Jakub Krzyszton, Lista 2

zad. 1.

Zał. nie wprost, że $p \cdot 1 + q \cdot \sqrt{2} = 0$, ale $p \neq 0$ lub $q \neq 0$ ($p, q \in \mathbb{Q}$).

Wtedy gdy:

I. $p \neq 0$: $\underset{\mathbb{Q}}{\frac{q}{p}} = -\underset{\notin \mathbb{Q}}{\frac{1}{\sqrt{2}}}$ $\lightning$

II: $q \neq 0$: $\underset{\mathbb{Q}}{\frac{p}{q}} = -\sqrt{2} \notin \mathbb{Q}$ $\lightning$

Zatem $p = q = 0$, czyli $\{1, \sqrt{2}\}$ lnz (nad $\mathbb{Q}$).

zad. 2.

(a) $F(0) = F(0 \cdot v) \overset{F}{\underset{\text{jedn.}}{=}} 0 \cdot F(v) \overset{\text{dla pewnego } w \in W}{=} 0 \cdot w = 0$

(b)

1. $F(0) = 0$, więc $0 \in F^{-1}[\{0\}]$.

2. Weźmy $v, w \in F^{-1}[\{0\}]$. Wtedy ~~$F(v)+F(w)=$~~
$F(v+w) = F(v) + F(w) = 0 + 0 = 0$, więc $v + w \in F^{-1}[\{0\}]$.

3. Weźmy $v \in F^{-1}[\{0\}]$ i $\alpha \in K$. Wtedy $F(\alpha v) = \alpha \cdot F(v) = \alpha \cdot 0 = 0$, więc $\alpha v \in F^{-1}[\{0\}]$.

Zatem $F^{-1}[\{0\}] < V$.

zad. 3.

$B = \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix} \right\}$, $A = \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix}$, więc $B_1 := \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix} \right\}$

$\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix}$, więc $B_2 := \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\}$

zad. 4.

$\dim \mathbb{R}^2 = 2$, więc $\max\{|A| : A \in \mathcal{X}\} = 2$.

- $\mathcal{X} \subseteq \{A \subseteq \mathbb{R}^2 : |A| \leq 2\}$ i $|\{A \subseteq \mathbb{R}^2 : |A| \leq 2\}| = \mathfrak{c}$, więc $|\mathcal{X}| \leq \mathfrak{c}$.
- $\{\{\binom{0}{1}, \binom{x}{0}\} : x \in \mathbb{R} \setminus \{0\}\} \subseteq \mathcal{X}$ i $|\{\{\binom{0}{1}, \binom{x}{0}\} : x \in \mathbb{R} \setminus \{0\}\}| = \mathfrak{c}$, więc $|\mathcal{X}| \geq \mathfrak{c}$.

Zatem $|\mathcal{X}| = \mathfrak{c}$.

Maksymalna dł. łańcucha wynosi 3: np. $\emptyset \subseteq \{\binom{1}{0}\} \subseteq \{\binom{1}{0}, \binom{0}{1}\}$.
Nie uzyskamy dłuższego łańcucha, gdyż zbiory z $\mathcal{X}$ mają moc 0, 1 lub 2, a w tym przypadku $\subseteq$ zwiększa moc.

zad. 5.

(1) $F\left(\binom{x}{y} + \binom{x'}{y'}\right) = F\left(\binom{x+x'}{y+y'}\right) = \begin{pmatrix} 2(x+x') + 3(y+y') \\ -1(x+x') + 5(y+y') \end{pmatrix} = \begin{pmatrix} 2x + 2x' + 3y + 3y' \\ -x - x' + 5y + 5y' \end{pmatrix} =$

$= \begin{pmatrix} (2x+3y) + (2x'+3y') \\ +(-x+5y) + (-x'+5y') \end{pmatrix} = F\binom{x}{y} + F\binom{x'}{y'}$.

(2) $F\left(\alpha\binom{x}{y}\right) = F\left(\binom{\alpha x}{\alpha y}\right) = \begin{pmatrix} 2\alpha x + 3\alpha y \\ -\alpha x + 5\alpha y \end{pmatrix} = \alpha \begin{pmatrix} 2x + 3y \\ -x + 5y \end{pmatrix} = \alpha F\binom{x}{y}$

zad. 7.

~~Niech~~ Weźmy standardową bazę K^2 tj. $\{\binom{1}{0}, \binom{0}{1}\}$. Wtedy

$\binom{x}{y} = x\binom{1}{0} + y\binom{0}{1}$. Wtedy

$F\binom{x}{y} = F\left(x\binom{1}{0} + y\binom{0}{1}\right) \overset{F \text{ lin.}}{=\!=} x \cdot \underbrace{F\binom{1}{0}}_{a} + y\underbrace{F\binom{0}{1}}_{b} = ax + by$ oraz

$a, b \in K$, gdyż $F\binom{1}{0}, F\binom{0}{1} \in K$.

zad. 9.

(a) $\mathrm{Lin}(A\cup B) = \mathrm{Lin}\,A + \mathrm{Lin}\,B$

d-d:

$x \in \mathrm{Lin}(A\cup B) \iff x \in \{\sum_{y\in A\cup B} t_y \cdot y : \text{prawie wszystkie } t_y = 0\} \iff$

$\iff x \in \{\sum_{y_1\in A} t_{y_1} \cdot y_1 + \sum_{y_2 \in B} t_{y_2} y_2 : -||-\} \iff$

$\iff x \in (\mathrm{Lin}\,A + \mathrm{Lin}\,B)$

(b) $\mathrm{Lin}(A\cap B) = \mathrm{Lin}\,A \cap \mathrm{Lin}\,B$

Kontrprzykład:

$A = \{\binom{1}{0}\}, \; B = \{\binom{2}{0}\}$

$\mathrm{Lin}(A\cap B) = \mathrm{Lin}\,\emptyset = \{\binom{0}{0}\}$

$\mathrm{Lin}\,A \cap \mathrm{Lin}\,B = \{t\cdot\binom{1}{0} : t \in \mathbb{R}\}$

zad. 10.

~~Weźmy w, w' ∈~~

Niech $F: V \to W$. Weźmy $w, w' \in W$.

F jest bijekcją więc ist. $v, v' \in V$ t.że $F(v) = w$ i $F(v') = w'$.

1. $F^{-1}(w+w') = F^{-1}(F(v)+F(v')) \overset{F\,\mathrm{lin}}{=\!=} F^{-1}(F(v+v')) =$

$= v + v' = F^{-1}(w) + F^{-1}(w')$

2. $F^{-1}(\alpha\cdot w) = F^{-1}(\alpha \cdot F(v)) \overset{F\,\mathrm{lin}}{=\!=} F^{-1}(F(\alpha v)) = \alpha v = \alpha \cdot F^{-1}(w)$