CS217 - Data Structures & Algorithm Analysis (DSAA)

Lecture #3

➢ **Divide-and-Conquer**

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Reading: Section 2.3 and Section 4.5

(optional: lots more details in Chapter 4)

Aims of this lecture

- To introduce the divide-and-conquer design paradigm.
- To introduce the MergeSort algorithm – a recursive algorithm using divide-and-conquer.
- To show how to prove correctness for a recursive algorithm
- To show how to analyse the runtime of recursive algorithms using recurrence equations.
- To show how to solve recurrence equations

Problem: Find a number in a sorted array

- I have a sorted array of integers;

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

- Is the number 40 in the array?

- If we scan the array from the beginning to the end what is the worst case runtime? ***θ(n) – linear search***

- What if we always check the middle point and discard the "wrong" half of the subarray? $2^k = n$ => ***θ(log n) – binary search***

- By **dividing** the problem size by half at each step we have reduced the runtime of the algorithm from linear to **logarithmic**!

➢ Design Paradigms

- InsertionSort used an incremental approach:
  - Having sorted the subarray A[1..j-1], we inserted A[j] into its proper place, yielding the sorted subarray A[1..j].
  - **Idea:** **incrementally build up** a solution to the problem.
- Alternative design approach: **divide-and-conquer**
  1. **Divide:** Break the problem into smaller subproblems, smaller instances of the original problem.
  2. **Conquer:** Solve these problems recursively.
  3. **Combine** the solutions to subproblems into the solution for the original problem.

MergeSort

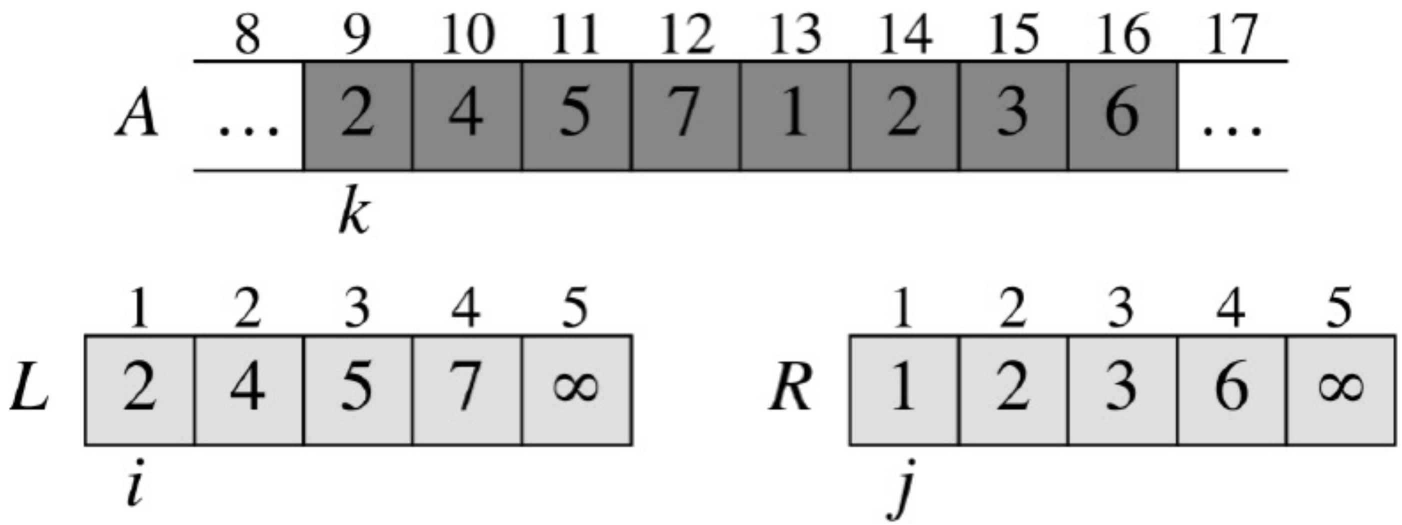
- MergeSort - sorting using **divide-and-conquer:**

  1. **Divide** the n-element sequence to be sorted into two subsequences of $n/2$ elements each.
  2. **Conquer:** Sort the two subsequences **recursively** using MergeSort.
  3. **Combine:** **merge** the two subsequences to produce the sorted answer.

