Probability and Statistics

Y-DATA School of Data Science

P&P 3

Due: 23.11.2022

PROBLEM 1. Given the joint PMF, answer the following questions.

X/Y	0	1	2
1	0.08	0.2	0.12
2	0.06	0.15	0.09
3	0.04	0.12	0.04
4	0.02	0.03	0.05

(1) Compute the marginal PMF's of X and Y.
(2) Compute $E(X)$,$E(Y)$, $E(X+Y)$,$E(XY)$,$Var(X)$, $Var(Y)$.
(3) Find the conditional distributions of $X|\{Y=0\}$, $Y|\{X=1\}$, $Y|\{X=2\}$, $Y|\{X=3\}$.
(4) Calculate $P(X=1,Y=2|X+Y<5)$.

(1) The marginal PMF of X is obtained by summing the joint PMF over the values of Y:

$$P(X=x) = \sum_{y \in supp(Y)} P_{X,Y}(x,y)$$

In our case,

$$P(X=x) = \begin{cases} 0.4, x=1 \\ 0.3, x=2 \\ 0.2, x=3 \\ 0.1, x=4 \\ 0, \text{otherwise} \end{cases}$$

Similarly,

$$P(Y=y) = \begin{cases} 0.2, y=0 \\ 0.5, y=1 \\ 0.3, y=2 \end{cases}$$

(2) Using the marginal PMF's, we get:

$$E(X) = 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 + 4 \cdot 0.1 = 2$$

$$E(X^2) = 1 \cdot 0.4 + 4 \cdot 0.3 + 9 \cdot 0.2 + 16 \cdot 0.1 = 5$$

$$Var(X) = E(X^2) - E^2(X) = 5 - 4 = 1$$

$$E(Y) = 0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3 = 1.1$$

$$E(Y^2) = 0 \cdot 0.2 + 1 \cdot 0.5 + 4 \cdot 0.3 = 1.7$$

$$Var(Y) = E(Y^2) - E^2(Y) = 1.7 - 1.1^2 = 0.49$$

$$E(X+Y) = E(X) + E(Y) = 3.1$$

Lastly,

$$\begin{aligned} E(XY) &= \sum_{x=1}^{4}\sum_{y=0}^{2} xyP_{X,Y}(x,y) \\ &= 1\cdot 0\cdot 0.08 + 1\cdot 1\cdot 0.2 + 1\cdot 2\cdot 0.12 + \\ &\quad 2\cdot 0\cdot 0.06 + 2\cdot 1\cdot 0.15 + 2\cdot 2\cdot 0.09 + \\ &\quad 3\cdot 0\cdot 0.04 + 3\cdot 1\cdot 0.12 + 3\cdot 2\cdot 0.04 + \\ &\quad 4\cdot 0\cdot 0.02 + 4\cdot 1\cdot 0.03 + 4\cdot 2\cdot 0.05 = 2.22 \end{aligned}$$

(3) In general, we obtain the conditional PMF of X given Y using the formula

$$P(X = x|Y = y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

Therefore,

$$P(X = x|Y = 0) = \begin{cases} 0.4, x = 1 \\ 0.3, x = 2 \\ 0.2, x = 3 \\ 0.1, x = 4 \\ 0, \text{otherwise} \end{cases}$$

$$P(Y = y|X = 1) = \begin{cases} 0.2, y = 0 \\ 0.5, y = 1 \\ 0.3, y = 2 \end{cases}$$

$$P(Y = y|X = 2) = \begin{cases} 0.2, y = 0 \\ 0.5, y = 1 \\ 0.3, y = 2 \end{cases}$$

$$P(Y = y|X = 3) = \begin{cases} 0.2, y = 0 \\ 0.6, y = 1 \\ 0.2, y = 2 \end{cases}$$

(4)

$$\begin{aligned} P(X = 1, Y = 2|X+Y < 5) &= \frac{P(X = 1, Y = 2, X+Y < 5)}{P(X+Y < 5)} \\ &= \frac{P(X = 1, Y = 2)}{\sum_{(x,y):x+y<5} P(X = x, Y = y)} \\ &= \frac{0.12}{0.88} = 0.136 \end{aligned}$$

PROBLEM 2. The joint PDF of X and Y is,

$$f_{X,Y}(x,y) = cxy^2,\ 0 < x < y < 1$$

(1) Find c.
(2) Find the marginal PDF and CDF of X.
(3) Find the marginal PDF and CDF of Y.
(4) Compute the mean and variance of X and Y.

