Assignment XV - DSAA(H)

Name: Yuxuan HOU (侯宇轩)

Student ID: 12413104

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Question 15.1 (0.25 marks)

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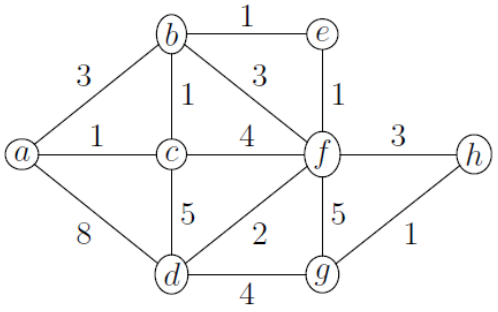
Compute the minimum spanning tree of the following weighted graph both with Prim's and Kruskal's algorithm. List the edges in the order considered, draw the tree and calculate its weight.



Sol:

1. Prim (start at a)

   Selected edges (in order): $ac(1), cb(1), be(1), ef(1), fd(2), fh(3), hg(1)$.

   MST edges: $\{ac(1), cb(1), be(1), ef(1), fd(2), fh(3), hg(1)\}$.

   Total weight: $1+1+1+1+2+3+1=10$.

   Sketch (connectivity): $a-c-b-e-f-h-g$ and $f-d$.

2. Kruskal (ties broken lexicographically)

   Edges considered (take/skip):

   $ac(1)$ take;

   $bc(1)$ take;

   $be(1)$ take;

   $ef(1)$ take;

   $gh(1)$ take;

   $df(2)$ take;

   $ab(3)$ skip (cycle);

   $bf(3)$ skip (cycle);

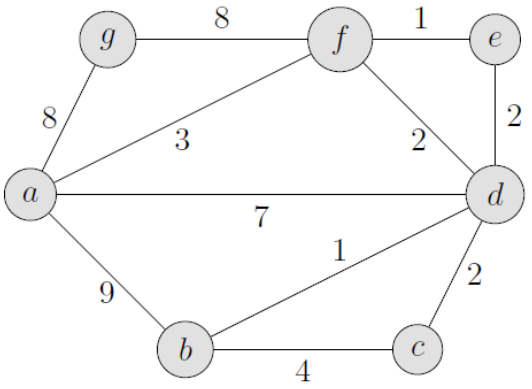
   $fh(3)$ take (stop).

   MST edges: $\{ac(1), bc(1), be(1), ef(1), gh(1), df(2), fh(3)\}$.

   Total weight: $1 + 1 + 1 + 1 + 1 + 2 + 3 = 10$.

Question 15.2 (0.25 marks)

Execute Dijkstra's algorithm on the following weighted graph to find a shortest path from vertex a to c. Show for each iteration of the while loop which vertex is added to the set S and how the distance estimates of adjacent vertices are being refined.



Sol:

Dijkstra from a to c

Initialization: $d[a] = 0$, all other $d[\cdot] = \infty$, $S = \varnothing$.

Relax from a: $d[f] = 3,\ d[d] = 7,\ d[g] = 8,\ d[b] = 9$ (each predecessor is a).

Iteration 1: extract f (min, $d[f] = 3$), add to S.

Relax via f:

- e: $3 + 1 = 4 < \infty \Rightarrow d[e] = 4,\ \pi[e] = f$
- d: $3 + 2 = 5 < 7 \Rightarrow d[d] = 5,\ \pi[d] = f$
- g: $3 + 8 = 11 > 8$ (no change)

Iteration 2: extract e ($d[e] = 4$), add to S.

Relax via e:

- d: $4 + 2 = 6 > 5$ (no change)

Iteration 3: extract d ($d[d] = 5$), add to S.

Relax via d:

- b: $5 + 1 = 6 < 9 \Rightarrow d[b] = 6,\ \pi[b] = d$
- c: $5 + 2 = 7 < \infty \Rightarrow d[c] = 7,\ \pi[c] = d$

Iteration 4: extract b ($d[b] = 6$), add to S.

Relax via b:

- c: $6 + 4 = 10 > 7$ (no change)

Iteration 5: extract c ($d[c] = 7$), add to S and stop.

Shortest path (by predecessors): $a \to f \to d \to c$.

Distance: $3 + 2 + 2 = 7$.

Question 15.3 (0.5 marks)

A precondition for Dijkstra's algorithm is that all edges of the directed graph under consideration have non-negative weight.

Somebody on the internet claims that the algorithm works for graphs with negative edge weights as well: just add an appropriate constant c to each edge weight to make all weights positive, then run Dijksta's algorithm, and finally remove the constants from the shortest paths computed.

Give a directed acyclic graph as a counterexample to falsify this claim. Explain in your own words what goes wrong.

Sol:

1. Counterexample DAG

   Vertices: a, b, c.

