Recursion
Reduce and Conquer
Divide (Evenly) and Conquer
Sorting

Recursive algorithms - Divide and Conquer

- General idea:
  - **Divide** a large problem into smaller ones
  - Solve each smaller one *recursively*
  - **Combine** the solutions of smaller ones to form a solution for the original problem
- Examples:
  - Tower of Hanoi (Reduced and Conquer)
  - Fibonacci sequence (Reduced and Conquer, Dynamic Programming)
  - Euclidean's Algorithm (GCD) (Reduced and Conquer)
  - Merge Sort, Quick Sort (Divide and Conquer)

The Recursive Factorial Function

- The factorial function:

  `n! = n*(n-1)*(n-2)*...*3*2*1` if `n > 0`

  `n! = 1` if `n = 0`

- Can compute factorial of `n` if the factorial of `(n-1)` is known:

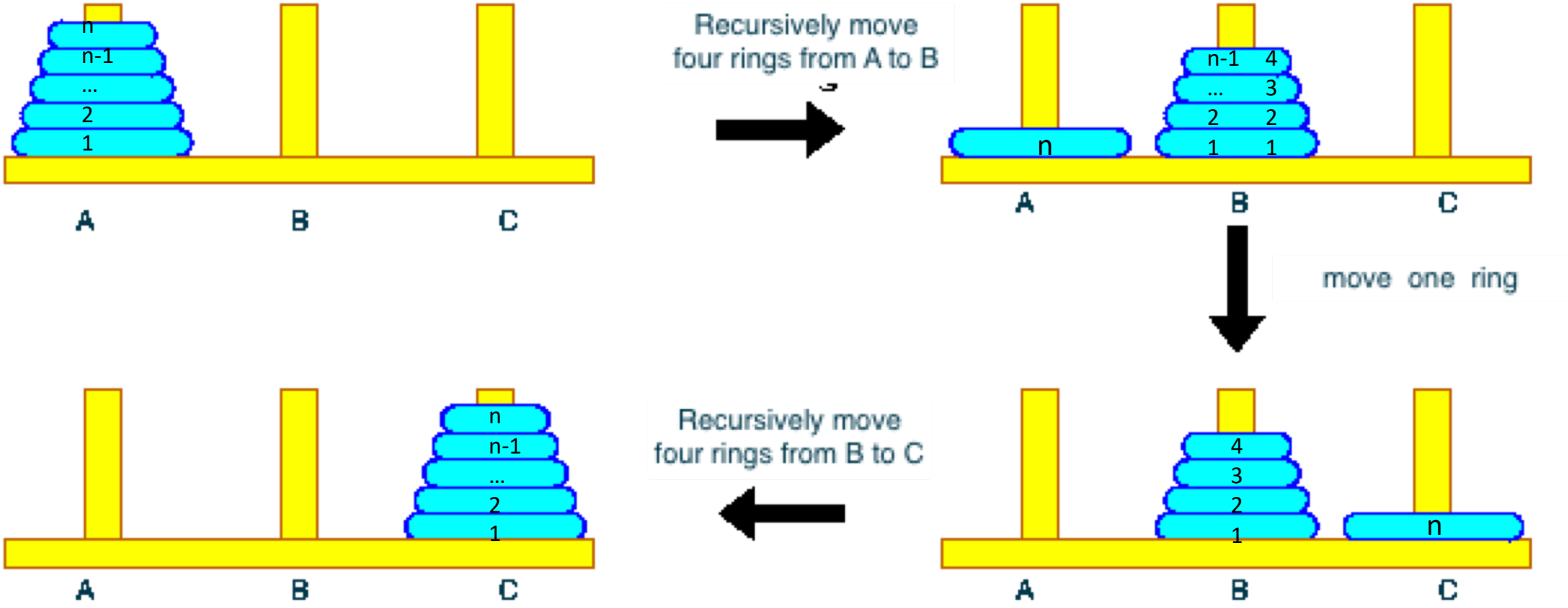
  `n! = n * (n-1)!`

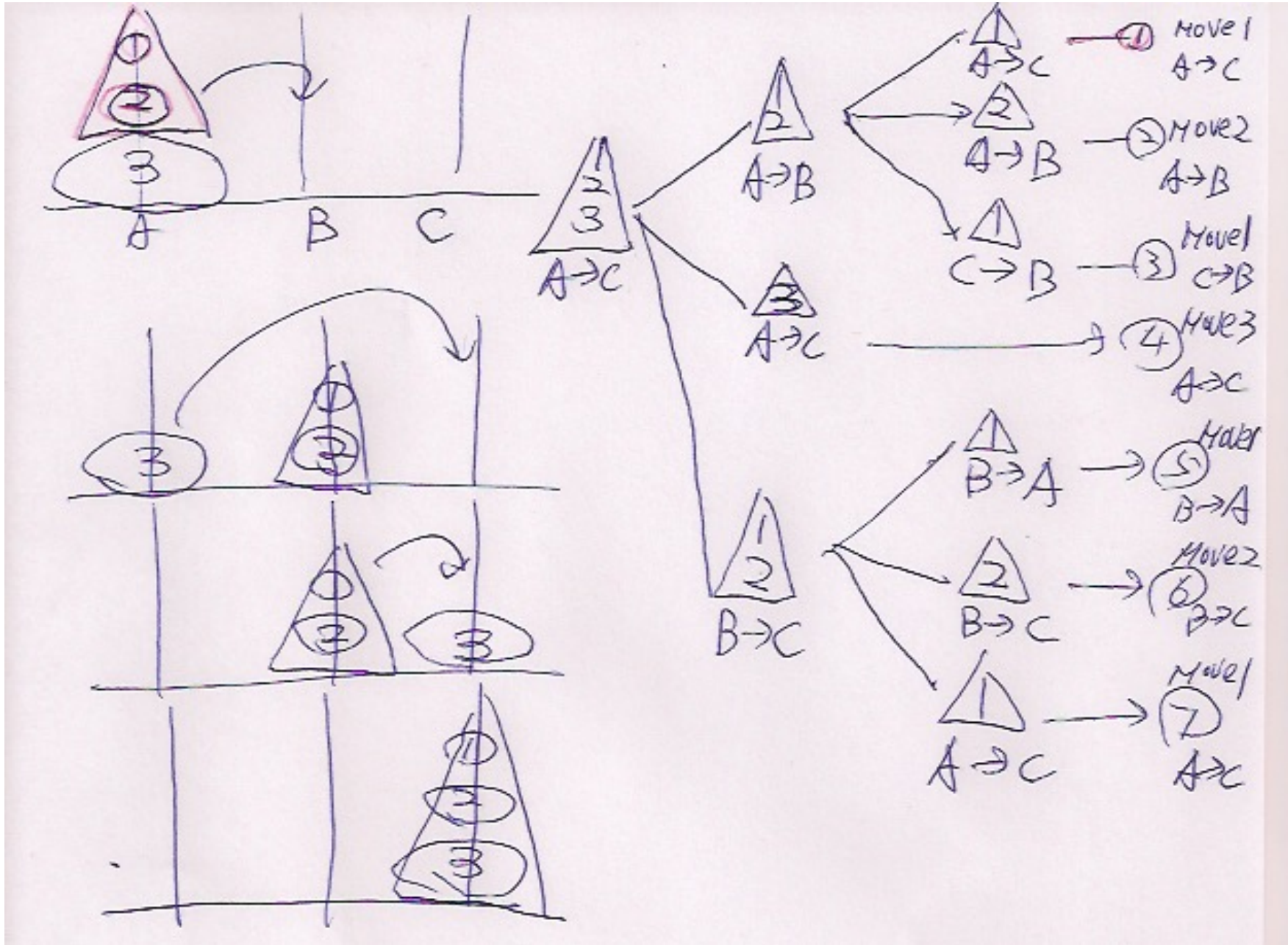
- `n = 0` is the base case

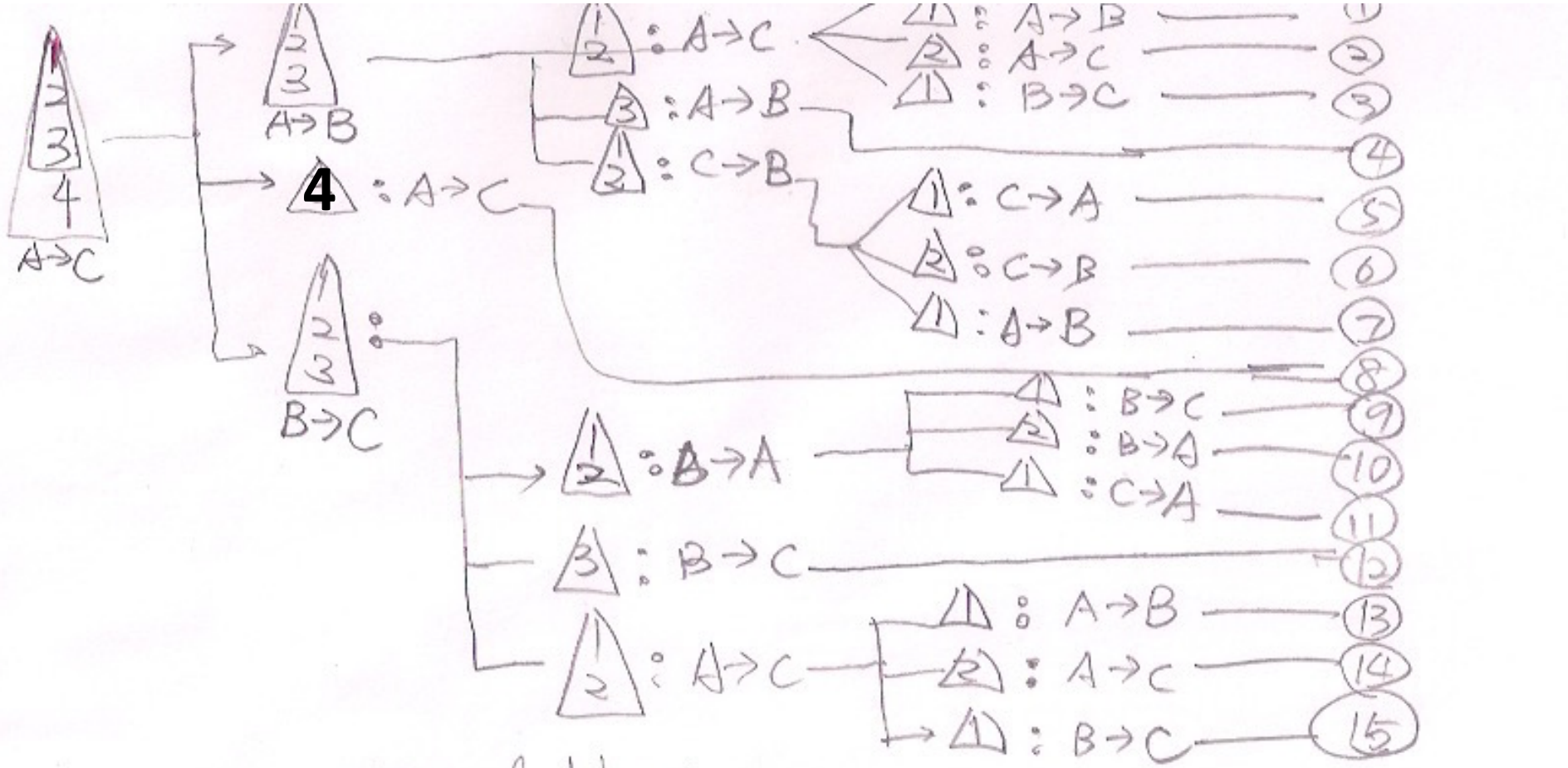
The Recursive Factorial Function

```
def factorial (num):
{
  if (num > 0)
     return num * factorial(num - 1)
  else
     return 1
}
```

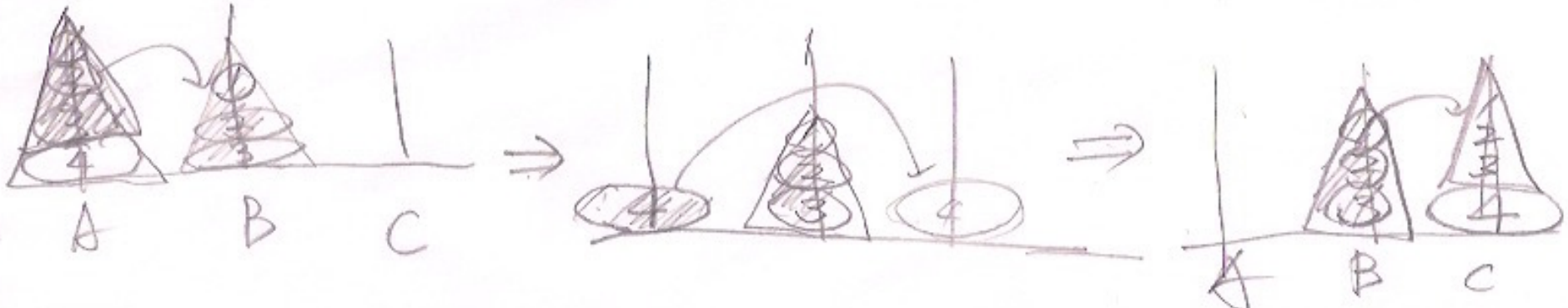
- To move n discs from peg A to peg C:
  - move n−1 discs from A to B. This leaves disc n alone on peg A
  - move disc n from A to C
  - move n−1 discs from B to C so they sit on disc n




A
B
C
A→C
A→B
A→C
C→B
A→C
B→A
B→C
A→C
Move1 A→C
Move2 A→B
Move1 C→B
Move3 A→C
Move1 B→A
Move2 B→C
Move1 A→C



Tower of Hanoi



Recursive Algorithms for Tower of Hanoi

```
def TowerOfHanoi(n , from_rod, to_rod, aux_rod):
    if n == 0:
        return
    TowerOfHanoi(n-1, from_rod, aux_rod, to_rod)
    print ("Move disk",n,"from",from_rod,"to",to_rod )
    TowerOfHanoi(n-1, aux_rod, to_rod, from_rod)


TowerOfHanoi(6, "A","B","C")
```

$$T(n) = \begin{cases} 0 \text{ if } n = 0; \\ 2T(n\text{-}1) + 1 \text{ if } n > 1. \end{cases}$$

Recurrence Function

$$T(n) = \begin{cases} 0 \text{ if } n = 0; \\ 2T(n-1) + 1 \text{ if } n > 1. \end{cases}$$

