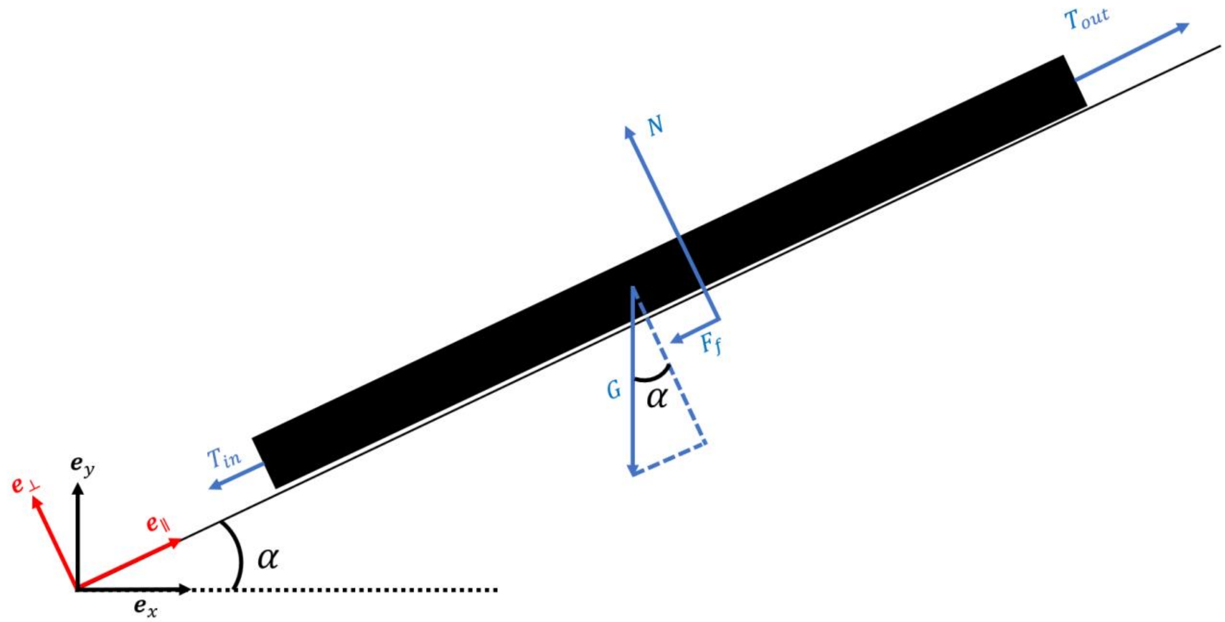
Straight section

The force diagram is as follows:



Note that for movement in the direction of T_{out}, the friction force must be in the opposite direction. The normal force is calculated as

$$\sum F_\perp = N - G_\perp = 0$$

$$N = G_\perp = \rho_c Lg \sin\alpha$$

, where L is the length of the cable and ρ_c is the 1D cable density. Demanding positive movement in the direction of $\boldsymbol{e}_\parallel$ gives:

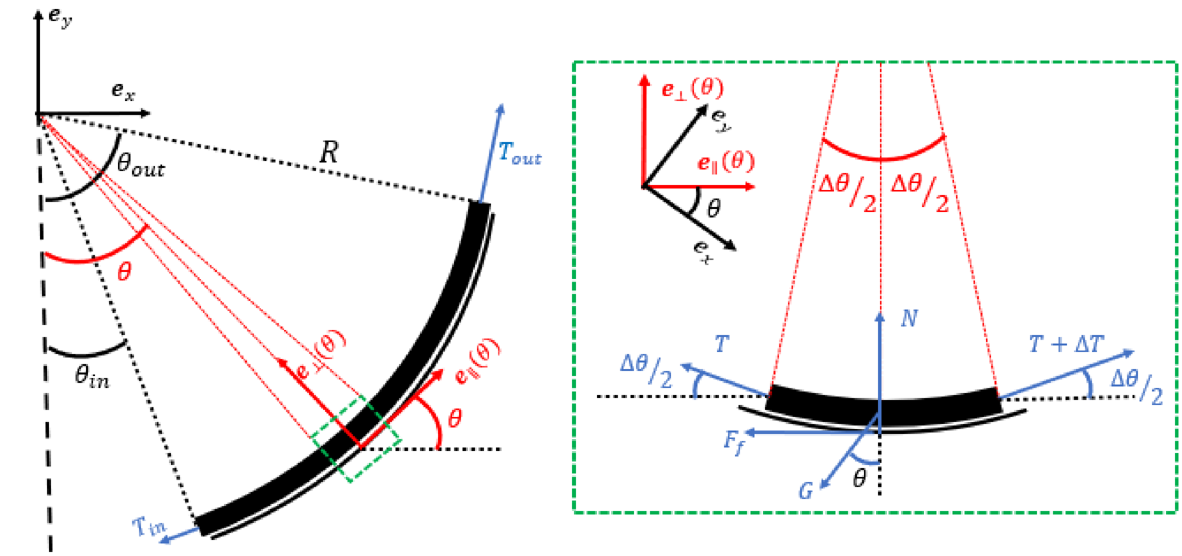
$$\sum F_\parallel = T_{out} - G_\parallel - F_f - T_{in} > 0,$$

where $G_\perp = \rho_c Lg\, sin\,\alpha$ and $F_f = \mu_s N.$ Substituting yields

$$T_{out} > T_{in} + \rho_c Lg(\mu_s \cos\alpha + \sin\alpha)$$

Vertical bottom bend

Consider a cable in a pipe in a vertical bend with radius R. Temporarily assume the cable rests on the floor of the pipe, which is to say the cable is positioned in-between the bend surface and the center of curvature. The force diagram is then as follows:



Now, the unit vectors are not constant like previously, but rather a function of the θ, i.e. the pitch of the coordinate system. A small subsection of the cable is highlighted in green and a zoomed in version of the subsection is shown in the local coordinate system, which is rotated θ from the world coordinates. Summing forces in the normal direction to zero gives

$$\sum F_\perp = N - G_\perp + T_\perp + (T + \Delta T)_\perp = 0$$

$$\Rightarrow \sum F_\perp = N - mg\cos\theta + T\sin\frac{\Delta\theta}{2} + (T + \Delta T)\sin\frac{\Delta\theta}{2} = 0$$

The mass m of the section is $m = \rho_c \Delta s = \rho_c\left(\pi R\frac{\Delta\theta}{2\pi}\right) = \rho_c R\Delta\theta$. For small angles, the approximation $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$ holds. Substituting and solving for the normal force yields:

$$N = \rho_c g R\Delta\theta\cos\theta - T\Delta\theta$$

The force imbalance in the parallel direction reads:

$$\sum F_\parallel = -T_\parallel + (T + \Delta T)_\parallel - G_\parallel - F_f > 0$$

$$\Rightarrow \sum F_\parallel = -T\cos\frac{\Delta\theta}{2} + (T + \Delta T)\cos\frac{\Delta\theta}{2} - mg\sin\theta - \mu_s N > 0$$

Again using the small angle approximation gives $\cos\frac{\Delta\theta}{2} \approx 1$. Substituting we get:

$$\Delta T - mg\sin\theta - \mu_s N \ > 0$$

Substituting in for the normal force and solving for ΔT gives

$$\Delta T > \ \mu_s(\rho_c g R \Delta\theta\cos\theta - T\Delta\theta) + \rho_c g R \Delta\theta \sin\theta$$

$$\Rightarrow \frac{\Delta T}{\Delta\theta} > \ \mu_s(\rho_c g R\cos\theta - T) + \rho_c g R\sin\theta$$

We are interested in finding the minimum tension in the cable such that there is positive moment in the $\boldsymbol{e}_{\parallel}$ direction. We therefore replace the inequality with an equation to find this minimum value. Letting $\Delta\theta \to 0$ gives the first order differential equation:

$$\frac{dT}{d\theta} = \ \mu_s(\rho_c g R\cos\theta - T) + \rho_c g R\sin\theta$$

$$\Rightarrow \frac{dT}{d\theta} + \mu_s T = \ \rho_c g R(\mu_s\cos\theta + \sin\theta)$$