- The recursion stops when the sequence is just 1 element.
- Key here is the procedure **Merge**
- Tedious bit: copying elements between arrays.

Merge(A, p, q, r)

- Assume subarrays $A[p \dots q]$ and $A[q+1 \dots r]$ are sorted.
- Copy these subarrays to new arrays L and R.
- Both L and R contain an additional element ∞ at the end ("sentinel"), so we don't have to check for end of array.
- Merge L and R back into A by comparing $L[i]$ and $R[j]$.



MERGE(A, p, q, r)

```
n1 = q - p + 1
n2 = r - q
let L[1 ... n1 + 1] and R[1 ... n2 + 1] be new arrays
for i = 1 to n1 do
        L[i] = A[p + i - 1]
for j = 1 to n2 do
        R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r do
        if L[i] ≤ R[j] then
                A[k] = L[i]
                i = i + 1
        else
                A[k] = R[j]
                j = j + 1
```

Set up arrays L and R (boring)

Actual merge

New book pseudo-code without sentinels

```
MERGE(A, p, q, r)
n_L = q - p + 1          // length of A[p : q]
n_R = r - q              // length of A[q + 1 : r]
let L[0 : n_L - 1] and R[0 : n_R - 1] be new arrays
for i = 0 to n_L - 1     // copy A[p : q] into L[0 : n_L - 1]
    L[i] = A[p + i]
for j = 0 to n_R - 1     // copy A[q + 1 : r] into R[0 : n_R - 1]
    R[j] = A[q + j + 1]
i = 0                    // i indexes the smallest remaining element in L
j = 0                    // j indexes the smallest remaining element in R
k = p                    // k indexes the location in A to fill
// As long as each of the arrays L and R contains an unmerged element,
//     copy the smallest unmerged element back into A[p : r].
while i < n_L and j < n_R
    if L[i] ≤ R[j]
        A[k] = L[i]
        i = i + 1
    else A[k] = R[j]
        j = j + 1
    k = k + 1
// Having gone through one of L and R entirely, copy the
//     remainder of the other to the end of A[p : r].
while i < n_L
    A[k] = L[i]
    i = i + 1
    k = k + 1
while j < n_R
    A[k] = R[j]
    j = j + 1
    k = k + 1
```

Runtime of Merge

$T(n) = \Theta(n)$

MERGE(A, p, q, r)

```
n1 = q - p + 1
n2 = r - q
let L[1 ... n1 + 1] and R[1 ... n2 + 1] be new arrays
for i = 1 to n1 do
    L[i] = A[p + i - 1]
for j = 1 to n2 do
    R[j] = A[q + j]
L[n1 + 1] = ∞
R[n2 + 1] = ∞
i = 1
j = 1
for k = p to r do
    if L[i] ≤ R[j] then
        A[k] = L[i]
        i = i + 1
    else
        A[k] = R[j]
        j = j + 1
```

$\Theta(n)$

Set up arrays L and R (boring)

only 1 loop

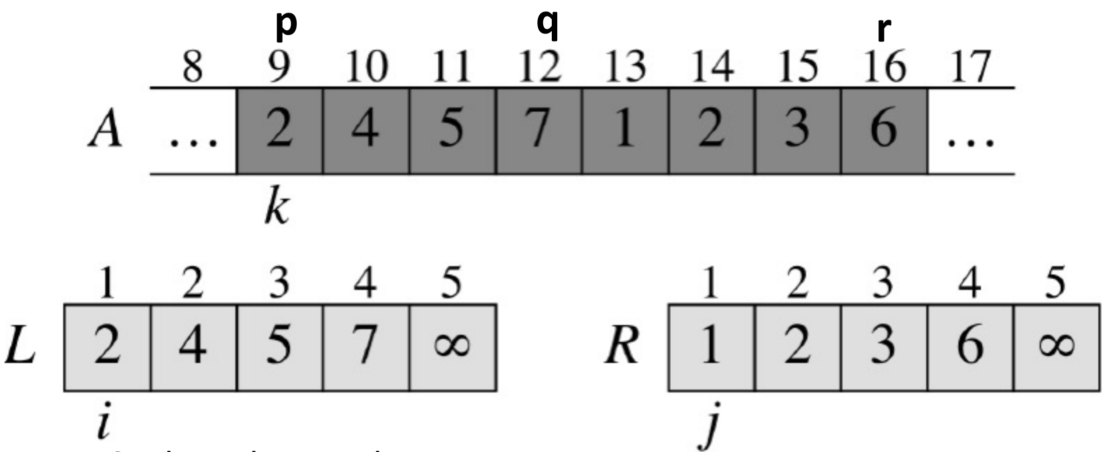
$\Theta(n)$

Actual merge

➢ Correctness of Merge (1)

