Pull-in calculations

Here, derivations for the minimum necessary pulling force are shown for four different pull-in arrangement geometries. In the following sections, the following parameters are used:

$\boldsymbol{w}$ is cable weight [kg/m]

$\boldsymbol{\mu_s}$ is static coefficient of friction [-]

$\boldsymbol{m}$ is cable mass [kg]

$\boldsymbol{g}$ is gravitational acceleration [m/s^2]

$\boldsymbol{G}$ is gravitational force [N]

$\boldsymbol{N}$ is normal force [N]

$\boldsymbol{F_f}$ is frictional force [N]

$\boldsymbol{T_{in}}$ is input pulling force [N]

$\boldsymbol{T_{out}}$ is output pulling force [N]

$\boldsymbol{L}$ is cable length [m]

$\boldsymbol{\theta}$ is angle in vertical plane (pitch) [deg]

$\boldsymbol{\psi}$ is angle in horizontal plane (yaw) [deg]

$\boldsymbol{R}$ is radius of bend length [m]

A summary of the derived equations are listed in below

Section type	Equation
Straight	$T_{out} > T_{in} + wLg(\mu_s \cos\theta + \sin\theta)$
Horizontal Bend	$T_{out} > T_{in} e^{\mu_s(\psi_{out} - \psi_{in})}$
Top Bend	$T_{out} > T_{in} e^{\mu_s(\theta_{out}-\theta_{in})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(-\cos\theta_{out} + \cos\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right) + 2\mu_s\left(\sin\theta_{out} - \sin\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right)\right]$
Bottom bend	$T_{out} > \begin{cases} T_{in} e^{\mu_s(\theta_{in}-\theta_{out})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta_{out} - \cos\theta_{in}\, e^{\mu_s(\theta_{in}-\theta_{out})}\right) + 2\mu_s\left(\sin\theta_{out} - \sin\theta_{in}\, e^{\mu_s(\theta_{in}-\theta_{out})}\right)\right], \lvert\theta\rvert < \theta_c \\ T_{in} e^{\mu_s(\theta_{out}-\theta_{in})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta_{out} - \cos\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right) - 2\mu_s\left(\sin\theta_{out} - \sin\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right)\right], \lvert\theta\rvert > \theta_c \end{cases}$

Table 1: Summary of equations for different pull-in arrangements

Straight section

Consider a cable section lying on an inclined straight plane with a pulling force at each end. The force diagram is then as illustrated in Figure 1.

