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Problem 4.4:

To construct a strict inequality in Fatou's lemma (Capinski & Kopp, Theorem 4.7), we can consider the sequence of non-negative measurable functions defined on $(0,1)$ by:

$$f_n(x) = n\,\mathbf{1}_{(0,\,1/n]}(x), \qquad n = 1, 2, \ldots$$

Since each f_n is measurable (because the indicator function $\mathbf{1}_{(0,\,1/n]}$ is measurable) as in (Capinski & Kopp, Theorem 3.2), and scalar multiplication by n preserves measurability, then since $(0, 1/n] \subset [0,1]$, and f_n is defined to vanish outside $[0,1]$, the support of each f_n lies entirely within the unit interval.

To compute the integral of f_n, observe that f_n is a simple function taking the constant value n on the measurable set $(0, 1/n]$, which has Lebesgue measure $1/n$. By (Capinski & Kopp, Definition 4.2 and Theorem 4.2), this gives:

$$\int_0^1 f_n\,dm = n \cdot m((0, 1/n]) = n \cdot \frac{1}{n} = 1 \quad \text{for all } n$$

Next, if we examine the pointwise limit inferior, fix $x \in (0,1]$ and choose $N > 1/x$. Then for all $n \geq N$, we have $1/n < x$, so $x \notin (0, 1/n]$ and $f_n(x) = 0$. It follows that $\liminf_{n\to\infty} f_n(x) = 0$ for all $x \in (0,1]$. At $x = 0$, we may define $f_n(0) = 0$ for all n; since this single point has measure zero, it does not affect the integral. Therefore, $\liminf_{n\to\infty} f_n(x) = 0$ for m-almost every $x \in [0,1]$. Now, we can compute both sides of Fatou's inequality. For the left-hand side:

$$\int_0^1 \liminf_{n\to\infty} f_n\,dm = \int_0^1 0\,dm = 0$$

For the right-hand side:

$$\liminf_{n\to\infty} \int_0^1 f_n\,dm = \liminf_{n\to\infty} 1 = 1$$

Which yields the strict inequality as claimed:

$$\int_0^1 \liminf_{n\to\infty} f_n\,dm = 0 < 1 = \liminf_{n\to\infty} \int_0^1 f_n\,dm$$

So, this shows that Fatou's lemma may be strict even when each f_n is a simple function supported on a compact interval. The discrepancy arises because, although the functions maintain constant L^1 norm, their mass concentrates on sets whose measure shrinks to zero, illustrating the subtlety of limiting operations under the integral sign.

Problem 4.8:

To analyze the behavior of the sequence:

$$I_n = \int_0^\infty \left(1 + \frac{x}{n}\right)^n \cdot \frac{n}{\sqrt{x}}\,dx, \qquad n = 1, 2, \ldots$$

We can begin by applying the substitution $x = ny$, so that $dx = n, dy$ and $x^{-1/2} = (ny)^{-1/2}$. Substituting into the integral gives:

$$I_n = \int_0^\infty (1+y)^n \cdot \frac{n}{\sqrt{ny}} \cdot n\,dy = n^{3/2} \int_0^\infty (1+y)^n y^{-1/2}\,dy$$

For convergence of this integral, observe that for $y \geq 1$ we have $(1+y)^n \geq y^n$, so:

$$\int_0^\infty (1+y)^n y^{-1/2}\,dy \geq \int_1^\infty y^{n-1/2}\,dy$$

The right-hand side is an improper integral of a function behaving like y^α with exponent $\alpha = n - \frac{1}{2} > -1$. So, the integral diverges for all integers $n \geq 1$, and so the entire expression in (**??**) is infinite:

$$\int_0^\infty (1+y)^n y^{-1/2}\,dy = \infty \qquad n \geq 1$$

Then, since $n^{3/2}$ is finite for each n, we have:

$$I_n = \int_0^\infty \left(1 + \frac{x}{n}\right)^n \cdot \frac{n}{\sqrt{x}}\,dx = \infty \qquad \text{for all } n \geq 1$$

So, the sequence I_n does not converge; every term is already divergent. This shows the necessity of the domination hypothesis in the Dominated Convergence Theorem (Capinski & Kopp, Theorem 4.18). Since each I_n is infinite, there can be no integrable function g such that $|f\_n(x)| \leq g(x)$ on $(0, \infty)$, where $f\_n(x) = n(1+x/n)^n/\sqrt{x}$. The failure of the domination condition prevents any application of the theorem and highlights that even point-wise convergence or uniform bounds on compact subsets are insufficient for interchanging limit and integral in the absence of a globally integrable envelope.

Problem 4.11:

To evaluate the integral $\int_{-\infty}^\infty n(x)\,dx, \qquad n(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\,e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, we can first consider the standard normal case $\mu = 0$ and $\sigma = 1$. We can first define:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\,e^{-x^2/2} \qquad I = \int_{-\infty}^\infty \varphi(x)\,dx$$

To compute I, square it and apply Tonelli's (or Fubini's) theorem (Capinski & Kopp, Theorem 4.12) to obtain

$$I^2 = \iint_{\mathbb{R}^2} \frac{1}{2\pi} e^{-(x^2+y^2)/2}\,dx\,dy$$

Switching to polar coordinates via $x = r\cos\theta$, $y = r\sin\theta$, with $dx, dy = r, dr, d\theta$, transforms the integral into:

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty e^{-r^2/2}\,r\,dr = \int_0^\infty e^{-r^2/2}\,r\,dr$$

Using the substitution $u = r^2/2$, so that $du = r, dr$, the inner integral becomes $\int\_0^\infty e^{-u}, du = 1$. Therefore, $I^2 = 1$, and since $\varphi(x) \geq 0$, it follows that $I = 1$. For the general case, let $x = \mu + \sigma z$, so that $dx = \sigma, dz$. Applying the change-of-variables formula (Capinski & Kopp, Theorem 4.32), we have:

$$\int_{-\infty}^{\infty} n(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-z^2/2} \cdot \sigma\,dz = \int_{-\infty}^{\infty} \varphi(z)\,dz = 1$$

This confirms that the Gaussian density integrates to one for all $\mu \in \mathbb{R}$ and $\sigma > 0$.

Problem 4.14:

To prove that the distribution function $F_X(y) = P\{\omega \in \Omega : X(\omega) \leq y\}$ is continuous if and only if $P_X y = 0$ for all $y \in \mathbb{R}$, we may consider first the expression of jump size. Note that for any $y \in \mathbb{R}$, define the left limit $F_X(y^-) = \lim_{t\uparrow y} F_X(t)$, which exists by monotonicity. Since the sets $(-\infty, t]$ increase to $(-\infty, y)$ as $t \uparrow y$, continuity from below and countable additivity imply that:

$$F_X(y^-) = P_X((-\infty, y))$$

Subtracting yields:

$$F_X(y) - F_X(y^-) = P_X((-\infty, y]) - P_X((-\infty, y)) = P_X\{y\}$$

This shows that the jump of F_X at any point y equals the probability mass at y. From this, it follows from the right-continuity and monotonicity of F_X (Capinski & Kopp, Proposition 4.30). If F_X is continuous, then $F_X(y) = F_X(y^-)$ for all y, and relation $(*)$ forces $P_X y = 0$ for all y. Conversely, if $P_X y = 0$ for every $y \in \mathbb{R}$, then $(*)$ yields $F_X(y) = F_X(y^-)$ for all y, so F_X has no jumps, combined with right-continuity (Capinski & Kopp, Proposition 4.30), which implies that F_X is continuous everywhere. So, F_X is continuous if and only if the measure P_X assigns zero mass to all singleton sets.