Lista 3 ZAD 4K

$f: G \to H$ homomorfizm grup, $g \in G$, $n \in \mathbb{Z}$. Udowodnić:

(a) $f(g^n) = (f(g))^n$

1° $n > 0$

$$f(g^n) = f(\underbrace{g \cdot \ldots \cdot g}_{n}) \overset{f:\text{ homom.}}{=} \underbrace{f(g) \cdot \ldots \cdot f(g)}_{n} = (f(g))^n \quad \text{ok}$$

2° $n = 0$

$$f(g^0) = f(e_G) \overset{?}{=} e_H = f(g)^0$$

e_G: elem. neutralny w G; e_H: elem. neutralny w H

Pokazujemy ?: $f(e_G) f(e_G) = f(e_G \cdot e_G) = f(e_G)$

$\Downarrow$ ZAD 15(a) (dla $a = f(e_G)$)

$f(e_G) = e_H$ ok. ?

3° $n = -1$

$$\left.\begin{array}{l} f(g^{-1}) f(g) = f(g^{-1} g) = f(e_G) = e_H \\ \text{Analogicznie} \quad f(g) f(g^{-1}) = e_H \end{array}\right] \Rightarrow f(g^{-1}) = \underbrace{f(g)^{-1}}_{\text{el. odwrotny do } f(g) \text{ w } H}$$

4° $n \leq -1$

$$f(g^n) = f((g^{-1})^{-n}) \underset{-n>0}{\overset{1°}{=}} f(g^{-1})^{-n} \overset{3°}{=} (f(g)^{-1})^{-n} = f(g)^{(-1)(-n)} = f(g)^n \quad \text{ok}$$

(b) f „1-1" $\Rightarrow \operatorname{ord}_G(g) = \operatorname{ord}_H(f(g))$

$\forall n > 0 \quad f(g)^n \overset{(a)}{=} f(g^n) = e_H$

$\Updownarrow$ f „1-1"

$g^n = e_G \quad (f(e_G) = e_H)$

Stąd: $\forall n > 0 \quad \boxed{f(g)^n = e_H \iff g^n = e_G}$

Stąd: $\{n > 0 \mid f(g)^n = e_H\} = \{n > 0 \mid g^n = e_G\}$

$\min\{n > 0 \mid f(g)^n = e_H\} = \operatorname{ord}_H(f(g)) = \operatorname{ord}_G(g) = \min\{n > 0 \mid g^n = e_G\}$ (dla wygody $\min \emptyset = \infty$)

(c) $\left.\begin{array}{l}\operatorname{ord}(g) = k \text{ : skończony} \\ n \in \mathbb{Z}\end{array}\right] \Rightarrow (g^n = e \iff k \mid n)$

$\Leftarrow$ $k \mid n \Rightarrow \exists l \in \mathbb{Z} \;\; n = kl \Rightarrow g^n = g^{kl} = (g^k)^l \underset{\operatorname{ord}(g) = k}{=} e^l = e$ ok

$\Rightarrow$ Załóżmy, że $g^n = e$.

Zapiszmy $n = kl + r$ gdzie $l \in \mathbb{Z}$ i $\underset{}{r} \in \{0, 1, \ldots, k-1\}$ $\quad (r = r_k(n))$

$$e = g^n = g^{kl + r} = (g^k)^l g^r \underset{\operatorname{ord}(g) = k}{=} e^l g^r = g^r$$

Mamy: $\left.\begin{array}{l} r \in \{0, 1, \ldots, k-1\} \\ g^r = e \\ k = \operatorname{ord}(g) \end{array}\right] \Rightarrow \left.\begin{array}{l} r = 0 \\ n = kl + r \end{array}\right] \Rightarrow k \mid n$ ok ∎