Historical aspects on Lévy's Brownian motion

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Abstract

Lévy のブラウン運動は半世紀以上も前に提唱されたが、その重要性の認識には多くの研究者の成果を待たなければならなかった。その経過をみながら、このブラウン運動の深い確率論的な性質を考えてみたい。Lévy のブラウン運動はパラメータの変動に応じて複雑ではあるがきわめて基本的な従属性を示すが、それは種々のアプローチによって知ることができる。変分をみることはその方法の一例である。さらに、ブラウン運動についてのいくつかのアプローチを再考する。

1 Introduction

In this note we wish to discuss the historical aspects on Lévy's Brownian motion. Paul Lévy introduced the multi-dimensional parameter Brownian motion. We now call it Lévy Brownian motion to discriminate from one dimensional parameter Brownian motion which is simply called Brownian motion. We first recall the discovery of Brownian motion and its development in mathematical approach and then review some results on Lévy's Brownian motion.

I. Brownian motion

In 1827 Robert Brown had found the movement of grains of pollen suspended in a liquid. Then Einstein introduced the theoretical and quantative approach to Brownian motion in 1905.

By supposing that the movement occurs uniformly in both time and space, and the proportion of the pollen grains moved from x to $x+y$ in a time interval of length s is written $\varphi(s,y)$ then the density $u(x,t)$ of the pollen grains per unit length at time t satisfies the following equation:

$$u(x,t+s)dx = dx\int_{-\infty}^{\infty} u(x-y,t)\varphi(s,y)dy,$$

where the functions u and φ can be assumed to be smooth.

The Taylor equation gives the heat equation

$$u_t = \frac{1}{2}Du_{xx}.$$

If α is the initial state of a grain so that $u(x,0) = \delta(x-y)$ then we have

$$u(x,t) = \frac{1}{\sqrt{2\pi Dt}}\exp\left[-\frac{(x-y)^2}{2Dt}\right]$$

which turns out to be the transition probability function of Brownian motion.

II. Lévy's Brownian motion

The multi-dimensional parameter Brownian motion $\{X(A), A \in R^d\}$ was introduced in 1945 in his paper " Sur le movement brownien dépendent de plusieurs paramètres, C.R. Acad. Sci. vol.220, 420-422".

Definition A system $\{X(A,\omega), A \in R^d\}$ of real valued random variables $X(A,\omega)$ with parameter space R^d is called a Lévy's Brownian motion or a Brownian motion with n dimensional parameter if it satisfies

1 $\{X(A,\omega)\} = X(A)$ is a Gaussian system,

2 $E[X(A)] = 0$,

3 Variance of $X(A) - X(B)$ is $\rho(A,B)$, where $\rho(A,B)$ denotes the distance between A and B.

Lévy discussed the multi-dimensional parmeter Brownian motion in the following literatures.

1948 Processus Stochastiques et Mouvement Brownien, Gauthier Villars, Paris

1950 Wiener's random function, and other Laplacian random functions, 2nd Berkeley

1953 Random Functions, General Theory with special reference to Laplacian random functions, Univ. of California publ.

1954 Le mouvement brownien. Fasc. CXXVI du Mémorial des Sciences mathématiques. Gauthier-Villars, Paris.

1954 A Special Problem of Brownian motion, and a general theory of Gaussian random functions, Proc. of the third Berkely Symp. on Math. Stats and Prob.

1955 Brownian motion depending on n-parameters : the particular case $n = 5$, Proc. of the 7th Symp. in Appl. Math. of AMS,1955, pp1-20.

1956 Random functions: A Laplacian random function depending on a point of Hilbert space, Univ. Califonia publ. Vol 2, No. 10, pp195-206

1959 Le movement Brownien fonction d'un point de la sphere de Riemann , Rend. Circ.Mat. Parlemo, Ser. 2, Vol. 8, pp297-310

1962 Le déterminisme de la fonction brownienne dans l'espace de Hilbert, Ann. Sciient. de l'Ec. Norm. Suppremier mémoire, 79, 377-98: second mémoire, 80, 193-212.

