Probability and Statistics
Y-DATA School of Data Science
P&P 1
Due: 8.10.2022

PROBLEM 1. Prove that for a finite sample space Ω and a subset $A \subseteq \Omega$,

$$P(A) = \frac{|A|}{|\Omega|}$$

is a probability function.

PROBLEM 2. Three ants are sitting at the three corners of an equilateral triangle. Each ant randomly picks a direction and starts to move along the edge of the triangle (each direction has equal probability). What is the probability that none of the ants collide?

PROBLEM 3. Out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. Determine the probability that a randomly selected student is neither genius nor a chocolate lover.

PROBLEM 4. The probability of at least one car passing a certain road intersection in a 20-minute window is 0.9. What is the probability of at least one car passing the intersection in a 5-minute window, assuming a constant probability throughout?

PROBLEM 5. Prove that for the events A and B such that $P(A|B) < P(A|B^c)$,

$$P(A|B) < P(A) < P(A|B^c)$$

(You may assume that all probabilities are positive)

PROBLEM 6. Consider n people who are attending a party. We assume that every person has an equal probability of being born on any day during the year, independent of everyone else and ignore the additional complication presented by leap years. What is the probability that each person has a distinct birthday?

PROBLEM 7. Verify that for a fixed event B, the conditional probabilities $P(A|B)$ form a legitimate probability function.
Hint: Use the distributive property of set operations.

PROBLEM 8. Prove that if the events A and B are independent, then the events A and B^c are independent.
Hint: Explain why the identity $P(A \cap B^c) = P(A) - P(A \cap B)$ holds.

PROBLEM 9. It is given that 5% of the population has COVID19. If a person has COVID, a rapid antigen test will be positive with probability 0.95. If he does not have COVID, the rapid test is positive with probability 0.1.

(1) A person is randomly selected and tested. If the result is positive, what is the probability that he has COVID?

(2) It was decided to conduct a second rapid test (the tests are independent given the person's condition). If the test shows positive again, what is the probability that he has COVID?

PROBLEM 10. In this problem, you will learn about independence of a collection of events and pairwise independence. We will begin with a definition, followed by an explanation, a statement and a demonstration of the statement trough a guided exercise.

DEFINITION 1. *We say that the events $A_1, A_2, ..., A_n$ are independent if for every subset S of $\{1, 2, ..., n\}$,*

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$$

For the case of three events A_1, A_2, A_3, independence amount to satisfying the four conditions

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)P(A_2) \\ P(A_1 \cap A_3) &= P(A_1)P(A_3) \\ P(A_2 \cap A_3) &= P(A_2)P(A_3) \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3) \end{aligned}$$

The first three conditions assert that any two events are independent, a property known as **pairwise independence**. The fourth condition is also important and does not follow from the first three. Conversely, the fourth condition does not imply the first three. We will demonstrate it through the following examples.

(1) Consider two independent fair coin tosses and the following events:

$$\begin{aligned} H_1 &= \{\text{1st toss is a head}\} \\ H_2 &= \{\text{2nd toss is a head}\} \\ D &= \{\text{the two tosses have different results}\} \end{aligned}$$

Show that the events are pairwise independent, but the three events are not independent.

(2) Consider two independent rolls of a fair six-sided die, and the following events:

$$\begin{aligned} A = \{\text{1st roll is 1, 2 or 3}\}, B &= \{\text{1st roll is 3, 4 or 5}\} \\ C &= \{\text{the sum of the two rolls is 9}\} \end{aligned}$$

Show that the fourth condition holds, but the events are not pairwise independent.