CS217 - Data Structures & Algorithm Analysis (DSAA)

Lecture #14

▶Depth First Search & Applications

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Reading: Chapter 20 and
I. Wegener. A simplified correctness proof for a well-known algorithm computing strongly connected components. Information Processing Letters 83(1), pages 17–19 – On Blackboard

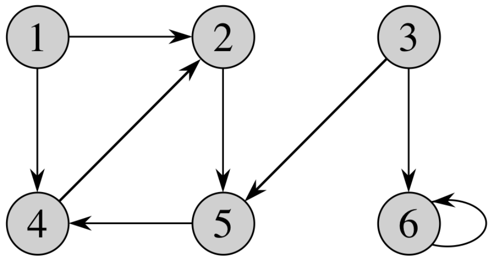
▶Aims for this lecture

- Introduce **depth-first search (DFS)** and depth-first trees.
- To show how DFS can **classify edges** for additional information about the graph.
- To show how to use DFS to
  - Check whether a graph contains cycles
  - Put tasks in the right order (topological sorting)
  - Compute strongly connected components in graphs
- To show the **correctness** of some remarkable algorithms.

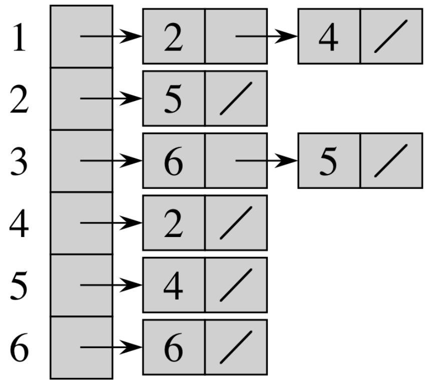
Representations of graphs

- Using terminology for graphs G = (V, E) from Appendix B
- **Adjacency-list representation**:
  - Array Adj of |V| lists, one for each vertex.
  - The list Adj[u] contains all vertices v adjacent to u in G, i.e. there is an edge $(u, v) \in E$.
  - The sum of all adjacency list lengths equals |E|.
- **Adjacency-matrix representation**:
  - Assume that vertices are numbered $1, 2, \ldots, n$.
  - Adjacency matrix is a $|V| \times |V|$ matrix with entries $a_{ij} = 1$ if $(i, j) \in E$ and $a_{ij} = 0$ otherwise.

Example for a directed graph



(a)

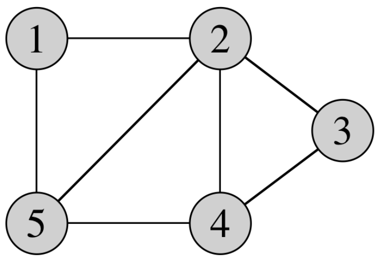


(b)

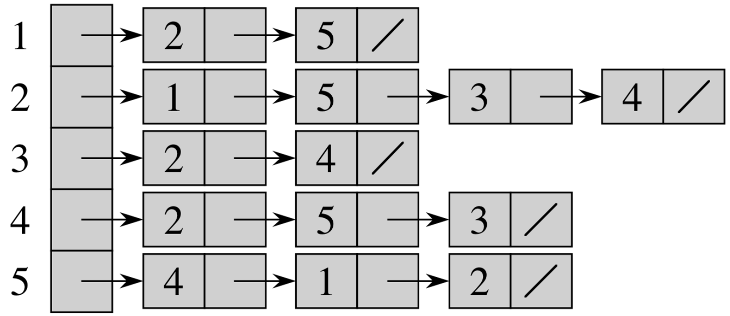
	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

(c)

Example for an undirected graph



(a)



(b)

	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

(c)

- For every undirected edge $\{u, v\}$, v is in u's adjacency list and u is in v's adjacency list.
- Note the symmetry in the adjacency matrix along the main diagonal. It's sufficient to store the entries on and above the diagonal.

Depth-first search (DFS)

- Works for undirected and directed graphs.
- Ideas:
  - Go into depth by exploring edges out of the most recently discovered vertex and backtrack when stuck.
  - Continue until all vertices reachable from the start vertex are discovered.
  - If any undiscovered vertices remain, continue with one of them as new source.
- As for BFS, define predecessors $v.\pi$ that represent several **depth-first trees**.
- These trees form a **depth-first forest.**

DFS: Colours and timestamps

- DFS uses colours white, gray, black as for BFS:
  - **White**: vertex has not been discovered yet
  - **Gray**: vertex has been discovered, but is not finished yet.
  - **Black**: vertex has been finished (finished scan of adjacency list).
- Also uses **timestamps**:
  - **v.d** is the time v is first **discovered** (and grayed)
  - **v.f** is the time v is **finished** (and blackened)
  - Global variable time is incremented with each event
  - Hence for all vertices $v.d < v.f$

