



NATIONAL OPEN UNIVERSITY OF NIGERIA
14/16 AHMADU BELLO WAY, VICTORIA ISLAND, LAGOS
SCHOOL OF SCIENCE AND TECHNOLOGY
MARCH/APRIL 2014 EXAMINATION

COURSE CODE: MTH 412

COURSE TITLE: FUNCTIONAL ANALYSIS II

TIME ALLOWED: TIME: 3 HOURS

**INSTRUCTION: COMPLETE ANSWERS TO ANY FIVE (5)
QUESTIONS BEAR FULL MARKS**

1(a) Define Linear Maps. -
5marks

1(b) Let $X = l_2$. For each $x = (x_1, x_2, x_3, \dots, x_k, \dots)$ in l_2 . Show that
if $Tx = T(x_1, x_2, x_3, \dots, x_k, \dots) = (0, x_1, x_2/2, x_3/3, \dots, x_k/k, \dots)$, then T is a linear
map on l_2 . - 9marks

2(a) Let (X, ρ) be a metric space. Define Cauchy sequence.
-5marks

2(b) Let (X, ρ) be a complete metric space, and let $E \subset X$. Show that
 (E, ρ_E) is complete if and only if it is closed. (Where ρ_E is the
subspace metric induced
by ρ). -9marks

3(a) Define the convergence of a sequence $\{x_n\}$ of elements of X to
a point $x \in X$?

-5marks

3(b) Let $X = [-3, 3]$ with $\|f\|_2 = \left(\int_{-3}^3 |f(t)|^2 \right)^{1/2}$ show that X is not complete. -9marks

4(a) The surface of a unit sphere centered around the origin of a linear space with the ℓ^p -norm is the locus of points $\{(x_1, x_2, \dots)\}$. Show that

$$\left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}} = 1.$$

-7marks

4(b) Let $X = \mathbf{R}^2$. For each vector $x = (x_1, x_2) \in X$. Define $\| \cdot \|_2 : X \rightarrow \mathbf{R}$ by $\|x\|_2 = \left(\sum_{k=1}^2 x_k^2 \right)^{1/2}$. Show that $\| \cdot \|$ is a norm on X . -7marks

5(a) Let $X = C[0,1] = Y$, where $C[0,1]$ is endowed with the supnorm. Let $D = \{f \in C[0,1] : f' \in C[0,1]\}$ where the prime denotes differentiation. Let $T : C[0,1] \rightarrow C[0,1]$ be a map with domain D defined by $Tf = f'$ (i.e. differentiation operator). Show that:
 i) T is linear -2½marks
 ii) T is closed. -2½marks

5(b) Show that an inner product space E becomes a normed linear space when equipped with the norm $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in E$. -9marks

6(a) Let X be a linear space over a scalar field $K = (\mathbf{R} \text{ or } \mathbf{C})$. When is a function $\|\cdot\| : X \rightarrow \mathbf{R}$ said to be a norm (in X)?
-5marks

6(b) Show that the real line \mathbf{R} becomes a normed linear space if you set $\|x\| = |x|$ for every number $x \in \mathbf{R}$.
- 9marks

7(a) What is a convex set? If $x \in \mathbf{R}^n$ and if $r > 0$; show that the ball $B(x^*, r) = \{ y \in \mathbf{R}^n : \|y - x^*\| < r \}$ centred at x^* of radius r is a convex set -7marks

7(b) Let x , and v be vectors in \mathbf{R}^n , show that the line L through x in the direction of v given by $L = \{ x + \alpha v : \alpha \in \mathbf{R} \}$ is a convex set.
-7marks