# Development of variational calculus in Europe and its appplication to random fields

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**Abstract**

最近、パラメータが多次元の確率場の研究が重要となってきた。その研究には研究場の変分方程式による方法が重要である。特に、確率場がホイトノイズの汎関数である場合は $S$ 変換により、古典的な変分法が一つの有力な手段として応用できる。

ここでは、古典変分の歴史、特に曲線や，曲面をパラメータに持つ場の変分問題を振り返ってみたい。それは最近の確率場の研究に有効と思われる。今回はヨーロッパの変分について、これまで調べた範囲に報告する。特に Hadamard, Tonelli, Volterra, Lévy などの仕事に注目したい。

## 1 Introduction

The classical variational calculus has been developed、although it has been considered only for non-random functionals. We now wish to think of the variational problem for random fields $X(C)$, where $C$ is a closed contour. Classical theory on the variation of functionals depending on contour leads us to consider the variation of random fields depending on contour.

It is important, from the view point of information theory, to consider random fields $X(C)$ and their variations since $X(C)$ can carry or create more information as $C$ deforms compared to the case where a stochastic process $X(t)$ does as $t$ runs. In addition, the anlysis of $X(C)$ in line with stochastic calculus becomes essentially infinite dimensional, and various kinds of profound new results will be discovered.

We, therefore, wish to establish the variational calculus for random fields $X(C)$ by applying the classical theory of variational calculus. However, it is necessary to prepare some basic theory to handle the random fields, and thus we are concerned with only some particular random fields.

On the other hand, among Volterra's several outstanding work, we are interested in his contribution to biological application, Logistic curve and Lotkka-Volterra equation where the variable is not a curve like as mentioned above, but a function. Various approaches have been made for these equations. In some what different method from them we are interested in a probablistic formulation of these problems.

This report involves a short review of the work in the classical theory of variational calculus and the stochastic variational calculus related to white noise approach.

## 2 Lévy's work on variational calculus

P. Lévy discussed the variation of functionals depending on a contour. We can see his results them in the literatures [7] and others. It can be also seen that his work on functional analysis was much influenced by those of Volterra and Hardamard.

Lévy generalized the Hadamard equation in his desertation. There, he discussed the Neumann problem too. Most of the results are included in the book "Problèmes concret d'analyse fonctionnelle", 1951, mainly in Part II. We discuss some of his work appeared in his book in the following.

In the above mentioned book, Part I consists of various significant examples of functions of contour or surface, $f(C)$ , with the deformation of $C$.

**Example 1.** The functional $U(C)$ of a curve $C$ is given by the integral over the domain enclosed by a curve $C$ as follows.

$$U(C) = \int\int_{(C)} f(u,v)dudv, \ (C) \ \text{ domain enclosed by } C.$$

Then its variation is obtained as

$$\delta U(C) = \int_C f(u(s), v(s))\delta n(s)ds, \ (u(s), v(s)) \in C.$$

**Example 2.** The functional $V(C)$ is defined as the integral over a curve $C$,

$$V(C) = \int_C g(s)ds.$$

Then we have

$$\delta V(C) = \int_C \{\frac{\partial g}{\partial n}(s) - \kappa(s) g(s)\} \delta n(s) ds, \ \kappa \ : \ \text{curvature},$$
$$\frac{\partial}{\partial n} \ : \ \text{outward normal vector}.$$

Following his idea, we study the variation of random field depends on a curve $C$. The well known $S-$transform which is an important key in white noise calculus is a main tool in our work transforming the random functional to non-random functional to be able to apply Lévy's formula,

**Example 3.** Let $X(A)$ be Lévy's Brownian motion which can be expressed by

$$X(A) = c(d) \int_{S(A)} |u|^{-\frac{d-1}{2}} W(u) du; u \in R^d,$$

interms of white noise $W$,

$$S(A) = \{u; (u, OA) \geq |u|^2\}$$

and

$$c(d) = \{\frac{2(d-1)}{|S^{d-1}|} \int_0^{\frac{\pi}{2}} sin^{d-2}\theta d\theta\}^{\frac{1}{2}}.$$

