PROBABILITY AND STATISTICS - P&P 3

SUBMISSION BY: ELIA YAKIN & ITAY KOREN

Problem 1. Given the joint PMF, answer the following questions.

TABLE 1

X/Y	0	1	2
1	0.08	0.2	0.12
2	0.06	0.15	0.09
3	0.04	0.12	0.04
4	0.02	0.03	0.05

(1) Compute the marginal PMF's of X and Y.
(2) Compute EX,EY, $E(X+Y)$,$E(XY)$,$Var(X)$, $Var(Y)$.
(3) Find the conditional distributions of $X|\{Y=0\}$, $Y|\{X=1\}$, $Y|\{X=2\}$, $Y|\{X=3\}$
(4) Calculate $P(X=1, Y=2 \mid X+Y<5)$.

Answer: (1) $P_X(x) = \sum_y P(x,y)$

$$\begin{aligned}
P_X(1) &= 0.08 + 0.2 + 0.12 = 0.4 \\
P_X(2) &= 0.06 + 0.15 + 0.09 = 0.3 \\
P_X(3) &= 0.04 + 0.12 + 0.04 = 0.2 \\
P_X(4) &= 0.02 + 0.03 + 0.05 = 0.1
\end{aligned}$$

Similarly

$$\begin{aligned}
P_Y(0) &= 0.08 + 0.06 + 0.04 + 0.02 = 0.2 \\
P_Y(1) &= 0.2 + 0.15 + 0.12 + 0.03 = 0.5 \\
P_Y(2) &= 0.12 + 0.09 + 0.04 + 0.05 = 0.3
\end{aligned}$$

(2)

$$\begin{aligned}
EX &= \sum_x x \cdot P_X(x) = 0.4 + 2 \cdot 0.3 + 3 \cdot 0.2 + 4 \cdot 0.1 = 2 \\
EY &= \sum_y y \cdot P_Y(y) = 0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3 = 1.1 \\
E(X+Y) &\underset{linearity}{=} E(X) + E(Y) = 2 + 1.1 = 3.1 \\
E(X \cdot Y) &= 0 \cdot [\cdots] + 1 \cdot 1 \cdot 0.2 + 1 \cdot 2 \cdot 0.12 + 2 \cdot 1 \cdot 0.15 + 2 \cdot 2 \cdot 0.09 \\
&\quad + 3 \cdot 1 \cdot 0.12 + 3 \cdot 2 \cdot 0.04 + 4 \cdot 1 \cdot 0.03 + 4 \cdot 2 \cdot 0.05 \\
&= 2.2 \\
Var(X) &= EX^2 - E^2X = 0.4 + 2^2 \cdot 0.3 + 3^2 \cdot 0.2 + 4^2 \cdot 0.1 - 2^2 = 1 \\
Var(Y) &= 0 \cdot 0.2 + 1^2 \cdot 0.5 + 2^2 \cdot 0.3 - 1.1^2 = 0.49
\end{aligned}$$

(3) Reminder

$$P_{X|Y}(x|y) = \frac{P_{XY}(x,y)}{P_Y(y)}$$

$$\Longrightarrow$$

$$\begin{aligned}
P_{X|Y}(1|0) &= \frac{0.08}{0.2} = 0.4;\ P_{X|Y}(2|0) = \frac{0.06}{0.2} = 0.3 \\
P_{X|Y}(3|0) &= \frac{0.04}{0.2} = 0.2;\ P_{X|Y}(4|0) = \frac{0.02}{0.2} = 0.1
\end{aligned}$$

similarly

$$\begin{aligned}
P_{Y|X}(0|1) &= \frac{0.08}{0.4} = 0.2;\ P_{Y|X}(1|1) = \frac{0.2}{0.4} = 0.5;\ P_{Y|X}(2|1) = \frac{0.12}{0.4} = 0.3 \\
P_{Y|X}(0|2) &= \frac{0.06}{0.3} = 0.2;\ P_{Y|X}(1|2) = \frac{0.15}{0.3} = 0.5;\ P_{Y|X}(2|2) = \frac{0.09}{0.3} = 0.3 \\
P_{Y|X}(0|3) &= \frac{0.04}{0.2} = 0.2;\ P_{Y|X}(1|3) = \frac{0.12}{0.2} = 0.6;\ P_{Y|X}(2|3) = \frac{0.04}{0.2} = 0.2
\end{aligned}$$

