Assignment VI - DSAA(H)

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Date: 2025.10.21

Question 6.1 (marks 0.5)

Consider the following algorithm. Does it sort correctly? (You might want to work out your own example to understand this better.)

```
DO-I-SORT(A, n)
for i = 1 to n do
    for j = 1 to n do
        if A[i] < A[j] then
            exchange A[i] with A[j]
```

1. If the algorithm is correct prove its correctness by loop invariant. Otherwise argue why it is not correct eg., provide an instance where it fails.
2. State the runtime of the algorithm in asymptotic notation. Justify your answer.

Sol: The algorithm is **correct**.

1.

Inner Loop Invariant: Before each iteration j, the element $A[i]$ is greater than every elements in $A[1, j-1]$.

- **Initialization**: At the beginning, $j - 1 = 0$, the loop invariant is trivially satisfied.
- **Maintenance**: In the iteration, if $A[j] > A[i]$, they will be swaped, leading that $A[i]$ remains to be the greatest.

- **Termination**: When $j = n$, the loop terminate, and $A[i]$ will be the largest element of $A[1, n]$.

Outer Loop Invariant: Before each iteration i, the prefix $A[1, i-1]$ is non-decreasing.

- **Initialization**: At the beginning, $i - 1 = 0$, the loop invariant is trivially satisfied.
- **Maintenance**: In the iteration, after the inner loop, $A[i]$ will become the greatest. And the larger elements will only be moved rightwards. Therefore $A[1, i]$ will be non-decreasing.
- **Termination**: When $i = n$, the loop terminate, and $A[1, n]$ will become non-decreasing, which is ordered.

2. The runtime is $\Theta(n^2)$. For the two loops are both from 1 to n, which leads to n^2 swaps or comparisons which is $\Theta(1)$, thus the total runtime is $\Theta(n^2)$.

Question 6.2 (0.5 marks)

Consider the following input for RANDOMIZED-QUICKSORT:

12	10	4	2	9	6	5	25	8

What is the probability that:

1. The elements $A[2] = 10$ and $A[3] = 4$ are compared?
2. The elements $A[1] = 12$ and $A[8] = 25$ are compared?
3. The elements $A[4] = 2$ and $A[8] = 25$ are compared?
4. The elements $A[2] = 10$ and $A[7] = 5$ are compared?

PF:

Lemma I: Denoting the elements in sorted array as z_i, the probility of comparison between $z_i, z_j, i < j$ is $\dfrac{2}{j - i + 1}$.

Proof:

- If pivot x s.t. $x < z_i$ or $x > z_j$ then the decision whether to do comparison is postponed to a recursive call.

- If pivot is $x = z_i$ or $x = z_j$, both z_i, z_j will be compared.
- If pivot is $z_i < x < z_j$ then both z_i, z_j will be compared and become separated, then never be compared!
- Thus the probility is $\dfrac{2}{j-i+1}$.

According to Lemma I:

We can sort A, we have $2, 4, 5, 6, 8, 9, 10, 12, 25$.

Then for z_i, we have $8, 7, 2, 1, 6, 4, 3, 9, 5$.

Therefore:

1. $i = 2, j = 7$, $\dfrac{2}{7-2+1} = \dfrac{1}{3}$.
2. $i = 8, j = 9$, $\dfrac{2}{9-8+1} = 1$.
3. $i = 1, j = 9$, $\dfrac{2}{9-1+1} = \dfrac{2}{9}$.
4. $i = 3, j = 7$, $\dfrac{2}{7-3+1} = \dfrac{2}{5}$.

Question 6.3 (1 mark)

Prove that the expected runtime of RANDOMIZED-QUICKSORT is $\Omega(n \log n)$.

(HINT: It may be useful to consider how long it takes to compare $n/2$ elements to achieve a lower bound on the runtime.)

PF:

Lemma I: Denoting the elements in sorted array as z_i, the probility of comparison between $z_i, z_j, i < j$ is $\dfrac{2}{j-i+1}$.

Proof:

- If pivot x s.t. $x < z_i$ or $x > z_j$ then the decision whether to do comparison is postponed to a recursive call.
- If pivot is $x = z_i$ or $x = z_j$, both z_i, z_j will be compared.
- If pivot is $z_i < x < z_j$ then both z_i, z_j will be compared and become separated, then never be

compared!

- Thus the probility is $\dfrac{2}{j-i+1}$.

Obviously, we can represent the runtime by comparison times X_n, for the other works are all low-order.

Denoting the elements in sorted array as z_i, let $X_{i,j}$ be the comparison times between i, j.

We have $X_{i,j} = \dfrac{2}{j-i+1}, i < j$, according to Lemma I.