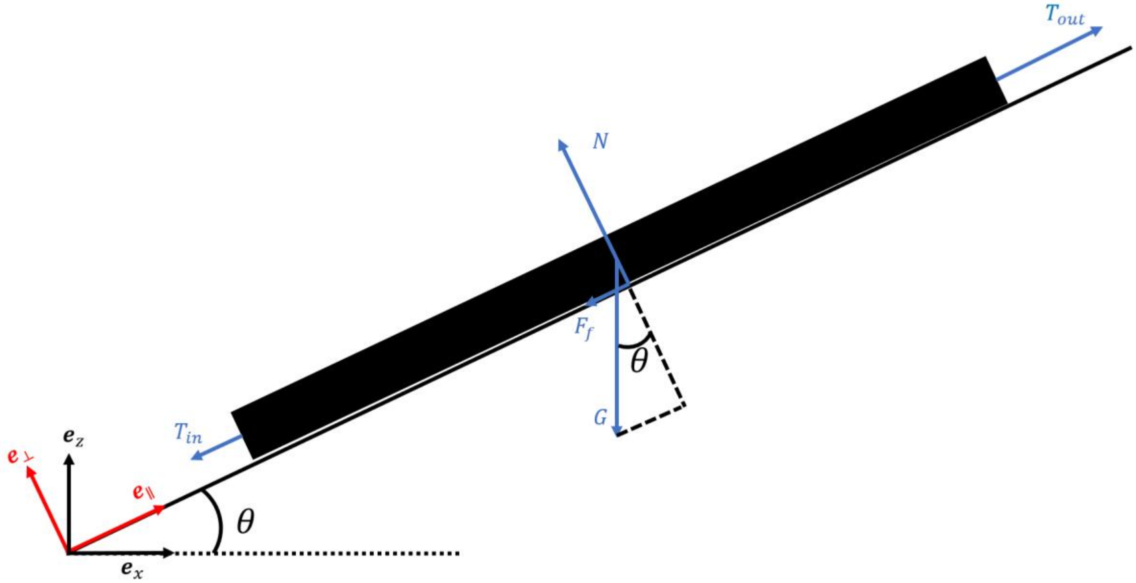

Figure 1: Force diagram for straight section

Note that for movement in the direction of T_{out}, the friction force must be in the opposite direction. The normal force is calculated with a force balance in the $\boldsymbol{e}_\perp$ direction as such

$$\sum F_\perp = N - G_\perp = 0$$

$$\Rightarrow N = G_\perp = wLg\cos\theta$$

Demanding positive movement in the direction of $\boldsymbol{e}_\parallel$ gives

$$\sum F_\parallel = T_{out} - G_\parallel - F_f - T_{in} > 0,$$

where $G_\parallel = wLg\, sin\,\theta$ and $F_f = \mu_s N.$ Substituting yields

$$T_{out} > T_{in} + wLg(\mu_s \cos\theta + \sin\theta)$$

Top bend section

Now, consider the case in which the cable is resting on top of a vertical bend, as illustrated in Figure 2. The angle θ is defined such that it represents the pitch of the $\boldsymbol{e}_\parallel, \boldsymbol{e}_\perp$ coordinate system. The arrangement is therefore from right to left, as to keep with the convention of increasing θ values in the pulling direction. This is an arbitrary choice, and the resulting pulling tension is off course the same for the mirror image.

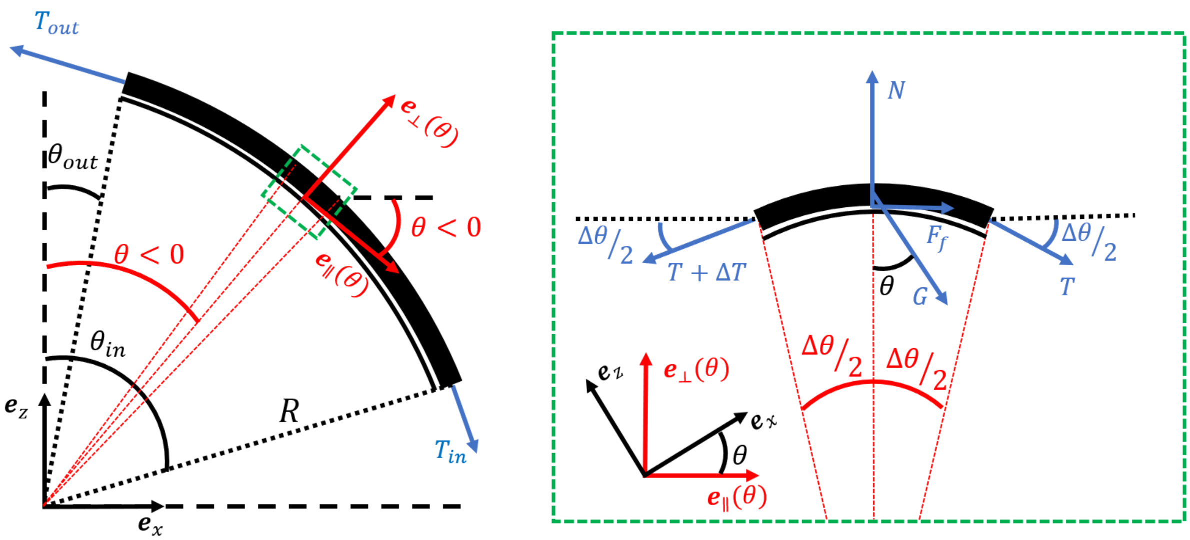

Figure 2: Force diagram for top bend section. To the left is the global view, and the green box to the right is a view of the forces in the local $\boldsymbol{e}_\parallel, \boldsymbol{e}_\perp$ coordinate system for a small section of the bend

