Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Compound Propositions

- A *proposition* is a declarative statement that is either true or false. More complex propositions can be built from elementary statements using logical connectives.

- Logical connectives:
  - ◇ $\neg p$ (*Negation*)
  - ◇ $p \wedge q$ (*Conjunction*)
  - ◇ $p \vee q$ (*Disjunction*)
  - ◇ $p \oplus q$ (*Exclusive or*)
  - ◇ $p \rightarrow q$ (*Implication*)
  - ◇ $p \leftrightarrow q$ (*Biconditional*)

Conditional Statement (implication)

- Let p and q be propositions. The *conditional statment* (a.k.a. *implication*) $p \to q$, is the proposition "if p, then q", is false when p is true and q is false, and true otherwise. In $p \to q$, p is called the *hypothesis* and q is called the *conclusion*.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- $p \rightarrow q$ is read in a variety of equivalent ways:
  - ◇ if p then q
  - ◇ p implies q
  - ◇ p is sufficient for q
  - ◇ q is necessary for p
  - ◇ q follows from p
  - ◇ q unless $\neg p$
  - ◇ p only if q

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Example:

- ◇ If you get 100 on the final, then you will get an A.

p q

Implication

- The *converse* of $p \to q$ is $q \to p$.
  The *contrapositive* of $p \to q$ is $\neg q \to \neg p$.
  The *inverse* of $p \to q$ is $\neg p \to \neg q$.

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  Examples:

  ◇ If you get 100 on the final, then you will get an A. ($p \rightarrow q$)

  ◇ If you get an A, then you get 100 on the final. ($q \rightarrow p$)

  ◇ If you don't get an A, then you don't get 100 on the final. ($\neg q \rightarrow \neg p$)

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  ◇ If you don't get 100 on the final, then you don't get an A. ($\neg p \to \neg q$)

  $\neg q \to \neg p$ is *equivalent* to $p \to q$

Constructing the Truth Table

- Construct a truth table for $p \vee q \rightarrow \neg r$

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p	q	r	¬r	p ∨ q	p ∨ q → ¬r
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Computer Representation of True and False

- A *bit* is sufficient to represent two possible values: 0 (false) or 1 (true).
- A varaiable that takes on values 0 and 1 is called a *Boolean variable*.
- A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

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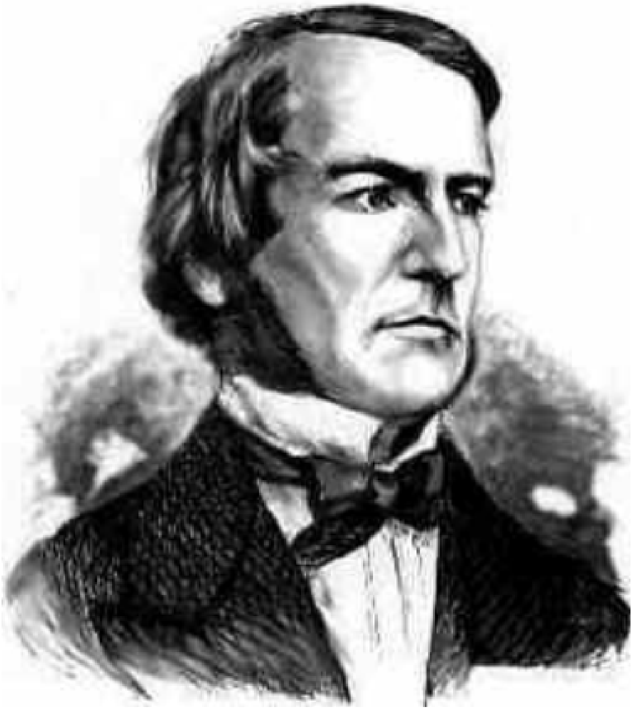
- A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.
- *bitwise operations*: replace "T" and "F" with 1 and 0.

$$\begin{array}{r} 1011\ 0011 \\ \vee\ \underline{0110\ 1010} \\ 1111\ 1011 \end{array}$$

George Boole

- British mathematician (b. 1815, d. 1864)
  - ◇ The inventor of Boolean algebra
  - ◇ Truth tables are an example of B.A.



