Well Reservoir Simulation

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1 Coupled PDE in space & time with Discontinuity

Consider the following Problem

$$\begin{aligned} &\frac{\partial^2 z_1}{\partial x^2} + \frac{\partial^2 z_1}{\partial y^2} + \delta(x-1)\delta(y) = \frac{\partial z_1}{\partial t} \\ &\frac{\partial^2 z_2}{\partial x^2} + \frac{\partial^2 z_2}{\partial y^2} = \frac{\partial z_2}{\partial t} \end{aligned} \tag{1}$$

with the following initial and boundary conditions

$$\begin{aligned} &\text{IC} \\ &z_1(x,y,t=0) = 0 \\ &z_2(x,y,t=0) = 0 \\ \\ &\text{BC} \\ &z_1(x,\pm\infty,t) = 0 \\ &z_2(x,\pm\infty,t) = 0 \\ &z_1(\infty,y,t) = 0 \\ &z_2(-\infty,y,t) = 0 \\ &\frac{\partial z_1}{\partial x} = \alpha\left(z_1\left(0,y,t\right) - z_2\left(0,y,t\right)\right) \\ &\frac{\partial z_1}{\partial x}(0,y,t) = \frac{\partial z_2}{\partial x}(0,y,t) \end{aligned} \tag{2}$$

We begin solving this problem by taking Laplace transform in time as follows:

$$\bar{z}_1 = \mathscr{L}\left\{z_1\right\} = \int_0^\infty e^{-st} z_1 dt$$

Taking the Laplace of Eqn(1) yields

$$\begin{aligned} &\frac{\partial^2 \bar{z}_1}{\partial x^2} + \frac{\partial^2 \bar{z}_1}{\partial y^2} + \frac{1}{s}\delta(x-1)\delta(y) = s\bar{z}_1 \\ &\frac{\partial^2 \bar{z}_1}{\partial x^2} + \frac{\partial^2 \bar{z}_1}{\partial y^2} = s\bar{z}_1 \end{aligned} \tag{3}$$

Taking the Laplace of the boundary conditions yield the following

$$\begin{aligned} &\bar{z}_1(x,\pm\infty,s) = 0 \\ &\bar{z}_2(x,\pm\infty,s) = 0 \\ &\bar{z}_1(\infty,y,s) = 0 \\ &\bar{z}_2(-\infty,y,s) = 0 \\ &\frac{\partial z_1}{\partial x} = \alpha\left(z_1\left(0,y,s\right) - z_2\left(0,y,s\right)\right) \\ &\frac{\partial z_1}{\partial x}(0,y,s) = \frac{\partial z_2}{\partial x}(0,y,s) \end{aligned} \tag{4}$$

Our problem is still a PDE so to convert to an ODE, we take a Fourier transform in the y coordinate as follows

$$\bar{\bar{z}}_1 = \mathcal{F}\left\{\bar{z}_1\right\} = \int_{-\infty}^{\infty} e^{-\omega y}\bar{z}_1 dy$$

Thus, the fourier transform of Eqn(3) gives

$$\begin{aligned} &\frac{d^2\,\bar{\bar{z}}_1}{dx^2} - \omega^2\bar{\bar{z}}_1 + \frac{1}{s}\delta\left(x-1\right) = s\bar{\bar{z}}_1 \\ &\frac{d^2\,\bar{\bar{z}}_2}{dx^2} - \omega\bar{\bar{z}}_2 = s\bar{\bar{z}}_2 \end{aligned} \tag{5}$$

Again, the Fourier transform of the boundary conditions lead to the following

$$\begin{aligned} &\bar{\bar{z}}_1(\infty,\omega,s) = 0 \\ &\bar{\bar{z}}_2(-\infty,\omega,s) = 0 \\ &\frac{\partial\,\bar{\bar{z}}_1}{\partial x} = \alpha\left(\bar{\bar{z}}_1\left(0,\omega,s\right) - \bar{\bar{z}}_2\left(0,\omega,s\right)\right) \\ &\frac{\partial\,\bar{\bar{z}}_1}{\partial x}(0,\omega,s) = \frac{\partial\bar{\bar{z}}_2}{\partial x}(0,\omega,s) \end{aligned} \tag{6}$$