This can be solved by the method of integrating factor as such:

$$\frac{d}{d\theta}(Te^{\mu_s\theta})e^{-\mu_s\theta} = \ \rho_c g R(\mu_s\cos\theta + \sin\theta)$$

$$\Rightarrow \int d(Te^{\mu_s\theta}) = \ \rho_c g R\int(\mu_s\cos\theta + \sin\theta)e^{\mu_s\theta}d\theta$$

$$\Rightarrow Te^{\mu_s\theta} = \ \frac{e^{\mu_s\theta}\rho_c g R}{\mu_s^2+1}\left((\mu_s^2-1)\cos\theta + 2\mu_s\sin\theta\right) + D.$$

$$\Rightarrow T(\theta) = \ \frac{\rho_c g R}{\mu_s^2+1}\left((\mu_s^2-1)\cos\theta + 2\mu_s\sin\theta\right) + De^{-\mu_s\theta}$$

for some unknown constant D. We introduce some constants to make the calculation more orderly:

$$A = \ \frac{\rho_c g R}{\mu_s^2+1}$$

$$B = \ (\mu_s^2 - 1)$$

$$C = 2\mu_s$$

Such that

$$T(\theta) = \ A(B\cos\theta + C\sin\theta) + De^{-\mu_s\theta}$$

Using the boundary condition $T(\theta_{in}) = T_{in}$ we can solve for D:

$$T(\theta_{in}) = T_{in} = \ A(B\cos\theta_{in} + C\sin\theta_{in}) + De^{-\mu_s\theta_{in}}$$

$$\Rightarrow D = \ [T_{in} - A(B\cos\theta_{in} + C\sin\theta_{in})]e^{\mu_s\theta_{in}}$$

Such that the full equation reads:

$$T(\theta) = \ A(B\cos\theta + C\sin\theta) + [T_{in} - A(B\cos\theta_{in} + C\sin\theta_{in})]e^{\mu_s(\theta_{in}-\theta)}$$

$$\Rightarrow T(\theta) = T_{in} e^{\mu_s(\theta_{in}-\theta)} + AB(\cos\theta - \cos\theta_{in} e^{\mu_s(\theta_{in}-\theta)}) + AC(\sin\theta - \sin\theta_{in} e^{\mu_s(\theta_{in}-\theta)})$$

Substituting the constants back in we get the expression

$$T(\theta) = T_{in} e^{\mu_s(\theta_{in}-\theta)} + \frac{\rho_c g R}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in} e^{\mu_s(\theta_{in}-\theta)}\right) + 2\mu_s\left(\sin\theta - \sin\theta_{in} e^{\mu_s(\theta_{in}-\theta)}\right)\right]$$

Then to find the minimum pulling tension at the bend outlet is to evaluate at $T_{out} = T(\theta_{out})$ which gives the final expression

$$T_{out} > T_{in} e^{\mu_s(\theta_{in}-\theta_{out})} + \frac{\rho_c g R}{\mu_s^2+1}\left[(\mu_s^2-1)\left(cos\,\theta_{out} - cos\,\theta_{in} e^{\mu_s(\theta_{in}-\theta_{out})}\right) + 2\mu_s\left(sin\,\theta_{out} - sin\,\theta_{in} e^{\mu_s(\theta_{in}-\theta_{out})}\right)\right]$$

It is evident that for a massless cable, i.e. $\rho_c \to 0$, the result is simply the inverse catenary equation $T_{out} = T_{in} e^{\mu_s(\theta_{in}-\theta_{out})}$. This is to be expected, because the bend leads to a slight lifting effect of the cable, thus resulting in a smaller normal force for the friction to act upon. Note that for the above calculations to make sense, contact with the bend floor is a requirement, i.e.