◇ Although Boole's work was not originally perceived as particularly interesting, even by other mathematicians, he is now seen as one of the founders of the field of Computer Science.

Applications of Propositional Logic

- Translation of English sentences
  - ◇ use atomic (elementary) propositions
- Inference and reasoning
  - ◇ new true propositions are inferred from existing ones
  - ◇ used in *Artificial Intelligence*:
    - – rule based (expert) systems, automatic theorem provers, ...
- Design of logic circuit

Translation

- If you are older than 13 or you are with your parents then you can watch this movie.

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  Atomic (**elementary**) propositions:

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  Translation: $p \vee q \rightarrow r$

  Given that p is true.

  With the help of the logic, we can infer the following statement:

  – You can watch this movie.

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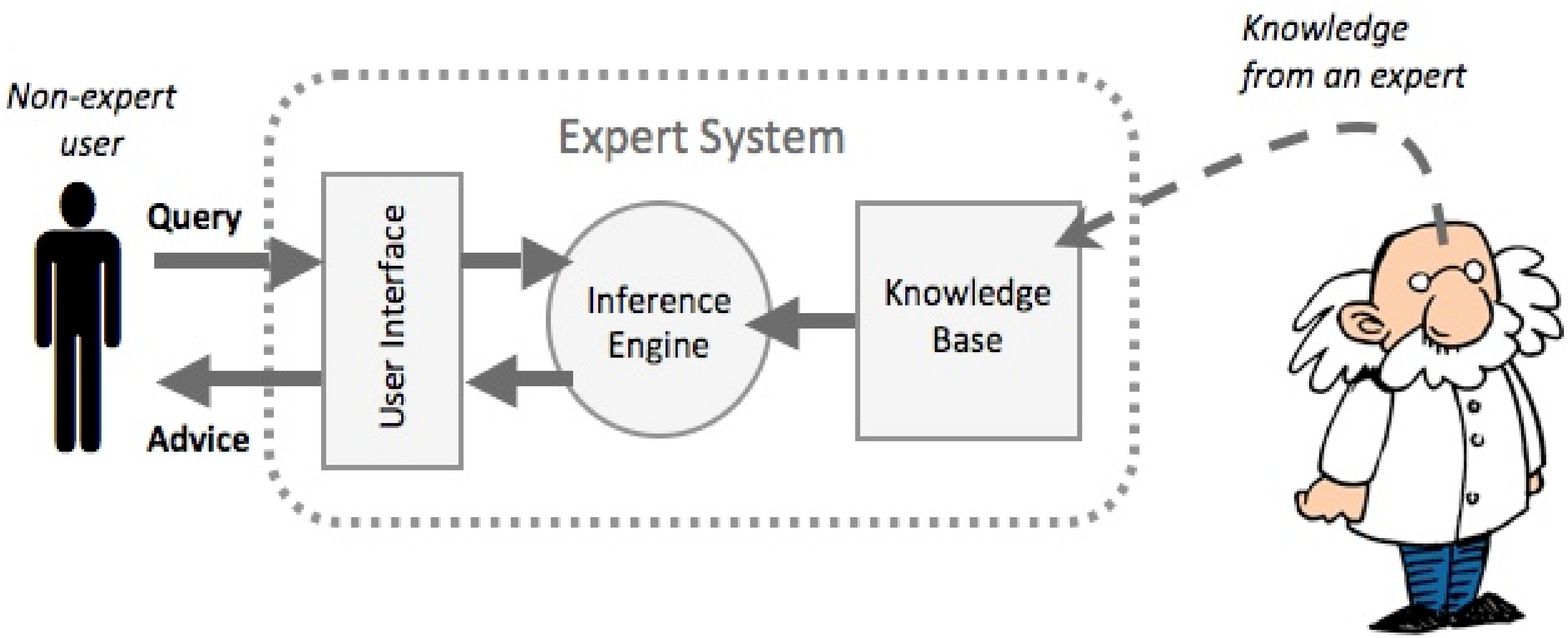
- *Expert systems*
  - ◇ encode knowledge about the world in logic
  - ◇ support inferences where new facts are inferred from existing ones following the semantics of logic

