# ポアソンズの話題

**Si Si**

**愛知県立大学**

**数学史シンポシウム、 津田塾大学**

**概 要**

ポアゾンノイズについて、歴史をガウス型のノイズ（ホワイトノイズ）と比較しながら述べてみた。

## 1 Siméon Denis Poisson (1781-1840)

Poisson was born on 21 June 1781 in Pithiviers, France and passed away on 25 April 1840 in Sceaux (near Paris), France.

In **1789 July** French revolution started and then the situation of Poisson's family had changed since his father became the President of the district of Pithiviers as a consequence of his support on the revolution . From this position Poisson's father was able to influence the future carrier of his son.Poisson was sent to Fontainebleau,where his uncle was a surgeon, to become an apprentice srgeon by his father. However, he had no interest in the medical profession. So Poisson returned home from Fontainebleau having essentially failed to make the grade in his apprenticeship and his father had to think again to find a career for him.

In **1796** Poisson was sent back to Fontainebleau by his father to enroll in the **Ecole Centrale** there. He now showed that he had great talents for learning, especially **mathematics**. His teachers at the Ecole Centrale were extremely impressed and encouraged him to sit the entrance examinations for the **Ecole Polytechnique** in Paris. Although he had far less formal education than most of the young men taking the examinations he achieved the top place.

Few people can have achieved academic success as quickly as Poisson did. When he began to study mathematics in **1798** at the Ecole Polytechnique he was therefore in a strong position to cope with the rigours of a hard course, yet overcome the deficiencies of his early education.

His teachers **Laplace and Lagrange** quickly saw his mathematical talents. They were to become friends for life with their extremely able young student and they gave him strong support in a variety of ways.

A memoir on finite differences, written when Poisson was 18, attracted the attention of **Legendre**. However, Poisson found that **descriptive geometry**, an important topic at the Ecole Polytechnique because of Monge, was impossible for him to succeed with because of his inability to draw diagrams. This would have been an insurmountable problem had he been going into public service, but those aiming at a career in pure science could be excused the drawing requirements, and Poisson was not held back.

In his final year of study he wrote a paper on **the theory of equations and Bezout's theorem**, and this was of such quality that he was allowed to **graduate in 1800** without taking the final examination.

He proceeded immediately to **the position of repetiteur**in the Ecole Polytechnique, mainly on the strong recommendation of Laplace. It was quite unusual for anyone to gain their first appointment in Paris, most of the top mathematicians having to serve in the provinces before returning to Paris.

Poisson was named **deputy professor at the Ecole Polytechnique in 1802** and in **1806 he was appointed to the professorship at the Ecole Polytechnique** which Fourier had vacated when he had been sent by Napoleon to Grenoble.

In **1808** he published *Sur les inegalites des moyens mouvements des planetes* where he looked at the mathematical problems which Laplace and Lagrange had raised about perturbations of the planets. His approach to these problems was to use series expansions to derive approximate solution.

In **1809** he published two papers,

*Sur le mouvement de rotation de la terre* and

*Sur la variation des constantes arbitraires dans les questions de mecanique* which was a direct consequence of developments in Lagrange's method of variation of arbitrary constants which had been inspired by Poisson's 1808 paper.

In addition he published a new edition of *Clairaut's Theorie de la figure de la terre* in **1808**. The work had been first published by Clairaut in 1743 and it confirmed the Newton-Huygens belief that the Earth was flattened at the poles.

Poisson published his two volume treatis *Traite de mecanique* in **1811**which was an exceptionally clear treatment based on his course notes at the Ecole Polytechnique and in **1831** *Théorie mathématique de la Chaleur* was published.

In **1837**, the book *Recherches sur la probabilité des jugements en matiere criminelle et matiére civile*, an important work on probability was published, the Poisson distribution first appears. The Poisson distribution describes the probability that a random event will occur in a time or space interval under the conditions that the probability of the event occurring is very small, but the number of trials is very large so that the event actually occurs a few times. He also introduced the expression "law of large numbers". Although we now rate this work as of great importance, it found little favour at the time, the exception being in Russia where Chebyshev developed his ideas.