- **Loop invariant:** At the start of the iteration of the last for loop,
  - the subarray $A[p \dots k-1]$ contains the $k-p$ smallest elements of $L[1 \dots n_1+1]$ and $R[1 \dots n_2+1]$, in sorted order and
  - $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back to A.

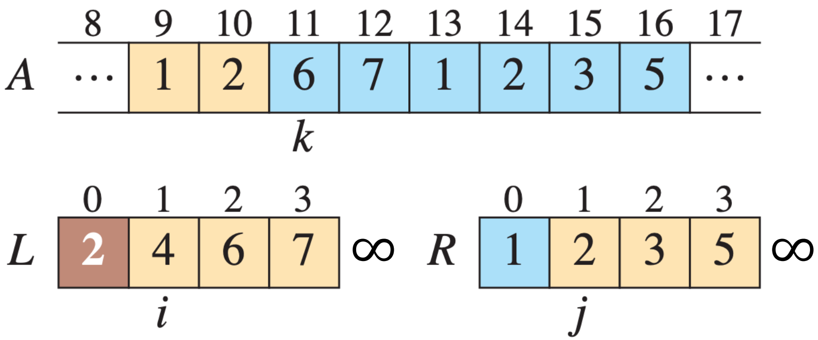
- **Initialisation:** the loop starts with $k=p$, hence $A[p \dots k-1]$ is empty and contains the $k-p=0$ smallest elements of L,R. As $i=j=1$, $L[i]$ and $R[j]$ are the smallest uncopied elements.



➢ Correctness of Merge (2)

- **Loop invariant:** At the start of the iteration of the last for loop,
  - the subarray $A[p \dots k-1]$ contains the $k-p$ smallest elements of $L[1 \dots n_1 + 1]$ and $R[1 \dots n_2 + 1]$, in sorted order and
  - $L[i]$ and $R[j]$ are the smallest elements of their arrays that have not been copied back to A.

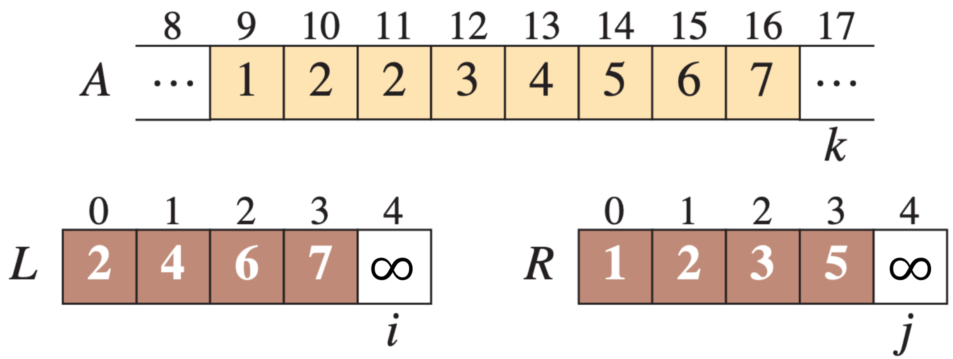
- **Maintenance:** suppose $L[i] \leq R[j]$. Then $L[j]$ is the smallest element not copied back. $A[p \dots k-1]$ contains the $k-p$ smallest elements, and after copying $L[j]$ into $A[k]$, $A[p \dots k]$ contains the $k-p+1$ smallest elements. Incrementing k and j re-establishes the loop condition.
  Argue similarly for $R[j] < L[i]$.



