MATHEMATICAL THEORIES DESCRIBING CAPILLARY ACTION BY LAPLACE, GAUSS AND POISSON
LAPLACE, GAUSS 及び *POISSON* による毛細管現象記述の数学的理論

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1. Abstract

We discuss the mathematical theory of deduction of the capillary action by Laplace, Gauss, Poisson. These share the common concept of attraction and repulsive force on continuum, which is realized with two constants. The former two are deduce the equations of the capillary surface, and the latter confirms the formulae, in another analytical problems.

2. **The "two-constant" theory in capillarity**

Gauss didn't mention the following fact, and Bowditch [1] also didn't comment on Gauss' work in Laplace's total works [6] except for only one comment of the name "Gauss" [6, p.686]. [2]

N.Bowditch comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat taken into consideration, because he supposed it to be unnecessary. But in 1819[3] he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the combined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$ [6, p.685].

In his historical descriptions about the study of capillary action, we would like to recognize that there is no counterattack to Gauss, but the correct valuation. Gauss [2] stated his conclusions about the papers by Laplace as follows :

> ($\Leftarrow$) To this cardinal proposition of the total theory with calculation for demonstration, we can not accept the papers by Mr. Laplace ; in p.5, since not only he developed clearly incorrect argument but also showed even the false proofs : we consider that *his calculations in the pages and the following after p.44 are the vain effects.*[4] [2, pp.33-34] (italic and trans. mine.)

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[1] ($\Downarrow$) The present work is a reprint, in four volumes, of Nathaniel Bowditch's English translation of volumes I, II, III and IV of the French-language treatise *Traité de Mécanique Céleste* by P.S.Laplace. The translation was originally published in Boston in 1829, 1832, 1834, and 1839, under the French title, "*Mécanique Céleste*", which has now been changed to its English-language form, "*Celestial Mechanics.*"

[2] ($\Downarrow$) Bowditch's comment number [9173g].

[3] ($\Downarrow$) Poisson comments this fact in [7, p.19].

[4] ($\Downarrow$) There are 35 pages of calculation between p.44 and p.78 in his *Supplément.*

3. Laplace papers of the capillary action

3.1. Laplace's conclusions of theory of the capillary action.

Laplace stated his "*complete theory*" of attraction which have an effct on the capillary action in the introduction [3], as follows : From the translation by Bowditch [6], for brevity, we show the corresponding part with above as follows :

> A long while ago, I endevored in vain to determine the laws of attraction which would represent these phenomena ; but same late researches have rendered it evident that the whole may be represented by the same laws, which satisfy the phenomena of refraction ; that is, by laws in which the attraction is sensible only at insensible distances ; and from this principle we can deduce a complete theory of capillary attraction. [6, p.688]
>
> . . .
>
> From these results, relative to bodies terminated by sensible segments of a spherical surface, I have deduced this general theorem. *"In all the laws which render the attraction insensible at sensible distance, the action of body terminated by a curve surface, upon an infinitely narrow interior canal, which is perpendicular to that surface, at any point whatever, is equal to the half sum of the actions upon the same canal, of two spheres which have the same radii as the greatest and the least radii of curvature of the surface at that point."* By means of this theorem, and of the laws of the equilibrium of fluids, we can determine the figure which a fluid must have, when it is included whithin a vessel of a given figure, and acted upon by gravity. [6, p.689]

3.2. Laplace's theory of the capillary action.

Laplace's theories of the capillary action are described in the 14 articles. We cite only the contents of *no* 1 ([4, pp.10-14]) of theory of [4] pointed out by Gauss:

¶ *no* 1 of the theory of capillary action :

> Pour avoir l'action de la sphère entière dont le rayon est b, supposons $b-u=z$; cette action sera égale à l'integrale
>
> $$2\pi\int\frac{(b-z)}{b}.dz.\Psi(z),$$
>
> prise depuise $z=0$ jusqu'à $z=b$. Soit donc K l'integrale $2\pi.\int dz.\Psi(z)$ prise dans ces limites, et H l'integrale $2\pi.\int zdz.\Psi(z)$ prise dans les mêmes limites ; l'action précédente deviendra
>
> $$K-\frac{H}{b}.$$
>
> On doit observer ici que K et H peuvent être considérés comme étant indépendans de b ; car $\Psi(z)$ n'étant sensible qu'à des distances insensibles, il est indifférent de prendre les intégrales preécédentes, depuis $z=0$ jusqu'à $z=b$, ou depuis $z=0$ jusqu'à $z=\infty$; ensort qu'on peu supposer que K et H répondent à ces dernières limites. [4, p.13]