$T(n) = 2*T(n-1) + 1$

$= 2*(2*T(n-2)+1) + 1 = 4*T(n-2) + 3 = 2^2*T(n-2) + 2^2 - 1$

$= 4*(2*T(n-3)+1) + 3 = 8*T(n-3) + 7 = 2^3*T(n-3) + 2^3 - 1$

$= 8*(2*T(n-4)+1) + 7 = 16*T(n-4) + 15 = 2^4*T(n-4) + 2^4 - 1$

$= \ldots = \ldots$

$= = 2^n*T(0) + 2^n - 1$

$= 2^n - 1$

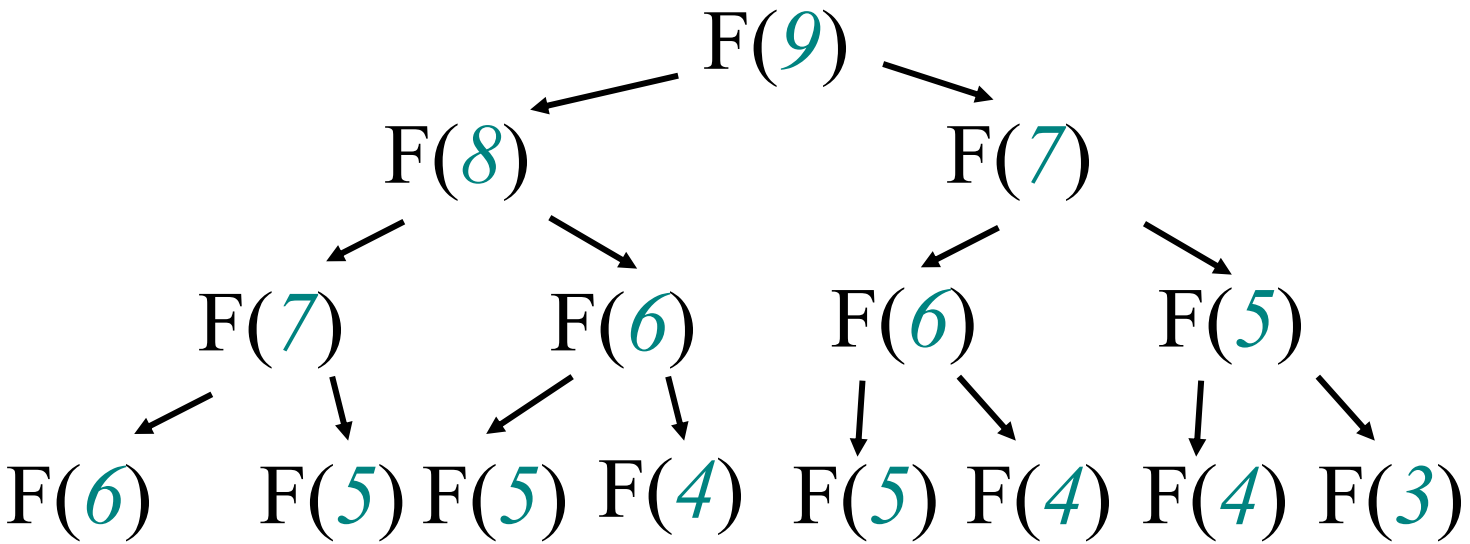
Fibonacci numbers

An **recursive** algorithm

fib(n) // recursive

$$T(n) = T(n-1)+T(n-2) + 1$$

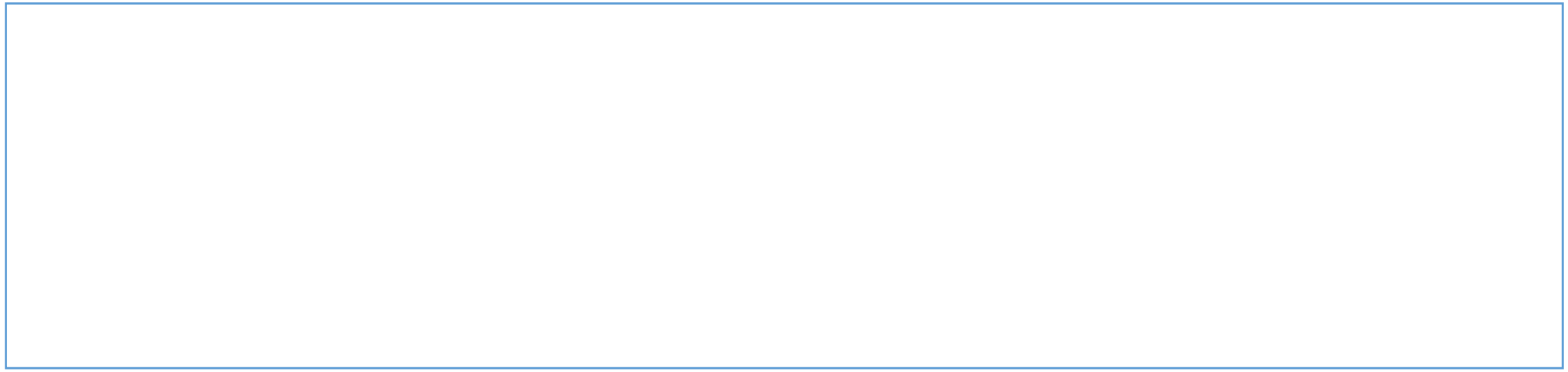
$$T(n) = \Theta(F_n) = \Theta(\phi^n)$$



Fibonacci numbers

An **iterative** algorithm

fib(n): // iterative



Time complexity? T(n)

The Recursive gcd Function

- Greatest common divisor (gcd) is the largest factor that two integers have in common
- Computed using Euclid's algorithm:

  `gcd(x, y) = y` if `y` divides `x` evenly

  `gcd(x, y) = gcd(y, x % y)` otherwise
- `gcd(x, y) = y` is the base case

Iteration vs. Recursion

```
def gcd(x, y):
        while y != 0:
                (x, y) = (y, x % y)
        return x
```

```
def gcd_recursive(a, b):
        if b == 0:
                return a
        else:
                return gcd_recursive(b, a % b)
```

Recursion vs. Iteration

- Benefits (+), disadvantages(-) for recursion:
  - + Models certain algorithms most accurately
  - + Results in shorter, simpler functions
  - • May not execute very efficiently
- Benefits (+), disadvantages(-) for iteration:
  - + Executes more efficiently than recursion
  - • Often is harder to code or understand

Binary Search

To find an element in a sorted array, we

1. Check the middle element
2. If ==, we've found it
3. else if less than wanted, search right half
4. else search left half

Example: Find 9

3	5	7	8	9	12	15

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To find an element in a sorted array, we

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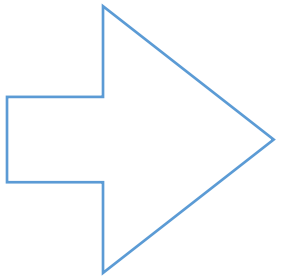
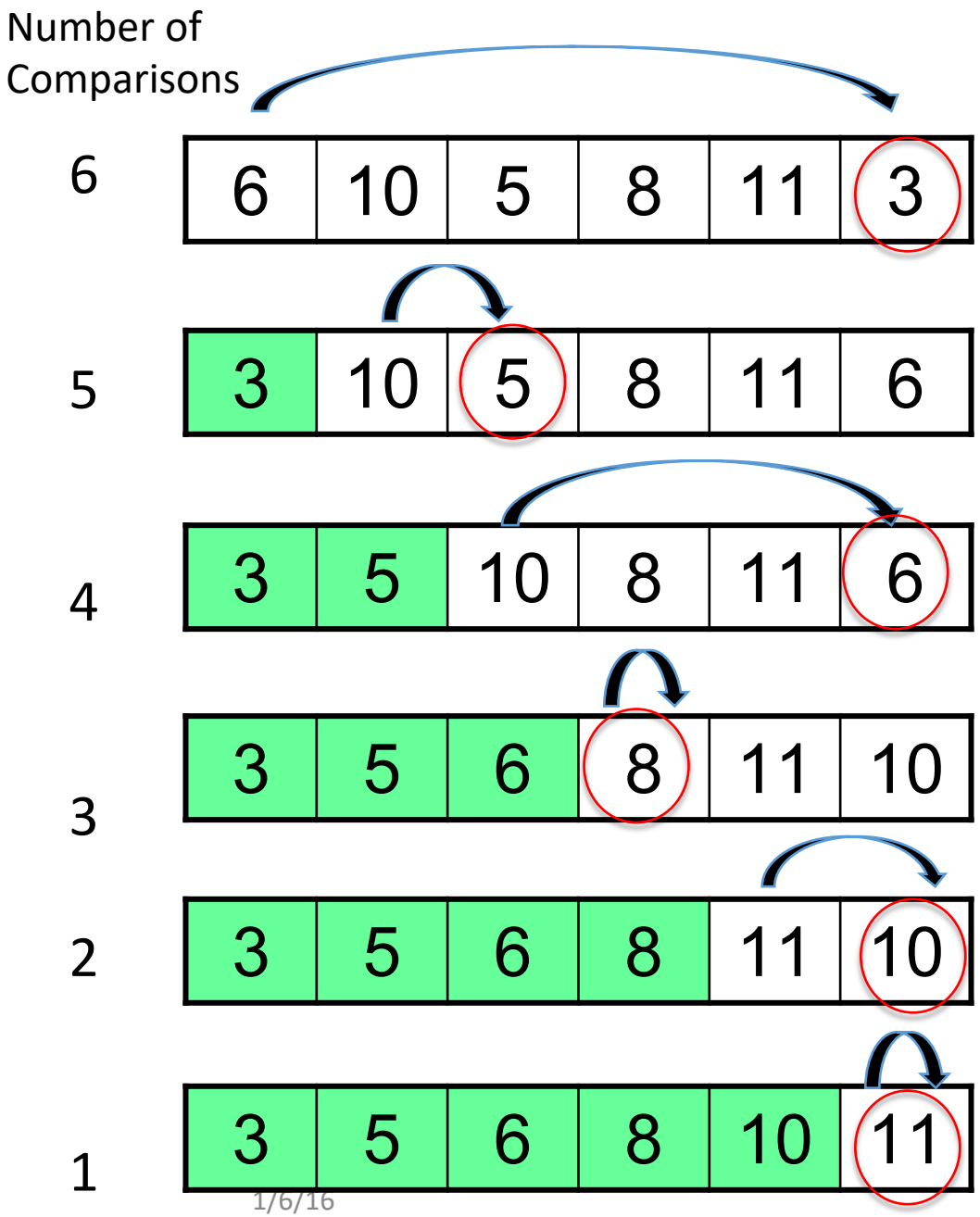
$$T(n) = T\left(\frac{n}{2}\right) + \Theta(1)$$

Sorting Algorithms



Sorting Algorithms

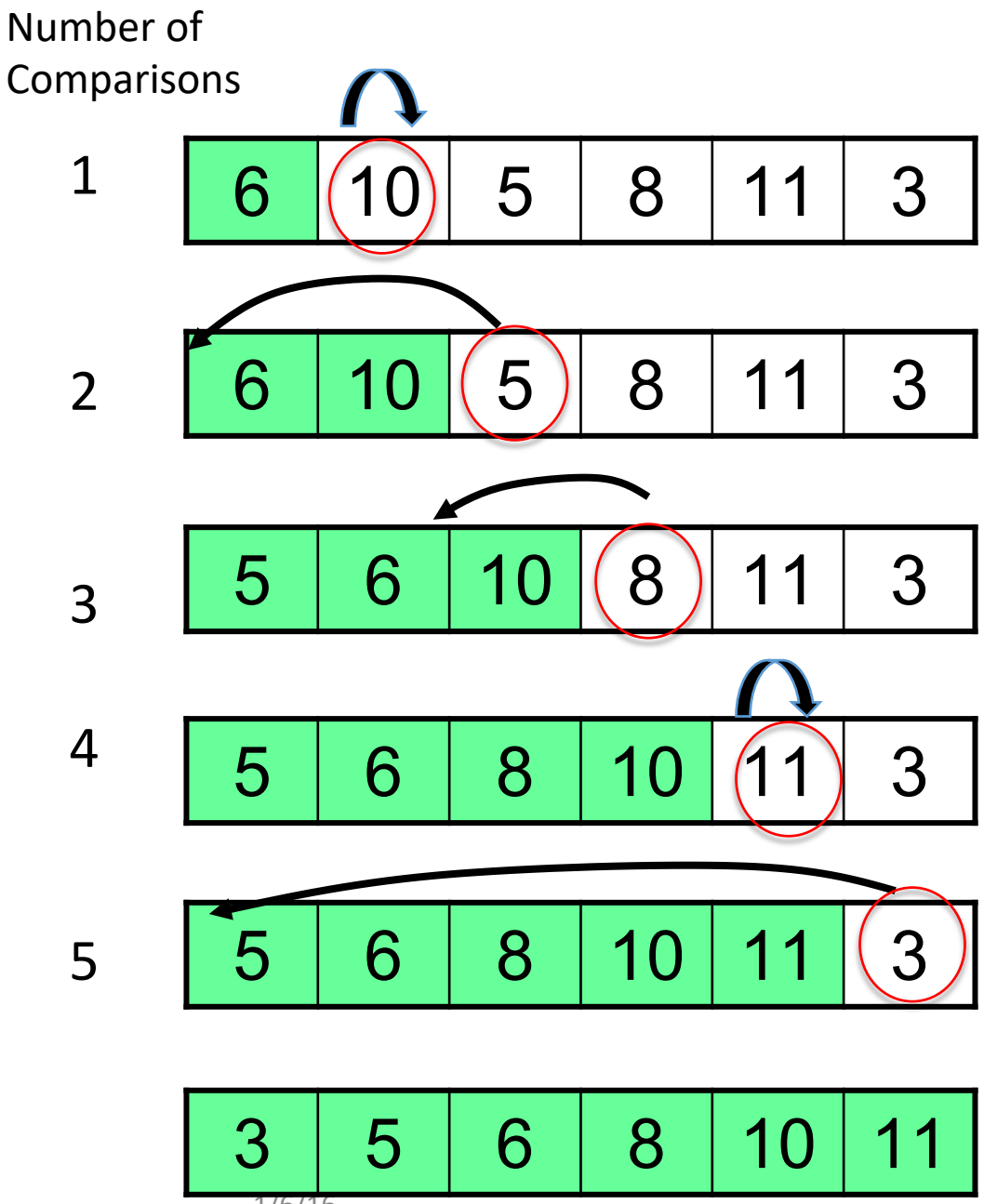
- Selection and Insertion Sort:
  - Incremental approach (Reduced and Conquer)
  - Simple but slow
- Merge and Quick Sort
  - Divide and Conquer



Selection Sort (Incremental)

- Find smallest item, move to 1st location (n comparisons).
- Find next smallest item, move to 2nd location (n-1 comparisons).
- …
- Repeat until the final location is reached.