During this period Poisson studied problems relating to ordinary differential equations and partial differential equations. In particular he studied applications to a number of physical problems such as the pendulum in a resisting medium and the theory of sound. His studies were purely theoretical, however, for as we mentioned above, he was extremely clumsy with his hands.

Poisson was content to remain totally unfamiliar with the vicissitudes of experimental research. It is quite unlikely that he ever attempted an experimental measurement, nor did he try his hand at drafting experimental designs.

His first attempt to be elected to the Institute was in 1806 when he was backed by Laplace, Lagrange, Lacroix, Legendre and Biot for a place in the Mathematics Section. Bossut was 76 years old at the time and, had he died, Poisson would have gained a place. However Bossut lived for another seven years so there was no route into the mathematics section for Poisson. He did, however, gain further prestigious posts.In addition to his professorship at the Ecole Polytechnique, in 1808 Poisson became an astronomer at Bureau des Longitudes. In 1809 he added another appointment, namely that of the chair of mechanics in the newly opened Faculte des Sciences.

Malus was known to have a terminal illness by 1811 and his death would leave a vacancy in the physics section of the Institute. The mathematicians, aiming to have Poisson fill that vacancy when it occurred, set the topic for the Grand Prix on electricity so as to maximise Poisson's chances.

Poisson's name is attached to a wide variety of ideas, for example:- Poisson's integral, Poisson's equation in potential theory, Poisson brackets in differential equations, Poisson's ratio in elasticity, and Poisson's constant in electricity. However, he was not highly regarded by other French mathematicians either during his lifetime or after his death. His reputation was guaranteed by the esteem that he was held in by foreign mathematicians who seemed more able than his own colleagues to recognise the importance of his ideas. Poisson himself was completely dedicated to mathematics. Arago reported that Poisson frequently said:-

*Life is good for only two things, discovering mathematics and teaching mathematics.*

**Honours awarded to Simeon-Denis Poisson**

Fellow of the Royal Society Elected 1818
Fellow of the Royal Society of Edinburgh Elected 1820
Royal Society Copley Medal Awarded 1832
Lunar features Crater Poisson

Paris street names Rue Denis Poisson (17th Arrondissement)
Commemorated on the Eiffel Tower

**Publications**

He published between 300 and 400 mathematical works in all. Despite this exceptionally large output, he worked on one topic at a time.

# 2 Poisson approximation, Poisson distribution, Poisson process

Poisson introduced a particular sequence of Bernoulli cases of independent events $A_{nk}, k = 1, 2, ..n; n = 1, 2, ...,$ of the same proability $p_n$ which varies with the number $n$ of trails in such a way that the expectation of the number of occurences $S_n = \sum_{k=1}^{n} I_{A_{nk}}$ remains constant : $E[S_n] = np_n = \lambda$. Then, as $n \to \infty$,

$$P(S_n = j) \to \frac{\lambda^j}{j!} e^{-\lambda}.$$

Noting that

$$\frac{n!}{j!(n-j)!} p^j q^{n-j} \sim e^{-np} \frac{(np)^j}{j!}$$

for the probability of an event occuring $j$ times in $n$ trails with $n \geq 1, p \leq 1,$.
( 少確率の法則 or 少数の法則)

**Poisson process**

Let $X_1, X_2, \cdots$ be mutually independent random variables with the common exponential distribution . Thus the density function

$$f(x) = \alpha e^{-\alpha x}, \quad x \geq 0.$$

Set $S_0 = 0$, and

$$S_n = X_1 + X_2 + \cdots + X_n, \quad n = 1, 2, \cdots.$$

Let $N(t)$ be the number of indicies $k \geq 1$ such that $S_k \leq t$. The event $\{N(t) = n\}$ occurs iff $S_n \leq t$ and $S_{n+1} > t$. Since $S_n$ has the distribution function

$$G_n(x) = 1 - e^{-\alpha x} \left(1 + \frac{\alpha x}{1!} + \cdots + \frac{(\alpha x)^n}{n!}\right),$$

$$P\{N(t) = n\} = G_{n-1} - G_n = e^{-\alpha t} \frac{(\alpha t)^n}{n!}.$$

The process $N(t)$ is called a Poisson process.

