ARGUMENTS AMONG THE *GÉOMÈTRES* ON THE ORIGINAL NAVIER'S EQUATIONS IN THE PRIME OF THE SECOND MOLECULAR PERIOD

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ABSTRACT. The original Navier-Stokes equations or Navier equations were introduced in the prime of *the second period of molecules.* The heated arguments to build up the formulations of the various material based on the then current topics on actions of molecules between Navier and Poisson or between Navier and Arago or others were published in several times in various jounals.

From the viewpoint of Navier-Stokes equations, we would like to report on the questions discussed then among Poisson, Navier, Cauchy, Stokes and etc. This contents are not only on the mathematical history, but also on the modeling problem of the functional equations, especially on the original Navier-Stokes equations.

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1. Back bround

1.1. The Navier-Stokes equations as the productions introduced in the prime of 2nd period of molecules. We can name the period of 1687-1750 : between Newton[28] and Laplace[18] as "*the first period of molecules*". (This classification is due to C.Truesdell, whose editorial introduction to *Leonhardi Euleri Opera Omnia*).[1] Before Laplace, Newton's physics dominated in also fluid mechanics, whose "*Philosophiae naturalis principia mathematica*"[28] was published in 1687. The original Navier-Stokes equations or Navier equations are one of the productions introduced in the prime of this period.

The heated dispute among Poisson, Navier, Arago, Cournot over the then current topics on actions of molecules were published in several journals, above all, on *Annales de chimie et de physique* (ACP) in 1828-29, which was started by Navier[24] against Poisson[47]. A.Cournot[6] also critisized Navier, and Navier argued against Poisson and Arago in another journal (BSM). Cauchy [5] and Stokes [60] also proposed their tensors and equations in 1828 and in 1845 respectively, which are more likely to Poisson's rather than Navier's.

At last, Arago, the then co-editor of the journal, [2] stopped their discussions from the position of the editor, and therefore these problems were regretably remained as *the reserved problems* without the coincidence of each other after Navier and Poisson passed away in 1836 and 1840 respectively. Hence we call these problems, containing until Stokes, some *reserved problems* of the original Navier-Stokes equations. We show the directly related papers on the controversies over the molecular actions are from No.23 to 36 in Table 10. By the way, Poisson[53, 55] and Navier[21, 23] are the main papers or monographies on fluid.

1.2. The then situations of the academician depending on MAS & ACP. F.Graber[12, pp.146-148] analyzes the differences of the situation between MAS and ACP.

> (MAS) *Les Mémoires de l'Academie des sience de l'Institute de France* sont une publication officielle de l'Academie des sience. Ils accueillent uniquement des texts de membres, sélectionnés de surcroît par une commision. La nouvelle série, commencée sous la Restauration, se place dans la continuité de la précédente (qui correspond à la fondation de l'Institut en 1795) et des séries de *Mémoires de l'Academie des sience* du XVIII^e siècle, dont elle reprend l'exact principe. Elle comporte une volume par an et le délai de publication est en général de plusieurs années. Certes, cette publication reste au début du XIX^e siècle la plus prestigieuse dans le domaine des science, mais elle commence à être concurrencée par l'apparition de nombreuses revues scientifiques de qualité. ···
>
> (ACP) *Annales de chimie et de physique* sont par excellence une de ces productions jeunes et rapides (certains articles paraissent dans le mois même de leur communication), adoptées à la nouvelle circulation du savoir et aux impératifs de faire connaître le plus vite possible le lein entre le nom et la production du savant. Fondé en 1816, ce journal connaî dès ses premières années un grand succèes en France, mais aussi à l'étrangers. Il accueille un éventail très large de savants, dont un grand nombre d'étranges

[1] [61], Ser.2 Vol.12, p.81.

[2] This is titled "*Note du Rédacteur*", of which the autor seems to be Arago, is following the paper by Navier [26].

> et non-académiciens dont les textes n'ont pas été acceptés dans la publication de l'Academie réservée aux non-membres ; typiquement Fourier et sa théorie de la chaleur.

Graber cites also Navier's academic situation :

> *L'article* de Navier est publiè le même mois qu'a lieu la première lecture à l'Académie (mars 1822), *le mémoire* ne paraît que cinq ans plus tard (le fait que Navier soit élu académicien en 1824 est capital puisqu'il faut être membre pour être publié dans les *Mémoire*) $\cdots$

where Graber abridges *l'article* : [21] and *le mémoire* : [23].

2. Two parameters in elastic solid and fluid equations by Poisson, Navier, Cauchy and Stokes

We show the tables from Table 1 to Table 9, in which we show the equations, tensors and relations of coefficients in each other. We can summarize such as :

- The partial differential equations of the elastic solid or elastic fluid are expressed by using one or the pair of C_1 and C_2 such that : in the elastic solid :

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - (C_1 T_1 + C_2 T_2) = \mathbf{f}.$$

  In the elastic fluid :

$$\frac{\partial \mathbf{u}}{\partial t} - (C_1 T_1 + C_2 T_2) + \cdots = \mathbf{f},$$

  where $T_1,\ T_2, \cdots$ are the tensors or terms consisting our equations. For example, in modern notation of the incompressible Navier-Stokes equations, the kinetic equation and the equation of continuity are conventionally described as follows :

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{div } \mathbf{u} = 0. \tag{1}$$

- Moreover, C_1 and C_2 are described as follows :

$$\begin{cases} C_1 \equiv \mathcal{L} r_1 g_1 S_1, \\ C_2 \equiv \mathcal{L} r_2 g_2 S_2, \end{cases} \qquad \begin{cases} S_1 = \iint g_3 \rightarrow C_3, \\ S_2 = \iint g_4 \rightarrow C_4, \end{cases} \qquad \Rightarrow \qquad \begin{cases} C_1 = C_3 \mathcal{L} r_1 g_1 = \frac{2\pi}{15} \mathcal{L} r_1 g_1, \\ C_2 = C_4 \mathcal{L} r_2 g_2 = \frac{2\pi}{3} \mathcal{L} r_2 g_2. \end{cases}$$

- C_1 and C_2 are two coefficients, for example, k and K by Poisson, or ε and E by Navier, or R and G by Cauchy, and which are expressed by the infinite operator $\mathcal{L}$ ($\sum_0^\infty$ or $\int_0^\infty$) by personal principles or methods, where r_1 and r_2 are the functions related to the radius of the active sphere of the molecules, rised to the power of n, for Poisson's and Navier's case, the relation of function in expressing by logarithm to the base of r exists such that : $\log_r \frac{r_1}{r_2} = 2$.
- g_1 and g_2 are the certain functions which are dependent on r and are described with attraction &/or repulsion.
- S_1 and S_2 are the two expressions which describe the surface of active unit-sphere at the center of a molecule by the double integral (or single sum in case of Poisson's fluid).[3]
- g_3 and g_4 are certain compound trianglar-functions to compute the moment in the unit sphere.

[3] We use "$\rightarrow$" which means that it can not be described in using "$=$", because C_3 and C_4 are not always deduced directly, but as the common factor of the tensor.

TABLE 1. C_1, C_2, C_3, C_4 : the constant of definitions and computing of total moment of molecular actions by Poisson, Navier, Cauchy & Stokes

no	name	elastic solid	elastic fluid
1	Poisson	$C_1 = k \equiv \frac{2\pi}{15}\sum \frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = K \equiv \frac{2\pi}{3}\sum \frac{r^3}{\alpha^5} fr$ $C_3 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_3 \Rightarrow \{\frac{2\pi}{5}, \frac{2\pi}{15}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{2\pi} d\gamma \int_0^{\frac{\pi}{2}} \cos\beta \sin\beta d\beta\ g_4 \Rightarrow \frac{2\pi}{3}$ Remark: C_3 is choiced as the common factor of $\{\cdot, \cdot\}$	$C_1 = -k \equiv -\frac{1}{30\varepsilon^3}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}$ $= -\frac{2\pi}{15}\sum \frac{r^3}{4\pi\varepsilon^3}\frac{d.\frac{1}{r}fr}{dr}$ $C_2 = -K \equiv -\frac{1}{6\varepsilon^3}\sum rfr$ $= -\frac{2\pi}{3}\sum \frac{r}{4\pi\varepsilon^3} fr$ $C_3 : \begin{cases} G = \frac{1}{10}\sum r^3 \frac{d.\frac{1}{r}fr}{dr}, \\ E = F = \frac{1}{30}\sum r^3 \frac{d.\frac{1}{r}fr}{dr} \end{cases}$ $\Rightarrow \{\frac{1}{10}, \frac{1}{30}\} \Rightarrow \frac{1}{30}$ $C_4 : (3\text{–}2)_{Pf} \quad N = \frac{1}{6\varepsilon^3}\sum rfr \Rightarrow \frac{1}{6}$
2	Navier	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f\rho$ $C_3 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\psi \int_0^{2\pi} \cos\varphi d\varphi\ g_3 \Rightarrow \{\frac{16}{15}, \frac{4}{15}, \frac{2}{5}\}$ $\Rightarrow \frac{1}{2}\frac{\pi}{4}\frac{16}{15} = \frac{2\pi}{15}$	$C_1 = \varepsilon \equiv \frac{2\pi}{15}\int_0^\infty d\rho \cdot \rho^4 f(\rho)$ $C_2 = E \equiv \frac{2\pi}{3}\int_0^\infty d\rho \cdot \rho^2 F(\rho)$ $C_3 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_3$ $\Rightarrow \{\frac{\pi}{10}, \frac{\pi}{30}\} \Rightarrow \frac{2\pi}{15}$ $C_4 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\psi d\psi\ g_4 \Rightarrow \frac{2\pi}{3}$
3	Cauchy	$C_1 = R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r) dr$ $= \pm\frac{2\pi\Delta}{15}\int_0^\infty [r^4 \mathrm{f}'(r) - r^3 \mathrm{f}(r)] dr$ $C_2 = G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3 \mathrm{f}(r) dr$ $C_3 = \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2\alpha \cos^2\beta dp,$ $= \frac{1}{2}\int_0^{2\pi} \cos^2 q dq \int_0^\pi \cos^2 p \sin^2 p \sin p dp = \frac{2\pi}{15},$ $C_4 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi \cos^2\alpha \sin p dq dp$ $= \pi\int_0^\pi \cos^2 p \sin\ p dp = \frac{2\pi}{3},$	
4	Stokes		$C_1 = \mu, \quad C_2 = \frac{\mu}{3}$

- C_3 and C_4 are indirectly determined as the common coefficients from the invariant tensor. Except for Poisson's fluid case, C_3 of C_1 is $\frac{2\pi}{3}$, and C_4 of C_2 is $\frac{2\pi}{15}$, which are computed from the total moment of the active sphere of the molecules in computing only by integral, and which are independent on personal manner. In Poisson's case, after multiplying by $\frac{1}{4\pi}$, we get the same as above.
- The ratio of the two coefficients including Poisson's case is always same as : $\frac{C_3}{C_4} = \frac{1}{5}$.

3. The reserved problems between Navier and others including Poisson, Cauchy, etc. on the molecular actions

3.1. Poisson vs. Navier vs. Cauchy. Some reserved problems on the molecular actions in elastic solid/fluid are :

(1) [Priority :] Navier's anger as one of the then *géomètres*[4], from who Navier's works are underestimated.
(2) [C_1, C_2 :] Navier's ε and E vs. Poisson's k and K vs. Cauchy's G and R.
(3) [g_1, g_2 :]

[4]This means the mathematicians, which is used only in old French. G.Green uses this equivalent word in English such as follows : "This hypothesis, at first advanced by M.Cauchy, has since been adopted by several *philosophers*" , in his paper[13, p.305].

TABLE 2. Polar system in computing C_3 and C_4

no	name	polar system
1	Poisson	$x_1 = r\cos\beta\cos\gamma$, $y_1 = r\sin\beta\sin\gamma$, $\zeta = -r\cos\beta$
2	Navier	$\alpha = \rho\cos\psi\cos\varphi$, $\beta = \rho\cos\psi\sin\varphi$, $\gamma = \rho\sin\psi$
3	Cauchy	$\cos\alpha = \cos p$, $\cos\beta = \sin p\cos q$, $\cos\gamma = \sin p\sin q$
4	Stokes	

TABLE 3. The expression of the total moment of molecular actions by Poisson, Navier, Cauchy & Stokes

no	name	problem	C_1	C_2	C_3	C_4	$\mathcal{L}$	r_1	r_2	g_1	g_2	S_1, S_2, g_3, g_4	remark
1	Poisson [50]	elastic solid	k		$\frac{2\pi}{15}$		$\sum\frac{1}{\alpha^5}$	r^5		$\frac{d.\frac{1}{r}fr}{dr}$		cf.Table 4,5	
				K		$\frac{2\pi}{3}$	$\sum\frac{1}{\alpha^5}$		r^3		fr		
2	Poisson [55]	fluid	k		$\frac{1}{30}$		$\sum\frac{1}{\varepsilon^3}$	r^3		$\frac{d.\frac{1}{r}fr}{dr}$			$C_3 = \frac{1}{4\pi}\frac{2\pi}{15} = \frac{1}{30}$
				K		$\frac{1}{6}$	$\sum\frac{1}{\varepsilon^3}$		r		fr		$C_4 = \frac{1}{4\pi}\frac{2\pi}{3} = \frac{1}{6}$
3	Navier [22]	elastic solid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f\rho$			ρ : radius
4	Navier fluid [23]	fluid	ε		$\frac{2\pi}{15}$		$\int_0^\infty d\rho$	ρ^4		$f(\rho)$			ρ : radius
				E		$\frac{2\pi}{3}$	$\int_0^\infty d\rho$		ρ^2		$F(\rho)$		
5	Cauchy [5]	system	R		$\frac{2\pi}{15}$		$\int_0^\infty dr$	r^3		$f(r)$			$f(r) \equiv \pm[r\mathrm{f}'(r) - \mathrm{f}(r)]$
				G		$\frac{2\pi}{3}$	$\int_0^\infty dr$		r^3		$\mathrm{f}(r)$		$f(r) \neq \mathrm{f}(r)$
6	Stokes	fluid	μ	$\frac{\mu}{3}$									

- By which should we describe the fuction between two molecules, *attraction* &/or *repulsion* ?
- Navier's $f(\rho)$ and $F(\rho)$ vs. Poisson's fr vs. Cauchy's $\mathrm{f}(r)$ and $f(r)$.
- Navier's $e^{-k\rho}$ for an exponential function as the example of fr vs. Poisson's $ab^{(-\frac{r}{n\alpha})^m}$.

(4) [$\mathcal{L}$:] Navier's integral vs. Poisson's summation with mean value of the molecular intervals in the range from 0 to ∞.

(5) [Compressiblity :] Navier's *incompressible* vs. Poisson's *compressible.*

3.1.1. *Navier's anger as one of the then géomètres, from who Navier's works are underestimated.* Navier's anger maybe come from his pride as one of the *géomètres* (i.e. mathematicians) to Poisson[47], in which Navier's works ([21, 22, 23]) have not benn

TABLE 4. S_1, S_2, g_3, g_4 : the trianglar functions computing of total moment of molecular actions in unit sphere by Poisson, Navier, Cauchy & Stokes

<table>
<tr><th>no</th><th>name</th><th>S_1, S_2, g_3, g_4</th></tr>
<tr><td>1</td><td>Poisson</td><td>

$\cdot g_3$ and g_4 are in the following tensor :

$$\begin{cases} g = a\sin\beta\cos\gamma + b\sin\beta\sin\gamma - c\cos\beta, & g' = g\frac{du}{dx} + h\frac{du}{dy} + l\frac{du}{dz}, \\ h = a'\sin\beta\cos\gamma + b'\sin\beta\sin\gamma - c'\cos\beta, & h' = g\frac{dv}{dx} + h\frac{dv}{dy} + l\frac{dv}{dz}, \\ l = a''\sin\beta\cos\gamma + b''\sin\beta\sin\gamma - c''\cos\beta, & l' = g\frac{dw}{dx} + h\frac{dw}{dy} + l\frac{dw}{dz} \end{cases}$$

$$\begin{cases} P = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(g+g')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')g\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ Q = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(h+h')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')h\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \\ R = \int_0^{\frac{\pi}{2}}\int_0^{2\pi}\left[(l+l')\sum\frac{r^3}{\alpha^5}fr + (gg'+hh'+ll')l\sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}\right]\Delta, \end{cases}$$

i.e.

$$\Rightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\Delta\left(\begin{bmatrix} g+g' & (gg'+hh'+ll')g \\ h+h' & (gg'+hh'+ll')h \\ l+l' & (gg'+hh'+ll')l \end{bmatrix} \begin{bmatrix} \sum\frac{r^3 fr}{\alpha^5} \\ \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr} \end{bmatrix} \right)$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} d\beta\, d\gamma\left(\begin{bmatrix} g_4 & g_3 \end{bmatrix} \begin{bmatrix} K' \\ k' \end{bmatrix} \right),$$

where $\Delta := \cos\beta\cdot\sin\beta\; d\beta\; d\gamma$, $\quad K' := \sum\frac{r^3 fr}{\alpha^5}$, $\quad k' := \sum\frac{r^5}{\alpha^5}\frac{d.\frac{1}{r}fr}{dr}$.

