Problem 11.1

a) The Feynman-Hellman theorem states that

$$\frac{\partial E_n}{\partial \beta} = \langle \psi_n | \frac{\partial H}{\partial \beta} | \psi_n \rangle. \tag{1}$$

Use ω as our parameter β. Since, for a harmonic oscillator, $E_n = (n + \frac{1}{2})\hbar\omega$, the Feynman-Hellman theorem thus gives

$$\frac{\partial E_n}{\partial \omega} = (n + \frac{1}{2})\hbar = \langle \psi_n | \frac{\partial H}{\partial \omega} | \psi_n \rangle = \langle \psi_n | m\omega x^2 | \psi_n \rangle. \tag{2}$$

Dividing by $m\omega$ we thus get

$$\langle \psi_n | x^2 | \psi_n \rangle = (n + \frac{1}{2}) \frac{\hbar}{m\omega}. \tag{3}$$

c) Now use $\beta = m$ and note $\frac{\partial E_n}{\partial m} = 0$. Thus

$$\frac{\partial E_n}{\partial m} = 0 = \langle \psi_n | \frac{\partial H}{\partial m} | \psi_n \rangle = \langle \psi_n | \left(-\frac{p^2}{2m^2} + \frac{1}{2}\omega^2 x^2 \right) | \psi_n \rangle. \tag{4}$$

This gives

$$\langle \psi_n | \frac{p^2}{2m} | \psi_n \rangle = \langle \psi_n | \frac{1}{2} m\omega^2 x^2 | \psi_n \rangle \tag{5}$$

i.e. $\langle T \rangle = \langle V \rangle$.

Problem 11.2

We start by rewriting the potential as

$$V(r) = \frac{e}{4\pi\epsilon_0 r} + \Theta(b - r) \left[\frac{e}{4\pi\epsilon_0 b} \left(\frac{3}{2} - \frac{r^2}{2b^2} \right) - \frac{e}{4\pi\epsilon_0 r} \right] \equiv \frac{e}{4\pi\epsilon_0 r} + V'(r) \tag{6}$$

where $\Theta(x)$ is the Heaviside step function. The first term is the unperturbed potential (corresponding to a point particle), while $H'(r) = -eV(r)$ is the perturbation. In order to compute the first order correction to the ground state energy, we need the ground state wave function,

$$\psi_{100}(r, \phi, \theta) = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0} \tag{7}$$

with

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{me^2}. \tag{8}$$

The first order correction is then, noting that the integrals over θ and ϕ are trivial,

$$E^{(1)} = \int_0^\infty 4\pi r^2 \left| \psi_{100} \right|^2 (-eV'(r)) dr. \tag{9}$$

The step function cuts the upper limit of integration to b, and we get

$$\begin{aligned} E^{(1)} &= 4 \frac{e^2}{4\pi\epsilon_0} \frac{1}{a_0^3} \int_0^b dr\, r^2 \left[\frac{1}{r} - \frac{1}{b} \left(\frac{3}{2} - \frac{r^2}{2b^2} \right) \right] e^{-2r/a_0} && (10) \\ &= 4 \frac{e^2}{4\pi\epsilon_0} \frac{1}{a_0} \int_0^{b/a_0} dx\, x^2 \left[\frac{1}{x} - \frac{1}{b/a_0} \left(\frac{3}{2} - \left(\frac{a_0}{b} \right)^2 \frac{x^2}{2} \right) \right] e^{-2x} && (11) \end{aligned}$$

where we have changed variables to $x = r/a_0$. Now, we know that $b \ll a_0$, and in the whole range of integration, $x \leq b/a_0$ i.e. $x \ll 1$. Therefore, to leading order, we write $e^{-2x} \approx 1$. This simplifies the integral to

$$\begin{aligned} E^{(1)} &\approx 4\,\frac{e^2}{4\pi\epsilon_0 a_0}\left\{\int_0^{b/a_0} x\,dx - \frac{1}{(b/a_0)^2}\frac{3}{2}\int_0^{b/a_0} x^2\,dx + \frac{1}{(b/a_0)^3}\frac{1}{2}\int_0^{b/a_0} x^4\,dx\right\} && (12)\\ &= \frac{e^2}{4\pi\epsilon_0 a_0}\,\frac{2}{5}\left(\frac{b}{a_0}\right)^2. && (13)\end{aligned}$$

Now, recall that the unperturbed ground state energy is

$$E^{(0)} = -\frac{1}{2}\frac{e^2}{4\pi\epsilon_0 a_0} \tag{14}$$

so that the relative correction is $\sim (b/a_0)^2 \approx 10^{-10}$. For comparison, the relative correction due to fine structure is of the order $\alpha^2 \approx 10^{-4}$, while that due to hyperfine structure is $\sim (m_e/m_p)\alpha^2 \approx 10^{-7}$ (see table in Griffith page 267). Thus, our finite size correction is considerably smaller than both of these.