- *Theorem provers*
  - ◇ encode existing knowledge (e.g., math) using logic
  - ◇ show that some hypothesis is true

Expert System



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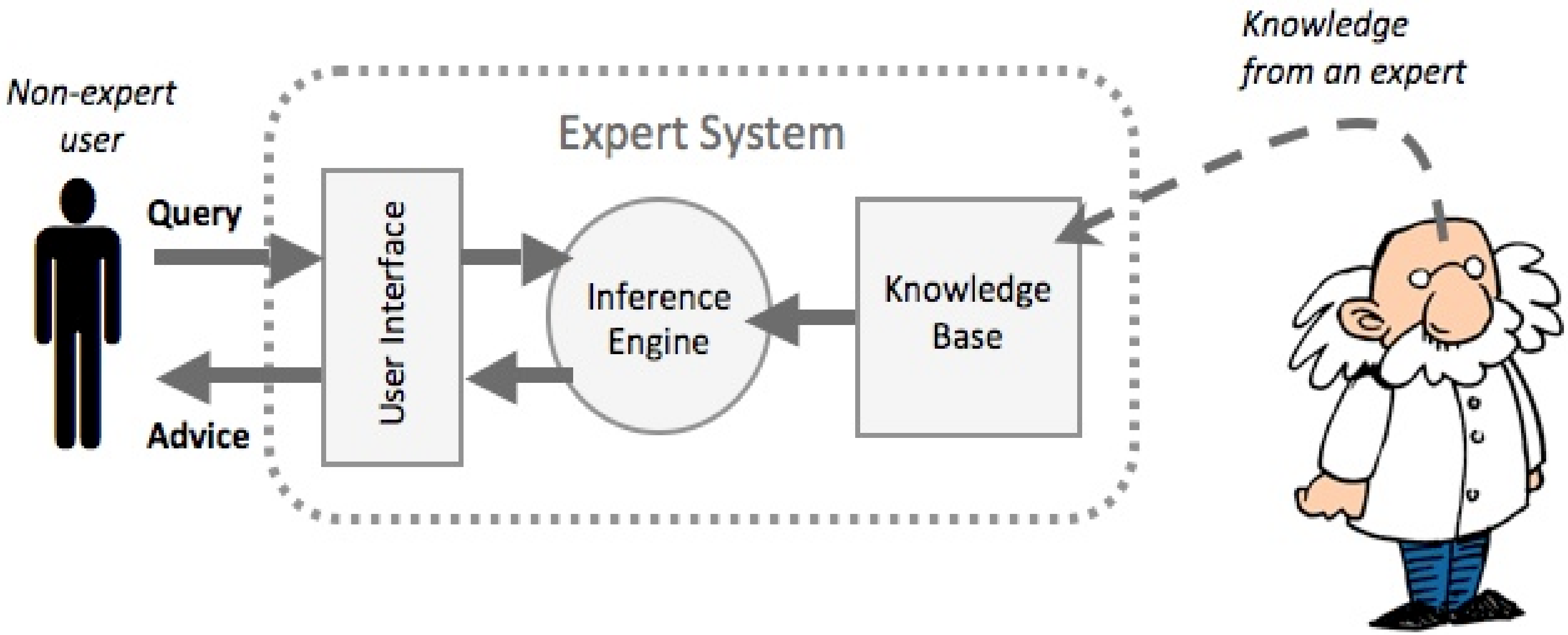

Theorem Prover

Contents lists available at ScienceDirect

Artificial Intelligence

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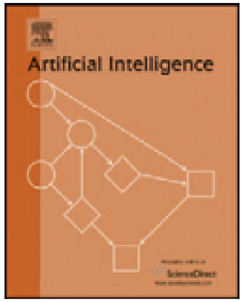