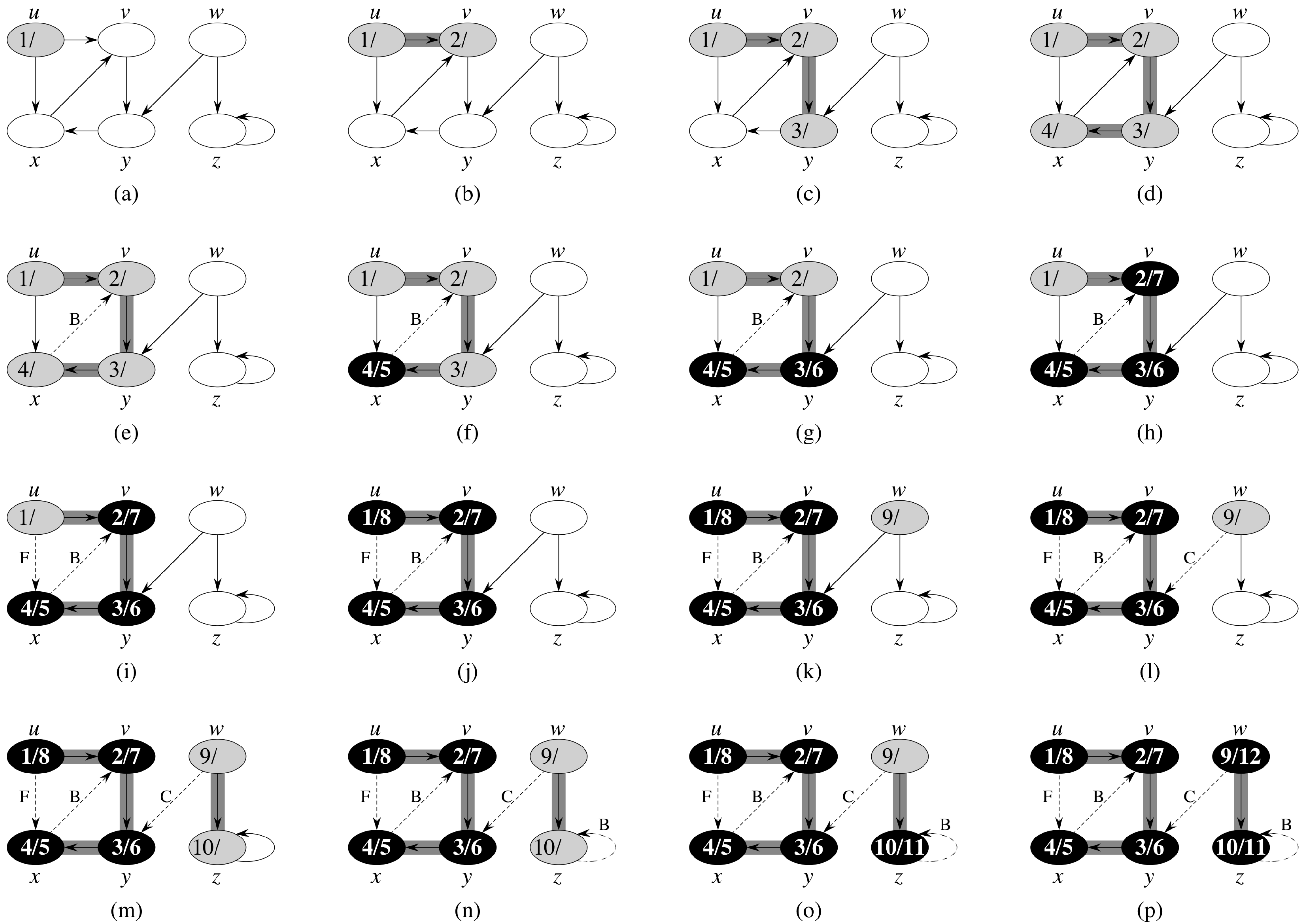
DFS: Pseudocode

DFS(G)

```
for each vertex u ∈ V do
    u.colour = white
    u.π = NIL
time = 0
for each vertex u ∈ V do
    if u.colour == white then
        DFS-VISIT(G, u)
```

DFS-VISIT(G, u)

```
time = time+1
u.d = time
u.colour = gray
for each v ∈ Adj[u] do
    if v.colour == white then
        v.π = u
        DFS-VISIT(G, v)
u.colour = black
time = time+1
u.f = time
```



u v w x y z
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(a)
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(g)
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(h)
1/ 2/7 F B 4/5 3/6
(i)
1/8 2/7 F B 4/5 3/6
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(l)
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1/8 2/7 9/12 F B C B 4/5 3/6 10/11
(p)

▶ DFS: Pseudocode and runtime

```
DFS(G)
for each vertex u ∈ V do
        u.colour = white
        u.π = NIL
time = 0
for each vertex u ∈ V do
        if u.colour == white then
                DFS-VISIT(G, u)
```

```
DFS-VISIT(G, u)
time = time+1
u.d = time
u.colour = gray
for each v ∈ Adj[u] do
        if v.colour == white then
                v.π = u
                DFS-VISIT(G, v)
u.colour = black
time = time+1
u.f = time
```

Runtime?

- Runtime is $\Theta(|V| + |E|)$:
  - DFS runs in time $\Theta(|V|)$ exclusive of the time for DFS-Visit.
  - DFS-Visit is only called once for each vertex v as *v* must be white and is grayed immediately. The loop executes |Adj[u]| times.
  - Since $\sum_{v \in V} |\mathrm{Adj}[v]| = \Theta(|E|)$, the total cost for loop is $\Theta(|E|)$.

Properties of DFS

Parenthesis structure: In any DFS of a (directed or undirected) graph, for any two vertices $u \neq v$, either

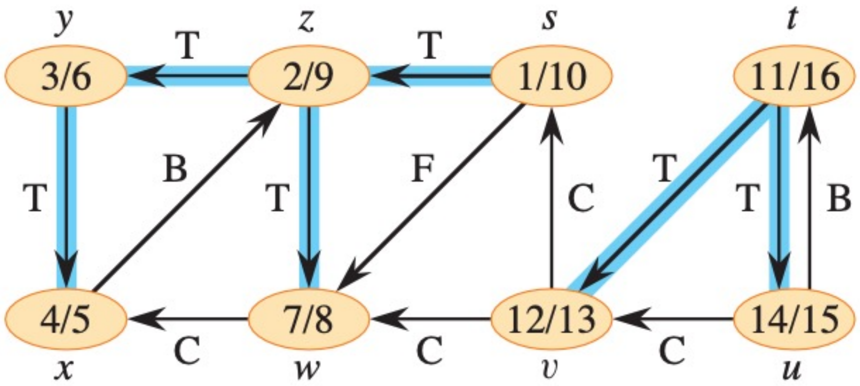
- DFS-Visit(v) is called during DFS-Visit(u), then v is a descendant of u and DFS-Visit(v) finishes earlier than u:

$$u.d < v.d < v.f < u.f$$

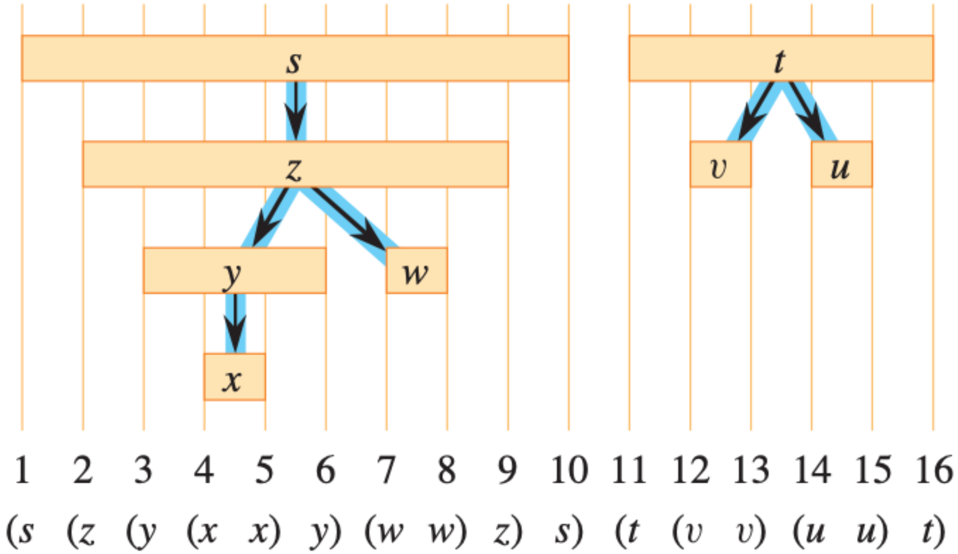
- the same happens with roles of v and u swapped, or
- the intervals $[u.d, u.f]$ and $[v.d, v.f]$ are entirely disjoint, and neither u nor v is a descendant of the other.

NB: Recursive calls mean that DFS implicitly uses a **stack** to store vertices while exploring the graph (cf. BFS using a queue).

Parenthesis structure: example



(a)



(b)

▶White-path theorem

Theorem 22.9: In a depth-first forest of a (directed or undirected) graph, vertex v is a descendant of a vertex u **if and only if** at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

- This means: If and only if there is a white path from u to v, DFS will create a DFS tree with edges from u to v.
- "If and only if" indicates a statement like "$A \Leftrightarrow B$"
- We split this into two steps:
  1. Prove that $A \Rightarrow B$
  2. Prove that $A \Leftarrow B$
- It is often easier to focus on proving one implication.