Assuming the proton has all its charge in the surface will change the potential for $r \leq b$ to the constant $e/(4\pi\epsilon_0 b)$, while the $r > b$ part is unchanged. The integrals for $E^{(1)}$ will thus be very similar to the above, except for the factor 3/2, and that the x^4 integral drops out. This only changes the prefactor of the final answer, from 2/5 to 2/3. In other words, for estimating the order of magnitude of the energy correction due to the finite size of the proton, it is not important how we model the charge distribution inside the proton.

Problem 11.3

First we need to identity the eight possible states $|2, l, j, m_j\rangle$. Since $n = 2$, l can be 0 or 1. For $l = 0$ we must have $j = 1/2$ with $m_j = \pm 1/2$. For $l = 1$, j is either 1/2 (with $m_j = \pm 1/2$) or 3/2 (with $m_j = \pm 3/2, \pm 1/2$). Thus, we label the states as follows:

$$\begin{aligned} |1\rangle &= \left|2, 0, \frac{1}{2}, \frac{1}{2}\right\rangle && (15)\\ |2\rangle &= \left|2, 0, \frac{1}{2}, -\frac{1}{2}\right\rangle && (16)\\ |3\rangle &= \left|2, 1, \frac{1}{2}, \frac{1}{2}\right\rangle && (17)\\ |4\rangle &= \left|2, 1, \frac{1}{2}, -\frac{1}{2}\right\rangle && (18)\\ |5\rangle &= \left|2, 1, \frac{3}{2}, \frac{3}{2}\right\rangle && (19)\\ |6\rangle &= \left|2, 1, \frac{3}{2}, \frac{1}{2}\right\rangle && (20)\\ |7\rangle &= \left|2, 1, \frac{3}{2}, -\frac{1}{2}\right\rangle && (21)\\ |8\rangle &= \left|2, 1, \frac{3}{2}, -\frac{3}{2}\right\rangle && (22)\end{aligned}$$

We need to calculate the following energies [6.76]

$$E = E_{nj} + \mu_B B_{ext} \cdot g_J m_j \tag{23}$$

where

$$E_{nj} = -\frac{13.6eV}{n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right] \tag{24}$$

is the fine structure energy [6.67], and

$$g_J = \frac{1 + j(j+1) - l(l+1) + 3/4}{2j(j+1)} \tag{25}$$

as given in [6.75]. First note that states $|1\rangle - |4\rangle$ all have $E_{nj} = -3.4eV(1 + \frac{5}{16}\alpha^2) \equiv E_A$, while states $|4\rangle - |8\rangle$ have $E_{nj} = -3.4eV(1 + \frac{1}{16}\alpha^2) \equiv E_B$. To find the slopes of the Zeeman splitting we note that states $|1\rangle$ and $|2\rangle$ have $g_J = 2$, $|3\rangle$ and $|4\rangle$ have $g_J = 2/3$, while the remaining four states have $g_J = 4/3$. Combining all of this we get

$$\begin{aligned}
E_1 &= E_A + \mu_B B_{ext} && (26)\\
E_2 &= E_A - \mu_B B_{ext} && (27)\\
E_3 &= E_A + \frac{1}{3}\mu_B B_{ext} && (28)\\
E_4 &= E_A - \frac{1}{3}\mu_B B_{ext} && (29)\\
& && (30)\\
E_5 &= E_B + 2\mu_B B_{ext} && (31)\\
E_6 &= E_B + \frac{2}{3}\mu_B B_{ext} && (32)\\
E_7 &= E_B - \frac{2}{3}\mu_B B_{ext} && (33)\\
E_8 &= E_B - 2\mu_B B_{ext} && (34)\\
& && (35)
\end{aligned}$$