Now the above equation can be solved in a straightforward manner either by variation of parameters, Laplace or similar methods. Solve the above to obtain the following

$$\begin{aligned} &\bar{\bar{z}}_1 = \frac{1}{2sA}\left(e^{-A|x-1|} + \frac{A}{A+2\alpha}e^{-A(x+1)}\right) \\ &\bar{\bar{z}}_2 = \frac{1}{2sA}\left(\frac{2\alpha}{A+2\alpha}\right)e^{A(x-1)} \end{aligned} \tag{7}$$

where $A = \sqrt{s+\omega^2}$
For simplicity let us assume $\alpha = 1$, Now we wish to find the analytical expressions of z_1, z_2 as follows via an inverse Fourier and an inverse Laplace as follows
First we observe the following, the above equations can be rewritten as

$$\begin{aligned} &\bar{\bar{z}}_1 = \frac{1}{2}\left(\frac{e^{-A|x-1|}}{sA} + \frac{e^{-A(x+1)}}{s(A+2)}\right) \\ &\bar{\bar{z}}_2 = \frac{1}{2}\left(\frac{e^{A(x-1)}}{sA} - \frac{e^{A(x-1)}}{s(A+2)}\right) \end{aligned} \tag{8}$$

Hence according to Eqn(8), only two terms need to be inverted both in the Fourier and Laplace space.
For simplicity, Let $x^* = |x_-|, x_- = x-1, x_+ = x+1$, The first term of the RHS of $\bar{\bar{z}}_1$ as

$$\frac{e^{-A|x_-|}}{2sA} = \frac{e^{-x^*\sqrt{s+\omega^2}}}{2s\sqrt{s+\omega^2}}$$

From the property of Laplace, we know that

$$\mathscr{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(t)dt \tag{9}$$

$$\mathscr{L}^{-1}\left\{f(s+b)\right\} = \exp(-bt)\mathscr{L}^{-1}\left\{f(s)\right\} \tag{10}$$

Given that the inverse Laplace of

$$\mathscr{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{\sqrt{s}}\right\}=\frac{e^{-a^2/4t}}{\sqrt{\pi t}}$$

It becomes evident from eqn(10) that the following ensues

$$\mathscr{L}^{-1}\frac{e^{-x^*\sqrt{s+\omega^2}}}{\sqrt{s+\omega^2}}=\frac{e^{-x^{*2}/4t}}{\sqrt{\pi t}}e^{-\omega^2 t} \tag{11}$$

From the above, it can be concluded also from eqn(9) that

$$\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s+\omega^2}x^*}}{s\sqrt{s+\omega^2}}\right\}=\mathscr{L}^{-1}\left\{\frac{e^{-\sqrt{s+\omega^2}x^*}}{\sqrt{s+\omega^2}}/s\right\}=\int_0^t\frac{e^{-x^{*2}/4t}}{\sqrt{\pi t}}e^{-\omega^2 t}dt \tag{12}$$

integrating the above leads to the following

$$\frac{1}{2\omega}\left[\left(\operatorname{erf}\left(\frac{x^*+2\omega t}{2\sqrt{t}}\right)-1\right)e^{\omega x^*}-\left(\operatorname{erf}\left(\frac{x^*-2\omega t}{2\sqrt{t}}\right)-1\right)e^{-\omega x^*}\right] \tag{13}$$

To find the inverse Fourier transform, we employ the following from mathematica

`https://www.wolframalpha.com/input/?i=integrate+exp%28%28a%2Bb%29*w%29*+%28erf%28%28b+%2B2*c%5E2*w%29%2F%282*c%29%29+-1%29+dw`

and `https://www.wolframalpha.com/input/?i=integrate+exp%28%28a%2Bb%29*w%29*+%28erf%28%28-b+-2*c%5E2*w%29%2F%282*c%29%29+-1%29+dw`

First we observe the following from inverse Fourier transform theory

$$\mathcal{F}^{-1}\left\{f(\omega)\right\}=\int_{-\infty}^{\infty}e^{i\omega y}f(\omega)d\omega \tag{14}$$

The following is also known, see Appendix for a formal proof.

