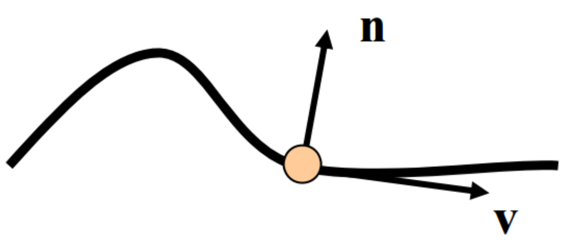
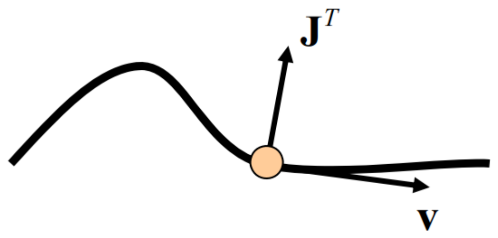
Lecture VII: Position-Based Dynamics

Previously: Constraints Calculus

- Constraint: $C(p, p', p'') = 0.$
- Velocity always tangent: $\mathrm{J}\vec{v} = 0$
- Constraint force always normal (virtual work):

$$\overrightarrow{F_c} = \sum_i \lambda_i \mathrm{J}_{i,.} = J^T \lambda$$





Previously: Sequential Impulses

- ```
  while (!done) {
          for all constraints c do solve c;
  }
  ```

- Computing the impulse:
  - For old velocity: $J_i v \neq 0$
  - For new velocity: $J_i(v + \Delta v) = 0$.
  - Apply impulse: $P = M\Delta v = {J_i}^T \lambda_i$, M is mass matrix.
- To compute λ_i

$$J_i\left(v + M^{-1}{J_i}^T \lambda_i\right) = 0$$

$$\lambda_i = \frac{-J_i v}{J_i M^{-1} {J_i}^T}$$
  ```

Previously: Semi-implicit Integration

- Also known as sympletic.
- Velocity integrated forward:

$$v(t + \Delta t) = v(t) + \frac{F}{m}\Delta t$$

- Position integrated backwards:

$$x(t + \Delta t) = x(t) + \Delta t v(t + \Delta t)$$

Position-Based Dynamics Principles

- Velocity always set to match position change at each iteration:

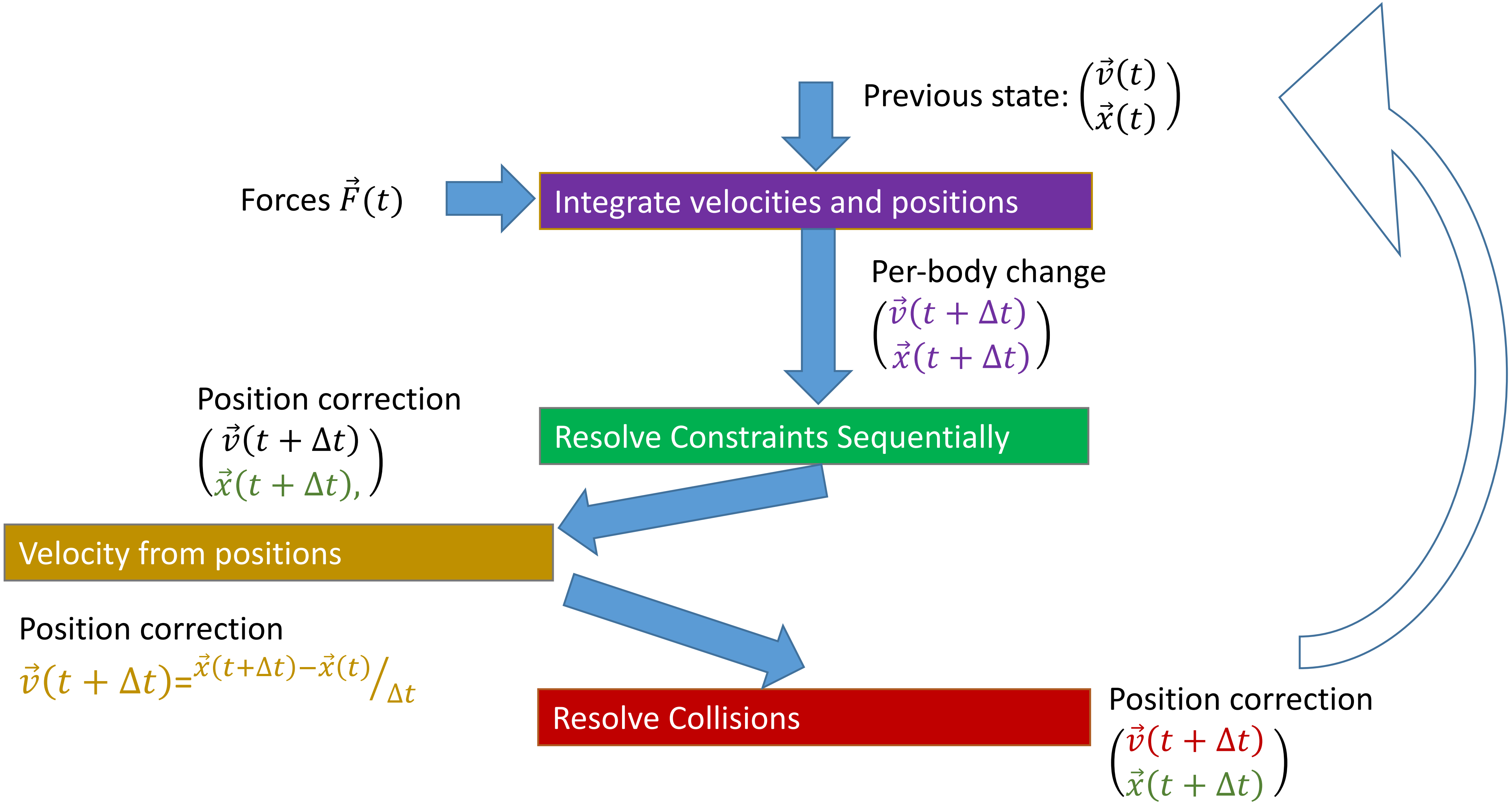
$$v = {}^{\Delta p}/_{\Delta t}$$

- Constraints expressed by positions alone

$$C(p, p', p'') = C(p) = 0$$

