Robotikk oblig3

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Task 1: Dynamics I (45)

a)

$$m\ddot{y} = f - mg$$

$$m\ddot{y} = \frac{d}{dt}(m\dot{y}) = \frac{d}{dt}\frac{\partial d}{\partial \ddot{y}}(\frac{1}{2}m\dot{y}^2) = \frac{d}{dt}\frac{\partial \kappa}{\partial \dot{y}} \qquad mg = \frac{\partial}{\partial y}(mgy) = \frac{\partial P}{\partial y}$$

$$L = K - P = \frac{1}{2}m\dot{y}^2 - mgy$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{\partial K}{\partial \dot{y}} \qquad \frac{\partial L}{\partial y} = \frac{\partial P}{\partial y}$$

$$f = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y}$$

$$\tau_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k}$$

b)

$$J = \begin{bmatrix} -s_1 c_2 L_2 & -c_1 s_2 L_2 \\ c_1 c_2 L_2 & -s_1 s_2 L_2 \\ 0 & c_2 L_2 \\ 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix}$$

$$K_i = \left[m\boldsymbol{v_i}^T\boldsymbol{v_i} + \boldsymbol{\omega} RIR^T\boldsymbol{\omega}\right]$$

$$K_1 = \frac{1}{2}m_1\boldsymbol{v_1}^T\boldsymbol{v_1} + \frac{1}{2}\boldsymbol{\omega_1} RI_1R^T\boldsymbol{\omega_1}$$

$$\boldsymbol{v_1} = Jv_1\dot{\boldsymbol{q}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{\omega_1} = J\omega_1\dot{\boldsymbol{q}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix}$$

$$R_1^0 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R_1^0)^T = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{m_1 r_1^2}{2} \end{bmatrix}$$

$$K_1 = \frac{1}{2} m_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix}^T \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{m_1 r_1^2}{2} \end{bmatrix} \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{q_1} \end{bmatrix}$$

$$= \frac{\dot{q_1}^2 m_1 r_1^2}{4}$$

$$K_2 = \frac{1}{2} m_2 \boldsymbol{v_2}^T \boldsymbol{v_2} + \frac{1}{2} \boldsymbol{\omega_2} R I_2 R^T \boldsymbol{\omega_2}$$

$$\boldsymbol{v_2} = J v_2 \dot{\boldsymbol{q}} = \begin{bmatrix} -s_1 c_2 L_2 & -c_1 s_2 L_2 \\ c_1 c_2 L_2 & -s_1 s_2 L_2 \\ 0 & c_2 L_2 \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} -c_2 s_1 L_2 \dot{q_1} - c_1 s_2 L_2 \dot{q_2} \\ c_1 c_2 L_2 \dot{q_1} - s_1 s_2 L_2 q_2 \\ c_2 L_2 \dot{q_2} \end{bmatrix}$$

$$\boldsymbol{\omega_2} = J \omega_2 \dot{\boldsymbol{q}} = \begin{bmatrix} 0 & s_1 \\ 0 & -c_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} s_1 \dot{q_2} \\ -c_1 \dot{q_2} \\ \dot{q_1} \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$--> RIR = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$K_2 = \frac{1}{2} m_2 \begin{bmatrix} -c_2 s_1 L_2 \dot{q_1} - c_1 s_2 L_2 \dot{q_2} \\ c_1 c_2 L_2 \dot{q_1} - s_1 s_2 L_2 q_2 \\ c_2 L_2 \dot{q_2} \end{bmatrix}^T \begin{bmatrix} -c_2 s_1 L_2 \dot{q_1} - c_1 s_2 L_2 \dot{q_2} \\ c_1 c_2 L_2 \dot{q_1} - s_1 s_2 L_2 q_2 \\ c_2 L_2 \dot{q_2} \end{bmatrix}$$

$$+ \begin{bmatrix} s_1 \dot{q_2} \\ -c_1 \dot{q_2} \\ \dot{q_1} \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_1 \dot{q_2} \\ -c_1 \dot{q_2} \\ \dot{q_1} \end{bmatrix}^T$$

