CS217 - Data Structures & Algorithm Analysis (DSAA)

Lecture #7

Sorting in Linear Time

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Reading: Chapter 8

Aims of this lecture

- To show how to sort numbers in a **bounded range** in **linear time**.
- Two algorithms use operations other than comparisons so the $\Omega(n \log n)$ runtime will not apply to them
- **CountingSort**
- **RadixSort**

Linear-Time Sorting

- The lower bound of $\Omega(n \log n)$ is bad news for applications where comparisons are the only source of information.
- However, it suggests a way out: where possible, **use more information** than mere comparisons!
- Elements to be sorted are often **numbers or strings**, which reveal more information.

CountingSort: Idea

- Assume that the input elements are integers in $\{0, \dots, k\}$.
- For each element x, CountingSort **counts the number of elements less than x**.
  - For instance, if 17 elements are smaller than x, then x belongs in output position 18.
- Caveat: need to make sure that equal elements are put in **different** output positions.
- CountingSort uses an array $C[\mathbf{0} \dots k]$ for counting and an array $B[\mathbf{1} \dots n]$ for writing the output.

CountingSort

- Initialise counter array
- Count elements
- Running sum: #elements $\leq i$
- Write elements to output

```
CountingSort(A, B, k)
let C[0...k] be a new array
for i = 0 to k do
    C[i] = 0
for j = 1 to A.length do
    C[A[j]] = C[A[j]] + 1
for i = 1 to k do
    C[i] = C[i] + C[i - 1]
for j = A.length downto 1 do
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```


CountingSort

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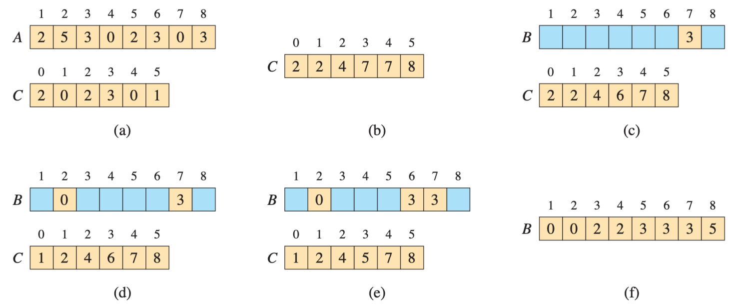
```



CountingSort: Runtime

- Initialise counter array
- Count elements
- Running sum: #elements $\leq i$
- Write elements to output

CountingSort(A, B, k)	Time
let $C[0\dots k]$ be a new array	
for $i = 0$ to k **do**	$\Theta(k)$
$\quad C[i] = 0$	
for $j = 1$ to A.length **do**	$\Theta(n)$
$\quad C[A[j]] = C[A[j]] + 1$	
for $i = 1$ to k **do**	$\Theta(k)$
$\quad C[i] = C[i] + C[i-1]$	
for $j = A$.length downto 1 **do**	
$\quad B[C[A[j]]] = A[j]$	$\Theta(n)$
$\quad C[A[j]] = C[A[j]] - 1$	

- Runtime is $\Theta(n + k)$
  - Depends on two input parameters instead of just the problem size n.
  - This is $O(n)$ if $k = O(n)$.

CountingSort: Correctness

```
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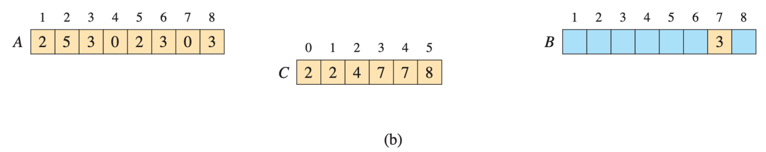
for j = 1 to A.length do
    C[A[j]] = C[A[j]] + 1
for i = 1 to k do
    C[i] = C[i] + C[i - 1]
for j = A.length downto 1 do
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

- Loop Invariant:

"At the start of each iteration j of the last for loop, the elements $A[j+1..n]$ are in the right position in B and the last element in A that has not yet been copied in *B*, with value $A[j] = i$, belongs to *B[C[i]]*."

Initialisation:
At the start of the loop $j = n$ and no elements have been copied. The array C provides for each element, the number of elements in A that are smaller or equal to it. So the last element of A, $A[n] = i$, naturally goes in position *B[C[i]]*.



CountingSort: Correctness

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```
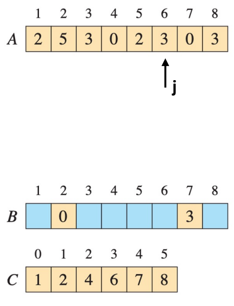

- Loop Invariant:

"At the start of each iteration j of the last for loop, the elements $A[j+1..n]$ are in the right position in B and the last element in A that has not yet been copied in *B,* with value $A[j]=i$, belongs to *B[C[i]]*."

Maintenance:
At iteration j, the loop invariant tells us that the element $A[j]=i$ goes in $B[C[i]]$ and we copy it in.
Since the next element equal to i in A that has not yet been copied in B should go in position $B[C[i]-1]$, we decrement $C[i]$ re-stablishing the loop invariant (the array C is updated such that each element i of A still to be copied in is indexed correctly again)



(d)

CountingSort: Correctness


