PROBABILITY AND STATISTICS - P&P 1

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Problem 1. .

Proposition. *let Ω be a finite sample space and $A \subseteq \Omega$ then $P(A) = \frac{|A|}{|\Omega|}$ is a probability function*

Proof. We need to show that $P(A) := \frac{|A|}{|\Omega|}$ satisfy the probability function defenition

1. let $A \subseteq \Omega$. if $A = \emptyset$ then $\frac{|A|}{|\Omega|} = \frac{0}{|\Omega|} = 0$ and otherwise $\frac{|A|}{|\Omega|} > 0$.
2. $P(\Omega) = \frac{|\Omega|}{|\Omega|} = 1$
3.

$$P(A \cup B) = \frac{|A \cup B|}{|\Omega|} \underset{*}{=} \frac{|A| + |B|}{|\Omega|} = \frac{|A|}{|\Omega|} + \frac{|B|}{|\Omega|} = P(A) + P(B)$$

*:if A, B disjoint then :$|A \cup B| = |A| + |B|$. The other direction is the same. □

Problem 2. Three ants are sitting at the three corners of an equilateral triangle. Each ant randomly picks a direction and starts to move along the edge of the triangle (each direction has equal probability). What is the probability that none of the ants collide?

Answer: For each ant the set of outcomes are $\{R, L\}$ with R, L denotes turning right an left respectively. The sample space $\Omega = \{R, L\}^3$. The set of outcomes in which the ants dont collide is $\{R, R, R\}$ and $\{L, L, L\}$. As stated in the problem, each direction has equal probability, and thus we can use the (uniform) probability function from problem 1:

$$P(\{R, R, R\} \cap \{L, L, L\}) = \frac{|\{R, R, R\}| + |\{L, L, L\}|}{|\Omega|} = \frac{2}{2^3} = \frac{1}{4}$$

Problem 3. Out of the students in a class, 60% are geniuses ($P(A) = 0.6$), 70% love chocolate ($P(B) = 0.7$), and 40% fall into both categories ($P(A \cap B) = 0.4$). Determine the probability that a randomly selected student is neither genius nor a chocolate lover ($P(A^c \cap B^c)$).

Answer:

$$\begin{aligned} P(A^c \cap B^c) &\underset{1}{=} 1 - P((A^c \cap B^c)^c) \\ &\underset{2}{=} 1 - P(A \cup B) \\ &\underset{3}{=} 1 - (P(A) + P(B) - P(A \cap B)) \\ &= 1 - (0.6 + 0.7 - 0.4) \\ &= 0.1 \end{aligned}$$

1: $P(X) = 1 - P(X^c)$; 2: De-Morgan;

3: $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$

Problem 4. The probability of at least one car passing a certain road inter-section in a 20-minute window is 0.9. What is the probability of at least one car passing the intersection in a 5-minute window, assuming a constant probability throughout?

Answer: It is easier to find the compliment where we caclulate the probability where no car is passing in the road. We can write that the probability of no car passing in a 20-minute window is $P(t > 20) = 1 - P(t \leq 20) = 0.9$ hence:

$$\begin{aligned}
0.1 &= P(t \leq 20) \\
&= P(t \leq 5 \cap 5 < t \leq 10 \cap 10 < t \leq 15 \cap 15 < t \leq 20) \\
&\underset{1}{=} P(t \leq 5)\, P(5 < t \leq 10)\, P(10 < t \leq 15)\, P(15 < t \leq 20) \\
&\underset{2}{=} P(t \leq 5)^4 \\
&\iff
\end{aligned}$$

$$P(t \leq 5)^4 = 0.1 \iff P(t \leq 5) = 0.1^{\frac{1}{4}} = 0.56$$

1: $P(\cdot)$ is constant thus each part in the hour is independent. 2: $P(\cdot)$ we can write $P(t \leq 5) = P(5 < t \leq 10) = P(10 < t \leq 15) = P(15 < t \leq 20)$

Thus

$$P(t > 5) = 1 - 0.56 = 0.44$$

Problem 5. .

Proposition. $P(A|B) < P(A|B^c) \implies P(A|B) < P(A) < P(A|B^c)$

Proof. .

$$\begin{aligned}
P(A) &\underset{LTP}{=} P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c) \\
&\underset{given}{>} P(A|B) \cdot P(B) + P(A|B) \cdot P(B^c) \\
&= P(A|B) \cdot (P(B) + P(B^c)) \\
&= P(A|B) \cdot (P(\Omega)) \\
&= P(A|B)
\end{aligned}$$

Similarly

$$\begin{aligned}
P(A) &= P(A|B) \cdot P(B) + P(A|B^c) \cdot P(B^c) \\
&< P(A|B^c) \cdot P(B) + P(A|B^c) \cdot P(B^c) = P(A|B^c)
\end{aligned}$$

□

Thus $P(A|B) < P(A) < P(A|B^c)$

Problem 6. Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independent of everyone else and ignore the additional complication presented by leap years. What is the probability that each person has a distinct birthday?