$$= \frac{1}{2} m_2 \left[c_2^2 s_1^2 L_2^2 \dot{q_1}^2 + c_1^2 c_2^2 L_2^2 \dot{q_1}^2 + s_1^2 s_2^2 L_2^2 \dot{q_2}^2 + c_1^2 s_2^2 L_2^2 \dot{q_2}^2 + c_2^2 L_2^2 \dot{q_2}^2 \right]$$

$$= \frac{1}{2} m_2 \left[c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2 \right]$$

$$K = K_1 + K_2$$

$$= \frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2} m_2 \left[c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2\right]$$

$$P_i = m_i g^T r_{ci}$$

$$g^T = \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}^T$$

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} m_i g^T r_{ci}$$

$$= \left(m_1 g^T r_{c1} + m_2 g^T r_{c2}\right)$$

$$r_{c1} = \begin{bmatrix} 0 \\ 0 \\ \frac{L1}{2} \end{bmatrix}$$

$$P_1 = m_1 \begin{bmatrix} 0 & 0 & g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{L1}{2} \end{bmatrix} = m_1 g \frac{L_1}{2}$$

$$r_{c2} = \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 s_2 \end{bmatrix}$$

$$P_2 = m_2 \begin{bmatrix} 0 & 0 & g \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ L_1 + L_2 s_2 \end{bmatrix}$$

$$= m_2 g(L_1 + L_2 s_2)$$

Se nederst for figur til utregning av r_{c2} til P_2 over. (figur 1.)

$$
\begin{aligned}
P &= P_1 + P_2 \\
&= m_1 g \frac{L_1}{2} + m_2 g (L_1 + L_2 s_2)
\end{aligned}
$$

$$
\begin{aligned}
L &= K - P \\
&= \frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2} m_2 \left[c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2 \right] \left[m_1 g \tfrac{L_1}{2} + m_2 g (L_1 + L_2 s_2) \right]
\end{aligned}
$$

c)

Finner euler- lagrange, τ_1 og τ_2:

$$\tau_k = \frac{d}{dt}\frac{\partial L}{\partial \dot{q_k}} - \frac{\partial L}{\partial q_k}; \qquad k = 1, 2$$

$$\tau_1 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} - \frac{\partial L}{\partial q_1}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_1}} = \frac{d}{dt}\frac{\partial \left[\frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2}m_2(c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2)\right]}{\partial \dot{q_1}}$$

$$= \frac{d\left[\frac{1}{2}\dot{q_1} m_1 r_1^2 + m_2 c_2^2 L_2^2 \dot{q_1}\right]}{dt}$$

$$= \frac{1}{2}\ddot{q_1} m_1 r_1^2 + m_2 c_2^2 L_2^2 \ddot{q_1} - 2 m_2 c_2 s_2 L_2^2 \dot{q_1}\dot{q_2}$$

$$\frac{\partial L}{\partial q_1} = \frac{\partial \left[\frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2}m_2(c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2)\right]}{\partial q_1}$$

$$= 0$$

$$\tau_1 = \frac{1}{2}\ddot{q_1} m_1 r_1^2 + m_2 c_2^2 L_2^2 \ddot{q_1} - 2 m_2 c_2 s_2 L_2^2 \dot{q_1}\dot{q_2} - 0$$

$$= \frac{1}{2}\ddot{q_1} m_1 r_1^2 + m_2 c_2^2 L_2^2 \ddot{q_1} - 2 m_2 c_2 s_2 L_2^2 \dot{q_1}\dot{q_2}$$

$$\tau_2 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q_2}} - \frac{\partial L}{\partial q_2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q_2}} = \frac{d}{dt}\frac{\partial \left[\frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2}m_2(c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2)\right]}{\partial \dot{q_2}}$$

$$= \frac{d\left[m_2 L_2^2 \dot{q_2}\right]}{dt}$$

$$= m_2 L_2^2 \ddot{q_2}$$

$$\frac{\partial L}{\partial q_2} = \frac{\partial \left[\frac{\dot{q_1}^2 m_1 r_1^2}{4} + \frac{1}{2}m_2(c_2^2 L_2^2 \dot{q_1}^2 + L_2^2 \dot{q_2}^2)\right]}{\partial q_2}$$