To find the normal force, we can do a force balance in the local $\mathbf{e}_\perp$ coordinate as such

$$\sum F_\perp = N - G_\perp - T_\perp - (T + \Delta T)_\perp = 0$$

$$\Rightarrow \sum F_\perp = N - mg\cos\theta - T\sin\frac{\Delta\theta}{2} - (T + \Delta T)\sin\frac{\Delta\theta}{2} = 0$$

Assuming the section is sufficiently small, i.e. $\Delta\theta \approx 0$, the first order Taylor expansion $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$ holds. The mass of the section is simply

$$m = w(2\pi\Delta\theta/2\pi)R = wR\Delta\theta$$

Inserting these expressions into the equation above gives the following expression for the normal force.

$$N = wgR\Delta\theta\cos\theta + T\Delta\theta$$

The normal force is thus divided into two parts, the first resulting from the gravitational force onto the bend, and the second resulting from the pulling forces of adding an orthogonal force on the bend. The latter is known as the capstan effect. Doing a similar force balance in the local $\mathbf{e}_\parallel$ coordinate system gives

$$\sum F_\parallel = T_\parallel - (T + \Delta T)_\parallel + G_\parallel + F_f = 0$$

$$\Rightarrow \sum F_\parallel = T\cos\frac{\Delta\theta}{2} - (T + \Delta T)\cos\frac{\Delta\theta}{2} - mg\sin\theta + \mu_s N = 0$$

Note that $G_\parallel$ has a negative sign, which comes from

$$G_\parallel = \boldsymbol{G}\cdot\boldsymbol{e}_\parallel = -mg\boldsymbol{e}_z\cdot\boldsymbol{e}_\parallel = -mg\sin\theta$$

The ostensible deviance from the illustration comes from the fact that the illustration is shown for a value of $\theta < 0$. With the same small section argument as above, we can use the first order Taylor expansion $\cos\frac{\Delta\theta}{2} \approx 1$ to give

$$\Delta T = \mu_s N - mg\sin\theta$$

Inserting gives previously defined relations gives

$$\Delta T = [wgR(\mu_s\cos\theta - \sin\theta) + \mu_s T]\Delta\theta$$

Rearranging and letting $\Delta\theta \to 0$ gives the ODE

$$\frac{dT}{d\theta} - \mu_s T = \ wgR(\mu_s\cos\theta - \sin\theta)$$

This can be solved by the method of integrating factor as such:

$$\frac{d}{d\theta}\left(Te^{-\mu_s\theta}\right)e^{\mu_s\theta} = \ wgR(\mu_s\cos\theta - \sin\theta)$$

$$\Rightarrow \int d(Te^{-\mu_s\theta}) = \ wgR\int(\mu_s\cos\theta - \sin\theta)e^{-\mu_s\theta}d\theta$$

$$\Rightarrow Te^{-\mu_s\theta} = \ \frac{e^{-\mu_s\theta}wgR}{\mu_s^2+1}(-(\mu_s^2-1)\cos\theta + 2\mu_s\sin\theta) + D$$

$$\Rightarrow T(\theta) = \ \frac{wgR}{\mu_s^2+1}(-(\mu_s^2-1)\cos\theta + 2\mu_s\sin\theta) + De^{\mu_s\theta}$$

Where D is some constant. This constant can be found by using the boundary condition $T(\theta_{in}) = T_{in}$ which after some algebra gives

$$D = \left[T_{in} - \frac{wgR}{\mu_s^2+1}(-(\mu_s^2-1)\cos\theta + 2\mu_s\sin\theta)\right]e^{-\mu_s\theta_{in}}$$

$$\Rightarrow T(\theta) = T_{in}e^{\mu_s(\theta-\theta_{in})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(-\cos\theta + \ \cos\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right) + 2\mu_s\left(\sin\theta - \ \sin\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right)\right]$$