The $S-$transform of $X$ is obtained as

$$(SX)(\xi) = c(d) \int \chi_{S(A)}(u) |u|^{-\frac{d-1}{2}} \xi(u) du.$$

Consider the conditional expectation $E[X(P)|X(A), A \in C]$, denoted by $Y(C)$. It can be expressed in the form

$$Y(C) = \int_C f(C, s) X(s) ds,$$

with suficiently smooth $f$ for $C \in \mathbf{C}$, where $X(s)$ denotes $X(A(s))$, $s$ being the parameter taken as an arc length running on $C$.

First take the $S-$transform of $Y(C)$, evaluate its variation and take back the inverse transform we obtain the variation of $Y(C)$ in the form

$$\begin{aligned}\delta Y(C) = \quad & \int_C \{\delta f(C, s) - \kappa f(C, s) \delta n(s)\} X(s) ds \\ & + \int_C f(C, s) \frac{\partial}{\partial n} X(s) \delta n(s) ds,\end{aligned}$$

where $\frac{\partial}{\partial n}$ stands for the normal operator, $\delta n(s)$ denotes the distance between $C$ and $C + \delta C$ at $s$, $\delta f(C, s)$ denotes the variation of kernel $f$ and $\kappa$ is the curvature of $C$.

**Remark.** The normal derivative $\frac{\partial}{\partial n}X(s)$ has singularity in $s$, and it is even not a generalized process. However, we can consider it as a two dimensional process $\frac{\partial}{\partial v}X(u,v)$ and then it is well defined as a generalized process over $R^2$. (Refer [10] for detail)

In Part II, P. Lévy discussed equations of first order functional derivatives. It is started with the variation of $U(x)$, a functional of an $L^2[0,1]-$function $x$ :

$$\delta U = \int_0^1 f[x(t);U,\tau]\delta x(\tau)d\tau \tag{2.1}$$

The kernel $f$ is denoted by $U'_x$.

If

$$\int f\left[x(t);U,\tau\right]^2 d\tau < \infty,$$

and Lipshitz's condition holds in $U$ then there exists a solution.

The variation formula (2.1) leads us to think of *Lévy's infinitesimal equation* for a stochastic process, where $x$ is viewed as a *white noise*.

We can see his idea in his well known stochastic **infinitesimal equation**

$$\delta X(t) = \varphi(X(s), s \le t, Y(t), t, dt)$$

where $\{Y(t)\}$ is the **innovation** of $X(t)$. Note that each $Y(t)$ is associated with infinitesimal interval $[t, t+\delta t)$ and that $Y(t)$ contains the same information as that gained by $X(t)$ during the time interval $(t, t+\delta t)$.

This equation was generalized by T. Hida to the stochastic variational equation for random field $X(C)$ as

$$\delta X(C) = \varphi(X(C'), C' < C, Y(s), s \in C, \delta C),$$

where $C' < C$ means that $C'$ is inside of $C$. That is the domain $(C')$ enclosed by a contour $C'$ is a subset of the domain $(C)$ and the system $Y = \{Y(s), s \in C; C \in \mathbf{C}\}$ is the *innovation* of $X(C)$.

Integrability condition for (2.1) is formally given by

$$\delta\delta_1 U = \delta_1\delta U \tag{2.2}$$

which is analogous to the finite dimensional case.

This condition can be considered also for the stochastic variational equation for a random field $X(C); C \in \mathbf{C}$ where $X(C) = X(C,x) \in (S)^*$, in which $(S)^*$

is the space of generalized white noise functionals and

$$\mathbf{C} = \{C; \text{smooth, convex, homeomorphic to } \mathrm{S}^1\}.$$

The $S-$transform $U(C,\xi)$ of $X(C)$ can be given by $U(C,\xi)$, where $\xi$ runs through the Schwartz space $S(R^2)$. We can rephrase $C$ by $C^2-$ function $\eta$. Then, $U(C,\xi)$ can also be expressed as $U(\eta,\xi)$. In what follows we omit $\xi$ since it is arbitrary and fixed.

