The Place of Archimedes' Quadrature
in the History of Greek *mathēmata*

— Triangle of properties of line segments representing mathematical objects—

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The Greek mathematicians always discuss the mathematical objects in terms of their essential (i.e., geometrical, arithmetical, ...) properties, and for them, the objects in question give their inquiry its direction, whereas, in modern mathematics, objects themselves become accessible through a general method. And it is well known that many Greek mathematicians had the custom of representing any magnitudes, of whatever nature, diagrammatically by line segments.

Then, we can find three fundamental distinctions in the properties of line segments in the history of Greek mathematics.

First, the distinction between "numerical length" and "continuous geometric magnitude" was fundamental for antiquity. This fundamental distinction can be seen well from the comparison between Babylonian mathematics and Euclidean geometry. By "Euclidean geometry", I mean the Greek pure geometry represented by Euclid's *Elements* Books I ~ VI, XI, XII and Apollonius' *Conica*, which have no incorporation of "geometry" and "arithmetic".

The Babylonians measured line segment as "numerical length." The length of a line segment is a continuous quantity. So, in order to measure it, we have to find how many "units" (i.e., a standard quantity for practical use or theoretical use) it has. Although the Babylonians used sexagesimal system in strictly mathematical or astronomical contexts, here I would like to explain their method using our units (i.e., metre, decimetre, etc.) instead of Babylonian unit of length, *nindan* or *gar*, because the kind of units is not essential to the following argument.

Now let us consider two line segments A and B. In order to compare the lengths of A and B, we can use a ruler of one metre. If the length of A is four metres and some remainder left over and the length of B is two metres and some remainder, the Babylonians seemed to use the method of making a smaller unit (e.g., decimetre) and to measure the remainder with it. In this way, for example, the length of A is measured as 4 m 6 dm (i.e., 4.6m), and the length of B 2 m 8 dm (i.e., 2.8m). Thus, when a measuring "ruler" is divided into smaller units, numbers with terminating decimal (or sexagesimal) expansions appear. Although there are no proper words in English to express these numbers because in English a "decimal" means a "decimal fraction", it is worthy of notice that these numbers used by Babylonians are not "fractions" as we know them. The original concept of "fraction" came from a different way of thinking from Babylonian sexagesimal expansions with "positional number systems." Babylonians applied the same system of representation (i.e., sexagesimal system) to numbers smaller than 1 as well as to numbers larger than 1. It is not correct historically to regard the Babylonian system as the same as "common fraction" whose denominator is a power of 60.

In the so-called Babylonian "algebra" the specific units are in many cases removed. Then, with regard to the properties of line segments in Babylonian mathematics we may take either of the following two standpoints.

a) Although Babylonians illustrated unknown numbers by means of line segments, rectangles and squares, they always remained numbers. In Babylonian "algebra" a lot of examples show clearly that no real geometrical situation is envisaged.

b) In Babylonian "algebra" line segments are the entities representing "measuring number." Babylonians dealt with measured or measurable line segments and areas through a "naive" geometric analysis.

The former view is maintained by O. Neugebauer, B.L. van der Waerden and A. Aaboe, and the latter by J. Høyrup. We shall first discuss the latter view. J. Høyrup has examined in minute detail the language of Babylonian "algebra", and he asserts that this language demonstrates a basis of Babylonian "algebra" in "naive" geometry. Thus, according to him, Babylonian "algebra" was not a science about pure numbers, but a science about geometrical entities. If so, we may say that Babylonian mathematicians were always conscious of concrete units even in the case when they omitted their units. By contrast, according to the former view, in Babylonian "algebra" both area and length are considered as numbers without units. Since a length and an area are essentially different kinds of quantity, we cannot do "addition" and "subtraction" between them. However, in Babylonian "algebra" lines had a numerical length without units and plane figures had a numerical "area" without units. Therefore Babylonian mathematicians could add the area and the line segment without concern.