Recall that this equation constitutes the minimum necessary pulling force along any θ. The final equation can thus be found by replacing the equality with an inequality and evaluating at θ_{out}:

$$T_{out} > T_{in}e^{\mu_s(\theta_{out}-\theta_{in})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(-\cos\theta_{out} + \ \cos\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right) + 2\mu_s\left(\sin\theta_{out} - \ \sin\theta_{in}\, e^{\mu_s(\theta_{out}-\theta_{in})}\right)\right]$$

The first term is easily recognizable as the capstan equation. There are two other similar exponential terms, which stem from the effect gravity has on the normal force. As a matter of fact, in the absence of cable weight, i.e. $w = 0$, the entire square bracket term vanishes and we are simply left with the capstan equation. This is sensical as the capstan equation assumes a massless rope. Furthermore, if we ignore the effect of friction by setting $\mu_s = 0$, we are left with $T_{out} = wgR\ (\cos\theta_{out} - \ \cos\theta_{in}) = wg\Delta h$, another well-known result for the lifting force.

Bottom bend section

Now, consider the case in which the cable is pushed against the roof of a vertical bend, such as illustrated in Figure 3.

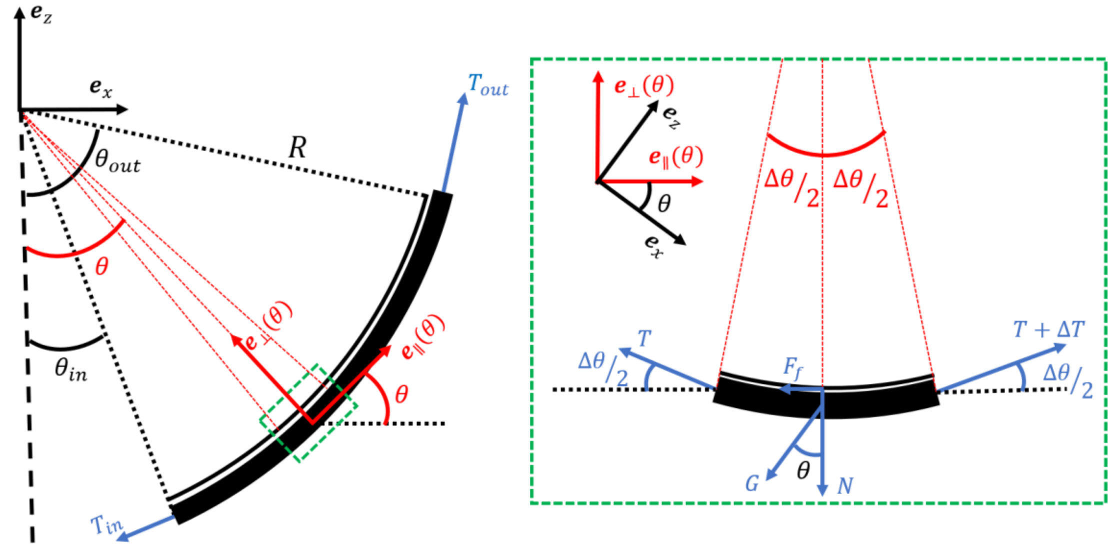

Figure 3: Force diagram for bottom bend section. To the left is the global view, and the green box to the right is a view of the forces in the local $\mathbf{e}_\parallel$,$\mathbf{e}_\perp$ coordinate system for a small section of the bend

The force balance is now similar to the top bend example from above. However, contrary to above, the gravitational pull of the cable acts to *decrease* the normal force and thus *reduces* the capstan effect. The normal force is like before found with a force balance in the $\mathbf{e}_\perp$ direction

$$\sum F_\perp = -N - G_\perp + T_\perp + (T + \Delta T)_\perp = 0$$

$$\Rightarrow \sum F_\perp = -N - mg\cos\theta + T\sin\frac{\Delta\theta}{2} + (T + \Delta T)\sin\frac{\Delta\theta}{2} = 0$$

$$\Rightarrow N = -wgR\Delta\theta\cos\theta + T\Delta\theta$$

The force balance in the $\mathbf{e}_\parallel$ direction gives

$$\sum F_\parallel = -T_\parallel + (T + \Delta T)_\parallel - G_\parallel - F_f = 0$$

$$\Rightarrow \sum F_\parallel = -T\cos\frac{\Delta\theta}{2} + (T + \Delta T)\cos\frac{\Delta\theta}{2} - mg\sin\theta + \mu_s N = 0$$