Computer-aided proofs of Arrow's and other impossibility theorems ☆

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A R T I C L E   I N F O

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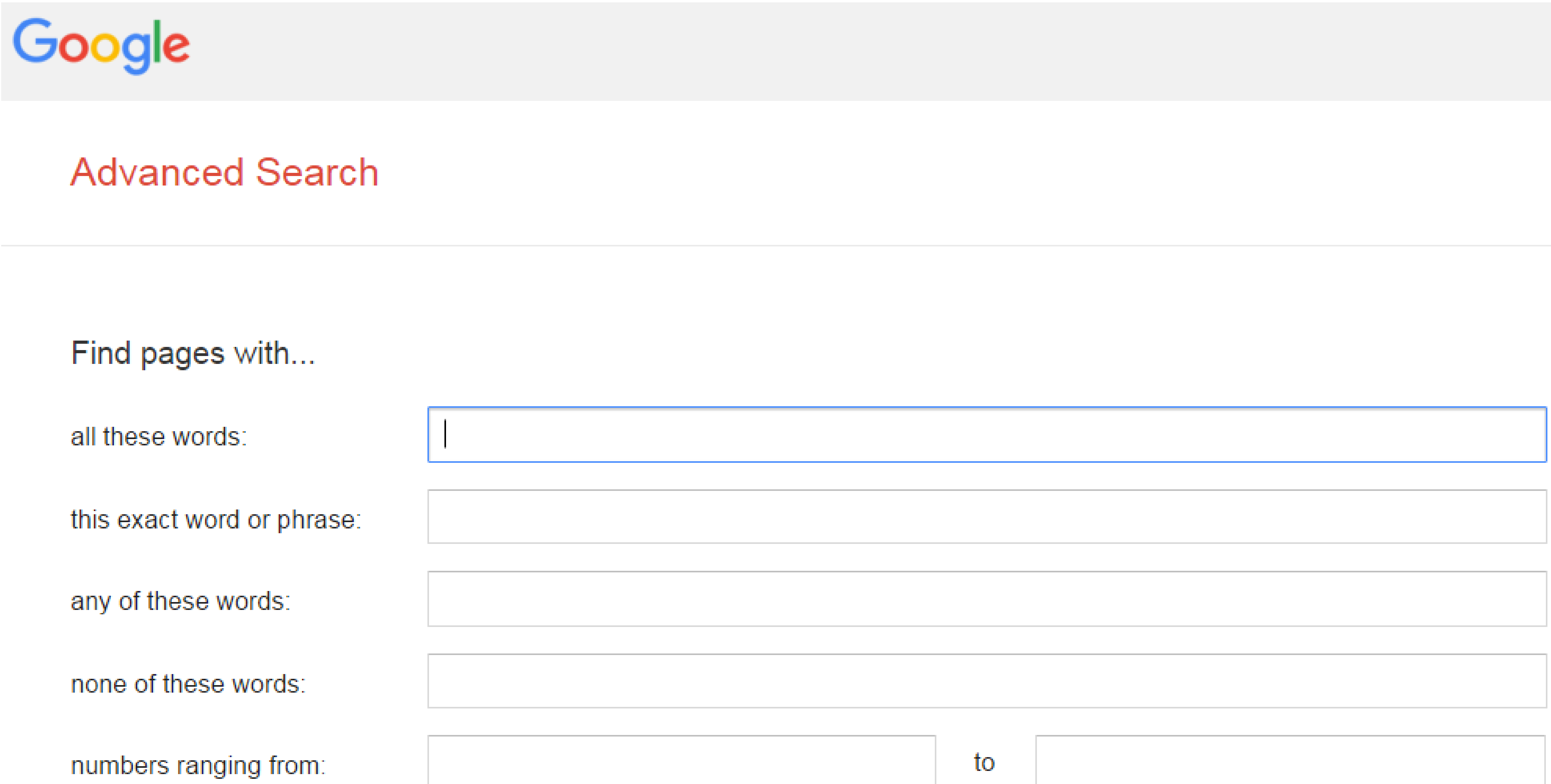
A B S T R A C T

