確率関数の変分解析の歴史

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概 要

We are interested in the variation of random fields. As is known, the characteristic functional plays an important role in the theory of stochastic processes and in the theory of random fields. Thus we study the variational calculus for characteristic functionals of stochastic process arising from hydrodynamics in historical approach.

1 Introduction

Our recent interest is the variation calculus of random fields and now we would like to revisit some of the original ideas of pioneers. This report will focus our attention to the variaional calculus for characteristic functionals of stochastic process arising from hydrodynamics.

The characteristic functions and functionals play an important role in the theory of stochastic processes and in the study of random fields, respectively.

On the other hand the variational equations of the characteristic functions and functionals are widely discussed in the field of hydrodynamics. We can find them in many literatures. Here we would like to discuss only on some important literatures among them.

1937 E. Hopf, Ergodentheorie. Ergebnisse der Mathmatik und ihrer Grenzebiete, Springer-Verlag.

1951 P. Lévy, Problèmes concrets d'analyse functionelle, Gauthier Villars, Paris .

1952 E. Hopf, Statistical hydromechanics and functionals calculus. J. Rat Mech. Anal. 1, 87-123. (Also, see Hamilton-Jacobi equation. 1964.)

1962 M.D. Donsker and J. L. Lions, Fréchet- Volterera variational equations, boundary value problems, and function space integrals, Acta Math. 108 (1962), 147-228.

1965 A.S. Momin and A.M. Yaglom, Statistical fluid mechanics: Mechanics of turbulence. vol. 2 English translation 1975. MIT Press.

2 Hydrodynamics

Hydrodynamics concerns with the equillibrium and motion where one deal with gas and liquid without considering their molecular structure. As is known there are two metnods of describing the motion of a fluid, Lagrange's method and Euler's method.

Lagrange's method

Consider fluid as a system of infinite number of particles, and the motion of each particle is discussed as a function of time. (ref. Encyclopedic Dictionary of Mathematics)

For instance, if a fluid particle is at the point with the coordinate $(x_1, x_2, x_3) = (a, b, c)$ at the moment $t = 0$ then its position at an arbitrary time t is expressed by the coordinate (x_1, x_2, x_3), with $x_i = f_i(a, b, c, t)$.

Euler's method

This method is to discuss the values of the velocity $v(u_1, u_2, u_3)$, the density ρ, the pressure p, etc., of the fluid at arbitrary times and positions. From this stand point, each quantity of the fluid is regarded as a function of space-time point (x_1, x_2, x_3, t).

Analysis of Dynamical system

I. Partial differential equations

Example. Navier-Strokes equation

If we take the vicosity into account, the equation of motion of an incompressible fluid (a fluid of constant density) becomes

$$\rho\frac{\partial v}{\partial t} = \rho\mathbf{K} - \text{grad } p + \mu\Delta v.$$

where ρ is density and μ is viscosity of a flow (Reynold's numbers).

II. Analysis of random fileds

Weintroduce the probability measure to the set of trajectories and analyse the characteristic functionals (i.e. to analyse random fields).

Example

Consider the velocity of a particle moving in the liquid. Then its velocity satisfies the Langevin equation,

$$\frac{dX(t)}{dt} = -\lambda X(t) + a\dot{B}(t), \quad \lambda > 0.$$

3 E. Hopf's theory

Consider an incompressible fluid of constant density $\rho = 1$ and viscosity of μ which moves between given material walls. The boundary condition is that the velocity vector

$$u = (u_1, u_2, u_3)$$

of the fluid coincides on a wall with the velocity of that wall.

Let R denote the region of x space, $x = (x_1, x_2, x_3)$ which is occupied by the fluid at all times and B be its boundary. Consider any sufficiently smooth velocity field $u = u(x)$ defined in $R + B$. It can be thought as a possible instantaneous state or phrase of the fluid provided that

$$\text{div } v = 0. \tag{3.1}$$

The basic equations of flow are the Navier-Strokes equations

$$u_{\alpha,t} + u_\beta u_{\alpha,\beta} = -p_\alpha + \mu u_{\alpha,\beta\beta} \quad \alpha = 1, 2, 3. \tag{3.2}$$

with u satisfying the equation(3.1).

