In this article we endeavor to test the validity of Euler's method against analytical means. As a subject, a system of decaying nuclei were used with given Ordinary Differential Equations. Eventually, it was found that Euler's method is a useful algorithm in certain situations, but not perfect.

Section I - Introduction

When discussing stochastic behavior, one must envision a process that is governed by random probability.

As oxymoronic as this may seem, where governing implies structure and randomness implies chaos, in this way it is possible to see how order and disorder interact in the natural world. For this article, the following closed system was considered:

Imagine that there are two balls confined to one vertical axis of motion, *x*. Both start at rest and are subjected to the force of gravity, falling at a rate designated by *g*. A diagram is provided below as Figure One, and it shows that Ball 2 (green) is higher than Ball 1 (blue). It also shows that there is a floor in this system.

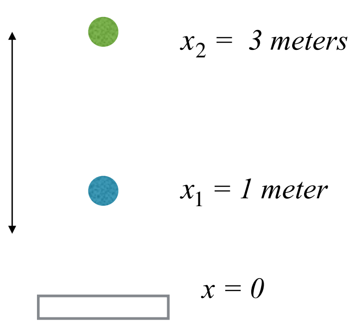

Figure One

So the two balls are falling. Clearly this scenario will involve a series of collisions, but it is important to consider the constraints involved here.

To start off, it is a fact that Ball 2 will never hit the ground. After all, since the balls are confined to the x-axis Ball 1 is always located between Ball 2 and the ground. This fact becomes important later on, when modeling the interactions.

Next, consider the collisions that Ball 1 may experience. As a result of being on the bottom, it may collide with either and the ground or Ball 2. Collisions with the ground can be described by traditional bouncing, which is governed by conservation laws.

Closed systems have the noteworthy property of conserving momentum. This, coupled together with the Law of Conservation of Energy, allow us to make useful calculations about the state of the system as time processes.

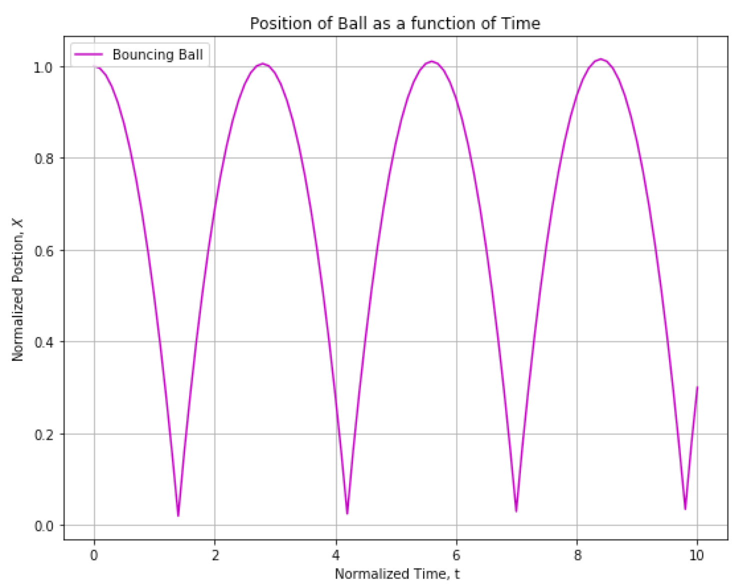

Figure Two: Trajectory of a bouncing ball.

Finally, the topic of chaotic behavior must be addressed. Traditional and unimpeded bouncing is periodic when in situations like this, where we only consider elastic colliding. This is shown in Figure Two, which is just provided as an example and does not relate to the system. In Figure Two, the bouncing is periodic, but also *discontinuous*. For every collision with the ground, the derivative has a jump discontinuity.

Something similar occurs between ball-to-ball collisions as a consequence of the momentum equation. It does, however, change the position graph significantly. Those equally spaced bounces are lost when another ball is introduced. Indeed, when we go further and even vary the relative masses of the two balls, the more changes can be seen.

The goal, then, becomes simple: design a model with which one can analyze the 2-ball system. From there, seek to understand how the system can be made chaotic. So far, nothing has indicated a lack of order. Jump discontinuities ruin the smoothness of the position function, but the motion is still predictable.

Section II - Method

To begin, it must be known that our interest is in general models. Naturally, this means that our variables need to be normalized. To figure out how to go about this, it is essential to acknowledge the parameters that we'll use.