1965 Le mouvement brownien fonction d'un ou de plusiers paramètres. rend. di Matematica, 22, 24-101

1966 Fonctions Browniennes dans l'espace Euclidean et dans l'espace de Hilbert.

2 Levy's contribution

Lévy introduced n parameter Brownian motion in his 1945 paper. There the existence of Brownian motion was also breifly discussed although it is a very short paper.

By taking the parameter A

(ii) in the Hilbert space,

(iii) on the sphere,

he expressed $X(A) = \mu + \sigma\xi$, where μ is the conditional expectation, μ and ξ are independent.

In his 1948 book, *Processus Stochastiques et Mouvement Brownien, Gauthier Villars, Paris, 1948*, P. Lévy introduced the multi-dimensional parameter Brownian motion precisely.

For the simpler calculation, additional assumption $X(O) = 0$ was made in the definition of multi-dimensional parameter Brownian motion $X(A), A \in R^d$. Then the covariance is obtained as

$$\frac{1}{2}\{r(O,A) + r(O,B) - r(A,B)\}.$$

The existence of Lévy's Brownian motion $X(A)$ was proved by using the I. J. Schoenberg and L. Schwartz theorem.

In his paper, *Wiener's random function, and other Laplacian random functions, 2nd Berkeley, 1950*, $X(A) = X(t)$, where A runs through on the Euclidean plane and A is taken to be $(\cos t, \sin t)$, was mentioned that it is a stationary and periodic function.

Further properties of the periodic case was discussed in the paper *Random Functions, General Theory with special reference to Laplacian random functions, Univ. of California publ. 1953,*

Example. The Periodic Brownian motion was given by using the Lévy Brownian motion. Define

$X_1(t) = X(A), \ A = (cost, sint)$

$X_0(t) = X_1(t) - \mu$ is a period Laplacian function, stationary, where $\mu = \sigma\zeta$, the mean of $X_1(t)$ in a period.

It is important to mention the stochastic infinitesimal equation, introduced by P. Lévy,

$$\delta X(t) = \Phi\{X(\tau), \tau \leq t, Y, t, dt\}$$

$$\delta X(t) \sim \mu dt + \xi\sqrt{d\omega(t)}, \quad \text{for Gaussian case}$$

where

$$d\omega(t) = \sigma^2 dt + 0(dt).$$

In the paper *Brownian motion depending on n-parameters : the particular case* $n = 5$, *Proc. of the 7th Symp. in Appl. Math. of AMS,1955, pp1-20,* he introduced $M(t)$ process as follows.

Let A, B be in E_n, $X(A)$ be Lévy's Brownian motion and suppose $\mathcal{E} \subset E_n$ is given and that the values of $X(A)$ are known at every $A \in \mathcal{E}$.

The Lévy Brownian motion $X(A)$ is written in the canonical form $m + \sigma\xi$ where $m = E[X(A)|X(B)], B \in \mathcal{E}]$ and $\sigma = \sigma\{X(A)|\mathcal{E}\}$. If $\mathcal{E}$ increases (i.e obtain new information), σ is nonincreasing.

The mean value of $X(A)$ on the sphere Ω_t with center O and radius r is denoted by $M(t)$. Lévy discussed the continuity of $M(t)$, its multiple Markov property and stochastic differential equation.

For the case of even number n the covariance of $M(t)$ involves an elliptic integral. So he considered for the case of $n = 2p + 1$, especially for $n = 5$.

He generalized the above problem by taking A in a Hilbert space in the paper : *Random functions; A Laplacian random function depending on a point of Hilbert space, Univ. Califonia publ. Vol 2, No. 10, pp195-206, 1956.*

Let $X(A), A \in E_\omega$, be a Brownian function on the Hilbert space E_ω.

Assume that $\{A_n\}$ is everywhere dense in E_ω (coordinates are rational numbers and only a finite number of them are not zero). Define $X_n = X(A_n), n = 1, 2, \cdots$ and form

$$X_n = \sum_0^{n-1} a_{n,\nu} X_\nu + \sigma_n \xi_n,$$

where X_0 is given and ξ_n are independent standard Gaussian variables. $a_{n,\nu}$ and σ_n are known functions of A_n, such that

$$\sum_1^n a_n, \nu = 1, \ \sigma_n > 0.$$

$X(A)$ is not continuous in E_ω but may be defined as the limit of $X(H_n)$, as $n \to \infty$, where $A = (a_1, a_2, \cdots)$ and $H_n = (a_1, a_2, \cdots, a_n)$. $X(A)$ is almost surely continuous in any n dimensional space.