($\Downarrow$) This means that

$$K=2\pi\int\Psi(z),\qquad H=2\pi\int z\Psi(z)dz\ ; \tag{1}$$

where the limits are from $z=0$ to $z=b$ or from $z=0$ to $z=\infty$. These two constants are the original of what we called the two constants, in the 1805's paper [4] by Laplace, so that we think, it is noteworthy. ($\Uparrow$)

¶ *no* 4 ([4, p.18-23]) of the theory of capillary action :

Soit donc O(*fig. 3*) le point le plus bas de la surface AOB de l'eau renfermée dans un tube. Nommonz z la coordonnée verticale OM; x et y, les deux coordonnées horizontales d'un point quelconque N de la surface. Soient R et R' le plus grand et le plus petit des rayons osculateurs de la surface à ce point.

R et R' seront les deux racines de l'équation [5]

$$R^2(rt-s^2) - R\sqrt{(1+p^2+q^2)}\{(1+q^2)r - 2pqs + (1+p^2)t\} + (1+p^2+q^2)^2 = 0, \tag{2}$$

équation dans laquelle

$$p = \frac{dz}{dx}; \quad q = \frac{dz}{dy}; \quad r = \frac{d^2z}{dx^2}; \quad s = \frac{d^2z}{dxdy} =^* \frac{dp}{dy} =^* \frac{dq}{dx}; \quad t = \frac{d^2z}{dy^2}. \tag{3}$$

On aura donc

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2)\frac{dp}{dx} - pq\left(\frac{dp}{dy} + \frac{dq}{dx}\right) + (1+p^2)\frac{dq}{dy}}{(1+p^2+q^2)^{\frac{3}{2}}} = \frac{(1+q^2)r - 2pqs + (1+p^2)t}{(1+p^2+q^2)^{\frac{3}{2}}} \tag{4}$$

Cela posé, si l'on conçoit un canal quelconque infiniment étroit NSO ; on doit avoir par la loi de l'équilibre du fluide renfermé dans ce canal,

$$K - \frac{H}{2}\left(\frac{1}{R} + \frac{1}{R'}\right) + gz = K - \frac{H}{2}\left(\frac{1}{b} + \frac{1}{b'}\right); \quad \Rightarrow \quad \left(\frac{1}{R} + \frac{1}{R'}\right) - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'}; \tag{5}$$

b et b' étant le plus grand et le plus petit des rayons osculateurs de la surface au point O, et g étant la pesanteur. En effet, l'action du fluide sur le canal, au point N, est par ce qui précède, $K - \frac{H}{2}\left(\frac{1}{R} + \frac{1}{R'}\right)$; et de plus, la hauteur du point N audessus du point O est z. L'équation précèdente donne, en y substituant pour $\frac{1}{R} + \frac{1}{R'}$, sa valeur, [6]

$$(a) \qquad \frac{(1+q^2).r - 2pqs + (1+p^2).t}{(1+p^2+q^2)^{\frac{3}{2}}} - \frac{2gz}{H} = \frac{1}{b} + \frac{1}{b'}; \tag{6}$$

[4, p.19]

4. **Gauss' papers of the capillary action**

Gauss states common motivations with Laplace about MD equations. For example, in §10, §11, §12, which we mention below, he states the difficulties of integral $\int r^2\varphi r.dr$, in which he confesses that he also is included in the person who feels difficulties to calculate the MD integral.

5. ***Principia generalia theoriae figurae fluidrum in statu aequilibrii. (General principles of theory on fluid figure in equilibrium state)***

Gauss introduces his expression of curved surface.

$$\xi = -\zeta.\frac{dz}{dx}, \qquad \eta = -\zeta.\frac{dz}{dy}, \qquad d\zeta = \xi\zeta^2 d\frac{dz}{dx} + \eta\zeta^2 d\frac{dz}{dy} \tag{7}$$

$$\begin{aligned} \frac{d\xi}{dx} &= -\zeta\frac{d^2z}{dx^2} - \frac{dz}{dx}.\frac{d\zeta}{dx} = -\zeta\frac{d^2z}{dx^2} \underbrace{-\zeta\frac{dz}{dx}}_{=\xi}\xi\zeta\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \\ &= -\zeta(1-\xi^2)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta\left(\eta^2+\zeta^2\right)\frac{d^2z}{dx^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} \end{aligned}$$

[5]($\Downarrow$) (2) is a quadratic equation with respect to R.

