Columbia University
IEOR4703 – Monte Carlo Simulation Methods (Hirsa)
Term Project 3
State Estimation in an Economic Model using Particle Filters

Introduction

This case study demonstrates the use of the `Particle Filter (PF)` technique to estimate unobservable states in an economic model. We focus on the `output gap`[1], which represents the difference between actual and potential output of an economy. The goal is to estimate this latent state from noisy observations, such as `GDP` data. The output gap helps policymakers assess whether an economy is underperforming (negative gap) or overheating (positive gap). Here, we apply the particle filter to estimate the output gap using a nonlinear state-space model.

The objective is to estimate the `output gap` of a country based on noisy `GDP` growth data and compare the accuracy of the particle filter's estimates with methods like the `Kalman Filter (KF)` and `Maximum Likelihood Estimation (MLE)`.

Particle Filter

A `Particle Filter` is a `Bayesian` sequential Monte Carlo method used for state estimation in nonlinear and non-Gaussian systems. It approximates the posterior distribution of a hidden state by representing it with weighted particles that evolve over time using prediction (state transition model) and update (measurement likelihood) steps. The key steps include:

1. Initialization: Initialize particles (representing possible values of θ and ϕ) with random values from prior distributions. Each particle i has an associated weight w_i.

2. Prediction: Use the state transition model $x_t = f(x_{t-1}, \theta) + \epsilon_t$ to propagate each particle forward, generating predicted values for the output gap x_t at each time step.

3. Update: For each particle, compute the likelihood of the observed data y_t based on the predicted values of x_t using the measurement equation $y_t = h(x_t, \phi) + \eta_t$. Update the weights w_i of the particles based on the likelihood of the observed data.

4. Resampling: Resample particles with higher weights and discard particles with lower weights to focus on the most likely particles.

[1]https://en.wikipedia.org/wiki/Output_gap

5. Estimation: After several iterations of prediction, update, and resampling, the particles with the highest weights represent the most likely values for the parameters θ and ϕ. The final estimate is obtained by taking the mean (or weighted average) of the particles.

Particle filters are widely used in tracking, robotics, and financial modeling for real-time estimation.

Model Setup

We assume a nonlinear `state-space model` for the output gap, where the latent state x_t evolves according to the system:

$$x_t = f(x_{t-1}, \theta) + \epsilon_t$$

where:

- x_t is the latent `output gap` at time t,
- $f(\cdot)$ is a nonlinear function,
- θ is the set of parameters to be estimated (e.g., autoregressive parameter, noise variance),
- $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$ is the process noise.

The observed data y_t (e.g., GDP, inflation) is related to the latent output gap by:

$$y_t = h(x_t, \phi) + \eta_t$$

where:

- $h(\cdot)$ is the measurement function,
- $\eta_t \sim \mathcal{N}(0, \tau^2)$ is the measurement noise,
- ϕ represents the parameters of the measurement equation.

For simplicity, assume:

$$x_t = \phi \cdot x_{t-1} + \epsilon_t \quad \text{where} \quad \epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

and the observed data y_t (GDP growth) is related to the latent state by:

$$y_t = \alpha \cdot x_t + \nu_t$$

where $\nu_t \sim \mathcal{N}(0, \tau^2)$ is the measurement noise, and α represents the linear relationship between the output gap and GDP growth.

Generate Synthetic Data

The true values of the output gap x_t are not directly observable, so we simulate them using the state-space model. We then generate noisy observations y_t using the relationship between GDP growth and the output gap, with added noise.

Implementing Particle Filter

To estimate the parameters θ and ϕ, we use the Particle Filter technique:

- Initialize a set of particles $x_t^{(i)}$ for $i = 1, \ldots, N$, where N is the number of particles.
- For each time step t, propagate the particles according to the system model, and update the particles based on the likelihood of the observed data y_t.
- Resample the particles based on their weights.

Comparison with Other Methods

We compare the performance of the Particle Filter with:

- `Kalman Filter`: A linear filter for estimating the state of a linear system with Gaussian noise.
- Maximum Likelihood Estimation (`MLE`): A method for estimating model parameters by maximizing the likelihood of the observed data.

Evaluation

We evaluate the methods using:

- `RMSE`: Compares the estimated output gap with the true output gap.
- Visualization: Plots the true output gap alongside estimates from different methods.
- Statistical Tests: Hypothesis tests on parameter estimates to evaluate method performance.

Example: Estimating the Output Gap

Consider the model where x_t evolves as:

$$x_t = 0.9x_{t-1} + \epsilon_t$$

with $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$, and the observed data y_t is related to x_t by:

$$y_t = 1.5x_t + \eta_t$$

where $\eta_t \sim \mathcal{N}(0, \tau^2)$.

We estimate the parameters $\theta = 0.9$ and $\phi = 1.5$ using the particle filter technique.

Results, Discussion, and Findings

The `Particle Filter` estimates the output gap by tracking particles through noisy observations. We compare the results of the Particle Filter, Kalman Filter, and MLE, using `RMSE` and statistical tests to evaluate performance.

The `Particle Filter` is effective for state estimation in nonlinear models, particularly in non-Gaussian systems. Comparing it with traditional methods like the `Kalman Filter` and `MLE`, we assess its advantages and limitations. This case study demonstrates the flexibility and robustness of the `Particle Filter` for economic modeling.