The notations used in the above euquation are defined as follows.

$$\begin{aligned}
u_{\alpha,\beta} &= \frac{\partial u_\alpha}{\partial x_\beta} \\
u_{\alpha,\beta\gamma} &= \frac{\partial u_\alpha}{\partial x_\beta \partial x_\gamma} \\
u_{\alpha,t} &= \frac{\partial u_\alpha}{\partial t} \\
u_t &= \frac{\partial u(x,t)}{\partial t} = (u_{1,t}, u_{2,t}, u_{3,t}) : \\
&\quad \text{rate of change of phase } u = u(x,t).
\end{aligned}$$

Let Ω be the u-space or phase space asociated with the flow problem.That is

$$\Omega = \{u = u(x) | u = u(x) \in R + B\}.$$

Let the transform T^t be

$$T^t : u \to u_t; u_t = T^t u,$$

where $u^t = u(x,t), u = u(x) = u(x,0)$.

For fixed u and variable $t \geq 0$, T^t describes the flow of the fluid in R starting from the initial velocity field $u = u(x)$.

Under the assumption that the boundary data of the flow problem be stationary,

$$T^tT^s = T^{t+s}, \quad T^0 = \text{identity}.$$

Characteristic functionals

The characteristic functional

$$\Phi(y,t) = \int_\Omega \exp(i(y,u))p^t(du)$$

is a functional of arbitrary vector field $y = y(x)$ in $R + B$ and is a function of t (the characteristic functional of a phase distribution p^t). Thus

$$\Phi(y,t) = \int_\Omega \exp(i(y,T^tu))p(du).$$

If $y = (y_1, y_2, \cdots, y_n), u = (u_1, u_2, \cdots, u_n)$, the variation of Φ is

$$\delta\Phi = \sum_{i=1}^{n} \frac{\partial\Phi}{\partial y_\nu}\delta y_\nu.$$

The continuous analogue of the above variation is

$$\delta\Phi = \int_0^1 \frac{\partial\Phi}{\partial y(x)dx}\delta y(x)dx.$$

Consider the characteristic functional $\Phi(y)$ of a vector field $y = (y_1(x), y_2(x), y_3(x))$ defined in some region R of the three dimensional x-space.

$$\delta\Phi = \int_R A_\alpha(x)\delta y_\alpha(x)dx.$$

where $dx = dx_1dx_2dx_3$ and A_α is called the functional derivative of Φ and denote it by

$$A_\alpha(x) = \frac{\partial\Phi}{\partial y_\alpha(x)dx}.$$

We consider variations of functionals.

Example 1. Let a functional be given

$$\Phi_1 = \int_R \int_R K_{\alpha\beta}(x, x') y_\alpha(x) y_\beta(x') dx dx',$$

with $\quad K_{\alpha\beta}(x, x') = K_{\beta\alpha}(x', x).$

$$\frac{\partial \Phi_1}{\partial y_\alpha(x) dx} = \int_R K_{\alpha\beta}(x, x') y_\beta(x') dx',$$

and

$$\frac{\partial^2 \Phi_1}{\partial y_\beta(x') \partial y_\alpha(x) dx} = 2K_{\alpha\beta}(x, x').$$

Example 2. $\Phi_2 = \int_R K_{\alpha\beta}(x) y_\alpha(x) y_\beta(x) dx$, with $K_{\alpha\beta}(x) = K_{\beta\alpha}(x)$, then

$$\frac{\partial \Phi_2}{\partial y_\alpha(x') dx'} = 2K_{\alpha\beta}(x, x') y_\beta(x').$$

Remark The second order derivative of the above functional Φ_2 does not exist. However, if

$$\Phi = \int K(x) y(x)^2 dx$$

and if we consider y as a sample fuction of white noise in the domain of the Lévy Laplacian,

$$\Delta_L \Phi = 2 \int K(x) dx,$$

where Δ_L is the Lévy Laplacian.