- "Classic" version: works on particles.
  - Connection (like rigidity) modeled with constraints.
  - Does not explicitly handle rotations

The Position-Based Game-Engine Loop



Position-Based Dynamics

- Advantages
  - Unconditional stability
  - Modularity and uniformity
- Disadvantages
  - Not physically accurate (but visually OK)
  - Resolves rotation by constraint solving
    - But then uniformly handles non-rigidity.

https://www.youtube.com/watch?v=j5igW5-h4ZM

Constraints Solving

- For body positions $\mathrm{X} = x_1, x_2, \cdots, x_m$.
- Set of equality constraints $C_i(X) = 0$.
  - lengths, connectivity.
- Set of inequality constraints $C_i(X) \geq 0$.
  - Collision, deformation limits.
- Sequential solution:
  - For each violating $C_i(X)$ in turn, compute ΔX s.t. $C_i(X + \Delta X)$ is valid.

Conservation

- Conserving linear momentum:

$$\sum_{i=1}^{m} m_i \Delta x_i = 0$$

- In matrix writing: $M\Delta X = 0$.
  - $M = diag(m_1, m_1, m_1, \cdots, m_n, m_n, m_n)$.
- Conserving angular momentum (r_i to a fixed origin)

$$\sum_{i=1}^{m} r_i \times m_i \Delta x_i = 0$$

$$\lambda_i = \frac{-J_i v}{J_i M^{-1} {J_i}^T}$$

Resolving Constraints

- Linear approximation:

$$C_i(p + \Delta p) \approx C_i(p) + \nabla C_i \cdot \Delta p$$

- Closest point: move in the gradient direction.