- Time complexity: $n+(n-1)+(n-2)+\ldots+1 = (n^2+n)/2 = O(n^2)$

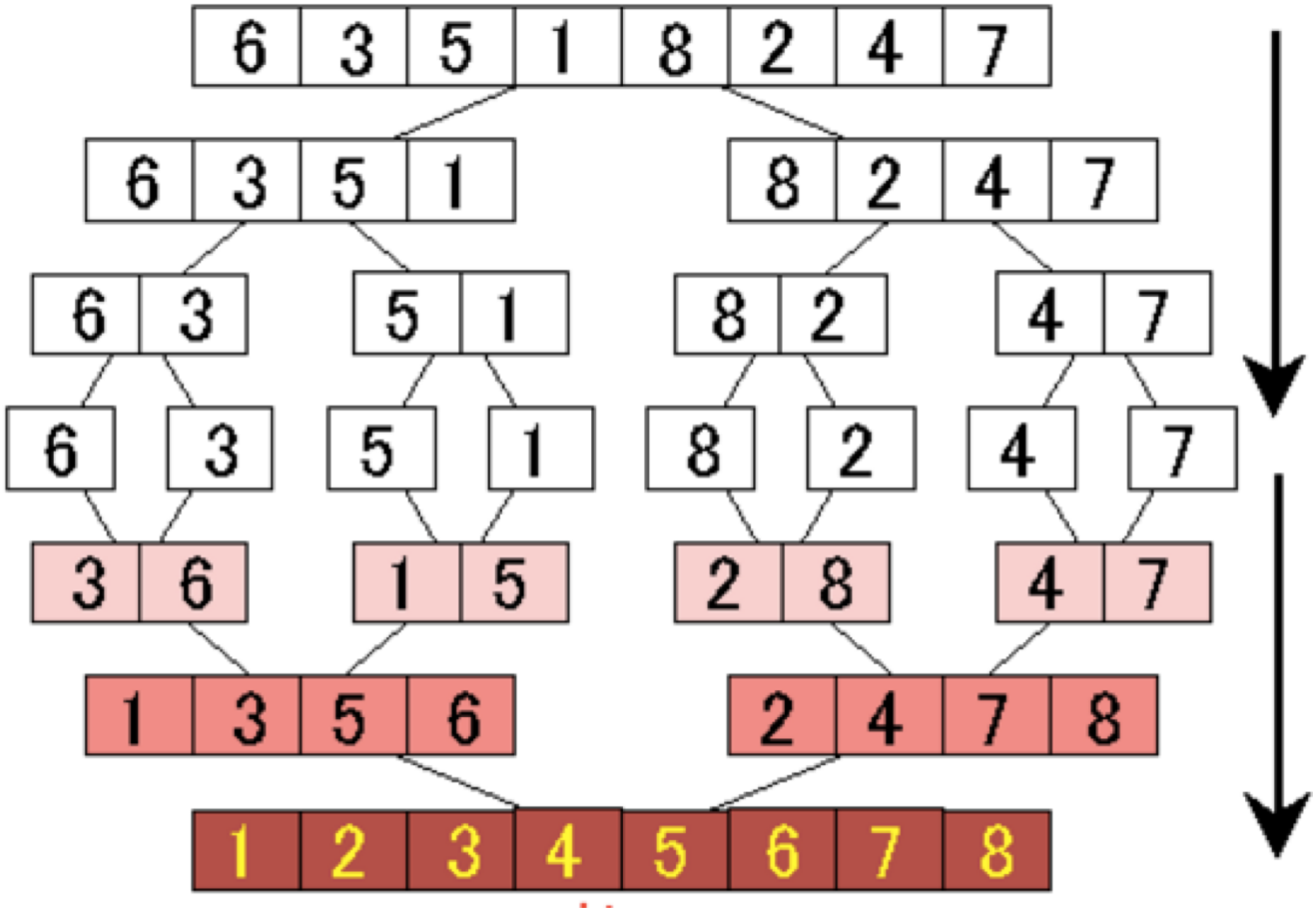


Insertion Sort (Incremental)

- Begin with 1 sorted item.
- Insert 2nd item properly onto the sorted list
  - 2 sorted items
- Insert 3rd item properly ontothe sorted list
  - 3 sorted item
- …
- Repeat util we have all (n) sorted iterms.

- Time complexity: $1+2+\ldots+(n-1) = (n^2-n)/2 = O(n^2)$

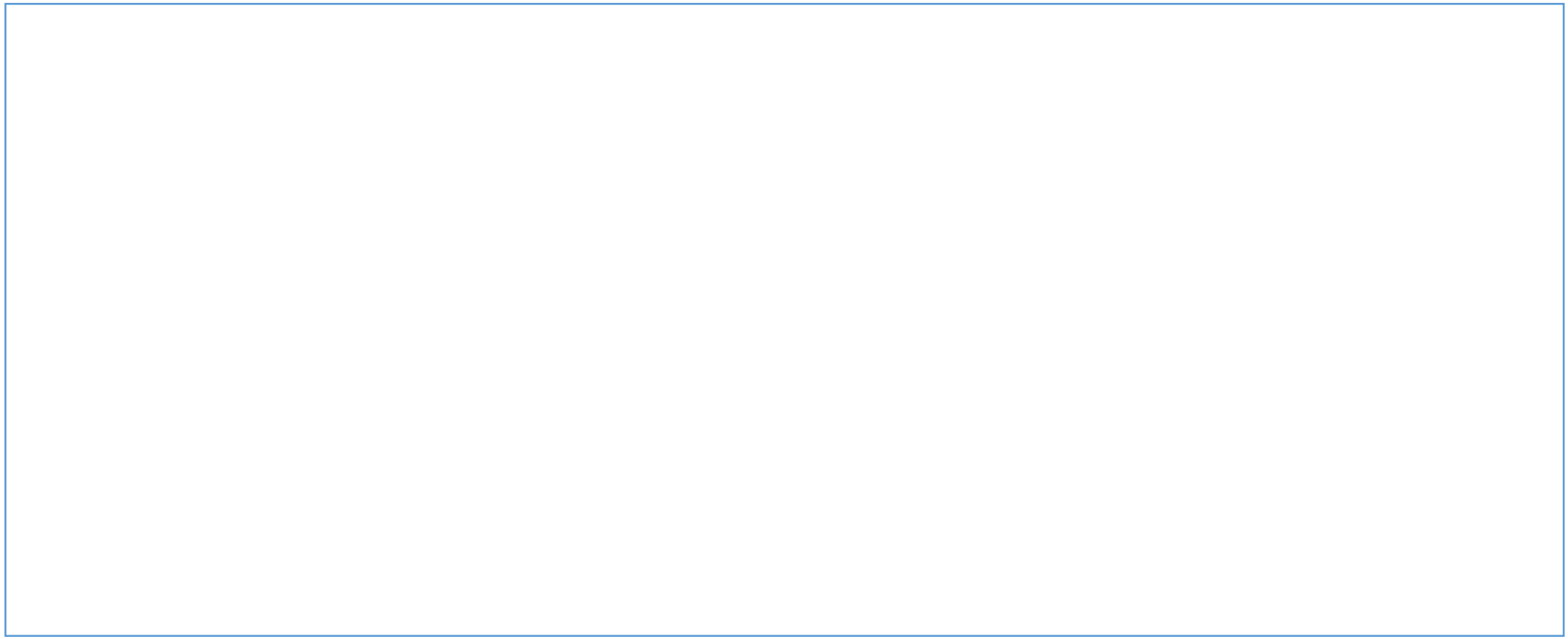
Merge Sort



- Divide-and-conquer
- Recursively sort ½ the set
- Then merge them together: O(n)
- O(n *lg* n) for both best, average, and worst cases
- Can be made to be in-place

Codes: Merge Sort

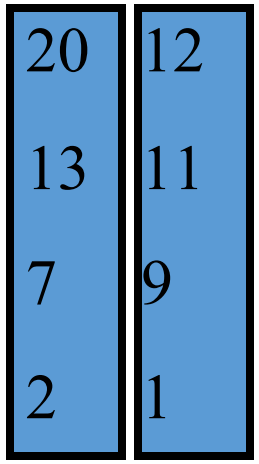
Key subroutine: MERGE O(n)



Merging two sorted arrays

Array 1

k numbers



Array 2

m numbers

Time: $O(k + m)$.

Merging two sorted arrays

20 12

13 11

7 9

2 1

Merging two sorted arrays

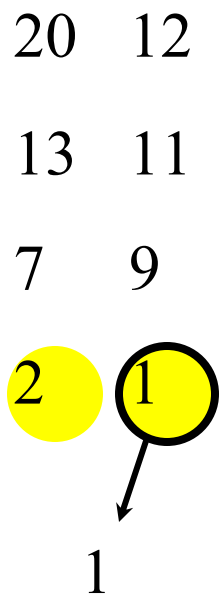
20 12

13 11

7 9

2 1

Merging two sorted arrays



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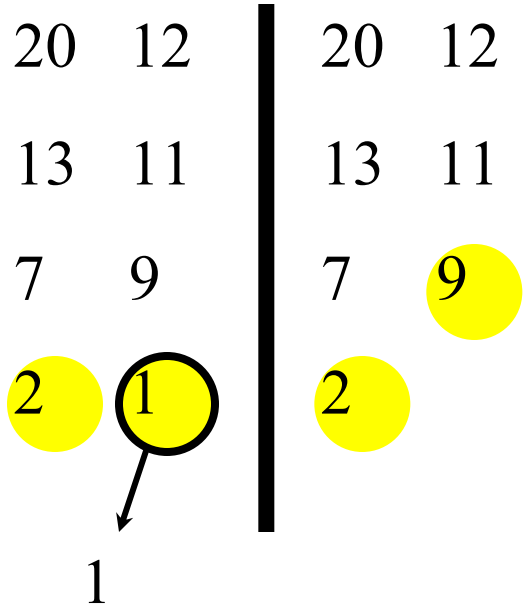

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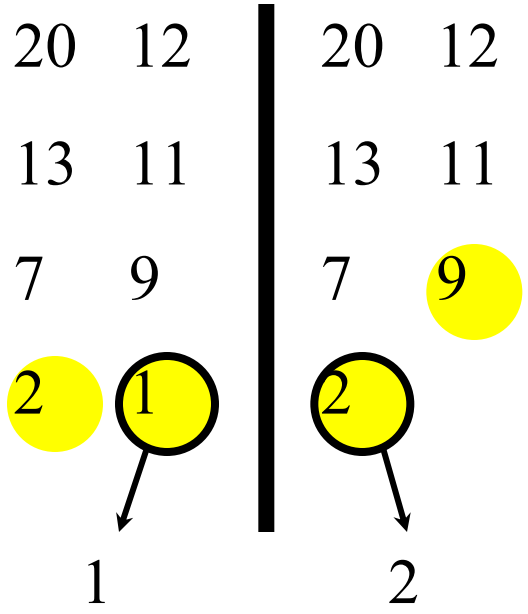

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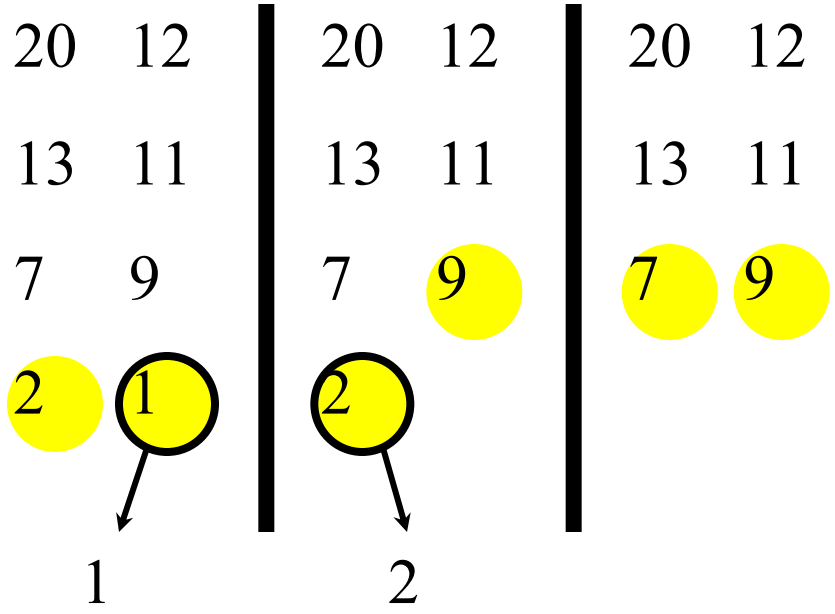

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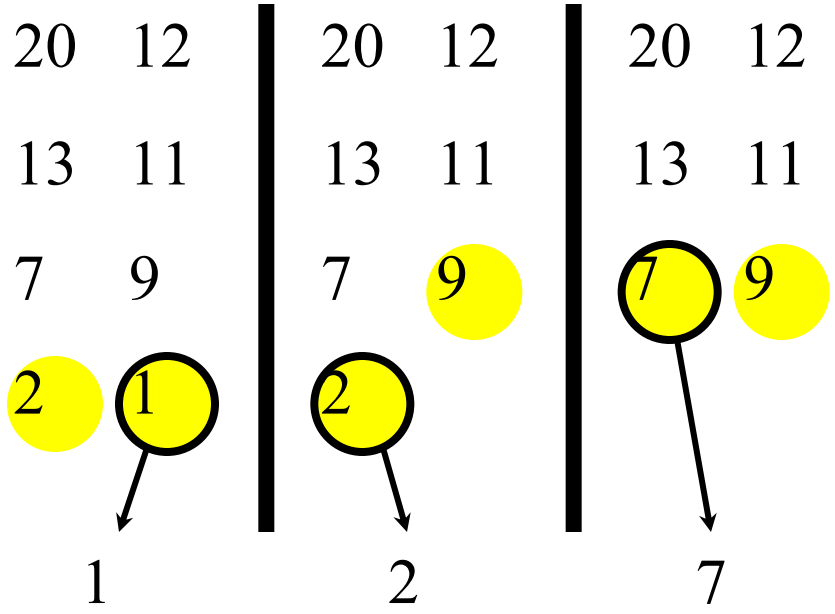

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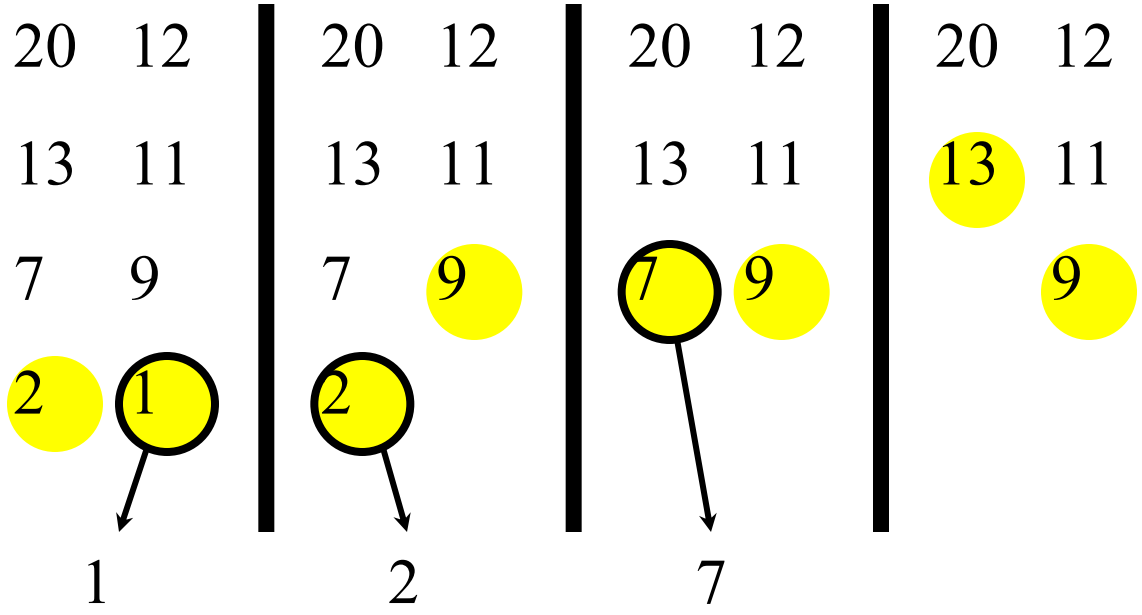
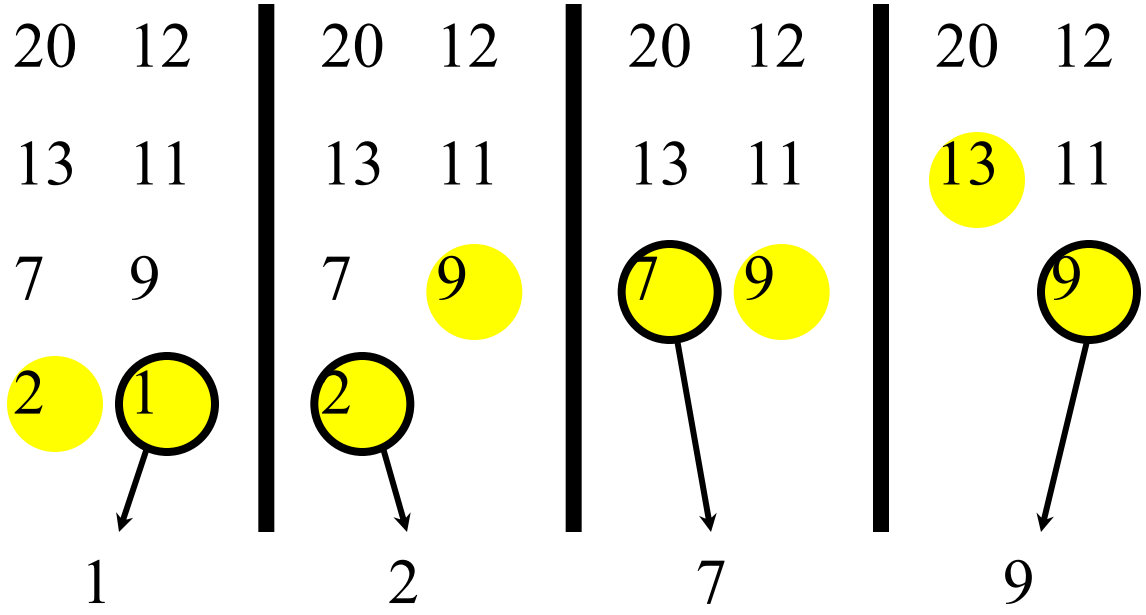


