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God does not care about our mathematical difficulties; He integrates empirically.

Albert Einstein, quoted by Leopold Infeld in *Quest: an autobiography*, 1941

Chapter 6

Discussion

TODO: introductory paragraph In Chapters 3–5, I presented a complete proof of concept for data-driven parametrisation in Rayleigh-Bénard convection, from the initial formulation of the problem to the assessment of the parametrised model in an online setting. In the final chapter of this thesis, I will discuss in greater detail the main obstacles that were encountered in the process (§ 6.1) and the concrete outcomes of this work (§ 6.3) before drawing final conclusions in § 6.4.

6.1 Practical issues and their implications

This work has highlighted several obstacles to the practical implementation data-driven parametrisation that were initially not obvious from a purely theoretical point of view or not applicable to earlier work using simpler dynamical systems such as Lorenz '96. Fundamentally, many of these arise because the Rayleigh-Bénard problem is spatially continous—a PDE, as opposed to ODEs like Lorenz '96.

6.2 Coarse model instability

One of the earliest issues encountered in this work was the potential for low-resolution models to become numerically unstable, making it difficult to obtain a baseline for evaluating parametrised models. Aside from the need for a baseline, it seems unlikely that a simple parametrisation scheme like the one developed in Chapter 4 would have improved the skill of a coarse model on the brink of instability. Since this work was not concerned with accurately representing physical reality, I chose to artificially add hyperviscosity terms to the equations. Another option for future work could be to use stress-free boundary conditions on the top and bottom plates (i.e., $w(z = 0, 1) = \partial u/\partial z|_{z=0,1} = 0$) rather than the no-slip condition. This could reduce the large near-wall temperature and velocity gradients that often triggered instability at low resolutions.

However, when modelling real-world systems such as the atmosphere, one does not have the freedom to simply alter the governing equations. Successful data-driven parametrisation in these cases will depend on the careful choice of the numerical methods used to discretise and solve the equations such that the underlying coarse model is stable, even if it is inaccurate. Futher work with the Rayleigh-Bénard system should investigate whether other types of solvers (finite difference/volume/element, etc.) are more suitable.

6.2.1 The coarse-graining problem

The predictability of the subgrid tendencies was found to depend strongly on the method used to coarse-grain the high-resolution solutions, and the development of an an appropriate method was non-trivial. These findings can be justified with further discussion.

The Lorenz ’96 system (2.1) explicitly separates its degrees of freedom into “coarse” and “fine” variables, allowing one to unambiguously define a reduced model by simply truncating the fine variables from the full model. The corresponding “coarse-graining” operation is trivial: just discard the fine variables. For spatially continuous systems like Rayleigh-Bénard, however, standard practice is to construct high- and low-resolution models independently from each other by applying (possibly different) discretisation schemes to the governing equations. Since the low-resolution model is not, in general, obtained by explicitly truncating the high-resolution model, it is up to the modeller to choose a coarse-graining operation that maps the high-resolution state space to the low-resolution one. While instructive, Lorenz ’96 and its coarse/fine paradigm are not helpful analogues in this respect.

As explained in § 4.2, it was essential for the coarse-graining operation to smooth the high-resolution fields to make them resolvable on the coarse grid. This requirement can be connected to the issue of scale separation, which was discussed in Chapter 1 as one of the main constraints on parametrisability. For a system whose energy spectrum has clear scale separation, as sketched in Figure 6.1a, a coarse model that cannot resolve the high-wavenumber behaviour can still capture the low-wavenumber features with good resolution (Figure 6.1b). Such a coarse model is therefore likely to be numerically stable, and the system’s parametrisability relatively insensitive to the choice of coarse-graining operation; simple averaging over each coarse grid-box will easily remove the high-wavenumber fluctuations. On the other hand, a system lacking scale separation (Figure 6.1c) presents a far greater challenge because coarse-graining with insufficient smoothing will leave too much power at the highest wavenumbers resolvable by the coarse model (darker blue dots in Figure 6.1d), leading to numerical instability.