$$\Delta p = \lambda \nabla C_i$$

- To conserve momenta: Alternative weighting

$$\Delta p = \lambda_i M^{-1} \nabla C_i$$

  - Intuitive explanation: exchanging impulses and not velocities.

- For equality constraints:

$$C_i(p) + \lambda_i \nabla {C_i}^T M^{-1} \nabla C_i = 0 \Rightarrow$$

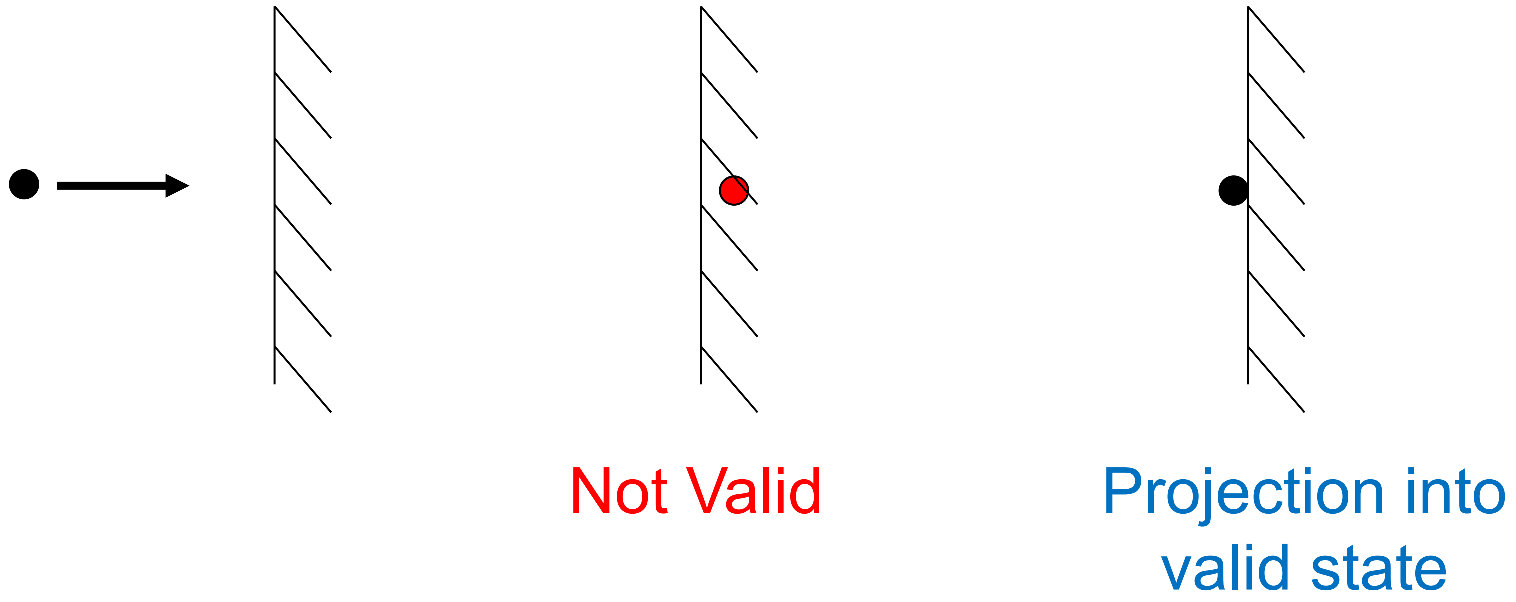
$$\lambda_i = \frac{-C_i(p)}{\nabla {C_i}^T M^{-1} \nabla C_i}$$

Inequality Constraints

- Essentially resolved the same

$$C_i(p + \Delta p) \approx C_i(p) + \nabla C_i \cdot \Delta p \geq 0$$

- Main difference: projection only performed if constrained is violated
  - Example: if collision happened.