Thus:

$$
\begin{aligned}
E(X) &= \sum_{i<j} X_{i,j} \\
&= \sum_{i<j} \frac{2}{j-i+1} \\
&= \sum_{k=1}^{n-1} (n-k) \cdot \frac{2}{k+1} \\
&\geq \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{2} \cdot \frac{1}{k} \\
&= n \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor + 1} \frac{1}{k} \\
&= n \cdot \Theta(\log n) \\
&= \Theta(n \log n)
\end{aligned}
$$

Therefore, $E(X) \geq \Theta(n \log n)$, i.e., $\Omega(n \log n)$.

`Q.E.D.`.

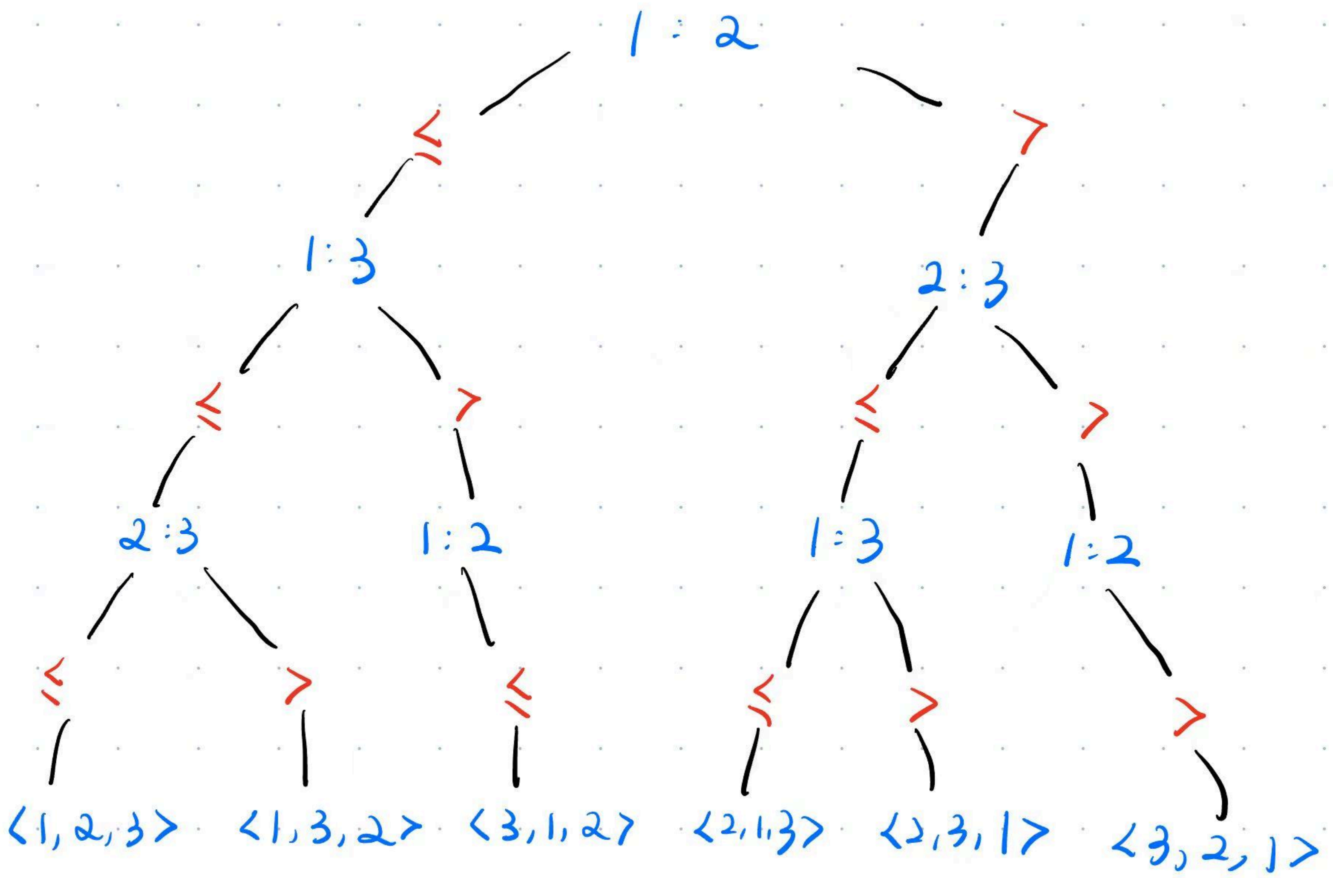
Question 6.4 (1 mark)

Draw the decision tree that reflects how SelectionSort sorts $n = 3$ elements. Assume that all elements are mutually distinct.

For convenience here's the pseudocode again:

```
SELECTION-SORT(A)
n = A.length
for j = 1 to n - 1 do
    smallest = j
    for i = j + 1 to n do
        if A[i] < A[smallest] then smallest = i
    exchange A[j] with A[smallest]
```

Sol:



Question 6.5 (0.5 marks)

What is the smallest possible depth of a leaf in a decision tree for a comparison sort?

Sol: Obvoiusly it's $n-1$.

