University of Patras

Computer Engineering and Informatics Department

Probabilistic Techniques and Randomized Algorithms

Author:
Tsamopoulos Angelos
AM:1072591

Professor:
Nikoletseas Sotiris

Contents

1 Problem 1 **1**

2 Problem 2 **3**

3 Problem 3 **4**

4 Problem 4 **5**

5 Problem 5 **6**

6 Problem 6 **8**

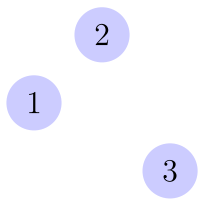
7 Problem 7 **9**

8 Problem 8 **11**

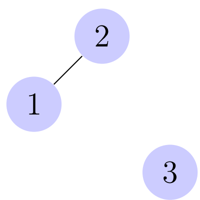
1 Problem 1

The graph $G_{3,\frac{2}{3}}$ has 3 vertices and the probability for an edge to exist is $p = \frac{2}{3}$. Therefore, the probability for an edge not to exist is $1 - p = \frac{1}{3}$. In this case, the amount of edges that can exist is $\binom{3}{2}$ (i.e. for an edge to occur we need 2 vertices). The sample space of this graph will consist of $2^3 = 8$ sample points:

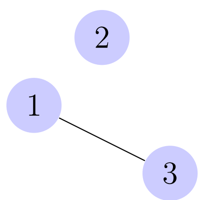
(Let q_i i $\in [1, 8]$ be the probability of each sample point.)

1. 

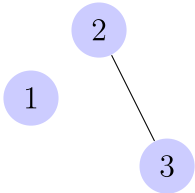
$$q_1 = (1 - p) \cdot (1 - p) \cdot (1 - p) = (1 - p)^3 = \frac{1}{27}$$

2. 

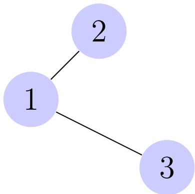
$$q_4 = p \cdot (1 - p) \cdot (1 - p) = p \cdot (1 - p)^2 = \frac{2}{27}$$

3. 

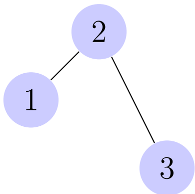
$$q_4 = p \cdot (1 - p) \cdot (1 - p) = p \cdot (1 - p)^2 = \frac{2}{27}$$

4. 

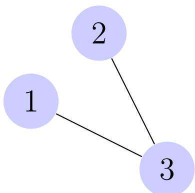
$$q_4 = p \cdot (1-p) \cdot (1-p) = p \cdot (1-p)^2 = \frac{2}{27}$$

5. 

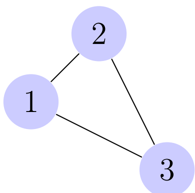
$$q_5 = p \cdot p \cdot (1-p) = p^2 \cdot (1-p) = \frac{4}{27}$$

6. 

$$q_6 = p \cdot p \cdot (1-p) = p^2 \cdot (1-p) = \frac{4}{27}$$

7. 

$$q_7 = p \cdot p \cdot (1-p) = p^2 \cdot (1-p) = \frac{4}{27}$$

8. 

$$q_8 = p \cdot p \cdot p = p^3 = \frac{8}{27}$$

2 Problem 2

For a graph $G_{4,p}$ we have 4 vertices and p is the probability of an edge to exist.

a.

The distribution of the degree of any vertex in this graph, is binomial. For each vertex we have $n-1$ (where $n=4$) possible connections to other vertices (edges) with probability p each. Therefore we have $n-1$ sample points on our sample space for each vertex. Let's define a random Bernoulli variable $x_i, i \in [1,3]$ (we have focused on some vertex u) where:

$$x_i = \begin{cases} 1, & \text{when edge (u,i) exists} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Then we define the binomial variable $X = x_1 + x_2 + x_3$ that corresponds to the degree of the vertex. This proves that he distribution is binomial and the average degree of any vertex is:

$$E(x) = (n-1) \cdot p = 3 \cdot p$$

b.

