

Lecture #11

► Dynamic Programming

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Reading: Chapter 14.1

► Aims of this lecture

- To discuss the dynamic programming paradigm for solving optimisation problems.
- To work through an example of a problem solved efficiently with dynamic programming.
- To discuss properties of problems where dynamic programming is efficient.
- To discuss how to implement dynamic programming algorithms.

► How to compute Fibonacci numbers?

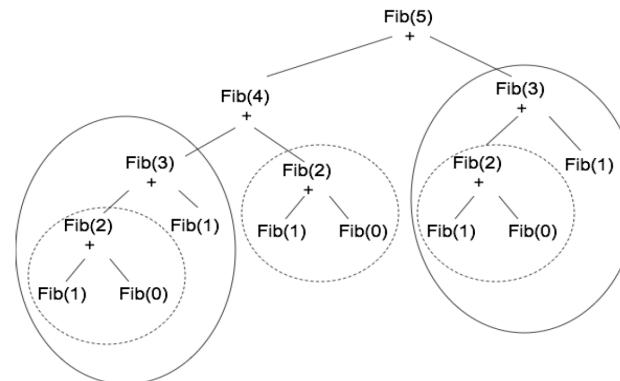
- Fibonacci numbers:
 - $Fib(0) = Fib(1) = 1$
 - $Fib(k) = Fib(k - 1) + Fib(k - 2)$
 - Handy closed form lower bound:

$$Fib(k) \geq \frac{1}{\sqrt{5}} \left[\left(\frac{\sqrt{5} + 1}{2} \right)^{k+1} - 1 \right]$$

- Let's try to compute $Fib(n)$ exactly using the recursive definition.

► What happened??

- The same values are **computed from scratch many times!**



►What happened??

- Let's call $T(n)$ the time to compute $Fib(n)$.
- Let's ignore constants for simplicity so that

$$T(0) = T(1) = 1.$$

- Then $T(n) = T(n - 1) + T(n - 2) + 1$.
- Let's ignore the "+1" and take

$$T(n) = T(n - 1) + T(n - 2).$$

- Then $T = Fib$! And from closed formula $Fib(n) = \Omega\left(\left(\frac{\sqrt{5} + 1}{2}\right)^n\right)$
- $T(90)=Fib(90)=4660046610375530309$.
- Larger than the age of the Universe in seconds.

►A smarter way

- Compute Fibonacci numbers **bottom-up in a table**.
- Refer to table instead of re-calculating!
- (Bottom-up ensures we refer to entries already calculated.)
- Time $O(n)$ instead of $\Omega\left(\left(\frac{\sqrt{5} + 1}{2}\right)^n\right)$

►Dynamic Programming

- A **general algorithm design method** that can be used when the solution to a problem may be viewed as the result of a **sequence of decisions**.
 - Developed back in the day when "**programming**" meant "**tabular method**".
- **Idea:** solve **subproblems** of the original problem and **save the answers in a table**. Solve subproblems of increasing size until we can solve the original problem.
 - Avoids the work of recomputing the answer every time it solves a subproblem.
 - Solving subproblems is similar to divide and conquer, but for Dynamic Programming **subproblems typically overlap**.
- Optimisation problems: find a solution with the optimal value.

►Properties of Dynamic Programming

- **Optimal substructure:** The solutions to the subproblems used within the optimal solution must themselves be optimal.
 - Often: making a first decision in an optimal way, and then being left with a smaller problem that needs to be solved optimally.
- Dynamic Programming is usually efficient if the problem has optimal substructure and the space of subproblems is small.

► Rod Cutting Problem

- How to cut a steel rod of length n into pieces in order to maximise the revenue from selling all pieces?



- Each cut is free. Rod lengths are an integral number of cm.
- Each rod length i has its own price p_i .
- Output: maximum revenue obtainable from rods whose lengths sum to n , according to the price list.

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► Rod Cutting Problem: One way

	(a)	(b)	(c)	(d)
	9	1 8	5 5	8 1
length i	1	2	3	4
price p_i	1	5	8	9
	10	17	17	20
	24	30		

- Let r_i be the maximum revenue for a rod of length i

$$r_n = \max\{p_n, p_1 + r_{n-1}, p_2 + r_{n-2}, p_3 + r_{n-3}, \dots, p_{n-1} + p_1\}$$

(Bellman equation)

- If we knew the solutions of the smaller r_i values we would be done, because the optimal solution incorporates the optimal solutions to the smaller subproblems (max rev. of the two pieces) **(optimal substructure)**
- These subproblems may be solved independently of the original (larger) problem

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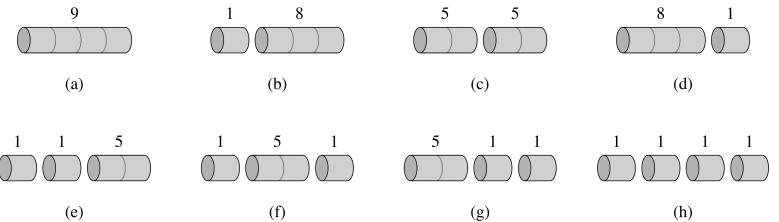
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► Rod Cutting Problem: Example

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30

There are 2^{n-1} different ways to cut up a rod, because we can choose to cut or not cut after each of the first $n - 1$ cm.