The Poisson process has been found a good approximation to the process governing the instants of emissions of radioactive material. (see e.g. [5])

**Poisson bracket**

Let $u$ and $v$ be any functions of a set of variables . Then the expression

$$(u, v) = \sum_{r=1}^{n} \left( \frac{\partial u}{\partial q_r} \frac{\partial v}{\partial p_r} \right) - \left( \frac{\partial u}{\partial p_r} \frac{\partial v}{\partial q_r} \right)$$

is called a Poisson bracket (Poisson 1809; Whittaker 1944,). Plummer (1960) uses the alternate notation .

The Poisson brackets are anticommutative,

$$(u_l, u_m) = -(u_m, u_l)$$

If $A$ and $B$ are physically measurable quantities (observables) such as position, momentum, angular momentum, or energy, then they are represented as non-commuting quantum mechanical operators in accordance with Heisenberg's formulation of quantum mechanics. In this case,

$$[A, B] = AB - BA = i\hbar(A, B)$$

where $[A, B]$ is the commutator and $(A, B)$ is the Poisson bracket. Thus, for example, for a single particle moving in one dimension with position $q$ and momentum $p$,

$$[q, p] = qp - pq = i\hbar(q, p) = i\hbar,$$

where $\hbar = h/2\pi$.

**Poisson integral, Poisson kernel**

There are at least two integrals called the Poisson integral. The first is also known as Bessel's second integral,

$$J_r(z) = \frac{(\frac{1}{2}z)^r}{\Gamma(n+\frac{1}{2})\Gamma(\frac{1}{2})} \int_0^{\pi} \cos(z\cos\theta)\sin^{2r}\theta d\theta,$$

where $J_r(z)$ is a Bessel function of the first kind and $\Gamma(x)$ is a gamma function. It can be derived from Sonine's integral. With $n = 0$, the integral becomes Parseval's integral.

In complex analysis, let $u : U \to R$ be a harmonic function on a neighborhood of the closed disk $\bar{D}(0, 1)$, then for any point $z_0$ in the open disk $D(0, 1)$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\psi}) \frac{1 - |z_0|^2}{|z_0 - e^{i\psi}|^2} d\psi$$

In polar coordinates on $\bar{D}(0, R)$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} K(r, \theta)\phi(z_0 + re^{i\theta}) d\theta,$$

where $R = |z_0|$ and $K(r, \theta)$ is the **Poisson kernel.**

**Poisson summation formula** (Functional analyssis)

Let $f : R \to R$ be an integrable function and let

$$\hat{f}(\xi) = \int_R e^{-2\pi\xi x} f(x)dx, \quad \xi \in R$$

be its Fourier transform. The Poisson summation formula is the assertion that

$$\sum_{n\in Z} f(n) = \sum_{n\in Z} \hat{f}(n).$$

whenever $f$ is such that both of the above infinite sums are absolutely convergent. This is useful because it establishes a correspondence between Fourier series and Fourier integrals. (See Feller II, Chapt XIX and K. Yosida : Functional analysis, Chap. VI)

# 3 Poisson noise

### Mathematical approach

We have discussed some historical aspects of the theory of innovation at the symposium of History of Mathematics, Tsudajuku University in 2004, and discussed on Lévy process, Lévy decomposition, Compound Poisson process reffering Lévy's literature :

*Théorie de l'addition des variables aléatoires. Gauthier-Villars. (1954. in particular Chapt. VII.)*

We now skip a general theory of innovation, and consider possible kinds of innovations of a stochastic process. Innovation should consists of independent random variables indexed by the time $t$.