[6]($\Downarrow$) From (4) and (5) we get it.

TABLE 1. Comparison of Q and V in $\delta U = \int QdP + \int VdU$ between analytic and geometric method

analytic method	geometric method
$Q = \left(\frac{\xi\eta}{\zeta}\delta x - \frac{\xi^2+\zeta^2}{\zeta}\delta y - \eta\delta z\right)X \quad + \left(\frac{\eta^2+\zeta^2}{\zeta}\delta x - \frac{\xi\eta}{\zeta}\delta y - \xi\delta z\right)Y$	$Q = -\delta e.\cos(5,7)$
$V = \left(\frac{d\frac{\xi\eta}{\zeta}}{dy} - \frac{d\frac{\eta^2+\zeta^2}{\zeta}}{dx}\right)\zeta\delta x \quad + \left(\frac{d\frac{\xi\eta}{\zeta}}{dx} - \frac{d\frac{\xi^2+\zeta^2}{\zeta}}{dy}\right)\zeta\delta y \quad + \left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)\zeta\delta z$	$V = \delta e.\cos(4,5).\left(\frac{d\xi}{dx} + \frac{d\eta}{dy}\right)$

$$\begin{aligned}\frac{d\eta}{dy} &= -\zeta\frac{d^2z}{dy^2} + \eta^2\zeta\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy} = -\zeta(1-\eta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}\\ &= -\zeta(\xi^2+\zeta^2)\frac{d^2z}{dy^2} + \xi\eta\zeta\frac{d^2z}{dx.dy}\end{aligned}$$

$$\begin{aligned}\frac{d\xi}{dx} + \frac{d\eta}{dy} &= -\zeta^3\Big[\frac{d^2z}{dx^2}\Big\{1 + \Big(\frac{dz}{dy}\Big)^2\Big\} - \frac{2d^2z}{dx.dy}.\frac{dz}{dx}.\frac{dz}{dy} + \frac{d^2z}{dy^2}\Big\{1 + \Big(\frac{dz}{dx}\Big)^2\Big\}\Big],\\ \text{where,}\quad \zeta^3 &= \Big[1 + \big(\frac{dz}{dx}\big)^2 + \big(\frac{dz}{dy}\big)^2\Big]^{-\frac{3}{2}}.\end{aligned} \tag{8}$$

6. Poisson's paper of capillarity

6.1. Poisson's comments on Gauss [1].

Poisson [7] commented in the preface about Gauss [1]:

- Gauss' success is due to the merit of his $\prec$ *characteristic* $\succ$
- even Gauss uses the same method as the given physics by Laplace.
- Gauss calculates by the condition only the same density and incompressibility

6.2. Poisson's two constants : K and H in capillary action.

We cite Poisson's K and H from [7, 12-14].

$$K = 2\pi\rho^2 q\int_0^\infty r^3\varphi r dr$$

where,

$$q \equiv \int_0^\infty\int_0^\infty \frac{(y+z)dydz}{[1+(y+z)^2]^{\frac{3}{2}}} = \frac{1}{3}\int_0^\infty \frac{dy}{(1+y^2)^{\frac{3}{2}}} = \frac{1}{3}$$

$$(1)_P \quad K = \frac{2}{3}\pi\rho^2\int_0^\infty r^3\varphi r\ dr \tag{9}$$

$$\eta = u\sin v, \quad \eta' = u\cos v, \qquad \zeta = Q\eta^2 + Q'(\eta')^2 + Q''\eta\eta'$$

We denote λ and λ' radii of two principle curvatures.

$$\frac{1}{\lambda} = \frac{d^2\zeta}{d\eta^2} = 2Q, \quad \frac{1}{\lambda'} = \frac{d^2\zeta}{d(\eta')^2} = 2Q',$$

