PROBABILITY AND STATISTICS - P&P 4

SUBMISSION BY: ELIA YAKIN & ITAY KOREN

Problem 1. Let $X_1, ..., X_n \overset{iid}{\sim} Geo(\theta)$, $\theta \in [0,1]$. Find the MLE for θ

Answer:

$$\mathcal{L}(\theta; \boldsymbol{X}) = P(x_1 \cap x_2 \cap \cdots \cap x_n; \theta) \underset{iid}{=} \prod_{i=1}^{n} (1-\theta)^{x_i - 1} \theta$$

$$\iff$$

$$\ell(\theta; \boldsymbol{X}) = \sum_i [(x_i - 1) log(1-\theta) + log(\theta)] = n \cdot log(\theta) + log(1-\theta) \sum_i (x_i - 1)$$

.

$$\frac{\partial \ell(\theta; \boldsymbol{X})}{\partial \theta} = \frac{n}{\theta} + \frac{1}{1-\theta}(-1) \sum_i (x_i - 1) = \frac{n}{\theta} - \frac{\sum_i x_i - n}{1-\theta} \underset{FOC}{=} 0$$

$$\iff n(1-\theta) = \theta \left(\sum_i x_i - n \right) \iff \widehat{\theta}_{MLE} = \frac{n}{\sum_i x_i} = \frac{1}{\overline{X}}$$

.

Problem 2. Let $X_1, ..., X_n$ be an i.i.d with $f_\theta(x) = \frac{1}{x} e^{-\pi(log(x)-\theta)^2}$Find the MLE for θ

Answer:

$$\mathcal{L}(\theta; \boldsymbol{X}) = f_X(x_1 \cap x_2 \cap \cdots \cap x_n; \theta) \underset{iid}{=} \prod_{i=1}^{n} \frac{1}{x_i} e^{-\pi(log(x_i)-\theta)^2}$$

$$\iff$$

$$\begin{aligned}
\ell(\theta; \boldsymbol{X}) &= \sum_{i=1}^{n} \left[log\left(e^{-\pi(log(x_i)-\theta)^2} \right) - log(x_i) \right] \\
&= \sum_{i=1}^{n} -\pi (log(x_i) - \theta)^2 - \sum_{i=1}^{n} log(x_i) \\
&= -\pi \sum_{i=1}^{n} \left(log^2(x_i) - log(x_i)\theta + \theta^2 \right) - \sum_i log(x_i) \\
&= -\pi \left(\sum_{i=1}^{n} log^2(x_i) - 2 \sum_{i=1}^{n} log(x_i)\theta + n\theta^2 \right) - \sum_{i=1}^{n} log(x_i)
\end{aligned}$$

•

$$\begin{aligned}\frac{\partial\ell(\theta;X)}{\partial\theta} &= \pi 2\sum_{i=1}^{n} log(x_i) - \pi 2n\theta \underset{FOC}{=} 0 \\ \Longleftrightarrow \widehat{\theta}_{MLE} &= \frac{1}{n}\sum_{i=1}^{n} log(x_i)\end{aligned}$$

Problem 3. Let $X_1, ..., X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, where μ and σ^2 are unknown . Find the MLE for μ and σ^2.

Answer:

$$\begin{aligned}\mathcal{L}(\mu,\sigma;\boldsymbol{X}) &= f_X(\mu,\sigma;x_1\cap x_2\cap\cdots\cap x_n) \\ &\underset{iid}{=} \prod_{i=1}^{n}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} = n\cdot\left(\sigma\sqrt{2\pi}\right)^{-1} e^{-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2} \\ \Longleftrightarrow & \\ \ell(\mu,\sigma;\boldsymbol{X}) &= -n\cdot log\left(\sigma\sqrt{2\pi}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell(\mu,\sigma;\boldsymbol{X})}{\partial\mu} &= -\frac{1}{2\sigma^2}2\sum_{i=1}^{n}(x_i-\mu)\cdot(-1) \underset{FOC}{=} 0 \Longleftrightarrow n\mu = \sum_{i=1}^{n} x_i \Longrightarrow \widehat{\mu}_{MLE} = \overline{X} \\ \frac{\partial\ell(\mu,\sigma;\boldsymbol{X})}{\partial\sigma} &= -n\cdot\frac{1}{\sigma\sqrt{2\pi}}\cdot\sqrt{2\pi} - \frac{(-2)}{2\sigma^3}\sum_{i=1}^{n}(x_i-\mu)^2 \underset{FOC}{=} 0 \\ &\Longrightarrow \widehat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(x_i-\overline{X}\right)^2\end{aligned}$$

•

Problem 4. Let $X_1, ..., X_n \overset{iid}{\sim} U(\theta+2, \theta+10)$ (continuous).