▶White-path theorem (2)

Theorem 22.9: In a depth-first forest of a (directed or undirected) graph, vertex v is a descendant of a vertex u **if and only if** at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

Proof of "⇒" (being descendant implies white path):

- If $u = v$ then u is still white when $u.d$ that is set, thus a white path from u to v exists (just one vertex $u = v$).
- If v is a proper descendant of u, then $u.d < v.d$ and therefore v is white at time $u.d$. This holds for all descendants of u, hence a white path from u to v exists at time $u.d$.

▶White-path theorem (3)

Theorem 22.9: In a depth-first forest of a (directed or undirected) graph, vertex v is a descendant of a vertex u **if and only if** at the time $u.d$ that the search discovers u, there is a path from u to v consisting entirely of white vertices.

Proof of "⇐" (white path implies descendancy) **by contradiction**:

- Suppose there is a white path from u to v at time $u.d$.
- Assume v is the **first vertex on the path** which is not a descendant of u (otherwise we consider this first vertex instead).
- Let w be the predecessor of v on the path (could be $w = u$). Hence w must be a descendant of u (by above assumption). Thus $w.f \leq u.f$.
- v is discovered after u but before w is finished (as there is an edge from w to v), so we get: $u.d < v.d < w.f \leq u.f$.
- How large is $v.f$? Parenthesis structure tells us that $u.d < v.d < v.f < u.f$ is the only feasible case for $v.f$ and so v must be a descendant of u.

▶Classification of edges in directed graphs

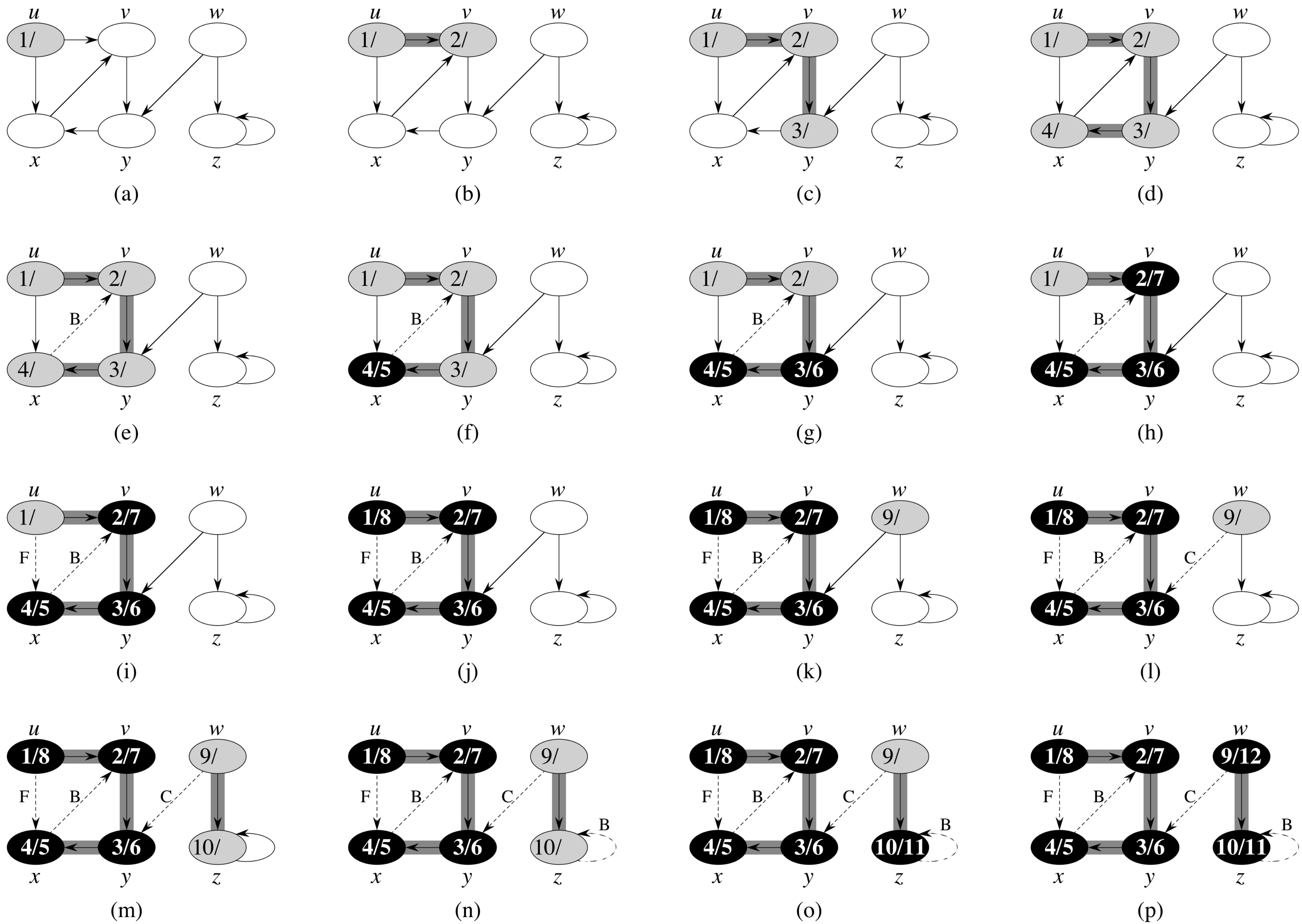
DFS can be used to classify edges of the input graph.

1. **Tree edges** are edges in the depth-first forest. Edge (u, v) is a tree edge if, v was first discovered by exploring edge (u, v).
   *An edge (u, v) is a tree edge if at the time of exploration **v is white**.*

2. **Back edges** are edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree (or self-loops in directed graphs).
   *An edge $(u$, v) is a back edge if at the time of exploration **v is gray**.*

3. **Forward edges** are nontree edges (u, v) connecting a vertex u to a descendant v in a depth-first tree (pointing forward in the tree).
   *$(u$, v) is a forward edge if **v is black and was discovered later**:*
   $u.d < v.d$.