$$= -m_2 c_2 s_2 L_2^2 \dot{q_1}^2 - m_2 g c_2 L_2$$

$$\tau_2 = m_2 L_2^2 \ddot{q_2} - \left(-m_2 c_2 s_2 L_2^2 \dot{q_1}^2 - m_2 g L_2 c_2\right)$$

$$= m_2 L_2^2 \ddot{q_2} + m_2 c_2 s_2 L_2^2 \dot{q_1}^2 + m_2 g c_2 L_2$$

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q), \quad \tau \,\epsilon\, R^2$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}m_1 r_1^2 \ddot{q}_1 + m_2 c_2^2 L_2^2 \ddot{q}_1 - 2m_2 c_2 s_2 L_2^2 \dot{q}_1 \dot{q}_2 \\ m_2 L_2^2 \ddot{q}_2 + m_2 c_2 s_2 L_2^2 \dot{q}_1{}^2 + m_2 g c_2 L_2 \end{bmatrix}$$

$$= \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}m_1 r_1^2 + m_2 c_2^2 L_2^2 & 0 \\ 0 & m_2 L_2^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -2m_2 c_2 s_2 L_2^2 \dot{q}_2 & 0 \\ m_2 c_2 s_2 L_2^2 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ m_2 g c_2 L_2 \end{bmatrix}$$

Task 2: Dynamics II (55)

a)

Løst med Python

b)

Løst med Python

c)

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q)$$

$D(q)$:

$$\tau = \begin{bmatrix} \tau_k \\ . \\ . \\ . \\ \tau_n \end{bmatrix} = \begin{bmatrix} d_{kj} & . & . & . & d_{kn} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ d_{nj} & . & . & . & d_{nn} \end{bmatrix} (q) \begin{bmatrix} \ddot{q}_j \\ . \\ . \\ . \\ \ddot{q}_n \end{bmatrix}, \quad k,j = 1,...,n$$

$rad\ k = d_{kj}$

$$\tau_k = \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + (...) + (...), \quad k = 1,...,n$$

$C(q,\dot{q})\dot{q}$:

$$\tau = \begin{bmatrix} \tau_k \\ . \\ . \\ . \\ \tau_n \end{bmatrix} = \begin{bmatrix} c_{kj} & . & . & . & c_{kn} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ c_{nj} & . & . & . & c_{nn} \end{bmatrix} (q,\dot{q}) \begin{bmatrix} \dot{q}_j \\ . \\ . \\ . \\ \dot{q}_n \end{bmatrix}, \quad k,j = 1,...,n$$

$$= \sum_{j=1}^{n} c_{kj}\dot{q}_j$$

$$c_{kj} = \begin{bmatrix} c_{ikj} & . & . & . & c_{ikn} \\ . & & & & . \\ . & & & & . \\ . & & & & . \\ c_{inj} & . & . & . & c_{nnn} \end{bmatrix} (q) \begin{bmatrix} \dot{q}_i \\ . \\ . \\ . \\ \dot{q}_n \end{bmatrix}, \quad i,k,j = 1,...,n$$

$$= \sum_{i=1}^{n} c_{kji}(q)\dot{q}_i$$

$$\tau_k = (...) + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{kji}(q)\dot{q}_i\dot{q}_j + (...), \quad k = 1,...,n$$

$g(q)$:

$$\tau = \begin{bmatrix} \tau_k \\ . \\ . \\ . \\ \tau_n \end{bmatrix} = \begin{bmatrix} g_k \\ . \\ . \\ . \\ g_n \end{bmatrix} (q), \quad k, j = 1, ..., n$$

$$\tau_k = (...) + (...) + g_k(q), \quad k = 1, ..., n$$

$$\tau_k = \sum_{j=1}^{n} d_{kj}(q)\ddot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q), \quad k = 1, ..., n$$

d)

Løst med Python

e)

Løst med Python