(SL 1)

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$$Y = X\beta + \varepsilon$$

$$\hat{\beta}_{LS} = \underset{\beta \in \mathbb{R}^p}{\arg\min} \|Y - X\beta\|^2 = (X'X)^{-1} X'Y$$

We have

$$(Y - X\beta)^T (Y - X\beta) = Y^T Y - \beta^T X^T Y - Y^T X \beta + \beta^T X^T X \beta \quad / \partial\beta$$

$$-X^T Y - Y^T \cancel{X} + 2X^T X \beta = 0 \quad / \partial\beta \quad 2X^T X \geq 0$$

$$X^T X \beta = X^T Y$$

$$\hat{\beta} = (X^T X)^{-1} X^T Y$$

$$v^T X^T X v = (Xv)^T Xv = \|Xv\|^2 \geq 0$$

$$\hat{\beta}_{LS} \sim N(\beta, \sigma^2 (X'X)^{-1})$$

We have

$$\hat{\beta} = (X^T X)^{-1} X^T (X\beta + \varepsilon) = (X^T X)^{-1} X^T X \beta + (X^T X)^{-1} X^T \varepsilon$$

$$= \beta + (X^T X)^{-1} X^T \varepsilon$$

$$\mathbb{E}\hat{\beta} = \beta$$

$$\mathrm{Cov}(\hat{\beta}) = \mathrm{Cov}((X^T X)^{-1} X^T \varepsilon) = (X^T X)^{-1} X^T \underbrace{\mathrm{Cov}(\varepsilon)}_{\sigma^2 I} X (X^T X)^{-1}$$

$$\sigma^2 (X^T X)^{-1}$$

$$\hat{\sigma}^2 = S^2 = \frac{\|Y - X\hat{\beta}_{LS}\|^2}{n-p} = \frac{RSS}{n-p}$$

unbiased estimate of σ^2

$$\hat{Y} = X\hat{\beta} = X\beta + X(X'X)^{-1}X'\varepsilon = X\beta + H\varepsilon$$

$$\hat{\varepsilon} = Y - \hat{Y} = X\beta + \varepsilon - X\beta - H\varepsilon = (I-H)\varepsilon$$

$$\mathbb{E}\left[(n-p)\hat{\sigma}^2\right] = \mathbb{E}\left(\hat{\varepsilon}^T\hat{\varepsilon}\right) = \mathbb{E}\left[\varepsilon^T(I-H)^T(I-H)\hat{\varepsilon}\right]$$

$$= tr\left[(I-H)\mathbb{E}(\varepsilon\varepsilon^T)\right] = tr\left[(I-H)\sigma^2 I\right] =$$

$$tr\left[(I-H)\sigma^2\right] = \sigma^2 tr(I-H) = \sigma^2 rank(I-H)$$

$$\sigma^2(n-p)$$

Testing

$H_0: \beta_1 = \dots = \beta_p = 0$

$H_1: \sim H_0$

$SST = \sum_{i=1}^{n}(Y_i - \bar{Y})^2 \qquad n-1$

$SSE = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 \qquad n-p$

$SSM = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2 \qquad p-1$

Test statistic

$$F = \frac{\frac{1}{p-1} SSM}{\frac{1}{n-p} SSE} =$$

$Q = Q_1 + \dots + Q_k$

$Q_1, \dots, Q_{k-1} \quad \chi^2_{r_i}$

$Q_k \geq 0 \quad \chi$

$T: Q_1, \dots, Q_k \quad i.i.d.$

$Q_k \sim \chi^2(r - \sum_{i=1}^{k-1} r_i)$

$$R^2 = \frac{SSM}{SST}$$

$$T_i = \frac{\hat{\beta}_i}{s(\hat{\beta}_i)}, \quad s(\hat{\beta}_i) = s^2 (X^T X)^{-1}_{i,i}$$

$$\beta_i \in \left(\hat{\beta}_i \pm q_{t(1-\frac{\alpha}{2}, n-p)}\, s(\hat{\beta}_i)\right)$$

Def

- Wishart distribution

random vectors

$X_1, \dots, X_m$ i.i.d $\mathcal{N}_n(0, \Sigma)$, $|\Sigma| > 0$

$$A = \sum_{j=1}^{m} X_j X_j^T \sim W_{\underset{\text{dim}}{n}}(\underset{\text{degrees of freedom}}{m}, \Sigma)$$

Remark (Corallary)

$\mathbb{E} A = m\Sigma$

If $n = 1$, $\Sigma = 1$, then $W_1(m, 1) \sim \chi^2_m$

- inverse Wishart distribution $W_n^{-1}(m, \Sigma)$

$\mathbb{B}$ has inverse Wishart distribution if

$\mathbb{B}^{-1}$ ~~tist~~ $\sim$ — " — $W_n(m, \Sigma^{-1})$

$$\mathbb{E}\mathbb{B} = \frac{\Sigma}{m-n-1} \quad \text{for } m > n+1$$

X iid $\mathcal{N}(0, \underset{(sd)}{\frac{1}{\sqrt{n}}})$ $\quad \underset{(p\times n)}{X'}\underset{(n\times p)}{X} \sim W_p(n, \operatorname{diag}(\frac{1}{n}, \dots, \frac{1}{n}))$

$$(X'X)^{-1} \sim W_p^{-1}(n, \operatorname{diag}(n, \dots, n))$$

$$\mathbb{E}(X'X)^{-1} = \frac{\operatorname{diag}\{n, \dots, n\}}{n-p-1}$$