$$\mathcal{F}^{-1}\left\{\frac{f(\omega)}{i\omega}\right\}=\frac{1}{2\pi}\int_{-\infty}^{u}f(y)\,dy\quad\forall\, f(y)=\mathcal{F}^{-1}f(\omega)$$

Now we begin with the inverse Fourier of first term in Eqn(13) as follows

$$\begin{aligned}&\mathcal{F}^{-1}\left\{\frac{1}{2\omega}\left(\operatorname{erf}\left(\frac{x^*+2\omega t}{2\sqrt{t}}\right)-1\right)e^{\omega(x^*)}\right\}=\\&\frac{i}{4\pi}\int_{-\infty}^{u}\int_{-\infty}^{\infty}\left(\operatorname{erf}\left(\frac{x^*+2\omega t}{2\sqrt{t}}\right)-1\right)e^{\omega(x^*+iy)}d\omega dy\end{aligned} \tag{15}$$

Integrating with respect to ω leads to the following result

$$\begin{aligned}&\frac{i}{4\pi}\int_{-\infty}^{u}\frac{1}{x^*+iy}\left\{\exp\left(\frac{-x^*(x^*+iy)}{2t}\right)\left[\exp\left(\frac{(x^*+iy)^2}{4t}\right)\operatorname{erf}\left(\frac{iy-2\omega t}{2\sqrt{t}}\right)\right.\right.\\&\left.\left.+\exp\left(\frac{(x^*+iy)(x^*+2t\omega)}{2t}\right)\left(\operatorname{erf}\left(\frac{x^*+2\omega t}{2\sqrt{t}}\right)-1\right)\right]\Bigg|_{-\infty}^{\infty}dy\right\}\end{aligned} \tag{16}$$

Evaluating the inner indefinite integral at its limits of $[-\infty,\infty]$ leads to the following result

$$\begin{aligned}&=\frac{i}{4\pi}\int_{-\infty}^{u}\frac{1}{x^*+iy}\exp\left(\frac{-x^*(x^*+iy)}{2t}\right)\left[-\exp\left(\frac{(x^*+iy)^2}{4t}\right)+0-\left(\exp\left(\frac{(x^*+iy)^2}{4t}\right)+0\right)\right]\\&=-\frac{i}{2\pi}\int_{-\infty}^{u}\frac{1}{x^*+iy}\exp\left(\frac{-\left(x^{*2}+y^2\right)}{4t}\right)dy\end{aligned}$$

$$(17)$$

Now we move to the inverse Fourier transform of the second term of Eqn(13)

$$\begin{aligned}&\mathcal{F}^{-1}\left\{-\left(\operatorname{erf}\left(\frac{x^*-2\omega t}{2\sqrt{t}}\right)-1\right)e^{-\omega(x^*)}\right\}=\\&-\frac{i}{4\pi}\int_{-\infty}^{u}\int_{-\infty}^{\infty}\left(\operatorname{erf}\left(\frac{x^*-2\omega t}{2\sqrt{t}}\right)-1\right)e^{-\omega(x^*+iy)}d\omega dy\end{aligned} \tag{18}$$

Again, integrating with respect to ω leads to the following result

$$\begin{aligned}&\frac{i}{4\pi}\int_{-\infty}^{u}\frac{1}{-x^*+iy}\exp\left(\frac{-(x^{*2}+y^2)}{4t}\right)\operatorname{erf}\left(\frac{iy-2\omega t}{2\sqrt{t}}\right)\\&+\exp\left(\omega\left(iy-x^*\right)\right)\left(\operatorname{erf}\left(\frac{-x^*+2\omega t}{2\sqrt{t}}\right)+1\right)\Bigg|_{-\infty}^{\infty}dy\end{aligned} \tag{19}$$

Finally, evaluating the inner indefinite integral at its limits of $[-\infty,\infty]$ leads to the following

$$-\frac{i}{2\pi}\int_{-\infty}^{u}\exp\left(\frac{-(x^{*2}+y^2)}{4t}\right)\frac{1}{-x^*+iy}dy \tag{20}$$