Here are all $2^{4-1} = 8$ ways to cut a rod of length 4, with above prices:



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► The journey of 1000 miles begins with one step



- The rod cutting of n cm begins with one cut.
- Let r_i be the maximum revenue for a rod of length i .
 - Boundary case: $r_0 = 0$ (no rod to sell).
- If we make a first cut of length i , **the revenue from the first piece is p_i** and we are left with a rod of length $n-i$.
- **Optimal substructure:** we get an optimal revenue if
 - we make an optimal decision for the first cut length i and
 - we get optimal revenue for the remaining rod of length $n-i$.
- Leads to the following **Bellman equation**:

$$r_n = \max\{p_i + r_{n-i} \mid 1 \leq i \leq n\}$$

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► Bellman equations

$$r_n = \max\{p_i + r_{n-i} \mid 1 \leq i \leq n\}$$

- The **Bellman equation** tells us **how an optimal solution for a problem depends on solutions to smaller subproblems.**
 - It captures an **optimal decision** (e.g. which cut length i for 1st cut?)
 - The precise equation depends on the problem being solved. Different problems have different Bellman equations.
 - Named after **Richard Bellman**, the inventor of dynamic programming.
 - (Strangely, the book refuses to mention the term “Bellman equation”.)
- The Bellman equation is **at the heart of a dynamic programming algorithm.**
 - Working it out can be hard work; implementation is usually straightforward once you have worked out the Bellman equation!

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► Same mistake again...

$$r_j = \max\{p_i + r_{j-i} \mid 1 \leq i \leq j\}$$

CUT-ROD(p, n)

```

1  if n == 0
2      return 0
3  q = -∞
4  for i = 1 to n
5      q = max {q, p[i] + CUT-ROD(p, n - i)}
6  return q
    
```

$p[1..n]$: array of prices; n length

Revenue = 0

Minimum revenue is negative

Recursively calculates Bellman eq.
Returns max

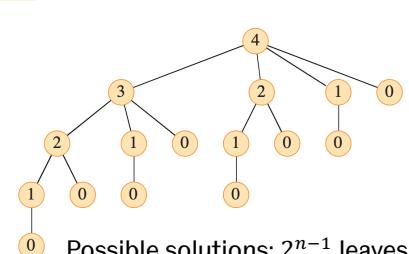
• Correctness: simple induction.

• Runtime? ($T(n)$: n. of times Cut-Rod is called for size n)

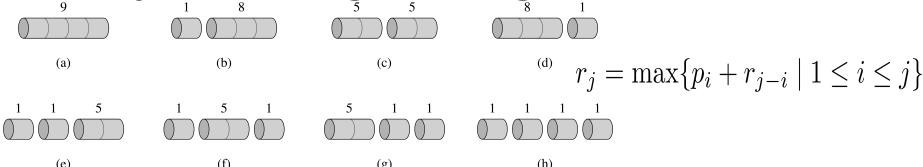
• $T(0) = 1$

• $T(n) = 1 + \sum_{j=0}^{n-1} T(j) = 2^n$
(again by induction)

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(A path from root one of the possible ways to cut the rod)



► Dynamic Programming



- Arrange for the problems to be **solved only once**
- If the rod had length $n=1$, what would be the optimal solution? $r_1 = p_1$
- If the rod had length $n=2$? $r_2 = \max(p_2, r_1 + r_1) = \max(p_2 + r_{2-1} \mid 1 \leq i \leq 2)$
- Sort the subproblems by size, solve the smaller ones first, and **store the solutions**
 - That way, when solving a subproblem, we have already solved (and tabulated) the smaller subproblems we need.
- How many subproblems do we have when $n = 4$?

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► Bottom-up implementation

- Sort the subproblems by size and solve the smaller ones first.
 - That way, when solving a subproblem, we have already solved (and tabulated) the smaller subproblems we need.