Merging two sorted arrays



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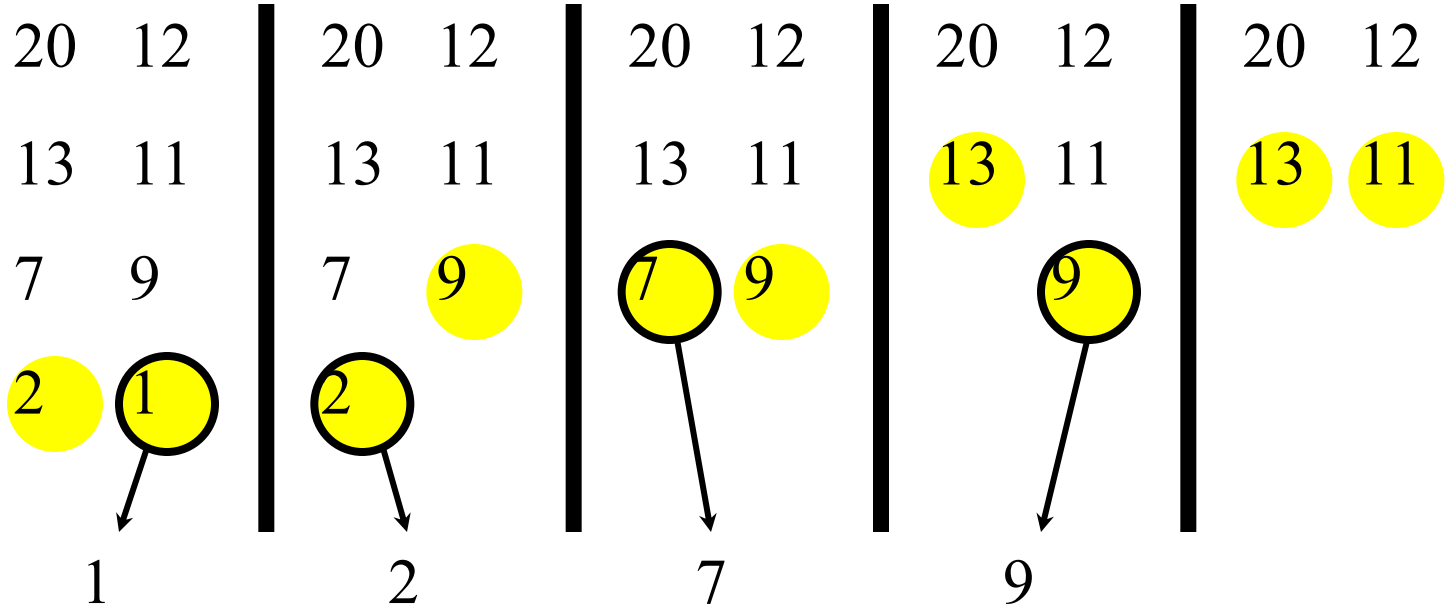

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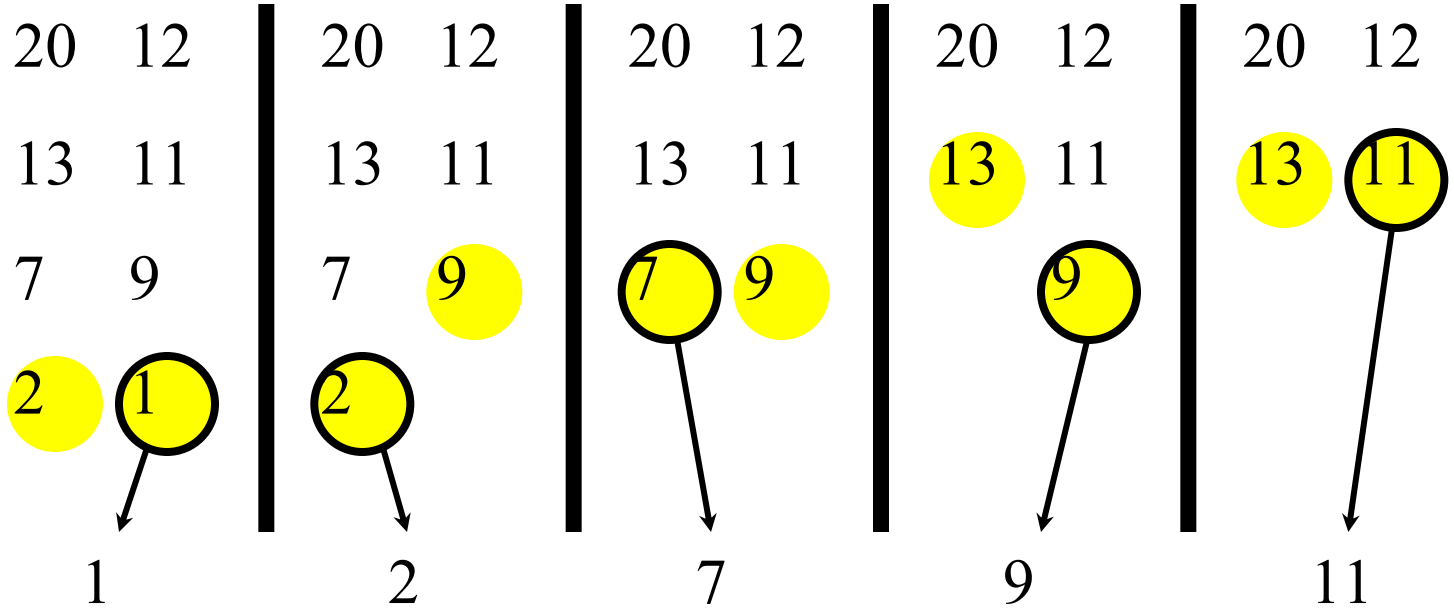

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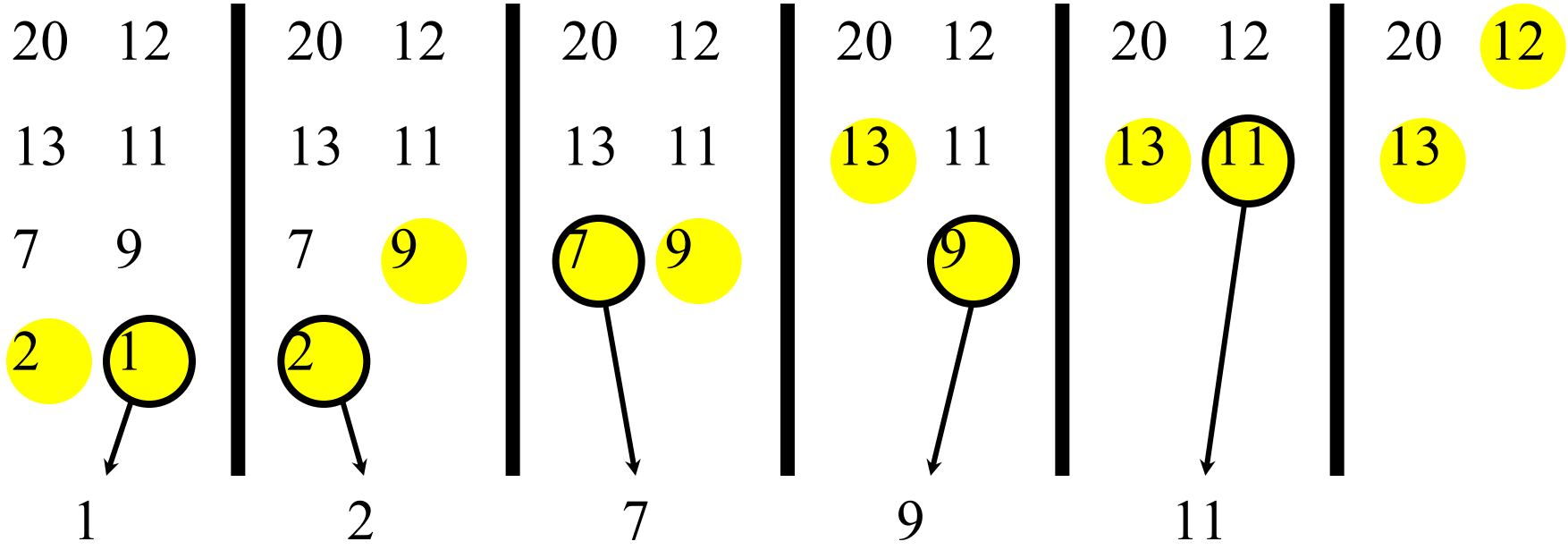

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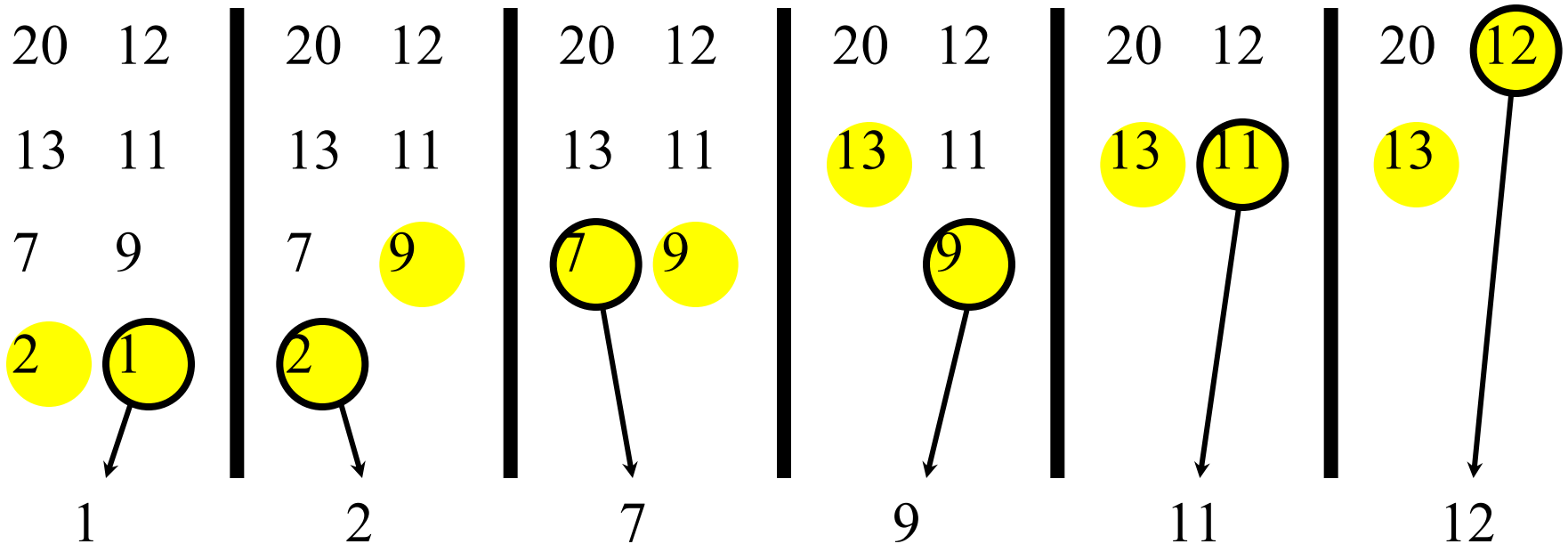


Analyzing merge sort

```
Def mergesort(data, first, last):

    if last > first:
        mid = (first+last)/2 ;
        mergesort(data, first, mid );
        mergesort(data[], mid+1,last);
        merge ( data, first, last );
```