$\cdot S_1$ and S_2 are given from above.

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<tr><td>2</td><td>Navier elastic solid</td><td>

$\cdot g_3$:

$$g_3 = \frac{1}{2}\delta f^2$$

$$f \equiv \rho\Big[\frac{dx}{da}\cos^2\psi\cos^2\varphi + \left(\frac{dx}{db}+\frac{dy}{da}\right)\cos^2\psi\sin\varphi\cos\varphi + \left(\frac{dx}{dc}+\frac{dz}{da}\right)\cos\psi\sin\psi\cos\varphi$$
$$+\frac{dy}{db}\cos^2\psi\sin^2\varphi + \left(\frac{dy}{dc}+\frac{dz}{db}\right)\sin\psi\cos\psi\sin\varphi + \frac{dz}{dc}\sin^2\psi\Big].$$

</td></tr>
<tr><td>3</td><td>Navier fluid</td><td>

$\cdot g_3$:

$$\alpha = \rho\cos\psi\cos\varphi, \quad \beta = \rho\cos\psi\sin\varphi, \quad \gamma = \rho\sin\psi$$

$$g_3 = V\delta V = \left[\alpha\left(\frac{du}{dx}\alpha + \frac{du}{dy}\beta + \frac{du}{dz}\gamma\right) + \beta\left(\frac{dv}{dx}\alpha + \frac{dv}{dy}\beta + \frac{dv}{dz}\gamma\right) + \gamma\left(\frac{dw}{dx}\alpha + \frac{dw}{dy}\beta + \frac{dw}{dz}\gamma\right)\right] \times$$
$$\left[\alpha\left(\frac{\delta du}{dx}\alpha + \frac{\delta du}{dy}\beta + \frac{\delta du}{dz}\gamma\right) + \beta\left(\frac{\delta dv}{dx}\alpha + \frac{\delta dv}{dy}\beta + \frac{\delta dv}{dz}\gamma\right) + \gamma\left(\frac{\delta dw}{dx}\alpha + \frac{\delta dw}{dy}\beta + \frac{\delta dw}{dz}\gamma\right)\right],$$

$\cdot g_4$:

$$\alpha' = \rho\cos\psi\cos\varphi = \rho\cos r\cos s, \quad \beta' = \rho\cos\psi\sin\varphi = \rho\cos r\sin s, \quad \gamma' = \rho\sin\psi = \rho\sin r.$$

$$g_4 = V\delta\, V = \begin{cases} \alpha'^2 & \{\ (u\sin^2 r - v\sin r\cos r)\delta u, \\ & \quad (-u\sin r\cos r + v\cos^2 r)\delta v\ \}, \\ \beta'^2 & \{\ (u\cos^2 r\sin^2 s + v\sin r\cos r\sin^2 s + w\cos r\sin s\cos s)\delta u, \\ & \quad (u\sin r\cos r\sin^2 s + v\sin^2 r\sin^2 s + w\sin r\sin s\cos s)\delta v, \\ & \quad (u\cos r\sin s\cos s + v\sin r\sin s\cos s + w\cos^2 s)\delta w\ \}, \\ \gamma'^2 & \{\ (u\cos^2 r\cos^2 s + v\sin r\cos r\cos^2 s - w\cos r\sin s\cos s)\delta u, \\ & \quad (u\sin r\cos r\cos^2 s + v\sin^2 r\cos^2 s - w\sin r\sin s\cos s)\delta v, \\ & \quad (-u\cos r\sin s\cos s - v\sin r\sin s\cos s + w\sin^2 s)\delta w\ \} \end{cases}$$

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mentioned at all. (cf. Navier[24, p.304-305], Graber[12, p.154]). Poisson however cites Navier in his later paper[50].

Navier says :

> Mon travail a été présenté à l'Académie des sciences en 1821. Je suis fort loin d'en vouloir exagérer l'importance ; mais enfin j'ai eu l'idée de sortir des hypothèses fort restreintes, quant à la figure et au mode de flexion des corps, dans lesquelles on s'était toujours renfermé ; et *j'ai donné le premier*, d'une

TABLE 5. S_1, S_2, g_3, g_4 : the trianglar functions computing of total moment of molecular actions in unit sphere by Poisson, Navier, Cauchy & Stokes (continued.)

no	name	S_1, S_2, g_3, g_4
4	Cauchy	$\cdot g_3 = g_4 = \frac{v}{2}:$ $(44)_C \begin{cases} G = G(\cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1) \equiv GA_1, \\ L = L(\cos^4\alpha_1+\cos^4\beta_1+\cos^4\gamma_1) \\ \quad +6R(\cos^2\beta_1\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_1) \equiv LB+6RC, \\ R = R(\cos^2\beta_1\cos^2\gamma_2+\cos^2\beta_2\cos^2\gamma_1+\cos^2\gamma_1\cos^2\alpha_2 \\ \quad +\cos^2\gamma_2\cos^2\alpha_1+\cos^2\alpha_1\cos^2\beta_2\cos^2\alpha_2\cos^2\beta_1) \\ \quad +4R(\cos\beta_1\cos\beta_1\cos\gamma_1\cos\gamma_2+\cos\gamma_1\cos\gamma_2\cos\alpha_1\cos\alpha_2 \\ \quad +\cos\alpha_1\cos\alpha_2\cos\beta_1\cos\beta_2) \\ \quad +L(\cos^2\alpha_1\cos^2\alpha_2+\cos^2\beta_1\cos^2\beta_2+\cos^2\gamma_1\cos^2\gamma_2) \equiv RD+4RE+LF \end{cases}$ where $\begin{cases} \cos^2\alpha_1+\cos^2\beta_1+\cos^2\gamma_1=1, \quad \cos^2\alpha_2+\cos^2\beta_2+\cos^2\gamma_2=1, \\ \cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2=0 \end{cases}$ · From $(49)_C$ $G = \frac{\Delta}{2}\mathbf{S}[\pm r\cos^2\alpha f(r)v]$, $R = \frac{\Delta}{2}\mathbf{S}[r\cos^2\alpha\cos^2\beta f(r)v]$ and $(50)_C$ $\begin{cases} G = \pm\frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\sin p\,dr\,dq\,dp, \\ R = \frac{\Delta}{2}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^3 f(r)\cos^2\alpha\cos^2\beta\sin p\,dr\,dq\,dp \end{cases}$ $\begin{cases} S_1 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\cos^2\beta\sin\ p\,dp = \frac{1}{2}\int_0^{2\pi}\cos^2 q\,dq\int_0^\pi \cos^2 p(1-\cos^2 p)\sin p\,dp \\ \quad = \frac{1}{2}\pi(\frac{2}{3}-\frac{2}{5}) = \frac{2\pi}{15} \equiv C_3, \\ S_2 = \frac{1}{2}\int_0^{2\pi}\int_0^\pi\cos^2\alpha\sin p\,dq\,dp = \frac{1}{2}2\pi\int_0^\pi\cos^2 p\sin\ p\,dp = \frac{2\pi}{3} \equiv C_4. \end{cases}$

manière exacte et rationnelle, les équations différentielles qui expriment les lois génerales auxquelles sont assujétis les déplacemens de points d'un corps solide élastique, soit dans l'état d'equilibre, soit lorsque ce corps est en vibration.

Cette matière a attiré dans ces derniers temps l'attention de plusieurs personnes. Elle a été traitée en France par MM.Cauchy et Poisson, en Russie par MM Lamé et Clapeyron. Ces derniers géomètres, sans connaître mon mémoire (puisqu'ils ne l'ont pas cité), ont adopté les mêmes principes, et sont parvenus aux mêmes résultats, dont ils ont fait d'ailleurs plusieurs applications impotrantes. M.Cauchy a considéré la question dans diverses hypothèses différentes les unes des autres, dont quelques-unes l'ont ramené aux équations que j'avais données. M.Poisson a égalment reproduit ces mêmes équations, en s'attachant à les déduire d'autres principes. : Navier [27, pp.243-244]

TABLE 6. $T_1 \& T_2$: tensors & equations by Poisson, Navier, Cauchy & Stokes in elastic solid

no	name	tensor & equations
1	Poisson elastic solid [50]	· Poisson's tensor : $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ · Poisson's equations : $\begin{cases} P = -K\left(c + \frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c''\right) \\ -k\left(3\frac{du}{dx}c + \frac{du}{dy}c' + \frac{du}{dz}c'' + \frac{dv}{dx}c' + \frac{dv}{dy}c + \frac{dw}{dx}c'' + \frac{dw}{dz}c\right), \\ Q = -K\left(c' + \frac{dv}{dx}c + \frac{dv}{dy}c' + \frac{dv}{dz}c''\right) \\ -k\left(\frac{dv}{dx}c + 3\frac{dv}{dy}c' + \frac{dv}{dz}c'' + \frac{du}{dx}c' + \frac{du}{dy}c + \frac{dw}{dy}c'' + \frac{dw}{dz}c'\right), \\ R = -K\left(c'' + \frac{dw}{dx}c + \frac{dw}{dy}c' + \frac{dw}{dz}c''\right) \\ -k\left(\frac{dw}{dx}c + \frac{dw}{dy}c' + 3\frac{dw}{dz}c'' + \frac{du}{dx}c'' + \frac{du}{dz}c + \frac{dv}{dy}c'' + \frac{dv}{dz}c'\right). \end{cases}$ If $K = 0 \Rightarrow$ $(6)_{Pe} \quad \begin{cases} X - \frac{d^2u}{dt^2} + a^2\left(\frac{d^2u}{dx^2} + \frac{2}{3}\frac{d^2v}{dydx} + \frac{2}{3}\frac{d^2w}{dzdx} + \frac{1}{3}\frac{d^2u}{dy^2} + \frac{1}{3}\frac{d^2u}{dz^2}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + a^2\left(\frac{d^2v}{dy^2} + \frac{2}{3}\frac{d^2u}{dxdy} + \frac{2}{3}\frac{d^2w}{dzdy} + \frac{1}{3}\frac{d^2v}{dx^2} + \frac{1}{3}\frac{d^2v}{dz^2}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + a^2\left(\frac{d^2w}{dz^2} + \frac{2}{3}\frac{d^2w}{dxdz} + \frac{2}{3}\frac{d^2w}{dydz} + \frac{1}{3}\frac{d^2w}{dx^2} + \frac{1}{3}\frac{d^2w}{dy^2}\right) = 0, \end{cases}$ where $a^2 = \frac{3k}{\rho}$, these are equal to Navier's elastic solid equations.
2	Navier elastic solid [22]	· Navier's tensor : $t_{ij} = -\varepsilon(\delta_{ij}u_{kk} + u_{ji} + u_{ij})$ · Navier's equations : $\begin{cases} X - \frac{d^2u}{dt^2} + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dydx} + 2\frac{d^2w}{dzdx}\right) = 0, \\ Y - \frac{d^2v}{dt^2} + \varepsilon\left(\frac{d^2u}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dzdy}\right) = 0, \\ Z - \frac{d^2w}{dt^2} + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2w}{dxdz} + 2\frac{d^2w}{dydz}\right) = 0 \end{cases}$ Same as $(6)_{Pe}$ which is deduced from Poisson when $K = 0$ and $a^2 = \frac{3k}{\rho}$.
3	Cauchy [5]	· Cauchy's tensor : $t_{ij} = \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$ · Cauchy's equations : $(68)_C \begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$ where $G = H = I, \quad L = M = N, \quad P = Q = R, \quad L = 3R.$ If $G = 0 \Rightarrow$ $(84)_C \quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2} + R\frac{\partial^2\xi}{\partial y^2} + Q\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = \frac{\partial^2\xi}{\partial t^2}, \\ R\frac{\partial^2\eta}{\partial x^2} + M\frac{\partial^2\eta}{\partial y^2} + P\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = \frac{\partial^2\eta}{\partial t^2}, \\ Q\frac{\partial^2\zeta}{\partial x^2} + P\frac{\partial^2\zeta}{\partial y^2} + N\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = \frac{\partial^2\zeta}{\partial t^2} \end{cases}$ In the end, it turns out only G and R of the 6 parameters.

3.1.2. *By which should we describe the fuction between two molecules, attraction &/or repulsion ?* Laplace[5] in 1819 : $\varphi(f) = A(f) - R(f)$, Poisson[6] : $R = Fr - fr$, where

[5] N.Bowditch[19, p.685] comments as follows :

> This theory of capillary attraction was first published by La Place in 1806 ; and in 1807 he gave a supplement. In neither of these works is the repulsive force of the heat of fluid taken into consideration, because he supposed it to be unnecessary. But in 1819 he observed, that this action could be taken into account, by supposing the force $\varphi(f)$ to represent the difference between the attractive force of the particles of the fluid $A(f)$, and the repulsive force of the heat $R(f)$ so that the conbined action would be expressed by, $\varphi(f) = A(f) - R(f)$; $\cdots$

[6] Poisson[53, p.73], [55, p.6]

TABLE 7. T_1&T_2 : the tensors & equations by Poisson, Navier, Cauchy & Stokes in fluid

no	name	tensor & equations
1	Poisson fluid [55]	· Poisson's tensor : $t_{ij} = -p\delta_{ij} + \lambda v_{k,k}\delta_{ij} + \mu(v_{i,j} + v_{j,i})$: $\begin{bmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{bmatrix} = \begin{bmatrix} \epsilon\left(\frac{du}{dz}+\frac{dw}{dx}\right) & \epsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\epsilon'}{\chi t}\frac{d\chi t}{dt}+2\epsilon\frac{du}{dx} \\ \epsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & p-\alpha\frac{d\psi t}{dt}-\frac{\epsilon'}{\chi t}\frac{d\chi t}{dt}+2\epsilon\frac{dv}{dy} & \epsilon\left(\frac{du}{dy}+\frac{dv}{dx}\right) \\ p-\alpha\frac{d\psi t}{dt}-\frac{\epsilon'}{\chi t}\frac{d\chi t}{dt}+2\epsilon\frac{dw}{dz} & \epsilon\left(\frac{dv}{dz}+\frac{dw}{dy}\right) & \epsilon\left(\frac{du}{dz}+\frac{dw}{dx}\right) \end{bmatrix}$ $(k+K)\alpha = \epsilon, \quad (k-K)\alpha = \epsilon', \quad p = \psi t = K, \quad \text{then} \quad \epsilon + \epsilon' = 2k\alpha.$ $\varpi \equiv -\alpha\frac{d\psi t}{dt} - \frac{\epsilon+\epsilon'}{\chi t}\frac{d\chi t}{dt}.$ · Poisson's equations in incompressible fluid : If ψt(= density), χt(=pressure) $\equiv$ const. and incompressible $\Rightarrow (9)_{Pf} \begin{cases} \rho(X - \frac{d^2x}{dt^2}) = \frac{d\varpi}{dx} + \epsilon(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}) = 0, \\ \rho(Y - \frac{d^2y}{dt^2}) = \frac{d\varpi}{dy} + \epsilon(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}) = 0, \\ \rho(Z - \frac{d^2z}{dt^2}) = \frac{d\varpi}{dz} + \epsilon(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}) = 0. \end{cases}$ i.e. $\frac{\partial \mathbf{u}}{\partial t} + \frac{\epsilon}{\rho}\Delta\mathbf{u} + \frac{1}{\rho}\nabla\varpi = \mathbf{f}$ · Coincidence with Stokes' equations : by $\varpi = p + \frac{\alpha}{3}(K+k)\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$ $\Rightarrow \nabla\varpi = \nabla p + \frac{\epsilon}{3}\nabla\cdot(\nabla\cdot\mathbf{u}), (9)_{Pf} \cong (12)_S$ of Stokes.
2	Navier incomp. fluid [23]	· Navier's tensor : $t_{ij} = -\varepsilon(\delta_{ij}u_{kk} + u_{ji} + u_{ij})$ · Navier's equations : $\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\left(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\right) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\left(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\right) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\left(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\right) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases}$ i.e. $\frac{\partial \mathbf{u}}{\partial t} - \varepsilon\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \mathbf{f},\ \text{div}\ \mathbf{u} = 0$ · Equations with ε and E : $0 = \iiint dxdydz \begin{cases} [P - \frac{dp}{dx} - \rho\left(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\right)]\delta u \\ [Q - \frac{dp}{dy} - \rho\left(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\right)]\delta v \\ [R - \frac{dp}{dz} - \rho\left(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\right)]\delta w \end{cases}$ $-\varepsilon \iiint dxdydz \begin{cases} 3\frac{du}{dx}\frac{\delta du}{dx} + \frac{du}{dy}\frac{\delta du}{dy} + \frac{du}{dz}\frac{\delta du}{dz} + \frac{dv}{dy}\frac{\delta du}{dx} + \frac{dv}{dx}\frac{\delta du}{dy} + \frac{dw}{dz}\frac{\delta du}{dx} + \frac{dw}{dx}\frac{\delta du}{dz} \\ \frac{du}{dx}\frac{\delta dv}{dy} + \frac{du}{dy}\frac{\delta dv}{dx} + \frac{dv}{dx}\frac{\delta dv}{dx} + 3\frac{dv}{dy}\frac{\delta dv}{dy} + \frac{dv}{dz}\frac{\delta dv}{dz} + \frac{dw}{dy}\frac{\delta dv}{dz} + \frac{dw}{dz}\frac{\delta dv}{dy} \\ \frac{du}{dx}\frac{\delta dw}{dz} + \frac{du}{dz}\frac{\delta dw}{dx} + \frac{dv}{dy}\frac{\delta dw}{dz} + \frac{dv}{dz}\frac{\delta dw}{dy} + \frac{dw}{dx}\frac{\delta dw}{dx} + \frac{dw}{dy}\frac{\delta dw}{dy} + 3\frac{dw}{dz}\frac{\delta dw}{dz} \end{cases}$ $+\mathbf{S}ds^2E(u\delta u + v\delta v + w\delta w).$ · Boundary condition with ε and E : $\begin{cases} Eu + \varepsilon[\cos l 2\frac{du}{dx} + \cos m\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos n\left(\frac{du}{dz} + \frac{dw}{dx}\right)] = 0, \\ Ev + \varepsilon[\cos l\left(\frac{du}{dy} + \frac{dv}{dx}\right) + \cos m 2\frac{dv}{dy} + \cos n\left(\frac{dv}{dz} + \frac{dw}{dy}\right)] = 0, \\ Ew + \varepsilon[\cos l\left(\frac{du}{dz} + \frac{dw}{dx}\right) + \cos m\left(\frac{dv}{dz} + \frac{dw}{dy}\right) + \cos n 2\frac{dw}{dz}] = 0, \end{cases}$

