NATIONAL OPEN UNIVERSITY OF NIGERIA
14-16 AHMADU BELLO WAY, VICTORIA ISLAND LAGOS
SCHOOL OF SCIENCE AND TECHNOLOGY
MAY/JUNE 2012 EXAMINATION

MTH 412 NORMED LINEAR SPACES (3 CR)
TIME: 3 HOURS

INSTRUCTION: COMPLETE ANSWERS TO ANY FIVE (5) QUESTIONS BEAR FULL MARKS

1(a) Let X be a linear space over a scalar field K = (R or C). When is a function $||.|| : X \rightarrow R$ said to be a norm (in X)? - 5marks

1(b) Show that the real line R becomes a normed linear space if you set $||x|| = |x|$ for every number x ε R. - 9marks

2(a) The surface of a unit sphere centered around the origin of a linear space with the ℓ^p-norm is the locus of points $\{(x_1, x_2, \cdots\}$. Show that

$$\left(\sum_{k=1}^{\infty} |x_k|^p\right)^{\frac{1}{p}} = 1 .$$

2(b) Let X = R^2. For each vector $\bar{x} = (x_1, x_2)$ ε X. Define $||.||_2 : X \rightarrow R$ by $||x||_2 = (\sum_{k=1}^{2} X_k^2)^{1/2}$. Show that $||.||$ is a norm on X. -14marks

3(a) What is a Convex set? If xε R^n and if r > 0; show that the ball $B(x^*, r) = \{ y \in R^n : ||y - x^*|| < r\}$ centred at x* of radius r is a convex set -7marks

3(b) Let x, and v be vectors in R^n, show that the line L through x in the direction of v given by $L = \{ x+\alpha v : \alpha \in R\}$ is a convex set. -7marks

4(a) Let (X, ρ) be a metric space. Define Cauchy sequence. -5marks

4(b) Let (X, ρ) be a complete metric space, and let E $\subset$ X. Show that (E, ρE) is complete if and only if it is closed. (Where ρE is the subspace metric induced by ρ). -9marks

5(a) Define the convergence of a sequence { x_n} of elements of X to a point x ε X?

-5marks

5(b) Let X = [-3, 3] with $\|f\|_2 = \left(\int_{-3}^{3}|f(t)|^2\right)$ show that X is not complete.

-9marks

6(a) Define Linear Maps. -5marks

6(b) Let X = l_2. For each $\bar{x}$ = (x_1, x_2, x_3, . . . x_k, ...) in l_2. Show that if

T $\bar{x}$ = T (x_1, x_2, x_3, . . . x_k, ...) = (0, x_1, $x_2/2$, $x_3/3$, . . . x_k/k, ...), then T is a linear map on l_2. -9marks

7(a) Let X = c[0,1] = Y, where c[0,1] is endowed with the supnorm. Let D = {f ε c'[0,1] : f' ε c[0,1]} where the prime denotes differentiation. Let T : c[0,1] → c[0,1] be a map with domain D defined by Tf = f' (i.e. differentiation operator). Show that:

i) T is linear -2½marks

ii) T is closed. -2½marks

7(b) Show that an inner product space E becomes a normed linear space when equipped with the norm $\|x\| = \sqrt{\langle x, x\rangle}$ for all x ε E. -9marks