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CHEMICAL ENGINEERING 502 (APC 502)
Mathematical Methods of Engineering Analysis II

**Homework 7**

**DUE:** *Friday, December 14, in class;*
*exercise 4 can be given/emailed any time WITHIN 2012 !*

1. **7.12**
2. **To do 7.17 you need to use information about Bessel-type eigenproblems; this is contained in section 7.5, but uses material from Chapter 4 that we did not do in detail. So I want you to read on your own Section 4.16 (enough of it to be able to solve the problem) and write me a sentence that "you did a good effort to read and understand 4.16, also looking up other material from Chapter 4 as necessary" -- This should take at least as much time as doing Exercise 3.**
3. **7.17**

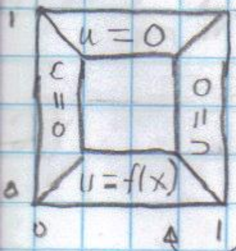
4. **IMPORTANT and different (you are encouraged to do this in pairs, and it is acceptable to hand in "the same" solution per pair).**

   **I want you to solve the one-dimensional finite element problem in the notes - to do it in MATLAB. Choose an epsilon (let's say 0.5), choose an f(x) and solve it analytically first (with the methods in the book). Then, for different number of elements in the domain (say 5, 20, 100) formulate and solve the finite element problem, and plot how the error changes as a function of the size of the discretization (you have to define a "good" error, comparing with the "true" solution you found analytically !). When I say "solve in MATLAB" I mean formulate it as a set of linear eqations for the 5, 20, 100 unknowns, and then ask MATLAB to solve the equations and plot the solutions.**

I

$\boxed{7.12}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad x \in (0,1) \quad y \in (0,1)$$




*need to remove nonhomogeneity.*

$$\varphi = \left[\sqrt{2}\sin(n\pi x)\right]\left[\sqrt{2}\sin(m\pi y)\right]$$

Use Duhamel's formula:

(7.4.10) $$u(x,y) = \int_0^y W(x-\tau, y) f_1(\tau)\,d\tau$$

$$W(x-\tau, y) = \lim_{a \to 0} v(x-\tau, y; a)$$ Infinitisimal Impulse Response

$$\int_0^1\int_0^1 \varphi\, L[x,y]\,dx\,dy = \int_0^1\int_0^1 \varphi\, 0\,dx\,dy$$ $\cancel{\partial x}$

$$\int_0^1\int_0^1 \sin(n\pi x)\sin(m\pi y)\frac{\partial^2 u}{\partial x^2}\,dx\,dy + \iint \varphi\, u_{yy}\,dx\,dy = 0$$

$\int u\,dv = uv - \int v\,du$

$$\int_0^1\left[\int_0^1 \frac{\partial^2 u}{\partial x^2}\sin(n\pi x)\,dx\right]\sin(m\pi y)\,dy + \quad '' \quad = 0$$

Useful!!!

1) Step response — replace $f(x)$ with 1 & solve original problem — like example in §7.4.2 (but 1 on bottom, not left edge.)

$$-(n\pi)^2 W_{nm} + \frac{2m}{n}\left[1-(-1)^n\right] - (n\pi)^2 W_{nm} = 0$$

Solve for $W_{nm} = -(-1+(-1)^n)m / n^3\pi^2$

1) $W_{nm} = -(-1+(-1)^n)\, m / n^3 \pi^2$

FFT of solution to the step-BC problem

$$w(x,y) \equiv \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} W_{nm}\, \varphi_{nm} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{-(-1+(-1)^n)\, m\, 2 \sin(n\pi x) \sin(m\pi y)}{n^3 \pi^2}$$

2)
$$w(x,y) = \int_0^x W(x-\tau, y) \cdot 1 \cdot d\tau$$

$$= -\int_x^0 W(\eta, y)\, d\eta \qquad (x-\tau = \eta)$$

differentiate wrt. x

$$= \int_0^x W(\eta, y)\, d\eta \Rightarrow \frac{\partial w}{\partial x} = W(x,y) \quad (7.4.11)$$

$$W(x,y) = 2\frac{\partial}{\partial x}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{-(-1+(-1)^n) m^2 \sin(n\pi x) \sin(m\pi y)}{n^3 \pi^2}\right]$$

$$= \sum\sum \frac{-(-1+(-1)^n) m^2}{\pi \quad n^2} \cos(n\pi x) \sin(m\pi y)$$

3) soln: $u(x,y) = \int_0^x W(x-\tau, y) f(\tau)\, d\tau \quad (7.4.10)$

$$= \int_0^x \left[\sum\sum \left[\frac{-(-1+(-1)^n) m^2}{\pi \quad n^2} \cos[n\pi(x-\tau)] \sin[m\pi y]\right] f(\tau)\, d\tau\right.$$

(1)

[7.17] Reading §4.16 . . . Done.