Arrow's impossibility theorem is one of the landmark results in social choice theory. Over the years since the theorem was proved in 1950, quite a few alternative proofs have been put forward. In this paper, we propose yet another alternative proof of the theorem. The basic idea is to use induction to reduce the theorem to the base case with 3 alternatives and 2 agents and then use computers to verify the base case. This turns out to be an effective approach for proving other impossibility theorems such as Muller–Satterthwaite and Sen's theorems as well. Motivated by the insights of the proof, we discover a new theorem with the help of computer programs. We believe this new proof opens an exciting prospect of using computers to discover similar impossibility or even possibility results.

Other Applications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Tautology and Contradiction

- A compound proposition that is always true for all possible truth values is called a *tautology*.
- A compound proposition that is always false for all possible truth values is called a *contradiction*.
- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

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Equivalent Propositions

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Examples:

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(1) if ((i+j ≤ p+q) && (i ≤ p) &&
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(2)   List3[k] = List1[i]
(3)   i = i+1
(4) else
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Consider the two pieces of codes taken from two different versions of *Mergesort*. Do they do the same thing?

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  $s \sim (i + j \leq p + q)$, $t \sim (i \leq p)$, $u \sim (j > q)$
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(1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

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  (1) $s \wedge t \wedge (u \vee v)$ (1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

  Now set $w \sim (s \wedge t)$

  (1) $w \wedge (u \vee v)$ (1') $(w \wedge u) \vee (w \wedge v)$

Truth Tables

(1) $w \wedge (u \vee v)$

w	u	v	$u \vee v$	$w \wedge (u \vee v)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(1') $(w \wedge u) \vee (w \wedge v)$

w	u	v	$w \wedge u$	$w \wedge v$	$(w \wedge u) \vee (w \wedge v)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
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  Example $p \to q \equiv \neg q \to \neg p$

De Morgan's Laws

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p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

- *Identity laws*
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- *Domination laws*
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- *Idempotent laws*
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$

Important Logical Equivalences

- *Double negation laws*
  - $\neg(\neg p) \equiv p$

- *Commutative laws*
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- *Associative laws*
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Important Logical Equivalences

- *Distributive laws*
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's laws*
  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$
  - $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- *Others*
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

    Absorption laws
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

    Negation laws
  - $p \rightarrow q \equiv \neg p \vee q$

    Useful law

Using Logical Equivalences

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- **Example**: Show that $(p \wedge q) \to p$ is a tautology.

  Proof:

$$
\begin{aligned}
(p \wedge q) \to p &\equiv \neg(p \wedge q) \vee p && \text{Useful} \\
&\equiv (\neg p \vee \neg q) \vee p && \text{De Morgan's} \\
&\equiv (\neg q \vee \neg p) \vee p && \text{Commutative} \\
&\equiv \neg q \vee (\neg p \vee p) && \text{Associative} \\
&\equiv \neg q \vee T && \text{Negation} \\
&\equiv T && \text{Domination}
\end{aligned}
$$

Using Logical Equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

- **Example**: Show that $(p \wedge q) \rightarrow p$ is a tautology.

  Proof (alternatively):

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

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- **Example**: Show that $p \to q \equiv \neg q \to \neg p$.

Proof:

$\neg q \to \neg p \equiv \neg(\neg q) \vee (\neg p)$	Useful
$\equiv q \vee (\neg p)$	Double negation
$\equiv (\neg p) \vee q$	Communtative
$\equiv p \to q$	Useful

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What is a more natural solution to express the knowledge?

Solution: make statements with variables

– If x is an integer, then $x^2 \geq 0$.

– x is an integer $\rightarrow x^2 \geq 0$.

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  - "The square of every integer is ≥ 0"
  - "Some of the integers are prime."

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Example 2: (statements that define the property of a group of objects)

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- ◇ "Some of the integers are prime."