(1) Find $\hat{\theta}_{MOM}$ (method of moments estimator for θ).

(2) Evaluate $\hat{\theta}_{MOM}$ for the sample: $\{12.3, 17.5, 15.1, 14.7\}$

Answer: (1)

$$\begin{aligned}\mu_1(\theta) = EX &= \frac{(\theta+2)+(\theta+10)}{2} = \theta + 6 = m_1 = \overline{X} \\ \Longleftrightarrow \hat{\theta}_{MOM} &= \overline{X} - 6\end{aligned}$$

(2)

$$\hat{\theta}_{MOM} = \overline{X} - 6 = \left(\frac{12.3+17.5+15.1+14.7}{4} - 6\right) \qquad = 8.9$$

Problem 5. It is assumed that the daily amount of rain (in mm) that falls in London during January is distributed $N(\mu, 25)$. We are interested in estimating $P(X > 75)$. Two approaches were suggested:

A. Estimate μ using the method of moments, and then estimate the probability using $\hat{\mu}_{MOM}$ instead of μ in the normal distribution.

B. Don't assume normality. Estimate the probability by calculating the proportion of observations that are greater than 75.

In a random sample of 10 observations, the following results were received:

$$\{68.49, 63.61, 71.22, 76.38, 75.99, 78.66, 59.08, 68.82, 75.47, 64.56\}$$

(1) Estimate the required probability using both methods and compare the results.

(2) Estimate the probability $P(X > 72)$ using both methods and compare the results.

Answer: (1) with MEM:

$$\hat{\mu}_{MEM} = \frac{68.49 + 63.61 + 71.22 + 76.38 + 75.99 + 78.66 + 59.08 + 68.82 + 75.47 + 64.56}{10} = 70.22$$

$$P(X > 75) = 1 - P(X \leq 75) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{75-\mu}{\sigma}\right) = 1 - \phi(0.9564) = 1 - 0.8306 = 0.1694$$

$$P(X > 75) = \frac{1}{10}\sum_{i=1}^{10} I\{X > 75\} = 0.4$$

(2)

$$P(X > 72) = 1 - P(X \leq 72) = 1 - P\left(\frac{X-\mu}{\sigma^2} \leq \frac{72-70.22}{5}\right) = 1 - \phi(0.0712) \approx 0.47$$

$$P(X > 72) = 1 - P(X \leq 72) = 1 - P\left(\frac{X-\mu}{\sigma} \leq \frac{72-70.22}{5}\right) = 1 - \phi(0.6391) \approx 0.36$$

$$P(X > 72) = \frac{1}{10}\sum_{i=1}^{10} I\{X > 72\} = 0.4$$

Problem 6. Let $X_1, ..., X_n \overset{iid}{\sim} Poiss(\lambda)$

(1) Compute the MSE of the MLE for λ.

(2) A researcher believes that λ is approximately 3, so he suggests to use the estimator which is the average between the MLE and 3: $T = \frac{\overline{X_n}+3}{2}$. Compute the MSE of T.

(3) Compare the bias and the variance of the estimators as functions of λ.

(4) Compare the MSE of the estimators as a function of λ and find for which values of λ each estimator is better than the other. Note that the range of λ might depend on n.

Answer:

$$\mathcal{L}(\lambda; \boldsymbol{X}) = P(x_1 \cap x_2 \cap \cdots \cap x_n; \lambda) \underset{iid}{=} \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

$$\iff \ell(\lambda; \boldsymbol{X}) = \sum_{i=1}^{n} x_i \cdot log(\lambda) - n \cdot \lambda - \sum_{i=1}^{n} log(x_i!)$$

$$\iff$$

$$\frac{\partial \ell(\lambda; X)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n \underset{FOC}{=} 0 \implies \hat{\lambda}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{X}$$

Notice that

$$E\left(\hat{\lambda}_{MLE}-\lambda\right)=E\left(\overline{X}-\lambda\right)=E\overline{X}-\lambda=0$$
$$\implies Bias\left(\hat{\lambda}_{MLE}\right)=0$$

$$\begin{aligned}MSE\left(\hat{\lambda}_{MLE}\right)&=Var\left(\hat{\lambda}_{MLE}\right)+Bias^2\left(\hat{\lambda}_{MLE}\right)=Var\left(\hat{\lambda}_{MLE}\right)+0\\&=Var\left(\frac{1}{n}\sum_i X_i\right)=\frac{1}{n^2}\cdot Var\left(\sum_i X_i\right)\underset{iid}{=}\frac{1}{n^2}\cdot n\cdot Var\left(X_i\right)\\&=\frac{1}{n}\cdot\lambda\end{aligned}$$
$$\iff$$
$$MSE\left(\hat{\lambda}_{MLE}\right)=\frac{\lambda}{n}$$