$\varphi(f)$ & R of the left hand side : a function depends on distance : f & r between two molecules, $A(f)$ & fr : attraction, $R(f)$ & Fr : repulsion. Navier introduces both $f(\rho)$ and $F(\rho)$ in the other meaning of a function on the culculation in partial moment and in total moment, in which Navier mentiones about the relations without showing the defference between the two molecular forces as above, and intensifies only repulsion, as follows :

> The force which brings up between these two molecules depend on the situation of the point M, must be balanced with the pression, which can

TABLE 8. $T_1\&T_2$: the tensors & equations by Poisson, Navier, Cauchy & Stokes in fluid (continued.)

no	name	tensor & equations
3	Stokes fluid [60]	· Stokes' tensor : $t_{ij} = \{p - 2\mu(v_{k,k} - \delta) + \gamma\}\delta_{ij} - \gamma$, $3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$, $\gamma = \mu(v_{i,j} + v_{j,i})$ i.e. $\begin{bmatrix} P_1 & T_3 & T_2 \\ T_3 & P_2 & T_1 \\ T_2 & T_1 & P_3 \end{bmatrix} = \begin{bmatrix} p - 2\mu\Big(\frac{du}{dx} - \delta\Big) & -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) \\ -\mu\Big(\frac{du}{dy} + \frac{dv}{dx}\Big) & p - 2\mu\Big(\frac{dv}{dy} - \delta\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) \\ -\mu\Big(\frac{dw}{dx} + \frac{du}{dz}\Big) & -\mu\Big(\frac{dv}{dz} + \frac{dw}{dy}\Big) & p - 2\mu\Big(\frac{dw}{dz} - \delta\Big) \end{bmatrix}$, *where* $3\delta = \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}$. · Stokes' equations with μ : $(12)_S \quad \begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \mu\Big(\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dx}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \mu\Big(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dy}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \mu\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + \frac{d^2w}{dz^2}\Big) - \frac{\mu}{3}\frac{d}{dz}\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big) = 0. \end{cases}$ i.e. $\rho\big(\frac{D\mathbf{u}}{Dt} - \mathbf{f}\big) + \nabla p - \mu\Big(\Delta\mathbf{u} + \frac{1}{3}\nabla(\nabla\cdot\mathbf{u})\Big) = 0$ i.e. $\begin{cases} \rho(\frac{Du}{Dt} - X) + \frac{dp}{dx} - \frac{\mu}{3}\Big(4\frac{d^2u}{dx^2} + 3\frac{d^2u}{dy^2} + 3\frac{d^2u}{dz^2} + \frac{d^2v}{dxdy} + \frac{d^2w}{dxdz}\Big) = 0, \\ \rho(\frac{Dv}{Dt} - Y) + \frac{dp}{dy} - \frac{\mu}{3}\Big(3\frac{d^2v}{dx^2} + 4\frac{d^2v}{dy^2} + 3\frac{d^2v}{dz^2} + \frac{d^2u}{dxdy} + \frac{d^2w}{dydz}\Big) = 0, \\ \rho(\frac{Dw}{Dt} - Z) + \frac{dp}{dz} - \frac{\mu}{3}\Big(3\frac{d^2w}{dx^2} + 3\frac{d^2w}{dy^2} + 4\frac{d^2w}{dz^2} + \frac{d^2u}{dxdz} + \frac{d^2w}{dydz}\Big) = 0, \end{cases}$

TABLE 9. The correlation of conditions in reducing to others

1	name	Poisson	Navier	Cauchy	Stokes
1	Poisson		if $K = 0, \varepsilon = \frac{a^3}{3} = \frac{k}{\rho} \Rightarrow \varepsilon \cong k$		
2	Navier	even if $K = 0 \Rightarrow \varepsilon \neq k$			
3	Cauchy	if $G = 0 \Rightarrow R \cong a^2$	if $G = 0 \Rightarrow R \cong \varepsilon$		
4	Stokes	if $\alpha(K + k) \equiv \beta \Rightarrow \mu \cong \epsilon$ i.e. $\varpi = p + \frac{\alpha}{3}(K + k)\Big(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\Big)$ $\Rightarrow \nabla\varpi = \nabla p + \frac{\epsilon}{3}\nabla\cdot(\nabla\cdot\mathbf{u}), (9)_{Pf} \cong (12)_S$.	$\neq$		

vary in the various particle of fluid. They depend on the distance ρ, and all the molecular actions, attenuate very rapidly when these distance increase. We call these force by the function $f(\rho)$. : Navier[23, p.392]

Navier poses the question about Poisson's $(r'-r)fr$ which is already appeared in Fourier[9, p.35]. [7] In relating to last paragraph, Poisson describe :

[7]We cite the paragraph on this point by D.H.Arnold, who is the leading researcher of Poisson :

By being somewhat casual in his selection of an example, Poisson succeeds in exposing himself to the sharp criticism of Navier. Still stinging from the abuse that his research on elasticity had suffered at the hand of Poisson, Navier is quick to point out that such an exponential function must be either always positive or always negative. Hence, he argues, the resultant force between molecules would have to be always atractive or always repulsive. He concludes that the "*nature de la fonction présentée par l'auteur semble*

> Cela posé, appelons m et m' les messes de deux molécules voisines, c et c' leurs quantités de calorique, M et M' leurs centres de gravité, et r la distance MM' ; et considérons l'action exercée par m' sur m, laquelle est égale et contraire à la réaction de m sur m'. Supposons d'abord les dimensions de m et de m' très-petites par rapport à la distance qui les sépare. L'action dont il s'agit se réduira alors à une force unique, dirigée suivant la droite MM', et dont l'intensité sera une fonction de r que nous représenterons par R. En même temps, leur répulsion mutuelle sera propotionnelle au produit de c et c', et leur attraction, au produit de m et m'. En considérant la force R comme positive ou négative, selon qu'elle tendra à augmenter ou à diminuer la distance r, sa valeur sera excés de la répulsion sur l'attraction ; et si l'on suppose que l'attraction réciproque de la matière et du calorique qui retient celui-ci dans chaque molécule s'étent au-dehors, il faudra retrancher de cet excès l'attraction du calorique de m' sur la matière de m, et celle de la matière de m' sur le calorique de m ; $\cdots$
>
> En réunissant les trois derniers termes de R en un seul, on pourra écrire sa valeur sous cette forme : $R = Fr - fr$. Chacune des deux fonctions Fr et fr n'aura que des valeurs positives ; ces valeurs décroîtront trés-rapidement et sans alternative, à mesure que la variable r augmentera : elles deviendront insensibles pour toute valeur sensible de r. : Poisson [55, p.6, §2.]

3.1.3. *Navier's integral vs. Poisson's summation with mean value of the molecular intervals.* Navier's infinite integral vs. Poisson's *definite integral*

Poisson wites many pure-mathematical papers on *definite integral* and the functions by series . Poisson says it will be the epoch-making attention among the *géomètres* to introduce his idea over the summation on the molecular intervals :

> Il s'agit ici de décider une question importante, sur laquelle je crois avoir appelé le premier l'attention des géomètres, et qui consiste à savoir si l'on pourra continuer de représenter les résultantes des actions mutuelles de molécules disjointes, par des intégrales définies, ou si l'on devra leur conserver la forme de sommes dans lesquelles la variable croît par différences finies et égales aux intervalles moléculaires. Poisson [49, p.206, §1].

Here we remark that the notations by Poisson in following : $d.\frac{1}{r}fr$ means $d\left(\frac{1}{r}fr\right)$ in modern notation. After taking $\varepsilon \equiv \frac{a^2}{3} = \frac{k}{\rho}$, we get a equivalence of $(6)_{Pe}$ in Table 6, which is in Navier[22, pp.388-391] as in Table 6.

Poisson writes the letter to Arago as follows :

> Mais si l'on exprime avec lui les forces moléculaires par des intégrales, on peu voir par une simple intégration par partie, que le coëfficient k ou ε

---

donc entièrement incompatible avec la notion d'un corps solide."

In spite of Navier's observation, Poisson's general discussion makes it clear that he was thinking of his resultant function as being represented as a difference of two functions of the kind described above. In his "Extrait" discussed above, he actually reduces his function R to the form $R = Fr - fr$, where both F and f are functions having only positive values that become insensible at sensible values for r. Presumably Poisson regarded his thoughts on his subject as being without need of further clarification, as he never bothered to answer Navier's objections. : D.H.Arnold [2, VI, p.355]

> s'evanouit en même temps que K ; en sort que les équations d'équilibre ne renferment plus rien qui dépende de l'action des molécules ; résultat absurde que l'on ne peut éviter qu'en exprimant les résultantes de cette action, par des sommes non réductibles à des des intégrales, ce qui empêche qu'on n'ait nécessairement $\varepsilon = 0$ par suite de $K = 0$. : Poisson [49, p.207, §2]

Navier says on Poisson's summation :

> Donc la difficulté d'accorder l'état naturel du corps avec l'état varié, c'est-à-dire, de faire en sorte que k conserve une valeur, tandis que K est nul, n'existe véritablement pas ; ou du moins il n'est pas nécessaire, pour résoudre, de suppooser que *les quantités k, K sont de sommes* plutôt que *des intégrales* : ils suffit de supposer que $r^4 fr$ n'est pas nul quand $r = 0$. Donc encore on peut être rassuré sur l'exactitude des solutions données par M. de Laplace et par d'autres géomètres, dans lesquelles *les actions moléculaires sont représentés par des intégrales.* : Navier[26, p.103, §7].

After multiplying $\frac{dr}{\alpha}$ to K and k in Table 1, we get new K and k as follows. At first, we show here $k = -K$ holds, and when $K = 0$ then $k = 0$. Using $d\Big(\frac{1}{r}f(r)\Big) = \frac{1}{r}f'(r)dr - f(r)(\frac{1}{r^2})dr$, we get :

$$\begin{aligned} k = & \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6} d\Big(\frac{1}{r}f(r)\Big) = \frac{2\pi}{15}\int_0^\infty \frac{r^5}{\alpha^6}\Big[\frac{1}{r}f'(r)dr - f(r)(\frac{1}{r^2})\Big]dr \\ = & \frac{2\pi}{15}\int_0^\infty \frac{r^4}{\alpha^6} f'(r)dr - \frac{2\pi}{15}\int_0^\infty \frac{r^3}{\alpha^6} f(r)dr \equiv I - \frac{1}{5}K \end{aligned}$$

$$I = \frac{2\pi}{15\alpha^6}\Big\{\Big[r^4 f(r)\Big]_0^\infty - 4\int_0^\infty f(r)r^3 dr\Big\} = -\frac{8\pi}{15}\int_0^\infty \frac{r^3}{\alpha^6}f(r)dr = -\frac{4}{5}K$$

From $k = -\frac{4}{5}K - \frac{1}{5}K$, we get the relation of $k = -K$. Then we get that when $K = 0$ then $k = 0$.

Moreover, we see also $K = -k$ holds, and when $k = 0$ then $K = 0$.

D.H.Arnold, the leading researcher for Poisson, cites one paragraph of Navier's letter to Arago as follows :[8]

> As a matter of fact, it is necessary first of all that he [Poisson] compensate for what is probably only an oversight or a misprint. It must read " If one observes that $r^4 fr$ is null at both limits etc.... " Moreover, the writer [again Poisson] does not show the necessity that $r^4 fr$ be null at the limit corresponding to $r = 0$. There are infinitely many forms that could be adopted for the unknown function fr for which this circumstance [vanishing faster than r^{-4} at the lower limit] would not occur. : Navier [26, p.103, §6]

Cauchy says by using $f(r)$ and $\mathrm{f}(r)$ which does not change the sign, namely for $\mathrm{f}(r)$ is always plus, the expression with this function is always necessary for the sign :[9]

---

[8] Arnold [2, Part IX, p.360]

[9] D.H.Arnold, who is the most famous researcher of Poisson, cites this paragraph in his paper[2, VI, pp.362-363]. It seems to be a simple misprint in his citation of G and R in Arnold[2, VI, p.363]. In Cauchy[4], he always uses $\mathrm{f}(r)$ with $\pm$ sign but $f(r)$ with no sign.

Concevons maintenant que, dans l'etat primitif du système des molécules $m, m', m'', \cdots$, et, du point (a, b, c) comme centre avec un rayon l convenablement choisi, on décrive une sphère qui renferme toutes les molécules dont l'action sur la masse $\mathbf{m}$ a une valeur sensible. Divisions le volume V de cette sphère en éléments très petits $v, v', v'', \cdots$, mais dont chacun renferme encore un très grand nombre de molécules. Soient M la somme des masses des molécules comprises dans la sphère, et $(48)_C \quad \Delta = \frac{M}{V} \cdots$

Alors, si la fonction $\mathrm{f}(r)$ est telle que, sans altérer sensiblement les sommes désignées par G et par R, $\cdots$

$$(52)_C \quad \begin{cases} G = \pm\frac{2\pi\Delta}{3}\int_0^\infty r^3\mathrm{f}(r)dr, \\ R = \frac{2\pi\Delta}{15}\int_0^\infty r^3 f(r)dr = \pm\frac{2\pi\Delta}{15}\int_0^\infty \Big[r^4\mathrm{f}'(r) - r^3\mathrm{f}(r)\Big]dr \end{cases} \tag{2}$$

D'ailleurs, si, pour des valeurs croissantes de la distance r, la fonction $\mathrm{f}(r)$ décroit plus rapidment que la fonction que $\frac{1}{r^4}$, si de plus le produit $r^4\mathrm{f}(r)$ s'évanouit pour $r = 0$, on trouvera, en supposant la fonction $\mathrm{f}'(r)$ continue, et en intégrant par parties,

$$(53)_C \quad \int_0^\infty r^4\mathrm{f}'(r)dr = -4\int_0^\infty r^3\mathrm{f}(r)dr$$

On aura donc alors $(54)_C \quad R = -G, \quad \cdots$ [10] : Cauchy[4, pp.242-243]

These contents are contradictory to Navier[26, p.103, §7]. In this point, Navier does not comment at all, although he cites Cauchy[27, p.244].

3.1.4. *Navier's ε and E vs. Poisson's k and K vs. Cauchy's G and R.* (Poisson says $\varepsilon = k$, but Navier denies it. Navier [26, p.103].) Navier cites the Euler's equations of incompressible fluid ([22, p.399]) :

$$\begin{cases} P - \frac{dp}{dx} = \rho\Big(\frac{du}{dt} + u\frac{du}{dx} + v\frac{du}{dy} + w\frac{du}{dz}\Big), \\ Q - \frac{dp}{dy} = \rho\Big(\frac{dv}{dt} + u\frac{dv}{dx} + v\frac{dv}{dy} + w\frac{dv}{dz}\Big), \\ R - \frac{dp}{dz} = \rho\Big(\frac{dw}{dt} + u\frac{dw}{dx} + v\frac{dw}{dy} + w\frac{dw}{dz}\Big), \end{cases} \qquad \frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

In Navier [21, p.252], owing to more Laplace than D'Alembert and Euler, Navier used his own equations published in 1821 ([21, p.250])[11]but which seem not to be the formation such today's equations as what we called Navier-Stokes equations on the incompressible fluid : (1). His first equations appeared in the history are as follw :

$$\begin{cases} \frac{1}{\rho}\frac{dp}{dx} = X + \varepsilon\Big(3\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} + 2\frac{d^2v}{dxdy} + 2\frac{d^2w}{dxdz}\Big) - \frac{du}{dt} - \frac{du}{dx}\cdot u - \frac{du}{dy}\cdot v - \frac{du}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dy} = Y + \varepsilon\Big(\frac{d^2v}{dx^2} + 3\frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} + 2\frac{d^2u}{dxdy} + 2\frac{d^2w}{dydz}\Big) - \frac{dv}{dt} - \frac{dv}{dx}\cdot u - \frac{dv}{dy}\cdot v - \frac{dv}{dz}\cdot w\ ; \\ \frac{1}{\rho}\frac{dp}{dz} = Z + \varepsilon\Big(\frac{d^2w}{dx^2} + \frac{d^2w}{dy^2} + 3\frac{d^2w}{dz^2} + 2\frac{d^2u}{dxdz} + 2\frac{d^2v}{dydz}\Big) - \frac{dw}{dt} - \frac{dw}{dx}\cdot u - \frac{dw}{dy}\cdot v - \frac{dw}{dz}\cdot w\ ; \end{cases} \tag{3}$$

and the equation of continuity (3). He explains ε from various concepts in [21, p.251] :

[10]From $(52)_C$ and $(53)_C$, it turns as follows : $R = -5\frac{2\pi\Delta}{15}\int_0^\infty r^3\mathrm{f}(r)dr = -G$. This corresponds to Poisson's discussion of $k = -K$.