(& §4.7.2, §4.8, §4.9, ~§4.10 also)

NOTES (unhelpful ones):

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - \alpha) = 0$$

Soln. $y(x) = c_1 J_\alpha(x) + c_2 Y_\alpha(x)$

w/ $x = 0$ in $\Omega$ & $y(x)$ $x \in \Omega$ finite,

$\Rightarrow c_2 = 0$

(4.16.26b) $d_n = \mathcal{H}[f(x)] \equiv \int_0^a x f(x) X_n(x)\, dx$

(4.16.25) $X_n(x) = \dfrac{\Phi_n(x)}{\|\Phi_n(x)\|} = \dfrac{J_0(x\sqrt{\lambda_n})}{(a/\sqrt{2}) J_1(a\sqrt{\lambda_n})}$

(4.16.9) $\lambda_n$ satisfy $J_0(a\sqrt{\lambda_n}) = 0$

$\boxed{7.17}$ $-\lambda \rho(s) u$ $\rho(s) = s/c^2$ b/c cylindrical.

if spherical, has $s^2$ instead of $s$

(7.6.1) $\dfrac{\partial^2 u}{\partial t^2} = c^2 \left[ \dfrac{\partial^2 u}{\partial s^2} + \dfrac{1}{s}\dfrac{\partial u}{\partial s} \right]$ $\begin{matrix} s \in (0,1) \\ t > 0 \end{matrix}$

(7.6.2) $\left.\begin{matrix} u \text{ finite} \\ u = 0 \Leftarrow s = 1,\ t > 0 \end{matrix}\right\}$ BC

$\left.\begin{matrix} u = 0 \Leftarrow 0 < s < 1,\ t = 0 \\ \dfrac{\partial u}{\partial t} = g(s) \Leftarrow 0 < s < 1,\ t = 0 \end{matrix}\right\}$ IC

2

$\boxed{7.17}$

$$L[u] = \frac{\partial}{\partial s}\left[s\frac{\partial u}{\partial s}\right] = \left(\frac{s}{c^2}\right)\frac{\partial^2 u}{\partial t^2} \qquad (1)$$

$$\rho(s) = s/c^2$$

eigenproblem:

$$L[\phi] = -\lambda\rho(s)\,\phi \qquad (2)$$

$\phi$ finite (within domain)

$\phi = 0$ when $s = 1$, $t > 0$

Take FHT of (1)

$$-\lambda_n u_n(t) = \frac{1}{c^2}\frac{d^2 u_n}{dt}$$

$$\longrightarrow u_n(t) = \int_0^1 s\,\phi_n(s)\,u(s,t)\,ds \equiv \mathscr{H}[u(s,t)]$$

Get ICs for $u_n(t)$ from problem ICs:

from $u(s,0) = 0$

$$u_n(0) = \lim_{t\to 0} u_n(t) = \ell\int_0^1 s\,\phi_n\,u\,ds = \lim_{t\to 0}\int_0^1 s\,\phi_n(s)\,0\,ds = 0$$

from $\frac{\partial u(0)}{\partial t} = g(s)$

$\boxed{7.17}$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = g(s)$$

$$\left.\frac{\partial u_n}{\partial t}\right|_{t=0} = \lim_{t\to 0}\left[\frac{\partial u_n(t)}{\partial t}\right] = \lim_{t\to 0}\int_0^1 \left[s\phi_n(s)\frac{\partial u(s,t)}{\partial t}\right] ds$$

$$= \lim_{t\to 0}\int_0^1 s\phi_n(s)\, g(s)\, ds = g_n$$

General Solution to eigenproblem (2):

$$u_n = d_1\cos(c\sqrt{\lambda_n}\,t) + d_2\sin(c\sqrt{\lambda_n}\,t)$$

$$u_n(0) = 0 \qquad \frac{\partial u_n(0)}{\partial t} = g_n$$

$$u_n(0) = d_1\cdot 1 + d_2\cdot 0 = 0 \Rightarrow d_1 = 0$$

$$\left.\frac{\partial u_n}{\partial t}\right|_{t=0} = \left. c d_2\sqrt{\lambda_n}\cos(c\sqrt{\lambda_n}\,t)\right|_{t=0} = g_n = c d_2\sqrt{\lambda_n}$$

$$\Rightarrow d_2 = \frac{g_n}{c\sqrt{\lambda_n}}$$

$$u_n(t) = \frac{g_n}{c\sqrt{\lambda_n}}\sin(c\sqrt{\lambda_n}\,t)$$

Invert the FHT ($u = \mathcal{H}^{-1}[u_n]$)

$$u(s,t) = \sum_{n=1}^{\infty}\frac{g_n}{c\sqrt{\lambda_n}}\sin(c\sqrt{\lambda_n}\,t)\frac{\phi_n(s)}{\|\phi_n(s)\|}$$

Where $X_n(s) = \dfrac{\phi_n(s)}{\|\phi_n(s)\|} = \sqrt{2}\,\dfrac{J_0(\sqrt{\lambda_n}\,s)}{J_1(\sqrt{\lambda_n})}$ Bessel functions o.t. 1st kind