Combining Eqn(17) and Eqn(20), the following is obtained

$$\begin{aligned}&-\frac{i}{2\pi}\int_{-\infty}^{u}\exp\left(\frac{-\left(x^{*2}+y^2\right)}{4t}\right)\left(\frac{1}{x^*+iy}+\frac{1}{-x^*+iy}\right)dy\\&=-\frac{i}{2\pi}\int_{-\infty}^{u}\exp\left(\frac{-\left(x^{*2}+y^2\right)}{4t}\right)\left(\frac{2iy}{-\left(x^{*2}+y^2\right)}\right)dy\\&=-\frac{1}{\pi}\int_{-\infty}^{u}\exp\left(\frac{-\left(x^{*2}+y^2\right)}{4t}\right)\frac{y}{x^{*2}+y^2}dy\\&=\frac{1}{2\pi}\int_{u}^{\infty}\exp\left(\frac{-\left(x^{*2}+y^2\right)}{4t}\right)\frac{4t}{x^{*2}+y^2}d\frac{y^2}{4t}\\&=\frac{1}{2\pi}\int_{u}^{\infty}\frac{\exp\left(-p\right)}{p}dp\end{aligned} \tag{21}$$

where $p = \left(x^{*2}+y^2\right)/4t$.
Hence we have that

$$\mathcal{F}^{-1}\left\{\mathscr{L}^{-1}\left\{\frac{e^{-|x_-|\sqrt{s+\omega^2}}}{2s\sqrt{s+\omega^2}}\right\}\right\}=W\left[\frac{\left(x^{*2}+y^2\right)}{4t}\right] \tag{22}$$

$W(u)$ is the Wells function defined by

$$W(u)=\int_{u}^{\infty}\frac{e^{-x}}{x}dx \tag{23}$$

The second term of Eqn(8) of the RHS of $\overline{\overline{z}}_1$ can be written generically (with dummy variable x) as

$$\mathcal{F}^{-1}\left\{\mathscr{L}^{-1}\left\{\frac{e^{-Ax}}{s\left(A+2\right)}\right\}\right\}$$

Again, we begin by applying the laws of Laplace transforms as follows

$$
\begin{aligned}
&\mathscr{L}^{-1}\left\{\frac{e^{-Ax}}{s\left(A+2\right)}\right\}= \\
&\int_0^t \mathscr{L}^{-1}\left\{\frac{e^{-Ax}}{\left(A+2\right)}\right\}dt= \\
&\int_0^t \mathscr{L}^{-1}\left\{\frac{\exp\left(-x\sqrt{s+\omega^2}\,\right)}{\sqrt{s+\omega^2}+2}\right\}dt
\end{aligned}
\tag{24}
$$

From the theory of inverse Laplace theory, it is known that

$$
\mathscr{L}^{-1}\left\{\frac{\exp\left(-b\sqrt{s}\right)}{\sqrt{s}+d}\right\}=\frac{\exp\left(-b^2/4t\right)}{\sqrt{\pi t}}-d\exp\left(bd+d^2t\right)\,\mathrm{erfc}\left(\frac{b+2dt}{2\sqrt{t}}\right)
$$

Using the property of Laplace transforms in Eqn(10), the following is obtained

$$
\begin{aligned}
&\int_0^t \mathscr{L}^{-1}\left\{\frac{\exp\left(-x\sqrt{s+\omega^2}\,\right)}{\sqrt{s+\omega^2}+2}\right\}dt= \\
&\int_0^t\left(\frac{\exp\left(-\left(x^2+4\omega^2t^2/4t\right)\right)}{\sqrt{\pi t}}-2\exp\left(2x+t\left(4-\omega^2\right)\right)\,\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)\right)dt= \\
&-\frac{\exp\left(-x\omega\right)}{2\left(2+\omega\right)}\mathrm{erf}\left(\frac{x-2\omega t}{2\sqrt{t}}\right)-\frac{\exp\left(x\omega\right)}{2\left(2-\omega\right)}\mathrm{erf}\left(\frac{x+2\omega t}{2\sqrt{t}}\right)+ \\
&\frac{\cosh\left(x\omega\right)}{2-\omega}-\frac{2\exp\left(2x+\left(4-\omega^2\right)t\right)}{4-\omega^2}\mathrm{erfc}\left(\frac{x+2\omega t}{2\sqrt{t}}\right)
\end{aligned}
\tag{25}
$$