[11]Navier cited his paper as follows : dans un Mémoire sur les lois de l'équilibre et des mouvemens des corps solides élastiques, que j'ai présenté, le 14 mai 1821 (*sic.*). This title is none in Graber's citation [12].

> ε is the constant which we mentioned above. Many experiments teach that this constant takes the various values for each fluids, and varies with the temperature for each fluid. It is considerd also as variant with the pression ; but we have observed as the known facts, on the contrary, that ε is almostly independ of the force which tends to compress the partial diffrences of the fluid.

In [21], he did not describe in details, so that we can not substitute the operator Δ, which means $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$, for (3), because the second terms in the right hand-side are more than Δu with :

$$2\Big(\frac{d^2u}{dx^2}+\frac{d^2v}{dxdy}+\frac{d^2w}{dxdz}\Big),\quad 2\Big(\frac{d^2v}{dy^2}+\frac{d^2u}{dxdy}+\frac{d^2w}{dydz}\Big),\quad 2\Big(\frac{d^2w}{dz^2}+\frac{d^2u}{dxdz}+\frac{d^2v}{dydz}\Big). \tag{4}$$

In modern notation, the kinetic equation and the equation of continuity are conventionally described as (1) . Navier says citing Laplace (*Equilibrium of Fluid* [18, Vol.1, Chap. 4-8, p.90-239]) :

> The consideration of the repulsive force, which the pressure develoves between the molecules, which M. de Laplace deduced already the general equations of the motion of the fluid in the 12-th book of *Méchanique céleste*, seems to depend more immidiately on the physical notion which we can formulate on the property of this corps.

Navier ([22, p.414]) had described the equation samely as today's vectotial expression (1) above stated as follows :
from (3), after operating in such a way as, at first by $\frac{d}{dx}$, and by $\frac{d}{dy}$, and at last by $\frac{d}{dz}$, then the right hand-side of (4) are zero respectively, therefore (3) turns out :[12]

$$\begin{cases} P-\frac{dp}{dx}=\rho\Big(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\Big)-\varepsilon\Big(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\Big), \\ Q-\frac{dp}{dy}=\rho\Big(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\Big)-\varepsilon\Big(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\Big), \\ R-\frac{dp}{dz}=\rho\Big(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\Big)-\varepsilon\Big(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\Big). \end{cases} \tag{5}$$

and the equation of continuity which is the same as (3). Here, if we take $\mathbf{f}=(P,Q,R)$ and $\frac{1}{\rho}\mathbf{f}\equiv\mathbf{f}$, then this means :

$$\frac{\partial \mathbf{u}}{\partial t}-\frac{\varepsilon}{\rho}\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}+\frac{1}{\rho}\nabla p=\mathbf{f}.$$

If we put $\mu\equiv\frac{\varepsilon}{\rho}$ then (5) equals to (1). According to the today's conventional text on the fluid dynamics by I.Imai [14, p.186], $\nu=\frac{\mu}{\rho}$, where ν is the kinematic viscosity, μ is the coefficient of viscosity, and ρ is the density, and where μ and ρ are the physical constants.

Navier [22] deduces the following three types of expressions :

(1) the expressions of the equilibrium of the fluids
(2) the expressions of the forces of the molecular action which is under the state of motion
    - the expressions of a partial moment of the forces
    - the expressions of the total moments of the forces caused by the reciprocal actions of the molecules of a fluid

[12] Navier([22, p.413])

We introduce the last expressions (2) which are directly related to our Navier-Stokes equations. Poisson says :

> Mais, outre ces forces, la compression inégale du corps donne encore naissance à d'autres actions qui sont du même ordre de grandeur et ne sont pas comprises dans l'hypothèse de M.Navier. En ayant égard aux unes et aux autres, on obtient des équations d'équilibre, renformant deux coëfficiens dépendents de la matière du corps, savoir : les coëfficiens K et k de la page 25 de mon Mémoire, dont le second *k est le coëfficient ε de M.Navier.* : Poisson [49, p.206, §2].

But Navier denies it :

> En revenant maintenant à la manière dont j'ai envisagé la question des corps élastiques, je dois dire qu'elle est entièrement différente des principes adoptés par M.Poisson (principes que je ne connais bien que depuis la publication de son Mémoire) ; et c'est *à* tort que M.Poisson dit que *son coefficient k est la même chose que mon coefficient ε.* : Navier [26, p.103, §8].

Cauchy deduces his equations as follows :

$(41)_C \quad G = H = I, \quad L = M = N, \quad P = Q = R, \quad \cdots \quad (45)_C \quad L = 3R.$

As the equations in equiblium :

$$(67)_C \quad \begin{cases} (L+G)\frac{\partial^2\xi}{\partial x^2} + (R+H)\frac{\partial^2\xi}{\partial y^2} + (Q+I)\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = 0, \\ (R+G)\frac{\partial^2\eta}{\partial x^2} + (M+H)\frac{\partial^2\eta}{\partial y^2} + (P+I)\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = 0, \\ (Q+G)\frac{\partial^2\zeta}{\partial x^2} + (P+H)\frac{\partial^2\zeta}{\partial y^2} + (N+I)\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = 0, \end{cases}$$

and substituting respectively the right hand-side of $(67)_C$ as the equations in motion : $(68)_C \quad = \frac{\partial^2\xi}{\partial t^2}, \quad = \frac{\partial^2\eta}{\partial t^2}, \quad = \frac{\partial^2\zeta}{\partial t^2}.$

Here, we put : $(69)_C \quad \nu = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z}$, then

$$(70)_C \quad \begin{cases} (R+G)\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} + \frac{\partial^2\xi}{\partial z^2}\right) + 2R\frac{\partial\nu}{\partial x} + X = 0, \\ (R+G)\left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial z^2}\right) + 2R\frac{\partial\nu}{\partial y} + Y = 0, \\ (R+G)\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2} + \frac{\partial^2\zeta}{\partial z^2}\right) + 2R\frac{\partial\nu}{\partial z} + Z = 0, \end{cases}$$

and substituting respectively the right hand-side of $(70)_C$: $(71)_C \quad = \frac{d^2\xi}{dt^2}, \quad = \frac{d^2\eta}{dt^2}, \quad = \frac{d^2\zeta}{dt^2}.$

If $G = 0$ then we get the equations in equiblium by $(41)_C$ as follows :

$$(83)_C \quad \begin{cases} L\frac{\partial^2\xi}{\partial x^2} + R\frac{\partial^2\xi}{\partial y^2} + Q\frac{\partial^2\xi}{\partial z^2} + 2R\frac{\partial^2\eta}{\partial x\partial y} + 2Q\frac{\partial^2\zeta}{\partial z\partial x} + X = 0, \\ R\frac{\partial^2\eta}{\partial x^2} + M\frac{\partial^2\eta}{\partial y^2} + P\frac{\partial^2\eta}{\partial z^2} + 2P\frac{\partial^2\zeta}{\partial y\partial z} + 2R\frac{\partial^2\xi}{\partial x\partial y} + Y = 0, \\ Q\frac{\partial^2\zeta}{\partial x^2} + P\frac{\partial^2\zeta}{\partial y^2} + N\frac{\partial^2\zeta}{\partial z^2} + 2Q\frac{\partial^2\xi}{\partial z\partial x} + 2P\frac{\partial^2\eta}{\partial y\partial z} + Z = 0, \end{cases}$$

and substituting respectively the right hand-side of $(83)_C$ as the equations in motion : $(84)_C \quad = \frac{\partial^2\xi}{\partial t^2}, \quad = \frac{\partial^2\eta}{\partial t^2}, \quad = \frac{\partial^2\zeta}{\partial t^2}.$

Cauchy remarks the coincidence with Navier's equation.

> Les formules $(70)_C$ et $(71)_C$, au contraire, semblent devoir s'appliquer au cas *où* les corps est également élastique dans tous les sens; et alors on retrouvera *les formules de M.Navier*, si l'on attribue à la quantitè G une

> valeur null. Ajoutons que, si, dans les formules $(67)_C$ et $(68)_C$, on réduit à zéro, non seulement la quantité G, mais encore les quantités de même espèce H et I, ces formules deviendront. : Cauchy [5, pp.251-252]

In sum, the Cauchy's equations : $(84)_C$ is equivalent to Poisson's ($(6)_{Pe}$ in Table 6) and Navier's (in Table 6).

3.1.5. *Navier's critical view to an exponential function as the example of Poisson's fr : $ab^{(-\frac{r}{n\alpha})^m}$.* (Navier[26, p.101] against Poisson[50, p.369].) Navier shows his example as the function as follows :

> On a dit ci-dessus que les termes contenant des puissances de ρ supérieures à la seconde devaient être négligés. Cette proposition peut être rendue très-sensible en paticularisant la fonction $f\rho$, et lui donnant le caractère d'un fonction qui décroît très-rapidment quand ρ augmente. Si par exemple on prend pour cette fonction $e^{-k\rho}$, e étant le nomble dont la logarithme népérien est l'unité, et k un coefficient constant, on aura
>
> $$\int_0^\infty d\rho\cdot\rho^4 e^{-k\rho}=\frac{1\cdot2\cdot3\cdot4}{k^5},\quad \int_0^\infty d\rho\cdot\rho^6 e^{-k\rho}=\frac{1\cdot2\cdot3\cdot4\cdot5\cdot6}{k^7},\quad \text{etc.}$$
>
> Or pour que la quantité $e^{-k\rho}$ decroisse avec un trés-grande rapidité quand ρ augmente, il faut supposer que le coefficient k est un trés-grande nombre. [22, p.383, §3]

An example function by Navier : $e^{-k\rho}$ comes from the side of repulsion, because the more ρ is shorter, the more the force increases.

3.1.6. *Navier's $f(\rho)$ and $F(\rho)$ vs. Poisson's fr vs. Cauchy's* f(r) *and $f(r)$.* Laplace [19, p.685] has already used $\varphi(r)$ of this sort of the function in 1819 as above mentioned. Navier's $f(\rho)$ and $F(\rho)$ are the arbitrary functions which depend on the distance ρ between M and M' in calculating the moment of the two cases. As the common factor, taking $\frac{\pi}{30}$ of $\{\frac{\pi}{10},\frac{\pi}{30}\}$ for $f(\rho)$ in case of partial moment, and $\frac{\pi}{6}$ for $F(\rho)$ in case of total moment. Navier calculated ε in the case of elastic solid in 1821 as follows : [22, p.382, §3]

$$\varepsilon\equiv\int_0^\infty d\rho\cdot\rho^4 f\rho\frac{\pi}{4}\frac{4\cdot4}{15}\frac{1}{2}=\frac{2\pi}{15}\int_0^\infty d\rho\cdot\rho^4 f\rho \tag{6}$$

Here, ε is an unknown constant which depends on the intensity of the force of elasticity of the solid.
Also in the case of fluid, as the kinematic viscosity : μ of (1), Navier calculated ε and E in 1822. [21, p.404, p.411, §3], and later uses this ε in fluid [23].

Here, the constant ε shows the resistance occured each other by the smoothness of the two layers, for the equal area to the unit of the surface, with the unit of the weight. The another constant E shows, the resistance occured by the smoothness of this layer on the wall, for the equal area to the unit of the surface, with the unit of the weight. ρ is the distance between two molecules, and $f(\rho)$ and $F(\rho)$ are the functions which depends on the distance ρ between two molecules M and M'.

3.2. **Modeling of the original Navier equations.** Navier deduces the equations of equilibrium using both parameters : ε and E of (8), and deduces the original Navier's equations (5) from (9) and at last he gets (5). Moreover Navier shows his unique boundary condition(10)[13] using ε and E.

3.2.1. *Navier's E.* As the common factor, taking $\frac{\pi}{6}$ for $F(\rho)$ in case of total moment, he puts

$$\frac{4\pi}{6}\int_0^\infty d\rho\rho^2F(\rho)=\frac{2\pi}{3}\int_0^\infty d\rho\rho^2F(\rho)\equiv E. \tag{7}$$

And he defines : $E(u\delta u+v\delta v+w\delta w)$ for the expression which we seek for the sum of the moments of the total actions which caused between the molecules of the wall and the fluid, following the direction which pass by the point of the separation of the fluid and the wall and the fluid, and which can be regarded as the measure of its reciprocal action. Navier gets the following equilibrium of a fluid using ε of (6) and the above $E(u\delta u+v\delta v+w\delta w)$:

$$\begin{aligned}0 = & \iiint dxdydz\begin{cases}\left[P-\frac{dp}{dx}-\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)\right]\delta u\\ \left[Q-\frac{dp}{dy}-\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)\right]\delta v\\ \left[R-\frac{dp}{dz}-\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right)\right]\delta w\end{cases}\\ - & \ \varepsilon\iiint dxdydz\begin{cases}3\frac{du}{dx}\frac{\delta du}{dx}+\frac{du}{dy}\frac{\delta du}{dy}+\frac{du}{dz}\frac{\delta du}{dz}+\frac{dv}{dy}\frac{\delta du}{dx}+\frac{dv}{dx}\frac{\delta du}{dy}+\frac{dw}{dz}\frac{\delta du}{dx}+\frac{dw}{dx}\frac{\delta du}{dz}\\ \frac{du}{dx}\frac{\delta dv}{dy}+\frac{du}{dy}\frac{\delta dv}{dx}+\frac{dv}{dx}\frac{\delta dv}{dx}+3\frac{dv}{dy}\frac{\delta dv}{dy}+\frac{dv}{dz}\frac{\delta dv}{dz}+\frac{dw}{dy}\frac{\delta dv}{dz}+\frac{dw}{dz}\frac{\delta dv}{dy}\\ \frac{du}{dx}\frac{\delta dw}{dz}+\frac{du}{dz}\frac{\delta dw}{dx}+\frac{dv}{dy}\frac{\delta dw}{dz}+\frac{dv}{dz}\frac{\delta dw}{dy}+\frac{dw}{dx}\frac{\delta dw}{dx}+\frac{dw}{dy}\frac{\delta dw}{dy}+3\frac{dw}{dz}\frac{\delta dw}{dz}\end{cases}\\ + & \ \mathrm{S}ds^2E(u\delta u+v\delta v+w\delta w).\end{aligned} \tag{8}$$

Here, S means the integration in the total surface of the fluid, in varing the quantity E of (7), following the nature of the solid with which this surface is in contact. Shifting d to the front of δ of the middle term of the right hand-side of (8) and by Taylor expansion using the partial integral, and considering $\mathrm{S}ds^2E(u\delta u+v\delta v+w\delta w)$ of (8) and the total of the rest terms is zero, we get the last expression from (8) and the first term is as follows :

$$0=\iiint dxdydz\begin{cases}\left[P-\frac{dp}{dx}-\rho\left(\frac{du}{dt}+u\frac{du}{dx}+v\frac{du}{dy}+w\frac{du}{dz}\right)+\varepsilon\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right)\right]\delta u\\ \left[Q-\frac{dp}{dy}-\rho\left(\frac{dv}{dt}+u\frac{dv}{dx}+v\frac{dv}{dy}+w\frac{dv}{dz}\right)+\varepsilon\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\right]\delta v\\ \left[R-\frac{dp}{dz}-\rho\left(\frac{dw}{dt}+u\frac{dw}{dx}+v\frac{dw}{dy}+w\frac{dw}{dz}\right)+\varepsilon\left(\frac{d^2w}{dx^2}+\frac{d^2w}{dy^2}+\frac{d^2w}{dz^2}\right)\right]\delta w\end{cases} \tag{9}$$

At last, we get (5) from (9).