   Directed edges (weights): $a \to c : 0$, $a \to b : 1$, $b \to c : -2$.

   Original shortest path from a to c:

   - $a \to c$ has weight 0,
   - $a \to b \to c$ has weight $1 + (-2) = -1$,
     so the true shortest path is $a \to b \to c$ with distance -1.

   Add a constant $c_0 = 3$ to every edge to make all weights positive:
   $a \to c : 3$, $a \to b : 4$, $b \to c : 1$.
   Now $a \to c$ has weight 3 while $a \to b \to c$ has weight $4 + 1 = 5$,
   so Dijkstra on the modified graph returns $a \to c$.

---

2. What goes wrong

   If a path P has k edges, then after adding c_0 to every edge, $w'(P) = w(P) + k \cdot c_0$.
   Thus paths with different numbers of edges are shifted by different amounts, so the relative order of path weights can change.
   In the counterexample, the 2-edge path was shortest originally, but after shifting it becomes more expensive than the 1-edge path.
   Subtracting $k \cdot c_0$ from the path found in the modified graph cannot recover the true shortest path, because the choice of path has already been altered.

Question 15.4 (0.5 marks)

Implement DIJKSTRA, Robot and Road Repair problems on the Judge system.

Sol:

题目

状态	最后递交于	题目
✓ 100 Accepted	18 小时前	62 Dijkstra
✓ 100 Accepted	16 小时前	63 Robot
✓ 100 Accepted	10 分钟前	64 Road Repair

```cpp
struct Edge{
    Edge* nxt;
    int to;
    int val;
};

int main(){
    int N = read(), M = read(), S = read(), T = read();
    vector < Edge* > head(N + 10, nullptr);
    for(int i = 1; i <= M; ++i){
        int s = read(), t = read(), v = read();
        head[s] = new Edge{head[s], t, v};
    }
    vector < ll > dis(N + 10, LLONG_MAX >> 2);
    vector < int > pre(N + 10, -1);

    using PQ = __gnu_pbds::priority_queue <
        pair < ll, int >,
        greater < pair < ll, int > >,
        __gnu_pbds::pairing_heap_tag
    >;

    auto Dijkstra = [&](int S)->void{
        dis[S] = 0;
        PQ cur;
        vector < PQ::point_iterator > it(N, nullptr);
        for(int i = 0; i < N; ++i)it[i] = cur.push({dis[i], i});
        while(!cur.empty()){
            auto p = cur.top().second; cur.pop();
            for(auto i = head[p]; i; i = i->nxt)
                if(dis[p] + i->val < dis[i->to])
```

```
                    dis[i->to] = dis[p] + i->val,
                    cur.modify(it[i->to], {dis[i->to], i->to}),
                    pre[i->to] = p;
        }
    }; Dijkstra(S);

    if(dis[T] == LLONG_MAX >> 2){printf("-1\n"); return 0;}

    vector < pair < int, int > > route;
    int curp(T);
    while(curp != S)
        route.push_back({pre[curp], curp}),
        curp = pre[curp];

    printf("%lld %d\n", dis[T], (int)route.size());
    for(auto it = route.rbegin(); it != route.rend(); advance(it, 1))
        printf("%d %d\n", it->first, it->second);

    // fprintf(stderr, "Time: %.6lf\n", (double)clock() /
CLOCKS_PER_SEC);
    return 0;
}
```

```
struct Edge{
    Edge* nxt;
    int to;
    int val;
};

vector < ll > dis;

class Heap{
private:
    int N;
    vector < int > h;
    vector < int > pos;
    #define fa (p >> 1)
    #define ls (p << 1)
    #define rs ((p << 1) | 1)
```

```
public:
    Heap(int N){
        this->N = N;
        h.assign(N + 10, 0), pos.assign(N + 10, 0);
        for(int i = 1; i <= N; ++i)h[i] = i - 1, pos[h[i]] = i;
    }
    void Swap(int x, int y){
        swap(h[x], h[y]);
        pos[h[x]] = x, pos[h[y]] = y;
    }
    void Pushup(int p){
        while(p > 1){
            if(dis[h[fa]] > dis[h[p]])Swap(fa, p);
            else break;
            p = fa;
        }
    }
    void Pushdown(int p){
        while(true){
            int best(p);
            if(ls <= N && dis[h[best]] > dis[h[ls]])best = ls;
            if(rs <= N && dis[h[best]] > dis[h[rs]])best = rs;
            if(p == best)break;
            Swap(p, best);
            p = best;
        }
    }
    int ExtractTop(void){
        int ret(h[1]);
        Swap(1, N);
        pos[ret] = 0;
        if(--N >= 1)Pushdown(1);
        return ret;
    }
    bool empty(void){return N == 0;}
    void DecreaseKey(int p){Pushup(pos[p]);}
};

int main(){
    int N = read(), M = read(), S = read(), T = read();
```

```cpp
    vector < Edge* > head(N + 10, nullptr);
    for(int i = 1; i <= M; ++i){