As far as the original nature of the problems of the second degree dealing with surface and length are concerned, I tend to agree with Høyrup's opinion, because in the texts we can find the traces of geometrical conceptualization. However, if we survey the whole of Babylonian mathematics, I suppose that some of the things he says are a bit extreme. In Babylonian mathematics we can find some examples of procedures which disregard the reality, such as "multiplication of two areas" or "addition of number of days and of men." Further, Babylonian mathematicians solved some types of cubic equations by means of their tables. It is well known that Babylonian table texts have a purely numerical character. Thus, it seems to me that, along with the development of Babylonian mathematics, in some cases original geometrical concepts gradually become to play a secondary part in Babylonian "algebra." In the case that line segments and plane figures are considered simply as numbers without units, we need not take the kind of quantity into consideration. Although "multiplication" and "division" between continuous quantity will produce a new kind of quantity, such operations are freely able to be done if we deal with "pure" numbers without units. Anyway, whether line segments represented pure numbers or geometrical entities with hidden units, we may say that Babylonians considered line segments as numerical length.

On the other hand, the Greeks originally seemed to use a different method for measuring the length of a line segment. For example, when the length of line segment A is four metres and some remainder left over, they seemed to divide a ruler with one metre by the remainder in order to measure the remainder. If the answer is 3, then the length of A is 4·1/3m. Thus, when we make both the measured line segment and the measuring ruler smaller alternately, a "fraction" appears. Of course, it was difficult to indicate one number by means of two numbers, i.e., numerator and denominator, in antiquity. For the Greeks a "fraction" seemed to mean a fractional part or parts of an original quantity. This seemed to be the practical way of measuring length in the Greek world.

The Euclidean geometers removed a ruler and tried to compare two line segments directly. They confronted directly the geometrical objects with which they were concerned. They used the method of successive subtractions known as "*antanairesis*" (i.e., Euclidean algorithm). When they compare two line segments A and B, the smaller B is subtracted from the larger A, thus giving two new magnitudes B and A-B. Then the smaller of these magnitudes is again subtracted from the larger one, and so on. Numbers with terminating sexagesimal expansions used by Babylonians originally appeared from the use of an instrument for measuring, while a "fraction" is able to be considered without a ruler. Therefore it seems to me that it is natural for the Euclidean mathematicians to develop such a method, even though we find a little distance between an anthyphairetic process and the method of making both the measured line segment and the measuring ruler smaller successively. For Euclidean geometers line segments are not considered to have numerical lengths, but to be line segments in themselves. In Euclidean geometry, "magnitude" was a property of geometrical figures, and "sum", "difference", and "ratio" were considered only as geometrical manipulations on homogeneous magnitudes. Euclidean geometers are always conscious of the dimension, i.e., "line segment," "plane figure" and "solid". In the above "Euclidean algorithm," they subtract alternately, not dividing by a remainder. This is because they always retain the dimension. The transformation of dimension is able to be done only by means of "proportion", i.e., a condition that holds between four objects. Further, in rigorous theory represented by the *Elements* I-VI, XI and XII and Apollonius' *Conica*, even the concept of a product in the general sense is not used. For example, whenever we use a product of two factors to denote the area of a curvilinear figure, Euclidean geometers introduced a geometrical figure whose magnitude was equal to that of the figure under consideration. An expression such as "a product of two lengths" is senseless in Euclidean geometry. Thus we are able to see that the world of Euclidean geometry is completely different from that of Babylonian mathematics where in some cases "multiplication" and "division" are done freely.

Secondly, for the Greek arithmeticians, line segments with numerical length were completely different from line segments representing "numbers divisible into discrete units." This distinction can be seen by the comparison between Babylonian mathematics and Greek arithmetic. By "Greek arithmetic", I mean the Greek number lore including Pythagorean number theory.

It seems very probable that the Pythagoreans studied their arithmetical problems by means of "pebbles" or "dots." But the writer (or writers) of the arithmetical books of Euclid's *Elements* VII-IX formalized parts of Pythagorean theory of number by means of linear representation. In Greek arithmetic, "number" (*arithmos*) is a multitude composed of pure "units" (*monas*). And the "unit" is the source (*archē*) of number. The essential marks of the unit are its internal indivisibility and its external discreteness. In this paper, I will denote the "*monas*" by "u" and the sequence of "*arithmos*" by u2, u3, ...