3.2.2. *Navier's boundary condition.* About $\mathrm{S}ds^2E(u\delta u+v\delta v+w\delta w)$ of (8) and the total of the rest terms, Navier explains as follows : regarding the conditions which react at the points of the surface of the fluid, if we substitute

- $dydz \quad\rightarrow\quad ds^2\cos l,\ \ l$: the angles by which the plane tangents on the surface frame with the plane yz,
- $dxdz \quad\rightarrow\quad ds^2\cos m,\ \ m$: samely, the angles with the plane xz,
- $dxdy \quad\rightarrow\quad ds^2\cos n,\ \ n$: samely, the angles with the plane xy,
- $\iint dydz, \iint dxdz, \iint dxdy \quad\rightarrow\quad \mathrm{S}ds^2$,

[13]This unique suggestion about Navier's boundary conditionis is due to Darrigol[8, p.115]

then because the affected terms by the quantities $\delta u, \delta v$ and δw respectively reduce to zero, the following determinated equations should hold for any points of the surface of the fluid :

$$\begin{cases} Eu+\varepsilon[\cos l 2\frac{du}{dx}+\cos m\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)+\cos n\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)]=0, \\ Ev+\varepsilon[\cos l\Big(\frac{du}{dy}+\frac{dv}{dx}\Big)+\cos m 2\frac{dv}{dy}+\cos n\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)]=0, \\ Ew+\varepsilon[\cos l\Big(\frac{du}{dz}+\frac{dw}{dx}\Big)+\cos m\Big(\frac{dv}{dz}+\frac{dw}{dy}\Big)+\cos n 2\frac{dw}{dz}]=0, \end{cases} \tag{10}$$

here the value of the constant E which varifies following the nature of the solid with which the fluid is in contact. (10) express the boundary condition. The first terms of the left-hand side of (10) are defined by (7) for the expression which we seek for the sum of the moments of the total actions which caused between the molecules of the boundary and the fluid, and the second terms are the normal derivatives of the component of the fluid velocity parallel to the surface. Moreover, by using Darrigol's simple notation[14], we can express this condition as $E\mathbf{v}+\varepsilon\partial_\perp \mathbf{v}_\parallel=\mathbf{0}$, where $\partial_\perp$ is the normal derivative, and $\mathbf{v}_\parallel$ is the component of the fluid velocity parallel to the surface. [15]

4. Conclusions and Perspect

The development of the tensors by Poisson [55], Cauchy [5], Green [13] and Stokes [60] were succeeded by Odqvist, who made of the primary Stokes boundary problem [29, p.332], and which is the today's adopted expression : $t_{ik}=-p\delta_{ik}+\mu(u_{ki}+u_{ik})$.[16] These are supporting Poisson's validity as Arnold says.[17] In addition, in 1883, Reynolds [59] discoverd the physical parameter as "Reynolds Number"[18] with the physical experiment, saying " two principles, which are that the general character of the motion of fluids in contact with solid surfaces depends on the relation between a physical constant of the fluid and the product of the linear dimensions of the space occupied by the fluid and the velocity"[19], which was not deduced from the analytical aspect. It seems that this will do for the present. However, we think these discussed problems by the then mathematicians are to be continued in studying it also from the analysis. We shall study it further including the tensors by Green [13], Oseen [30], Leray [20], Odqvist [29], Ladyzhenskaya [17], etc., and the modern sense of the tensor from the algebraic aspect. Moreover we study the mean of the *two parameters* described in our paper.

[14]cf. Darrigol [8, p.115]

[15]In Table 10, the abbrevations means as follows :
L : Lu à l'Académie des sciences (i.e. recieved by judge). P or other : published/printed. S : author's signed date, which are described by the order of yy/mm/dd.
ACP : *Annales de chimie et de physique.* BSM : *Bulletin des sciences mathématiques, astromatiques, physiques et chimiques.* JEP : *J. école Polytech.* MAS : *Mémoires de l'Academie Royale des Siences de l'Institute de France.* MSM : *Mémoires des siences mathematiques et physiques.*
$\Rightarrow P/N/A \Leftarrow$ N : Letter to Navier, P : to Poisson , A : to Arago. N← : the citation of Navier's paper (ex. by Cauchy). ** : the word "*fluid*" is only explicit in the title of the paper.

[16]cf. H.Kozono [16, p.2]

[17]cf. D.H.Arnold [2, VI, p.361]

[18]Reynolds Number : $R=\frac{UL}{\nu}$, $\nu=\frac{\mu}{\rho}$, U, L : the average velocity and length, respectively, ν : the kinematic viscosity, i.e. ν of (1), μ : the coefficient of viscosity, ρ : the density.

[19]cf. Reynolds[59, p.935]

TABLE 10. Papers & monographies in the seconnd period of molecules

no	Laplace/ Fourier/ Gauss	Poisson	Navier	Cauchy/ Stokes/ et al.
	L.1749-827/ F.1768-830/ G.1777-835	1781-840	1785-836	C.1789-857/ S.1819-903
1	L.1798-05[18]			
2	L.1806-07[19]			
3		JEP**9**(13)[31]		
4		MSM**9**(14)[32]		
5		JEP**10**(15)[33]		
6		JEP**11**(20)[34]		
7		JEP**11**(20)[35]		
8		ACP**19**(21)[36]		
9	F.1822[9]			
10			ACP**19**(22)[21]	
11		JEP**12**(23)[37]		
12		JEP**12**(23)[38]		
13		ACP**22**(23)[39]		
14		ACP**26**(24)[40]		
15		ACP**28**(25)[41]		
16		L24/02/24MAS**5**(26)[42]		
17		L24/12/27MAS**5**(26)[43]		
18			L22/03/18MAS**6**(27)[23]	
19			L21/05/14MAS**7**(27)[22]	
20		L26/07/10MAS**7**(27)[44]		
21		L26/12/11MAS**7**(27)[45]		
22		L27/10/01ACP**36**(27)[46]		
23		S28/04/14ACP**37**(28)[47]		
24			P[47]⇐P28/07/ACP**38**(28)[24]	
25		P28/08/ACP**39**(28)[48]⇒N		
26			P⇐P28/10/ACP**39**(28)[25]	
27		P28/10/ACP**39**(28)[49]⇒A[1],N		
28				N[23]⇐BSM**10**(28)[6]
29				C.1828[3]
30				C.1828[4]
31				N[22]←C.1828[5]
32			P⇐S29/01/18ACP**39**(29)[26] ⇒A[1]	
33				N,P⇐ACP**39**(29)[1]
34		L28/04/14MAS**8**(29)[50]→N[22] L28/11/24MAS**8**(29)[52]		
35				N,P←BSM**11**(29)[15]
36			P⇐BSM**11**(29)[27]⇒A[1]→C[5]	
37		L28/07/07MAS**8**(29)[51]		
38	F.MAS**8**(29)[10]			
39		L28/11/24MAS**9**(30)[53]		
40		L28/12/24MAS**9**(30)[54]		
41	G.1830[11]			
42		L29/10/12JEP**13**(31)[55]		
43		1831[56]		
44		1835[57]		
45		1835[58]		
46				S.1849[60]

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Remark : we use *Lu* (: in French) in the bibliography on Navier or Poisson meaning "read" date by the judges of the journal : MAS. In citing the original paragraphs in our papaer, the underscoring are by ours.

確率超過程と超汎関数

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1 Introduction

Some aspects of generalized function theory started to appear in mathematics in the nineteenth century. We can see it in the definition of the Green's function, in the Laplace transform, and in Riemann's theory of trigonometric series, which were not necessarily the Fourier series of an integrable function.

The Laplace transform is intensively used in engineering and it leads to use symbolic methods which are called later operational calculus. There are used divergent series, these methods are not accepted from the point of view of pure mathematics. Later they are typical application of generalized function methods.

After the Lebesgue integral was introduced, a concept of generalized function became essential to mathematics. An integrable function, in Lebesgue's theory, is equivalent to any other which is the same almost everywhere. That means its value at a given point is not its most important feature. An evident formulation is given, in functional analysis, of the essential feature of an integrable function, such as the way it defines a linear functional on other functions. This allows a definition of weak derivative.

The Dirac delta function was defined by Paul Dirac; this was to treat measures, thought of as densities. Sobolev who was working in partial differential equation theory, defined the first suitable theory of generalized functions, from the mathematical view point, in order to work with weak solutions of partial differential equations.

Schwartz distributions

The theory of distributions was developed by Laurent Schwartz. It is based on duality theory for topological vector spaces.

The theory of distribution is widely affect the differential and integral calculus. Heaviside and Dirac had generalized the calculus with specific applications in mind, and other similar methods of formal calculation, however, profound mathematical foundation was not given. Schwartz developed the theory of distributions by putting methods of this type onto a thourough basis. The theory extended their range of application, providing powerful tools for applications in numerous areas.

2 Laurent Schwartz (1915-2002)

Laurent Schwartz entered the Ecole Normale Superieure in Paris in 1934 and graduated with the Agreation de Mathematique in 1937 and studied for his doctorate in the Faculty of Science at Strasbourg which he was awarded in 1943.

His teachers were Choquet, Frechet, Borel, Julia, Cartan, Lebesgue, Hadamard. Schwartz writes, "The life of the ENS was a marvel for a young person of my temperament. In one blow, the field of mathematics became infinitely wide."

Schwartz was lecturer at the Faculty of Science at Grenoble the year 1944-45 before moving to Nancy where he became a professor at the Faculty of Science. During this period he produced his famous work on the theory of distributions.

From 1953 to 1959, Schwartz was holding the position of Professor in Paris. He taught at the Ecole Polytechnique in Paris from 1959 to 1980. He then spent three years at the University of Paris VII before he retired in 1983.

Schwartz made the outstanding contribution in the theory of distributions to mathematics in the late 1940s. He published these ideas in the paper "*Generalisation de la notion de fonction, de derivation, de transformation de Fourier et applications mathematique et physiques* " in 1948.

And the other literatures on Theory of distributions are

Théoriedes distributions, Tome I. 1950, Tome II. 1951, Hermann & Cie Paris.

Théoriedes distributions, Nouvelle ed. 1966 Hermann

Schwartz got a Fields Medal, presented by Harald Bohr, at the International Congress in Harvard on 30 August 1950 for his work on the theory of distributions. Schwartz has received a long list of prizes, medals and honours in addition to the Fields Medal.

His autobiography "Un mathématicien aux prises avec le siècle. Editions odile Jacob. 1997, (Japanese translation by 弥永健一、闘いの世紀を生きた数学者・上、下、シュプリンガー・ジャパン、2006.) tells us how he had been living as a mathematician. We are surprised to see what we have never imagined.

Literatures below are specifically mentioned.

Geometry and probability in Banach space. LNM 852, Springer, 1981

We are particularly interested in his work on probability theory. Also see, Notice of AMS 50, no.9 (2003), 1072-1084.

The contents and his basic idea are found in those literature. We shall however mention some of his results, which are often used in white noise theory, in the next section.

3 Generalized function (Distribution)

The basic idea of generalized function is as follows. If $f : R \to R$ is an integrable function, and $\phi : R \to R$ is a smooth function with compact support, then $\int f\phi dx$ is a real number which depends on ϕ linearly and continuously. The function f can be thought as a continuous linear functional on the test functions space of ϕ.

This notion of **continuous linear functional on the space of test functions** is therefore used as the definition of a distribution.

Such distributions may be multiplied with real numbers and can by added together, so they form a real vector space. In general it is not possible to define a multiplication for distributions, but distributions may be multiplied with infinitely often differentiable functions.

To define the derivative of a distribution, we first consider the case of a differentiable and integrable function $f : R \to R$. If ϕ is a test function, then we have

$$\int_R f'\phi\,dx = -\int_R f\phi'\,dx$$

using integration by parts (note that ϕ is zero outside of a bounded set and that therefore no boundary values have to be taken into account). This suggests that if S is a distribution, we should define its derivative S' as the linear functional which sends the test function ϕ to -$S(\phi')$. It turns out that this is the proper definition; it extends the ordinary definition of derivative, every distribution becomes infinitely often differentiable and the usual properties of derivatives hold.

The Dirac delta function is the distribution which sends the test function ϕ to $-\phi'(0)$. It is the derivative of the

step function

$$H(x) = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0 \end{cases}$$

The derivative of the Dirac delta is the distribution which sends the test function ϕ to -$\phi'(0)$.

Compact support and convolution

We say that a distribution S has compact support if there is a compact subset $K$of U such that for every test function ϕ whose support is completely outside of K, we have $S(\phi) = 0$. Alternatively, one may define distributions with compact support as continuous linear functionals on the space $C^\infty(U)$; the topology on $C^\infty(U)$ is defined such that ϕ_k converges to 0 if and only if all derivatives of ϕ_k converge uniformly to 0 on every compact subset of U.

If both S and T are distributions on R^n and one of them has compact support, then one can define a new distribution, i.e. the convolution $S*T$ of S and T. It generalizes the classical notion of convolution of functions and is compatible with differentiation in the following sense:

$$\frac{d}{dx}(S*T) = (\frac{d}{dx}S)*T + S*(\frac{d}{dx}T).$$

Tempered distributions and Fourier transform

By using a larger space of test functions, one can define the tempered distributions, a subspace of $D'(R^n)$. These distributions are useful if one studies the Fourier transform in generality: all tempered distributions have a Fourier transform, but not all distributions have one.

The space of test functions employed here, the so-called Schwartz-space, is the space of all infinitely differentiable rapidly decreasing functions, where $\phi : R^n \to R$ is called rapidly decreasing if any derivative of ϕ, multiplied with any power of $|x|$, converges towards 0 for $|x| \to \infty$. These functions form a complete topological vector space with a suitably defined family of seminorms. More precisely, let

$$p_{\alpha,\beta}(\phi) = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \phi(x)|$$

for α, β multi-indices of size n. Then ϕ is rapidly-decreasing if all the values

$$p_{\alpha,\beta}(\phi) < \infty$$

The family of seminorms $p_{\alpha,\beta}$ defines a locally convex topology on the Schwartz-space. It is metrizable and complete.

The derivative of a tempered distribution is again a tempered distribution. Tempered distributions generalize the bounded locally integrable functions; all distributions with compact support and all square-integrable functions can be viewed as tempered distributions.

To study the Fourier transform, it is best to consider complex-valued test functions and complex-linear distributions. The ordinary continuous Fourier transform F yields then an automorphism of Schwartz-space, and we can define the Fourier transform of the tempered distribution S by $(FS)(\phi) = S(F\phi)$ for every test function, ϕ, FS is thus again a tempered distribution. The Fourier transform is a continuous, linear, bijective operator from the space of tempered distributions to itself. This operation is compatible with differentiation in the sense that

$$F(\frac{d}{dx}S) = ixFS$$

and also with convolution: if S is a tempered distribution and ψ is a slowly increasing infinitely often differentiable function on R^n (meaning that all derivatives of ψ grow at most as fast as polynomials), then $S\psi$ is again a tempered distribution and

$$F(S\psi) = FS * F\psi.$$

4 Rigged Hilbert space (Gel'fand triple)

The rigged Hilbert space appears in white noise analysis in various places, where the expression has variation depending on the purpose.

In mathematics, a rigged Hilbert space (Gel'fand triple, nested Hilbert space, equipped Hilbert space) is a construction designed to link the distribution and the test function, where the square-integrable aspects of functional analysis serves as a key role.

Since a function such as

$$x \mapsto e^{ix},$$

which is in an obvious sense an eigenvector of the differential operator

$$i\frac{d}{dx}$$

on the real line R, is not square-integrable for the usual Borel measure on R, this requires some way of stepping outside the strict confines of the Hilbert space theory. This was supplied by the apparatus of Schwartz distributions, and a generalized eigenfunction theory was developed in the years after 1950.

Functional analysis approach

The concept of rigged Hilbert space places this idea in abstract functional-analytic framework. Formally, a rigged Hilbert space consists of a Hilbert space H, together with a subspace Φ which carries a finer topology, that is one for which the natural inclusion

$$\Phi \subset H$$

is continuous. It is no loss to assume that Φ is dense in H for the Hilbert norm. We consider the inclusion of dual spaces H^* in Φ^*. The latter, dual to Φ in its 'test function' topology, is realised as a space of distributions or generalised functions of some sort, and the linear functionals on the subspace Φ of type

$$\phi \mapsto \langle v, \phi \rangle$$

for v in H are faithfully represented as distributions (because we assume Φ dense).

Now by applying the Riesz representation theorem we can identify H^* with H. Therefore the definition of rigged Hilbert space is :

$$\Phi \subset H \subset \Phi^*.$$

The most significant examples are for which Φ is a nuclear space; this comment is an abstract expression of the idea that Φ consists of test functions.

Formal definition

A rigged Hilbert space is a pair (H, Φ) with H a Hilbert space, Φ a dense subspace, such that Φ is given a topological vector space structure for which the inclusion map i is continuous. Identifying H with its dual space H^*, the adjoint to i is the map $i^* : H = H^* \mapsto \Phi$. The duality pairing between Φ and Φ^* has to be compatible with the inner product on H :

$$\langle u, v \rangle_{\Phi \times \Phi^*}.$$

whenever $u \in \Phi \subset H$ and $v \in H = H^* \subset \Phi^*$.

$$\Phi \subset H = H^* \subset \Phi^*.$$

Note that even though Φ is isomorphic to Φ^* if Φ is a Hilbert space in its own right, this isomorphism is not the same as the inclusion .

Example. Fourier integral

$$f(x) = \int_R e^{isx} \hat{f}(s) ds, \quad x \in R, \ f, \ \hat{f} \in L^2(R).$$

The system $\{e^{isx}, s \in R\}$ is a system of generalized eigenfunctions of the differentiation operator, acting on $L^2(R)$, arising under the natural rigging of this space by the Schwartz space $S(R)$.

These Hilbert spaces play an important role in the definition of generalized white noise functinal.

5 Hyperfunction

Hyperfunctions are sums of boundary values of holomorphic functions, and can be thought of as distriburtions of infinite order. Hyperfunctions were introduced by Miko Sato in 1958, building upon earlier work by Grothiendieck and others.