4. **Cross edges** are all other edges: either leading to a subtree constructed earlier or leading to a different (earlier) depth-first tree.
   *(u, v) is a cross edge if **v is black and was discovered earlier**: u.d > v.d.*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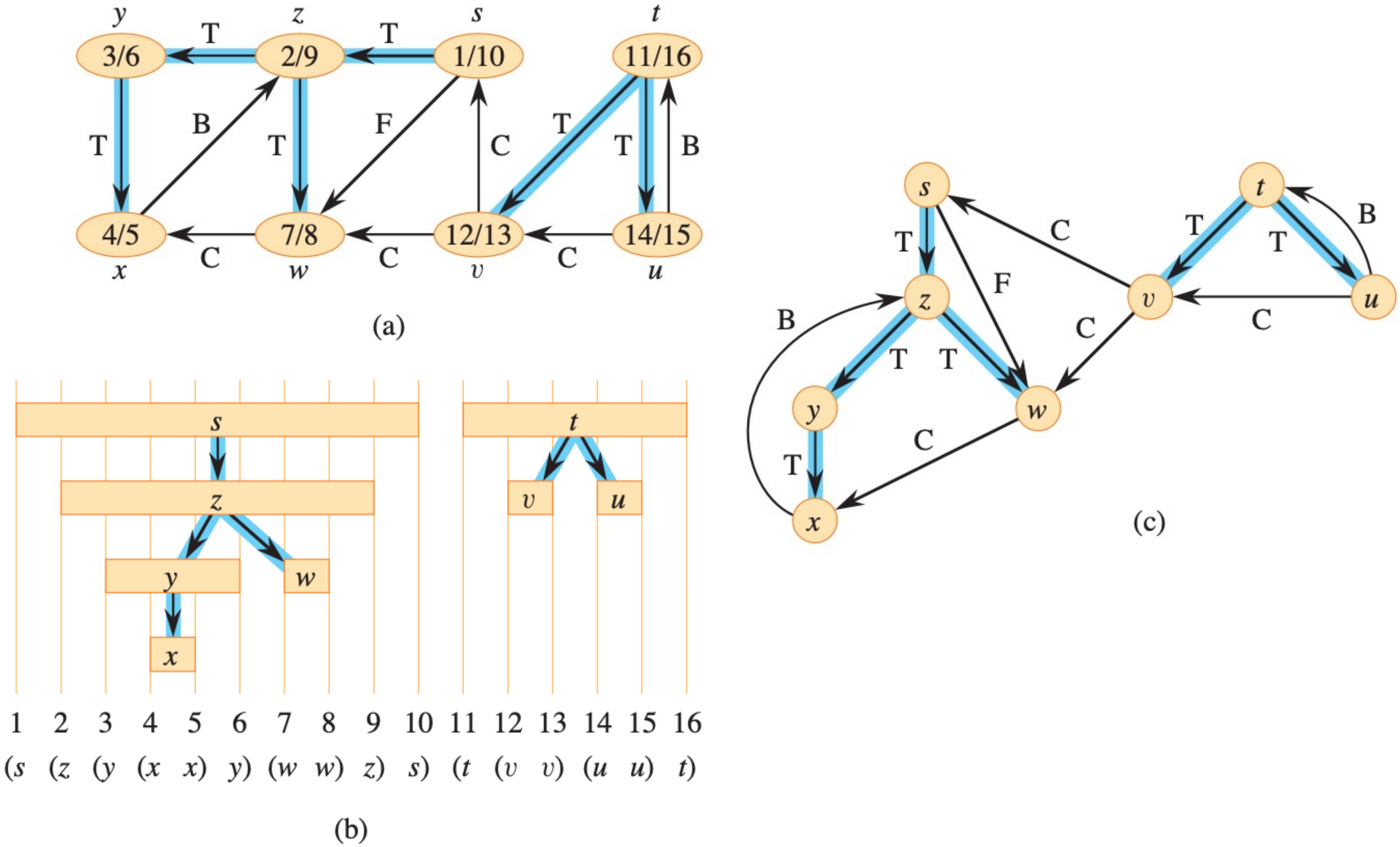
▶Edge classification: example



▶Edge classification in undirected graphs

Theorem 22.10: In a depth-first search of an undirected graph, every edge is either a tree edge or a back edge.

→ There are no forward/cross edges in undirected graphs.

Proof:

- Let $\{u, v\}$ be an arbitrary edge, and assume without loss of generality that $u.d < v.d$.
- Since v is on u's adjacency list, search must discover and finish v before it finishes u.
- If the first time the edge is explored, it is in the direction from u to v, then v is undiscovered and it becomes a **tree edge**.
- If the edge is first explored from v to u, then it becomes a **back edge**, since u is still gray.

Precedence graphs

- Graphs have many applications. One of them is modelling precedences:
  - Vertices represent tasks
  - A edge (u, v) means that task u has to be executed before task v.
- Coming up: how to order tasks such that all precedence constraints are respected.
- But this is only feasible if the precedence graph does not contain any cycles!
- Such a graph is called **acyclic**.

▶Application of DFS: testing for cycles

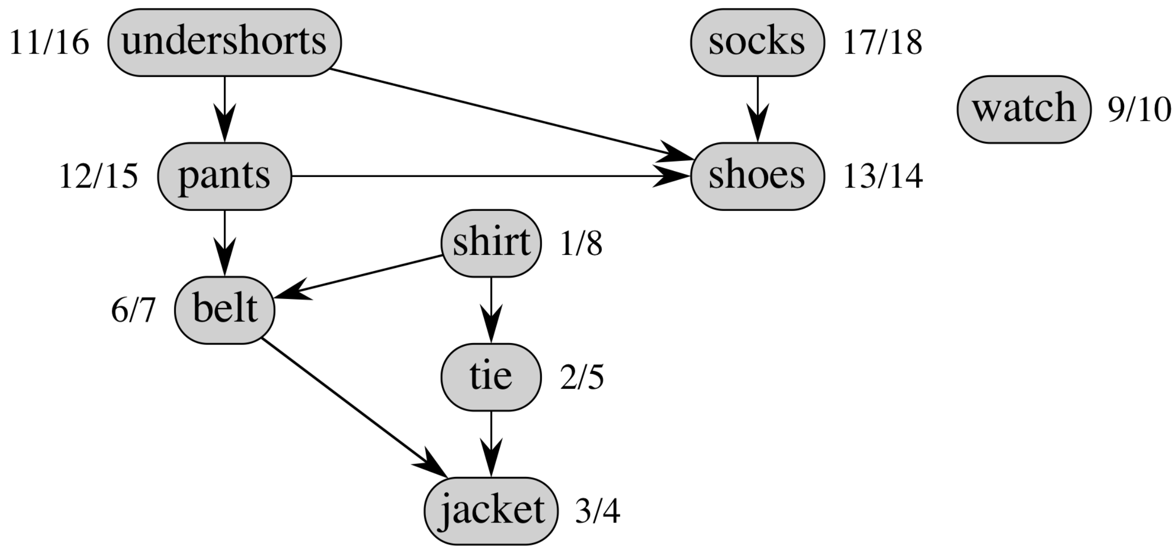
Theorem (adapted from Lemma 22.11): A graph G contains a cycle if and only if DFS finds at least one back edge.

Proof (for directed graphs):

- "⇐": Suppose DFS produces a back edge (u, v). Then v is an ancestor of u in the depth-first tree. Thus, G contains a path (of tree edges) from v to u, and the back edge completes a cycle.
- "⇒": Suppose that G contains a cycle C. We show that DFS yields a back edge. Let v be the first vertex to be discovered in C, and let (u, v) be the edge on C going into v. At time $v.d$, the vertices of C form a path of white vertices from v to u. By the white-path theorem, u becomes a descendant of v. Therefore, (u, v) is a back edge.

▶Topological sorting

- Consider a directed acyclic graph ("dag").
- A topological sort of a dag is a linear ordering of all its vertices such that for each edge (u, v), u appears before v.
- If vertices are arranged on a horizontal line, all edges go from left to right.
- Example: Professor Bumstead getting dressed.