The average value

$$\mu = -H(Q+Q') = -\frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big),$$

where, we denote H for convenience sake

$$H \equiv \pi\rho^2 \int_0^\infty \int_0^\infty \varphi r \frac{su^3}{r} duds$$

where,

$$s = ux, \quad ds = udx, \quad u = \frac{r}{\sqrt{1+x^2}}, \quad du = \frac{dr}{\sqrt{1+x^2}}$$

$$(2)_P \quad H = \pi\rho^2 \int_0^\infty r^4 \varphi r dr \int_0^\infty \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{4}\pi\rho^2 \int_0^\infty r^4 \varphi r \; dr \tag{10}$$

The normal action on this point :

$$(3)_P \quad N = K - \frac{1}{2}H\Big(\frac{1}{\lambda} + \frac{1}{\lambda'}\Big) \tag{11}$$

6.3. Coincidence of Poisson's K and H with Laplace's K and H.

Poisson proved Laplace's formulae as follows :

$$K = \frac{2\pi\rho^2}{3} h^3 \Pi h - \frac{2\pi\rho^2}{3} \int_0^h r^3 \frac{d\Pi r}{dr} dr = \frac{2\pi\rho^2}{3} \int_0^h r^3 \varphi r \; dr \tag{12}$$

$$H = \frac{\pi\rho^2}{4} h^4 \Pi h - \frac{\pi\rho^2}{4} \int_0^h r^4 \frac{d\Pi r}{dr} dr = \frac{\pi\rho^2}{4} \int_0^h r^4 \varphi r \; dr \tag{13}$$

ce qui coïncide avec les formes (9) et (10), en prenant $h = \infty$. [7, pp.14-15]

7. Conclusions

The formulae deduced by Laplace and Gauss are identical, Poisson uses as a commonly known formula. Poisson emphacizes the variation of density in the neighbor of wall, by which the fall or elavation occured. Today's explanation is by the surface tension, of which Poisson doesn't tell at all.

References

[1] C.F.Gauss, *Principia generalia theoriae figurae fluidorum in statu aequilibrii*, Gottingae, 1830, *Carl Friedrich Gauss Werke V*, Göttingen, 1867. (Similarly: "*Carl Friedrich Gauss Werke V*", Georg Olms Verlag, Hildesheim, New York, 1973, 29-77. Also, *Anzeigen eigner Abhandlungen, Götingische gelehrt Anzeigen*, 1829, as above in "*Werke V*", 287-293.)

[2] C.F.Gauss, *Carl Friedrich Gauss Werke. Briefwechsel mit F.W.Bessel. Gauss an Bessel (Göttingen den 27. Januar 1829), Bessel an Gauss (Königsberg 10. Februar 1829)*, Gottingae, 1830, Göttingen, 1880. Georg Olms Verlag, Hildesheim, New York, 1975.

[3] P.S.Laplace, *Traité de méchanique céleste*, Ruprat, Paris, 1798-1805, 1-66. (We use this original printed by Culture et Civilisation, 1967.)

[4] P.S.Laplace, *Supplément à la théorie de l'action capillaire*, Tome Quatrième, Paris, 1805, 1-78. (op. cit. [3].)

[5] P.S.Laplace, *Traité de méchanique céleste./* •§4 *On the equilibrium of fluids./* •§5 *General principles of motion of a system of bodies./* •§6 *On the laws of the motion of a system of bodies, in all the relations mathematically possible between the force and velocity./* •§7 *Of the motions of a solid body of any figure whatever./* •§8 *On the motion of fluids*, translated by N. Bowditch, Vol. I §4-8, pp. 90-95, 96-136, 137-143, 144-193, 194-238, New York, 1966.

[6] P.S.Laplace, *On capillary attraction, Supplement to the tenth book of the Méchanique céleste*, translated by N. Bowditch, Vol. IV, pp.685-1018, New York, 1966. (op. cit. [5].)

[7] S.D.Poisson, *Nouvelle théorie de l'action capillaire*, Bachelier Pére et Fils, Paris, 1831.
→ http://gallica.bnf.fr/ark:/12148/bpt6k1103201

Remark: we use *Lu* (: in French) in the bibliography meaning "read" date by the referees of the journals, for example MAS. In citing the original paragraphs in our paper, the underscoring are of ours.