Solutions: make statements with *quantifiers*

– universal quantifier: the property is satisfied by all members of the group

– existential quantifier: at least one member of the group satisfies the property

Predicate Logic

- Remedies the limitations of propositional logic:
  - ◇ explicitly models objects and their properties
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- Basic building blocks of the predicate logic:
  - ◇ *Constant* – models a specific object

    Examples: "1", "SUSTech", ...
  - ◇ *Variable* – represents object of specific type

    Examples: x, y, ... (universe can be people, numbers ...)
  - ◇ *Predicate* – represents properties or relations among objects

    Examples: Red(car23), student(x), married(A, B)...

Predicates

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  Example: Assume Prime(x) where the universe of discourse are integers

  - Prime(2) ... T
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  Is Prime(x) a proposition?

  No, but after the substitution it becomes one.

Predicates

- A *predicate* is a statement $P(x_1, x_2, \ldots, x_n)$ that contains n variables $x_1, x_2, \ldots, x_n$ and becomes a proposition when specific values are substituted for the variables x_i.

- The *universe* (*domain*) D of the *predicate variables* $(x_1, x_2, \ldots, x_n)$ is the set of all values that may be substituted in place of the variables.

- The *truth set* of $P(x_1, x_2, \ldots, x_n)$ is the set of all values of the predicate variables $(x_1, x_2, \ldots, x_n)$ such that the proposition $P(x_1, x_2, \ldots, x_n)$ is true.

Examples of Predicates

- **Example 1**: (Predicates with one variable)
  Let $P(x)$ be the predicate "$x^2 > x$" with universe of the real numbers.
  - ◇ What are the truth values of $P(2)$ and $P(1)$?
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  - ◇ $P(2) = \mathrm{T}$, $P(1) = \mathrm{F}$
  - ◇ $x > 1$ or $x < 0$

Examples of Predicates

- **Example 2**: (Predicates with two variables)
  Let $Q(x, y)$ be the predicate "$x = y + 3$" with universe of the real numbers.
  - ◇ What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?
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  - ◇ $Q(1, 2) = F$, $Q(3, 0) = T$
  - ◇ $(a, a - 3)$ for all real numbers a

Compound Statements in Predicate Logic

- Compound statements are obtained via logical connectives.

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Example:

- ◇ $\text{Prime}(2) \wedge \text{Prime}(3)$
  – Translation: "Both 2 and 3 are primes." (T)
- ◇ $\text{City}(\text{Shenzhen}) \vee \text{River}(\text{Shenzhen})$
  – Translation: "Shenzhen is a city or a river." (T)
- ◇ $\text{CS-major}(x) \rightarrow \text{Student}(x)$
  – Translation: "If x is CS-major then x is a student." (not a proposition)

Predicates

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- The statement $P(x)$ is not a proposition since there are many objects that it can be applied to.

- But the difference is:

  ◇ predicate logic allows us to explicitly manipulate and substitute for the objects

  ◇ predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

Quantified Statements

- Two types of quantified statements:
  - *Universal*

    Example: "All CS-major graduates have to pass CS201".
    (This is true for all CS-major graduates.)

  - *Existential*

    Example: "Some CS-major students graduate with honor."
    (This is true for some students.)

Universal Quantifier

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◇ Assume that the universe is all real numbers
◇ $\forall x\ P(x)$ is true.

◇ Is $P(x)$ a proposition? No. Many possible substitutions.
◇ Is $\forall x\ P(x)$ a proposition?
Yes. True if for all x from the universe $P(x)$ is ture.

Existential Quantifier

- The *existential quantification* of $P(x)$ is the proposition: "There exists an element in the universe of discourse such that $P(x)$ is true ." denoted by $\exists x\ P(x)$, and is expressed as there is an x such that $P(x)$.

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- ◇ $\exists x \; P(x)$ is true.
- ◇ Is $P(x)$ a proposition? No. Many possible substitutions.
- ◇ Is $\exists x \; P(x)$ a proposition?