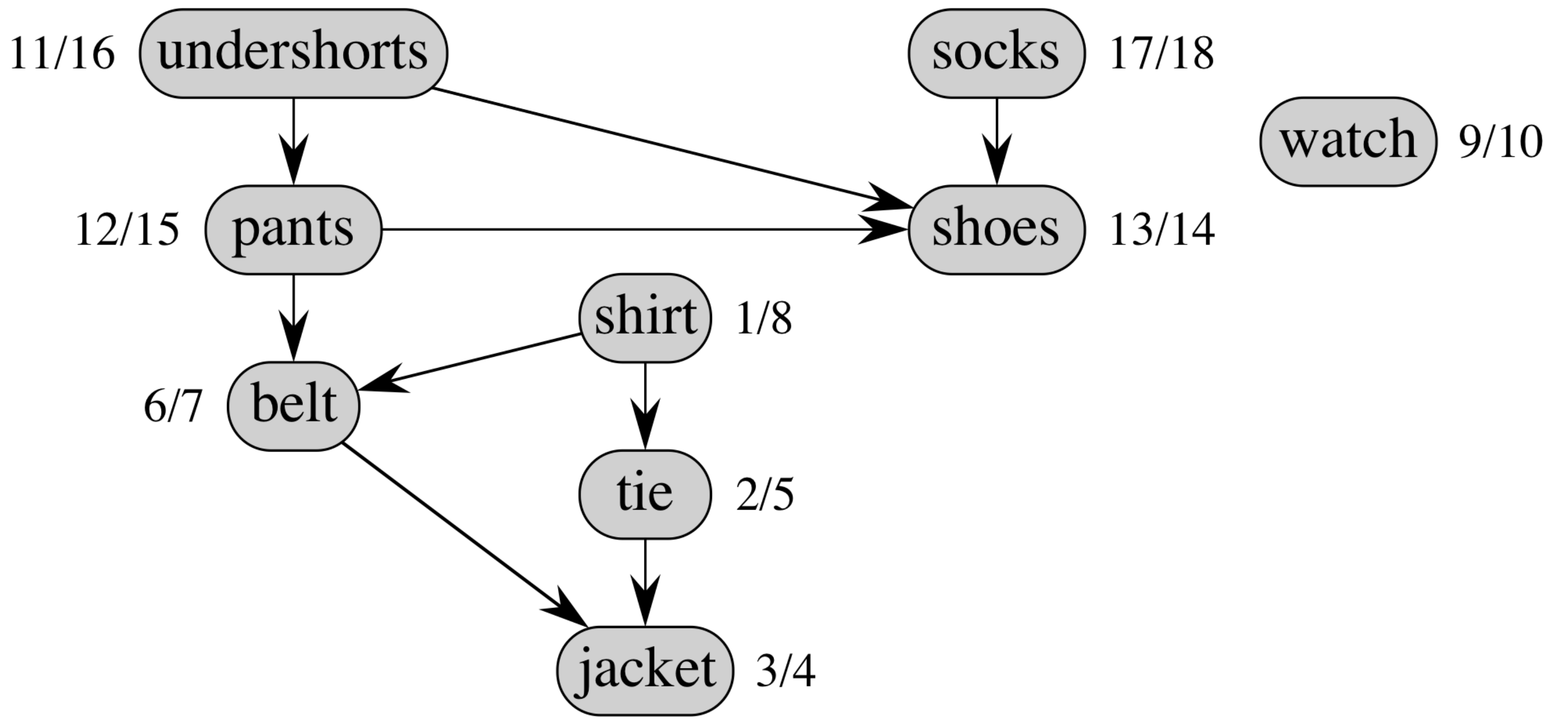


Computing a topological sort

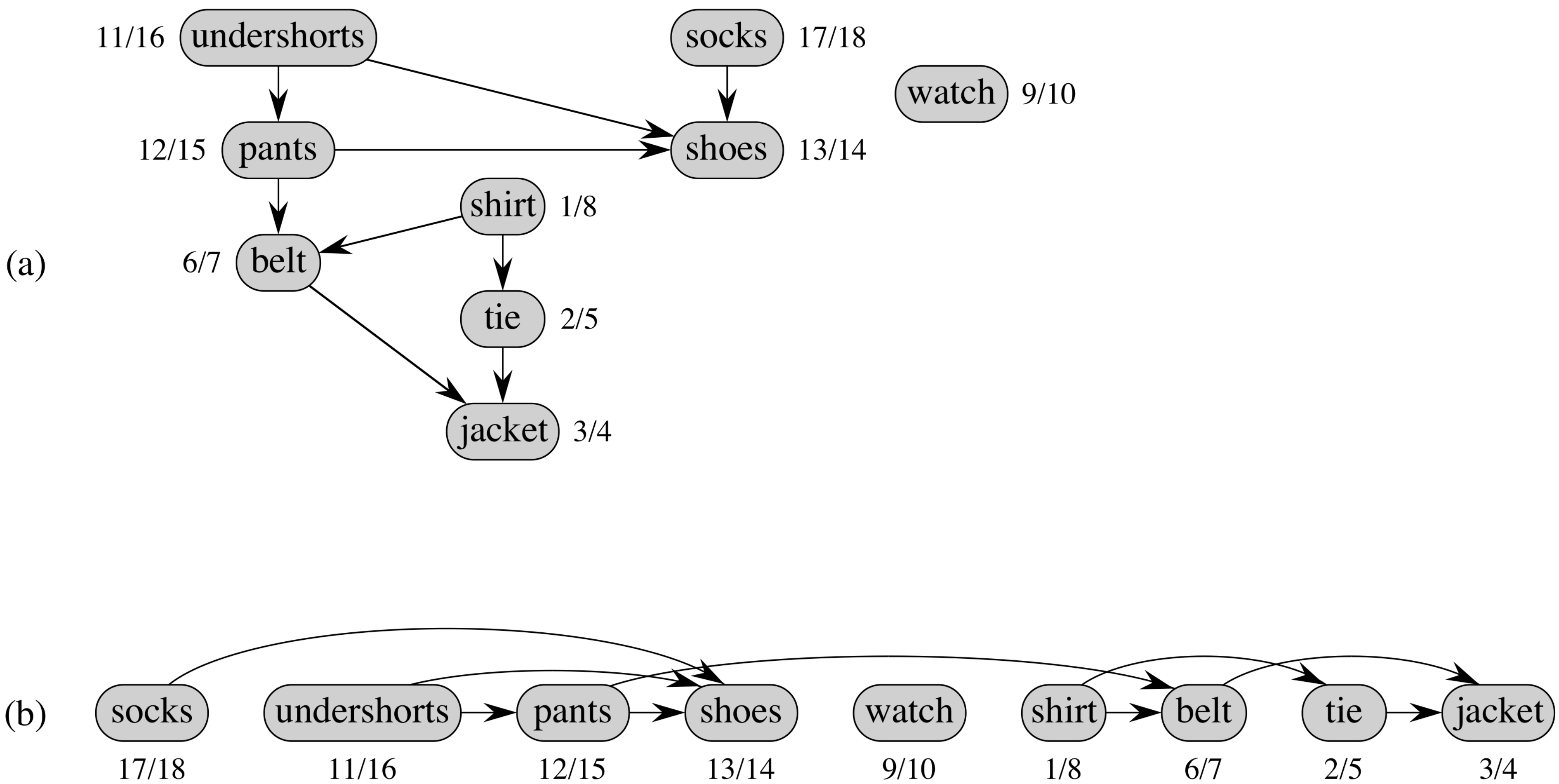
- Here's how to use DFS to compute a topological sort:

TOPOLOGICAL-SORT(G)

call DFS(G) to compute finishing times $v.f$ for each vertex v
as each vertex is finished, insert it onto the front of a linked list
return the linked list of vertices



Professor Bumstead getting dressed



Topological sort: Runtime

TOPOLOGICAL-SORT(G)

1: call DFS(G) to compute finishing times $v.f$ for each vertex v
2: as each vertex is finished, insert it onto the front of a linked list
3: **return** the linked list of vertices

- Runtime:
  - $\Theta(|V| + |E|)$ time for DFS
  - $+O(1)$ for each vertex inserted in to the linked list $\rightarrow +O(|V|)$
  - Total time $\Theta(|V| + |E|)$
- Why on earth does this work?!

