Zad. 3

Oznaczmy przez K_{n-1} liczbę ciągów dł. $2(n-1)$ spełniających war. zad.

Aby uzyskać ciąg dł. $2n$ spełniający war. zadania wystarczy dodać do ciągu dł $2(n-1)$ dwa nowe elementy n tak aby nie stały obok siebie. Możemy to zrobić na $(2(n-1)+1)\cdot(2(n-1))$ sposobów. Mamy:

$$K_n = K_{n-1} + (2(n-1)+1)(2(n-1)) = K_{n-1} + 4n^2 - 8n + 4 + 2n - 2 =$$

$$= K_{n-1} + 4n^2 - 6n + 2$$

$$K_n - K_{n-1} = 4n^2 - 6n + 2$$

$$K_n = \sum_{l=1}^{n} K_n - K_{n-1} = \sum_{l=1}^{n} 4n^2 - 6n + 2 = 2n + 4\sum_{l=1}^{n} n^2 - 6\sum_{l=1}^{n} n$$

bo $K_0 = 0$

$$= 2n + 4\frac{(n)(n+1)(2n+1)}{6} - 6\frac{n(n+1)}{2}$$

Zad. 4

$a_0 = 1$

$a_1 = 0$

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \longrightarrow 2a_n = a_{n-1} + a_{n-2} \longrightarrow a_{n-1} + a_{n-2} - 2a_n = 0$$

$$-2a_n + a_{n+1} + a_n = 0$$

Korzystając z metody anihilatorów:

$$(-2E^2 + E + 1)\langle a_n\rangle = 0$$

$\cancel{(x-1)(2x+1)}$ $\langle a_n\rangle$

$$(E-1)(E+\tfrac{1}{2}) = 0$$

Z lematu z wykładu:

$$a_n = \alpha 1^n + \beta\left(-\tfrac{1}{2}\right)^n$$

ale $a_0 = 1$, $a_1 = 0$

$$\begin{cases} a_0 = \alpha 1^0 + \beta\left(-\frac{1}{2}\right)^0 = \alpha + \beta = 1 \\ a_1 = \alpha 1 + \beta\left(-\frac{1}{2}\right) = \alpha - \frac{1}{2}\beta = 0 \end{cases} \longrightarrow \alpha = \frac{1}{3},\ \beta = \frac{2}{3}$$

Ostatecznie

$$a_n = \frac{1}{3}1^n + \frac{2}{3}\left(-\frac{1}{2}\right)^n$$

Zad. 5

a) $a_{n+2} = 2a_{n+1} - a_n + 3^n - 1 \quad a_0, a_1$

$a_{n+2} - 2a_{n+1} + a_n = (E^2 - 2E + 1)\langle a_n \rangle$

$(E-1)^2 \langle a_n \rangle - 3^n + 1 = 0$

dalej chcemy $u_n = 3^n - 1$

$(E-3)(E-1)\langle u_n \rangle = 0$

więc

$(E-1)^3 (E-3) \langle a_n \rangle = 0$

$a_n = \alpha + \beta n + \gamma n^2 + \varphi 3^n$

z war. pocz. $a_0 = a_1 = 0$

$a_0 = \alpha + 0 + 0 + \varphi = 0 \to \alpha = -\varphi$

$a_1 = \alpha + \beta + \gamma + 3\varphi = 0$

$-y + \gamma + 3(-x) = 0$

dla dowolnych x, y

$a_n = x + yn + (3x - y)n^2 + (-x)3^n$

b) $a_{n+2} = 4a_{n+1} - 4a_n + n2^{n+1}$

$n2^{n+1} = b_n$

$(E-2)^2 \langle b_n \rangle = 0$

$(E^2 - 4E - 4)\langle a_n \rangle = n2^{n+1}$

$(E-2)^4 \langle a_n \rangle = 0$

$a_n = \alpha 2^n + \beta n 2^n + \gamma n^2 2^n + \varphi n^3 2^n$

c) $a_{n+2} = 2^{n+1} - a_{n+1} - a_n$

$(E^2 + E + 1) = \left(E - \frac{1}{2} - \frac{\sqrt{3}i}{2}\right)\left(E - \frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$

$(E^2 + E + 1)(E-2)\langle a_n \rangle = 0$

$a_n = \alpha 2^n + \beta \left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^n + \gamma \left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^n$

