14:30-18:30 (4 hours)

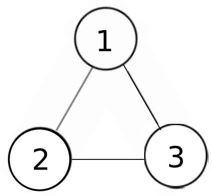
This examination paper consists of 3 pages.

Permitted material: Approved calculator, D.J. Griffiths: "Introduction to quantum mechanics", the handouts: "Time evolution of states in quantum mechanics" and "Symmetry and degeneracy", one handwritten A4-sheet (2 pages) with your own notes, and K. Rottmann: "Matematisk formelsamling".

Check that the problem set is complete before you start working.

Problem 1.

A spinless particle is located on one of the atoms of a three-atom molecule, see figure. The particle can jump from one atom to another. Do not consider other degrees of freedom than the position of the particle in this problem.



The Hamiltonian is

$$H = -g\left(|2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3| + |1\rangle\langle 2| + |2\rangle\langle 3| + |3\rangle\langle 1|\right)$$

where $|i\rangle$ is the state where the particle is located on atom number i. The states $|i\rangle$, $i \in \{1, 2, 3\}$ form an orthonormal set. g is a positive real number with dimension energy.

1.1 Write down a matrix representation for H. State explicitly the basis you are using.

The operator R is

$$R = |2\rangle\langle 1| + |3\rangle\langle 2| + |1\rangle\langle 3|$$

1.2 Is R Hermitean? Unitary? Does R describe a symmetry-transformation? Give reasons for your answers..

1.3 Show that $R^3 = I$, where I is the identity operator, and use this to find the eigenvalues of R. Then find the corresponding eigenstates of R. What are the eigenstates and eigenvalues of H?

1.4 The particle is located on atom 2 at time $t = 0$. Compute the probability of finding the particle on the same atom 2 at a later time t.

Problem 2

A particle with mass m is in a three-dimensional harmonic oscillator potential characterized by a frequency ω. The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + \frac{1}{2}m\omega^2\left(x^2 + y^2 + z^2\right).$$

where p_i is the momentum operator in the i-direction ($i \in \{x, y, z\}$). This Hamiltonian can be viewed a composite system of three independent one-dimensional harmonic oscillators, each with ladder operators a_i, $a_i^\dagger$, $i \in \{x, y, z\}$. That is, one one-dimensional oscillator for each direction x, y, z in space. In this problem you should use the basis states $|n_x n_y n_z\rangle$ ($\equiv |n_x\rangle \otimes |n_y\rangle \otimes |n_z\rangle$) where n_i is the eigenvalue for $a_i^\dagger a_i$ of the i'th one-dimensional oscillator ($i \in \{x, y, z\}$).

2.1 Express the Hamiltonian H using ladder operators for the three one-dimensional oscillators. Write down the energies and the degree of degeneracy for the *three* lowest energy levels. Assume here that the particle is spinless.

2.2 Express the z-component of the angular momentum operator $L_z = xp_y - yp_x$ using ladder operators for the one-dimensional oscillators.

The square of the total angular momentum operator can be written

$$L^2 = \hbar^2 \left[(n_x + n_y + n_z)(n_x + n_y + n_z + 1) - (a_x^{\dagger 2} + a_y^{\dagger 2} + a_z^{\dagger 2})(a_x^2 + a_y^2 + a_z^2) \right]$$

where $n_i = a_i^\dagger a_i$ og $a_i, a_i^\dagger$ $(i \in \{x, y, z\})$ are ladder operators for the one-dimensional oscillators.

2.3 Construct a common set of eigenstates for H, L^2 og L_z for each of the *two* lowest enegy levels. Give the quantum number l of the total angular momentum and the quantum number m of its z-component for all these eigenstates.

2.4 Assume here (2.4) that the particle in the oscillator potential has spin-1/2 and that H is being perturbed with an additional term $H_{so} = \beta \vec{L} \cdot \vec{S}$ where $\vec{S}$ is the spin-operator of the particle and β is a constant. Use perturbation theory to compute the splitting of the first excited energy level valid to first order in β. Compute both the energies and the degree of degeneracy of the new energy levels.

In the rest of this exam: $\beta = 0$.

2.5 Assume here (2.5) that *two* identical spin-1/2 fermions are present in the three-dimensional oscillator potential and that these have the lowest possible energy. The particles interact with each other via a contact interaction potential $V_k = \alpha\delta(x_1 - x_2)\delta(y_1 - y_2)\delta(z_1 - z_2)$ where $\delta(x)$ is the Dirac delta-function and α is a constant with dimension energi $\times$ volume. x_1 is the x-coordinate of particle 1 etc. Compute an expression for the total energy of the two particles valid to first order in α. Hint: Use the coordinate representasjon of the state.
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