Topological sort: correctness proof

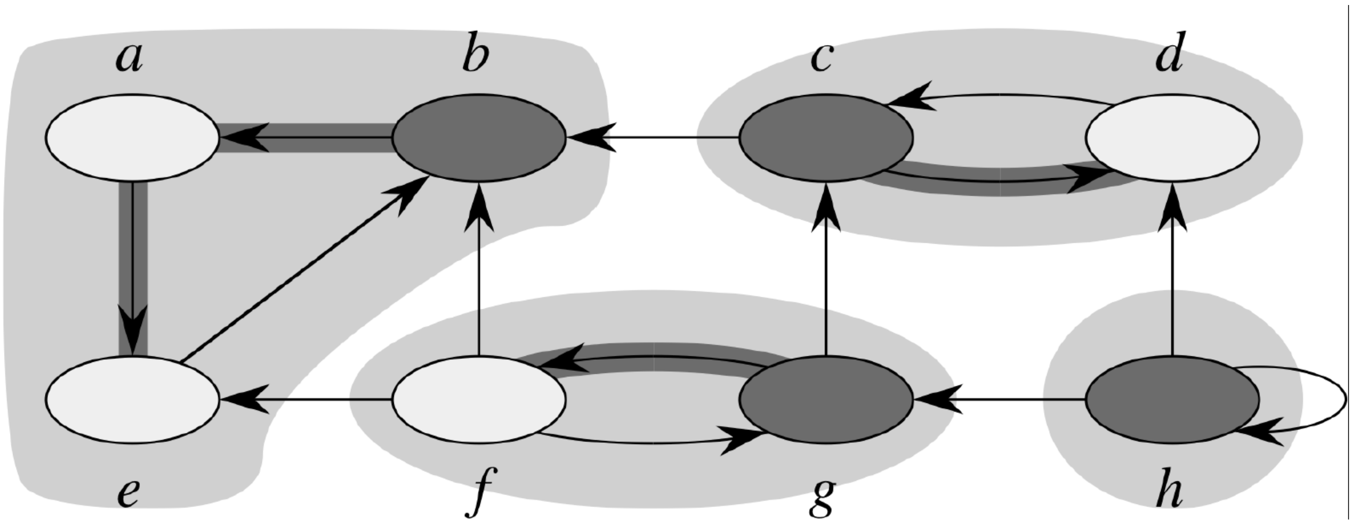
- Suffices to show that if G contains an edge (u, v), then $\boldsymbol{v.f < u.f}$. Then v is inserted to the list earlier and will come to rest after u.

- Consider any edge (u, v) explored by DFS. When this edge is explored, v **cannot be gray**, since then v would be an ancestor of u and (u, v) would be a back edge, contradicting the fact that G is acyclic.

- Therefore, v must be either white or black.
  - If v is white, it becomes a descendant of u, and so $v.f < u.f$ by parenthesis structure.
  - If v is black, it has been finished and $v.f$ has been set. Because we are still exploring from u, a timestamp $u.f$ will be assigned later and once we do, it will be larger: $v.f < u.f$.

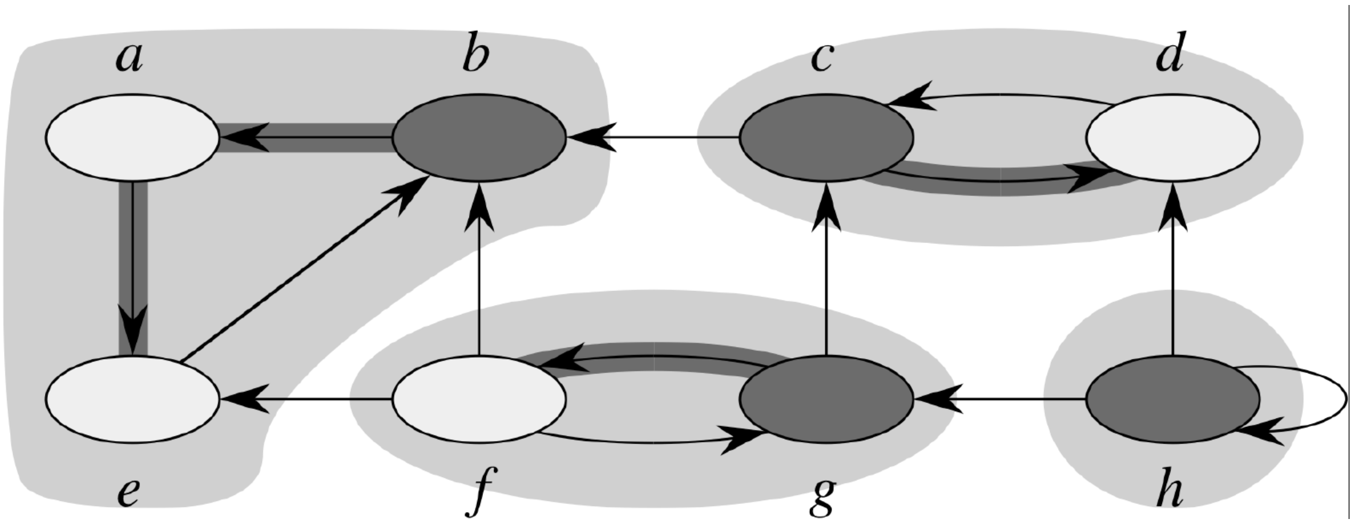
▶Strongly connected components

- A directed graph is called **strongly connected** if every two vertices are reachable from each other.
- The **strongly connected components (SCCs)** of a directed graph are the equivalence classes under the “mutually reachable” relation. In other words, they are maximal sets of vertices where all vertices in every set are mutually reachable.



Strongly connected components

- Applications:
  - Finding groups of friends in social network graphs.
  - Many algorithms working on directed graphs decompose the graph into its SCCs, run separately on all of them, and then combine solutions for all SCCs to one overall solution.



Computing SCCs with DFS

- Let $G^\top$ be the transpose of G, i. e. the graph where all edges have their direction reversed.
- Note that G and $G^\top$ have the same SCC as u and v are reachable in $G^\top$ if and only if they are reachable in G.
- $G^\top$ can be computed in time $O(|V| + |E|)$.

