3:

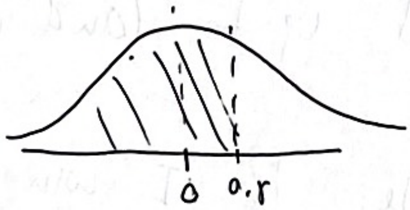
Problem 2: We see one observation X and we want to test $H_0: X \sim N(0,1)$ vs. $H_1: X \sim Exp(1)$

A. The ~~test~~ proposed test is: $R = \{X > 0.5\}$ (reject null if.
Calculate the significance α and Type II error β.

Sol:

$$\alpha = P_{H_0}(R) = P_{H_0}(X > 0.5) = 1 - P_{H_0}(X \le 0.5) = 1 - \Phi(0.5) = 0.308$$

$$X \overset{H_0}{\sim} N(0,1)$$



$$\beta = P_{H_1}(R^c) = P_{H_1}(X \le 0.5) = 1 - e^{-0.5} = 0.39$$

$$X \overset{H_1}{\sim} Exp(1)$$

B. Find a test of the form $R = \{X > c\}$ such that the significance would be 0.05.
Compute type II error for the test.

Sol:

We need to solve for c: $0.05 = P_{H_0}(X > c)$

$$\Leftrightarrow \quad 0.05 = 1 - \Phi(c) \Leftrightarrow \Phi(c) = 0.95 \Rightarrow c = \Phi^{-1}(0.95)$$

$$c = Z_{0.95} = 1.645$$

$$R = \{X > 1.645\}$$

$$\beta = P_{H_1}(R^c) = P_{H_1}(X \le 1.645) = 1 - e^{-1.645} = 0.807$$

Conclusion: lower α, higher β (therefore lower power).

3: It is known that the α level CI for the variance of $X_1, \dots, X_n$ RV's with $Var(X_i) = \sigma^2$ is

$$\left[\frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}}\right] \qquad (\chi^2_d = \Gamma(\tfrac{d}{2}, \tfrac{1}{2}))$$

Use this fact to test the hypothesis (two-sided)

$$H_0: \sigma^2 = \sigma_0^2 \quad \text{vs.} \quad H_1: \sigma^2 \neq \sigma_0^2$$

Sol: We saw in class the equivalence between CI's and two-sided hypotheses.

We reject H_0 if σ_0^2 is not in the CI and we accept H_0 if σ_0^2 is inside the CI.

The rejection set is therefore:

$$R = \left\{\frac{(n-1)S_n^2}{\chi^2_{n-1,1-\alpha/2}} > \sigma_0^2\right\} \cup \left\{\frac{(n-1)S_n^2}{\chi^2_{n-1,\alpha/2}} < \sigma_0^2\right\}$$

$$= \left\{\frac{(n-1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,1-\alpha/2}\right\} \cup \left\{\frac{(n-1)S_n^2}{\sigma_0^2} < \chi^2_{n-1,\alpha/2}\right\}$$

//

Problem 4: Let $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ (σ^2 known).

We want to test

$$H_0: \mu = \mu_0 \qquad (\mu_1 > \mu_0)$$
$$H_1: \mu = \mu_1$$

Find the MP test at level α.

Sol: We will use the N-P lemma and use the LR statistic.

$$\lambda(X) = \frac{L(\mu_1; X)}{L(\mu_0; X)} = e^{-\frac{1}{2\sigma^2}\left[\sum_{i=1}^n (X_i - \mu_1)^2 - \sum_{i=1}^n (X_i - \mu_0)^2\right]}$$

$$n(\mu_1+\mu_0)(\mu_1-\mu_0) \quad (\mu_0 - \mu_1)$$

$$X_i^2 - 2\mu_1 X_i + \mu_1^2 - (X_i^2 - 2\mu_0 X_i + \mu_0^2) = 2X_i(\mu_0 - \mu_1) + \mu_1^2 - \mu_0^2$$

$$\left\langle L(\mu; X) = \prod_{i=1}^n \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{(X_i - \mu)^2}{2\sigma^2}} = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (X_i - \mu)^2} \right\rangle$$

$$= e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (\mu_0 - \mu_1)(2X_i - \mu_0 - \mu_1)}$$

$$= e^{\frac{1}{\sigma^2}(\mu_1 - \mu_0) n \left(\bar{X}_n - \frac{\mu_1 + \mu_0}{2}\right)}$$

$\Rightarrow$ Reject H_0 if $\lambda(X) \geq C$, or equivalently,

since $\mu_1 > \mu_0$: if $\bar{X}_n \geq C^*$.

In that case,

$$\alpha = P_{H_0}(\bar{X}_n \geq C^*) = 1 - \Phi\left(\frac{C^* - \mu_0}{\sigma^2/\sqrt{n}}\right)$$

$$\Leftrightarrow \quad \Phi\left(\frac{C^* - \mu_0}{\sigma^2/\sqrt{n}}\right) = 1 - \alpha$$

$$\Leftrightarrow \quad C^* = \Phi^{-1}(1-\alpha) \cdot \frac{\sigma^2}{\sqrt{n}} + \mu_0 \;//$$