        int s = read(), t = read(), v = read();
        head[s] = new Edge{head[s], t, v};
    }
    vector < int > pre(N + 10, -1);
    dis.assign(N + 10, LLONG_MAX >> 2);


    auto Dijkstra = [&](int S)->void{
        dis[S] = 0;
        Heap cur(N);
        cur.DecreaseKey(S);
        while(!cur.empty()){
            auto p = cur.ExtractTop();
            for(auto i = head[p]; i; i = i->nxt)
                if(dis[p] + i->val < dis[i->to])
                    dis[i->to] = dis[p] + i->val,
                    cur.DecreaseKey(i->to),
                    pre[i->to] = p;
        }
    }; Dijkstra(S);

    if(dis[T] == LLONG_MAX >> 2){printf("-1\n"); return 0;}

    vector < pair < int, int > > route;
    int curp(T);
    while(curp != S)
        route.push_back({pre[curp], curp}),
        curp = pre[curp];

    printf("%lld %d\n", dis[T], (int)route.size());
    for(auto it = route.rbegin(); it != route.rend(); advance(it, 1))
        printf("%d %d\n", it->first, it->second);

    // fprintf(stderr, "Time: %.6lf\n", (double)clock() /
CLOCKS_PER_SEC);
    return 0;
}
```

```cpp
struct Edge{
    Edge* nxt;
    int to;
    ll val;
};

vector < ll > dis;

class Heap{
private:
    int N;
    vector < int > h;
    vector < int > pos;
    #define fa (p >> 1)
    #define ls (p << 1)
    #define rs ((p << 1) | 1)
public:
    Heap(int N){
        this->N = N;
        h.assign(N + 10, 0), pos.assign(N + 10, 0);
        for(int i = 1; i <= N; ++i)h[i] = i - 1, pos[h[i]] = i;
    }
    void Swap(int x, int y){
        swap(h[x], h[y]);
        pos[h[x]] = x, pos[h[y]] = y;
    }
    void Pushup(int p){
        while(p > 1){
            if(dis[h[fa]] > dis[h[p]])Swap(fa, p);
            else break;
            p = fa;
        }
    }
    void Pushdown(int p){
        while(true){
            int best(p);
            if(ls <= N && dis[h[best]] > dis[h[ls]])best = ls;
            if(rs <= N && dis[h[best]] > dis[h[rs]])best = rs;
            if(p == best)break;
            Swap(p, best);
```

```
            p = best;
        }
    }
    int ExtractTop(void){
        int ret(h[1]);
        Swap(1, N);
        pos[ret] = 0;
        if(--N >= 1)Pushdown(1);
        return ret;
    }
    bool empty(void){return N == 0;}
    void DecreaseKey(int p){Pushup(pos[p]);}
};

int main(){
    read();
    int N = read(), M = read(), K = read();

    vector < ll > V(K + 10, 0), W(K + 10, 0), sum(K + 10, 0);
    for(int i = 1; i <= K - 1; ++i)V[i] = read < ll >();
    for(int i = 2; i <= K; ++i)W[i] = read < ll >(), sum[i] = sum[i - 1] + W[i];

    vector < int > d(N + 10, 0), l(N + 10, 0), r(N + 10, 0);
    vector < vector < pair < int, int > > > Eout(N + 10);

    int tot(0);
    for(int i = 1; i <= N; ++i){
        d[i] = read();
        l[i] = tot, r[i] = tot + d[i], tot += d[i] + 1;
        Eout[i].push_back({0, 0});
        for(int j = 1; j <= d[i]; ++j){
            int y = read(), z = read();
            Eout[i].push_back({y, z});
        }
    }

    vector < Edge* > head(tot + 10, nullptr);

    auto AddEdge = [&](int s, int t, ll val)->void{
```

```
        head[s] = new Edge{head[s], t, val};
    };

    for(int i = 1; i <= N; ++i)
        for(int j = 2; j <= d[i]; ++j){
            int a = l[i] + (j - 2), b = l[i] + (j - 1);
            AddEdge(a, b, V[j - 1]);
            AddEdge(b, a, W[j]);
        }

    for(int i = 1; i <= N; ++i)
        for(int j = 1; j <= d[i]; ++j){
            int y = Eout[i][j].first, z = Eout[i][j].second;
            int from = l[i] + (j - 1);
            if(j <= d[y])AddEdge(from, l[y] + (j - 1), (ll)z);
            else{
                if(d[y] > 0)AddEdge(from, r[y] - 1, (ll)z + (sum[j] -
sum[d[y]]));
                AddEdge(from, r[y], (ll)z);
            }
        }

    const ll INF = LLONG_MAX >> 2;
    dis.assign(tot + 10, INF);

    int S = l[1];
    dis[S] = 0;

    auto Dijkstra = [&](int S)->void{
        Heap cur(tot);
        cur.DecreaseKey(S);
        while(!cur.empty()){
            int p = cur.ExtractTop();
            for(auto i = head[p]; i; i = i->nxt)
                if(dis[p] + i->val < dis[i->to])
                    dis[i->to] = dis[p] + i->val,
                    cur.DecreaseKey(i->to);
        }
    }; Dijkstra(S);