The functional differential equation for Φ (Hopf's equation)

The characteristic functional of any phase distribution which moves in accordance with the laws of flow in the x-region R satisfis the equation

$$\frac{\partial \Phi}{\partial t}=\int_R y_\alpha(x)\left[i\frac{\partial}{\partial x_\beta}\frac{\partial}{\partial^2\Phi}\partial y_\beta(x)dx\partial y_\alpha(x)dx+\mu\Delta_x\frac{\partial\Phi}{\partial y_\alpha(x)dx}-\frac{\partial\Pi}{\partial x_\alpha}\right]dx,$$

where

$$\Delta_x=\frac{\partial^2}{\partial x_\beta\partial x_\beta},$$

for every vector field $y=y(x)$ in $R+B$ and for every value of $t\geq 0$.

For a flow problem (entirely finite region R and sufficiently smooth boundary data),

$$\Phi=\sum_i \Phi^i,\quad \Phi^0=1,$$

where Φ^n is a homogeneous polynomial of degree n in $y=y(x)$,

$$\Phi^n=\int_R\cdots\int_R K_{\alpha\cdots\omega}(x_1,\cdots,x_n,t)y_\alpha(x_1)\cdots y_\omega(x_n)dx_1\cdots dx_n.$$

and where

$$K_{\alpha\cdots\omega}(x_1,\cdots,x_n)=\frac{1}{n!}\left[\frac{\partial^n\Phi}{\partial y_\alpha(x_1)\cdots\partial y_\omega(x_n)dx^n}\right]_{y=o}.$$

In the case of white noise analysis the above expression can be interpretted as the Wiener expansion (Fock space expansion), where Φ^n is the homogeneous chaos of degree n. Here we will make a quick review.

Let (S^*,μ) be the measure space of white noise, where μ is a probability measure determined by the characteristic functional

$$C(\xi)=\exp[-\frac{1}{2}\|\xi\|^2],\quad \xi\in E. \tag{3.3}$$

Let (L^2) denote the Hilbert space $L^2(E^*,\mu)$ which is the collection of a complex-valued functionals with finite variance. The functional

$C(\xi - \eta), (\xi, \eta) \in S \times S$, where $C(\xi)$ is given by (.) is positive definite. Thus it defines a reproducing kernel Hilbert space F.

The reproducing kernel Hilbert space F with kernel $C(\cdot)$ is isomorphic under the $S-$transform to the Hilbert space (L^2).

A direct decomposition of F is :

$$F = \sum_{n=0}^{\infty} \oplus F_n. \quad \text{(Fockspace)}$$

Then we have $\mathcal{H}_n = S^{-1}F_n$, the multiple Wiener integral of degree n and the decomposition

$$(L^2) = \sum_{n=0}^{\infty} \mathcal{H}_n.$$

For $\varphi_n \in \mathcal{H}_n$, the S-transform of which is obtained as

$$(S\varphi_n)(\xi) = \exp\langle x, \xi \rangle \int \cdots \int F_n(u_1, \cdots, u_n)\xi(u_1) \cdots \xi(u_n) d\mathbf{u},$$

where **du** denotes $du_1 \cdots du_n$.

Note : For $\varphi \in (L^2)$, the S-transform is defined as

$$(S\varphi) = \int \varphi(x) : e^{\langle x, \xi \rangle} : d\mu(x),$$

where ξ is a test function.

By taking the inverse transform of S-transform, we obtain

$$\varphi(x) = \int \cdots \int F(u_1, \cdots, u_n) : x(u_1) \cdots x(u_n) : d\mathbf{u}.$$

Thus, for a general $\varphi \in (L^2)$,

$$\varphi = \sum_{n=0}^{\infty} \varphi_n, \quad \varphi_n \in \mathcal{H}_n.$$

The developements of the Hopf theory

As the developements of the Hopf theory, there are some interesting results in the following literatures.

