Assignment I - Discrete Math(H)

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Date: 2025.10.12

Q.1

Q.1 Let p, q be the propositions
p: You get 100 marks on the final.
q: You get an A in this course.
Write these propositions using p and q and logical connectives (including negations).

(1) You get 100 marks on the final, but you do not get an A in this course.

(2) You will get an A in this course if you get 100 marks on the final.

(3) If you do not get 100 marks on the final, then you will not get an A in this course.

(4) Getting 100 marks on the final is sufficient for getting an A in this course.

(5) You get an A in this course, but you do not get 100 marks on the final.

Sol:

1. $p \land \neg q$.

2. $p \to q$.

3. $\neg p \to \neg q$.

4. $p \to q$.

5. $q \wedge \neg p$.

Q.2

Q.2 Construct a truth table for each of these compound propositions.

(1) $(p \oplus q) \to (p \wedge q)$

(2) $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

(3) $(p \oplus q) \to (p \oplus \neg q)$

Sol:

1. $(p \oplus q) \to (p \wedge q)$

p	q	$p \oplus q$	$p \wedge q$	$(p \oplus q) \to (p \wedge q)$
F	F	F	F	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

2. $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$

p	q	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
F	F	T	F	T
F	T	F	T	T
T	F	F	T	T
T	T	T	F	T

3. $(p \oplus q) \to (p \oplus \neg q)$

p	q	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \to (p \oplus \neg q)$
F	F	F	T	T
F	T	T	F	F
T	F	T	F	F
T	T	F	T	T

Q.3

Q.3 Use truth tables to decide whether or not the following two propositions are equivalent.

(1) $p \leftrightarrow q$ and $(p \wedge q) \vee (\neg p \wedge \neg q)$

(2) $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$

(3) $(p \to q) \to r$ and $p \to (q \to r)$

(4) $(p \vee q) \to r$ and $(p \to r) \wedge (q \to r)$

(5) $(p \to \neg q) \leftrightarrow (r \to (p \vee \neg q))$ and $q \vee (\neg p \wedge \neg r)$

Sol:

1.

p	q	$p \leftrightarrow q$	$(p \wedge q) \vee (\neg p \wedge \neg q)$	**Equivalent**
F	F	T	T	T
F	T	F	F	T
T	F	F	F	T
T	T	T	T	T

Thus it's **Equivalent**.

2.

p	q	r	$(p \to q) \vee (p \to r)$	$p \to (q \vee r)$	**Equivalent**
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	F	T	T	T	T
T	T	F	T	T	T
T	T	T	T	T	T

Thus it's **Equivalent**.

3.

p	q	r	$(p \to q) \to r$	$p \to (q \to r)$	**Equivalent**
F	F	F	F	T	F
F	F	T	T	T	T
F	T	F	F	T	F
F	T	T	T	T	T
T	F	F	T	T	T
T	F	T	T	T	T
T	T	F	F	F	T
T	T	T	T	T	T

Thus it's **Not Equivalent**.

4.

p	q	r	$(p \vee q) \to r$	$(p \to r) \wedge (q \to r)$	**Equivalent**
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	F	F	T
F	T	T	T	T	T
T	F	F	F	F	T
T	F	T	T	T	T
T	T	F	F	F	T
T	T	T	T	T	T

Thus it's **Equivalent**.

5.

p	q	r	$(p \to \neg q) \leftrightarrow (r \to (p \lor \neg q))$	$q \lor (\neg p \land \neg r)$	**Equivalent**
F	F	F	T	T	T
F	F	T	T	F	F
F	T	F	T	T	T
F	T	T	F	T	F
T	F	F	T	F	F
T	F	T	T	F	F
T	T	F	F	T	F
T	T	T	F	T	F

Thus it's **Not Equivalent**.

Q.4

Q.4 Use logical equivalences to prove the following statements.

(1) $\neg(p \to q) \to \neg q$ is a tautology.

(2) $\neg p \to (q \to r)$ and $q \to (p \lor r)$ are equivalent.

(3) $(p \to q) \to ((r \to p) \to (r \to q))$ is a tautology.

PF:

1.

$$
\begin{aligned}
\neg(p \to q) \to \neg q &\equiv \neg(\neg p \lor q) \to \neg q \\
&\equiv (p \land \neg q) \to \neg q \\
&\equiv \neg(p \land \neg q) \lor \neg q \\
&\equiv (\neg p \lor q) \lor \neg q \\
&\equiv \top
\end{aligned}
$$

Q.E.D..