It seems to me that Euclid presupposes the fundamental arithmetic notions of counting and adding between these numbers. And he defines only one operation on numbers, i.e., "multiplication". In Euclidean arithmetic, the multiplication is explicitly described in terms of addition; the *Elements* VII Definition 16 states as follows:

"A number is said to "multiply" a number when that which is multiplied is "added" to itself as many times as there are units in the other, and so some number is produced." (e.g., (u3)×(u2)=(u2)+(u2)+(u2))

It should be noticed that Euclid's treatment of multiplication in arithmetic is a little different from that of multiplication by an integer in modern mathematics, although we are able to interpret them as an analogous operation. The Greek arithmeticians always retained a concept of their units, although "pure" units could be grasped only by way of pure thought. In my opinion, in the realm of Greek arithmetic, "multiplication" must always be interpreted as "addition" of "definite objects."

Further, according to D.H. Fowler, the division is construed in terms of subtraction. He insists that the Greeks did not deploy anything corresponding to our conception of a common fraction n/m. I agree with his opinion, because in order to understand a fraction as a "nonintegral rational number," *arithmos* must itself already have been understood as "number" in the modern sense. Instead of a fraction, Euclid used the idea of "part" or "parts" of a number (the *Elements* VII, Def. 3 and Def. 4) in the arithmetical books in the *Elements*. This is because, in the realm of Greek arithmetic, "pure unit" was always retained, just as, in the realm of Euclidean geometry, "dimension" was always retained. In this paper, I will denote "an n-th part of a number um" by $\binom{p}{n}$ um.

(e.g., u2 is a part of u8; u2= $\binom{p}{4}$ u8,

u7 is parts of u20; u7=u5+u2= $\binom{p}{4}$ u20+ $\binom{p}{10}$ u20)

Finally, the distinction between "continuous geometric magnitude" and "numbers divisible into discrete units" was originally fundamental in Greek mathematics. Geometry is concerned with geometric magnitude at rest, whereas arithmetic is concerned with number itself. Geometric magnitudes can be infinitely divided, while in the realm of numbers, the "unit" provides the last limit of all possible partitions. Proclus stated that in geometry a "least magnitude" has no place at all, while Arithmetic is based on "numbers" and the unit. Thus, geometry and arithmetic dealt with completely different mathematical objects. In geometry, a method of proving mathematical existence was "construction". On the other hand, arithmetical proofs of existence depended upon logical considerations and were frequently indirect. It is interesting to notice that Archytas of Tarentum made the statement that not geometry but arithmetic alone could provide satisfactory proofs. Proclus also stated as follows:

> "That numbers are purer and more immaterial than the magnitudes and that the starting-point of numbers is simpler than that of magnitudes are clear to everyone."
>
> (*Procli Diadochi in primum Euclidis elementorum librum commentarii*, ed. F. Friedlein, p.95)

According to Szabó, at the beginning of Greek mathematics, the Greeks thought that arithmetic took precedence over geometry because "geometrical figures" are less abstract than "numbers". Anyway such ontological problems of mathematical objects seemed to determine fundamentally Greek mathematical presentation, especially in its beginnings. According to Anatolius(3c. A.D.), the followers of Pythagoras are said to have applied to term *μαθηματική* more particularly to the two subjects of geometry and arithmetic, which had previously been known by their own separate names only and not by any common designation covering both. Anatolius called them "two main branches of the prime and more honourable type of mathematics".

However, it is true that in fully developed Greek mathematics there were some sources which show a close connection between geometry and arithmetic.

In the *Elements* X (the theory of incommensurable lines) and XIII (the theory of "Platonic regular polyhedrons"), some arithmetic properties of numbers are used. For example, Book X Prop. 9 states: "The squares on straight lines commensurable in line segment have to one another the ratio which a square number has to a square number. ..." Such properties of numbers enabled the author of Book X to combine the geometric property of the incommensurability with an arithmetic property of the numbers. Thus, the author of Book X and XIII partly transcended the opposition of "geometry" and "arithmetic."

However, as far as laws of proportionality of Book X and Book XIII are concerned, I would like to draw your attention to one point. Modern commentators have usually not found anything particularly problematic about the application of the theory of proportion to Book X and Book XIII, perhaps because they regard the theory of proportion in the *Elements* V as a general theory of proportion which can treat general magnitudes including numbers.

