1.

Z zasady włączeń i wyłączeń

$$S=\sum 2^{-n}-\Big(\sum 2^{-2n}+\sum 2^{-3n}+\sum 2^{-5n}+\sum 2^{-7n}-\sum 2^{-6n}-\sum 2^{-10n}-\sum 2^{-14n}-$$
$$-\sum 2^{-15n}-\sum 2^{-21n}-\sum 2^{-35n}+\sum 2^{-30n}+\sum 2^{-105n}+\sum 2^{-70n}+\sum 2^{-42n}-$$
$$-\sum 2^{-210n}\Big)$$

dla $a_n=1$ $\quad A(x)=\sum x^{an}=\frac{1}{1-x}$ $\quad a'_n=(2^{-a})^{an}$ $\quad A'(x)=\sum (2^{-a})^n=\frac{1}{1-2^{-a}}=\frac{2^a}{2^a-1}$

$$S=\frac{2}{2-1}-\frac{2^2}{2^2-1}-\dots+\frac{2^{210}}{2^{210}-1}$$

2. $A(x)=\sum a_n x^n$ $\quad A'(x)=\sum a_{2n}x^{2n}$ $\quad A''(x)=\sum a_{3n}x^{3n}$

$$A'(x)=\frac{A(x)+A(-x)}{2}$$

0 1 2 3 4 5 6 7 8 9

Dla 3 nie wiedziałem jak zrobić.

zad.3 $\quad a_n=\sum_{i=1}^{n}F_iF_{n-i}=\sum_{i=1}^{n-2}F_i\left(F_{n-i-1}+F_{n-i-2}\right)+F_{n-1}\cdot F_1+F_nF_0=$

$$=\underbrace{\sum_{i=1}^{n-2}F_iF_{n-i-1}}_{a_{n-1}-\underbrace{F_{n-1}F_0}_{0}}+\underbrace{\sum_{i=1}^{n-2}F_iF_{n-i-2}}_{a_{n-2}}+\underbrace{F_1F_{n-1}}_{F_{n-1}}=a_{n-1}+a_{n-2}+F_{n-1}$$

F_{n+1}

$$a_n-a_{n-1}-a_{n-2}=F_{n-1}$$
$$\Downarrow$$
$$a_{n+2}-a_{n+1}-a_n=F_{n+1}$$

$$(E^2-E-1)\langle a_n\rangle=F_{n+1}$$

$$(E^2-E-1)(E^2-E-1)\langle a_n\rangle=0$$

Dalej jest tylko liczenie.

Zad. 4

Ze wzoru Taylora: $f(x) = \sum_{k=0}^{n} \left(\frac{(x-\alpha)^k}{k!} f^{(k)}(\alpha) \right) + R$

$\cancel{f(x) = (1+x)^a}$

rozwijamy wokół $\alpha = 1$

$f^{(k)}(x) = a\cdot(a-1)\dots(a-k) = a^{\underline{k}} x^{a-k}$

$f^{k}(1) = a^{\underline{k}}$

$$f(x) = x^a = \sum \frac{(x-\alpha)^k}{k!} f^{(k)}(\alpha) \overset{\alpha=1}{=} \sum \frac{(x-1)^k}{k!} a^{\underline{k}}$$

$$f(x+1) = (x+1)^a = \sum \frac{(x+1-1)^k}{k!} a^{\underline{k}} = \sum \frac{x^k}{k!} a^{\underline{k}}$$

Zad. 11 Z wykładu:

p_n – liczba podziałów n

$$P(x) = \sum_{i_1, i_2, \dots} x^{i_1 + 2i_2 + \dots} = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

Różne składniki: $(1+x)(1+x^2)\dots$

Niekoniecznie różne: $(1+x+x^2+\dots)(1+x^2+x^4+\dots)\dots$

a) $\prod_{i=1}^{\infty} \frac{1}{1-x^{2i}}$

b) $\prod_{i=1}^{m-1} \frac{1}{1-x^i}$

c) $\prod_{i=1}^{\infty} (1+x^i)$

d) $\prod_{i=1}^{\infty} (1+x^{2i})$

Zad. 13 a_n – liczba inwolucji n-elem.

$*$ $a_{n+1} = a_n + n a_{n-1}$

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & \dots & n \\ & & & \end{bmatrix} & \begin{array}{c} n+1 \\ n+1 \end{array} \end{array} \qquad \begin{array}{cc} \begin{bmatrix} 1 & 2 & \dots & n-1 \\ & & & \end{bmatrix} & \begin{array}{cc} n & n+1 \\ n+1 & n \end{array} \end{array}$$

inwolucja n-elem.   inwolucja n-1-elem.   ten element wybieramy na n sposobów

EGF

$**$ $A'(x) = \sum_{i=0} a_i \frac{i x^{i-1}}{i!} = \sum_{i=1} a_i \frac{x^{i-1}}{(i-1)!} = \sum_{i=1}^{\infty} a_{i+1} \frac{x^i}{i!} + a_1$

2*

$*$ $\sum_{i=1}^{\infty} a_{i+1} \frac{x^i}{i!} = \sum_{i=1}^{\infty} a_i \frac{x^i}{i!} + \sum_{i=1}^{\infty} i\, a_{i-1} \frac{x^i}{i!} = A(x) - a_0 + \sum_{i=1}^{\infty} a_{i-1} \frac{x^i}{(i-1)!} = A(x) - \underbrace{a_0}_{1} + xA(x)$

$\|**$

$A'(x) - \underbrace{a_1}_{1}$

$A'(x) - \cancel{a_1} = A(x) - \cancel{a_0} + xA(x)$

$$\frac{A'(x)}{A(x)} = 1 + x$$

$$\left.\begin{array}{l} \int_{x_0}^{x} \frac{A'(t)}{A(t)}\,dt = \ln A(x) + C \\ \int_0^x 1+t\,dt = \frac{t^2}{2} + t \Big|_0^x = \frac{x^2}{2} + x \end{array}\right\} \Rightarrow A(x) = e^{\frac{x^2}{2}+x}$$

$\frac{x^2}{2} + x = \ln(A(x))$

zad. 12

Z wykładu:

$$P(x) = \prod_{k=1}^{\infty} \frac{1}{1-x^k}$$

$$R(x) = \prod_{k=1}^{\infty} (1+x^k)$$

$$R(x)P(x^2) = \prod_{k=1}^{\infty} (1+x^k) \cdot \frac{1}{1-(x^2)^k} = \prod_{k=1}^{\infty} \frac{1+x^k}{1-x^{2k}} = \prod_{k=1}^{\infty} \frac{\cancel{1+x^k}}{(1-x^k)\cancel{(1+x^k)}} = \prod_{k=1}^{\infty} \frac{1}{1-x^k} = P(x) \quad \blacksquare$$