The distribution theory led to investigation of the idea of hyperfunction, in which spaces of holomorphic functions are used as test functions. A refined theory has been developed, in particular by Mikio Sato, using sheaf theory and several complex variables. This extends the range of symbolic methods that can be made into rigorous mathematics, for example Feynman integrals.

A hyperfunction is specified by a pair (f, g), where f is a holomorphic function on the lower half-plane and g is a holomorphic function on the upper half-plane. Informally, the hyperfunction (f, g) is the sum of the boundary values of f and g. If f is holomorphic on the whole complex plane, then it should have the same boundary values when considered as a function on either the upper or lower hyperplane.

For further reference, see

佐藤幹夫、超関数の理論、数学 10 巻、1-27.

6 Generalized stochastic process

White noise analysis started in 1975, more than three decades ago; this means it is now in the history. However, there are many serious misunderstandings. Anyhow it is a good opportunity to have a review of the history of white noise analysis. Before white noise, there is a history of generalized stochastic process.

Well known approaches

1) Stationary random distributions : by K. Itô. Mem. Univ. of Kyoto. Math. 1953, 209-223.

The stationary distributions were introduced as a generalization of stationary stochastic process and they are classified according to the specral density.

2) Generalized random processes : by Gel'fand. He defined a stochastic process, a sample function of which is a generalized function. Doklady Acad Nauk, 1955. 853-856.

   Literature : Gel'fand and Shirov, Generalized function Vol 4, 1955. Academic Press.

3) Generalized white noise function (started in 1975)

There are two ways to introduce a space of generalized white noise functionals. One is using the method of the Gel'fand's generalized functions (Hida 1975). The other is an infinite dimensional analogue of Schwartz distributions (1980- , Kubo-Takenaka then Potthoff-Streit).

The first method uses the Sobolev space structure which is familier for us, while the second method provides a characterization theorem of generalized white noise functionals in such a way that, let F be its S-transform, then it is necessary and sufficient that it is ray entire and it satisfies

$$|F(z\xi)| \leq K_1 \exp[K_2|z|^{\ 2}|\xi|_p^2].$$

Generalized functionals of white noise (Hida distribution)

White noise analysis started in 1975, more than three decades ago, where the time derivative $\dot{B}(t)$ of a Brownian motion $B(t)$ was introduced as a variable of white noise functionalsby T. Hida. Since $\dot{B}(t,\omega)$ is no more an ordinary random variable but an *idealized generalized random variable*. However, by many reasons $\dot{B}(t)$ is taken to be an elemental (atomic) variable, so that a sharp time description is given. He does not smear the $\dot{B}(t)$ by a smooth function ξ like that $\dot{B}(\xi) = -\int \xi'(t)B(t)dt$. He gives a rigorous meaning to each $\dot{B}(t)$ by introducing a new space of generalized random variable. It is therefore noted that the $\dot{B}(t)$ can not be approximated by members in the usual space of Brownian functionals.

For a general setup of the space of white noise functionals, $\dot{B}(t)$'s are taken to be variables, so that their functionals can be defined, not formal but in a correct manner. By doing so, we can see many applications, for instance, in the expression of propagator in quantum dynamics according to the Feynman integral.

To define $\dot{B}(t)$ for every t rigorously, He use the rigged Hilbert space in the following manner:

Start with smeared variables

$$\langle \dot{B}, \xi \rangle = \int \xi(t) \dot{B}(t) dt,$$

ξ being a member of E, say the Schwartz space.d Then extend ξ to be in the Hilbert space $L^2(R)$ to have

$$H_1 = \left\{ \dot{B}(f), f \in L^2(R) \right\} \cong L^2(R) \tag{1}$$

since $\dot{B}(f) = \langle \dot{B}, f \rangle$ is an ordinary Gaussian random variables $N(0, |f|^2)$.

Then, take a rigged Hilbert space

$$K \subset L^2(R) \subset K^*,$$

where K is taken so as K^* involves delta-functions, and the isomorphism (1) extends to a rigged Hilbert space

$$H_1^{(1)} \subset H_1 \subset H_1^{(-1)},$$

where $\dot{B}(t)$ is a well defined member of $H_1^{(-1)}$.

Note. It should be made clear that a delta function is not a member of $L^2(R)$, but it belongs to much wider class K^*. Similarly $\dot{B}(t)$ has an identity as a member of $H_1^{(-1)}$ and can not be reduced to an element of H_1.

Further, we can form functionals $\varphi(\dot{B}(t), t \in R)$ to use a rigged Hilbert space

$$H_n^{(n)} \subset H_n \subset H_n^{(-n)},$$

where H_n is a space of homogeneous chaos and $H_n^{(-n)}$ involves Hermite polynomials in $\dot{B}(t)$'s of degree n, which are rigorously defined.

The space of generalized white noise functionals is definded by

$$(L^2)^- = \bigoplus_{n\geq 0} H_n^{(-n)}.$$

As for the second method of defining the space of generalized white noise functionals (Hida distributions) denoted by $(S)^*$ can be defined as a member of the rigged Hilbert spaces

$$(S) \subset (L^2) \subset (S)^*,$$

where (L^2) is the space of ordinary white noise functionals and

$$(L^2) = \oplus H_n$$

is a Fock space. Actual method to form (S) uses the second quantization method, where the operator is $A = -\frac{d^2}{du^2} + u^2 + 1$.

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「翦管術」の流れ

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1 翦管術

1.1 「翦管術」とは

「翦管術」とは連立1次合同式 (以下, 剰余方程式と呼ぶ)

「$x \equiv r_1 \pmod{m_1}, \cdots, x \equiv r_n \pmod{m_n}$」

の解法のことをいう. 剰余方程式は中国の古算書『孫子算経』(400年頃, 著者不詳) の中に「物不知其総数」としてある1題が嚆矢とみられ, 暦学上の必要により中国では古くから剰余方程式の解法が確立し, 伝承されていた.『孫子算経』の後,『数書九章』(1247年, 秦九韶) では「大衍総数術(だいえんそうすうじゅつ)」として9題, 南宋の『楊輝算法』(1274〜1275年, 楊輝) の「續古摘奇算法」(1275年) では「翦管術」, 俗名「秦王暗點兵猶覆射之術」として5題,『算法統宗』(1592年, 程大位) では「物不知其総数」,「韓信點兵」として3題など, 様々な名称とともに剰余方程式が伝えられていた. 現在「中国剰余定理」(Chinese Remainder Theorem) と称されることもあるが, 中国では「孫子定理」と呼ばれ, 高校教科書で単元としても取り上げられている. 日本においては『孫子算経』の伝来とともに, 古く奈良時代から「剰余方程式」は知られていた. しかし, 時代を下り江戸時代になって『算法統宗』をもとにして書かれたとされる『塵劫記』(1627年, 吉田光由) で, 法が 3,4,5 の剰余方程式が「百五減算」として紹介され, 広く一般庶民にまで親しまれるようになった.

1.2 『楊輝算法』から和算「翦管術」へ

関孝和 (1642？〜1708) は, 中国から伝来し, 庶民の遊戯的問題としても親しまれていた剰余方程式を一般化し「翦管」と名付け, その解法を「翦管術」として,『括要算法』,『大成算経』に著した.

『括要算法』は1680年〜1683年に関が書いたものをもとに, 関の没後, 関

流の門弟,荒木村英,大高由昌によって編集され,1712年出された刊本である.『大成算経』(1683年〜1710年)以前に書かれたもので,誤植も見られるが,関流の伝本の中核となった書と見られている.

『大成算経』は建部賢明(かたあき),賢弘(かたひろ)兄弟との共著であり,1683年頃から始め,およそ28年の歳月をかけて編まれ,関の没後,『括要算法』の出版よりは2年ほど早い1710年頃に完成されたとみられる写本である.この『大成算経』第6巻の第6章「翦管」の冒頭の解説では,「翦管術」という名称ばかりでなく,「秦王暗點兵」という『楊輝算法』にある別称までもが紹介されている.関や建部兄弟が『楊輝算法』を読み,そこから「翦管術」,「秦王暗點兵」の名を引用したことは明らかといえよう.

1.3 関孝和と『楊輝算法』

関孝和は奈良の寺へ行き,寺にあった算書を写本し,江戸に持ち帰り3年の歳月をかけそれを読解し,会得したと斉東野人の『武林隠見録』(1738年)に記されている.『武林隠見録』は武士など当時の有名人の伝記集で,没後30年ほどになる関孝和の伝記も含まれており,今日語られている関に纏わる幾つかの逸話が載せられている.しかし,その真偽のほどは定かでなく,実際に奈良の寺へ行ったか否か,或いはその中国の算書が何であったかも確かな証は得られていない.しかし,この算書は『楊輝算法』であったと考えるのが通説となっている.その根拠の一つは,奥付に関孝和の署名がある『楊輝算法』の写本が現在に伝わっており,中国で編纂された『中国科学技術典籍通彙 數學卷 』(1993年,任継愈他)には,奥付に関孝和の署名がある『楊輝算法』写本が収められている.一方,その写本とは奥付の年紀が異なる別の写本も現存している.『中国科学技術典籍通彙 數學卷 』に収められているものには,「寛文辛丑仲夏下浣日訂写訖」とあり,また著者が目にする機会を得た故藪内清氏所蔵,現在川原秀樹氏(東大大学院人文社会系研究科)所蔵の写本には,「寛文癸丑仲夏下浣日訂写訖」とある.この奥付の中の1文字「癸」と「辛」の違いについては,寛文年間が寛文13年(癸丑,1673年)の9月21日に改元され延宝元年となったため,暦上は「寛文癸丑」はありえないものとの誤解から「寛文辛丑」(寛文元年,1661年)と書き換えられてしまったのではないかとの推察もあり(参考文献[A10] p 202),また「寛文辛丑」が正し

いとする説もある (参考文献 [B6]). 仮に, 関孝和の生年を 1640 年とすると,「寛文癸丑」であれば, 関 33 才,「寛文辛丑」であれば関 21 才のときに『楊輝算法』を写本したことになる.

1.4 中国伝来の剰余方程式の解法

剰余方程式

「$x \equiv r_i \pmod{m_i},\ i = 1, \cdots, n$ 但し $(m_i, m_j) = 1\ (i \neq j)$」$\cdots$ (1)

についての中国の算書に伝わる解法を以下に述べる[1].

まず $i = 1, 2, \cdots, n$ それぞれについて, 不定方程式

「$\dfrac{M}{m_i}x - m_i y = 1,$ 但し $M = m_1 \cdot m_2 \cdot \cdots \cdot m_n$」$\cdots$ (2)

を解き, 解を「$\dfrac{M}{m_i}x = x_i$」とする.

この x_i は,

「$x_i \equiv 1 \pmod{m_i},\ x_i \equiv 0 \pmod{m_j} (i \neq j)$」$\cdots$(3)

を満たすので, 剰余類「$x \equiv \sum_{i=1}^{n} r_i x_i \pmod{M}$」が剰余方程式 (1) の解であるが, これら条件を満たす x のうち正の最小数ひとつのみを答としている.

2 関孝和の「翦管術」

2.1 『括要算法』の「翦管術」

『括要算法』は, 元(げん), 亨(こう), 利(り), 貞(てい)の 4 巻からなっており, 4 巻それぞれは, 元巻 (第 1 巻) は累裁招差法, 垜(だ) 積(せき)総術 (パスカルの三角形, ベルヌイ数) など, 亨巻 (第 2 巻) は 諸約術, 剰一術から翦管術に至る整数論, 利巻 (第 3 巻) は角術 (正多角形の辺の長さと内接円, 外接円の半径に関する考察), そして貞巻 (第 4 巻) は利巻の結果を利用して円周率を求めるなど, 所謂円理と称される内容となっている. 更にその亨巻 (第 2 巻) は, 前編, 後編の 2 編からなっている. 前編は「諸約の法」と名付けられ,「互約, 逐約, 斉約, 遍約, 増約, 損約,

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[1] 整数 $m_1, m_2, \cdots, m_n$ の最大公約数を $(m_1, m_2, \cdots, m_n)$, 最小公倍数を $\{m_1, m_2, \cdots, m_n\}$ と表す.

零約, 遍通, 剰一」の約法九項目 (公約数, 公倍数の求め方, 1次不定方程式の解法など) の解説からなっている[2]. 続く後編は「翦管術解」と題され, 以下に挙げる9題の剰余方程式とその解法を示している.

$\boxed{1}$ $\begin{cases} x \equiv 1 \pmod{5} \\ x \equiv 2 \pmod{7} \end{cases}$ 答 $x = 16$

$\boxed{2}$ $\begin{cases} x \equiv 2 \pmod{36} \\ x \equiv 14 \pmod{48} \end{cases}$ 答 $x = 110$

$\boxed{3}$ $\begin{cases} x \equiv 2 \pmod{3} \\ x \equiv 1 \pmod{5} \\ x \equiv 5 \pmod{7} \end{cases}$ 答 $x = 26$

$\boxed{4}$ $\begin{cases} x \equiv 3 \pmod{6} \\ x \equiv 3 \pmod{8} \\ x \equiv 5 \pmod{10} \end{cases}$ 答 $x = 75$

$\boxed{5}$ $\begin{cases} x \equiv 3 \pmod{5} \\ x \equiv 2 \pmod{7} \\ x \equiv 2 \pmod{9} \\ x \equiv 7 \pmod{11} \end{cases}$ 答 $x = 128$

$\boxed{6}$ $\begin{cases} 35x \equiv 35 \pmod{42} \\ 44x \equiv 28 \pmod{32} \\ 45x \equiv 35 \pmod{50} \end{cases}$ 答 $x = 13$

$\boxed{7}$ $\begin{cases} 8x \equiv 2 \pmod{3} \\ 7x \equiv 3 \pmod{4} \\ 6x \equiv 3 \pmod{5} \end{cases}$ 答 $x = 13$

$\boxed{8}$ $\begin{cases} 34x \equiv 6 \pmod{8} \\ 34x \equiv 14 \pmod{20} \\ 34x \equiv 23 \pmod{27} \end{cases}$ 答 $x = 11$

$\boxed{9}$ $\begin{cases} 13x \equiv 3 \pmod{7} \\ 13x \equiv 8 \pmod{9} \end{cases}$ 答 $x = 11$

平易な数値を用いながら, それぞれに特色のある9題が例示されている.
『孫子算経』(400年頃) にはじまる中国の算書に伝わる剰余方程式,

「$x \equiv r_i \pmod{m_i},\ i = 1, \cdots, n$ 但し $(m_i, m_j) = 1\ (i \neq j)$」 $\cdots(1)$ を

「$a_i x \equiv r_i \pmod{m_i},\ i = 1, \cdots, n$」」 $\cdots(1')$

と一般化し, $(m_i, m_j) \neq 1$, 即ち, 法 m_i, m_j が互いに素でない問題を取り入れている. 第1問から第5問は $a_i = 1$, 第6問から第9問は $a_i \neq 1$, そしてさらに, 奇数番の問題は法が互いに素, $(m_i, m_j) = 1$ であり, 偶数番の問題は $(m_i, m_j) \neq 1$, 法が互いに素でない問題となっている.

この『括要算法』の「翦管術解」9題の中から2題, 第3問と第6問を現代の数式記法を用いて紹介する[3].

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[2]前編の諸約術, 剰一術に関しては参考文献 [A7][B5] 参照

[3]他の問題及び詳解は参考文献 [A7][B5][B7] 参照

2.2 『括要算法』の「翦管術解」第 3 問

第 3 問の原文は「今有物不知総数, 只云, 三除余二箇, 五除余一箇, 七除余五箇, 問総数幾何. 答日, 総数二十六箇」とある. 合同式を用いて表せば

「$x \equiv 2 \pmod 3, x \equiv 1 \pmod 5, x \equiv 5 \pmod 7$, 答 $x = 26$」

となる. 続く「術文」では, この剰余方程式を

「$2 \times \underline{70} + 1 \times \underline{21} + 5 \times \underline{15} = 236, 236 - 105 - 105 = 26 \cdots$答」と解いている.

ここで, アンダーラインのキーナンバー $\underline{70}, \underline{21}, \underline{15}$ それぞれは,

「$x_1 \equiv 1 \pmod 3, x_1 \equiv 0 \pmod 5, x_1 \equiv 1 \pmod 7$」

を満たす最小の正の整数 $\underline{70}$,

「$x_2 \equiv 0 \pmod 3, x_2 \equiv 1 \pmod 5, x_2 \equiv 0 \pmod 7$」

を満たす最小の正の整数 $\underline{21}$,

「$x_3 \equiv 0 \pmod 3, x_3 \equiv 0 \pmod 5, x_3 \equiv 0 \pmod 7$」

を満たす最小の正の整数 $\underline{15}$ となっている.

従って, これらキーナンバーの性質から,

$$
\begin{aligned}
236 = 2 \times \underline{70} + 1 \times \underline{21} + 5 \times \underline{15} &\equiv 2 \pmod 3 \\
&\equiv 1 \pmod 5 \\
&\equiv 5 \pmod 7
\end{aligned}
$$

であり, 236 は問題の条件に合う数のひとつであり, 一般解は「$x \equiv 236 \pmod{105}$」と分かる. ここでの法 105 は [3,5,7] の最小公倍数である. 和算では条件に合う正の最小数だけを答えるのが常であったため, 236 から [3,5,7] の最小公倍数 105 を 2 回引いた「26」だけを答としている.