2.

$$\begin{aligned}\neg p \to (q \to r) &\equiv p \vee (\neg q \vee r)\\ &\equiv \neg q \vee (p \vee r)\\ &\equiv q \to (p \vee r)\end{aligned}$$

Q.E.D..

3.

$$\begin{aligned}(r \to p) \to (r \to q) &\equiv \neg(\neg r \vee p) \vee (\neg r \vee q)\\ &\equiv (r \wedge \neg p) \vee \neg r \vee q\\ &\equiv \neg p \vee \neg r \vee q.\end{aligned}$$

Then:

$$\begin{aligned}(p \to q) \to ((r \to p) \to (r \to q)) &\equiv (p \to q) \to (\neg p \vee \neg r \vee q)\\ &\equiv \neg(\neg p \vee q) \vee (\neg p \vee \neg r \vee q)\\ &\equiv \top\end{aligned}$$

Q.E.D..

Q.5

Q.5 Let $F(x, y)$ be the statement "x can fool y", where the domain consists of all people in the world. Use quantifiers to express each of these statement.

(1) Everybody can fool somebody.

(2) There is no one who can fool everybody.

(3) Everyone can be fooled by somebody.

(4) Nancy can fool exactly two people.

(5) There is exactly one person whom everybody can fool.

(6) There is someone who can fool exactly one person besides himself or herself.

Sol:

1. $\forall x, \exists y, F(x, y)$.

2. $\forall x, \exists y, \neg F(x, y)$.

3. $\forall y, \exists x, F(x, y)$.

4. $\exists x \exists y, [x \neq y \wedge F(Nancy, x) \wedge F(Nancy, y) \wedge \forall u(F(Nancy, u) \rightarrow (u = x \vee u = y))]$.

5. $\exists u, [\forall x, F(x, u) \wedge \forall v(\forall x, F(x, v) \rightarrow v = u)]$.

6. $\exists x \exists y, [y \neq x \wedge F(x, y) \wedge \neg F(x, x) \wedge \forall z(F(x, z) \rightarrow z = y)]$.

Q.6

Q.6 Given the following predicate on the set P of all people who ever lived: $\text{Parent}(x, y)$ means x is the parent of y.

(1) Rewrite the following in the language of mathematical logic (you may use the equality/inequality operators):

"All people have two parents."

(2) Based on the definition of Parent, we will *recursively* define the concept of *ancestor*: "An ancestor of a person is one of the person's parents or the ancestor of (at least) one of the person's parents."

Rewrite this definition using the language of mathematical logic. Specifically, you need to provide a necessary and sufficient condition for the predicate $\text{Ancestor}(x, y)$ to be true. Note that you can *recursively* use the predicate $\text{Ancestor}(x, y)$ in the condition itself.

Sol:

1. (if it represent 'exactly 2')

$$\forall x, \exists y \exists z, [y \neq z \wedge Parent(y, x) \wedge Parent(z, x) \wedge \forall w(Parent(w, x) \rightarrow (w = y \vee w = z))]$$

2.

$$\forall x \forall y, [Ancestor(x, y) \leftrightarrow (Parent(x, y) \vee \exists z(Parent(z, y) \wedge Ancestor(x, z)))].$$

Q.7

Q.7 For the predicate $P(x, y)$ with two variables x, y, answer the following two questions.

(1) Give an example of a predicate $P(x, y)$ such that $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ have *different* truth values.

(2) If $\forall y \exists x P(x, y)$ is true, does it necessarily follow that $\exists x \forall y P(x, y)$ is true?

Sol:

1. Take the domain $\mathbb{N}$, and let $P(x, y) : x > y$, obviously $\exists x \forall y P(x, y)$ is false (we can trivally find some y that violates $P(x, y)$ forall x), but $\forall y \exists x P(x, y)$ is true, for we can let $x = y + 1$.
2. Not necessary. Same as the example in (1), it's not necessary.

Q.8

Q.8 Express the negations of each of these statements so that all negation symbols immediately precede predicates.