A stochastic process with independent values at every moment is, in fact, a generalized stochastic process. Assume that its probability distribution is invariant under the time shift, namely the process is stationary. The assumption that it is temporary homogeneous seems natural. If we sum up the random elements (that are elemental random elements), that is, if we integrate those elemental random elements, we are given an additive process. Assume that the additive process is continuous in probability. And we are given a Le'vy process. Let it be denoted by $Z(t)$. The Le'vy decomposition of $Z(t)$ is expressed as a sum

$$Z(t) = mt + \sigma \; B(t) + X(t),$$

where m is a constant, which may be deleted, $B(t)$ is a Brownian motion and $X(t)$ is a compound Poisson process. More precisely, it is a superposition of various Poisson processes $X_u(t)$ parametrised by $u$ that describes the magnitude of jump. Thus, we claim that innovation comes from elemental additive processes $B(t)$ and a system $X_u(t), u \in \mathrm{U}$, where $U$ is a subset of $R$. The innovation in question is obtained simply by taking the time derivative. Hence, we are let to investigate Gaussian white noise (simply called white noise) and Poissonj noise and their functions (in fact, functionals).

Poisson noise is called **Shot noise**, in the field of physics and engineering, which is discussed by the following scientists and others.

1909 N. Campbell, The study of discontnuous phenomena, Proc. Cambr. Phil. Soc., vol.15., 117-136.

Discontinuities in light emmision, Proc. Cambr. Phil. Soc., vol.15., 310-328.

(Basic shot-noise statistics, namely mean and variance)

1918 : W. Schottky, Shot noise was investigated on spontaneous current fluctuation in electric conductors.

1944-45 : S.O. Rice, Mathematical analysis of random noise, Bell Syst. Tech. J., vol. 23, 283-322.

He gave an extensive analysis when the underlying Possion process has a constant intensity, say $\lambda$. There it was shown that when $\lambda \to \infty$ the probability distribution of shot noise tends to a normal distribution.

We refer the following literatures in which Poisson noise is discussed in mathematical approach.

Gel'fand Vilenkin, Generalized functions Vol. IV ; Generalized random processes with independent values at every point1964 (in Russian 1961)

T. Hida, Generalized Brownian functionals (1970)

I. Kubo, Y. Ito and others study Poisson noise analysis following Hida approach like as White noise analysis.

1. A Poisson noise is the time derivative of Poisson process $P(t)$ and is denoted by $\dot{P}(t)$.
2. Its characteristic functional $C_P(\xi), \xi \in E$, is given by

$$C_P(\xi) = \exp[\lambda \int (e^{i\xi(t)} - 1)dt]; \ \xi \in E, \tag{1}$$

   where $E$ is a nuclear space which is dense in $L^2(R)$.
3. $\dot{P}(t)$ is a *stationary* generalized stochastic process with independent values at every instant $t$.

# 4 Recent developement of Poisson noise

To fix the idea, consider one dimensional parameter case and let the time parameter space be a compact set, say $I = [0,1]$. In this case, $C_P^I(\xi)$ is continuous in $E_1 = L^2(I)$, so that there is a Gel'fand triple of the form

$$E_1 \subset L^2(I) \equiv E_0 \subset E_1^*. \tag{4.1}$$

A Poisson measure is now introduced on the space $E_1^*$.

Define $P(t,x) = \langle x, \chi_{[0,t]}\rangle, 0 \le t \le 1,\ x \in E_1^*$, by a stochastic bilinear form, where $\chi$ is the indicator function. Then, $P(t,x)$ is a Poisson process with parameter set $[0,1]$.

Let $A_n$ be the event on which there are $n$ jump points over the time interval $I$. That is

$$A_n = \{x \in E_1^*; P(1,x) = n\}, \tag{4.2}$$

where $n$ is any non-negative integer.