A.S. Monin and A. M. Yaglom , Statistical Fluid Mechanics (English translation 1975).
O. A. Ladyzhenskaya and A.M. Vershik , Ann. della Scuola Norm. Sup. di Pisa(1977).
M.I. Visik and A.V. Foursikov, J. Math pures et appl (1977)
A. Inoue, On weak, strong and classical solutions of the Hopf equation (preprint).

Monin and Yaglom

Yaglon and Monin follows Hopf theory and consider the equations for the spatial characteristic functional of the velocity field. And then the spatial description of the velocity field is transformed to the spectral representation of this field and the spectral form of the Hopf equation is derived. To solve the equation they used the same method as Hopf.

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4 Fréchet Volterra Variation equations, Boundary value problems, and Function space integrals

Consider the function space integral (that is Wiener integral).

Denote that $C[0,t)$ be the space of continuous funcions,
$\{z(\sigma)|0 \leq \sigma \leq t, z(0) = 0\}$ and
$E_z^\omega[F(z)]$ be the Wiener integral of a functional $F(z)$ defined on $C(0,t)$.

Consider

$$\lim_{\epsilon \to 0} \epsilon^{-1} E_z^\omega[F(z); x < z(t) < x + \epsilon]$$

and denote it by $E_z^\omega[F(z)(\delta(z(t) - x)]$.

Let

$$u(x,t;q) \equiv E_z^{\omega}\left[\exp\left(i\int_0^t z(\sigma)q(\sigma)d\sigma - \frac{1}{2}\int_0^t z^2(\sigma)d(\sigma)\right)\delta(z(t)-x)\right], \tag{4.1}$$

where $q(\sigma)$ is continuous on $[0,t]$. The explicit solution can be easily obtaned from the actual calculation by using probabilistic technique.

It is shown by Donsker and Lions (1962) that (4.1) is the unique solution of the variational equation

$$\frac{\delta u}{\delta q(\tau)} = \left(-\int_0^t \min(\tau,s)q(s)ds\right)u - \int_0^t \min(\tau,s)\frac{\delta u}{\delta q(s)}ds - i\tau\frac{\partial u}{\partial x}, \tag{4.2}$$

where $\quad 0 < \tau < t.$

Thus we can see that the Wiener integral, in other words white noise integral gives the solution of F-V variational equation.

5 References for the history of variational calculus related to functional integration

As for the references for the history of variational calculus related to functional integration, we list some literatures below.

1948 R. P. Feynman, Soace time approach to non-relativstic quantum mechanics, Rev. Mod. Phys., 20, No.2,367-387.

1951 M. Kac, On some connections between probability theory and differential and integral equations, Proc. Second Berkeley Symposium. Math. Stat. and Probab., Univ. of California press, 189-215.

1959 M. Kac, Probability and Relatedf topics in Physical Sciences, Interscience Publ., London and New York.

1960 I. M. Gel'fand and A. M. Yaglom, Integration in functional spaces and in application to quantum physics, J. Math. Phys., 1, No. 1, 48-69.

1961 I. M. Gel'fand and and N. Ya. Vilenkin , Generalized stochastic process with independent solution in terms of Wiener integral to a variation equation, Generalized Functions: Vol. 4, Academic Press, New York-London.

1962 M. D. Donsker and J. L. Lions, Fréchet Volterra variational equations , boundary value problems, and Function space integrals.

1965 Monin and Yaglom, Statistical fluid mechanics(english translation).

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[4] P. Lévy, Problèmes concrets d'analyse functionelle, Gauthier Villars,1951 Paris .

[5] A.S. Momin and A.M. Yaglom, Statistical fluid mechanics: Mechanics of turbulence. vol. 2 , 1965, English translation 1975. MIT Press.