NATIONAL OPEN UNIVERSITY OF NIGERIA
14/16 AHMADU BELLO WAY, VICTORIA ISLAND, LAGOS
SCHOOL OF SCIENCE AND TECHNOLOGY
JUNE/JULY EXAMINATION

COURSE CODE: MTH 412
COURSE TITLE: Normed Linear Spaces
TIME ALLOWED:3 HOURS
INSTRUCTION: COMPLETE ANSWERS TO ANY FIVE (5) QUESTIONS BEAR FULL MARKS

INSTRUCTION:

1(a) Define Linear Maps.
5marks

1(b) Let X = l_2. For each $\bar{x}$ = ($x_1, x_2, x_3, \ldots x_k, \ldots$) in l_2. Show that if $T\bar{x} = T(x_1, x_2, x_3, \ldots x_k, \ldots) = (0, x_1, x_2/2, x_3/3, \ldots x_k/k, \ldots)$, then T is a linear map on l_2.
9marks

2(a) Let (X, ρ) be a metric space. Define Cauchy sequence.
5marks

2(b) Let (X, ρ) be a complete metric space, and let E $\subset$ X. Show that (E, ρ_E)is complete if and only if it is closed. (Where $\rho_E$is the subspace metric induced by ρ).
9marks

3(a) Define the convergence of a sequence $\{x_n\}$ of elements of X to a point x ε X?
5marks

3(b) Let X = [-3, 3] with $\|f\|_2 = \left(\int_{-3}^{3} |f(t)|^2\right)$ show that X is not complete.
9marks

4(a) The surface of a unit sphere centered around the origin of a linear space with the ℓ^p-norm is the locus of points $\{(x_1, x_2, \cdots\}$. Show that

$$\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} = 1 \ .$$

7marks

4(b) Let X = $\mathbf{R}^2$. For each vector $\bar{x}$ = (x_1, x_2) ε X. Define $||.||_2 : X \to R$ by $||x||_2 = \left(\sum_{k=1}^{2} x_k^2\right)^{½}$. Show that ||.|| is a norm on X.

7marks

5(a) Let X = c[0,1] = Y, where c[0,1] is endowed with the supnorm. Let D = {f ε c'[0,1] : f'ε c[0,1]} where the prime denotes differentiation.
Let T : c[0,1] → c[0,1] be a map with domain D defined by Tf = f' (i.e. differentiation operator). Show that:

i) T is linear 2½marks
ii) T is closed. 2½marks

5(b) Show that an inner product space E becomes a normed linear space when equipped with the norm $\|x\| = \sqrt{\langle x, x\rangle}$ for all x ε E.

9marks

6(a) Let X be a linear space over a scalar field K = (**R** or **C**). When is a function ||.|| : X →R said to be a norm (in X)?

5marks

6(b) Show that the real line **R** becomes a normed linear space if you set ||x|| =|x| for every number x ε **R.**

9marks

7(a) What is aConvex set? If x$\varepsilon$$R^n$and if r >0; show that the ball

$B(x_{*}, r) = \{ y \varepsilon R^n : ||y - x^*|| < r\}$centred at x* of radius r is a convex set
7marks

7(b) Let x, and v be vectors in $\mathbf{R}^n$, show that the line L through x in the direction of v given by $L = \{ x+\alpha v : \alpha \varepsilon \mathbf{R}\}$ is a convex set.
7marks