BOTTOM-UP-CUT-ROD(p, n)

```

1: Let  $r[0 \dots n]$  be a new array
2:  $r[0] = 0$ 
3: for  $j = 1$  to  $n$  do
4:      $q = -\infty$ 
5:     for  $i = 1$  to  $j$  do
6:          $q = \max(q, p[i] + r[j-i])$ 
7:      $r[j] = q$ 
8: return  $r[n]$ 
    
```

Outer loop solves problem of rod length j

Inner loop computes Bellman equation:

$r_j = \max\{p_i + r_{j-i} \mid 1 \leq i \leq j\}$

Correctness? Same as before

Runtime?

Runtime is $\Theta(n^2)$.

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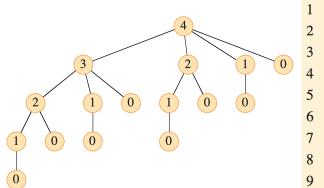
► Top down Implementation with Memoization

- Alternative to bottom-up:

- Write the recursive procedure naturally, but save the subproblem solutions somewhere
- Recursive procedure first checks if it knows the solution (if so returns it); Otherwise proceeds and saves it

Runtime?

Same arithmetic series: $\Theta(n^2)$



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```

MEMOIZED-CUT-ROD( $p, n$ )
1 let  $r[0 : n]$  be a new array      // will remember solution values in  $r$ 
2 for  $i = 0$  to  $n$ 
3    $r[i] = -\infty$ 
4 return MEMOIZED-CUT-ROD-AUX( $p, n, r$ )

MEMOIZED-CUT-ROD-AUX( $p, n, r$ )
1 if  $r[n] \geq 0$                   // already have a solution for length  $n$ ?
2   return  $r[n]$ 
3 if  $n == 0$ 
4    $q = 0$ 
5 else  $q = -\infty$ 
6 for  $i = 1$  to  $n$              //  $i$  is the position of the first cut
7    $q = \max\{q, p[i] + \text{MEMOIZED-CUT-ROD-AUX}(p, n - i, r)\}$ 
8    $r[n] = q$                   // remember the solution value for length  $n$ 
9 return  $q$ 

```

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No, that's
not a typo.

► Top down Implementation with Memoization

- Advantage:

- Only solves problem sizes that are actually needed.
- No better runtime for rod cutting, though.

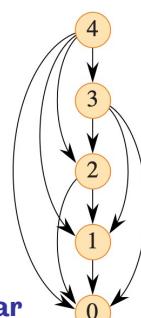
- Disadvantage:

- Bottom-up has better constant factors (lower overhead for recursive procedure calls)

No, that's
not a typo.

► Dynamic Programming: when to use

- The problem has Optimal Substructure.
- Runs in polynomial time when the number of **distinct** subproblems involved is polynomial in the input size **and** you can solve each subproblem in polynomial time.
- A subproblem graph indicates the subproblems that need to be solved before the larger problem can.
- Top-Down:** arrows indicate the recursive calls
- Bottom-Up:** solves the nodes “pointed at” before those “pointing to”
- Time to compute subproblem is proportional to degree of its node.
- Usually the runtime of dynamic programming is **linear in the number of vertices and edges**



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► Reconstructing a solution

- The algorithms only tell us the **value of the optimal revenue**, it **doesn't reveal how to cut!**
- Solution:** if we know how to compute the optimal value, we can **record additional information** about how we got there (that is, **recording decisions** made in Bellman equations).

```

EXTENDED-BOTTOM-UP-CUT-ROD( $p, n$ )
1 Let  $r[0 \dots n]$  and  $s[0 \dots n]$  be new arrays
2  $r[0] = 0$ 
3 for  $j = 1$  to  $n$  do
4    $q = -\infty$ 
5   for  $i = 1$  to  $j$  do
6     if  $q < p[i] + r[j - i]$  then
7        $q = p[i] + r[j - i]$ 
8        $s[j] = i$ 
9    $r[j] = q$ 
10 return  $r$  and  $s$ 

PRINT-CUT-ROD-SOLUTION( $p, n$ )
1 ( $r, s$ ) = EXTENDED-BOTTOM-UP-CUT-ROD( $p, n$ )
2 while  $n > 0$ 
3   print  $s[n]$           // cut location for length  $n$ 
4    $n = n - s[n]$         // length of the remainder of the rod

```

Current best solution cuts at i
Store this information in s .

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► Summary

- Dynamic Programming is a **general design paradigm** that breaks down a problem into **smaller subproblems**; these are solved first and the solutions are usually tabulated.
- Works for optimisation problems with **optimal substructure**: the optimal solution is composed of optimal solutions for subproblems.
- The **Bellman equation** describes how an optimal solution is derived from optimal solutions for subproblems.
- Bottom-up approach solves subproblems of increasing size; Top-down solves recursively asking when needed
- The solution can be reconstructed by **recording decisions** made in applying Bellman equations across subproblems.
- The rod cutting problem can be solved this way in time $\Theta(n^2)$ reducing the runtime from exponential to a small polynomial.