He proved a deterministic character of $X(A)$. If $X(A)$ is given in V_n where

$$V_n = \{A \mid R_1 \leq r(O, A) \leq R_2\}$$

then the information increases with n.

If $X(A)$ is given in V_ω it is known at every point of E_ω which has a distance to O less than R_1.

If $X(A)$ is given in a neighbourhood of sphere ($R_2 - R_1$ is very small), it is known on the interior of sphere.

Analytic character of $M(t)$ and generalized Dirichlet problem in a sphere Ω of the Hilbert space was also discussed.

He introduced white noise measure on the sphere in the paper: *Le movement Brownien fonction d'un point de la sphere de Riemann , Rend. Circ.Mat. Parlemo, Ser. 2, Vol. 8, pp 297-310, 1959.* Using white noise, the existence (indeed a realization) of Lévy's Brownian motion is guaranteed.

The Brownian motion $X(A), A$ being on the sphere, is defined as an integral over the semi-sphere with center A with respect to the white noise measure,

$$X(A) = cR^{\frac{2-N}{2}} \int_{S_A} \xi_M \sqrt{dS},$$

where $S(A) = \{M; r(A, M) \leq \frac{\pi}{2}R\}; r(A, B)$ is arc length between A and B, dS is the surface measure, i.e $dS = R^{N-1}d\omega$; $d\omega$ is a surface element of unit sphere and ξ_M is a standard Gaussian variable.

Thus

$$E[\{(X(A) - X(B)\}^2] = \frac{c^2}{\pi}\omega_N r(A, B),$$

where

$$c^2 = \frac{\pi}{\omega_N} = \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}-1}}.$$

Obviously $X(A)-X(B)$ is Gaussian $N(0, r(A, B))$, r being the distance measured along the geodesic. The existence of such $\{X(A)\}$ is obvious. Letting the radius tend to infinity, a Brownian motion is obtained. Some other properties are also obtained in this direction.

Later, let the radius tend to infinity to obtain Lévy's Brownian motion.

In the paper *Fonctions Browniennes dans l'espace Euclidean et dans l'espace de Hilbert, 1966*, he discussed the case where the parameter space is sphere. Since white noise indexed by a point of a sphere is well defined, white noise integral is well defined. In particular if integrand is constant, then we are given a random function with the covariance that agrees with Brownian motion. There $M(t)$ process is also discussed.

He mentioned the works of T. Hida, H.P. Mckean and N.N. Chentsov in this paper.

3 Some developements on Lévy's Brownian motion

There are many people who contributed on Lévy's Brownian motion. Among them we like to mention the following.

I. Hida's work on Lévy's Brownian motion

In Hida's 1960 paper Levy's $M(t)$ process with a parameter space $T = [0, \infty)$ was discussed.

$$M_N(t) = \int_{S_N(t)} X(A) d\sigma(A)$$

where $S_N(t)$ is the sphere in E^N with radius t and σ is a uniform measure on $\sigma(S_N(t)) = 1$.

For $N = 2p+1$, the canonical representation (due to Lévy)

$$M_N(t) = \int_0^t P_N(\frac{u}{t}) \dot{B}_0(u) du,$$

where

$$P_N(u) = \frac{2p}{\sqrt{\pi}} \sqrt{I_{2p}} \int_u^1 (1-x^2)^{p-1} dx$$

Hida studied the canonical representation including the case $N = 2p$. He discussed in the following procedure.

1. The properties of covariance functions of $M_N(t)$ for even and odd N was discussed.

2. Transformed $M_N(t)$ to $X(t) = e^{-t} M(e^{2t})$ to obtain a stationary Gaussian process.

3. Reduced $X_N(t)$ to $X_2(t)$ or $X_3(t)$ by the differential operators:

$$D_3 D_5 \cdots D_{2p+1} X_{2p+1}(t) = X_1(t),$$

$$D_4 D_6 \cdots D_{2p} X_{2p}(t) = X_2(t),$$

   where

$$D_N = C_N^{-1} e^{-(2N-3)t} \frac{d}{dt} e^{(2N-3)t}.$$

4. Proved that M_{2p+1} is $p+1$ multiple Markov for odd N.

5. Found the spectral density of $X_2(t)$ and obtain that of $X_{2p}(t)$ and used Fourier transform to have the canonical kernel.