The inverse Fourier of the above was impossible so a different approach is employed. i.e an inverse Fourier before the integral of the inverse Laplace as shown:

$$
\begin{aligned}
&\mathcal{F}^{-1}\left\{\mathscr{L}^{-1}\left\{\frac{e^{-Ax}}{s\left(A+2\right)}\right\}\right\} \\
&\mathcal{F}^{-1}\left\{\int_0^t\frac{\exp\left(-\left(x^2+4\omega^2t^2/4t\right)\right)}{\sqrt{\pi t}}-2\exp\left(2x+t\left(4-\omega^2\right)\right)\,\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)dt\right\} \\
&\mathcal{F}^{-1}\left\{\int_0^t\frac{\exp\left(-\left(x^2+4\omega^2t^2/4t\right)\right)}{\sqrt{\pi t}}dt\right\}-\mathcal{F}^{-1}\left\{\int_0^t 2\exp\left(2x+t\left(4-\omega^2\right)\right)\,\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)dt\right\} \\
&W\left[\frac{\left(x^2+y^2\right)}{4t}\right]-\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(i\omega\,y\right)\int_0^t 2\exp\left(2x+t\left(4-\omega^2\right)\right)\,\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)dt\,d\omega \\
&W\left[\frac{\left(x^2+y^2\right)}{4t}\right]-\frac{1}{\pi}\int_0^t\int_{-\infty}^{\infty}\exp\left(i\omega\,y\right)\,\exp\left(2x+t\left(4-\omega^2\right)\right)\,\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)d\omega\,dt \\
&W\left[\frac{\left(x^2+y^2\right)}{4t}\right]-\frac{1}{\pi}\int_0^t\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)\int_{-\infty}^{\infty}\exp\left(i\omega\,y\right)\,\exp\left(2x+t\left(4-\omega^2\right)\right)\,d\omega\,dt \\
&W\left[\frac{\left(x^2+y^2\right)}{4t}\right]-\frac{\exp(2x)}{\sqrt{\pi}}\int_0^t\mathrm{erfc}\left(\frac{x+4t}{2\sqrt{t}}\right)\frac{\exp\left(4t-y^2/4t\right)}{\sqrt{t}}\,dt
\end{aligned}
\tag{26}
$$

Therefore we have that

$$
\begin{aligned}
z_1=\frac{1}{2}\Bigg\{&W\left[\frac{\left(x^{*2}+y^2\right)}{4t}\right]+W\left[\frac{x_+^2+y^2}{4t}\right]- \\
&\frac{\exp(2x_+)}{\sqrt{\pi}}\int_0^t\mathrm{erfc}\left(\frac{x_++4t}{2\sqrt{t}}\right)\frac{\exp\left(4t-y^2/4t\right)}{\sqrt{t}}\,dt\Bigg\}
\end{aligned}
\tag{27}
$$

$$
z_2=\frac{\exp(-2x_-)}{2\sqrt{\pi}}\int_0^t\mathrm{erfc}\left(\frac{-x_-+4t}{2\sqrt{t}}\right)\frac{\exp\left(4t-y^2/4t\right)}{\sqrt{t}}\,dt
\tag{28}
$$

2 Appendix

Prove that

$$FT\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \frac{X(\tau)}{i\omega} \tag{29}$$

$$\begin{aligned}
FT\left\{x(\tau)d\tau\right\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{t} x(\tau)\,d\tau\,e^{-i\omega t}\,dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)u(t-\tau)\,d\tau\,e^{-i\omega t}\,dt \\
&= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} u(t-\tau)e^{-i\omega t}\,dt\right)d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\left(\int_{-\infty}^{\infty} u(p)e^{-i\omega p}\,dp\right)e^{-i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\left(\int_{0}^{\infty} e^{-i\omega p}\,dp\right)e^{-i\omega\tau}\,d\tau \\
&= \int_{-\infty}^{\infty} x(\tau)\frac{1}{i\omega}e^{-i\omega\tau}\,d\tau \\
&= \frac{X(\omega)}{i\omega}
\end{aligned} \tag{30}$$