Correctness of Merge (3)

- **Loop invariant:** At the start of the iteration of the last for loop,
  - the subarray $A[p \ldots k-1]$ contains the $k-p$ smallest elements of $L[1 \ldots n_1+1]$ and $R[1 \ldots n_2+1]$, in sorted order and
  - $L[i]$ and $R[i]$ are the smallest elements of their arrays that have not been copied back to A.

- **Termination:** at termination, $k = r + 1$. By the loop invariant, $A[p \ldots k-1] = A[p \ldots r]$ contains the $k - p = r - p + 1 = n_1 + n_2$ smallest elements of $L[1 \ldots n_1 + 1]$ and $R[1 \ldots n_2 + 1]$, in sorted order. That's all elements in L and R apart from the two ∞.



MergeSort: The Complete Algorithm

Notation: $\lfloor x \rfloor$ means "floor of *x*" (rounding down).

$\textsc{MergeSort}(A, p, r)$

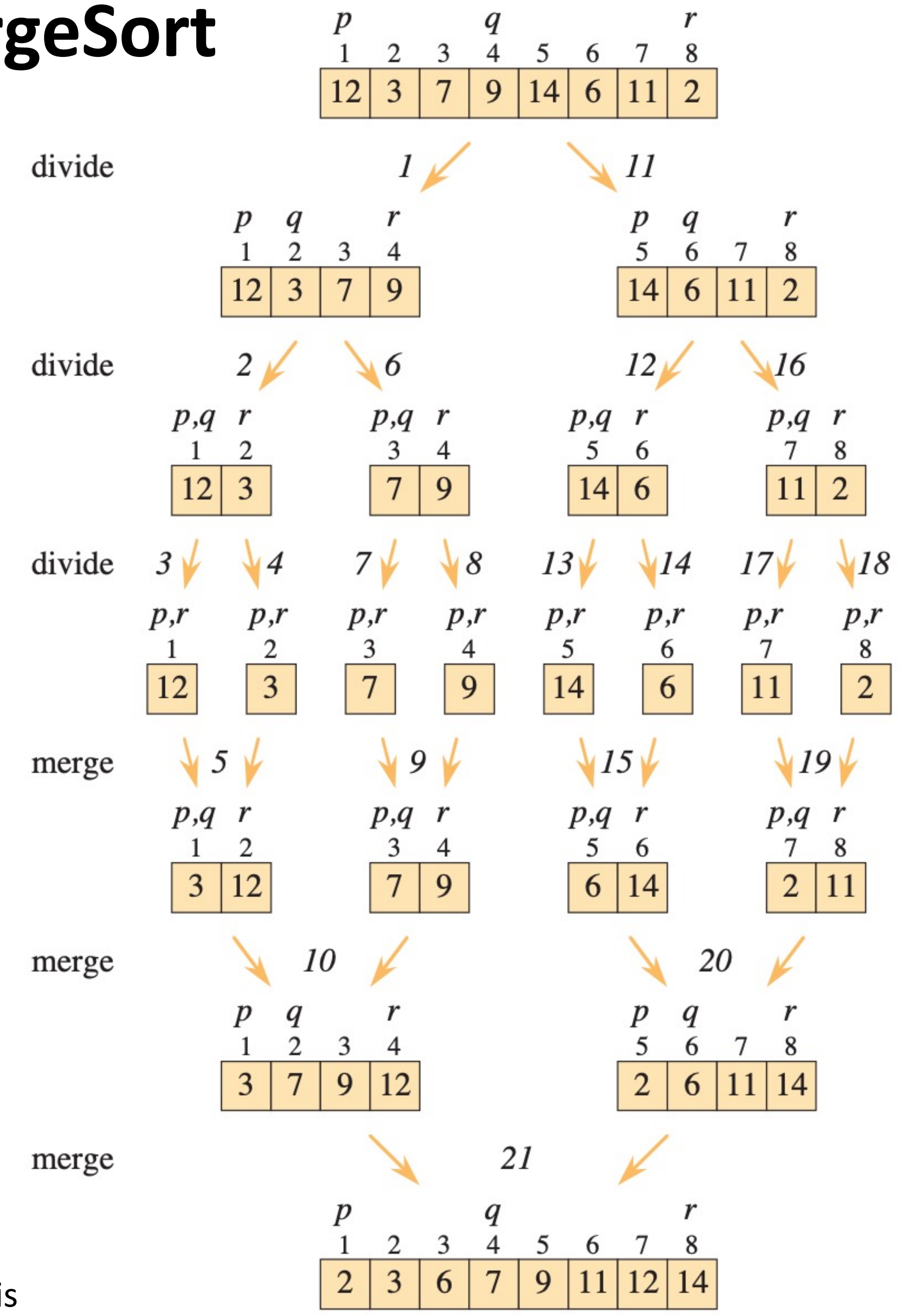
1: **if** $p < r$ **then**
2: $\quad q = \lfloor (p + r)/2 \rfloor$
3: $\quad \textsc{MergeSort}(A, p, q)$
4: $\quad \textsc{MergeSort}(A, q + 1, r)$
5: $\quad \textsc{Merge}(A, p, q, r)$