(1) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

(2) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

(3) $\forall x \exists y P(x, y) \vee \forall x \exists y Q(x, y)$

(4) $\forall x \exists y (P(x, y) \wedge \exists z R(x, y, z))$

Sol:

1.

$$\begin{aligned} &\neg[\exists x \exists y, P(x, y) \wedge \forall x \forall y, Q(x, y)] \\ \equiv &\neg \exists x \exists y, P(x, y) \ \vee \ \neg \forall x \forall y, Q(x, y) \\ \equiv &\forall x \forall y, \neg P(x, y) \ \vee \ \exists x \exists y, \neg Q(x, y) \end{aligned}$$

2.

$$
\begin{aligned}
&\neg\exists x\exists y(Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x\forall y, \neg(Q(x,y) \leftrightarrow Q(y,x)) \\
&\equiv \forall x\forall y, \big[(Q(x,y) \wedge \neg Q(y,x)) \;\vee\; (\neg Q(x,y) \wedge Q(y,x))\big]
\end{aligned}
$$

3.

$$
\begin{aligned}
&\neg[\forall x\exists y, P(x,y) \;\vee\; \forall x\exists y, Q(x,y)] \\
&\equiv \neg\forall x\exists y, P(x,y) \;\wedge\; \neg\forall x\exists y, Q(x,y) \\
&\equiv \exists x\forall y, \neg P(x,y) \;\wedge\; \exists x\forall y, \neg Q(x,y)
\end{aligned}
$$

4.

$$
\begin{aligned}
&\neg\forall x\exists y, [P(x,y) \wedge \exists z R(x,y,z)] \\
&\equiv \exists x\forall y, [\neg P(x,y) \vee \forall z, \neg R(x,y,z)]
\end{aligned}
$$

Q.9

Q.9

(a) Let P be a proposition in atomic propositions p and q. If $P = \neg(p \leftrightarrow (q \vee \neg p))$, prove that P is equivalent to $\neg p \vee \neg q$.

(b) If P is of any length, using any of the logical connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, prove that P is logically equivalent to a proposition of the from

$$A \Box B,$$

where $\Box$ is one of $\wedge$, $\vee$, $\leftrightarrow$, and A and B are chosen from $\{p, \neg p, q, \neg q\}$.

PF:

1.

$$
\begin{aligned}
P &\equiv \neg[(p \wedge (q \vee \neg p)) \vee (\neg p \wedge \neg(q \vee \neg p))] \\
&\equiv (\neg p \vee \neg(q \vee \neg p)) \wedge (p \vee q \vee \neg p) \\
&\equiv (\neg p \vee (\neg q \wedge p)) \\
&\equiv \neg p \vee \neg q.
\end{aligned}
$$

`Q.E.D..`

2.

We have a truth table like:

p	q	P
0	0	p_1
0	1	p_1
1	0	p_3
1	1	p_4

- When $p_1 = p_2 = p_3 = p_4 = 0$, we have $p \leftrightarrow p$.
- When $p_1 = p_2 = p_3 = p_4 = 1$, we have $p \wedge \neg p$.
- When there exists exactly one $p = 1$, we can use $\wedge$ to express.
- When there exists exactly three $p = 1$, we can use $\vee$ to express.
- When $p_1 = p_2 = 1, p_3 = p_4 = 0$, we have $\neg p \wedge \neg p$.
- When $p_1 = p_2 = 0, p_3 = p_4 = 1$, we have $p \wedge p$.
- When $p_1 = p_3 = 1, p_2 = p_4 = 0$, we have $\neg q \wedge \neg q$.
- When $p_1 = p_3 = 0, p_2 = p_4 = 1$, we have $q \wedge q$.
- When $p_1 = p_4 = 0, p_2 = p_3 = 1$, we have $p \oplus q \equiv \neg p \leftrightarrow q$.
- When $p_1 = p_4 = 1, p_2 = p_3 = 0$, we have $p \leftrightarrow q$.

`Q.E.D.`.

Q.10

Q.10 For the following argument, explain which rules of inference are used for each step.

"Each of five roommates, A, B, C, D, and E, has taken a course in discrete mathematics. Every student who has taken a course in discrete mathematics can take a course in algorithms. Therefore, all five roommates can take a course in algorithms next year."

Sol:

Let $D(x)$ denotes x has taken discrete math, and $A(x)$ denotes x can take algorithms next year.

We have: $D(A) \wedge D(B) \wedge D(C) \wedge D(D) \wedge D(E)$.

Also we have $\forall x(D(x) \rightarrow A(x))$.

Implement **UI(Universal Instantiation)** to A, B, C, D, E, then we have:
$D(A), D(B), D(C), D(D), D(E)$.

Implement **Modus Ponens**, then we have: $A(A), A(B), A(C), A(D), A(E)$.