Answer: For two individuals:

$$P\left(distinct|n=2\right)=\frac{365}{365}\cdot\frac{365-1}{365}=\frac{364}{365^2}$$

Hence for n ($n<365$) individuals:

$$P\left(distinct|n=n\right)=\left(\frac{1}{365}\right)^n\cdot\frac{365!}{(365-n)!}$$

Problem 7. fixed event probability function

Proposition. *let B be a fixed event, then $P\left(\cdot|B\right)$ is a probabilty function.*

Proof. we need to show that $P\left(\cdot|B\right):=\frac{P(\cdot\cap B)}{P(B)}$ suffices the defintion of a probability function.

we assum that $P\left(\cdot\right)$ is a probability function.

1. $P\left(A|B\right)=\frac{P(A\cap B)}{P(B)}\underset{A\supseteq\emptyset}{\geq}\frac{P(\emptyset\cap B)}{P(B)}=\frac{P(\emptyset)}{P(B)}=0$
2. $P\left(\Omega|B\right)=\frac{P(\Omega\cap B)}{P(B)}\underset{B\subseteq\Omega}{=}\frac{P(B)}{P(B)}=1$
3. let A,C be disjoint events. thus

$$\begin{aligned}P\left(A\cup C|B\right)\frac{P\left((A\cup C)\cap B\right)}{P\left(B\right)}&=\frac{P\left((A\cap B)\cup(C\cap B)\right)}{P\left(B\right)}\\&\underset{*}{=}\frac{P\left(A\cap B\right)+P\left(C\cap B\right)}{P\left(B\right)}\\&=P\left(A|B\right)+P\left(C|B\right)\end{aligned}$$

*: if $A\cap B\subseteq A$ and $C\cap B\subseteq C$ thus $\left(A\cap B\right),\left(C\cap B\right)$ are disjoint and by third property of $P\left(\cdot\right)$ we get the equation. □

Problem 8. complement independence

Proposition. *if A,B are independent events then A,B^c are independent.*

Proof. we need to show $P\left(A\cap B^c\right)=P\left(A\right)\cdot P\left(B^c\right)$

$$P\left(A\right)\underset{*}{=}P\left(A\cap B\right)+P\left(A\cap B^c\right)\underset{A\perp B}{=}P\left(A\right)P\left(B\right)+P\left(A\cap B^c\right)$$

*: $P\left(\cdot\right)$ prop. 5.

On the other hand:

$$P\left(A\right)=P\left(A\right)\cdot\left(P\left(B\right)+P\left(B^c\right)\right)=P\left(A\right)P\left(B\right)+P\left(A\right)P\left(B^c\right)$$

$$\Updownarrow$$

$$P\left(A\right)P\left(B\right)+P\left(A\cap B^c\right)=P\left(A\right)P\left(B\right)+P\left(A\right)P\left(B^c\right)$$

$$\Updownarrow$$

$$P\left(A\cap B^c\right)=P\left(A\right)P\left(B^c\right)$$

□

Problem 9. It is given that 5% of the population has COVID19. If a person has COVID, a rapid antigen test will be positive with probability 0.95. If he does not have COVID, the rapid test is positive with probability 0.1.

(1) A person is randomly selected and tested. If the result is positive, what is the probability that he has COVID?
(2) It was decided to conduct a second rapid test (the tests are independent given the person's condition). If the test shows positive again, what is the probability that he has COVID?

Answer: 1. we denote PT for positive test

$$\begin{aligned}
P\left(COVID|PT\right) &\underset{Bayes}{=} \frac{P\left(PT|COVID\right)P\left(COVID\right)}{P\left(PT\right)}\\
&\underset{LTP}{=} \frac{P\left(PT|COVID\right)P\left(COVID\right)}{P\left(PT|COVID\right)P\left(COVID\right)+P\left(PT|COVID^c\right)P\left(COVID^c\right)}\\
&= \frac{0.95\cdot 0.05}{0.95\cdot 0.05+0.1\cdot(1-0.05)} = \frac{0.95\cdot 0.05}{0.95\cdot 0.05+0.1\cdot 0.95} = \frac{0.05}{0.15} = \frac{1}{3}
\end{aligned}$$

2. we denote the two test as PT_1 and PT_2 and we denote C as $COVID$

$$\begin{aligned}
P\left(COVID|PT_1\cap PT_2\right) &\underset{Bayes}{=} \frac{P\left(PT_1\cap PT_2|C\right)P\left(C\right)}{P\left(PT_1\cap PT_2\right)}\\
&\underset{T_1\perp T_2}{=} \frac{P\left(PT_1|C\right)P\left(PT_1|C\right)P\left(C\right)}{P\left(PT_1\cap PT_2\right)}\\
&\underset{LTP}{=} \frac{P\left(PT_1|C\right)P\left(PT_1|C\right)P\left(C\right)}{P\left(PT_1\cap PT_2|C\right)P\left(C\right)+P\left(PT_1\cap PT_2|C^c\right)P\left(C^c\right)}\\
&= \frac{P\left(PT_1|C\right)P\left(PT_2|C\right)P\left(C\right)}{P\left(PT_1C\right)P\left(PT_2|C\right)P\left(C\right)+P\left(PT_1|C\right)P\left(PT_2|C^c\right)P\left(C^c\right)}\\
&= \frac{P\left(PT_1|COVID\right)^2P\left(COVID\right)}{P\left(PT_1|C\right)^2P\left(C\right)+P\left(PT_1|C^c\right)^2P\left(C^c\right)}\\
&= \frac{0.95^2 0.05}{0.95^2 0.05+0.1^2 0.95}\\
&= \frac{0.95\cdot 0.05}{0.95\cdot 0.05+0.1^2} = 0.826
\end{aligned}$$