```
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let C[0...k] be a new array
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for j = A.length downto 1 do
    B[C[A[j]]] = A[j]
    C[A[j]] = C[A[j]] - 1
```

- Loop Invariant:

"At the start of each iteration j of the last for loop, the elements $A[j+1..n]$ are in the right position in B and the last element in A that has not yet been copied in *B,* with value $A[j]=i$, belongs to *B[C[i]]*."

Termination:
When the loop terminates $j=0$. The loop invariant tells us that all the elements of A [1..n] are in the right position in B thus there are no more elements to be copied.

	1	2	3	4	5	6	7	8
B	0	0	2	2	3	3	3	5

Stability in Sorting

- CountingSort is **stable**: numbers with the same value appear in the output in **the same order as** they do **in the input** array.
  - The order of equal elements is preserved.
  - This property is relevant when **satellite data** (e.g. Java objects) is attached to keys being sorted.
  - We may think of the original order being used to break ties between elements with equal keys.
  - Works well for sorting emails according to (1) read/unread and (2) date.

- How do we prove stability of CountingSort?
- Can I be faster if I don't care about stability?

Counting Sort: advantages & disadvantages

- Sorts in linear time n integers in the range *{0..k}* if *k=O(n)*
- Is stable (preserves original ordering for breaking ties)
- Does not sort in place
- What if $k=\omega(n)$ (or $k \gg n$)? (eg., I have to sort n=100 numbers between 0 and 1 billion)
- Is there a way of limiting the size of *k*?

Radix Sort

- How many different integers can appear in a digit in a number of x digits?
- How many different letters can appear in a word written using a latin (eg., English) alphabet?
- Can we sort digit by digit (or letter by letter)?
- Stability helps for sorting numbers digit by digit (or English words letter by letter).

Radix Sort

- Assume that each array element has d digits (from lowest significance to highest significance)

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```


Radix Sort: Example

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```

329		720		720		329
457		355		329		355
657		436		436		436
839	→	457	→	839	→	457
436		657		355		657
720		329		457		720
355		839		657		839

Radix Sort: Correctness

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```

Correctness follows from stability and induction on columns.

Loop Invariant: "At each iteration of the **for** loop, the array is sorted on the last *i-1* digits"

Initialisation: The array is trivially sorted on the empty set of digits for i=0

329
457
657
839
436
720
355

Radix Sort: Correctness

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```

720
329
436
839
355
457
657

Correctness follows from stability and induction on columns.

Loop Invariant: “At each iteration of the **for** loop, the array is sorted on the last *i-1* digits”

Maintenance: The invariant tells us that the array is sorted on the last i-1 digits. Now we sort the i_th digit re-establishing the loop invariant, since our stable sort ensures that elements with same i_th digit remain in the same order as before sorting.

Radix Sort: Correctness

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```

329
355
436
457
657
720
839

Correctness follows from stability and induction on columns.

Loop Invariant: “At each iteration of the **for** loop, the array is sorted on the last *i-1* digits”

Termination: The loop terminates when i=d+1. Then the loop invariant states that the array is completely sorted.

Radix Sort: Runtime

```
RadixSort(A, d)
for i = 1 to d do
    use a stable sort to sort array A on digit i
```

- Given n d-digit numbers in which each digit can take up to k possible values, RadixSort using CountingSort sorts these numbers in time $\Theta(d(n+k))$.
  - This is just the runtime of running CountingSort *d* times.

- Advantage to CountingSort?
- The support array has only size [0..9] for numbers, [A..Z] for words with latin letters (*k* is not too large)

Radix Sort: Application

Task: Sort n integers in the range 0 to $n^3 - 1$

- Runtime of a ComparisonSort algorithm?
- Runtime of CountingSort?
- Runtime of RadixSort?

A number n^3-1 requires $\lceil \log_{10} n^3 \rceil$ = $\lceil 3\log_{10} n \rceil = O(\log n)$ digits

Eg. $n = 10$, then $n^3 - 1 = 999$, and $\lceil 3\log_{10} n \rceil = 3$

Eg. $n = 20$, then $n^3 - 1 = 7999$, and $\lceil 3\log_{10} n \rceil = 4$

So RadixSort has runtime $\Theta(d(n+k)) = \Theta(\log n(n+10)) = \Theta(n \log n)$

Turn the numbers to base n (eg., n=20 range: [0..JJJ]

$\Longrightarrow \lceil 3\log_n n \rceil = O(1)$, and $T(n) = O(n+k) = O(n+k) = O(n)$

Caveat? I have to make sure I can convert bases and back in time O(1)

Summary

- CountingSort sorts numbers in a bounded range $\{0, \dots, k\}$ in time $\Theta(n + k)$.
- RadixSort uses a **stable sorting algorithm** to sort digit by digit.
  - **Stability** preserves the order of equal elements.
  - The time for sorting d-digit numbers is $\Theta\big(d(n + k)\big)$.
  - This is $\Theta(n)$ when $d = O(1)$ and $k = O(n)$.