```

```
    for(int i = 1; i <= N; ++i){
        ll mn(INF);
        for(int j = l[i]; j <= r[i]; ++j)mn = min(mn, dis[j]);
        printf("%lld%c", mn == INF ? -1 : mn, i == N ? '\n' : ' ');
    }

    return 0;
}
```

```
struct Road{
    int u, v;
    ll w;
};

struct Edge{
    int u, v;
    ll w;
    int town;
};

class UnionFind{
private:
    int N;
    vector < int > fa;
    vector < int > siz;
    vector < int > cnt;

public:
    UnionFind(int N = 0){InitSize(N);}

    void InitSize(int N){
        this->N = N;
        fa.assign(N + 10, 0);
        siz.assign(N + 10, 0);
        cnt.assign(N + 10, 0);
    }

    int Find(int x){
        while(x != fa[x])x = fa[x] = fa[fa[x]];
```

```
        return x;
    }

    bool Union(int x, int y){
        int fx = Find(x), fy = Find(y);
        if(fx == fy)return false;
        if(siz[fx] < siz[fy])swap(fx, fy);
        fa[fy] = fx;
        siz[fx] += siz[fy];
        cnt[fx] += cnt[fy];
        return true;
    }

    void Reset(int n, int K, int mask){
        int lim = n + K;
        for(int i = 1; i <= lim; ++i)fa[i] = i, siz[i] = 1, cnt[i] =
0;
        for(int i = 1; i <= n; ++i)cnt[i] = 1;
        for(int j = 1; j <= K; ++j)cnt[n + j] = (mask >> (j - 1)) & 1;
    }

    int QueryCnt(int x){return cnt[Find(x)];}
};

int main(){
    int N = read(), M = read(), K = read();

    vector < Road > roads;
    roads.resize(M);
    for(int i = 0; i < M; ++i){
        int u = read(), v = read();
        ll w = read < ll >();
        roads[i] = Road{u, v, w};
    }

    vector < ll > townCost(K + 10, 0);
    vector < Edge > cand;
    for(int j = 1; j <= K; ++j){
        townCost[j] = read < ll >();
        for(int i = 1; i <= N; ++i){
```

```
            ll a = read < ll >();
            cand.push_back(Edge{i, N + j, a, j});
        }
    }

    sort(roads.begin(), roads.end(), [](const Road &a, const Road &b)->bool{return a.w < b.w;});

    UnionFind ufCity(N);
    ufCity.Reset(N, 0, 0);

    vector < Edge > mst;
    mst.reserve(N + 5);
    for(auto &e : roads){
        if(ufCity.Union(e.u, e.v))mst.push_back(Edge{e.u, e.v, e.w, 0});
        if((int)mst.size() == N - 1)break;
    }

    for(auto &e : mst)cand.push_back(e);
    sort(cand.begin(), cand.end(), [](const Edge &a, const Edge &b)->bool{
        if(a.w != b.w)return a.w < b.w;
        return a.town < b.town;
    });

    int full = 1 << K;
    vector < ll > sumC(full, 0);
    for(int mask = 1; mask < full; ++mask){
        int low = mask & -mask;
        int bit = __builtin_ctz(low);
        sumC[mask] = sumC[mask ^ low] + townCost[bit + 1];
    }

    ll ans = LLONG_MAX >> 2;
    UnionFind ufAll(N + K);

    for(int mask = 0; mask < full; ++mask){
        int need = N + __builtin_popcount((uint)mask);
        ll curCost = sumC[mask];
```

```
107
108         ufAll.Reset(N, K, mask);
109
110         for(auto &e : cand){
111             if(e.town && (((mask >> (e.town - 1)) & 1) == 0))continue;
112             if(ufAll.Union(e.u, e.v))curCost += e.w;
113             if(ufAll.QueryCnt(1) == need)break;
114         }
115
116         if(ufAll.QueryCnt(1) == need)ans = min(ans, curCost);
117     }
118
119     printf("%lld\n", ans);
120
121     // fprintf(stderr, "Time: %.6lf\n", (double)clock() /
    CLOCKS_PER_SEC);
122     return 0;
123 }
```