『塵劫記』で紹介された「百五減算」とは, この第 3 問と同様の問題, 即ち法が [3,5,7] である問題に限定して名付けられたもので, 最後に [3,5,7] の最小公倍数 105 を引くことに由来している.

この第 3 問の問題文の直前には,『算法統宗』の書名と共に「孫子歌曰」として七言絶句,

「三人同行七十稀　五樹梅花廿一枝
七子團圓正半月　除百令五便得知」

が紹介されている. 剰余方程式 (1) で「$n = 3, m_1 = 3, m_2 = 5, m_3 = 7$」の場合,(3) 式を満たすキーナンバー x_i $(i = 1, 2, 3)$ は「$x_1 = \underline{70}, x_2 = \underline{21}, x_3 = \underline{15}$(半月)」であること, さらに $\{m_1, m_2, m_3\} = 105$ となることを覚えやすく歌

に込めたものである.

また,「術文」の中ではこれらのキーナンバー, $x_1 = \underline{70}$ を「三除法」, $x_2 = \underline{21}$ を「五除法」, $x_3 = \underline{15}$ を「七除法」と呼び, さらに $m_1 \cdot m_2 \cdot m_3 = 35$ を「去法」と名付けている. 一般に和算では不定方程式 (3) を解いたキーナンバー x_i を「m_i 除法」, 法 $m_i (i = 1, \cdots, n)$ の最小公倍数 $\{m_1, m_2, \cdots, m_n\}$ を「去法」と呼んでいた.

「術文」に続く「解文」ではキーナンバー, 三除法 $x_1 = \underline{70}$, 五除法 $x_1 = \underline{21}$, 七除法 $x_2 = \underline{15}$ の求め方が述べられている.
三除法は, 不定方程式「$\overset{5\times 7}{35}x - 3y = 1$」を「剰一術」を用いて解き,
左段数 $x = 2$ から左総数 $35x = \underline{70}$(三除法) を得る.
五除法は, 不定方程式「$\overset{3\times 7}{21}x - 5y = 1$」を「剰一術」を用いて解き,
左段数 $x = 1$ から左総数 $21x = \underline{21}$(五除法) を得る.
七除法は, 不定方程式「$\overset{3\times 5}{15}x - 7y = 1$」を「剰一術」を用いて解き,
左段数 $x = 1$ から左総数 $15x = \underline{15}$(七除法) を得る.

「**剰一術**」とは, 2元1次不定方程式 (Diophantos 方程式)

「$Ax - By = 1$ 但し, A, B は自然数の定数, $(A, B) = 1$」$\cdots (*)$

の解法,「術」のことである. 関孝和は『括要算法』亨(こう)巻の前編「諸約の法」の中で, 不定方程式 $(*)$ およびその解法のことを「剰一」と名付け, 詳しく解説している.「剰一」とは「一を剰(あま)す」という意味で用いたものと思われる.
「剰一術」は後編の「翦管術」のための重要な道具立てとして置かれている.
特に「翦管術」の解法の中での「剰一術」は「x の最小の正の整数解」から「Ax の値」(m_i 除法) を求めることを専らとして用いられていた.

2.3 『括要算法』の「翦管術解」第6問

第6問の原文は
「今有物不知総数, 只云, 三十五乗四十二除余三十五箇. 四十四乗三十二除余二十八箇. 四十五乗五十除余三十五箇. 問総数幾何. 答曰, 総数一十三箇.」
とある. 合同式を用いて表せば
「$35x \equiv 35 \pmod{42}, 44x \equiv 28 \pmod{32}, 45x \equiv 35 \pmod{50}$

答 $x = 13$」$\cdots$ ①

となる.この第6問では,剰余方程式 ($1'$) に関して,

・$a_i \neq 1$ $(a_1 = 35,\ a_2 = 44,\ a_3 = 45)$

・$(m_i, m_j) \neq 1$ (法 $m_1 = 42, m_2 = 32, m_3 = 50$ が互いに素でない)

となっていることに注目したい.

問題文に続く「解文」では, 3本の合同式それぞれを約して,

「$5x \equiv 5 \pmod 6), 11x \equiv 7 \pmod 8), 9x \equiv 7 \pmod{10}$」$\cdots$ ①$'$

とし, さらに法 $[6, 8, 10]$ を $[3, 8, 5]$ と**逐約**(ちくやく)[4] し,

「$5x \equiv 5 \pmod 3, 11x \equiv 7 \pmod 8, 9x \equiv 7 \pmod 5$」$\cdots$ ①$''$

とした後, 以下第3問と同様の手続きで解いている. すなわち, 第6問の初めに与えられた剰余方程式①を①$'$ に, そして更に法を逐約し①$''$ にと置き換えて解いている.

一般に, 不定方程式「$Ax - By = 1$」を解くためには, A, B(剰余方程式における各合同式の法) が互いに素でなければならない.「翦管術解」第6問の法は互いに素ではない. 従って逐約して得た互いに素である新しい法に置き換えて解いている.

しかし,法 $[m_1, m_2, m_3]$ が互に素でない剰余方程式 (♭) とその法を逐約して得た新たな法 $[m_1', m_2', m_3']$ に置き換えた剰余方程式 (♭$'$) は一般には同値とはいえない.

$$(\flat) \begin{cases} a_1 x \equiv r_1 \pmod{m_1} \\ a_2 x \equiv r_2 \pmod{m_2} \\ a_3 x \equiv r_3 \pmod{m_3} \end{cases} \qquad (\flat') \begin{cases} a_1 x \equiv r_1 \pmod{m_1'} \\ a_2 x \equiv r_2 \pmod{m_2'} \\ a_3 x \equiv r_3 \pmod{m_3'} \end{cases}$$

剰余方程式 (♭) と (♭$'$) が同値であるための条件は,

$$(\#) \begin{cases} a_1 r_2 \equiv a_2 r_1 & \pmod{(m_1, m_2)} \\ a_2 r_3 \equiv a_3 r_2 & \pmod{(m_2, m_3)} \\ a_3 r_1 \equiv a_1 r_3 & \pmod{(m_3, m_1)} \end{cases}$$

であり, 剰余方程式 (♭) が解を持つための必要十分条件もまた (#) である. さ

[4] 互いに素でない自然数の組について, その組の最小公倍数を変えずに, 各2数が互いに素となるように約すこと. [例](105,112,126) を逐約すると (5,16,63) となる. 逐約の結果は一意ではない.

らに, その時, 解は法の最小公倍数 $\{m_1, m_2, m_3\}$ において一意である.『括要算法』に取り上げられている剰余方程式9題のうち, 偶数番の4題は法が互いに素でない問題であるが, それら4題はみな悉く条件 (#) を満たしているものである. 多数の数値例を検証する中で, 剰余方程式が解を持つための条件 (#) について, どの程度の認識を持ったうえで, 条件 (#) に適ったものばかりを例示していたものであろうか. 条件 (#) に関する言及は見当たらない.

2.4 『大成算経』の「翦管術」

建部賢明, 賢弘兄弟との共著とされている『大成算経』であるが, 関はその完成を待たず1708年に没し, 賢弘自身も公務多忙となり, 編集の最終段階においては, 建部賢明が単独で仕上げたものであると『六角佐々木山内流建部氏伝記』(1715年, 建部賢明) には記されている. 従って, 関が関与したものである一方, 関だけの考察によるものではないことも確実である. 刊本でないこともあり, 関流直系の伝本とはならず, 研究, 利用された形跡の少ない算書であったとみられている. 従って, 関以後の「翦管術」は主に『括要算法』をもとに伝えられていった. しかし『大成算経』には質量共に『括要算法』を上回る「翦管術」問題17題が収められている. その『大成算経』は前集 (第1巻〜第3巻), 中集 (第4巻〜第15巻), 後集 (第16巻〜第20巻) の3部からなり, 中集に含まれる第6巻の第6章が「翦管」と題されている. その直前の第5章は「諸約」と題され,「互約, 逐約, 斉約, 遍約, 累約, 零約, 重約, 増約, 損約, 添約」の約術十項目が例題と共に解説されている. この第5章と第6章は, 丁度『括要算法』亨(こう)巻の前編「諸約術」と後編「翦管術解」にまさに対応している. しかし, 約術の項目立てには幾つかの相違がある. 例えば『括要算法』をはじめその後の関流を中心とした和算では「剰一術」と称されている一次不定方程式 (∗) の解法であるが,『大成算経』第5巻では右辺の値を1に限らないなど不定方程式 (∗) を変形した問題の解法を含め,「累約術」と称している. その『大成算経』第6巻「翦管」では, まず冒頭で「翦管術」の一般的な解説が述べられ, その後『括要算法』の「翦管」問題9題と同様の「総数を求める」, 標準的な剰余方程式8題が例示, 解説されている. その後, さらに手を加えた形の剰余方程式 (予め総数を与え, その総数に加減乗除を施した数に関する剰余方程式を与え,「加減乗除した数を求める」

問題など) 9題が紹介されている.『大成算経』の第6巻の第6章「翦管」の前半「求総数」と題された8題は以下のものである[5].

$\boxed{1}$ $\begin{cases} x \equiv 1 \pmod{5} \\ x \equiv 2 \pmod{7} \end{cases}$

答　$x = 16$

$\boxed{2}$ $\begin{cases} x \equiv 3 \pmod{6} \\ x \equiv 3 \pmod{8} \\ x \equiv 5 \pmod{10} \end{cases}$

答　$x = 75$

$\boxed{3}$ $\begin{cases} x + 6 \equiv 3 \pmod{3} \\ x - 9 \equiv 6 \pmod{7} \end{cases}$

答　$x = 22$

$\boxed{4}$ $\begin{cases} x/2 \equiv 3 \pmod{5} \\ x/3 \equiv 4 \pmod{7} \\ x/4 \equiv 6 \pmod{19} \end{cases}$

答　$x = 96$

$\boxed{5}$ $\begin{cases} 35x \equiv 35 \pmod{42} \\ 44x \equiv 28 \pmod{32} \end{cases}$

答　$x = 13$

$\boxed{6}$ $\begin{cases} 24x \equiv 12 \pmod{30} \\ 35x \equiv 7 \pmod{42} \\ 44x \equiv 28 \pmod{32} \end{cases}$

答　$x = 53$

$\boxed{7}$ $\begin{cases} \frac{2}{3}x \equiv 4 \pmod{7} \\ \frac{3}{4}x \equiv 4 \pmod{8} \end{cases}$

答　$x = 48$

$\boxed{8}$ $\begin{cases} (x+5)/2 \equiv 3 \pmod{6} \\ 3(x-4) \equiv 4 \pmod{7} \\ \frac{3}{5}(x+2) \equiv 5 \pmod{8} \end{cases}$

答　$x = 73$

ここに挙げた8題はすべて『括要算法』の「翦管術解」と同じ「総数を求める」問題である. $\boxed{1}$ $\boxed{2}$は『括要算法』と全く同じ数値の問題, $\boxed{5}$ $\boxed{6}$は『括要算法』にある問題を少し変形したものである. しかし, その他の問題 $\boxed{3}$ $\boxed{4}$ $\boxed{7}$ $\boxed{8}$のように総数に加減或いは除法を施したものは『括要算法』にはない. また, 具体的な「翦管」問題の前に, 一般的な解法の解説を述べているところは, 中国の『数書九章』の「大衍類」(第1巻、第2巻) と同様の組み立てになっている.

[5]原文に「総数二約」とあるものは「$x/2$」,「総数取三分之二」は「$\frac{2}{3}x$」と表した.

2.5 関孝和と『数書九章』

関孝和に始まる和算の「翦管術」の特筆すべき点は，日本に伝来したと確認されている中国の算書では，法 $m_i(i=1,\cdots,n)$ が互いに素である剰余方程式(1)に限っている中で，剰余方程式(1)を一般化して剰余方程式(1′)としたうえ，「法 m_i, m_j が互いに素でない」問題をも扱い，「剰一術」を用いて解法を示していることである．『括要算法』以前の和算にあっては『股勾弦鈔(ここうげんしょう)』(1672年，星野実宣(されのぶ))に1題，法が互いに素でない問題「$x\equiv 5 \pmod 6$,$x\equiv 7 \pmod 8$,$x\equiv 5 \pmod{10}$」が紹介されている．ただし，「答95」だけが書かれてあり，解法は示されていない．

このような状況の中で関は『括要算法』で「翦管術解」として，それぞれに特徴のある9題を取り上げ，法が互いに素でない問題を含めた剰余方程式の一般的な解法を明快に示した．「翦管術」という剰余方程式の名称は中国，南宋の『楊輝算法』(1275年，楊輝)にあるものを引用したと見られるが，「剰一術」という名称は中国の算書には見当たらない．『孫子算経』をはじめ『楊輝算法』，『算法統宗』など，日本に伝わったと認められている中国の算書は，剰余方程式を扱っているものの，問題と解答そして簡略な解法が述べられているだけで，剰余方程式解法の核心である不定方程式(∗)の解法，「剰一」にあたる術の解説はない．

不定方程式(∗)の解法を述べている中国の算書は南宋，秦九韶の『数書九章』(1247年)である．『数書九章』では不定方程式(∗)の解法「剰一術」のことを「大衍求一術(たいえんきゅういちじゅつ)[6]」，「翦管術」のことは「大衍総数術」と呼んでいる．そして，「連環求等」→「大衍求一術」→「大衍総数術」という流れの中で系統立てて語られている剰余方程式解法は，「互約・逐約」→「剰一術」→「翦管術」という関が示した和算における剰余方程式の解法と吻合するものであり，更に術の実質的組立てに於いても，『数書九章』と『括要算法』は同様の手続きで進められている(参考文献[A8][B3][B5][B7])．

『数書九章』は日本に伝来した形跡が認められていない．「我国に入った形跡がない」ことから「実際に日本に伝来しなかった」と判断する説も多くある(参考文献[A2][A3][A10])．しかし一方，会田安明(あいだやすあき)(1747年〜1817年)はその著書『豊島算経評林』の中で，「関は中国より伝来の書物を見て，『括要

[6]衍(余り)が一となる術，衍(余り)から総数を求める術という意と考えられる．

算法』を書き, 自らの功績とし, もとの書物を焼き捨てた」と語っている. 果たして関孝和は『数書九章』ないしはその流れを汲む書物を手にし, 感化を受けていたのであろうか. 或いは, 時空の隔たりを超越した, 数学の普遍性ゆえの必然的帰結であったのであろうか.

3　関孝和以後の「翦管術」

関孝和の没後出版された『括要算法』は,「三部抄」「七部書」と称される算書と共に関流の中核的な伝書となり, その『括要算法』によって「剰一術」,「翦管術」はよく知られるようになった. 特に法が互いに素である, 平易な剰余方程式の問題とその簡略な解法は数多くの和算書で扱われた. 例えば『勘者御伽雙紙』(1743 年, 中根彦循) では「百五減算」ばかりでなく, 法が [5,7,9] である問題を「三百十五減」, 法が [7,9] である問題を「六十三減」と名付け紹介している. ただし, 和算で常用された「翦管術」は中国伝来の解法に倣い,「最小の正の整数解」を求めることを目的としていた. 江戸初期の『塵劫記』に始まる「百五減算」或いは『勘者御伽雙紙』の「三百十五減」「六十三減」などという名称は, 剰余類を視野に収めているものとも窺えるが, 無限個ある解すべてを全体として答えてはいない.

中根彦循は『勘者御伽雙紙』の中で, 連立不定方程式,

$$\begin{cases} x+y+z=t \\ ax+by+cz=u \end{cases} \quad (\text{但し } a,b,c,t,u \text{ は有理数の定数.})$$

の正の整数解 x,y,z を求める問題に関して,『改算記』(1659 年, 山田正重) および『改算記綱目』(1687 年, 持永豊次, 大橋宅清) が一組の解だけを答えていることを取り上げ,「一組だけではなく, 当てはまる数の組すべてを答えるべきである」と指摘し, 実際複数組の解すべてを求め, 答えている. しかし, 一方「百五減算」などの問題の答えはそれまでの算書同様「最小の正の整数解」のみを答としている.「負の数」の概念がなかったわけではないが, 限定的な認識に留まり, 答の対象として「負の数」を認識するには至っていなかった. また現在のような記号法を持たず, 数についての集合論的な認識および無限の概念もない中での和算の営みの特徴, 数の概念の有りようを語っているものといえよう.