Constraint Stiffness

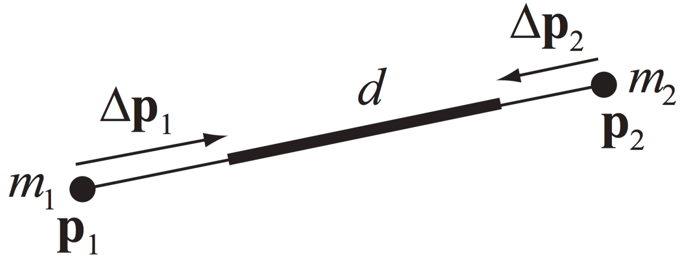
- Scaling each constraint by a scalar $k_i \in [0,1]$.
- Then we use: $\Delta p = \lambda_i k_i M^{-1} \nabla C_i$
- Makes the constraint "less" or "more" important.
  - The solution space is biased towards a solution that is closest to the components of the previous solution that satisfy the important constraints.

Stretch Constraint

- Between two points: $C(x_1, x_2) = |x_1 - x_2| - d_{12}$.
- Gradient components: $\nabla_1 C = \hat{n}, \nabla_2 C = -\hat{n}, \hat{n} = \frac{x_1 - x_2}{|x_1 - x_2|}$
- Resulting movement ($w = m^{-1}$) :

$$\Delta x_1 = \frac{-w_1}{w_1 + w_2} C(x_1, x_2)\hat{n}$$

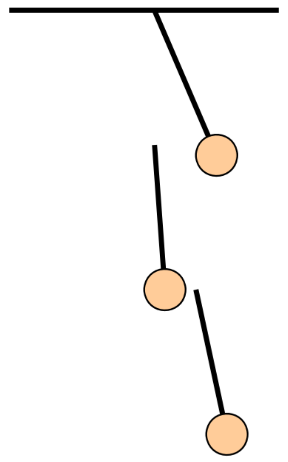