Which with the same arguments as above simplifies to the ODE

$$\frac{dT}{d\theta} - \mu_s T = wgR(-\mu_s\cos\theta + \sin\theta)$$

Except for the signs deviating on the right hand side, this is the same ODE as before. We can use the same method as before to give the final inequality

$$T_{out} > T_{in}e^{\mu_s(\theta_{out}-\theta_{in})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta_{out} - \cos\theta_{in}\,e^{\mu_s(\theta_{out}-\theta_{in})}\right) + 2\mu_s\left(-\sin\theta_{out} + \sin\theta_{in}\,e^{\mu_s(\theta_{out}-\theta_{in})}\right)\right]$$

Again, we are left with the catenary equation for $w = 0$ and the lifting equation for $\mu_s = 0$. Note that so far in this section, we have assumed tension in the cable large enough for it to be pinned against the roof of the bend. The above equation is off course only valid insofar as this assumption holds. We can check whether or not there is contact with the roof by the condition that the normal force is greater than zero for all angles, i.e.

$$T(\theta) > wgR\cos\theta$$

For sections where this inequality does not hold, the cable will sag like a catenary in the absence of any bottom support. In most cases for geometries like these though, there will be a supporting structure immediately beneath the cable, like in a pipe for example. In that case, the normal force points in the diametrically opposite direction, such that

$$N = wgR\Delta\theta \cos\theta - T\Delta\theta$$

I.e., the tension in the cable will provide a lifting force on the cable and thus *decrease* the normal force. Inserting the normal force in the $\mathbf{e}_{\parallel}$ force balance, solving the ODE and inserting the boundary condition $T(\theta_{in}) = T_{in}$, then

$$T_{out} > T_{in}e^{\mu_s(\theta_{in}-\theta_{out})} + \frac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(cos\,\theta_{out} - cos\,\theta_{in}\,e^{\mu_s(\theta_{in}-\theta_{out})}\right) + 2\mu_s\left(sin\,\theta_{out} - sin\,\theta_{in}\,e^{\mu_s(\theta_{in}-\theta_{out})}\right)\right]$$

It is now obvious that the capstan-like effect here has a negative contribution, revealing the tension lifting effect mentioned earlier. For a bottom bend section, the minimum pulling force along the cable can thus be summarized as follows

$$T(\theta) = \begin{cases} T_{in}e^{\mu_s(\theta_{in}-\theta)} + \dfrac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in}\,e^{\mu_s(\theta_{in}-\theta)}\right) + 2\mu_s\left(\sin\theta - \sin\theta_{in}\,e^{\mu_s(\theta_{in}-\theta)}\right)\right], |\theta| < \theta_c \\ T_{in}e^{\mu_s(\theta-\theta_{in})} + \dfrac{wgR}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in}\,e^{\mu_s(\theta-\theta_{in})}\right) - 2\mu_s\left(\sin\theta - \sin\theta_{in}\,e^{\mu_s(\theta-\theta_{in})}\right)\right], |\theta| > \theta_c \end{cases}$$

where θ_c signifies the critical angle where normal force is zero.

Horizontal bend section

With a bend in the horizontal direction, the pulling forces are unaffected by gravity. This is then equivalent to the two cases above in which tension in the cable adds to the normal force, only now ignoring gravity, and using the angle in the horizontal direction. As we have seen before, the minimum pulling force then reduces to the capstan equation

$$T_{out} > T_{in}e^{\mu_s(\psi_{out}-\psi_{in})}$$