Similarly to the sub-question a. the distribution will be binomial since we examine the possibility of the existence of an edge with probability p, in case of a success, and $1-p$ in case of a failure. We repeat this experiment $\binom{4}{2}$ (i.e. the total number of possible edges) times. Therefore:

$$E(x) = \binom{4}{2} \cdot p = 6 \cdot p$$

3 Problem 3

We have a complete bipartite graph which means, we have two sets of variables, V_1 , V_2 where $V_1 = m$ and $|V_2| = n$ ($K_{m,n}$). Since it is a complete graph, every vertex of V_1 is connected to every vertex of V_2. We want to prove that there exists a monochromatic $K_{a,b}$ with $a+b$ nodes.

Moreover, we construct a probability space by two-coloring the edges at random, equiprobably and independently for each color. Let $K_{a,b}$ be any fixed set of $a+b$ vertices with induced vertices considered. Let's define the event:

$$M_S : \{K_{a,b} \text{ is monochromatic}\}$$

The subgraph $K_{a,b}$ has $a \cdot b$ two-colored edges. So:

$$Pr(M_S) = (\frac{1}{2})^{a\cdot b} + (\frac{1}{2})^{a\cdot b} = 2^{(1-a\cdot b)}$$

Now, we define the event

$$M : \{\exists \text{ at least one } K_{a,b}\}$$

Hence:

$$M \bigcup_{K_{a,b}} M_S$$

Now, using Boole's inequality we can compute Pr(M):

$$Pr(M) \leq \sum Pr(M_S) \ (1)$$

In order to define the amount of $K_{a,b}$ we can see that we have 2 possible scenarios. We have $a+b$ where a is a subset of V_1 and b is a subset of V_2 (or vice versa). So:

$$K_{a,b} = \binom{m}{a} \cdot \binom{n}{b} + \binom{m}{b} \cdot \binom{n}{a}$$

Therefore from (1) we have:

$$Pr(M) \leq Pr\{\bigcup_{K_{a,b}} M_S\} \leq \sum Pr(\bigcup_{K_{a,b}} M_S) = \left[\binom{m}{a}\cdot\binom{n}{b}+\binom{m}{b}\cdot\binom{n}{a}\right]\cdot 2^{(1-a\cdot b)}$$

If: $Pr(M) < 1 \Rightarrow Pr(\bar{M}) > 0 \Leftrightarrow$ If: $\left[\binom{m}{a}\cdot\binom{n}{b}+\binom{m}{b}\cdot\binom{n}{a}\right]\cdot 2^{(1-a\cdot b)} < 1$

Thus, there exists at least on point in the sample space $\Rightarrow$ there is a two-coloring edge with at most $\left[\binom{m}{a}\cdot\binom{n}{b}+\binom{m}{b}\cdot\binom{n}{a}\right]\cdot 2^{(1-a\cdot b)}$ monochromatic $K_{a,b}$.

4 Problem 4

Let S be any fixed arithmetic progression of k terms. Let there be an event M_S : {S is monochromatic} so that all k terms of the event S, have the same color. Every term is colored with one color from a palette of r colors with probability $p = \frac{1}{r}$. All k terms have the same color with probability $p_k = (\frac{1}{r})^k$. Consequently, we have:

$Pr(M_S) = Pr\{\text{all one color (for r colors)}\} = r \cdot (\frac{1}{r})^k = r \cdot r^{(-k)} = r^{(1-k)}$ (for a specific event)

Now, let's define the event M where:

$\bigcup_{|S|} M_S = \{\exists$ at least one monochromatic arithmetic progression of k terms$\}$