$$\Delta x_2 = \frac{w_2}{w_1 + w_2} C(x_1, x_2)\hat{n}$$

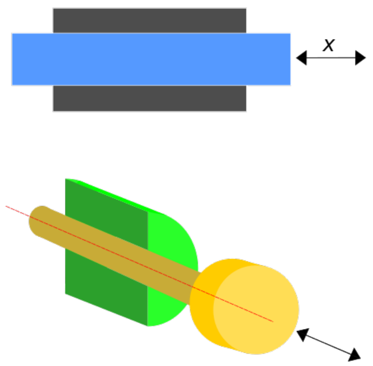


Connectors (Attachment Constraint)

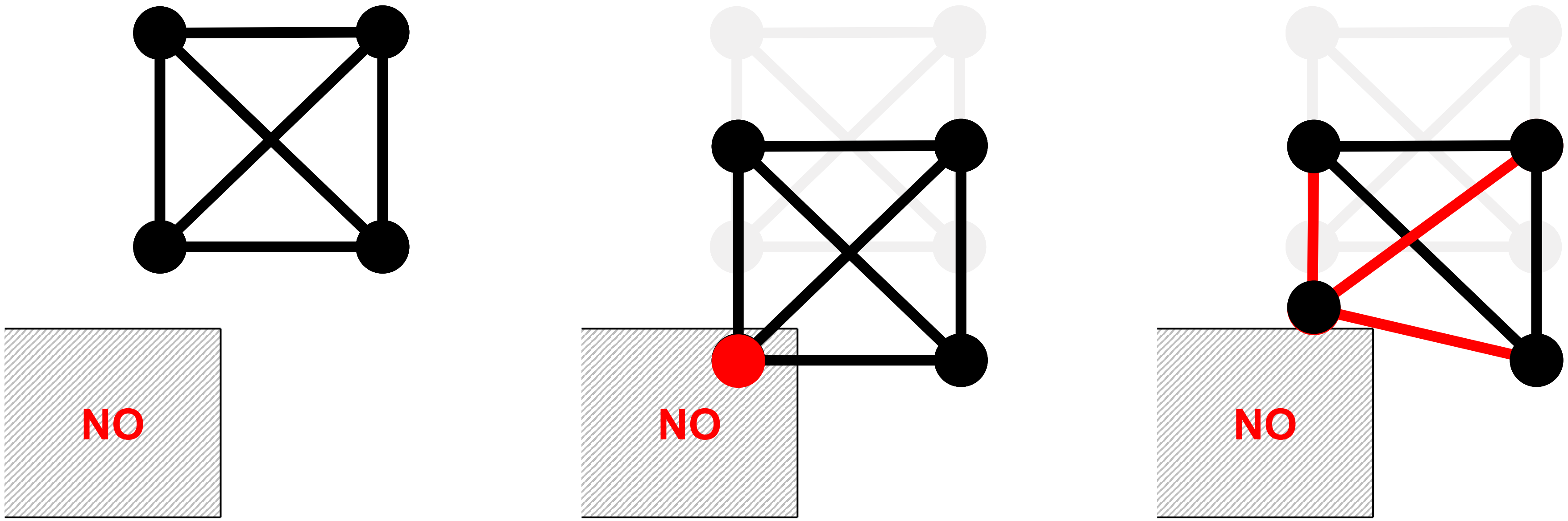
- If two (rigid) objects need to connect at two points x_1, x_2.
- Add connector constraint $C_i(x_1, x_2)$=$x_1 - x_2$.
  - Usually with the highest stiffness $k_i = 1$.
- Extensions: 1D translational constraint along $\hat{n}$:

$$C_i(x_1, x_2) = (x_1 - x_2)\hat{n} = 0$$





Example (rigid collision)



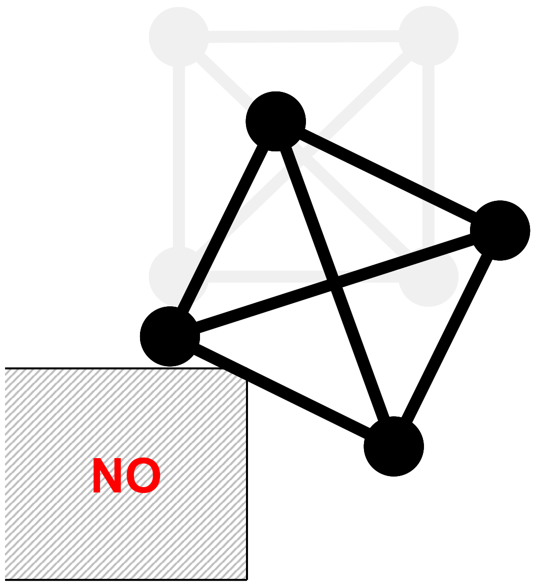
STEP 0

STEP 1
before constraints

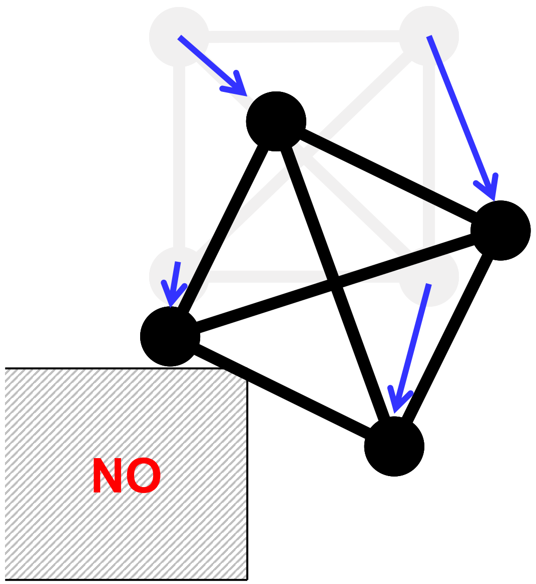
STEP 1
after 1st constraint

Example

Rotation is induced by rigidity!