Namely, when talking of trajectory it should be apparent that we'll need a position variable, *x*. Along that line of thinking, there should be quantities for time, *t,* and velocity, *v*.

The two balls also have their own respective masses, m_1 and m_2.

Now, there are also parameters that are defined by these quantities: energy and momentum, *E* and *p*. The current task is to ensure that each of these variables are unit-less.

The system itself has initial conditions. As stated before: $v_1 = v_2 = 0$ m/s, but also $x_1 = 1$ meter and $x_2 = 3$ meters. The mass of Ball 2 will fluctuate relative to to Ball 1's, which will stay fixed as $m_1 = 1\,\text{kg}$. To normalize, the following variables will be applied.

$$x_0 = \frac{E}{mg}, v_0 = \sqrt{E/m}, \quad m = m_1 + m_2,$$

$$t_0 = \sqrt{\frac{E}{mg^2}} \qquad \text{Eq. 1}$$

Now the normalized variables will take the form $\bar{i} = i/i_0$, where i represents a general variable. Since the variables E and p are composed of the other normalized parameters, their adjusted versions are equations. However, it is important to note that energy is normalized as $\bar{E} = 1$. In order to normalize the other variables, we need a value for E. Finding E requires that we fully define the variables. So far, the only thing that remains undefined is m_2. For now, allow $m_2 = 0.5m_1 = 0.5$ kg. Now, we split Energy into two parts: Kinetic (T) and Potential (U) Energy.

$$E = T + U$$

$$E = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + m_1gx_1 + m_2gx_2 \quad \text{Eq. 2}$$

Well, since the system conserves energy, we know that the maximum energy can be found when T = 0 or U = 0. Initially, both velocities are 0, so using that condition leaves us with potential energy terms.

$$E = g(m_1 x_1 + m_2 x_2) = g(1 * 1 + \frac{1}{2} * 3) \quad \text{Eq. 3}$$

$$E = \frac{5}{2} g$$

$$\bar{x_1} = \frac{x_1 * mg}{E} = \frac{1 * 3/2 * g}{5/2g} \quad \text{Eq. 4}$$

$$\bar{x_1} = 3/5, \quad \bar{x_2} = 9/5$$

This method is used throughout this discussion to find new energies for new cases. Notice that the acceleration of gravity cancels out in the normalized values. Constants tend to normalize to 1. Also of note are the new adjusted equations.

$$\bar{E} = \frac{m_1}{2m}\bar{v_1}^2 + \frac{m_2}{2m}\bar{v_2}^2 + \frac{m_1}{m}\bar{x_1} + \frac{m_2}{m}\bar{x_2} = 1$$

$$\bar{p} = \frac{m_1}{m}\bar{v_1} + \frac{m_2}{m}\bar{v_2} \quad \text{Eqs. 5 and 6}$$

From here, it should be clear that this is a system of equations. To simplify things, consider how the equations change during collisions. When Ball 1 and 2 collide, they must be at the same location, $\bar{x}$. Also, using the stipulation that $\bar{p_0} = \bar{p_f}$ due to the conservation of momentum, the second equation arises.

$$\bar{v_2}^2 = \frac{2m}{m_2}(1 - \bar{x}) - \frac{m_1}{m_2}\bar{v_1}^2 \quad \text{Eq. 7}$$

$$\bar{v_2} = \frac{m}{m_2}\bar{p_0} - \frac{m_1}{m_2}\bar{v_1} \quad \text{Eq. 8}$$

So, the system is defined. With two equations and two variables we can solve it and move forward. To monitor trajectories between collisions, the following is used, found through the application of the difference midpoint method:

$$v_{1,i+1} = v_{1,i} - g\Delta t \quad \text{Eq. 9}$$

$$x_{1,i+1} = x_{1,i} - \frac{v_{1,i+1} + v_{1,i}}{2}\Delta t \quad \text{Eq. 10}$$

$$v_{2,i+1} = v_{2,i} - g\Delta t \quad \text{Eq. 11}$$

$$x_{2,i+1} = x_{2,i} - \frac{v_{2,i+1} + v_{2,i}}{2}\Delta t \quad \text{Eq. 12}$$

The algorithm, now, is straightforward. Use Eqs. 9-12 to trace paths until a collision occurs, then use Eqs. 7 and 8 to solve for the resultant velocities. The trajectory equations don't care about the constraints on our system. If left alone, the x_1 and x_2 paths will cross each other. It is vital that a clause is implemented to ensure that if Ball 2's location is ever less than or equal to Ball 1, a collision must have occurred and the loop must stop.