Then, the collection $\{A_n, n \ge 0\}$ is a partition of the entire space $E_1^*$. Namely, up to measure 0, the following relations hold:

$$A_n \bigcap A_m = \phi,\ n \ne m;\ \bigcup A_n = E_1^*. \tag{4.3}$$

Given $A_n$, the conditional probability $\mu_P^n$ is defined :

$$\mu_P^n(A) = \frac{\mu_P(A_n \cap A)}{\mu_P(A_n)},\ A \subset E_1^*.$$

For $C \subset A_k$, the probability measure $\mu_P^k$ on a probability measure space $(A_k, \mathbf{B}_k, \mu_P^k)$, is such that

$$\mu_P^k(C) = \mu_P(C|A_k) = \frac{\mu_P(C)}{\mu_P(A_k)},$$

where $\mathbf{B}_k$ is the sigma field generated by measurable subsets of $A_k$, determined by $P(t,x)$.

For $k = 0$, the measure space is $(A_0, \mathbf{B}_0, \mu_P^0)$ is trivial, where

$$\mathbf{B}_0 = \{\phi, A_0\} \bmod \mu_P^0,$$

and $\mu_P^0(A) = 1$.

Let $x \in A_n, n \ge 1$, and let $\tau_i \equiv \tau_i(x), i = 1,2,...,n$, be the order statistics of jump points of $P(t)$:

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = 1.$$

(The $\tau_k$'s are strictly increasing almost surely.) Set

$$X_i(x) = \tau_i(x) - \tau_{i-1}(x),$$

so that

$$\sum_1^{n+1} X_i = 1.$$

**Proposition** (Ref. [13]) On the space $A_n$, the (conditional) probability distribution of the random vector $(X_1, X_2, ..., X_{n+1})$ is uniform on the simplex

$$\sum_{j=1}^{n+1} x_j = 1, \quad x_j \geq 0.$$

In this sense, Poisson process enjoys a sort of *optimality* (maximum entropy).

**Cororally** (Ref. [13]) The probability distribution function of each $X_j$ is

$$1 - (1-u)^n, \quad 0 \leq u \leq 1.$$

**Proposition** (Ref. [13]) The *conditional characteristic functional*

$$C_{P,n}(\xi) = E[e^{i\langle \dot{P}, \xi\rangle} | A_n] \tag{4.4}$$

is obtained as

$$C_{P,n}(\xi) = \left(\int_0^1 e^{i\xi(t)} dt\right)^n. \tag{4.5}$$

**Proposition** (Ref. [15]) The conditional probability measure $\mu_P^n$, defined on the measure space $(A_n, \mathbf{B}_n, \mu_P^n)$, is invariant under the symmetric group $S(n+1)$ acting on $(X_1, X_2, ..., X_{n+1})$.

**Remark** Noting that $\mathrm{P}(A_n) = \frac{\lambda^n}{n!} e^{-\lambda}$, it is proved that

$$\begin{aligned} \sum_0^\infty C_{P,n}^I(\xi) \frac{\lambda^n}{n!} e^{-\lambda} &= \sum_0^\infty \left(\int_0^1 e^{i\xi(t)} dt\right)^n \frac{\lambda^n}{n!} e^{-\lambda} \\ &= \exp\left(\lambda \int_0^1 (e^{i\xi(t)} - 1) dt\right). \end{aligned}$$

which is in agreement with the characteristic functional of Poisson noise without any restriction

# 5 Characterization of Poisson noise

Some properties reviewed in the previous section are taken to be characteristics of Poisson noise. Among others, uniform probability distribution on a simplex and the invariance of the conditional Poisson distribution under the symmetric group are significant to characterize a Poisson noise.

With this fact in mind, we are going to construct a Poisson noise in such a way that we decompose a measure space defining the noise and on each component of the decomposition we can see those characteristics in a visualized manner.