6. Observed the representation and knew that $M_{2p}(t)$ was not multiple Markov, but a limiting process of n-ple Markov process with homogenious canonical kernel of degree zero.

II. Chentsov's integral

In the famous paper of Chentsov, he considered the Euclidean plane as a collection of lines and introduce measure $\mu(V) = \int_V d\varphi ds$ where V is a set of lines. And then Lévy Brownian motion is defined as a random field $X(A)$ by using Integral Geometry.

III. McKean's expansion

McKean (1963) has given the expansion of the Lévy Brownian motion as

$$X(A) = \sum_{k \leq \, dim \mathcal{B}_n} M_{nk}(r)\varphi_{nk}(\theta), \quad |OA| = r,$$

where $\mathcal{B}_n$ is a subspace of $L^2(S^{d-1})$, spanned by all spherical harmonic functions $\varphi_{nk}(\theta)$ of degree n ,

$$\varphi_{nk}(\theta) = e^{\pm in\theta} \prod_{i=1^{d-2}} (\sin\theta_{i+1})^{n_i} C^{\frac{n}{2}+\frac{1}{2}}_{n_{i+1}-n_i}(\cos\theta_{i+1}), \; n \geq n_{d-1} \geq \cdots n_1 \geq 0,$$

and $\sum n_i = n$ and $C_n^\nu(x)$ is a Gegenbauer polynomial, given by

$$C_n^\nu(x) = \frac{(-1)^n\Gamma(\nu+\frac{1}{2})\Gamma(n+2\nu)(1-x^2)^{\frac{1}{2}-\nu}}{2^n\Gamma(2\nu)\Gamma(n+\nu+\frac{1}{2})n!} D^n(1-x^2)^n + \nu - \frac{1}{2},$$

$$M_{nk}(r) = \int_{S^{d-1}} X(r,\theta)\varphi_{nk}(\theta)d\theta.$$

For $k=0, M_{n0}(r)$ is Lévy $M(t)$ process which is the average of $X(A)$, A runs through a sphere S_t with radius t;

Remark. Chentsov's representation is canonical, but Mc Kean's representation is non-canonical.

IV. Gangoli

Gangoli discussed the case where the parameter space is a homogeneous space. He is awarded Lévy's prize by this result.

V. Canonical representation of Lévy's Brownian motion on the manifold

Consider $X(A), A \in C$ where C is a simple smooth curve in C^3. $X(A)$ can be taken as $X(t)$ where a parameter t is an arc length. It can be expressed as

$$X(A) = X(t) = \int_C F(t,u)\dot{B}(u)du.$$

1. First, C is taken to be a circle. The canonical representation of a Brownian motion on a circle is obtained explicitly and it is a double Markov.

2. The Brownian motion on a simple curve C has a canonical representation and the kernel $F(t,u)$ is obtained.

3. The conditional expectation of the Lévy B.M at a point P

   $$E[X(P)|X(A), A \in C] = \int_{A\in C} f(P,A)X(A)dA = \int_{A\in C} f(P,x)X(x)dx$$

   The interesting result is that if C is open curve then $f(P,x)$ involves the delta function at the boundary. That is, if C is not closed, singularity arises at the end points.

4. Let $Y(C) = E[X(P)|X(A), A \in C]$.
   The variation $\delta Y(C)$ exists and it is obtained as

   $$\delta Y(C) = \int_C \{\delta f(C,s) - \kappa f(C,s)\delta n(s)\}X(s)ds + \int_C f(C,s)\frac{\partial}{\partial n}X(s)\delta n(s)ds,$$

5. The normal derivative of Lévy B.M. on C is neither an ordinary process nor a generalized process, however it is well defined as a generalized process over R^n.

(For details, we refer [18],[19].)

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