STRONGLY-CONNECTED-COMPONENTS(G)

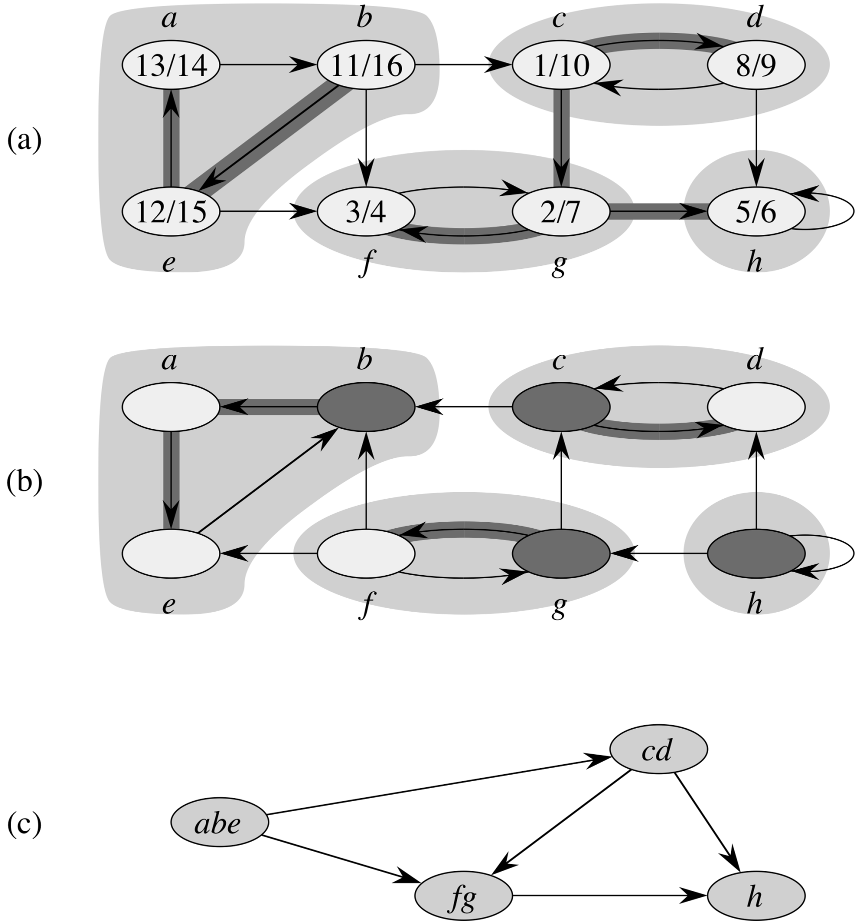
1: call DFS(G) to compute finishing times $v.f$ for each vertex v
2: compute $G^\top$
3: call DFS($G^\top$), but in the main loop of DFS, consider the vertices in order of decreasing $u.f$ (as computed in line 1)
4: output the vertices of the tree in the depth-first forest formed in line 3 as a separate SCC

Strongly connected components: example

STRONGLY-CONNECTED-COMPONENTS(G)

1: call DFS(G) to compute finishing times $v.f$ for each vertex v
2: compute $G^\top$
3: call DFS($G^\top$), but in the main loop of DFS, consider the vertices in order of decreasing $u.f$ (as computed in line 1)
4: output the vertices of the tree in the depth-first forest formed in line 3 as a separate SCC

Runtime?

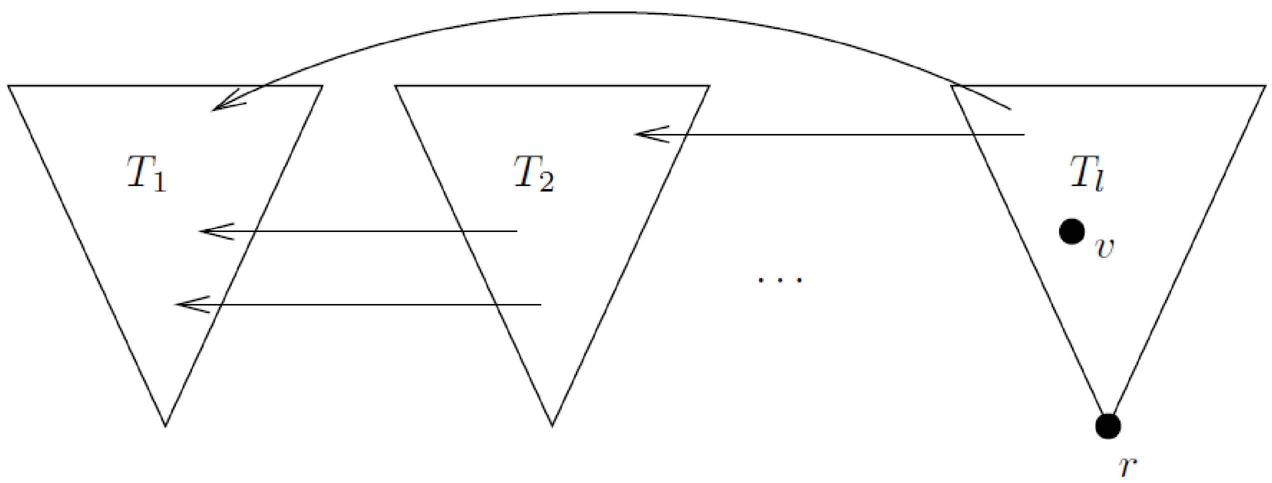


▶Correctness of the SCC algorithm

- Why on earth does this work? It's a miracle!
- Proof in the book is 3 pages of lemmas and not very intuitive.
- Let's use a simpler and more intuitive proof by Ingo Wegener:
- *A simplified correctness proof for a well-known algorithm computing strongly connected components*, Information Processing Letters 83(1), pages 17–19 (on Blackboard)

▶Correctness (2)

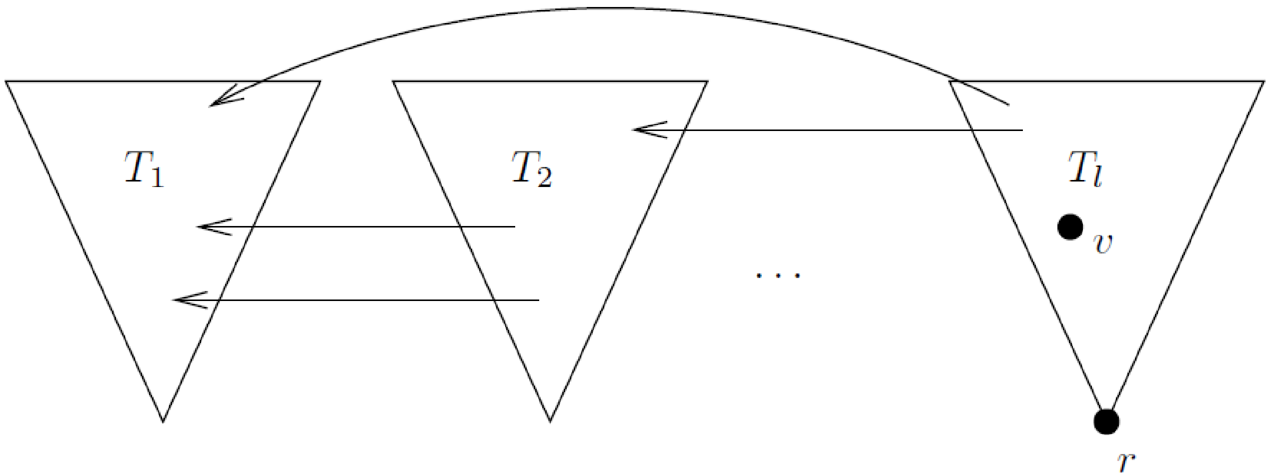
- Draw constructed depth-first trees from left to right and name them $T_1, T_2, \dots, T_l$.
- Then edges between trees can only go right to left (otherwise, e.g. if there is an edge from T_1 to T_2, parts of T_2 would have been included in the depth-first tree T_1)



- Hence each SCC must be contained in one of the trees.