Also our method can be compared to the approximation to Gaussian white noise by inductive limit of spheres, where one can see the characteristics; rotation invariant and maximum entropy. The comparison is successful to observe the two noises step by step. There latent properties appear explicitly in front of us.

We first observe the Gaussian case by reviewing the result by Hida-Nomoto (1964).

G.0) Start with a measure space $(S_1, d\theta)$, where $S_1$ is a circle and $d\theta = \frac{1}{2\pi}$.

G.1) Take a measure space $(\tilde{S}_n, \sigma_n)$, where $\tilde{S}_n$ is the $n$-dimensional sphere excluding north and south poles; and where $d\sigma_n$ is the uniform probability measure on $\tilde{S}_n$.

G.2) Define the projection $\pi_n : \tilde{S}_{n+1} \to \tilde{S}_n$ in such a way that each longitude is projected to the intersection point of the longitude and the equator. Thus, $\tilde{S}_n$ is identified with the equator of $\tilde{S}_{n+1}$. Take the radius of $\tilde{S}_{n+1}$ be $\sqrt{n}$.

G.3) Given the measure $\sigma_n$ on $\tilde{S}_n$ which is the uniform probability measure, i.e. invariant under the rotation group $SO(n+1)$. Then the measure $\sigma_{n+1}$ is defined in such a way that

$$\sigma_{n+1}(\pi_n^{-1}(E)) = \sigma_n(E), E \subset \tilde{S}_n$$

and that $\sigma_{n+1}$ gives the maximum entropy under the above restriction.

G.4) As a result $\sigma_{n+1}$ is proved to be the uniform distribution on $\tilde{S}_{n+1}$ and is invariant under the rotation group $SO(n+2)$.

G.5) The inductive limit of the measure space is defined by the projections $\pi_n$, and actually a measure space $(\tilde{S}_\infty, \sigma)$ is defined

$$(\tilde{S}_\infty, \sigma) = \text{ind}\cdot\lim\ (\tilde{S}_n, \sigma),$$

where "ind·lim" denotes the inductive limit.

G.6) The limit can be identified with the white noise measure space $(E^*, \mu)$ with the characteristic functional $e^{-\frac{1}{2}\|\xi\|^2}$, where $E^*$ is the space of generalized functions.

In parallel with Gaussian case, we can form a Poisson noise by an approximation where optimality (maximum entropy) and invariance under the symmetric group are effectively used.

P.0) Start with a probability space $(\Delta_1, \mu_1)$,
where $\Delta_1 = \{(x_1, x_2); x_1 + x_2 = 1\}$ and $\mu_1$ is the Lebesgue measure,

P.1) Define a probability space $(\Delta_n, \mu_n)$, where $\Delta_n$ is an Euclidean $n-$simplex,

$$\Delta_n = \{(x_1, \cdots, x_{n+1}),\ x_i \geq 0,\ \sum_1^{n+1} x_i = 1\} \subset R^{n+1}.$$

P.2) Define $\pi_n$ to be the projection of $\Delta_{n+1}$ down to $\Delta_n$ which is a side simplex of $\Delta_{n+1}$, determined as follows. Given a side simplex $\Delta_n$ of $\Delta_{n+1}$. Then, there is a vertex of $\Delta_{n+1}$ which is outside of $\Delta_n$, let it be denoted by $v_n$. The projection $\pi_n$ is a mapping defined by

$$\pi_n : \overline{v_n x} \to x.$$

where $\overline{v_n x}$ is defined as a join connecting the $v_n$ and a point $x$ in $\Delta_n$.

P.3) We introduce a probability measure $\mu_{n+1}$ under the requirements that

$$\mu_{n+1}(\pi_n^{-1}(B)), = \mu_n(B), \quad B \subset \Delta_n$$

and that $\mu_{n+1}$ has maximum entropy.