With a step of dt = 0.1, and maximum walk of lastStep = 100, the plots will resemble Figure 2 in that they'll terminate at roughly $\bar{t} = 10$. This is how the comparisons will be made. To envision chaos, a Poincare section will be used.

This is a plot in which velocity and position are plotted against each other. The question is how chaos will manifest in such a graph.

Section III - Results

Case 1 defines the following relationship between Ball 1 and 2: $m_2 = 0.5m_1 = 0.5$. It's difficult to envision what this might mean. Intuition says that if the top ball is much heavier than the bottom 1, it's unlikely that Ball 1's motion will carry on for very long.

If the top ball is much lighter, there will be a lot more bouncing in the system since the bottom ball will sill have the floor to bounce off.

To maintain consistency, the three position plots will be given. From there, the three Poincare plots. From there the three autocorrelation plots. Each should offer information on whether we're dealing with stochastic behavior or not.

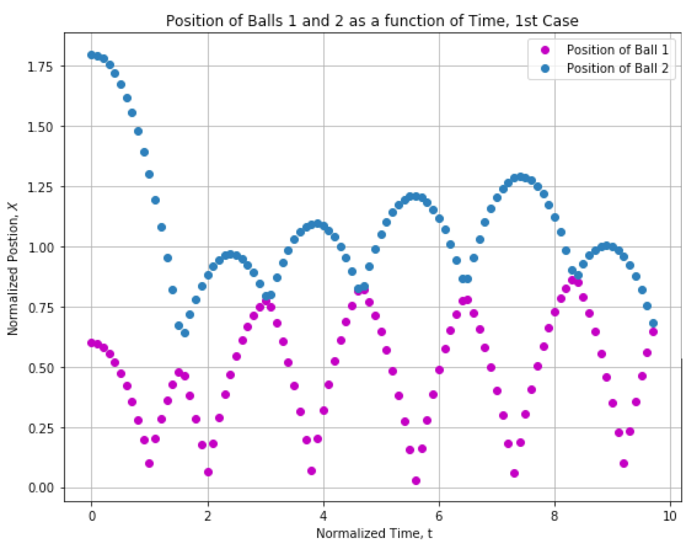

Figure Three: Positions of Balls 1 and 2 plotted against time. In which $m_2 = 0.5m_1$.

Figure Three depicts the collisions between Balls 1 and 2. Since the governing equations were normalized, the influence of mass is translated to the relative heights.

Ball 2, here, consistently collided with Ball 1 at its peak, where it's velocity would be nearly 0.

On the time interval, 5 collisions and their altered paths are shown. The type of collisions are are notably different from those in the next case. The method used to find new normalized initial heights is similar to what was outlined in Eqs. [], though adjusted for a different Energy.

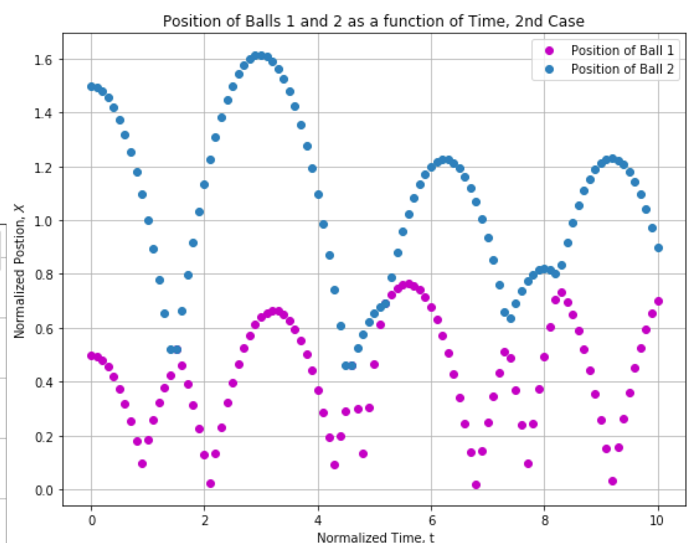

Figure Four: Positions of Balls 1 and 2 plotted against time. In which $m_2 = m_1$.