Initial call: $\textsc{MergeSort}(A, 1, \text{A.length})$

Operation of MergeSort

$\textsc{MergeSort}(A, p, r)$

```
if p < r then
    q = ⌊(p + r)/2⌋
    MergeSort(A, p, q)
    MergeSort(A, q + 1, r)
    Merge(A, p, q, r)
```



Correctness of MergeSort

$\textsc{MergeSort}(A, p, r)$

```
if p < r then
    q = ⌊(p + r)/2⌋
    MergeSort(A, p, q)
    MergeSort(A, q + 1, r)
    Merge(A, p, q, r)
```

Proof by Induction:

Weak induction

- **Base case:** Show statement true for initial case: $n = a$ (usually *n=0 or n=1)*
- **Inductive step:** If assumed true for *n* and can show true for *n+1* then true for all $n \geq a$

Strong induction

- **Base case:** Show statement true for initial case: $n = a$ (usually *n=0 or n=1)*
- **Inductive step:** If assumed true for all $a \leq k \leq n$ and can show true for *n+1* then true for all $n \geq a$

Strong induction can be proved using **Weak induction**

➢ Correctness of MergeSort

Proof by Induction:

$\textsc{MergeSort}(A, p, r)$

1: **if** $p < r$ **then**
2: $\quad q = \lfloor (p + r)/2 \rfloor$
3: $\quad \textsc{MergeSort}(A, p, q)$
4: $\quad \textsc{MergeSort}(A, q + 1, r)$
5: $\quad \textsc{Merge}(A, p, q, r)$

Assume MergeSort sorts correctly arrays of size <*n* and show that it sorts correctly an array of size *n*

- **Base case:** *n=1 =>* the algorithm returns at line 1 with the sorted array of a single element
- **Inductive step:** by inductive assumption lines 3 and 4 return two sub-arrays sorted correctly. We have already proved that **Merge** is correct hence after its execution the algorithm will return the array A sorted

$\blacksquare$

MergeSort: Runtime Analysis

- Looking for time ***T(n)****:* time for MergeSort to sort *n* elements.
- Assume for simplicity that n is an exact power of 2.

$\textsc{MergeSort}(A, p, r)$

if $p < r$ **then**	$\Theta(1)$
$\quad q = \lfloor (p + r)/2 \rfloor$	$\Theta(1)$
$\quad \textsc{MergeSort}(A, p, q)$	*T(n/2)*
$\quad \textsc{MergeSort}(A, q + 1, r)$	*T(n/2)*
$\quad \textsc{Merge}(A, p, q, r)$	$\Theta(n)$

Time for MergeSort to sort *n/2* elements.

Yields a recurrence equation where *T(n)* depends on *T(n/2):*

- If n=1, then $p = r$, and the algorithm terminates in constant time $\Theta(1)$

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 2^0 = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n = 2^k, \text{ for } k \geq 1 \end{cases}$$

- "The time for MergeSort to sort *n* elements is twice the time for MergeSort to sort *n/2* elements plus $\Theta(n)$ time (for Merge)."

Recurrence Equation (MergeSort)

- Looking for time ***T(n)***: time for MergeSort to sort *n* elements.

- Assume for simplicity that n is an exact power of 2.