STEP 1
after all constraints
multiple times



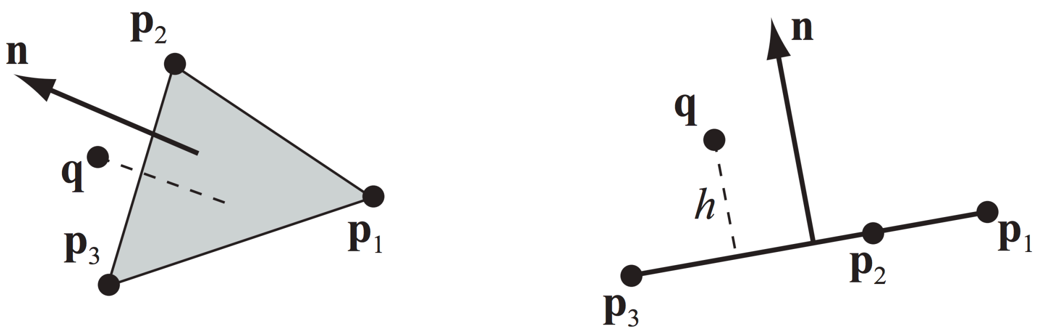
STEP 1
(implicit) velocities

Collision Constraint

- In previous iteration: x_i is out of the object.
- After position integration: x_{i+1} is penetrating.
- Penetration normal: $\hat{n}$ (pointing outwards)
- Must make sure: the projected point x_{i+1} is non-penetrating.
- Constraint: $C(x_{i+1}) = (x_{i+1} - x_i)\hat{n} \geq 0.$
  - with the highest stiffness $k_i = 1$
- Disadvantage: not re-colliding objects.
  - Artefacts usually negligible.

Specific Collision Constraint: Point through Triangle

- A point orientation vs. triangle normal:
- $C_{ptt}(q, x_1, x_2, x_3) = (q - x_1) \cdot \frac{(x_2 - x_1) \times (x_3 - x_1)}{|(x_2 - x_1) \times (x_3 - x_1)|} - h$
- h - thickness of triangle
  - Used for cloth simulation.



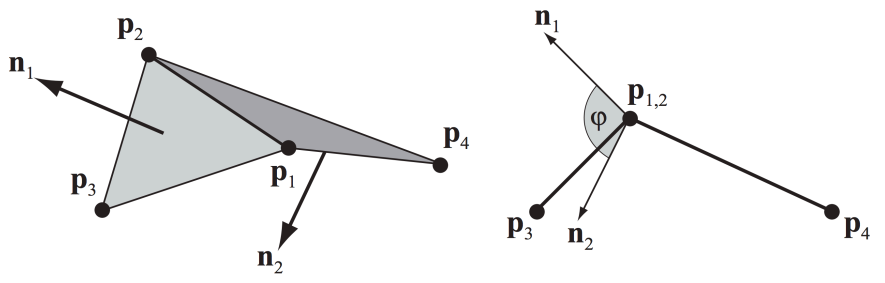
Bending Constraint

- Trying to preserve dihedral angle ϑ_{fg} between triangle f, g as much as possible.

$$C_{bend}(x_1, x_2, x_3, x_4) = acos\left(\hat{n}_f \cdot \hat{n}_g\right) - \vartheta_{fg}$$

- $\hat{n}_f$: normal to triangle f (resp. g)

$$\hat{n}_f = \frac{(x_2 - x_1)\times(x_3 - x_1)}{|(x_2 - x_1)\times(x_3 - x_1)|}$$



Velocity Damping

- For integrating velocities, before integrating positions.
- $v_i \leftarrow v_i + k_{damp} \Delta v_i$
- Many schemes exist.