An arithmetic progression is defined uniquely by its first two terms therefore there are at most $\binom{n}{2}$ arithmetic progressions with probability:

$$\text{Pr(M)} = Pr(\bigcup_{|S|} M_S) \leq \binom{n}{2} \cdot r^{1-k}$$

We know that:

$$Pr(M) \leq \binom{n}{2} \cdot 2^{(1-k)} = \frac{n \cdot (n-1)}{2} \cdot r^{(1-k)} < \frac{n^2}{2} \cdot r^{(1-k)}$$

If $Pr(M) < 1 \Rightarrow Pr(\bar{M}) > 0 \Rightarrow \frac{n^2}{2} \cdot r^{(1-k)} < 1 \Rightarrow n^2 < \frac{2}{r^{(1-k)}} \Rightarrow$
$n^2 < 2 \cdot r^{(1-k)} \Rightarrow n < \sqrt{2} \cdot r^{\frac{(k-1)}{2}}$

Therefore, there is a two-coloring without a monochromatic arethmetic progression of k terms when $n < \sqrt{2} \cdot r^{\frac{(k-1)}{2}}$

So: $n > \sqrt{2} \cdot r^{\frac{(k-1)}{2}}$

5 Problem 5

Let S be any fixed set of 3 vertices. We define a random variable X that counts the number of triangles.

$$X = \sum_{S,|S|=3} \text{where } X_S \text{ is an indicator variable}$$

$$X_S = \begin{cases} 1, & \text{S is a triangle} \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

$$E(X_S) = Pr(X_S = 1) = p^3 \text{ (i.e. all 3 vertices have to exist)}$$

By linearity of expectation:

$$E(X) = E(\sum_{S,|S|=3} X_S) = \binom{n}{3} \cdot p^3$$

(i.e there are $\binom{n}{3}$ ways for these kind of S sets to be chosen.)

$$\text{If } E(X) = n^3 \cdot p^3 \ll 1 \Leftrightarrow p^3 \ll \frac{1}{n^3} \Rightarrow p \ll \frac{1}{n}$$

$$\text{If } p \ll \frac{1}{n} \Rightarrow \text{E(X)} \to 0 \Rightarrow \text{Pr(X} = 0) \to 1 \Rightarrow \nexists \text{ triangles w.h.p}$$

$$\text{If } p \gg \frac{1}{n} \text{ then clearly, E(X)} \to \infty$$

We have to show that $\Delta* = o(E(X))$. For that we will assume that the events X_i, X_j are symmetrical (i.e. $Pr(X_i|X_j = 1) = Pr(X_j|X_i = 1$. That means that the sum of conditional probabilities due to the stochastic dependencies has to be less than the average value of X. $A_i \sim A_j$ means that A_i, A_j are stochastically dependant and different. Therefore, A_i, A_jmust have common edges.

$$\Delta* = \sum_{j\sim i} Pr(A_j|A_i), A_i \to \text{ The set } S_i \text{ is a triangle.}$$

That means that $S_i \cap S_j = 2$. If it was equal to 1 there would be no dependency because there would be no common edge. Same thing applies if it was equal to 3.

If S_i has a triangle with 2 of its vertices belonging in S_j there are 2 more edges belonging in S_j for a triangle to exist in S_j.

$$\text{So: } Pr(A_j|A_i) = p^2$$

There are $\binom{3}{2} \cdot \binom{n-3}{1} = O(n)$ different ways to choose the set S_j so that $(S_i \cap S_j) = 2$

$$\Delta* = \sum_{|S_i \cap S_j|=2} Pr(A_j|A_i) \sim n \cdot p^2$$

When $p = \frac{1}{n}$ then, $\frac{\Delta*}{E(X)} \sim \frac{n \cdot p^2}{n^3 \cdot p^3} = \frac{1}{n^2 \cdot p} = \frac{1}{n} \to 0$ (because $p = \frac{1}{n}$)

Consequently, for that value of p we have: $\Delta* = o(E(Q))$ and that also stands for larger p values because of the monotony.