From the above we can see that possible error of a False Positve is more than twice lower for two rapid tests.

Problem 10. Independence of a col-lection of events and pairwise independence.

Definition. We say that the events $A_1, A_2, \ldots, A_n$ are independent if for every subset S of $\{1, 2, \ldots, n\}$

$$P\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} P\left(A_i\right)$$

For the case of three events A_1, A_2, A_3, independence amount to satisfying the four conditions

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)\, P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)\, P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)\, P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)\, P(A_2)\, P(A_3) \end{aligned}$$

The first three conditions assert that any two events are independent, a property known as pairwise independence. The fourth condition is also important and does not follow from the first three. Conversely, the fourth condition does not imply the first three. We will demonstrate it through the following examples.

1. Consider two independent fair coin tosses and the following events:

$$\begin{aligned} H_1 &= \{\text{1st toss is a head}\} \\ H_2 &= \{\text{2nd toss is a head}\} \\ D &= \{\text{the two tosses have different results}\} \end{aligned}$$

Answer: For simplification we write h for *head* and t for *tail*:

$$\begin{aligned} P(H_1) &= \frac{\{h,t\} \cup \{h,h\}}{|\Omega|} = \frac{1}{2} = \frac{\{t,h\} \cup \{h,h\}}{|\Omega|} = P(H_2) \\ P(D) &= \frac{\{h,t\} \cup \{t,h\}}{|\Omega|} = \frac{1}{2} \\ P(H_1 \cap H_2) &= \frac{|\{h,h\}|}{|\Omega|} = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = P(H_1) \cdot P(H_2) \end{aligned}$$

$$P(D \cap H_1) = \frac{|(\{h,t\} \cup \{t,h\}) \cap (\{h,t\} \cup \{h,h\})|}{|\Omega|} = \frac{|\{h,t\}|}{|\Omega|} = \frac{1}{4} = P(D) \cdot P(H_1)$$

$$P(D \cap H_2) = \frac{|(\{h,t\} \cup \{t,h\}) \cap (\{t,h\} \cup \{h,h\})|}{|\Omega|} = \frac{|\{t,h\}|}{|\Omega|} = \frac{1}{4} = P(D) \cdot P(H_2)$$

On the other hand:

$$\begin{aligned} P(D \cap H_1 \cap H_2) &= \frac{|(\{h,t\} \cup \{h,t\}) \cap (H_1 \cap H_2)|}{|\Omega|} \\ &= \frac{|(\{h,t\} \cup \{t,h\}) \cap \{h,h\}|}{|\Omega|} = \frac{|\emptyset|}{|\Omega|} = 0 \end{aligned}$$

2. Consider two independent rolls of a fair six-sided die, and the following events:

$$\begin{aligned} A &= \{\text{1st roll is 1, 2 or 3}\} = \{\{1\} \cup \{2\} \cup \{3\} \times \{n \mid 1 \leq n \leq 6\}\} \\ B &= \{\text{1st roll is 3, 4 or 5}\} = \{\{3\} \cup \{4\} \cup \{5\} \times \{n \mid 1 \leq n \leq 6\}\} \\ C &= \{\text{the sum of the two rolls is 9}\} = \{\{3,6\}, \{4,5\}, \{5,4\}, \{6,3\}\} \end{aligned}$$

Show that the fourth condition holds, but the events are not pairwise in- dependent.

Answer:

$$P(A\cap B)=\frac{1}{6}\cdot 1=\frac{1}{6}\neq\frac{1}{4}=\frac{3}{6}\cdot\frac{3}{6}=P(A)\cdot P(B)$$
$$P(A\cap C)=\frac{|\{3,6\}|}{|\Omega|}=\frac{1}{6^2}\neq\frac{3}{6}\cdot\frac{4}{6^2}=P(A)\cdot P(C)$$
$$P(B\cap C)=\frac{|\{3,6\}|}{|\Omega|}=\frac{1}{6^2}\neq\frac{3}{6}\cdot\frac{4}{6^2}=P(B)\cdot P(C)$$
$$P(A\cap B\cap C)=\frac{|\{3,6\}|}{|\Omega|}=\frac{1}{6^2}\neq\frac{3}{6}\cdot\frac{3}{6}\cdot\frac{4}{6^2}=P(A)\cdot P(B)\cdot P(C)$$