$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

Recurrence for merge sort

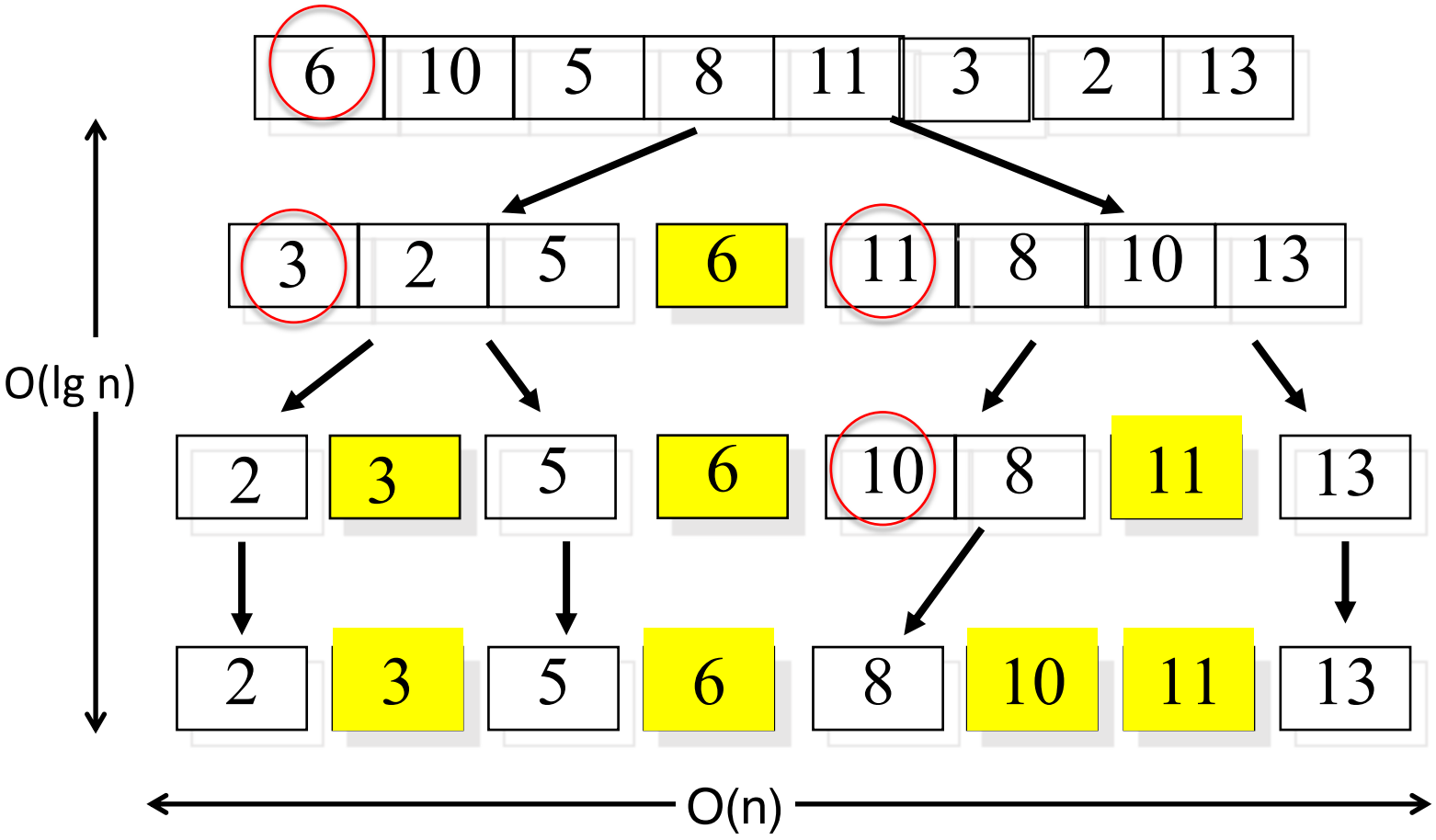
$$T(n) = \begin{cases} \Theta(1) \text{ if } n = 1; \\ 2T(n/2) + \Theta(n) \text{ if } n > 1. \end{cases}$$

T(n)	
	= 2 T(n/2)+ n
= 2(2T(n/4)+n/2)+n	= 4 T(n/4)+ 2*n
= 4 (2T(n/8)+n/4)+2*n	= 8 T(n/8)+ 3*n
= …	= 2^k T(n/2^k)+ k*n =
= …	= n* T(1) +k*n, where k = log n
	= n+ ($\log_2$ n) *n = O (n log n)

Quicksort

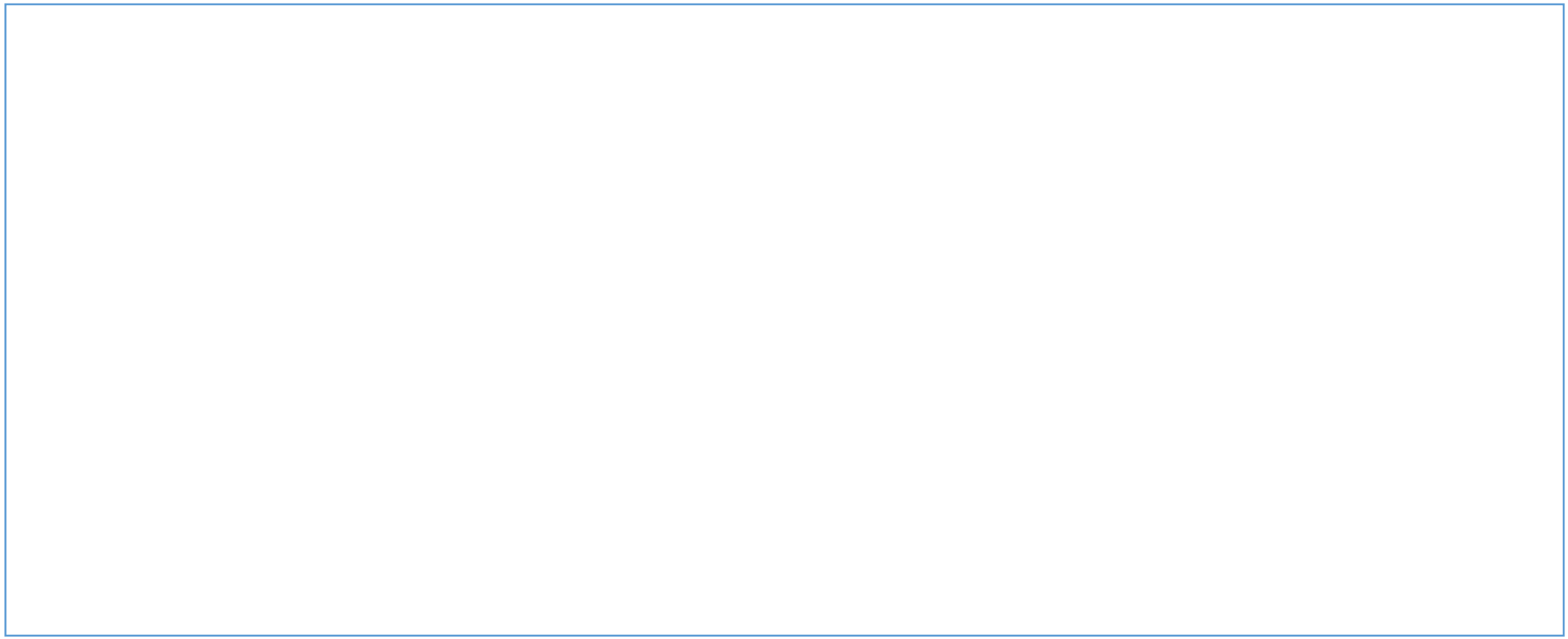
- Another divide and conquer sorting algorithm
  - like merge sort
- The worst-case and average-case running time
- Learn new algorithm analysis tricks

Quicksort (Divide & Conquer)

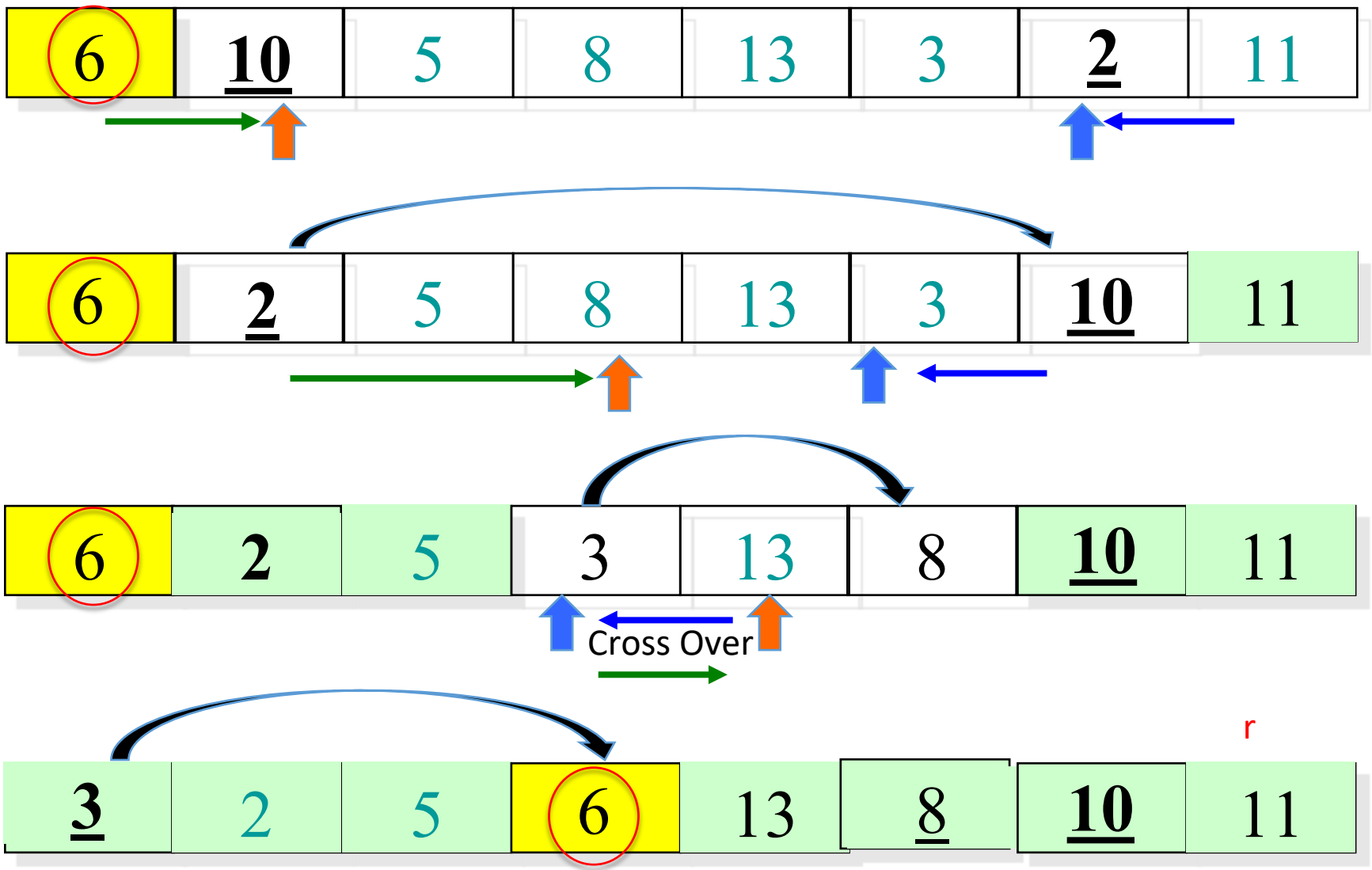


- Pick a pivot element
- Partition items to 2 groups
  - Items < pivot on the left
  - Items > pivot on the right
- Sort the left and the right partitions recursively
- …
- Done when problem trivial
- Time complexity: O(n lg n)

Pseudocode for quicksort

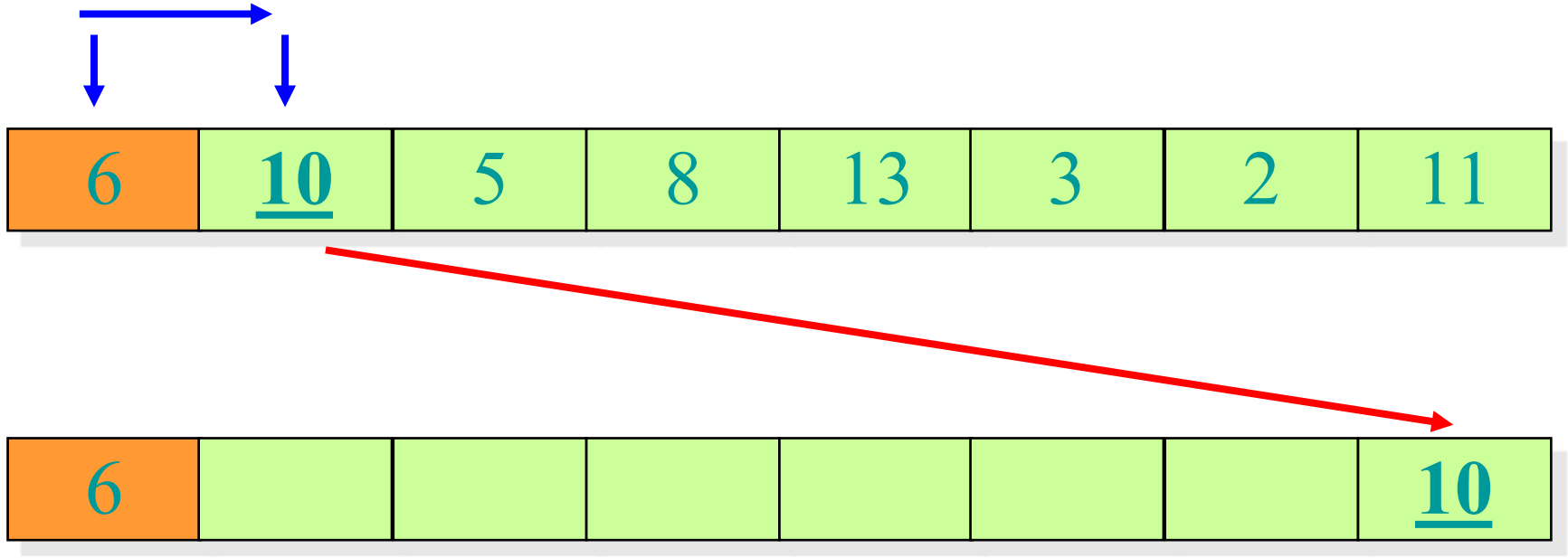


Key Step of Quick Sort: Partitioning O(n)



Idea of partition

- If we are allowed to use a second array, it would be easy



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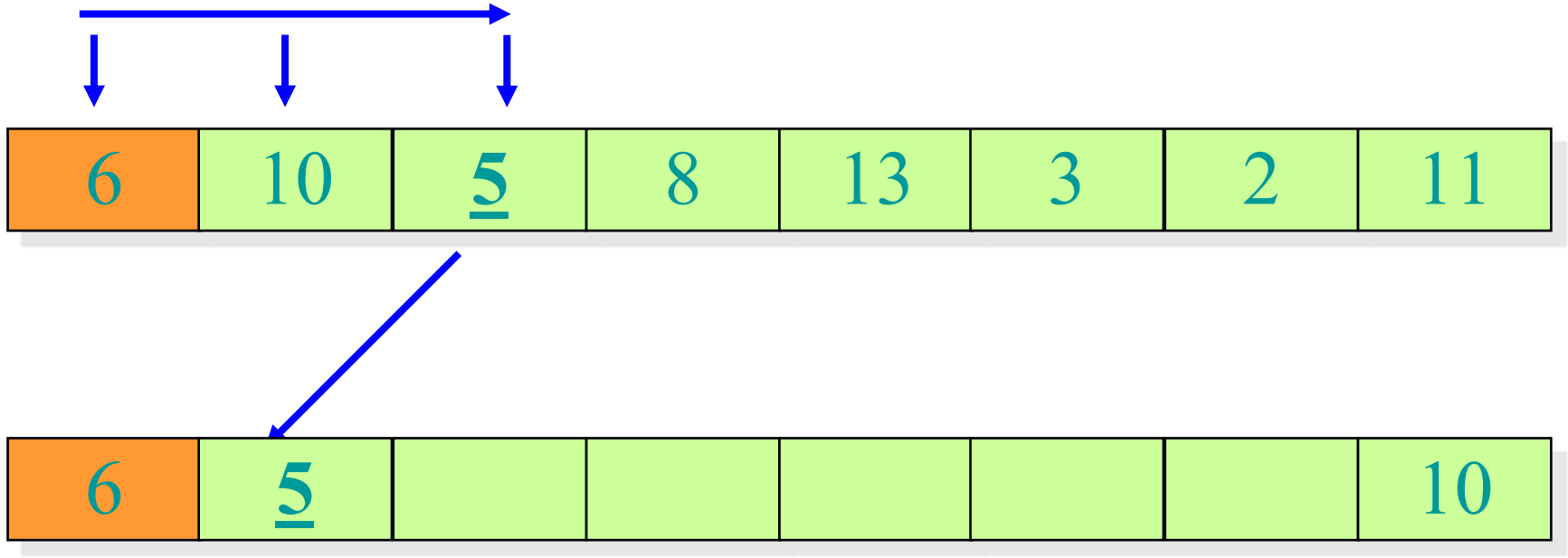