Implement $\vee_i$ **(Conjunction Introduction)**, then we have: $A(A) \wedge A(B) \wedge A(C) \wedge A(D) \wedge A(E)$.

Q.11

Q.11 Prove the **triangle inequality**, which states that if x and y are real numbers, then $|x| + |y| \geq |x + y|$ (where $|x|$ represents the absolute value of x, which equals x if $x \geq 0$ and equals $-x$ if $x < 0$.

PF:

$$\begin{aligned}
(\texttt{LHS})^2 - (\texttt{RHS})^2 &= (|x| + |y|)^2 - (|x + y|)^2 \\
&= x^2 + y^2 + 2|x||y| - (x + y)^2 \\
&= 2(|x||y| - xy) \\
&\geq 0
\end{aligned}$$

Thus $\texttt{LHS} \geq \texttt{RHS}$.

`Q.E.D.`.

Q.12

Q.12 Prove or disprove the following.

(1) For two irrational numbers a and b, a^b is also irrational.

(2) For an irrational number a, $\sqrt{a}$ is also irrational.

(3) There is a rational number x and an irrational number y such that x^y is irrational.

PF:

1. If $\sqrt{2}^{\sqrt{2}}$ is irrational, let $a = \sqrt{2}^{\sqrt{2}}, b = \sqrt{2}$, then $a^b = \sqrt{2}^2 = 2$, which is rational.

   If $\sqrt{2}^{\sqrt{2}}$ is rational, W.L.O.G., let $a^b = \sqrt{2}^{\sqrt{2}}$, where $\sqrt{2}$ is irrational, i.e., a, b is irrational.

   Therefore the statement is false.

   `Q.E.D..`

2. The contrapositive of the statement is if $\sqrt{a}$ is rational, then a is rational.

   Let $\sqrt{a} = \dfrac{m}{n}$, then $a = \dfrac{m^2}{n^2} \in \mathbb{Q}$.

   For the contrapositive is true, the original statement is true.

   `Q.E.D..`

3. Let $y = \log_{10}^{\pi}$, then $x^y = \pi$, which is irrational.

   And x is obvoiusly rational, for $y = \log_{10}^{\pi}$. Assuming y is rational, let $\log_{10}^{\pi} = \dfrac{a}{b}$, then $10^a = \pi^b$. Since π is a transcendental, every power of it is irrational, which leads to a controdiction. Thus y is irrational.

   Therefore, x is rational, y is irrational, x^y is irrational.

   `Q.E.D..`

Q.13

Q.13 Prove that $\sqrt[3]{2}$ is irrational.

PF: Assuming $\sqrt[3]{2}$ is rational, let $\sqrt[3]{2} = \dfrac{m}{n}$, where m, n are coprime. Then $2n^3 = m^3$, and $2 \mid m^3 \Rightarrow 2 \mid m$. Thus let $m = 2k$, then $n^3 = 4k^3 \Rightarrow 2 \mid n^3 \Rightarrow 2 \mid n$, leading to a contradiction. Therefore, $\sqrt[3]{2}$ is irrational.

`Q.E.D..`

Q.14

Q.14 Prove that between every two rational numbers there is an irrational number.

PF:

Let $p < q$ ans p, q are irrational.

Thus, let $r = p + \dfrac{q-p}{\sqrt{2}}$, obviously $p < r < q$. And since p, q are rational, assuming r is rational, then $r - p = \dfrac{q-p}{\sqrt{2}}$ is rational, then $\sqrt{2} = \dfrac{q-p}{r-p}$ is rational, which leads to a contradiction. Therefore r is irrational.

`Q.E.D..`

Q.15

Q.15 Let the coefficients of the polynomial $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ be integers. Show that any *real* root of the equation $f(x) = 0$ is either integral or irrational. Note that in your proof, you may direct use the following result without a proof. "**Fact.** If a prime p is a factor of some power of an integer, then it is a factor of that integer."

PF: Let the root $x = \frac{a}{b}$, where a, b are coprime, w.l.o.g., let $b > 0$.

Substitude and multiply by b^n, we have:

$$a^n + a_{n-1}a^{n-1}b + \cdots + a_0 b^n = 0$$

Then:

$$a^n = -b(a_{n-1}a^{n-1} + \cdots + a_0 b^{n-1})$$

Thus, $b \mid a^n$. Then $b \mid a$ for the Fact noticed in the question, which leads to a contradiction.

Therefore, $x = \frac{a/b}{1}$ which is an integer or x couldn't be represented as fraction, which is irrational.

`Q.E.D.`.