We obtain that each comparison represents one depth deeper, and if we want to finish a comparison sort, we need to clarify the minimum element at least. And the process of clarifying the minimum needs at least $n-1$ comparisons, thus the smallest possible depth is $n-1$.

Question 6.6 (0.25 marks)

Implement RANDOMIZED-QUICKSORT and solve the 'Yet Another Quicksort' Problem.

题目

状态	最后递交于	题目
✓ 100 Accepted	1 周前	**33** Quick Sort III
✓ 100 Accepted	1 周前	**30** Yet Another Quick Sort Problem

```
int main(){
    int N = read();
    vector < int > A(N + 10, 0);
    for(int i = 1; i <= N; ++i)A[i] = read();
    auto Partition = [](vector < int > &A, int l, int r)->int{
        int val(A[r]);
        int spl(l - 1);
        for(int i = l; i <= r - 1; ++i)
            if(A[i] <= val)swap(A[++spl], A[i]);
        swap(A[++spl], A[r]);
        return spl;
    };
    auto RandPartition = [&](vector < int > &A, int l, int r)->int{
        swap(A[r], A[rndd(l, r)]);
        return Partition(A, l, r);
    };
    auto RandQuickSort = [&](auto&& self, vector < int > &A, int l, int
r)->void{
        if(l >= r)return;
        int spl = RandPartition(A, l, r);
        self(self, A, l, spl - 1);
```

```
            self(self, A, spl + 1, r);
    }; RandQuickSort(RandQuickSort, A, 1, N);

    for(int i = 1; i <= N; ++i)printf("%d%c", A[i], i == N ? '\n' : ' ');

    // fprintf(stderr, "Time: %.6lf\n", (double)clock() / CLOCKS_PER_SEC);
    return 0;
}
```

```
int N;

class SegTree{
private:
    int mn[510000 << 2], mx[510000 << 2];
    #define LS (p << 1)
    #define RS (LS | 1)
    #define MID ((gl + gr) >> 1)
public:
    void Clear(void){
        for(int i = 0; i <= (N << 2) + 100; ++i)mn[i] = INT_MAX, mx[i] = INT_MIN;
    }
    void Pushup(int p){
        mn[p] = min(mn[LS], mn[RS]);
        mx[p] = max(mx[LS], mx[RS]);
    }
    void Modify(int pos, int val, int p = 1, int gl = 1, int gr = N){
        if(gl == gr)return mx[p] = mn[p] = val, void();
        if(pos <= MID)Modify(pos, val, LS, gl, MID);
        else Modify(pos, val, RS, MID + 1, gr);
        Pushup(p);
    }
    int QueryGreaterThan(int val, int l, int r, int p = 1, int gl = 1, int gr = N){
        if(gr < l || gl > r)return -1;
        if(gl == gr)return mx[p] >= val ? gl : -1;
        int ret(-1);
```

```cpp
        if(l <= MID && mx[LS] >= val)ret = QueryGreaterThan(val, l, r, LS, gl, MID);
        if(!~ret && r >= MID + 1)ret = QueryGreaterThan(val, l, r, RS, MID + 1, gr);
        return ret;
    }
    int QueryLessThan(int val, int l, int r, int p = 1, int gl = 1, int gr = N){
        if(gr < l || gl > r)return -1;
        if(gl == gr)return mn[p] <= val ? gl : -1;
        int ret(-1);
        if(r >= MID + 1 && mn[RS] <= val)ret = QueryLessThan(val, l, r, RS, MID + 1, gr);
        if(!~ret && l <= MID)ret = QueryLessThan(val, l, r, LS, gl, MID);
        return ret;
    }
}st;

int main(){
    int T = read();
    while(T--){
        ll res(0);
        N = read();
        st.Clear();
        vector < int > A(N + 10, 0);
        for(int i = 1; i <= N; ++i)st.Modify(i, A[i] = read());
        auto Partition = [&](vector < int > &A, int l, int r)->int{
            int pivot = A[(l + r) >> 1];
            int i(l - 1), j(r + 1);
            while(true){
                i = st.QueryGreaterThan(pivot, i + 1, r);
                j = st.QueryLessThan(pivot, l, j - 1);
                i = !~i ? r + 1 : i;
                j = !~j ? l - 1 : j;
                // printf("next i = %d, j = %d\n", i, j);
                if(i >= j)return j;
                ++res;
                st.Modify(i, A[j]);
                st.Modify(j, A[i]);
```

```cpp
                    swap(A[i], A[j]);
                }
            };
            auto QuickSort = [&](auto&& self, vector < int > &A, int l, int r)->void{
                if(l >= r)return;
                int spl = Partition(A, l, r);
                self(self, A, l, spl);
                self(self, A, spl + 1, r);
            }; QuickSort(QuickSort, A, 1, N);
            printf("%lld\n", res);
        }

    // fprintf(stderr, "Time: %.6lf\n", (double)clock() / CLOCKS_PER_SEC);
    return 0;
}
```