(1) We will integrate the joint PDF over the whole support of X and Y and equate to 1.

$$\begin{aligned}\int_0^1 \int_x^1 cxy^2 dydx &= c\int_0^1 x(\frac{y^3}{3})|_x^1 dx \\ &= c\int_0^1 x(\frac{1}{3} - \frac{x^3}{3})dx \\ &= c(\frac{x^2}{6} - \frac{x^5}{15})|_0^1 = \frac{c}{10} = 1\end{aligned}$$

therefore, $c = 10$.
Remark: Alternatively, the limits of integration could be $0 < y < 1$, $0 < x < y$.

(2) For any $0 < x < 1$, the marginal PDF of X is

$$\begin{aligned}f_X(x) &= \int_x^1 10xy^2 dy \\ &= 10x\int_x^1 y^2 dy \\ &= 10x(y^3/3)|_x^1 \\ &= \frac{10}{3}(x - x^4)\end{aligned}$$

For $0 < x < 1$ the marginal CDF of X is obtained by

$$F_X(x) = \int_0^x f_X(t)dt$$

otherwise, it is 0 or 1. That is,

$$\int_0^x \frac{10}{3}(t - t^4)dt = \frac{10}{3}(t^2/2 - t^5/5)|_0^x = \frac{1}{3}(5x^2 - 2x^5)$$

To sum up,

$$F_X(x) = \begin{cases} 0,\ x \leq 0 \\ \frac{1}{3}(5x^2 - 2x^5),\ 0 < x < 1 \\ 1,\ x \geq 1 \end{cases}$$

(3) For any $0 < y < 1$, the marginal PDF of Y is

$$f_Y(y) = \int_0^y 10xy^2 dx = 10y^2\int_0^y xdx = 10y^2 \cdot \frac{y^2}{2} = 5y^4$$

The marginal CDF of Y is then,

$$P_Y(y) = \begin{cases} 0, y \leq 0 \\ y^5, 0 < y < 1 \\ 1, y \geq 1 \end{cases}$$

(4) Following the formulas from the lecture,

$$E(X) = \int_0^1 x f_X(x) dx = \int_0^1 x \frac{10}{3}\left(x - x^4\right) dx = \frac{10}{3}\frac{x^3}{3}\Big|_0^1 - \frac{10}{3}\frac{x^6}{6}\Big|_0^1 = \frac{10}{9} - \frac{10}{18} = \frac{10}{18}$$

$$E\left(X^2\right) = \int_0^1 x^2 f_X(x) dx = \int_0^1 x^2 \frac{10}{3}\left(x - x^4\right) dx = \frac{10}{3}\frac{x^4}{4}\Big|_0^1 - \frac{10}{3}\frac{x^7}{7}\Big|_0^1 = \frac{10}{12} - \frac{10}{21} = \frac{5}{14}$$

$$\mathrm{Var}(X) = E\left(X^2\right) - E^2(X) = \frac{55}{1134}$$

Similarly, for Y we have,

$$E(Y) = \int_0^1 y f_Y(y) dy = \int_0^1 y 5y^4 dx = 5\frac{y^6}{6}\Big|_0^1 = \frac{5}{6}$$

$$E\left(Y^2\right) = \int_0^1 y^2 f_Y(y) dy = \int_0^1 y^2 5y^4 dx = 5\frac{y^7}{7}\Big|_0^1 = \frac{5}{7}$$

$$\mathrm{Var}(Y) = E\left(Y^2\right) - E^2(Y) = \frac{5}{252}$$

PROBLEM 3. Let $X \sim Ber(p)$ and it is given that $E(Y|X=0) = 1$, $E(Y|X=1) = 2$. Calculate $E(Y)$.

Using the law of total expectation, we get:

$$\begin{aligned} E(Y) &= E(E(Y|X)) \\ &= E(Y|X=0)P(X=0) + E(Y|X=1)P(X=1) \\ &= 1 \cdot (1-p) + 2p = 1 + p \end{aligned}$$

PROBLEM 4. Each morning, Hungry Harry eats some eggs. On any given morning, the number of eggs he eats is equally likely to be 1, 2, 3, 4, 5, or 6, independent of what he has done in the past. Let X be the number of eggs that Harry eats in 10 days. Find the mean and variance of X.