(4) Notice that $\{x, y \mid X + Y < 5\} = \{x, y \mid X < 4 \cup \{X = 4, Y = 0\} \cap \{X \neq 3, Y = 2\}\} = A$

$$\begin{aligned}
P_{XY}\left(\bigcup_{x,y \in X<4 \cup \{X=4,Y=0\}} x, y\right) &= \sum_{x \in A} \sum_{y \in A} P_{XY}(x,y) \\
&= 1 - \sum_{x \in A} \sum_{y \in A} P_{XY}(x,y) \\
&= 1 - (0.04 + 0.03 + 0.05) \\
&= 0.88
\end{aligned}$$

Thus

$$P(1, 2 \mid X + Y < 5) = \frac{0.12}{0.88} = 0.136$$

Problem 2. The joint PDF of X and Y is, $f_{XY}(x,y) = cxy^2$ s.t $0 < x < y < 1$

(1) Find c.
(2) Find the marginal PDF and CDF of X.
(3) Find the marginal PDF and CDF of Y.

(4) Compute the mean and variance of X and Y.

Answer: (1)

$$
\begin{aligned}
1 &= \int_0^1 \int_x^1 cxy^2 dydx = \int_0^1 cx \int_x^1 y^2 dydx \\
&= \int_0^1 cx \left[\frac{1}{3} - \frac{x^3}{3}\right] dx = \frac{c}{3}\int_0^1 x\left[1-x^3\right] dx \\
&= \frac{c}{3}\left(\frac{x^2}{2} - \frac{x^5}{5}\right) |_0^1 = \frac{c}{3}\left(\frac{1}{2} - \frac{1}{5}\right) \\
&= \frac{c}{10} \\
&\iff \\
c &= 10
\end{aligned}
$$

Thus

$$f_{XY}(x,y) = 10xy^2$$

(2)

$$f_X(x) = \int_x^1 f_{XY}(x,y)\, dy = \int_x^1 10xy^2 dy = 10x\frac{y^3}{3} \; |_x^1 = 10x\left(\frac{1}{3} - \frac{x^3}{3}\right) = \frac{10}{3}x\left(1-x^3\right)$$

$$\iff$$

$$f_X(x) = \frac{10}{3}x\left(1-x^3\right)$$

The CDF of X is:

$$F_X(x) = \int_0^x \frac{10}{3}t\left(1-t^3\right) dt \quad s.t \; x \in (0,1)$$

(3)

$$
\begin{aligned}
f_Y(y) &= \int_0^y f_{XY}(x,y)\, dx = \int_0^y 10xy^2 dx = 5x^2y^2 \; |_0^y = 5y^4 \\
&\iff \\
f_Y(y) &= 5y^4
\end{aligned}
$$

The CDF of Y is:

$$F_Y(y) = \int_0^y 5t^4 dt \quad s.t \; y \in (0,1)$$

(4)

$$\begin{aligned} EX &= \int_0^1 x \cdot f_x(x)\,dx = \int_0^1 x \cdot \frac{10}{3} x\left(1-x^3\right) \\ &= \frac{10}{3}\int_0^1 x^2 - x^5 = \frac{10}{3}\left(\frac{1}{3}-\frac{1}{6}\right) = \frac{10}{3}\cdot\left(\frac{1}{6}\right) = \frac{5}{9} \end{aligned}$$

$$EY = \int_0^1 y \cdot 5y^4 dy = \frac{5}{6}$$

Reminder $Var(X) = EX^2 - E^2X$ thus:

$$\begin{aligned} Var(X) &= \int_0^1 x^2 \cdot f_x(x)\,dx - \left(\frac{5}{9}\right)^2 \\ &= \frac{10}{3}\int_0^1 x^3 - x^6 - \left(\frac{5}{9}\right)^2 = \frac{10}{3}\left(\frac{1}{4}-\frac{1}{7}\right) - \left(\frac{5}{9}\right)^2 \end{aligned}$$

$$Var(Y) = \int_0^1 y^2 \cdot 5y^4 dy - \left(\frac{5}{6}\right)^2 = \frac{5}{7} - \left(\frac{5}{6}\right)^2$$

Problem 3. Let $X \sim Ber(p)$ and it is given that $E(Y \mid X=0) = 1$, $E(Y \mid X=1) = 2$. Calculate EY.