6 Problem 6

X_i : {The event that the random assignment does not satisfy clause i }

(We assume that X_i can be reenacted in the sample space as a node in a dependency graph)

We have k literals in each clause i.e. 2^k where one of them is problematic. $Pr(X_i) = \frac{1}{2^k}$. We want the $\bar{X}_i$ event to be true (we consider X_i the bad events).

We can say that two clauses have dependencies as long as they share at least one variable. The event X_i is mutually independent from all the events that do not share variables with the clause i (The maximum dependency a clause can have is to depend on $k \cdot \frac{2^{(k-2)}}{k}$ more clauses $\Rightarrow d(clause) = k \cdot \frac{2^{(k-2)}}{k}$) since we have k literals in each clause.

So, from the Symmetric Lovasvz Local Lemma we have:

$$4 \cdot d \cdot p \leq 1 \Rightarrow 4 \cdot k \cdot \frac{2^{(k-2)}}{k} \cdot \frac{1}{2^k} = 4 \cdot \frac{2^k}{4} \cdot \frac{1}{2^k} = 1$$

Let m clauses. Therefore, $Pr(\bar{X_1}, ...\bar{X_m}) > 0$ which means that the probability that all the clauses are satisfiable is positive (i.e. the probability an assignment is satisfiable $\Rightarrow \exists$ satisfiable assignment).

7 Problem 7

a.

The one-step transition matrix is the following:

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

b.

Since the sum of each column is 1, the matrix is doubly stochastic. Thus, the stationary distribution is uniform $\Pi_i = \frac{1}{n} = \frac{1}{3}$

c.

$$r_{1,3}{}^{(1)} = P_{1,3}$$

$$r_{1,3}{}^{(2)} = P_{1,2} \cdot P_{2,3}$$

$$r_{1,3}{}^{(3)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,3}$$

$$r_{1,3}{}^{(4)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,3}$$

$$r_{1,3}{}^{(5)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,3}$$

$$r_{1,3}{}^{(6)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,3}$$

$$r_{1,3}{}^{(7)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,3}$$

$$r_{1,3}{}^{(8)} = P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,1} \cdot P_{1,2} \cdot P_{2,3}$$

$$z = P_{1,2} \cdot P_{2,1} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

$$\text{When t is odd: } r_{1,3}{}^{2 \cdot t+1} = z^t \cdot P_{1,3}$$

$$\text{When t is even: } r_{1,3}{}^{2 \cdot t+2} = P_{1,2} \cdot z^t \cdot P_{2,3}$$

$$r_{1,3}{}^{(t)} = \sum_{\text{t:odd}} z^{2 \cdot t+1} \cdot P_{1,3} + \sum_{\text{t:even}} \cdot P_{1,2} \cdot z^{2 \cdot t+2} \cdot P_{2,3} = 9 + \frac{3}{2} = \frac{21}{2}$$

8 Problem 8

At first we are on a node that has only one edge. Since we have a symmetrical random walk, it is $h_{n,0} = h_{0,n}$. Since this graph is connected, non-bipartite and undirected, we define a Markov Chain MC_G corresponding to a random walk on the vertices of G, with transition probability:

$$P_{u,v} = \begin{cases} \frac{1}{d(u)}, & \text{if uv } \in E(G) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

The commute time between u and v nodes is $CT_{u,v} = h_{u,v} + h_{v,u}$. We also know that, for any two vertices u,v in G, the commute time is $CT_{u,v} = 2 \cdot m \cdot R_{u,v}$ (m is the number of edges of the graph and $R_{u,v}$ the effective resistance between u and v in the associated electrical network N(G)). Therefore, edge $C_{u,v} = 2 \cdot m \cdot R_{u,v} = 2 \cdot 3 \cdot n \cdot 2 \cdot n = 12 \cdot n^2$.

Because the graph is symmetric $h_{u,v} = h_{v,u} = 6 \cdot n^2$