Zad. 6

$$\left[a_n = n \bmod 3 \quad a_0 = 0,\ a_1 = 1,\ a_2 = 2,\ a_n = a_{n-3} \right]$$

$$a_n = a_{n-3}$$

$$(E^3 - 1)\langle a_n \rangle = 0$$

$$(E-1)(E^2 + E + 1)\langle a_n \rangle = 0$$

$$a_n = \alpha + \beta\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)^n + \gamma\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^n$$

$\beta = \gamma$

$$a_0 = \alpha + \beta + \gamma = 0$$

$$a_1 = \alpha - \frac{1}{2}\beta - \frac{\sqrt{3}i}{2}\beta - \frac{1}{2}\gamma + \frac{\sqrt{3}i}{2}\gamma = 1$$

$$a_2 = \alpha + \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\beta + \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)\gamma = 2$$

$\alpha - \frac{1}{2}\beta - \frac{1}{2}\gamma = 1$

$\alpha - \frac{1}{2}\beta - \frac{1}{2}\gamma = 2$

Rozwiązując ten układ wyliczymy α, β, γ. Otrzymamy wzór: ~~$M_3(n) = n \bmod 3$~~

$$\left\lfloor \frac{n}{3} \right\rfloor = n - \operatorname{sgn}(a_n)$$

Zad 7

C_{n-1} – liczby ciągu $n-1$ liter mający parz. zadań $C_0 = 1 \quad C_1 = 25$

$$C_n = \underbrace{(26^{n-1} - C_{n-1})}_{\text{ciągi dł. } n-1 \text{ z nieparz. liczbą } a} + 25 \cdot \underbrace{C_{n-1}}_{\text{bez „a”}} = 26^{n-1} + 24 C_{n-1}$$

$$(E-24)(E-26)\langle C_n \rangle = 0$$

$$C_n = \alpha \cdot 24^n + \beta \cdot 26^n$$

$C_0 = 1 \quad \alpha + \beta = 1$

$C_1 = 24\alpha + 26\beta = 25 \rightarrow \beta = \frac{1}{2},\ \alpha = \frac{1}{2}$

$$C_n = \frac{1}{2} 24^n + \frac{1}{2} 26^n$$

Zad. 8

$$s_n = \sum_{i=1}^{n} i 2^i$$

$$s_n = s_{n-1} + n2^n$$

$$(E-1)\langle s_n\rangle = n2^n$$

$$(E-1)(E-2)^2\langle s_n\rangle = 0$$

$$s_n = \alpha + \beta 2^n + \gamma n 2^n$$

$$s_1 = 2$$

$$s_2 = 2 + 2\cdot 2^2 = 10$$

$$s_3 = 2 + 2\cdot 2^2 + 3\cdot 2^3 = 10 + 3\cdot 8 = 34$$

$$\begin{cases} \alpha + 2\beta + \gamma 2 = 2 \\ \alpha + 4\beta + 8\gamma = 10 \\ \alpha + 8\beta + 24\gamma = 34 \end{cases} \Rightarrow \begin{array}{l} \alpha = 2 \\ \beta = -2 \\ \gamma = 2 \end{array}$$

$$s_n = 2 - 2\cdot 2^n + 2n2^n$$

Zad. 11. A - kapitał mat B - n-k z.n

p_k - p.p. wygranej startując z k mało

$$p_k = p\cdot p_{k+1} + (1-p)p_{k-1}$$

$$\left(E^2 - \frac{1}{p} + \frac{1-p}{p}\right)\langle p_k\rangle = 0$$

$$(E-1)\left(E - \left(\frac{1}{p} - p\right)\right)\langle p_k\rangle = 0$$

$p_0 = 0$ od razu przegrana

$p_n = 1$ ma wszystko

$$p_k = \alpha + \beta\left(\frac{1}{p} - p\right)^k$$

$$\begin{cases} p_0 = \alpha + \beta = 0 \\ p_n = \alpha + \beta\left(\frac{1}{p} - p\right)^n = 1 \end{cases}$$

$$\alpha = -\beta$$

$$-\beta + \beta\left(\frac{1}{p} - p\right)^n = 1$$

$$\beta\left(\left(\frac{1}{p} - p\right)^n - 1\right) = 1$$

$$\beta = \frac{1}{\left(\frac{1}{p} - p\right)^n - 1}$$

$$p_k = \frac{1}{1 - \left(\frac{1}{p} - p\right)^n} + \frac{1}{\left(\frac{1}{p} - p\right)^n - 1}\left(\frac{1}{p} - p\right)^k$$