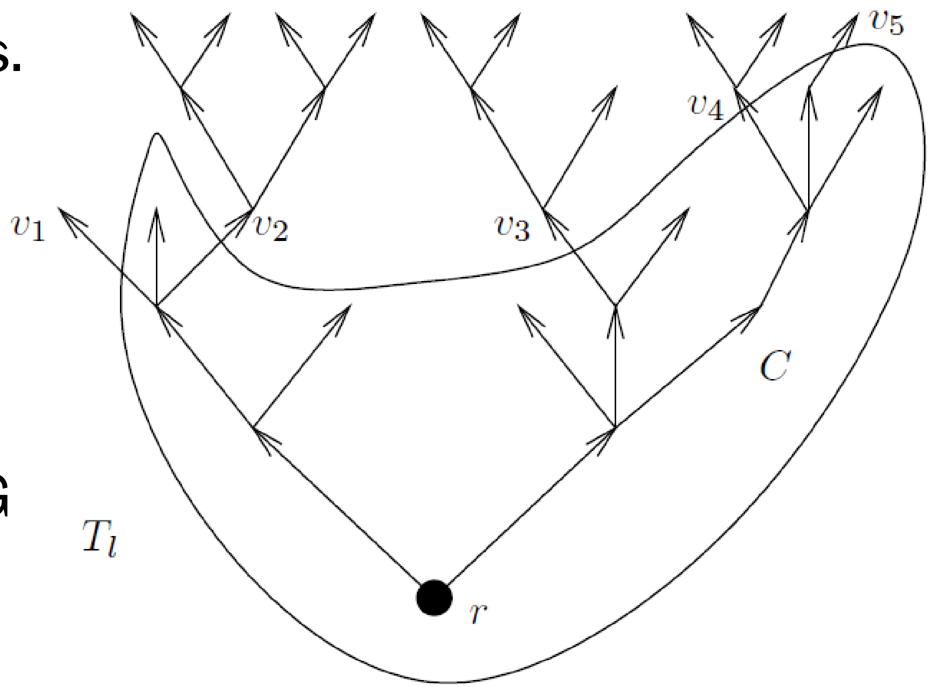
▶Correctness (3) – finding a first SCC

- The algorithm starts the second DFS on $G^\top$ computing the SCC C containing the **root r of the last tree** (as r finished last).
- We know that there is a path from r to all $v \in T_l$ (tree edges).
  So C is the set of all vertices v for which there is a path v to r in G.
  This is the set of **all vertices v reachable from r in $G^\top$**.
- After reversing all edges, DFS from r in $G^\top$ cannot leave T_l.
  Hence DFS in $G^\top$ from r outputs **exactly the SCC containing r**.



Correctness (4) – extracting a first SCC

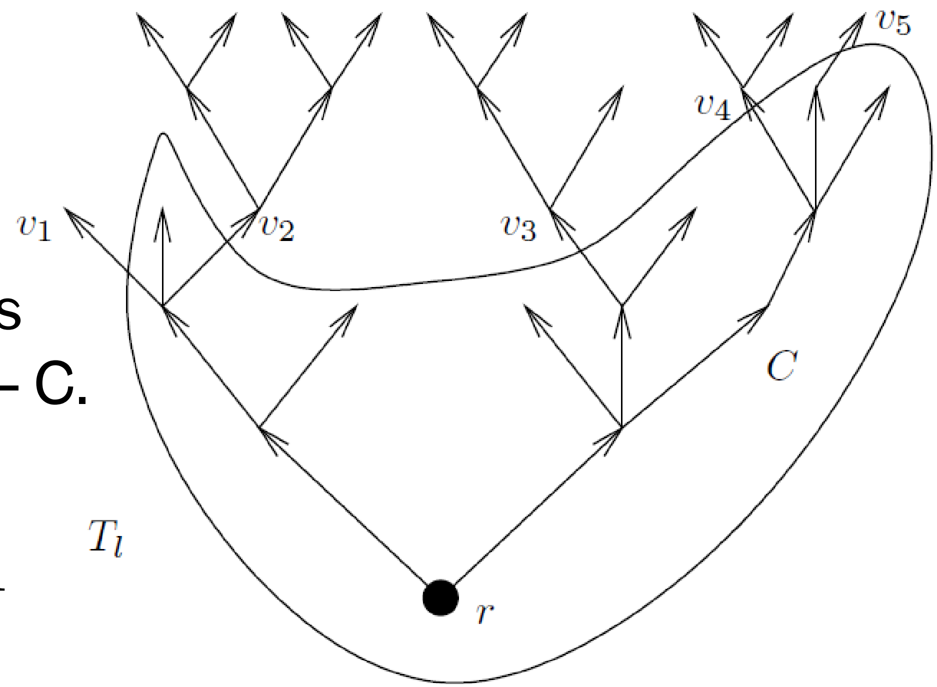
- How does the SCC C containing r look like?
- If v belongs to C, then all vertices on the path r to v must also belong to C (as there is a path from v back to r).
- Hence C is a connected part of T_l.
- T_l without C splits into subtrees.
- $T_1, \ldots, T_{l-1}$ along with these subtrees is a depth-first forest which is also the result of a DFS traversal of G – C.
- The time stamps from DFS on G also work as time stamps for DFS on G – C! (**main insight**)



Correctness (5) – repeated extraction

Proving correctness by induction over the number of SCCs:

- **Base case:** If the graph is a single SCC, the algorithm outputs it.
- Assume the algorithm is correct for graphs with $k - 1$ SCCs.
- For a graph with k SCCs, the algorithm correctly outputs the SCC C containing the root r of the last DFS tree.
- Algorithm continues with vertices and depth-first (sub-)trees in G – C.
- By the induction hypothesis, it then outputs the remaining $k - 1$ SCCs of G – C correctly as well.



▶Summary for Depth-First Search

- Depth-first search explores the graph going into depth and using backtracking in time $\boldsymbol{\Theta(|V| + |E|)}$.
- DFS classifies edges into **tree, back, forward**, and **cross edges**.
- DFS is used to test whether a graph is **acyclic** in time $\Theta(|V| + |E|)$. Can be improved to $O(|V|)$ for undirected graphs (exercise!).
- DFS is used for **topological sorting** in directed acyclic graphs in time $\Theta(|V| + |E|)$.
- DFS is used to determine **strongly connected components** in graphs in time $\Theta(|V| + |E|)$.
- Seen detailed **correctness proofs** to demystify algorithms that appear magical at first glance.