$\textsc{MergeSort}(A, p, r)$	Time
if $p < r$ **then**	$\Theta(1)$
$\quad q = \lfloor (p+r)/2 \rfloor$	$\Theta(1)$
$\quad \textsc{MergeSort}(A, p, q)$	$T(n/2)$
$\quad \textsc{MergeSort}(A, q+1, r)$	$T(n/2)$
$\quad \textsc{Merge}(A, p, q, r)$	$\Theta(n)$

Time for MergeSort to sort *n/2* elements.

Yields a recurrence equation where *T(n)* depends on *T(n/2)*:

- If n=1, then p=r, and the algorithm terminates in constant time $\Theta(1)$
- Otherwise: ***T(n) = D(n) + a T(n/b) + C(n)***
  - D(n) - time to *divide* into subproblems: $\Theta(1)$
  - *a T(n/b)* – time to solve *a* subproblems each of size *n/b*: *2 T(n/2)*
  - C(n) – time to *conquer* (to combine the obtained sub-solutions): $\Theta(n)$

How to Solve a Recurrence Equation

$$T(n) = \begin{cases} d & \text{if } n = 2^0 \\ 2T(n/2) + cn & \text{if } n = 2^k, \text{ for } k \geq 1 \end{cases}$$

1. **Substitution method** (Sec 4.3): guess a solution and verify using **induction** (over *k*).
   - Tutorial exercise.
2. Draw a **recursion tree** (Sec 4.4), add times across the tree.
3. Use the **Master Theorem** (Sec 4.5) to solve a general recurrence equation in the shape of:

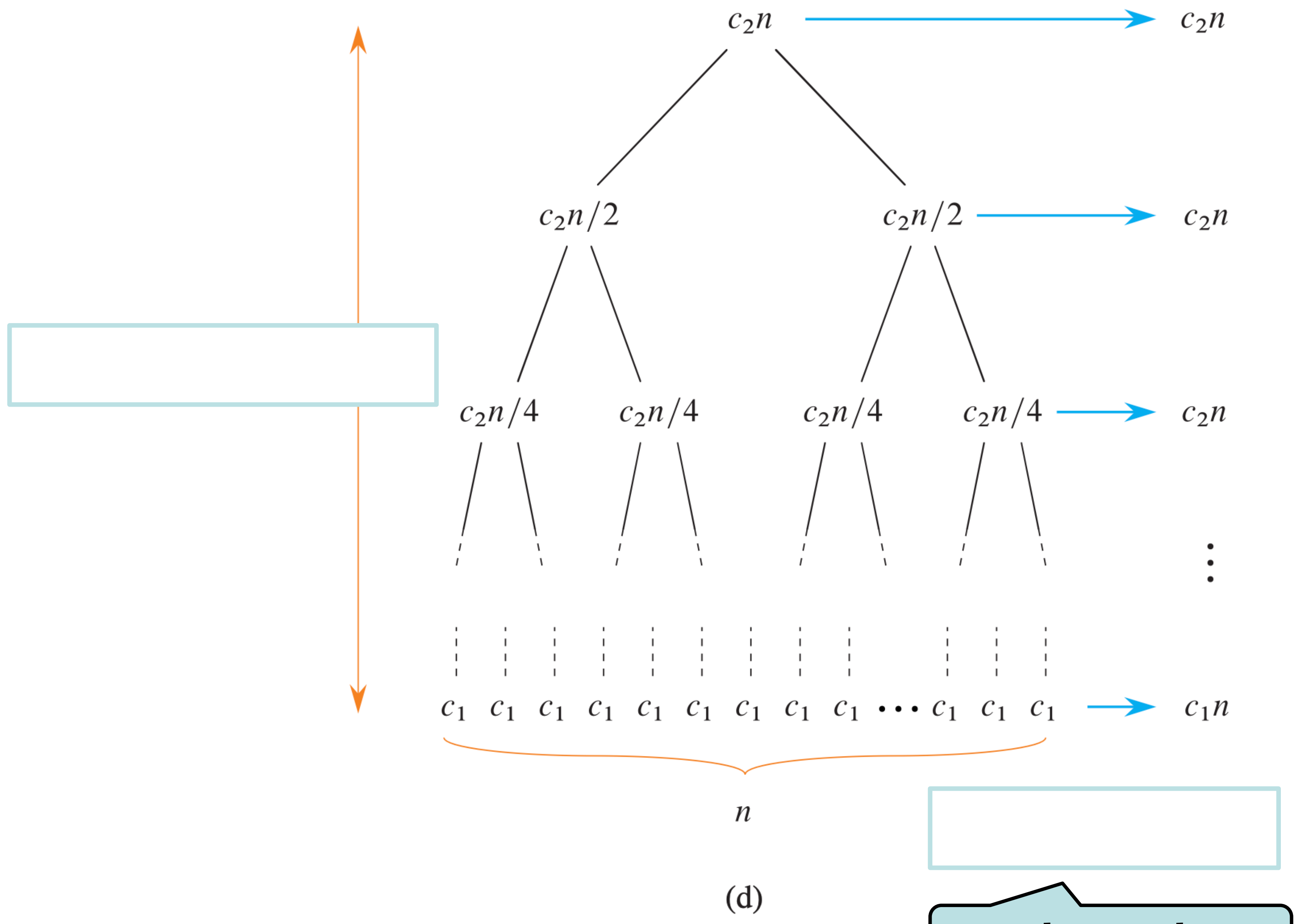
$$T(n) = aT(n/b) + f(n).$$

Runtime Visualised as Recursion Tree

$T(n)$

(a)

Runtime Visualised as Recursion Tree



I use "$\log$" for $\log_2$

Comparison with InsertionSort

- MergeSort **always runs in time $\boldsymbol{\Theta(n \log n)}$**.
- Way better than worst case and average case of $\Theta(n^2)$ for InsertionSort.
- Worse than the best-case time $\Theta(n)$ of InsertionSort.
  - InsertionSort might be faster if your array is almost sorted.
- MergeSort needs **more space** than InsertionSort:
  - MergeSort always stores $\Omega(n)$ elements outside the input.
  - InsertionSort only needs $O(1)$ additional space.
  - We say that InsertionSort sorts **in place**:

A sorting algorithm sorts **in place**
if it only uses $O(1)$ additional space.

➢ The Master Theorem (1)

- Provides a “cookbook” method for solving recurrences of the form ***T(n)=aT(n/b) + f(n)*** where a>0 and b>1

- ***f(n)*** is called the **driving function** and ***T(n)*** is called the **master recurrence**

- The master recurrence ***T(n)*** describes the running time of a divide and conquer algorithm that divides a problem of size n into ***a*** subproblems each of size ***n/b*** < *n*
  -> the algorithm solves each subproblem in time ***T(n/b)***

- The driving function ***f(n)*** describes the cost of dividing the problem before the recursion **(divide)**, as well as the cost of combining the results together **(conquer)**

Important term:

- $\boldsymbol{n}^{\log_{\boldsymbol{b}} \boldsymbol{a}}$ is called the **watershed function**

The Master Theorem (Statement)

Let *a > 0 and b > 1* be constants, and let *f(n)* be non-negative for large enough n. Then, the solution of the recurrence function defined over $n \in \mathbb{N}$

$$T(n) = a\,T(n/b) + f(n)$$

has the following asymptotic behaviour:

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

➢ The Master Theorem (Properties)

- Allows you to state the master recurrence T(n) without floors and ceilings even when you don't have problems of exactly the same size

  eg., $T(n) = T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor) + \theta(n)$

- The theorem does not apply to all possible recurrence equations but it does cover the vast majority of those that arise in practice

The Master Theorem: closer look

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

- $\boldsymbol{n^{\log_b a}}$ is called the **watershed function**
- **Case 1:** the watershed function must grow **polynomially faster** than f(n) – by at least a factor $\theta(n^{\epsilon})$ for some constant $\epsilon > 0$
- **Case 2:** watershed and driving (f(n)) functions grow asymptotically **nearly at the same rate** (you get the same growth for $k = 0$ – common situation)
- **Case 3:** the watershed function must grow **polynomially slower** than f(n) – by at least a factor $\theta(n^{\epsilon})$ for some constant $\epsilon > 0$ + **regularity condition** must hold

The Master Theorem: MergeSort example

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

- MergeSort: $\mathrm{T(n)} = 2\mathrm{T(n/2)} + \theta(\mathrm{n})$
- a=2, b=2, $\mathrm{f(n)} = \theta(\mathrm{n})$ watershed function: $\boldsymbol{n}^{\log_b a} = \boldsymbol{n}^{\log_2 2} = \boldsymbol{n}^1 = \boldsymbol{n}$
- Does **Case 1** hold? Does the watershed function grow **polynomially faster** than f(n) ?
- Does **Case 3** hold? Does the watershed function grow **polynomially slower** than f(n) ?

