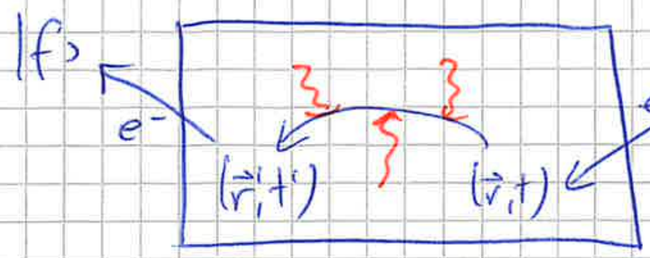
1.2 Time evolution pictures

How can we obtain information about interactions in systems $\Rightarrow$ Scattering experiments



information from overlap of initial and final states $|i\rangle, |f\rangle$

three ingredients needed:

-) many-body nature $\Rightarrow$ Second quantisation ✓
-) time-dependence $\Rightarrow$ pictures of time evolution
-) response of system $\rightarrow$ linear response theory

pictures of time evolution

-) Schrödinger picture ($\hbar = 1$)

  states: $|\psi(t)\rangle = e^{-i\hat{H}t}|\psi_0\rangle$, $i\partial_t|\psi(t)\rangle = \hat{H}|\psi(t)\rangle$

  operators: $\hat{A}$ can depend on time, $\hat{H}$ does not depend on time (we now drop the $\wedge$ for operators)

-) Heisenberg picture

  states: time-independent $|\psi_0\rangle = e^{iHt}|\psi(t)\rangle$

  operators: time-dependent $A(t) = e^{iHt}A e^{-iHt}$

  H time-independent

We demand the same matrix elements as in Schrödinger picture:

$$\langle\psi'(t)|A|\psi(t)\rangle \stackrel{!}{=} \langle\psi_0'|e^{iHt}Ae^{-iHt}|\psi_0\rangle =$$

Schrödinger

$$\stackrel{!}{=} \langle\psi_0'|A(t)|\psi_0\rangle$$

Heisenberg

$\Rightarrow$ equation of motion for operators

$$\dot{A}(t) = e^{iHt}(iHA - iAH + \partial_t A)e^{-iHt}$$

$$\Rightarrow \boxed{\dot{A}(t) = i[H, A(t)] + (\partial_t A)(t)}$$

Heisenberg equation of motion

•) Dirac ("interaction" picture)

$$H = H_0 + V(t) \quad , \quad H_0|n_0\rangle = \varepsilon_{n_0}|n_0\rangle$$

time-independent

"perturbation"

Key idea: Separate "trivial" time-evolution from H_0 and complicated from $V(t)$

states: $|\hat{\psi}(t)\rangle = e^{iH_0t}|\psi(t)\rangle$ ← Dirac picture

operators: $\hat{A}(t) = e^{iH_0t}Ae^{-iH_0t}$, H_0 does not depend on time

equation of motion for states

$$i\partial_t|\hat{\psi}(t)\rangle = (i\partial_t e^{iH_0t})|\psi(t)\rangle + e^{iH_0t}(i\partial_t|\psi(t)\rangle) =$$

$$= e^{iH_0t}(-H_0 + H)|\psi(t)\rangle = \hat{V}(t)|\hat{\psi}(t)\rangle$$

$$\Rightarrow \boxed{i\partial_t|\hat{\psi}(t)\rangle = \hat{V}(t)|\hat{\psi}(t)\rangle}$$

$\Rightarrow$ only depends on $\hat{V}(t)$

time evolution: $|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle$

$$\hat{U}^{-1} = \hat{U}^{+}, \quad \hat{U}(t_0,t_0) = \mathbb{1}$$

$\Downarrow$

$$i\partial_t \hat{U}(t,t_0) = \hat{V}(t)\,\hat{U}(t,t_0)$$

$$\Rightarrow \boxed{\hat{U}(t,t_0) = 1 + \frac{1}{i}\int_{t_0}^{t} dt'\,\hat{V}(t')\,\hat{U}(t',t_0)}$$

$\Rightarrow$ solve iteratively

$$\hat{U}(t,t_0) = 1 + \frac{1}{i}\int_{t_0}^{t} dt_1\,\hat{V}(t_1) + \frac{1}{i^2}\int_{t_0}^{t} dt_1\,\hat{V}(t_1)\int_{t_0}^{t_1} dt_2\,\hat{V}(t_2) + \dots$$

$$\int_{t_0}^{t} dt\,\hat{V}(t_1)\int_{t_0}^{t_1} dt_2\,\hat{V}(t_2) = \frac{1}{2}\int_{t_0}^{t} dt_1\,\hat{V}(t_1)\int_{t_0}^{t_1} dt_2\,\hat{V}(t_2) + \frac{1}{2}\int_{t_0}^{t} dt_2\,\hat{V}(t_2)\int_{t_0}^{t_2} dt_1\,\hat{V}(t_1) =$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\,\hat{V}(t_1)\hat{V}(t_2)\,\theta(t_1-t_2) + \frac{1}{2}\int_{t_0}^{t} dt_2\int_{t_0}^{t} dt_1\,\hat{V}(t_2)\hat{V}(t_1)\,\theta(t_2-t_1) =$$

$$= \frac{1}{2}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\left[\hat{V}(t_1)\hat{V}(t_2)\,\theta(t_1-t_2) + \hat{V}(t_2)\hat{V}(t_1)\,\theta(t_2-t_1)\right] =$$

$$=: \frac{1}{2}\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2\; T_t\left[\hat{V}(t_1)\hat{V}(t_2)\right]$$

time-order operator

generalization: permutations of n times t_j

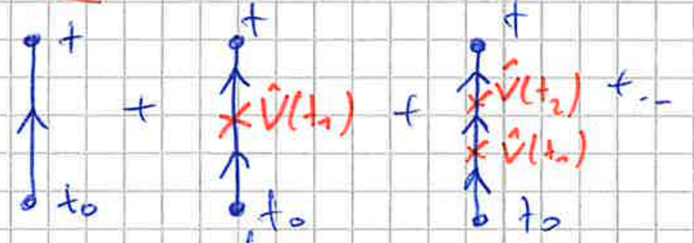
$$T_t\left[\hat{V}(t_1)\dots\hat{V}(t_n)\right] = \sum_{p\in S_n} \hat{V}(t_{p(1)})\,\hat{V}(t_{p(2)})\dots\hat{V}(t_{p(n)})\cdot$$

$$\cdot\,\theta(t_{p(1)}-t_{p(2)})\,\theta(t_{p(2)}-t_{p(3)})\dots\theta(t_{p(n-1)}-t_{p(n)})$$

$$\Rightarrow \hat{U}(t,t_0) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{i}\right)^n \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n \, T_t\left(\hat{V}(t_1)\dots\hat{V}(t_n)\right) =$$

$$= T_t\left(e^{-i\int_{t_0}^{t} dt' \hat{V}(t')}\right)$$

graphically: $\hat{U}(t,t_0) =$



linear response: $\hat{U}(t,t_0) \approx 1 + \frac{1}{i}\int_{t_0}^{t} dt' \hat{V}(t')$

(e.g.: Fermi's Golden rule!)