Let us consider a particular stochastic variational equation such that the $U-$functional $U(\eta)$ of which can be expressed in the Volterra form

$$\delta U(\eta) = \int_C f(\eta, U, s)\delta\eta(s)ds. \tag{2.3}$$

Assume that

1. $f(\eta, U, s)$ is continuous in three variables satisfying the Lipshitz condition in $U$
2. it has Fréchet derivative in $\eta$ in the $C^2(S^1)$-topology and partial derivative in $U$.

Then we are given the following theorem. (refer [11])

**Theorem.** *The integrability condition for the variational equation* (2.3) *is given by*

$$\frac{\delta f(\eta,U,s)}{\delta\eta(t)} + \frac{\partial f(\eta,U,s)}{\partial U} f(\eta,U,t) = \frac{\delta f(\eta,U,t)}{\delta\eta(s)} + \frac{\partial f(\eta,U,t)}{\partial U} f(\eta,U,s). \tag{2.4}$$

Part II of Lévy's 1951 book is centered on the Hadamard equation. Let $g(C,u,v); u,v \in (C), C \in \mathbf{C}$ be Green's function. For fixed $u$ and $v$, $g$ is a functional of a contour $C$. The variation of $g$ is expressed in the form

$$\delta g(C,u,v) = \frac{1}{2\pi}\int_C \frac{\partial g(C,u,s)}{\partial n_s}\frac{\partial g(C,s,v)}{\partial n_s}\delta n(s)ds,$$

for a slight deformation of $C$ outward and within $\mathbf{C}$.

This equation is first shown by Hadamard (1903) for a Green's function $g$. Hence the equation of the above form is to be integrable.

There is a probabilistic application given by the following formula. Let $x$ be a white noise. Set

$$X(C,u,x) = \int_{(C)} g(C,u,v)x(v)dv.$$

Then

$$\delta X(C,u,x) = \int_{(C)} \delta g(C,u,v)x(v)dv.$$

while

$$\Delta_u X(C,u,x) = x(u),$$

namely the original white noise is obtained.

P. Lévy discussed a general variational equation of the form

$$\delta\Phi(A,B) = \int_C f(\Phi(A,B),\Phi(A,M),\Phi(M,B))\delta n(s)ds. \tag{2.5}$$

It is shown that the Hadamard equation can be transformed to the equation of the above form.

The integrability condition for (2.5) is guaranteed only for the following particular cases :

1. $f = f(\Phi(A,B))$
2. $f = \Phi(A,M)\Phi(M,A)$.

**Example** Consider the conditional expectation

$$E[X(P)|X(A(\theta)), A(\theta) \in C_r] = \int_{C_r} f_{C_r}(r,\theta)X(\theta)d\theta \tag{2.6}$$

where $C_r$ is a circle with radius $r$.

We are particularly interested in the variation where $C$ is extending only in one direction.

We denote the conditional expectation (2.6) by $Y(C_r)$ since it is a random functional of the circle $C_r$ with radius $r$. The parameter $\theta$ of the Lévy Brownian motion $X(A(\theta)) = X(\theta)$ is taken as the inclination of the line segment between the origin and the point $A(\theta)$.

Then it is obtained as

$$Y(C_r) = \int_{C_r} f_{C_r}(t,\theta)X(\theta)d\theta \tag{2.7}$$

where

$$\begin{aligned} f_{C_r}(r,\theta) &= \frac{(l^2-2rl\cos\alpha)^2}{8r\rho(x(r),r,\theta)} \\ &\quad +\frac{1}{2\pi}\left(1-\frac{r+x(r)}{2t}E(\frac{\pi}{2},k(r))\right) \end{aligned} \tag{2.8}$$

in which

$l$ is the distance between the origin and a given point $P$,
$x(r)$ is the distance from the point $P$ to the center of $C_r$,
$\alpha = \angle OMP, M$ being the center of the circle $C_r$,
$\rho(a,b,\theta) = \sqrt{a^2+b^2-2ab\cos\theta}$,
$k(r) = \frac{2\sqrt{r\rho(l,t,\alpha)}}{r+x(r)}$ and
$E$ is the Elliptic function.