P.4) Since there is a freedom to choose a side simplex of $\Delta_{n+1}$, the rquirements on $\mu_{n+1}$ in P.3, the measure space $(\Delta_{n+1}, \mu_{n+1})$ should be invariant under symmetric group $S(n+2)$ which acts on permutations of coordinates $x_i$.

P.5) We can form $\{(\Delta_n, \mu_n)\}$ successively by using the projection $\{\pi_n\}$, where $\mu_n$ is to be the uniform probability measure on $\Delta_n$. Set

$$\Delta_\infty = \cup_n \Delta_n,$$

$$\mathbf{B}(\Delta_\infty) = \sigma\text{−field generated by} \cup A_n,\ A_n \in \mathbf{B}(\Delta_n),$$

and

$$\mu(A) = \sum_n p_n \mu_n(A \cap \Delta_n) \quad A \in \mathbf{B}(\Delta_\infty).$$

P.6) We have not yet specified $p_n$ in P.5. Thus we now set $p_n = \frac{\lambda^n e^{-\lambda}}{n!}, \quad \lambda > 0$, which has been determined in [15]. In fact, there we have used the reproducing property of a Poisson distribution, or equivalently Poisson variables can be imbedded in a Poisson process.

**Theorem**
i) The measure space $(\Delta_n, \mu_n)$ is isomorphic to the measure space $(A_n, \mu_p^n)$ defined in Section 2.
ii)The weighted sum $(\Delta_\infty, \mu)$ of measure spaces $(\Delta_n, \mu_n)$, $n = 1, 2, \cdots$ is identified with the Poisson noise space.

Proof. According to the facts P.1 to P.6, the theorem can be proved.

What have been investigated are explained as follows. Basic noises, that is Gaussian and Poisson noises, have

1) optimality in randomness, which is expressed in terms of *entropy*, and
2) invariance under the transformation group; each noise has its own characteristic *group* (one is continuous and another is discrete).

Such a principle has been discovered in the course of constructing finite dimensional approximation and partition for the two noises in question, respectively.

# 参考文献


[1] S. D. Poisson, Recherches sur la probabilité des jugements en matière criminelle et en maitière civile", 1837.

[2] R. von Mises, Probabuility, statistics and truth. 1928, J. Springer. 3rd German ed. 1951. english translation 1939 by J. Neyman et al; and Dover Pub. 1957.

[3] A. Einstein, Über einen die Erzeung Verwandlung des Lichtes betreffenden heuristischen Gesichtpukt. (光の発生と変換に関する一つの発見的な見地について). Ann. der physik. 17 (1905), 132-148.

[4] J. L. Doob, Stochastic processes, Wiley. 1953, Wiley.

[5] J.R. Klauder and E.C.G. Sudarshan, Fundamentals of Quantum Optics. Benjamin. 1968.

[6] W. Feller, An introduction to probabilty theory and its applications. voil. I and voil. II. Wiley.

[7] L. LeCam. Pacific Journal paper.

[8] H. Weyl, Classical group. Princeton Univ. Press. 1939.

[9] P. Lévy, Théorie de l'addition des variables aléatoires. Gauthier-Villars. 1937. 1954. in particular Chapt. VII.

[10] N. Wiener, The Discrete chaos, Ame. J. Math. 65 (1943), 279-298..

[11] T. Hida and Si Si, Lectures of white noise functionals. World Scientific Pub. Co. 2005.

[12] Si Si, Note on fractional power distribution. Lecture at Solvay Inst., March 2005.

[13] Si Si, Effective determination of Poisson noise. IDAQP 6 (2003), 609-617.

[14] Si Si, Note on Poisson noise, Quantum Information and Complexity, World Sci. 2004, 411-425.

[15] Si Si, A. Tsoi and Win Win Htay, Invariance of Poisson noise, Stochastic Analysis: Classical and Quantum. World Sci. 2004. (to appear)

[16] Si Si, Characterization of Poisson noise, (preprint)