For this problem, with whatever BCs:

$$L[y] = -f(x) \tag{1}$$

Get eigenfunctions $\phi_n(x)$ from the linear operator and boundary conditions, satisfying the eigenproblem

$$L[\phi_n(x)] = -\lambda_n \phi_n \tag{2}$$

Multiply the nonhomogeneous problem by the eigenproblem & integrate over the domain.

$$\int_a^b L[y(x)]\,\phi_n(x)\,\mathrm{d}x = \int_a^b -f(x)\,\phi_n(x)\,\mathrm{d}x \tag{3}$$

Where the RHS is the definition of the Fourier transform of the forcing function, $\mathcal{F}[-f(x)] = -f_n(x)$. By the self-adjointness of the operator (your operator is self-adjoint, right?), the LHS can be switched to

$$\int_a^b y(x)\,L[\phi_n]\,\mathrm{d}x = -f_n \tag{4}$$

by the definition of eigenfunctions, we can remove the operator from the LHS.

$$\int_a^b y(x)\,(-\lambda_n \phi_n(x))\,\mathrm{d}x = f_n \tag{5}$$

Where the constants λ_n can safely be taken outside the integral. We can again use the definition of the FFT (as the inner product of a function and n^{th} eigenfunctions), now on the LHS.

$$-\lambda_n\,\mathcal{F}[y(x)] = -f_n \tag{6}$$

$$-\lambda_n\,y_n = -f_n \tag{7}$$

This gives an expression for the Fourier coefficients of the solution:

$$y_n = f_n/\lambda_n \tag{8}$$

The solution is then defined by the inverse Fourier transform.

$$\mathcal{F}^{-1}[y_n] = \Sigma y_n \phi_n(x) \tag{9}$$

$$y(x) = \Sigma y_n \phi_n(x) \tag{10}$$