Title

Honours Literature Review

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Abstract

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5.1 Problem statement

Rayleigh-Bénard convection is the motion of a fluid confined between two horizontal isothermal plates, the temperature of the bottom plate being higher than that of the top plate.

The governing equations for the flow are derived from the Navier-Stokes equations of mass, energy and momentum conservation (see, e.g., Chandrasekhar (1961)). The density, ρ, of the fluid is related to its temperature T by the linear equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)],$$

where α is the (constant) volume coefficient of thermal expansion and ρ_0 and T_0 are the base-state density and temperature such that $\rho = \rho_0$ when $T = T_0$. The key assumption is that density variations are small ($\alpha(T - T_0) \ll 1$), which allows the governing equations to be simplified under the *Boussinesq approximation*. The Boussinesq approximation involves first writing the pressure, p, of the fluid as

$$p = p_0 - \rho_0 g z + p',$$

where p_0 is an arbitrary constant, g is the acceleration due to gravity and z is the vertical coordinate. p' is the (time-varying) deviation from a hydrostatically balanced background profile $p_0 - \rho_0 g z$ in which the upward pressure gradient force per unit volume $\rho_0 g$ cancels the downward weight force per unit volume $-\rho_0 g$. Since $\alpha(T - T_0) \ll 1$, density variations are neglected everywhere except in their contribution to the weight force, leading to a net buoyant (background pressure gradient plus weight) force per unit mass

$$\frac{\rho_0 - \rho}{\rho_0} g = \alpha(T - T_0)g.$$

I adopt the governing equations as they are derived by Chandrasekhar (1961):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{1}{\rho_0}\boldsymbol{\nabla} p' + \alpha(T - T_0)g\hat{\boldsymbol{z}} + \nu\nabla^2\boldsymbol{u} \quad \text{(momentum conservation)}, \tag{5.1}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \kappa\nabla^2 T, \quad \text{(energy conservation), and} \tag{5.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{(incompressibility)}. \tag{5.3}$$

$\boldsymbol{u}$ is the fluid velocity, t is time, $\hat{\boldsymbol{z}}$ is the upward unit vector, ν is the (constant) kinematic viscosity and κ is the thermal diffusivity (also constant).

The parametrisation test-bed developed in this work solves the governing equations in a two-dimensional domain $(x, z) \in [0, d] \times [0, L]$, subject to no-slip, isothermal boundary conditions on the top and bottom plates,

$$\boldsymbol{u} = \boldsymbol{0}, \quad T = T_0 + \frac{\delta T}{2} \qquad \text{at } z = 0 \text{ and} \tag{5.4}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad T = T_0 - \frac{\delta T}{2} \qquad \text{at } z = d, \tag{5.5}$$

and periodic boundary conditions in the horizontal,

$$\boldsymbol{u}(x = 0) = \boldsymbol{u}(x = L) \quad \text{and} \quad T(x = 0) = T(x = L). \tag{5.6}$$

δT is the constant temperature difference between the plates.

5.2 Nondimensionalisation and scale analysis

It is helpful to nondimensionalise the governing equations (5.1)–(5.6); this is not only useful for numerical work but also gives insight into the different flow regimes that are possible. A range of nondimensionalisations are used in fluid dynamics literature; I adopt a common one which is used, for example, by Ouertatani et al. (2008) and Stevens, Verzicco, and Lohse (2010) and is suitable for the turbulent convective regime.

For low-viscosity, turbulent flow, a suitable time scale is the *free-fall time* t_0, which is the time for a fluid element with constant temperature $T = T_0 - \delta T$ to fall from the top plate to the bottom plate under the influence of buoyancy alone. It is simple to show that

$$t_0 \sim \left(\frac{d}{g\alpha\delta T}\right)^{1/2},$$

ignoring a factor of $\sqrt{2}$. The obvious length and temperature scales are the plate separation d and temperature difference δT, respectively.

Making the substitutions $p'/\rho_0 \to \pi$ and $T - T_0 \to \theta$ in (5.1)–(5.6) and expressing all variables in units of t_0, d and δT leads to the dimensionless equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla}\pi + \left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2} \nabla^2\boldsymbol{u} + \theta\hat{\boldsymbol{z}}, \tag{5.7}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}\theta = (\mathrm{Ra}\,\mathrm{Pr})^{1/2}\,\nabla^2\theta, \quad \text{and} \tag{5.8}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{5.9}$$

with boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \theta = +\frac{1}{2} \qquad \text{at } z = 0, \tag{5.10}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \theta = -\frac{1}{2} \qquad \text{at } z = d, \tag{5.11}$$

$$\boldsymbol{u}(x=0) = \boldsymbol{u}(x=L) \quad \text{and} \quad \theta(x=0) = \theta(x=L). \tag{5.12}$$

There are only two parameters, both dimensionless: the *Prandtl number*

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa},$$

and the *Rayleigh number*

$$\mathrm{Ra} \equiv \frac{g\alpha d^3 \delta T}{\kappa\nu}.$$

The Rayleigh number can be interpreted as the ratio of the time scale for thermal transport by convection to the the time scale for thermal transport by conduction. Detailed theoretical analysis of the governing equations (see, e.g., Chandrasekhar (1961) and the seminal work by Lord Rayleigh (1916)) reveals that there exists a critical Rayleigh number, below which the equations have a stable equilibrium with the fluid at rest and a linear conductive temperature profile. Above the critical value, the equilibrium is unstable and small perturbations lead to the formation of a regular series of rotating convection cells.

6 Conclusion

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