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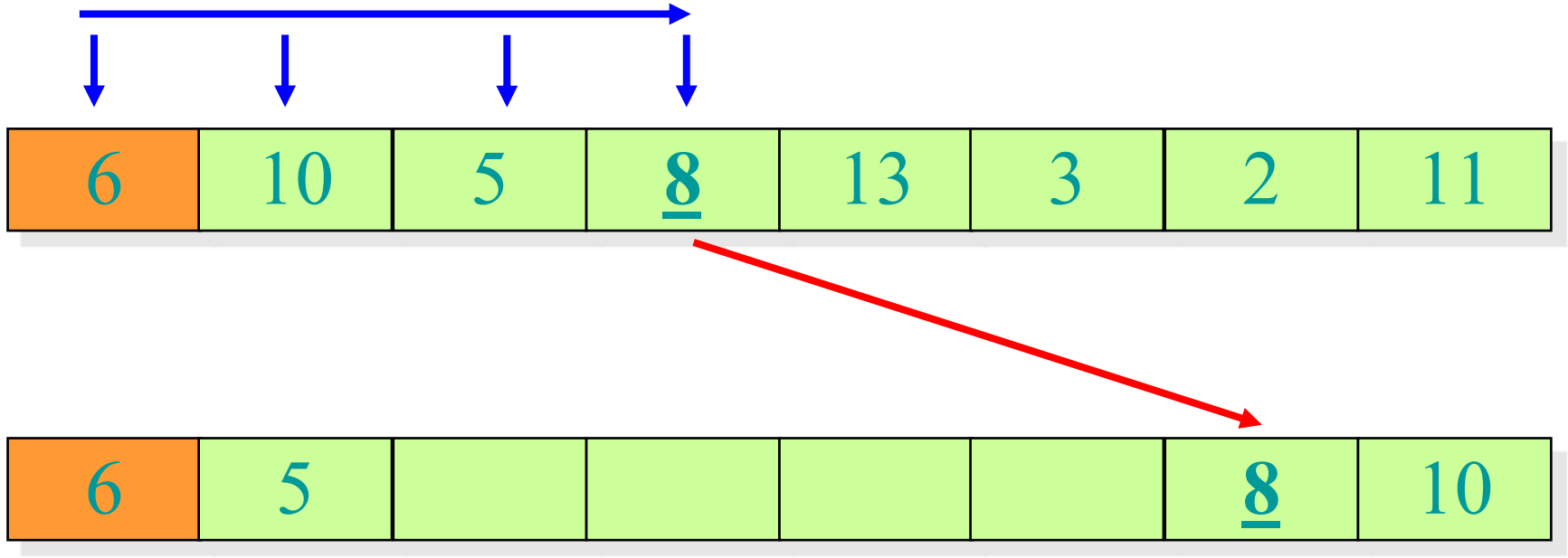

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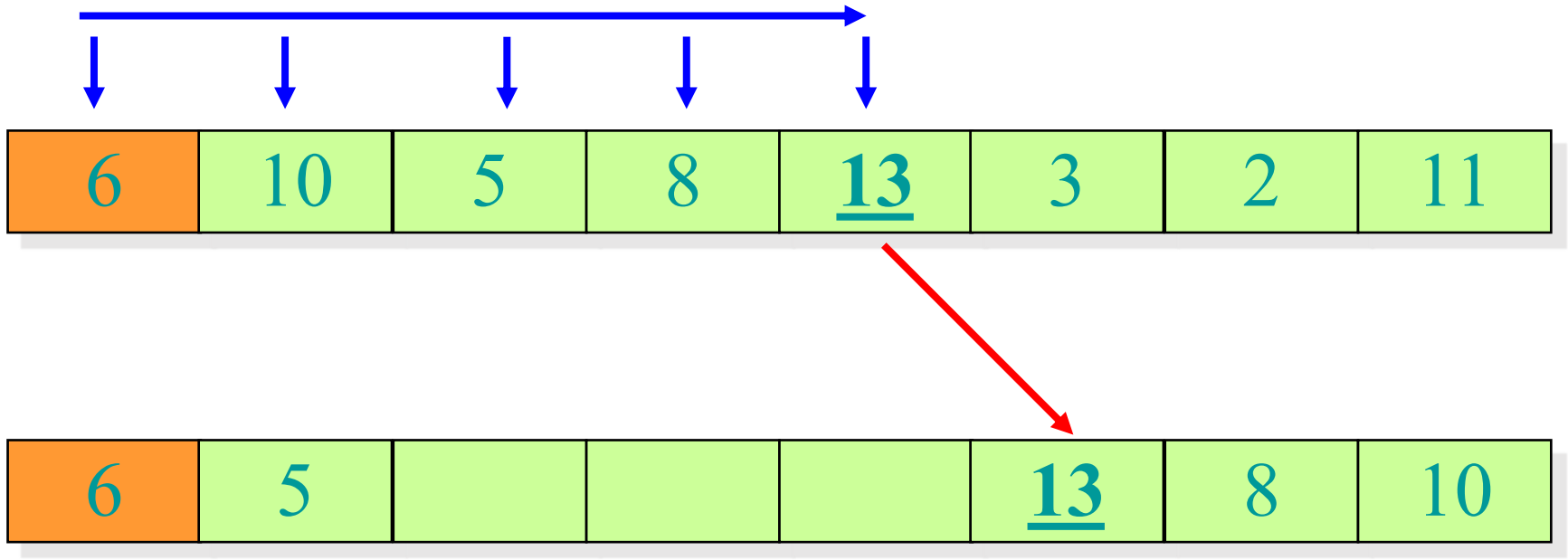

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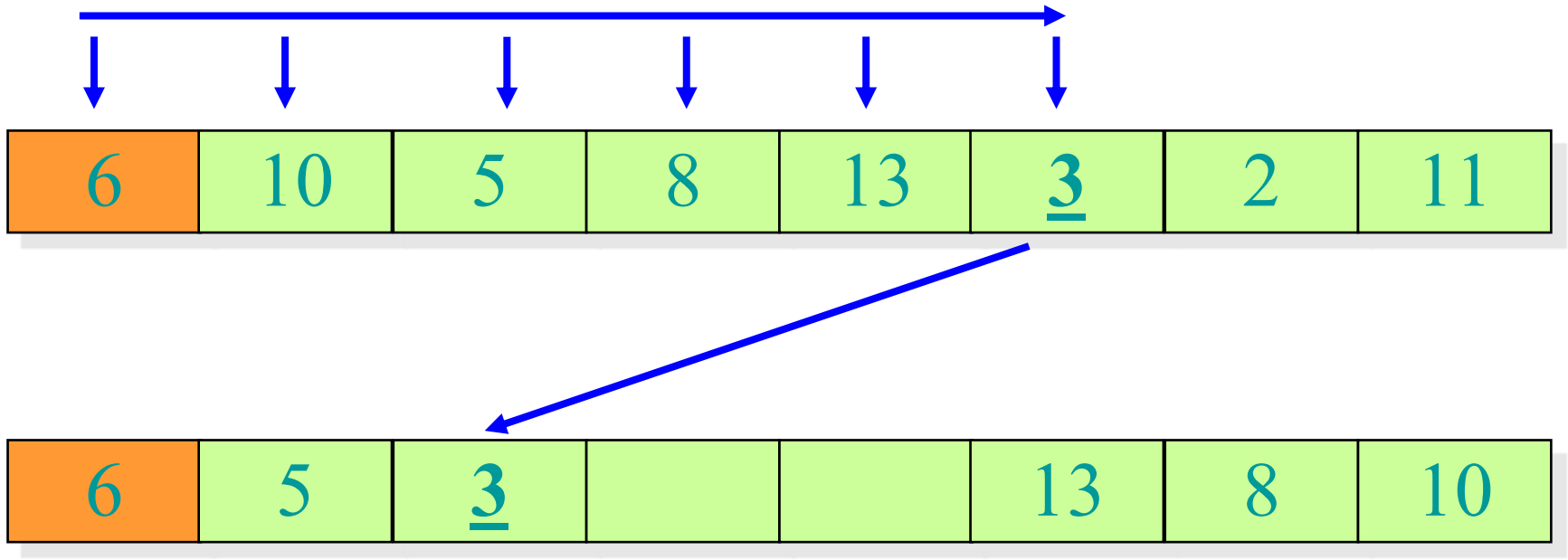

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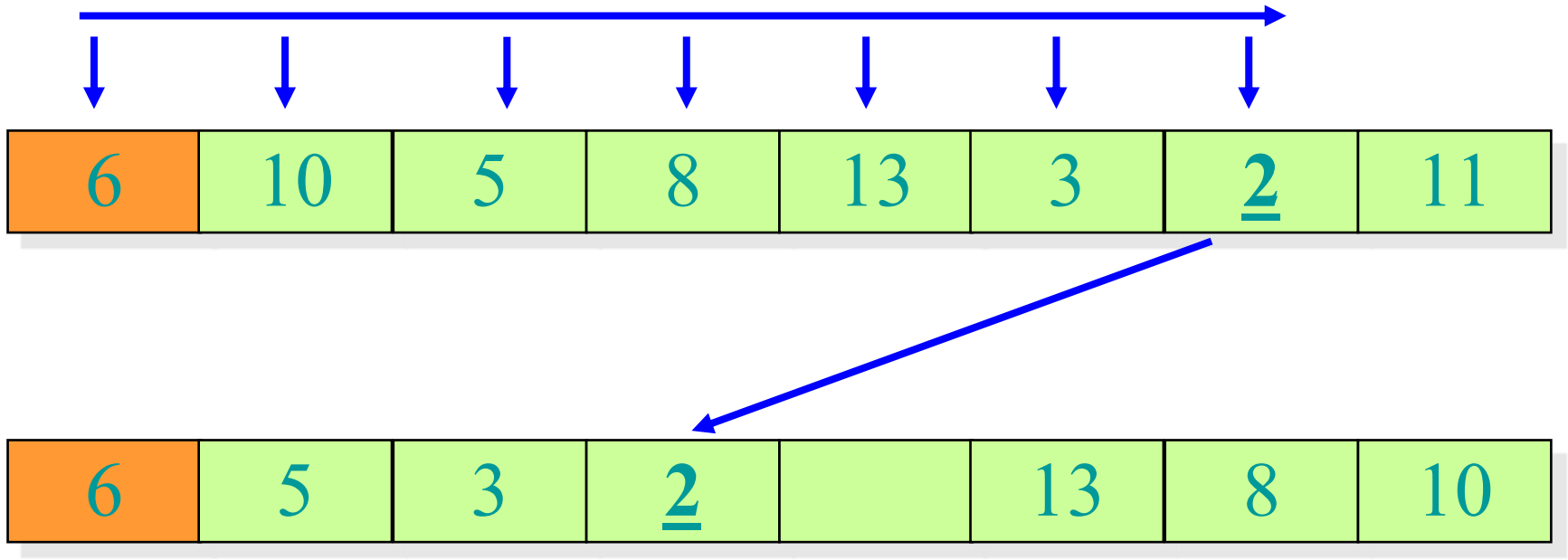

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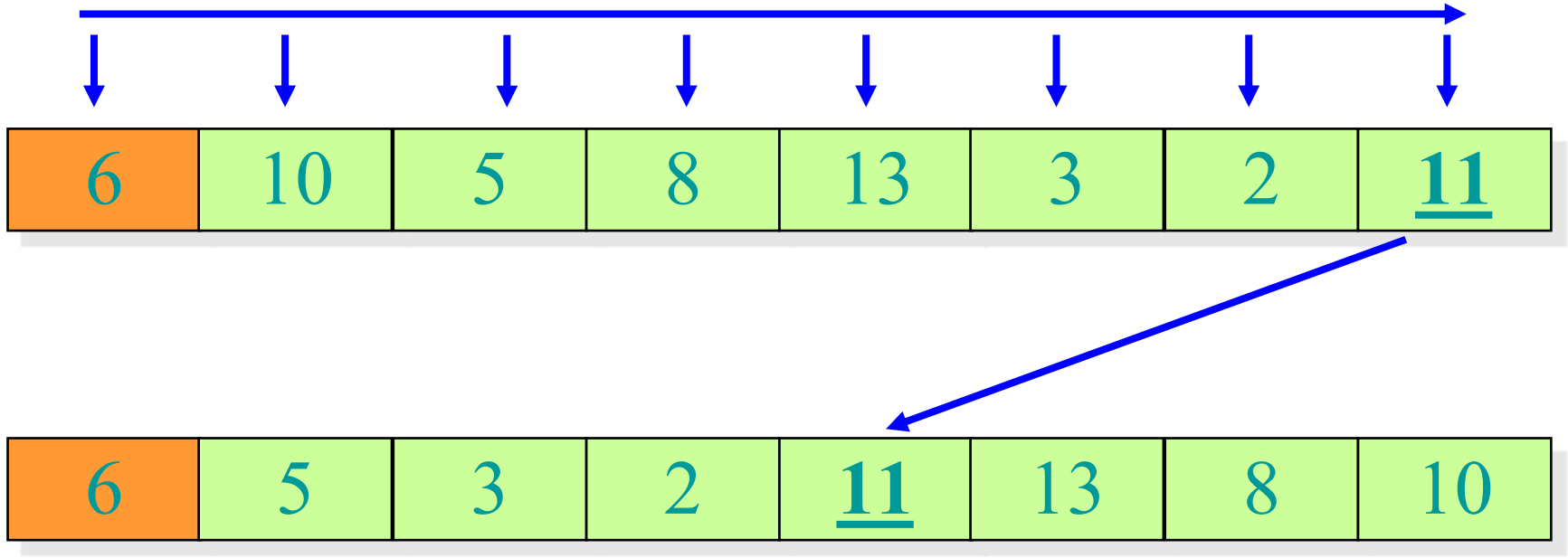