Answer:

$$EY \underset{LTP}{=} E(Y \mid X=0)\cdot P(X=0) + E(Y \mid X=1)\cdot P(X=1) = 1\cdot(1-p) + 2(p) = 1+p$$

Problem 4. Each morning, Hungry Harry eats some eggs. On any given morning, the number of eggs he eats is equally likely to be 1, 2, 3, 4, 5, or 6, independent of what he has done in the past. Let X be the number of eggs that Harry eats in 10 days. Find the mean and variance of X.

Answer: We notate $X_i \sim U(1,6)$ the RV of day i. hence

$$\begin{aligned} EX_i &= \sum_x x P_{X_i}(x) = \frac{1}{6}\sum_{x=1}^{6} x = 3.5 \\ Var(X_i) &= E[X - EX]^2 = \frac{1}{6}\sum_{x=1}^{6}(x-3.5)^2 = \frac{17.5}{6} \approx 2.91 \end{aligned}$$

given the 10 dayes $X = \sum_{i=1}^{10} X_i$

$$\begin{aligned} EX &= E\sum_{i=1}^{10} X_i \underset{linearity}{=} \sum_{i=1}^{10} EX_i \underset{iid}{=} 10\cdot 3.5 = 35 \\ Var(X) &= Var\left(\sum_{i=1}^{10} X_i\right) \underset{indep}{=} \sum_{i=1}^{10} Var(X_i) \underset{iid}{=} 10 Var(X_i) \approx 29.1 \end{aligned}$$

Problem 5. Let X be a continuous RV with PDF $f_X(x)$ and Y a continuous RV with PDF $f_Y(y)$. X and Y are independent. Denote their sum by $Z = X + Y$.

(1) Show that $f_{Z|X=x}(z) = f_Y(z-x)$. Hint: First show that $P(Z \le z \mid X = x) = P(Y \le z - x)$.

(2) Suppose now that X and Y are exponentially distributed with parameter λ (they are still independent). Find the conditional PDF $f_{X|Z=z}(x)$ for every $0 \leq x \leq z$.

Answer: $Y = Z - X$

Proof of the hint:

$$\begin{aligned} P(Z \leq z \mid X = x) &= P(X + Y \leq z \mid X = x) \\ &\underset{X=x}{=} P(x + Y \leq z \mid X = x) \\ &= P(Y \leq z - x \mid X = x) \\ &\underset{X \perp Y}{=} P(Y \leq z - x) \end{aligned}$$

$$\iff$$

$$\begin{aligned} \int_{-\infty}^{z} f_{Z|X=x}(t)\, dt &= \int_{-\infty}^{z-x} f_{Y|X=x}(t)\, dt = \int_{-\infty}^{z-x} \frac{f_{XY}(t,x)}{f_X(x)} dt \\ &\underset{X \perp Y}{=} \int_{-\infty}^{z-x} \frac{f_Y(t) f_X(x)}{f_X(x)} d = \int_{-\infty}^{z-x} f_Y(t)\, d \end{aligned}$$

Thus

$$f_{Z|X=x}(z \mid x) = \frac{f_{YX}(z-x, x)}{f_X(x)} \underset{X \perp Y}{=} \frac{f_Y(z-x) f_X(x)}{f_X(x)} = f_Y(z-x)$$

(2) We now assume $X, Y \sim Exp(\lambda)$

$$\begin{aligned} f_{X|Z=z}(x \mid z) &\underset{Bayes}{=} \frac{f_{Z|X=x}(z \mid x) f_X(x)}{f_Z(z)} = \frac{f_Y(z-x) f_X(x)}{f_Z(z)} \\ &= \frac{f_Y(z-x) f_X(x)}{f_Z(z)} \underset{*}{=} \frac{\lambda e^{-\lambda(z-x)} \lambda e^{-\lambda x}}{\lambda^2 z e^{-\lambda z}} \\ &= \frac{e^{-\lambda(z-x+x)}}{z e^{-\lambda z}} = \frac{1}{z} \end{aligned}$$

* Note that

$$\begin{aligned} f_Z(z) &= \int_0^z f_Z(z)\, dz = \int_0^z f_{YX}(z-x, x)\, dz \\ &\underset{X \perp Y}{=} \int_0^z f_Y(z-x) f_X(x)\, dz \\ &= \int_0^z \lambda e^{-\lambda(z-x)} \lambda e^{-\lambda x} dz \\ &= \lambda^2 z e^{-\lambda z} \end{aligned}$$