In this plot, we see five collisions and their resultant trajectories, and can see how the two cases compare. Where in case 1, Ball 2 has a steady pattern up until the last collision, the second case has Ball 2 bouncing in very odd ways. The two balls tend to meet in different locations as opposed to the trend of colliding at Ball 1's apex.

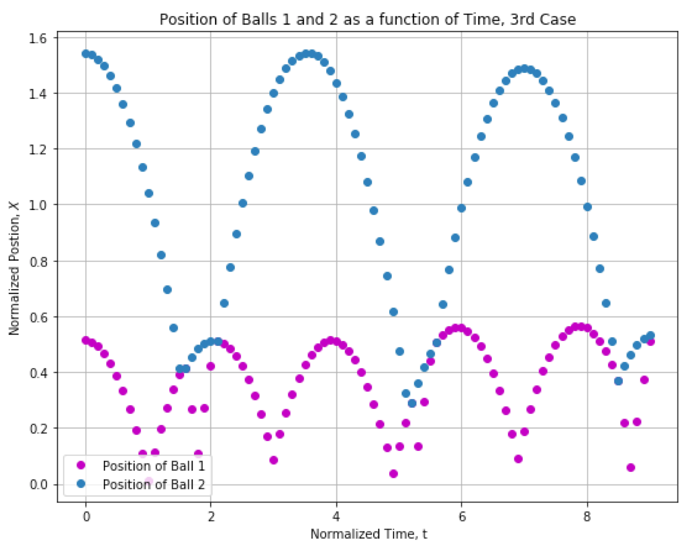

Figure Five: Positions of Balls 1 and 2 plotted against time. In which $m_2 = 0.9m_1$.

It's certainly comforting to see how, with a case so similar to case 2, the plots tend to resemble each other in certain aspects. There are five collisions here as well, none of them landing in the same spot. Already it's evident that the first case is the outlier in the trio. Still, nothing to concretely indicate a lack of order. The first case has collisions every 1.5 t0. The second and third case didn't have such a period, though the pattern Ball 2 having a big jump followed by a little jump is consistent in both.

Next up are the Poincare sections.