Let Y_i denote the number of eggs that Harry ate in day $i, i = 1, \ldots, 10$. Then, $Y_i \sim U(\{1, \ldots, 6\})$ and these random variables are independent. The total number of eggs that Harry ate is $X = \sum_{i=1}^{10} Y_i$. Recall that $E(Y_i) = \frac{6+1}{2} = 3.5$ and $Var(Y_i) = \frac{6^2-1}{12} = 35/12$.
From linearity of expectation,

$$E(X) = E(\sum_{i=1}^{10} Y_i) = \sum_{i=1}^{10} E(Y_i) = 10 \cdot 3.5 = 35$$

Since the Y_i's are independent, the variance of the sum equalls the sum of the variances:

$$Var(X) = Var(\sum_{i=1}^{10} Y_i) = \sum_{i=1}^{10} Var(Y_i) = 10 \cdot 35/12 = 29.16$$

PROBLEM 5. Let X be a continuous RV with PDF $f_X(x)$ and Y a continuous RV with PDF $f_Y(y)$. X and Y are independent. Denote their sum by $Z = X + Y$.

(1) Show that $f_{Z|X=x}(z) = f_Y(z-x)$.
Hint: First show that $P(Z \leq z|X = x) = P(Y \leq z-x)$.

(2) Suppose now that X and Y are exponentially distributed with parameter λ (they are still independent). Find the conditional PDF $f_{X|Z=z}(x)$ for every $0 \leq x \leq z$.

(1) As the hint suggests, we will first find the conditional CDF of $Z|X = x$.

$$\begin{aligned} F_{Z|X=x}(z) &= P(Z \leq z|X = x) = P(X+Y \leq z|X = x) \\ &\overset{\dagger}{=} P(x+Y \leq z) = P(Y \leq z-x) = F_Y(z-x) \end{aligned}$$

Differentiating both sides with respect to z, we get that $f_{Z|X=x}(z) = f_Y(z-x)$.
$\dagger$ follows from the fact that X and Y are independent. Formally speaking,

$$\begin{aligned} P(X+Y \leq z|X = x) &= \frac{P(X+Y \leq z, X = x)}{P(X = x)} \\ &= \frac{P(x+Y \leq z, X = x)}{P(X = x)} \\ &= \frac{P(X = x|Y \leq z-x)P(Y \leq z-x)}{P(X = x)} \\ &= \frac{P(X = x)P(Y \leq z-x)}{P(X = x)} = P(Y \leq z-x) \end{aligned}$$

(2) If $X, Y \sim Exp(\lambda)$ and they are independent, we know from the previous part that for $z \geq x$,

$$f_{Z|X=x}(z) = f_Y(z-x) = \lambda e^{-\lambda(z-x)}$$

Using Bayes theorem,

$$\begin{aligned} f_{X|Z=z}(x) &= \frac{f_{Z|X=x}(z) f_X(x)}{\int_0^z f_{Z|X=t}(z) f_X(t) dt} = \frac{\lambda e^{-\lambda(z-x)} \lambda e^{-\lambda x}}{\int_0^z \lambda e^{-\lambda(z-t)} \lambda e^{-\lambda t} dt} \\ &= \frac{e^{-\lambda z}}{\int_0^z e^{-\lambda z} dt} = \frac{1}{\int_0^z dt} = \frac{1}{z} \end{aligned}$$

PROBLEM 6. Let $X \sim Ber(p), Y \sim Geo(p)$ be two independent random variables and define $Z = XY$. Express in terms of p the covariance $Cov(Z,X)$ and $Cov(Z,Y)$.

$$\begin{aligned}
\mathrm{cov}(Z,X) &= E[ZX] - E[Z]E[X] = E[XYX] - E[XY]E[X] \\
&= E\left[X^2Y\right] - E[XY]E[X] \overset{(*)}{=} E\left[X^2\right]E[Y] - E[X]E[Y]E[X] \\
&= E[Y]\left(E\left[X^2\right] - E[X]E[X]\right) = E[Y]\,\mathrm{var}[X] = \frac{1}{p} \cdot p(1-p) = 1-p
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{cov}(Z,Y) &= E[ZY] - E[Z]E[Y] = E[XYY] - E[XY]E[Y] \\
&= E\left[XY^2\right] - E[XY]E[Y] \overset{(*)}{=} E[X]E\left[Y^2\right] - E[X]E[Y]E[Y] \\
&= E[X]\left(E\left[Y^2\right] - E[Y]E[Y]\right) = E[X]\,\mathrm{var}[Y] = p\frac{1-p}{p^2} = \frac{1-p}{p}
\end{aligned}$$

In both cases, $(*)$ follows from the fact that X and Y are independent, therefore for any functions $f(X), g(Y)$, it holds that

$$E(f(X)g(Y)) = E(f(X))E(g(Y))$$