Problem 6. Let $X \sim Ber(p)$, $Y \sim Geo(p)$ be two independent random variables and define $Z = XY$. Express in terms of p the covariance $Cov(Z, X)$ and $Cov(Z, Y)$.

$$
\begin{aligned}
Cov\left(Z,X\right) = E\left(Z-EZ\right)\left(X-EX\right) = E\left(XY-EXY\right)\left(X-\mu_X\right) &\underset{X\perp Y}{=} E\left[\left(XY-\mu_X\mu_Y\right)\left(X-\mu_X\right)\right] \\
&= E\left[XYX - XY\mu_X - X\mu_X\mu_Y + \mu_X^2\mu_Y\right] \\
&\underset{linearity}{=} E\left[X^2Y\right] - EXY\mu_X - \mu_X\mu_X\mu_Y + \mu_X^2\mu_Y \\
&\underset{X\perp Y}{=} E\left[X^2Y\right] - \mu_X^2\mu_Y - \mu_X^2\mu_Y + \mu_X^2\mu_Y \\
&= E\left[X^2Y\right] - \mu_X^2\mu_Y \\
&= Cov\left(X^2,Y\right)
\end{aligned}
$$

So we can say

$$
\begin{aligned}
Cov\left(Z,X\right) = Cov\left(XY,X\right) &\underset{X\perp Y}{=} Cov\left(Y,X^2\right) \\
&= E\left[X^2Y\right] - p^2\frac{1}{p} \\
&\underset{X\perp Y}{=} EX^2EY - p \\
&= p\cdot\frac{1}{p} - p \\
&= 1-p
\end{aligned}
$$

and similarly

$$
\begin{aligned}
Cov\left(Z,Y\right) = Cov\left(XY,Y\right) &= Cov\left(Y^2,X\right) \\
&= EY^2X - E^2YEX \\
&= EY^2\cdot p - \frac{1}{p} \\
&\underset{*}{=} \left(\frac{2-p}{p^2}\right)\cdot p - \frac{1}{p} \\
&= \frac{1-p}{p}
\end{aligned}
$$

* See the following:

$$
\begin{aligned}
EY^2 = \sum_{k=0}^{\infty} k^2 P\left(Y=k\right) &= \sum_{k=0}^{\infty} k\left(k-1+1\right) p\left(1-p\right)^{k-1} \\
&= \sum_{k=0}^{\infty} k\left(k-1\right) p\left(1-p\right)^{k-1} + \sum_{k=0}^{\infty} kp\left(1-p\right)^{k-1} \\
&= \sum_{k=0}^{\infty} k\left(k-1\right) p\left(1-p\right)^{k-1} + EY \\
&= \sum_{k=0}^{\infty} k\left(k-1\right) p\left(1-p\right)\left(1-p\right)^{k-2} + EY \\
&= p\left(1-p\right) \sum_{k=0}^{\infty} k\left(k-1\right)\left(1-p\right)^{k-2} + EY \\
&= p\left(1-p\right) \frac{\partial^2}{\partial^2 p}\left[\left(-1\right)\cdot\left(-1\right)\cdot \sum_{k=0}^{\infty}\left(1-p\right)^k\right] + EY \\
&= p\left(1-p\right) \frac{\partial^2}{\partial^2 p}\left[\frac{1}{p}\right] + \frac{1}{p} = p\left(1-p\right) \frac{\partial^2}{\partial^2 p}\left[p^{-1}\right] + \frac{1}{p} \\
&= p\left(1-p\right)\cdot\left(-1\right)\cdot\left(-2\right)\cdot p^{-3} + \frac{1}{p} \\
&= \frac{2\left(1-p\right)}{p^2} + \frac{1}{p} = \frac{2\left(1-p\right)}{p^2} + \frac{p}{p^2} = \frac{2\left(1-p\right)+p}{p^2} \\
&= \frac{2-p}{p^2}
\end{aligned}
$$

Note that $Var\left(Y\right) = \frac{1-p}{p^2} = EY^2 - E^2Y \iff EY^2 = \frac{1-p}{p^2} + \left(\frac{1}{p}\right)^2 = \frac{2-p}{p^2}$