(2) we are given $T=\frac{3+\lambda}{2}$

$$E\left(T\right)=E\left(\frac{\overline{X}+3}{2}\right)=\frac{1}{2}E\overline{X}+\frac{3}{2}=\frac{1}{2}\lambda+\frac{3}{2}\underset{\lambda\neq3}{\neq}\lambda\implies T\ is\ Baised$$

$$\begin{aligned}MSE\left(T\right)&=Var\left(T\right)+Bias^2\left(\hat{\lambda}_{MLE}\right)\\&=\frac{1}{4}Var\left(\overline{X}\right)+\left(\frac{1}{2}\lambda+\frac{3}{2}-\lambda\right)^2\\&\underset{iid}{=}\frac{1}{4}\cdot\frac{\lambda}{n}+\frac{1}{4}\left(3-\lambda\right)^2\end{aligned}$$
$$\iff$$
$$MSE\left(T\right)=\frac{1}{4}\left[\frac{\lambda}{n}+\left(3-\lambda\right)^2\right]$$

(3)

$$Var\left(T\right)=\frac{1}{4}Var\left(\overline{X}\right)<Var\left(\overline{X}\right)=Var\left(\hat{\lambda}_{MLE}\right)\ \forall n\in N$$

and

$$Bias^2\left(T\right)=\left(\frac{3-\lambda}{2}\right)^2\underset{\lambda\neq3}{>}0=Bias^2\left(\hat{\lambda}_{MLE}\right)$$
$$Bias^2\left(T\right)\underset{\lambda=3}{=}0=Bias^2\left(\hat{\lambda}_{MLE}\right)$$

We can say that if $\lambda\neq3$ than T has higher squared bias and lower variance.
(4) As $n\to\infty$ we get:

$$MSE\left(T\right)=\frac{1}{4}\left[\frac{\lambda}{n}+\left(3-\lambda\right)^2\right]\underset{n\to\infty}{=}\frac{\left(3-\lambda\right)^2}{4}\underset{\lambda\neq3}{>}0=MSE\left(\hat{\lambda}_{MLE}\right)$$

while only if $\lambda=3$ we get that

$$MSE\left(T\right)=0=MSE\left(\hat{\lambda}_{MLE}\right)$$

Problem 7. The weight of students in some university is normally distributed. A sample of 12 students is drawn with the following results (in Kg):

$$\{53.8, 67.34, 51.7, 52, 58.9, 74, 45.3, 53, 62.5, 48.87, 49, 55.6\}$$

(1) Assuming that the variance is known and equals 1.5 Kg, calculate the confidence interval for the expected value of the weight with confidence level 95%.

2) Repeat part 1, this time for a confidence level of 90%. What can you say about the difference between the results?

(3) Assuming that the variance is known and equals 2 Kg, calculate the confidence interval for the expected value of the weight with confidence level 95%. What can you conclude from the result?

(4) Repeat part 1, assuming that the variance is unknown.

Answer: (1)

$$\overline{X} = \frac{53.8 + 67.34 + 51.7 + 52 + 58.9 + 74 + 45.3 + 53 + 62.5 + 48.87 + 49 + 55.6}{12} \simeq 56$$

$$\frac{\sigma}{\sqrt{12}} = \sqrt{\frac{1.5}{12}} = 0.353$$

we can than write

$$\begin{aligned} P\left(C_1\left(\boldsymbol{X}\right) \leq 56 \leq C_2\left(\boldsymbol{X}\right)\right) &= 0.95 \\ &\Longleftrightarrow \\ P\left(-z_{97.5} \leq \frac{56}{\sigma/\sqrt{n}} \leq z_{97.5}\right) &= 0.95 \\ \Longrightarrow CI_{0.95} &= \left[56 - z_{97.5}\cdot\sigma/\sqrt{n}, 56 + z_{97.5}\cdot\sigma/\sqrt{n}\right] \\ &= [56 - 1.96\cdot 0.353, 56 + 1.96\cdot 0.353] \\ &= [55.30, 56.69] \end{aligned}$$

(2)

$$\begin{aligned} CI_{0.9} &= \left[56 - z_{95}\cdot\sigma/\sqrt{n}, 56 + z_{95}\cdot\sigma/\sqrt{n}\right] \\ &= [56 - 1.65\cdot 0.353, 56 + 1.65\cdot 0.353] \\ &= [55.41, 56.58] \end{aligned}$$

As we allow for lower confidence that μ_X is in the interval that interval itself get smaller:

$$\begin{aligned} L\left(CI_{0.95}\right) = C_{2,0.95} - C_{1,0.95} &= 56.69 - 55.30 \\ = 1.39 > 1.17 &= 56.58 - 55.41 = C_{2,0.9} - C_{1,0.9} = L\left(CI_{0.9}\right) \\ &\Longleftrightarrow \\ CI_{0.95} &> CI_{0.9} \end{aligned}$$