Future work must be mindful of the technicalities surrounding coarse-graining, especially for systems like Rayleigh-Bénard that lack scale separation. The approach used in this work has shown promise and could continue to be a useful tool. `TODO: any literature?`

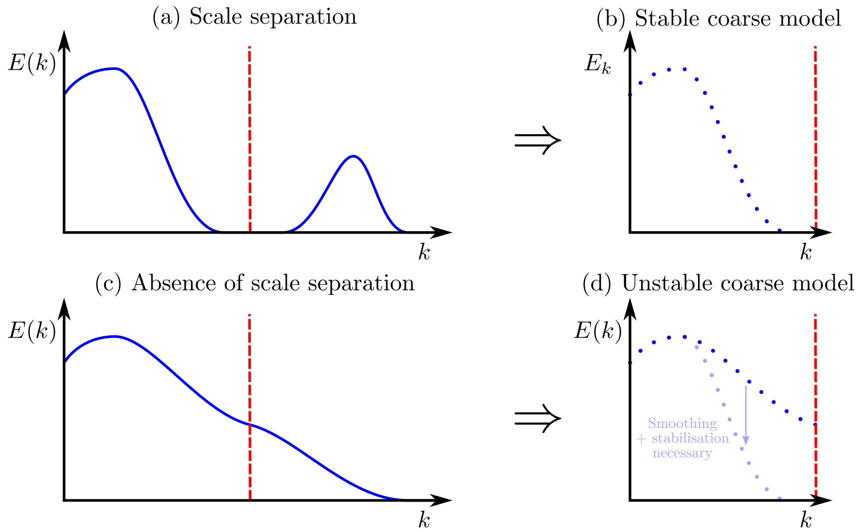

Figure 6.1: **(a):** Cartoon sketch of the energy spectrum for a system that exhibits clear scale separation. **(b):** The likely discrete energy spectrum of a coarse model of the system in (a), corresponding to a well-resolved and numerically stable solution. **(c-d):** Counterparts of (a-b) for a system lacking scale separation, resulting in a poorly-resolved and numerically unstable coarse solution solution unless action is taken to stabilise the model and apply smoothing when coarse-graining.

6.2.2 Subgrid tendency modelling

Another obstacle to data-driven parametrisation for spatially continuous problems is that there is a very large number of possible subgrid tendency predictors. For Rayleigh-Bénard, one not only has the three prognostic variables, but also their derivatives to arbitrary order in the two spatial directions. The position-dependence of the subgrid tendency statistics was a further complication, adding z to the list of predictors. One could even use spatially or temporally nonlocal predictors (i.e., the values of the variables at nearby points in space or previous time steps)—a possibility that was not even considered in this work. It would be impossible for a human to explore every possible combination. Supervised machine learning algorithms, discussed briefly in § 1.3.2, are much better-suited to regression problems with large numbers of predictors and could potentially capture hidden and/or nonlinear relationships between these predictors and the subgrid tendencies. There is no doubt, however, that this work was limited by the simplicity of the predictors and regression models that were considered, and future work using more sophisticated statistical models may indeed have greater success without needing to resort to machine learning.

This work was further limited by its use of purely deterministic parametrisation schemes despite the existence of considerable residuals in the subgrid tendency regressions Figures 4.8–4.10. Stochastic parametrisation, discussed in § 1.3.1, has the potential to reduce mean-state model biases by emulating the observed residuals. With the tools I have developed for the Rayleigh-Bénard problem, it would be relatively straightforward to experiment with various stochastic perturbations of the existing deterministic scheme (4.4), beginning with those that have been tested for Lorenz '96 (see § 2.2). This would include the use of time-correlated (e.g., AR(1)) noise to reflect the persistence (memory) of the subgrid tendencies.

6.3 Outcomes and opportunities for future work

6.3.1 Subgrid tendency correlations

The joint histograms shown in Chapter 4 clearly show that the subgrid tendencies are predictable given the coarse state.

6.3.2 Online performance

6.3.3 Framework and tools for future research

6.3.4 Unanswered questions

6.4 Conclusion