Yes. True if there is even an x s.t. $P(x)$ is ture (e.g. 10).

Existential Quantifier

- **Example**:
  - $Q(x)$ – "$x = x + 2$ where x is a real number".
  - What is the truth value of $\exists x\ Q(x)$?

Existential Quantifier

- **Example**:
  - $Q(x)$ – "$x = x + 2$ where x is a real number".
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  - $\exists x \; Q(x)$ is false.

Existential Quantifier

- **Example**:
  - ◇ $Q(x)$ – "$x = x + 2$ where x is a real number".
  - ◇ What is the truth value of $\exists x\ Q(x)$?
  - ◇ $\exists x\ Q(x)$ is false.

  - ◇ $C(x)$ – CS-major(x) $\wedge$ Honor-student(x).
  - ◇ What is the truth value of $\exists x\ C(x)$?
  - ◇ Translation: "There is a person who is a CS-major student and who also graduated with honor." (T)

Summary of Quantified Statements

- When $\forall x\ P(x)$ and $\exists x\ P(x)$ are true and false?

Statement	When true?	When false?
∀x P(x)	P(x) true for all x	There is an x where P(x) is false.
∃x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

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- Suppose that the elements in the universe can be enumerated as $x_1, x_2, \ldots, x_n$ then:
  - ◇ $\forall x\ P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ is true
  - ◇ $\exists x\ P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$ is true.

Properties of Quantifiers

- The truth values of $\exists x\ P(x)$ and $\forall x\ P(x)$ depend on both the propositional function $P(x)$ and the universe.

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Example: $P(x)$ – "$x < 2$"

◇ universe: the positive integers

$\exists x\ P(x)$ – T, $\forall x\ P(x)$ – F

◇ universe: the negative integers

$\exists x\ P(x)$ – T, $\forall x\ P(x)$ – T

◇ universe: $\{\ 3,\ 4,\ 5\ \}$

$\exists x\ P(x)$ – F, $\forall x\ P(x)$ – F

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have *higher precedence* than all the logical operators.

  ◇ $\forall x\ P(x) \vee Q(x)$ means $(\forall x\ P(x)) \vee Q(x)$ rather than $\forall x\ (P(x) \vee Q(x))$

Translation with Quantifiers

- Sentence: All SUSTech students are smart.
  - ◇ universe: SUSTech students

    translation: $\forall x \; Smart(x)$

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This means every student is at SUSTech and is smart!

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  - ◇ universe: people

    translation: $\forall x\ (Student(x) \wedge At(x, SUSTech) \rightarrow Smart(x))$

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$\mathcal{Q}$: What about this?

$\exists x\ (At(x, SUSTech) \rightarrow Smart(x))$

This is even true if there is anyone who is not at SUSTech!

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- Sentence: Nothing is perfect.
  - translation: $\neg \exists x \; Perfect(x)$

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  - ◇ translation: $\forall x \; \neg \mathit{Perfect}(x)$
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Negation of Quantifiers

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  - ◇ translation: $\forall x\ \neg Perfect(x)$
    (Everything is imperfect.)

Conclusion: $\neg\exists x\ P(x)$ is equivalent to $\forall x\ \neg P(x)$

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- Sentence: Not all horses are white.
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  - translation: $\neg\forall x\ (Horse(x) \to White(x))$
  - translation: $\exists x\ (Horse(x) \land \neg White(x))$
    (There is a horse that is not white.)

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  - logically equivalent to
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  - ◇ translation: $\neg \forall x\ (Horse(x) \to White(x))$
  - ◇ translation: $\exists x\ (Horse(x) \wedge \neg White(x))$
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Conclusion: $\neg \forall x\ P(x)$ is equivalent to $\exists x\ \neg P(x)$

Negation of Quantified Statements

- a.k.a. De Morgan laws for quantifiers

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x\ P(x)$	$\forall x\ \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg \forall x\ P(x)$	$\exists x\ \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Next Lecture

- Predicate logic, proofs, sets ...