However, it seems to me that this view is questionable. I incline to the belief that Euclid may have had different ways of thinking about "magnitudes." At the beginning of the *Elements* Book V, Euclid defines a "ratio" (*logos*) as follows:

> "A ratio is a sort of relation in respect of size between two magnitudes (*megethos*) of the same kind."

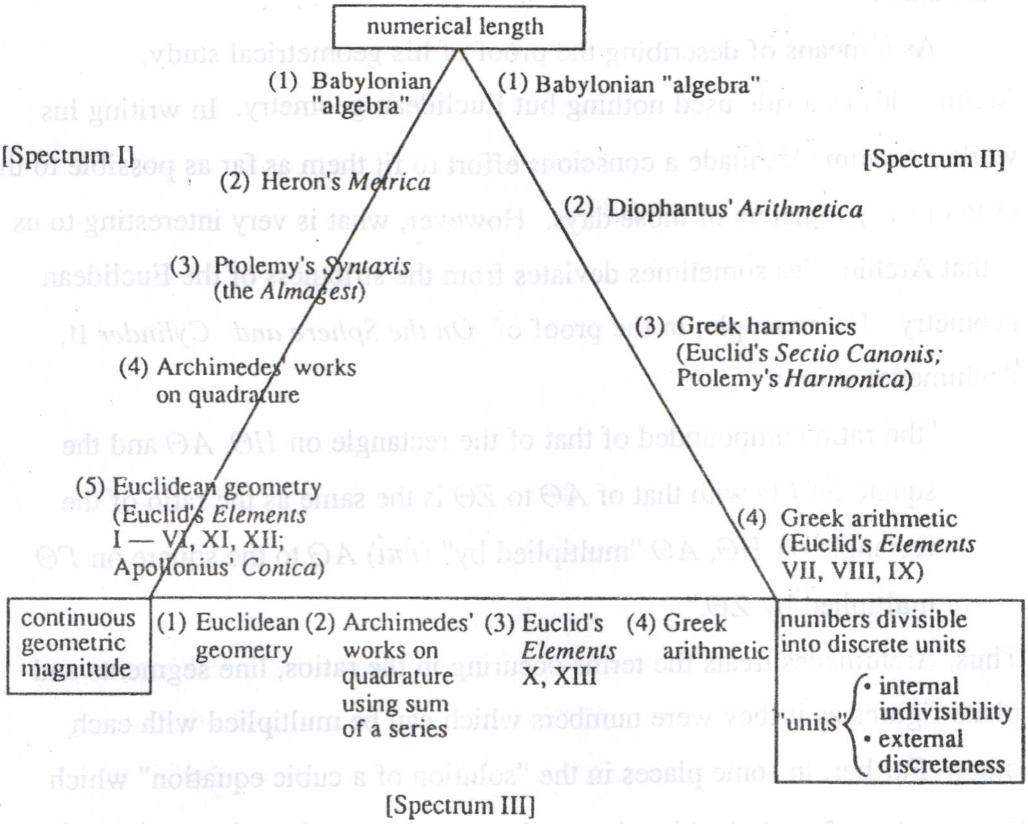
It is interesting to notice that, in the *Elements*, "ratios" are not "mathematical objects," and that "ratio" and "proportion" refer to different kinds of mathematical entities. A "proportion" is a condition between four magnitudes or four *arithmoi*. There has been some disagreement among scholars concerning the nature of *megethos* in Greek mathematics. This seems to be because the Greeks themselves used this word a little vaguely. It is true that, in his *Categories*, Aristotle explained *megethos* as including both the continuous (i.e., a line, a surface, and a body) and the discrete (i.e., a number). However, *megethos* for Aristotle seems to be generally connected with "body." In *De caelo* I.1 *megethos* is said to have one, two, or three dimensions, and in *Metaphysica* Δ, 13 it is "continuous one, or two, or three ways." Thus, in many passages Aristotle would not include numbers among magnitudes. And, as we have seen above, Proclus also contrasted *megethos* with *arithmos*. Although in our eyes the theory of proportion in the *Elements* Book V can be applied to "general magnitudes" including numbers, it should be noted that Greek *arithmos* is a little different from positive integers of modern theory. As we have seen above, Greek arithmeticians were always conscious of "pure units" and used the idea of "part" or "parts" of a number. In fact, concerning the theory of proportion of the *Elements* X, I. Mueller pointed out that Euclid needs some law or laws connecting proportionality involving magnitudes and proportionality as defined for numbers. In my opinion, Euclid does not appear to intend to apply any laws from Book V to *arithmoi*. The word "*megethos*" in the *Elements* Book V means only continuous geometrical magnitudes, namely, a line segment, a plane figure, and a solid. In fact, Euclid proved laws of proportion separately for geometric magnitudes (Book V) and numbers (Book VII). He applied two theories of proportion for different kinds of mathematical objects to his theory of incommensurability of the *Elements* X.