Let us take a variation of $Y(C)$ by letting vary the radius within the class **C** consisting the circles passing through the origin. Then its variation is expressed as

$$\delta Y(C_r) = \int_{C_r} \frac{\partial}{\partial r} f_{C_r}(r,\theta)X(\theta)d\theta + \int_{C_r} f_{C_r}(t,\theta)\delta_\theta X(\theta)d\theta \tag{2.9}$$

where $\delta_\theta X(\theta) = X'(\theta) - X(\theta)$ in which $X'(\theta)$ is a Brownian motion at the point $A(\theta)$ on the circle $C_{r+\delta r}$. We note that $\frac{\partial}{\partial t} f(t,\theta)$ includes the derivative of elliptic function which can be computed and is as follows.

$$\frac{\partial}{\partial t} E\left(\frac{\pi}{2}, k(r)\right) = \frac{k'(r)}{k(r)}\left\{E\left(\frac{\pi}{2}, k(r)\right) - F\left(\frac{\pi}{2}, k(r)\right)\right\} \tag{2.10}$$

where $F$ is the hypergeometric function.

**Remark** The equation (2.6) can be compared with the expression of a Harmonic function $Y$ such that

$$Y(P) = \frac{1}{2\pi}\int_C \frac{\partial G}{\partial n}(s)X(s)ds$$

where $G$ is a Green function, and $s$ is a parameter runs through on a boundary of $C$.

## 3 Volterra's work on variational calculus

As is well known, Volterra's work has been impressively active in several fields of Mathematics, theory of functions, integral equations, mathematical physics and analysis. He laid foundations on theory of functions, integral equations and reconstructed the chapters of mathematical physics and analysis. He was also one of the first person who was able to apply mathematics methods to biological problems.

Among his plenty work we wish to mention the following literatures, concerned with biology, which are still famous and being applied. Many mathematicians are interested in solving the equations, equation of Logistic curve and Lotka-Volterra equation, derived by him.

Here we wish to mention the topics in his following literatues;

1. Calculus of variations and the logistic curve, Human Biology Vol. II (1939), 173-178.
2. Principles de biologic mathematiques, Acta Biomathematica Vol III (1937).

We can see the significance of his approach in variational calculus :

1. The variable of the Lagrange function is taken to be the integral

$$X(t) = \int_0^t N(s)ds \qquad : \text{ quantity of life}$$

   of the poopulation size $N$ according to his biological thought.

2. Total vital action (see [8]) is formulated according to his axioms and has derived the Lotka-Volterra equation, although we do not give a good interpretation yet.

3. Application of Lotka-Volterra equation to thermodynamics, by Prigogine.

**Lotka-Volterra equation**

$$\frac{dN_1}{dt} = (b - aN_2)N_1$$
$$\frac{dN_2}{dt} = (-\beta + \alpha N_1)N_2$$

We see that the Lotka-Volterra equation is more general than the Logistic equation .

**Logistic curve**

Volterra gave the derivation of the logistic curve by using the method of variation in 1939. This problem is now rephrased to the variational problem by using entropy (refer Win Win Htay). The variable $X(t)$ in question is taken as the quantity of life which is the integral of population with respect to time.

The Lagrangian function $F$ is taken as

$$\begin{aligned} F(t) &= F(X(t), \dot{X}(t)) \\ &= m_1 \dot{X} \log \dot{X} + m_2(\varepsilon - \lambda\dot{X}) \log(\varepsilon - \lambda\dot{X}) + KX \end{aligned} \tag{3.1}$$

and define

$$P(X) = \int_0^T F(t)dt. \tag{3.2}$$

Volterra called $P(X)$ **total vital action** over the time interval $[0, T]$. It corresponds to the action in classical mechanics.