(3) we now assume that $\sigma^2 = 2$ thus

$$\begin{aligned}
\widetilde{CI}_{0.95} &= \left[56 - z_{97.5} \cdot \sigma/\sqrt{n}, 56 + z_{97.5} \cdot \sigma/\sqrt{n}\right] \\
&= \left[56 - 1.96 \cdot \sqrt{2/12}, 56 + 1.96 \cdot \sqrt{2/12}, \right] \\
&= [55.2, 56.8] \\
\implies L\left(\widetilde{CI}_{0.95}\right) &> L\left(CI_{0.95}\right)
\end{aligned}$$

A higher variance leeds to a larger CI

(4) Assuming the variance is unknow we have to use T-test confidence interval

$$S^2 = \frac{1}{12-1}\sum_{i=1}^{12}(x_i - 56)^2 = 69.64 \implies S = 8.34$$

$$\begin{aligned}
CI_{0.95} &= \left[56 - t_{11,97.5} \cdot S/\sqrt{n},\ 56 + t_{11,97.5} \cdot S/\sqrt{n}\right] \\
&= [56 - 2.201 \cdot 2.4,\ 56 + 2.201 \cdot 2.4] \\
&= [50.71, 61.28]
\end{aligned}$$

Problem 8. Let $X \sim N\left(\mu, \sigma^2\right)$ (both parameters are unknown). In a random sample of 10 observations we received that:

$$\sum_{i=1}^{10} x_i = 15 \sum_{i=1}^{10} x_i^2 = 27$$

and the CI for μ is $[1.09, 1.91]$. What is the confidence level of this confidence interval?

Answer:

$$\overline{X} = \frac{1}{10}\sum_{i=1}^{10} x_i = \frac{15}{10} = 1.5$$

$$\begin{aligned}
S^2 &= \frac{1}{9}\sum_{i=1}^{10}\left(x_i - \overline{X}\right)^2 = \frac{1}{9}\sum_{i=1}^{10}\left(x_i^2 - 2x_i\overline{X} + \overline{X}^2\right) = \frac{1}{9}\left(\sum_{i=1}^{10} x^2 - 2\overline{X}\sum_{i=1}^{10} x_i + \sum_{i=1}^{10}\overline{X}^2\right) \\
&= \frac{1}{9}\left[27 - 2 \cdot 1.5 \cdot 15 + 10 \cdot 1.5^2\right] = 0.5 \\
\implies S &= 0.707
\end{aligned}$$

$$\begin{aligned}
C_1 = \overline{X} - t_{n-1,1-\frac{\alpha}{2}} \cdot S/\sqrt{n} &= 1.5 - t_{n-1,\frac{\alpha}{2}} \cdot \sqrt{0.5}/\sqrt{10} = 1.09 \\
&\iff t_{9,\frac{\alpha}{2}} = \frac{1.5 - 1.09}{\sqrt{0.5/10}} = 1.833 \\
&\iff \alpha/2 = 0.05 \\
&1 - 0.1 \approx 0.9
\end{aligned}$$

the confidence level is 0.9

Problem 9. In a random sample of 100 students, it was found that 30 like Bamba.

(1) Compute an asymptotic confidence interval for the proportion of Bamba lovers among the students.

(2) Find the minimal sample size n for which the length of the CI will be at most 0.02.

Answer: (1) We assume all students like or deslike Bamba and thus $X_1, \ldots, X_{100} \sim Bin(\theta)$ given the information we can assume

$$\overline{X} = 0.3 = \hat{\theta}$$

and hence $E\hat{\theta} = 0.3$ $Var\left(\hat{\theta}\right) = 0.3(1-0.3) = 0.21$ and we get that The asymptotic confidence interval for $\hat{\theta}$ is:

$$\left[n \cdot \hat{\theta} \pm \frac{\sqrt{n\hat{\theta}\left(1-\hat{\theta}\right)}}{\sqrt{n}} z_{1-\frac{\alpha}{2}}\right] = \left[0.3 \pm \frac{\sqrt{0.21}}{10} z_{1-\frac{\alpha}{2}}\right] = \left[0.3 \pm 0.0458 \cdot z_{1-\frac{\alpha}{2}}\right]$$

(2)

$$n = 4 \cdot z_{1-\frac{\alpha}{2}}^2 \cdot \frac{\sigma^2}{L_0^2} = 4 \cdot z_{1-\frac{\alpha}{2}}^2 \cdot \frac{0.21}{0.02^2} = 0.21 \cdot 10^4 \cdot z_{1-\frac{\alpha}{2}}^2$$

for $z_{0.975} = 1.96$ we get $n \geq 0.21 \cdot 10^4 \cdot 1.96^2 \approx 8068$