The Master Theorem: MergeSort example

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a - \epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a + \epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

- MergeSort: $T(n) = 2T(n/2) + \theta(n)$
- a=2, b=2, $f(n) = \theta(n)$ watershed function: $\boldsymbol{n^{\log_b a}} = \boldsymbol{n^{\log_2 2}} = \boldsymbol{n^1} = \boldsymbol{n}$
- Does **Case 2** hold?

  Yes! for $k = 0$, $f(n) = \Theta(n^{log_b a}\ log^0 n) = \Theta(n)$
- So the solution is $T(n) = \Theta(n^{log_b a}\ log^{k+1} n) = \Theta(n \log n)$

➢ The Master Theorem: Further examples (1)

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a-\epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a+\epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ∎

- $\mathrm{T(n) = 9T(n/3) + n}$
- a=9, b=3, f(n) = n watershed function: $\boldsymbol{n^{\log_b a}} = \boldsymbol{n^{\log_3 9}}$ = $\boldsymbol{n^2}$
- Does **Case 1** hold? Does the watershed function must grow **polynomially faster** than f(n) ?

  Yes! f(n) = n = $O(\boldsymbol{n^{\log_b a -\epsilon}})$ =$O(n^{2-\epsilon})$ for any $\epsilon < 1$
- So the solution is $\mathrm{T(n)} = \theta\left(\boldsymbol{n^{\log_b a}}\right) = \boldsymbol{\theta(n^2)}$

➢ The Master Theorem: Further examples (2)

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a-\epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.

2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.

3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a+\epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

- $T(n) = 3T(n/4) + n \log n$
- a=3, b=4, f(n) = n log n, watershed function: $\boldsymbol{n}^{\log_b a} = \boldsymbol{n}^{\log_4 3} = \boldsymbol{n}^{0.793}$
- Does **Case 1** hold? Does the watershed function must grow **polynomially faster** than f(n) ?

The Master Theorem: Further examples (2)

1. If there exists a constant $\epsilon > 0$ such that $f(n) = O(n^{\log_b a-\epsilon})$, then $T(n) = \Theta(n^{\log_b a})$.
2. If there exists a constant $k \geq 0$ such that $f(n) = \Theta(n^{\log_b a} \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
3. If there exists a constant $\epsilon > 0$ such that $f(n) = \Omega(n^{\log_b a+\epsilon})$, and if $f(n)$ additionally satisfies the ***regularity condition*** $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$. ■

- $\mathrm{T(n) = 3T(n/4) + n \log n}$
- a=3, b=4, f(n) = n log n, watershed function: $\boldsymbol{n}^{\log_b a} = \boldsymbol{n}^{\log_4 3} = \boldsymbol{n}^{0.793}$
- Does **Case 3** hold? Does the watershed function must grow **polynomially slower** than f(n) ?

  Yes! f(n) = n log n = $\Omega(\boldsymbol{n}^{\log_b a+\epsilon})$ =$\Omega(n^{0.793+\epsilon})$ for any $0 < \epsilon$
  < 0.207

  and $af\left(\frac{n}{b}\right) = 3(\frac{n}{4})(\log n/4) \leq c\, n \log n$ for $c = 3/4$
- So the solution is $\mathrm{T(n)} = \boldsymbol{\theta(f(n))} = \boldsymbol{\theta(n \log n)}$

Summary

- The divide-and-conquer design paradigm
  - **Divides** a problem into smaller subproblems of the same kind
  - **Solves** these subproblems recursively, and then
  - **Combines** these solutions to an overall solution.
- MergeSort uses divide-and-conquer to sort in time $\Theta(n \log n)$ (best case = worst case).
- It’s possible to sort n elements in worst-case time $\Theta(n \log n)$!
- Drawback: MergeSort does not sort in place.
  - “In place”: sorting using only $O(1)$ additional space.
- The runtime of recursive algorithms can be analysed by solving a **recurrence equation**.