If we think of Greek *mathēmata* as made up of three spectra of properties of line segments, with Babylonian "algebra" at one corner, the Euclidean geometry at the second corner, and Greek arithmetic at the third, then we will be able to discover several lines of each spectrum in between each pair of corners.

Spectra I and II express to what degree each Greek mathematical work differed from Babylonian mathematicians' conception of line segments, i.e., numerical length.

Spectrum III expresses the degree of incorporation of geometrical way of thinking and an arithmetical way of thinking. This distinction is only relevant to the situation existing in Greek mathematics. In my "Triangle," I have divided Euclid's *Elements* into three parts; the *Elements* I-VI, XI, XII ("Euclidean geometry"), the *Elements* VII-IX (arithmetical books), and Books X, XIII.

Triangle of properties of line segments in the history of Greek *mathēmata*



Now let us examine the place of Archimedes in the history of Greek *mathēmata*.

As a means of describing the proof of his geometrical study, Archimedes as a rule used nothing but Euclidean geometry. In writing his works, Archimedes made a conscious effort to fit them as far as possible to the taste of the geometers of those days. However, what is very interesting to us is that Archimedes sometimes deviates from the strictness of the Euclidean geometry. For example, in the proof of *On the Sphere and Cylinder* II, Archimedes states:

> "the ratio compounded of that of the rectangle on $H\Theta$, $A\Theta$ and the square on $\Gamma\Theta$ with that of $A\Theta$ to $Z\Theta$ is the same as the ratio of the rectangle on $H\Theta$, $A\Theta$ "multiplied by" (ἐπί) $A\Theta$ to the square on $\Gamma\Theta$ multiplied by $Z\Theta$."

Thus, Archimedes treats the terms occuring in the ratios, line segments and plane figures as if they were numbers which can be multiplied with each other. Further, in some places in the "solution of a cubic equation" which Eutocius later found, Archimedes used expressions such as the product of a plane figure and a line segment. For example, he states:

> "(If (line segment EA):(line segment $A\Gamma$) = (surface Δ):(square on EB),) then, the square on EB multiplied by (ἐπί) EA is equal to the surface Δ multiplied by $A\Gamma$."

As we have seen above, an expression such as "a product of two line segments" is senseless in Euclidean geometry. Such way of thinking is contrary to the Euclidean geometry, although it is in agreement with the way of thinking such as in Heron's *Metrica*. Regarding signs of slackening in the strictness of the Euclidean geometry, E. J. Dijksterhuis concluded that it is the first step on the road to arithmetization of the geometrical argument. This unique daring may have something to do with his career, in which he started as a student of astronomy and mechanics under the influence of his father. Archimedes won fame in the ancient world as an inventor of marvelous machines. His famous statement "Give me a place to stand on and I will move the earth." (Δός μοι ποῦ στῶ καὶ κινῶ τὴν γῆν) is related to his study of mechanics. Archimedes' study on mechanics was destined to exert great influence on his later study of quadrature and hydrostatics. In his works on quadrature we find ideas such as the "centre of weight" of a plane figure without breadth, "motion of a point" describing a spiral, and ratio of times "in uniform motion", which were unacceptable to Euclidean geometers in those days. In the study of mechanics, Archimedes might have calculated the numerical "area" or "volume" of several concrete figures by means of the multiplication of the lengths of line segments. We have no evidence for this, but we can infer his approximate calculation from the story of the golden crown of King Hieron.