4 結び

関孝和によって一般化された剰余方程式の解法「翦管術」は, その後和算家たちによって大いに活用されるようになった.「剰余方程式を一般的に解いたこと」を関孝和の功績としている著述も多く見られる (参考文献 [A2][A3]). しかし,『括要算法』(1712年) に先んずること450年以上の南宋, 隣国中国に於て秦九韶が, 既に法が互いに素でない問題を含む剰余方程式9題を扱い, 解法を著している. その解法の流れは名称こそ異なれ, 関の解法の流れと同様の手順, 手法であり, 数値的には秦九韶の挙げた9題の方が遥かに複雑な, 難易度の高いものであった. しかし, たとえ関孝和が『数書九章』或いはその流れを汲む中国数学の何かしらを知り得ていたとしても,『数書九章』の「大衍総数術」と『括要算法』の「翦管術」は, 解法こそ類似しているが, それぞれの9題の問題の内容は実に趣きの異なるものとなっている.『数書九章』は易, 暦をはじめ米の分量など具体的, 実際的な問題を扱い, それぞれの問題の数値は極めて煩雑な, 難易度の高いものばかりである (参考文献 [A8][B3][B7]). 一方『括要算法』の9題は, みな完全に抽象化された単純な問題で, その数値はすべて整数, それも平易な数ばかりのものであった[7]. しかし,『括要算法』では問題が簡潔な形のもので, 数値も単純, 平易であるが故に,「剰一」や「翦管術」の本質が却って際立ち, 明快に, 鮮やかに説示されている. このことこそが関の数学的資質を物語っていると云えるのではないだろうか.「数学」の本質とも云える一般化する力, 抽象化する感性を備えていた. これが関の神髄であり, この抽象性の純度の高さによって,『括要算法』は芸術的と評価できるものといえよう.『数書九章』を見知っていたか, 否かという問題は残されているとしても, 少なくとも中国の算書『孫子算経』,『楊輝算法』,『算法統宗』などに伝わる「中国剰余定理」をもとに, 関孝和が抽象化して著した「翦管術」は純粋数学として, 文化としての和算の確かな稔り, その華の薫りを伝えているものといえよう.

[7]『大成算経』の17題には『数書九章』に近い, やや煩雑な形の問題も含まれている.

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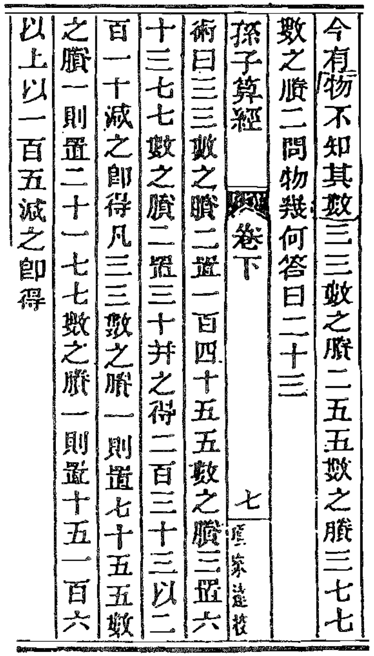
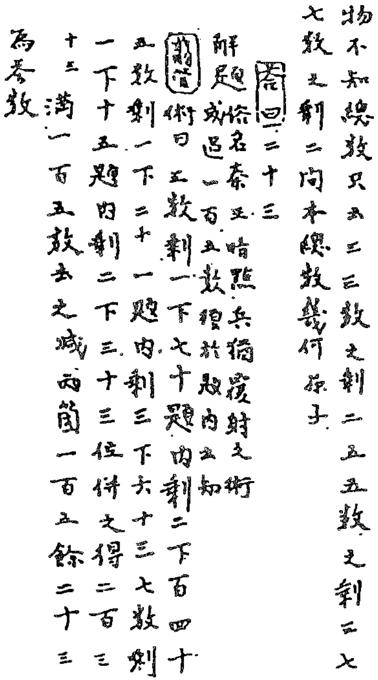


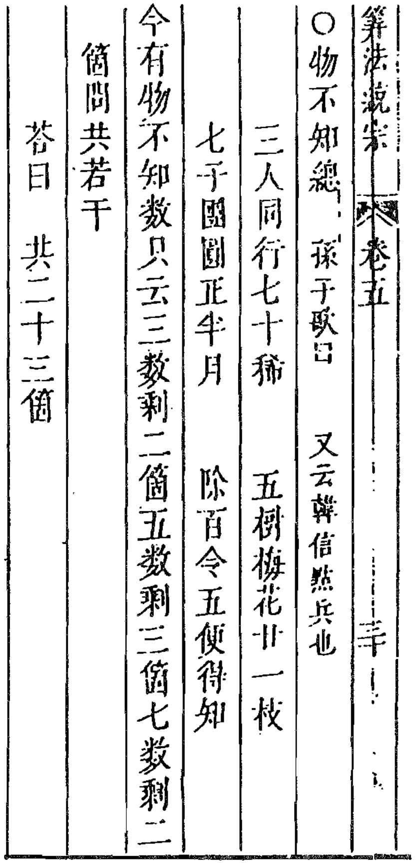
今有物不知其數三三數之賸二五五數之賸三七七
數之賸二問物幾何答曰二十三
孫子算經 卷下 七
術曰三三數之賸二置一百四十五五數之賸三置六
十三七七數之賸二置三十并之得二百三十三以二
百一十減之卽得凡三三數之賸一則置七十五五數
之賸一則置二十一七七數之賸一則置十五一百六
以上以一百五減之卽得

『孫子算経』
(400 年頃、著者不詳)



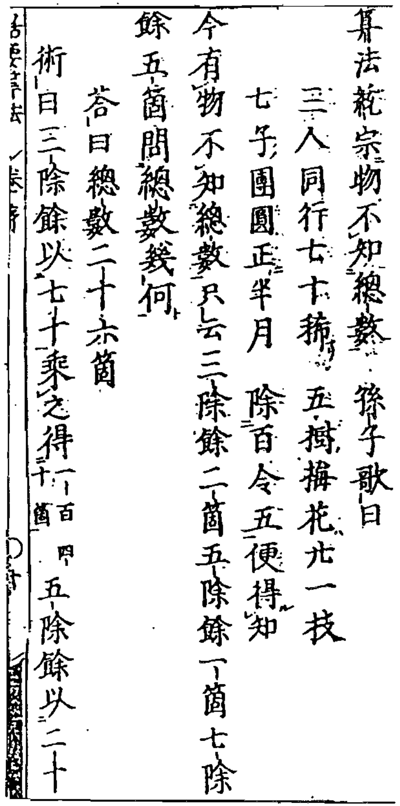
物不知總數只云三三數之剩二五五數之剩三七
七數之剩二問本總數幾何 孫子
答曰二十三
解題 俗名秦王暗點兵猶覆射之術
成邑一百五數得於題內去知
術曰三數剩一下七十題內剩二下百四十
五數剩一下二十一題內剩三下六十三七數剩
一下十五題內剩二下三十三位併之得二百三十三
十三 滿一百五數去之減兩箇一百五餘二十三
為答數

『楊輝算法—續古摘奇算法』(1275 年 楊輝)



算法統宗 卷五
○物不知總 孫子歌曰 又云韓信點兵也
三人同行七十稀 五樹梅花廿一枝
七子團圓正半月 除百令五便得知
今有物不知数只云三三数剩二箇五数剩三箇七数剩二
箇問共若干
答日 共二十三箇

『算法統宗』
(1592 年 程大位)



算法統宗物不知總數 孫子歌曰
三人同行七十稀 五樹梅花廿一枝
七子團圓正半月 除百令五便得知
今有物不知總數只云三除餘二箇五除餘一箇七除
餘五箇問總數幾何
答曰 總數二十六箇
術曰三除餘以七十乘之得一百四十箇 五除餘以二十

『括要算法』(1712 年 関孝和)

Éléments de géométrie の
平行線に関する命題 *

堀　井　政　信 † ‡

1　はじめに

昨年のシンポジウム（「*Éléments de géométrie* の定義・公理・命題」）[1] では，Adrien Marie Legendre（1752-1833）の *Éléments de géométrie avec des notes*/1812，蔵書印 École polytechnique(以下，*e.ge.notes*/1812) [2] の構成と内容について述べた．Charles Davies（1798-1876）の *Elements of geometry and trigonometry translated from the french of a.m.legendre*/1834，蔵書印 Harvard University (以下，*e.ge.tr.translated*/1834) [3] と比較対照し，次のことを明らかにした．定義は一致するが，公理「1 点を通り与えられた直線に平行な直線は，1 本のみ引かれる.」が，*e.ge.notes*/1812 には含まれず *e.ge.tr.translated*/1834 には含まれる．命題については，*e.ge.notes*/1812 の PROPOSITION XIX,XX，…，XXIII は，いずれも平行線に関する命題であるが，*e.ge.tr.translated*/1834 の対応する命題と図が異なったり，対応する命題がなかったりし，対応する命題の順序も入れ替わっている．そして，*e.ge.notes*/1812 の PROPOSITION XXI と PROPOSITION XXIII は互いに逆の命題であるが，用いられている図が異なっている．

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本報告では，*e.ge.notes*/1812に公理「1点を通り与えられた直線に平行な直線は，1本のみ引かれる.」が含まれないこととPROPOSITION XIX,XX, …, XXIIIの証明との関係について考える.

2 AXIOMES (*e.ge.notes*/1812) と *Axioms*(*e.ge.tr.translated*/1834)

e.ge.notes/1812のAXIOMESと*e.ge.tr.translated*/1834の*Axioms*は，公理の数が異なる．*e.ge.notes*/1812のAXIOMESは5項目であり，*e.ge.tr.translated*/1834の*Axioms*は13項目である．*e.ge.notes*/1812のAXIOME（5項目）については*e.ge.tr.translated*/1834の*Axioms*にそれぞれ対応するものがある．そして，*e.ge.tr.translated*/1834の*Axioms*には*e.ge.notes*/1812のAXIOMESにない公理が8項目ある.

その8項目の中で，12番目は「12. Through the same point, only one straight line can be drawn which shall be parallel to a given line.」である.「12. 1点を通り与えられた直線に平行な直線は，1本のみ引かれる.」を要請しており，平行線公理に相当する．これが*e.ge.notes*/1812のAXIOMESには含まれない.

3 『フロリアン・カジョリ初等数学史』

Florian Cajoriは『フロリアン・カジョリ初等数学史』[4]において非ユークリッド幾何学以前の歴史を述べ，A.M. Legendreについて次のように書いている.「あどりあん・まりー・るじゃんどる（1752-1833）ノ研究ハ，興味アルモノデアル．彼ハゆーくりっどノ公準ガ，“三角形ノ内角ノ和ハ二直角ニ等シ”ト云フ定理ト等値デアルコトヲ悟リ，之ニ解析的ノ證明ヲ與ヘタ．然シ彼ハ其ノ際相似形ノ存在ヲ假定シタノデアル．るじゃんどるハ是ヲ以テ満足シナカッタ」．この「解析的ノ證明」が本報告の主題である.

4 A History of Non-Euclidean Geometry

A History of Non-Euclidean Geometry [5] において，A.M. Legendre が *Éléments de géométrie avec des notes* の初版（1794 年）で与えた，parallel postulate（平行線公準）の証明の問題点が述べられている．それによると A.M. Legendre は，ある直線 AB に対して垂線 BD と斜めの線 AC がある場合，斜めの線 AC 上を点が移動するとその点から直線 AB に下ろした垂線の足はいずれ点 B と一致すると結論を下し，垂線と斜めの直線は必ず交わるから parallel postulate（平行線公準）の一般的な場合を演繹することは難しくないことを "proved"（証明）した．しかし，Semen Emel'yanovič Gur'ev（1746-1813）は，正の項の収束性の級数の部分和が単調増加であることが，その部分和が級数の和を越えることを意味しないように，垂線の足と点 A との距離が単調増加であることが，その距離を任意に大きくできることを意味しないと指摘した．

5 *e.ge.notes*/1812 の平行線に関する命題

5.1 PROPOSITION XIX

e.ge.notes/1812 の DÉFINITIONS XII は「Deux lignes sont dites *paralleles*, lorsque, étant situées dans le mème plan, elles ne peuvent se rencontrer à quelque distance qu'on les prolonge l'une et l'autre.」であり，平行の定義について述べている．

e.ge.notes/1812 の PROPOSITION XV は「*D'un point A donné hors d'une droite DE, on ne peut mener qu'une seule perpendiculaire à cette droite.*」であり，直線の外の点からその直線への垂線の数についての命題である．この命題は，PROPOSITION VI「2 辺と夾角が等しいとき，2 つの三角形は合同である.」, PROPOSITION IV「∠ ACD ＋∠ DCB ＝ 2 ∠ R →辺 AC，CB は 1 本の直線」, AXIOMES 4「1 点から他の点へはただ 1 本の直線が引ける.」により正しく証明されている．

e.ge.notes/1812 の PROPOSITION XIX は「*Si deux droites AC, BD, sont perpendiculaires à une troisieme AB, ces deux lignes seront paralleles,* *c'est-à-dire, qu'elles ne pourront se rencontrer à quelque distance qu'on les prolonge.*」，すなわち「AC ⊥ AB，BD ⊥ AB → AC∥BD」である．この命題は，DÉFINITIONS XII, PROPOSITION XVにより正しく証明されている．ただ，証明に出てくる点Oが巻末の図に記載されておらず，命題にも証明にも出てこない直線AEが書かれている．対応する *e.ge.tr.translated*/1834 のPROPOSITION XVIIIの図は，内容は共通するが別の図であり，点Oが書かれている．

5.2 PROPOSITION XX

e.ge.notes/1812のPROPOSITION XXは「*La droite BD étant perpendiculaire à AB, si une autre droite AE fait avec AB l'angle aigu BAE, je dis que les droites BD,AE, prolongées suffisamment, se rencontreront.*」，すなわち「BD ⊥ AB，∠BAE < ∠R → BDとAEは交わる」である．BD ⊥ AB，∠BAEが鋭角のとき，AE上の点がEの方向に移動すると，その点からAB上に下ろした垂線の足はBに近づくので，直線BDと直線AEは交わるとしている．しかし，Gur'evが *A History of Non-Euclidean Geometry* において，*e.ge.notes*の初版（1794年）に掲載されたparallel postulate（平行線公準）の証明について指摘しているように，証明は正しくない．巻末の図は *A History of Non-Euclidean Geometry* に掲載されている図とほぼ一致する．

e.ge.notes/1812は第9版であるが，この部分は初版と同じであることがわかる．巻末の図はPROPOSITION XIXと同じであり，直線AEが命題と証明に出てくる．この命題はTHÉORÈMEでなくLEMMEであり，*e.ge.tr.translated*/1834に対応する命題がない．

5.3 PROPOSITION XXI

e.ge.notes/1812のPROPOSITION XXIは「*Si deux droite AC,BD, font avec une troisieme AB, deux angles intérieurs CAB,ABD, dont la somme soit égale à deux droits, les deux lignes AC,BD, seront paralleles.*」，すな

わち「$\angle$ CAB + $\angle$ ABD = 2 $\angle$ R → AC ∥ BD」である．PROPOSITION II「直線 AB と直線 CD が交わるとき，$\angle$ ACD+$\angle$ BCD = 2 $\angle$ R である.」，PROPOSITION VII「1 辺と両端の角が等しいとき，2 つの三角形は合同である.」，PROPOSITION XIX「AC ⊥ AB，BD ⊥ AB → AC ∥ BD」により正しく証明されている．ただ，巻末の図には，命題にも証明にも表れない直線 AI が書かれている．対応する *e.ge.tr.translated*/1834 の PROPOSITION XIX の図は，内容は共通するが別の図である．

5.4 PROPOSITION XXII

e.ge.notes/1812 の PROPOSITION XXII は「*Si deux lignes droites AI,BD, font avec une troisieme AB, deux angles intérieurs BAI,ABD, dont la somme soit moindre que deux angles droits, les lignes AI,BD, prolongées, se rencontreront.*」，すなわち「$\angle$ BAI + $\angle$ ABD < 2 $\angle$ R → AI と BD は交わる」である．この命題は Euclid の fifth postulate（第 5 公準）に相当する．PROPOSITION XX を用いて証明しており，正しくない．巻末の図は PROPOSITION XXI と同じであり，直線 AI が命題と証明に出てくる．対応する *e.ge.tr.translated*/1834 の PROPOSITION XXI の図は，内容は共通するが別の図である．

5.5 PROPOSITION XXIII

e.ge.notes/1812 の PROPOSITION XXIII は「*Si deux lignes paralleles AB,CD, sont rencontrées par une sécante EF, la somme des angles intérieurs AGO,GOC, sera égale à deux angles droits.*」，すなわち「AB ∥ CD → $\angle$ AGO + $\angle$ GOC = 2 $\angle$ R」である．PROPOSITION XXII を用いて証明しており，正しくない．

6 終わりに

e.ge.notes/1812 の PROPOSITION XXI と PROPOSITION XXIII は互いに逆の命題であるが，用いられている図が異なっている．*e.ge.notes*/1812 の PROPOSITION XXI の図は，対応する *e.ge.tr.translated*/1834 の PROPOSITION XIX の図と，内容は共通するが別の図である．*e.ge.notes*/1812 の PROPOSITION XXIII の図は，対応する *e.ge.tr.translated*/1834 の PROPOSITION XX の図とほぼ同じである．*e.ge.tr.translated*/1834 の PROPOSITION XIX と PROPOSITION XX は同じ図が用いられており，命題の順序も連続している．

F. Cajori が書いているように，A.M. Legendre は証明が十分でないことを認識していた．*e.ge.notes* はよく売れ多くの版を重ねた．その中で A.M. Legendre と C. Davies は内容を書き直したと考えられる．*e.ge.notes*/1812 と *e.ge.tr.translated*/1834 はその過程の 1 冊である．

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