$$N > 0$$

$$\rho_c g R \cos\theta > T(\theta)$$

If this inequality is broken, so is the validity of the above equations. We must therefore limit these calculations to θ values in which contact with the bottom of the pipe holds, i.e. in the range $|\theta| < \theta_{crit}$. Beyond this range, the cable loses contact with the pipe floor. If the cable weren't then bounded from above, one would need to use catenary equations from this point on. Here however, we assume that the cable is bounded by the roof of the pipe. In reality there will be a section where the cable is floating in-between the bottom and top of the pipe without normal force because the pull-in forces are in balance with gravity. This piece of cable will have a slightly different bend characteristics, i.e. center of curvature, radius and angle. However, if we assume a pipe diameter only minutely larger than the cable diameter, this "catenary" will be negligible and we can assume contact with the pipe roof immediately after contact with the pipe floor is lost, i.e. $|\theta| > \theta_{crit}$. Finding θ_{crit} analytically can be challenging, however it is easily found using a numerical method such as Newtons method.

If the cable is bounded from above, the method is similar, but the normal force is then in the opposite direction, such that:

$$N = -\rho_c g R \Delta\theta \cos\theta + T\Delta\theta$$

Which by a similar force balance leads to the ODE

$$\frac{dT}{d\theta} - \mu_s T = \rho_c g R(-\mu_s \cos\theta + \sin\theta)$$

This is also solved with method of integrating factor to yield:

TODO: Write out full calculation

$$T(\theta) = A(B\cos\theta - C\sin\theta) + [T_{in} - A(B\cos\theta_{in} - C\sin\theta_{in})]e^{\mu_s(\theta-\theta_{in})}$$

$$T(\theta) = T_{in} e^{\mu_s(\theta-\theta_{in})} + \frac{\rho_c g R}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right) + 2\mu_s\left(-\sin\theta + \sin\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right)\right]$$

The procedure for calculating minimum required pulling force in a bottom bend section is thus to find the critical hangoff angle and use eq xx on the angles outside this range and eq xx within. In summary, the equation for minimum required pulling force for a bottom vertical bend is given by the following equation:

$$T(\theta) = \begin{cases} T_{in} e^{\mu_s(\theta_{in}-\theta)} + \frac{\rho_c g R}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in}\, e^{\mu_s(\theta_{in}-\theta)}\right) + 2\mu_s\left(\sin\theta - \sin\theta_{in}\, e^{\mu_s(\theta_{in}-\theta)}\right)\right], |\theta| < \theta_c \\ T_{in} e^{\mu_s(\theta-\theta_{in})} + \frac{\rho_c g R}{\mu_s^2+1}\left[(\mu_s^2-1)\left(\cos\theta - \cos\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right) + 2\mu_s\left(-\sin\theta + \sin\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right)\right], |\theta| > \theta_c \end{cases}$$

Vertical top bend

Now, consider the case in which the cable is resting on top of a vertical bend, such as illustrated below. Note that the arrangement is now from right to left, as to keep things orderly with increasing θ values in the pulling direction. This is an arbitrary choice, and the resulting pulling tension is off course the same for the mirror image.

TODO: Make illustration

For this case, both gravity and the capstan effect contributes to the normal force. As a consequence, contact is assured throughout the bend. The force balance is:

TODO: Write out force balance

Which leads to the ODE:

$$\frac{dT}{d\theta} - \mu_s T = \rho_c g R(\mu_s \cos\theta - \sin\theta)$$

Which is in the same manner as above solved to yield:

TODO: Write out intermediary steps

$$T(\theta) = T_{in} e^{\mu_s(\theta-\theta_{in})} + \frac{\rho_c g R}{\mu_s^2 + 1}\left[(\mu_s^2 - 1)\left(-\cos\theta + \cos\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right) + 2\mu_s\left(\sin\theta - \sin\theta_{in}\, e^{\mu_s(\theta-\theta_{in})}\right)\right]$$

Horizontal bend

Now, consider a bend in the horizontal plane. This is equivalent to the vertical bend cases above in which the cable is situated outside the bend, without the gravity component. By setting $\rho_c = 0\ kg/m$, both eq xx and xx simplifies to the well-known capstan equation:

$$T(\theta) = T_{in} e^{\mu_s(\theta - \theta_{in})}$$