Finally, in my previous paper, I have made it clear that, in his later years, Archimedes' method of discovery has a systematic and direct connection with the geometrical proof. Why could Archimedes develop his own systematic procedure connecting closely the method of discovery with that of proof? In Greek mathematics, in general, there seems to be a wide gap between the method of discovery and that of proof. For example, the Euclidean geometers proved that (cone)=$\left(\frac{p}{3}\right)$(cylinder which has the same base as the cone and equal height)

as follows.

"Suppose that

(cone)>$\left(\frac{p}{3}\right)$(cylinder).

Then, we can construct the inscribed solid in the cone, by the *Elements* X Prop.1,

such that

(cone)-(inscribed solid)<(cone)-$\left(\frac{p}{3}\right)$(cylinder).

Then we have

(inscribed solid)>$\left(\frac{p}{3}\right)$(cylinder).

But we can show by another derivation that

(inscribed solid)<$\left(\frac{p}{3}\right)$(cylinder), ... (*)

which is impossible.

Therefore the cone is not greater than a third part of the cylinder.

In the same way, it may be shown to be not less.

Therefore

(cone)=$\left(\frac{p}{3}\right)$(cylinder)."

This indirect proof named later "the method of exhaustion" has strong persuasive power, but it is not a method of discovery, since it is useful only as long as the conclusion has been reached. It is interesting to notice that the relation (*) in the above proof is related to the method of discovery of the theorem. However, when Euclidean geometers try to obtain the relation such as (*), they always applied an individual procedure in each problem.

On the other hand, although Archimedes obeyed the same formation in writing his proofs of the problems concerning quadrature, Archimedes, in his later years, was ingenious enough to use systematically the idea of the "sum of a series" in order to obtain the relation such as (*). Thus the use of the "sum of a series" is closely connected with his idea of "geometrical indivisibles" in his heuristic derivation in *The Method*. Although one of the merits of his method of discovery is that it is not necessary to obtain the troublesome sums of a series by using the concept of the "centre of gravity" of the whole solid, once the relations such as (*) are obtained by the use of the sum of a series, the proofs by the "method of exhaustion" are able to be constructed immediately. Consequently, the older Archimedes grew, the shorter become the distance between the method of discovery and that of proof.

Now let's examine further Archimedes' idea of the sum of a series which he introduced into his geometrical proof. The idea of the "sum of a series" is originally an arithmetic way of thinking. So, for example, Archimedes introduced the idea of sum of a series of "square numbers," i.e., $u1^2+u2^2+u3^2+...+un^2$, into a geometrical proof by changing it into a geometrical formation, namely, the sum of a series of "squares" whose sides are in arithmetic progression. (i.e., in modern notation, $d^2+(2d)^2+(3d)^2+...+(nd)^2$, d; side of the smallest square.)

However, in the case of Archimedes' sum of a series of squares, the side of the smallest square "d" itself cannot be completely interpreted as "unit" (*monas*) of Euclidean arithmetic. Although it looks like a unit which has external discreteness, it is a continuous geometric magnitude which has "not" internal indivisibility. Further, I suppose that in this case Archimedes seemed to regard each square as "a product of two sides." Thus, in my opinion, in his geometrical proofs of quadrature, Archimedes partly used the idea of the "multiplication" of line segments.

In the strict sense, Euclid's proof of quadrature has no connection with the modern conception of "area" and "volume," because the proof is capable of being applied only to "continuous geometric magnitude," not to "numerical area and volume." By introducing the arithmetical idea of sum of a series into the study of geometry, Archimedes' quadrature is in a subtly different position from Euclidean geometry. Thus, Archimedes partly transcends the opposition of "geometry" dealing with continuous geometric magnitudes and "arithmetic" dealing with numbers divisible into discrete units. As was seen above , this distinction was originally fundamental for antiquity. In this respect, we should once more bring to mind a kind of tension between the "method" and "objects" in Greek mathematics. Judging from the criterion of original nature of Euclidean geometry, Archimedes might be said to have exhibited a kind of "unique daring" by introducing an arithmetic way of thinking into geometrical problems. Since many modern scholars are apt to turn their attention to general methods found in Greek mathematics, such a viewpoint seems to be obscure. It seems to me that such unique daring enabled Archimedes to make use of his own systematic procedure connecting closely the method of discovery using the idea of indivisibles with the proof by the method of exhaustion.