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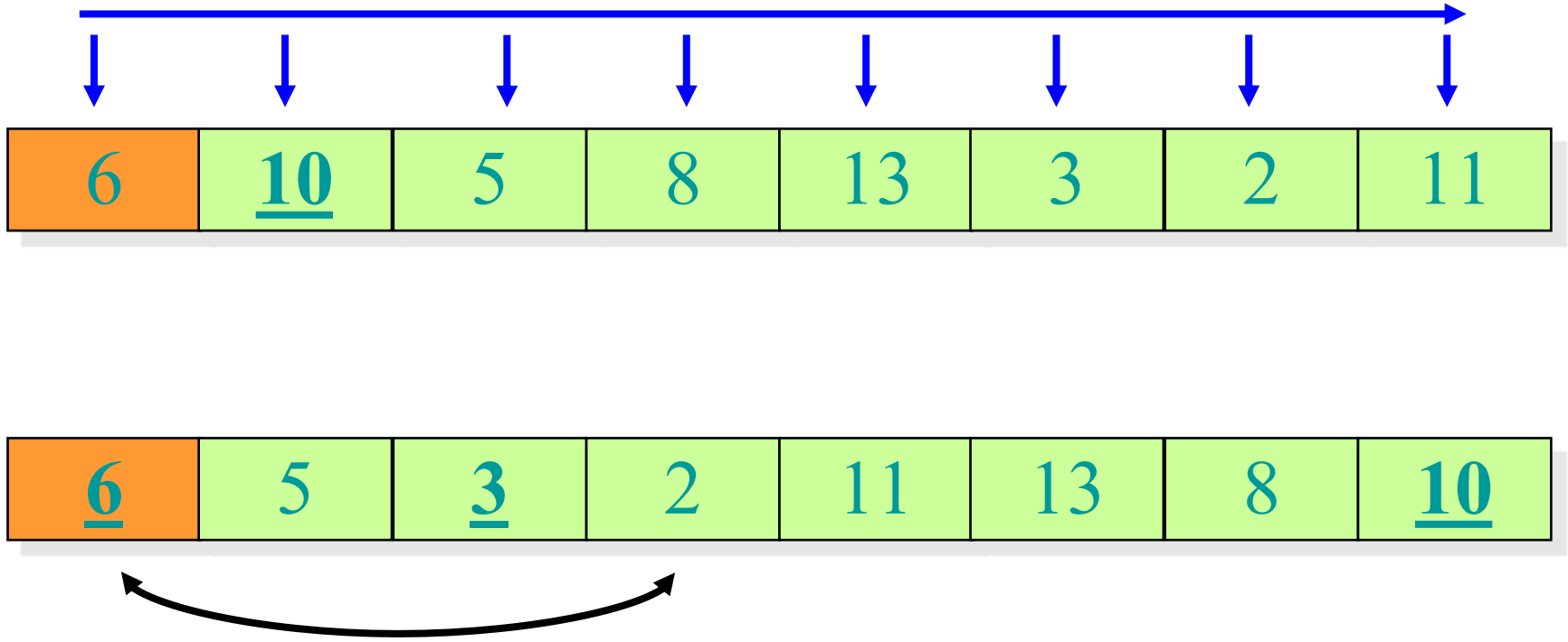
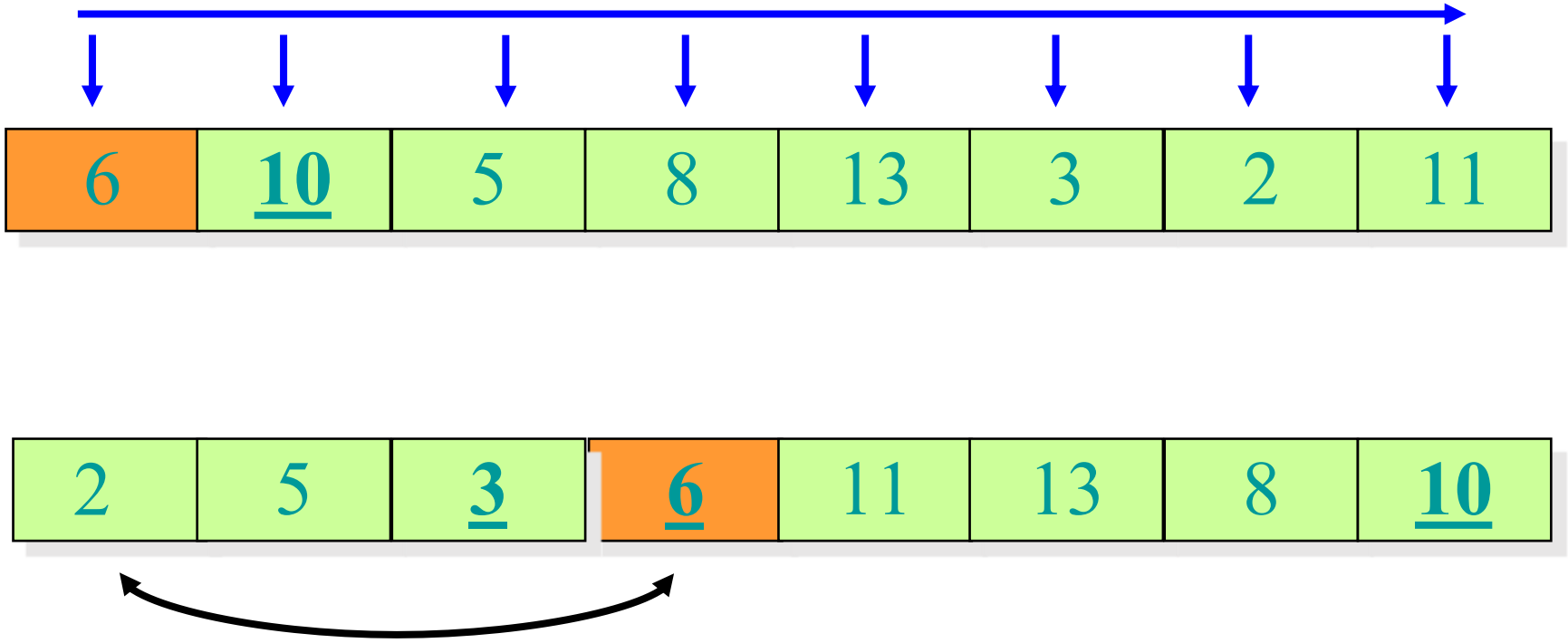
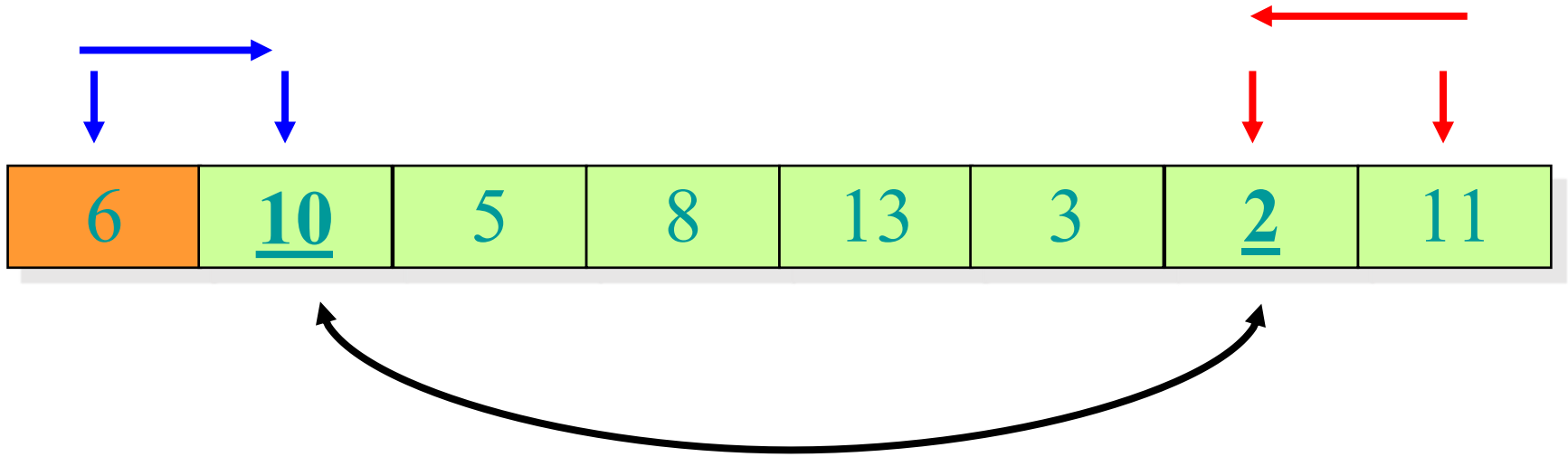


Idea of partition

- If we are allowed to use a second array, it would be easy

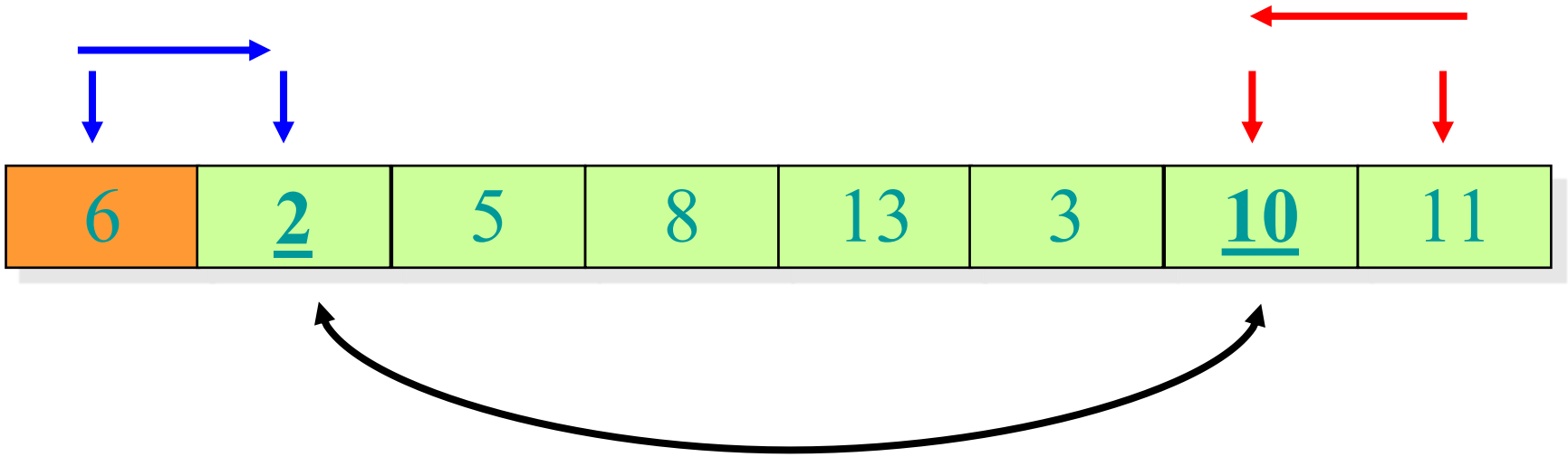


Idea of **in-place** partition



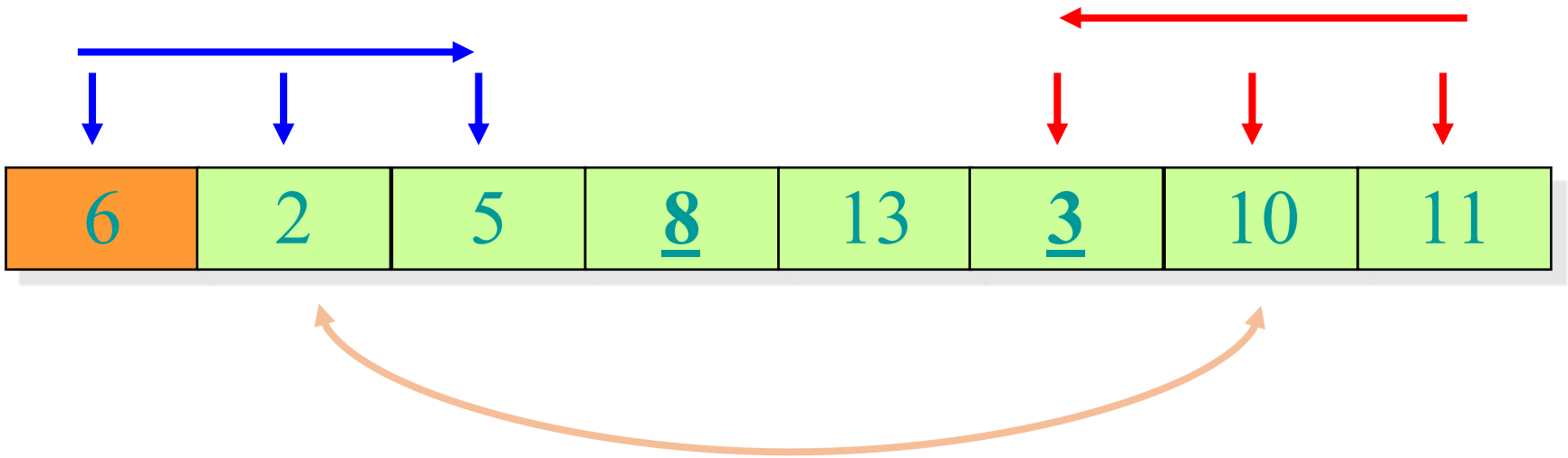
- Keep two iterators: one from head, one from tail
- Blue pointer stops at a number > 6(pivot)
- Red pointer stops at a number < 6 (pivot)
- Swap both numubers
  - Stop when BLUE and RED pointers cross over

Idea of **in-place** partition



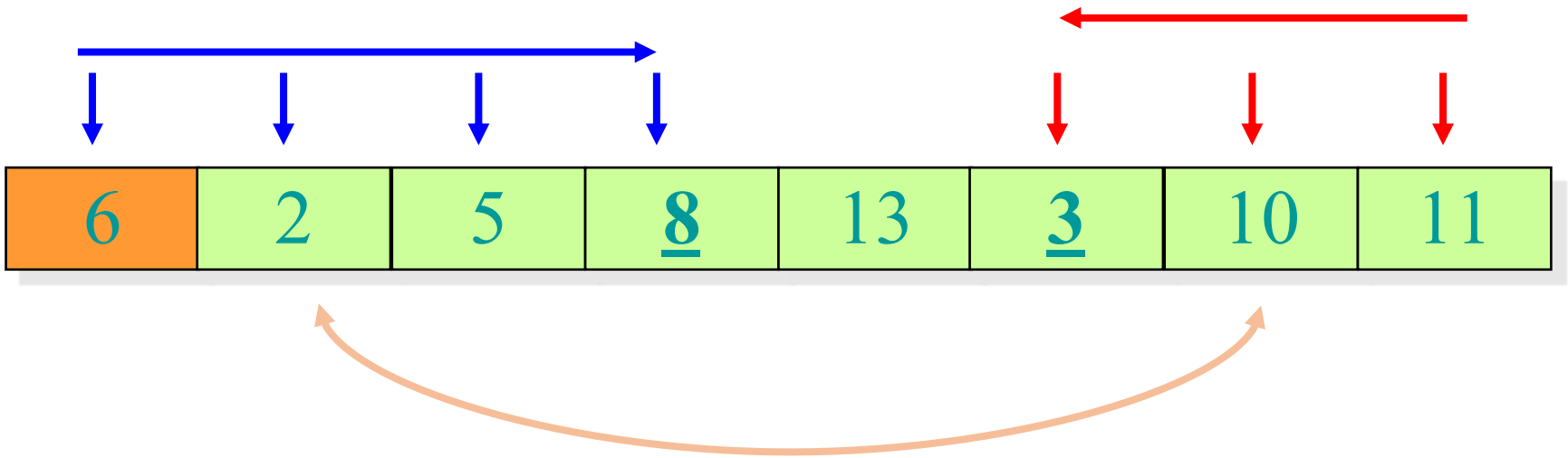
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