Since $X$ is a total population, it is always positive. However, it can be varied within the admitted region so we are able to apply the Euler equation

$$F_X - \frac{d}{dt} F_{\dot{X}} = 0.$$

Thus we have

$$K - \frac{d}{dt}\{m_1 \log \dot{X} + m_1 - m_2\lambda \log(\varepsilon - \lambda\dot{X}) - \lambda m_2\} = 0.$$

Let $m_2\lambda = m_1$, $K = m_1\varepsilon$, then we obtain

$$\frac{d}{dt} N = N(\varepsilon - \lambda N), \tag{3.3}$$

since $\dot{X}(t)$ is $N(t)$.

Then the Lagrangian function is obtained as

$$F(t) = \frac{1}{\lambda}(m_1\lambda N \log \lambda N + m_2\lambda(\varepsilon - N) \log(\varepsilon - \lambda N)) - m_1 N \log \lambda + KX.$$

Substitute $m_1 = m_2\lambda$ and $K = m_1\varepsilon$, then

$$F(t) = m_2\varepsilon[-H(p, q) + \log \varepsilon + \lambda X - p \log \lambda],$$

where $p = p(\dot{X}) = \frac{\lambda\dot{X}}{\varepsilon}$ and $q = q(\dot{X}) = 1 - p(\dot{X})$.

It is written in the form

$$F(t) = m_2 \widetilde{F}(X, \dot{X})) \tag{3.4}$$

where

$$\widetilde{F}(X, \dot{X}) = -H(p, q) + (\log \varepsilon + \lambda X - p \log \lambda) \tag{3.5}$$

**Remark.** We can see that the first term of $\widetilde{F}$ is the entropy.

**The case to which Euler equation can not be applied**

In the above discussion, Euler equation can be applied although the variable $X$ is positive since we take the variation within an admitted region. However, ther are some cases that Euler equation can not be applied. For such a case we need some othyer technique. This can be illustrated below.

Let $p(x), x \in R^d$, be the $d-$dimensional probability distribution then its entropy is given by

$$H(p) = -\int_{R^d} p(x) \log p(x) dx^d.$$

We wish to find the maximum entropy under the restriction on the correlation coefficient. Since $H(p)$ is $R^d-$translation invariant, the mean vector can be taken to be zero. We assume that the second order moment exists and the covariance matrix $V = (v_{ij})$ is defined.

Set $\mathbf{P}$ be the set of probability densities $p(x)$ such that

$$\begin{aligned} \int x_i p(x) dx^d &= 0;\ i = 1, 2, \cdots d, \\ \int x_i x_j p(x) dx^d &= v_{ij}, \end{aligned}$$

where $V = [v_{ij}]$ is the given covariance matrix.

**Theorem.** Gaussian distribution attains the maximum of $H(p),\ p \in \mathbf{P}$.

Let us note the following facts.

Let $p(x)$ and $q(x)$ be probability density functions of two arbitrary probability distributions. Then

$$H(p) \leq -\int p(x) \log q(x) dx.$$

According to Jensen's inequality, where we understand if $p > 0,\ p \log 0 = -\infty$.

Let the probability density function $q(x)$ be a Gaussian distribution with mean vector zero and the covariance matrix $V$ :

$$q(x) = (2\pi)^{\frac{-d}{2}} |V|^{\frac{1}{2}} exp[-\frac{1}{2}(V^{-1}x, x)].$$

Then the entropy is

$$H(q) = \frac{d}{2}(\log 2\pi + 1) + \frac{1}{2} \log |V|.$$

**Proof of the theorem**

Let $p(x)$ satisfy the condition of the theorem and $V^{-1} = (\gamma_{ij})$. According to Fact 1,

$$H(p) \leq \int p(x) \left( \frac{d}{2} \log 2\pi + \frac{1}{2} \log |V| - \frac{1}{2} \sum \gamma_{ij} x_i x_j \right) dx^d.$$

Since the right hand side is equal to $H(q)$, the assertion is proved.

Our Future problem is to think of randomizing the variables $N_1$ and $N_2$ in Lotka-Volterra equation. That is, $N_1$ and $N_2$ are going to be taken as sample functions of a pair of stochastic processes $(N_1, N_2)$.


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