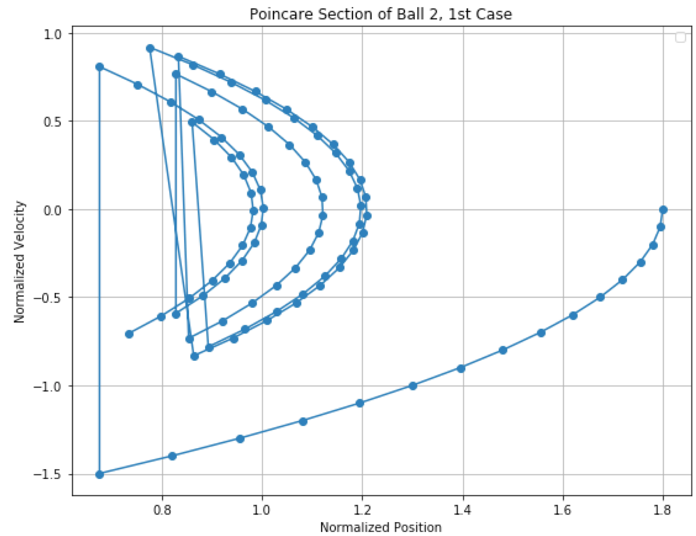

Figure Six: Poincare Plot of Ball 2, velocity plotted against position. Case 1

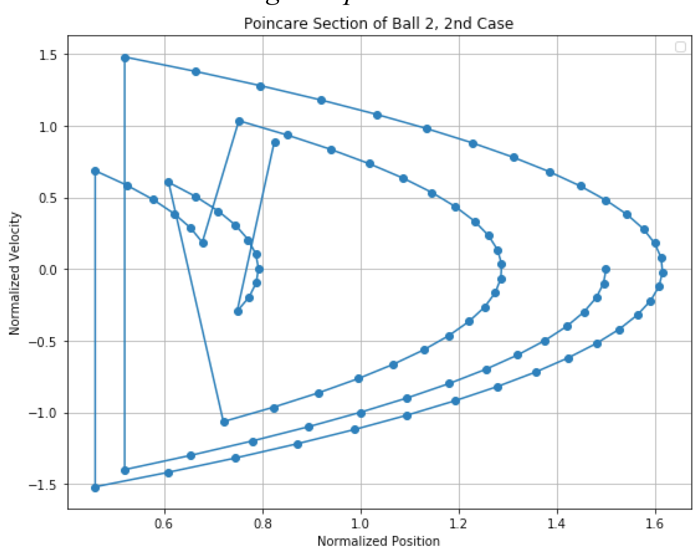

Figure Seven: Poincare Plot of Ball 2, velocity plotted against position. Case 2

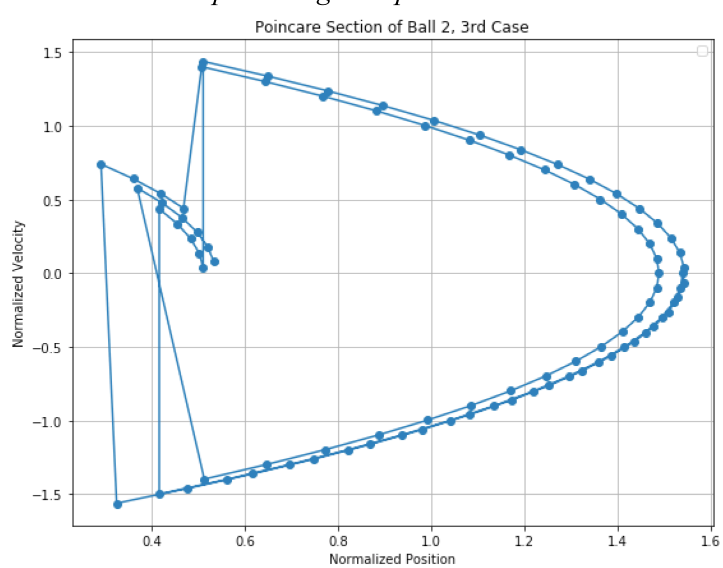

Figure Eight: Poincare Plot of Ball 2, velocity plotted against position. Case 3

Understand that our considerations right now are Ball 2. The Poincare plot is a bit tricky to read. It is, as I understand it, a method used describe the dynamics of a system. Technically, the Position vs Time plots strive to do the same thing. These plots remove the time aspect and directly compare location and velocity.

There is something to be said about what order and chaos even are. To be considered ordered, the behavior of the system should be, in some way, predictable. In this way, a look at Figure Six shows that there is a path being traced. A sort of spiral pattern is seen, wholly driven by the trends seen in Figure Three. Figure Eight seems to be beginning such a pattern, but lacked the time to complete it.

Figure Seven is important. It represents the case where Ball 1 and Ball 2 are the same mass, and shows a distinct lack of predictability. Tracing the line, there is no clear pattern between jumps. The discontinuities just seem to happen, and the resultant velocities differ due to the variety of interactions seen in Figure Four. From these plots, Case 2 shows stochastic behavior.

Next would be the autocorrelation function, but it was difficult to implement and so the theory will be addressed instead.

$$C(\tau) = \int_0^\infty [x(t) - \bar{x}][x(t+\tau) - \bar{x}]dt \quad \text{Eq. 13}$$

The goal of the autocorrelation function is to provide a method of comparison. With it, one would see how a signal (collection of data) compares with itself at a lagged time. This lagging parameter is the τ seen in Eq. 13. Depending on the result, something can be said about the stochasticity of the system.

With normal and ordered motion, the correlation function would be oscillatory. This makes sense, as the integrand is a convolution of two oscillatory functions. For the first case, referring back to Figure Three, it is expected that $C(\tau)$ would oscillate for both balls. The motion in that case is steady and predictable.

For case 2, $C(\tau)$ would take a different form. Being chaotic, the position functions are no longer very periodic, so the correlation function would decay or grow at a high rate. The point of this is to show that there is another, more definitive way, of finding chaos in systems.

Section III - Conclusion

For our very specific system of two balls bouncing on a restricted coordinate system, we found that the case in which the balls are identical provides chaotic motion.

However, that wasn't the goal of the experiment. The point was to see how chaotic motion looked like and the plots in Case 2 do a good job of doing so. The behavior seen in the Poincare plots is general. A lack of predictability between x and v values should always allude to chaotic motion.

The experiment would've been complete with a series of autocorrelation plots. That's on me.