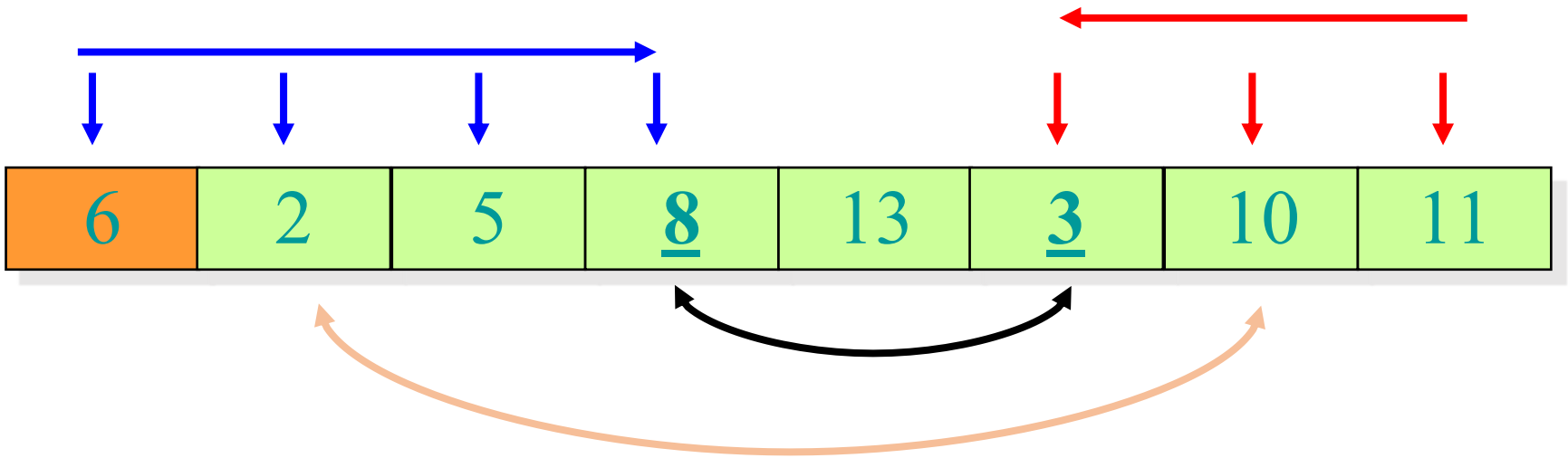
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



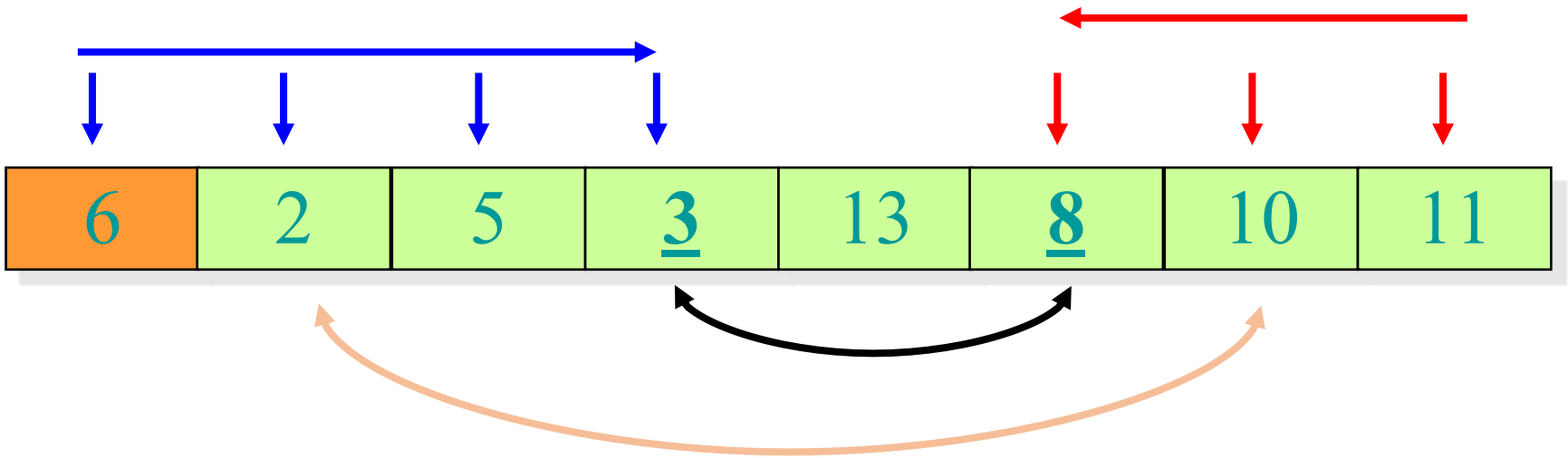
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



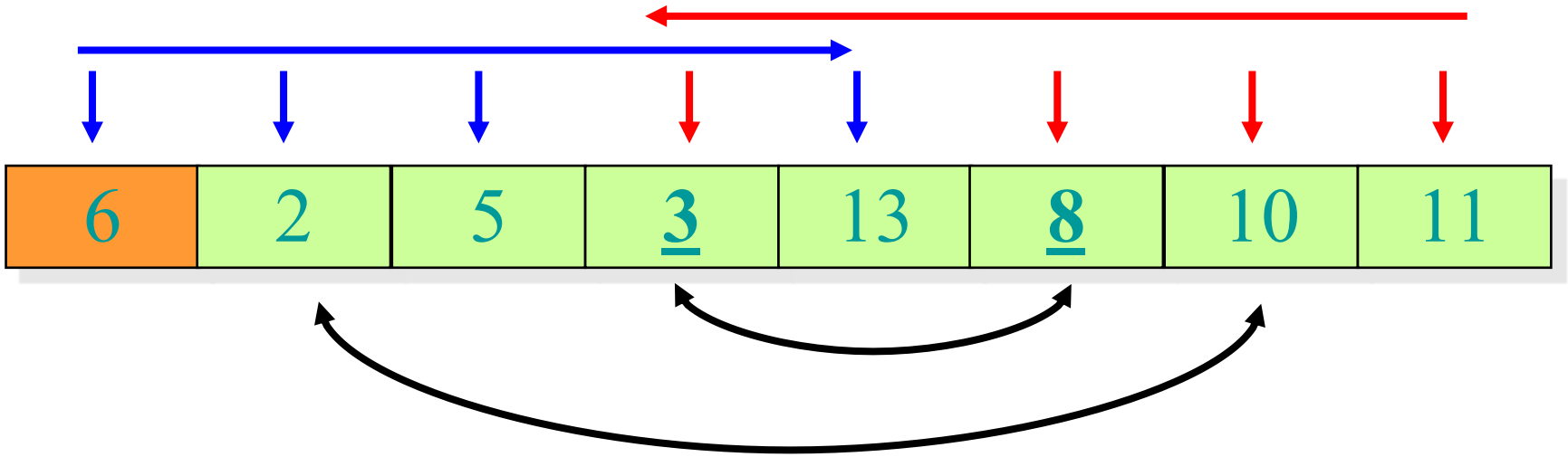
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



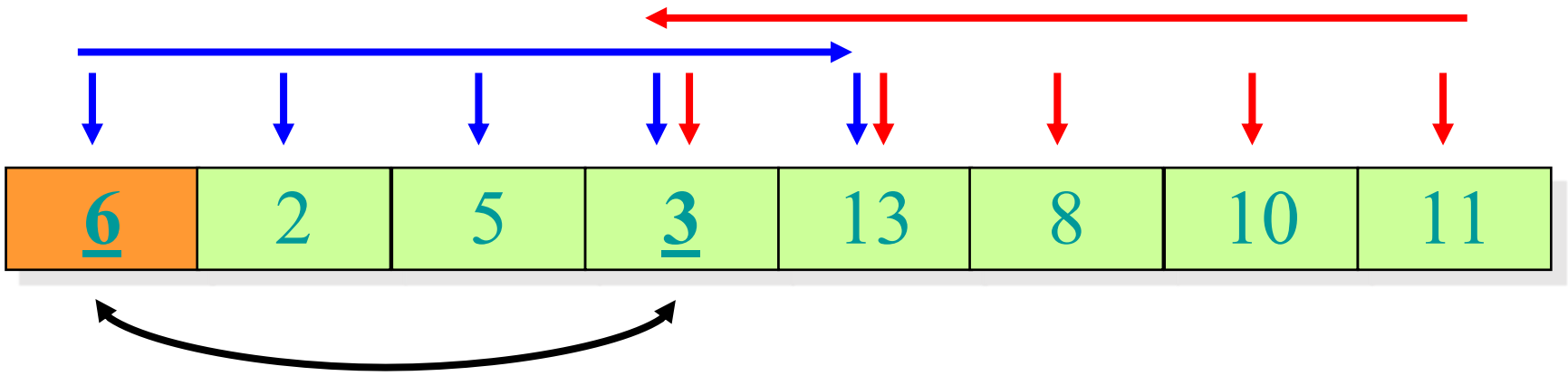
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



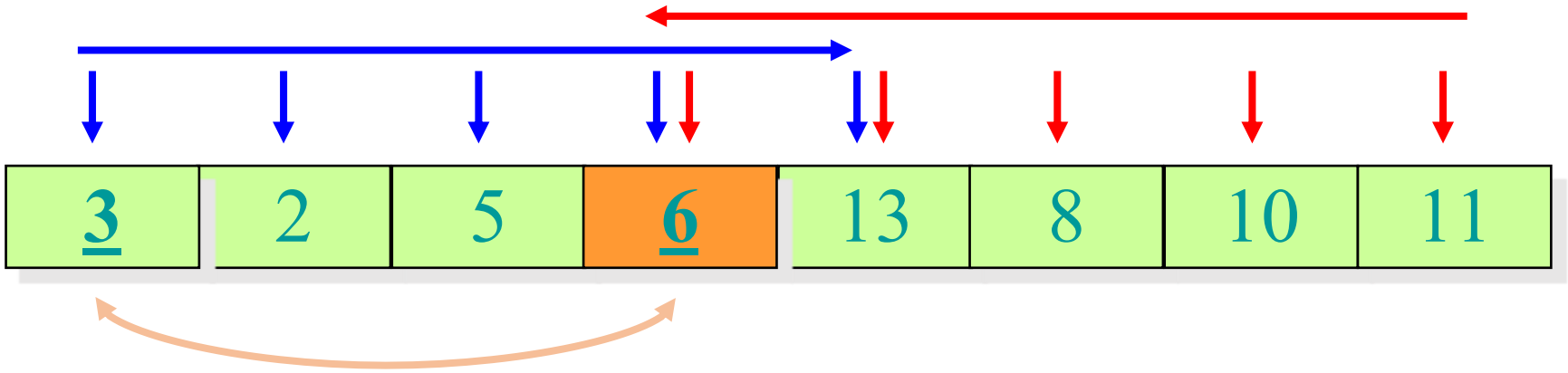
- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Idea of **in-place** partition



- Keep two iterators: one from head, one from tail
  - Stop when they cross over

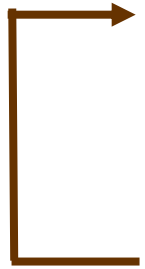
Idea of **in-place** partition



- Keep two iterators: one from head, one from tail
  - Stop when they cross over

Partition In Words

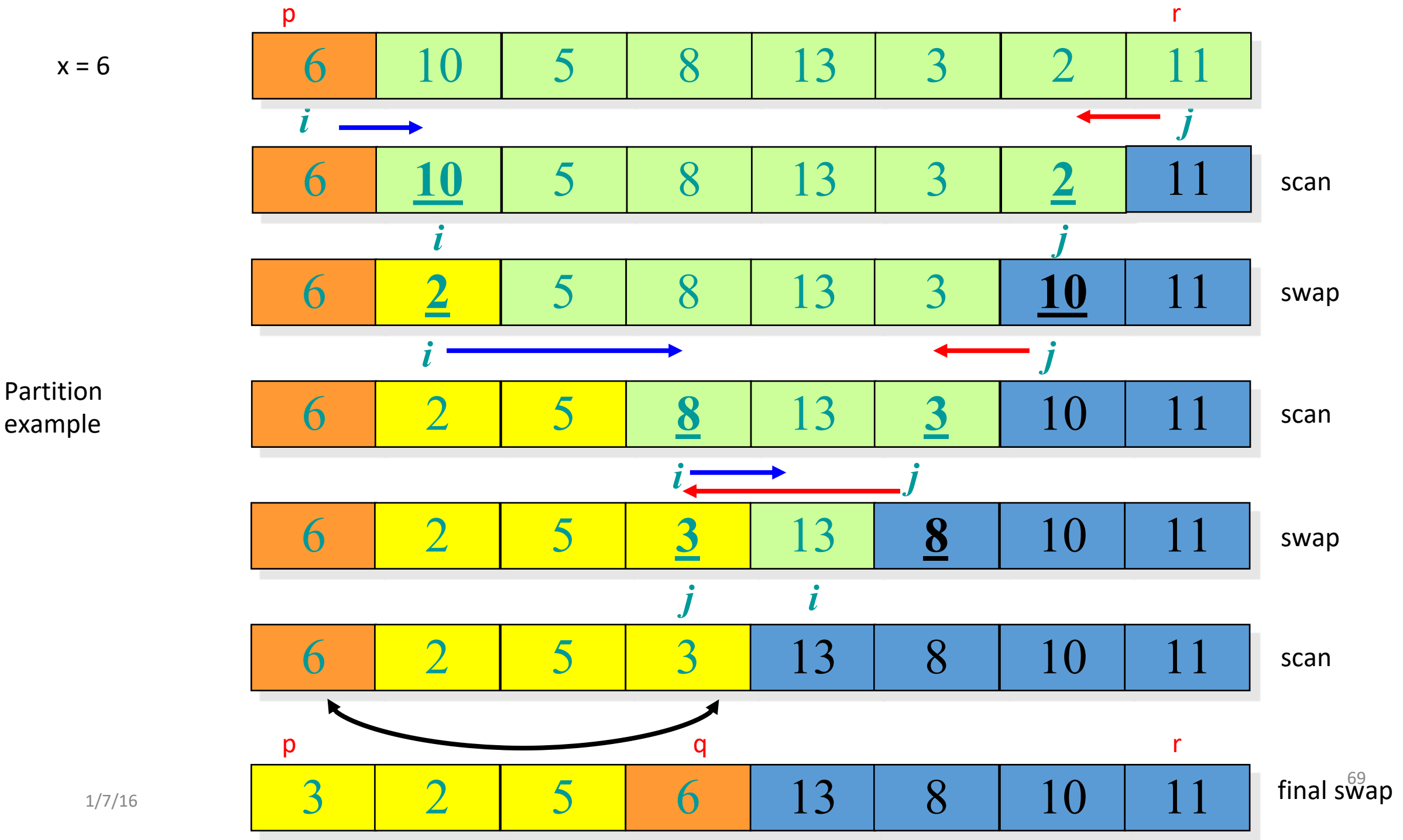
- Partition(A, p, r):
  - Select an element to act as the "pivot" (*which?*)
  - Grow two regions, A[p..i] and A[j..r]
    - All elements in A[p..i] <= pivot
    - All elements in A[j..r] >= pivot
  - Increment i until A[i] > pivot
  - Decrement j until A[j] < pivot
  - Swap A[i] and A[j]
  - Repeat until i >= j
  - Swap A[j] and A[p]
  - Return j



Note: different from book's **`partition()`**, **`which`** uses two iterators that both move forward.

Partition Code

Running time: $\Theta(n)$ *time*


x = 6
p
r
6 10 5 8 13 3 2 11
i
j
6 10 5 8 13 3 2 11
scan
i
j
6 2 5 8 13 3 10 11
swap
i
j
Partition example
6 2 5 8 13 3 10 11
scan
i
j
6 2 5 3 13 8 10 11
swap
j
i
6 2 5 3 13 8 10 11
scan
p
q
r
3 2 5 6 13 8 10 11
final swap

Quick sort example