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Good approximations often lead to better ones.

Mathematical Methods in Science
George Pólya, 1977

Chapter 1

Introduction

TODO: introductory paragraph with motivation (from proposal) and outline

1.1 Theoretical background

TODO: introductory paragraph

1.1.1 The necessity of parametrisation

The primary task of general circulation models (GCMs) for Earth's weather and climate is to simulate the dynamics of the atmosphere and ocean, which are governed by the Navier-Stokes equations. The algorithm chosen to solve these partial differential equations is known as a model's *dynamical core* (McFarlane 2011). Since analytical solutions to the equations do not exist, the dynamical core necessarily approximates the continuous equations with finite-dimensional, exactly solvable alternatives using one of several possible discretisation schemes (e.g., the finite difference, finite element, finite volume and spectral methods) (Christensen and Zanna 2022). In practice, this usually involves representing the prognostic variables (i.e., those that affect the evolution of the flow) with sets of discrete samples in space and time, whose resolution is constrained by the available computing resources. The unavoidable consequence of discretisation is the loss of information about processes occurring on spatial and temporal scales smaller than the corresponding sampling intervals. These processes are said to be *unresolved*.

It is tempting to naïvely accept the loss of fine-scale information as a necessary sacrifice and hope the dynamical core will still make accurate predictions for the larger, resolved scales. Unfortunately, this too is impossible due to the *nonlinearity* of the governing equations. The reason for this may be seen by considering linear differential equations as a counterexample. If the governing equations were linear, they would allow arbitrary superpositions of solutions, meaning that any given solution could be partitioned into high- and low-frequency components, themselves also solutions. One would thus have the freedom to solve for the low-frequency components alone without compromise. This property may be understood more formally using the Fourier transform, defined for a function f of space and time by

$$\tilde{f}(\omega, \boldsymbol{k}) = \int \mathrm{d}t \, \mathrm{d}^3x \, e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} f(t, \boldsymbol{x}),$$

which reduces any linear differential equation to an algebraic equation relating the frequency ω to the wave vector $\boldsymbol{k}$. Each wavenumber component of the initial state propagates trivially according to its own time dependence $e^{-i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega(\boldsymbol{k})t)}$, *independently of the other components*. If the equations of fluid dynamics were linear, one could safely neglect the fine-scale dynamics because they would have no influence on the coarse scales. In reality, the equations are nonlinear cannot be solved by Fourier transform.

The consequence of the nonlinearity of the equations governing atmospheric and oceanic flows is, therefore, a coupling of the resolved coarse scales to the unresolved fine scales (McFarlane 2011). This fact may be demonstrated more explicitly by applying so-called *Reynolds averaging* to the equations (Christensen and Zanna 2022). Reynolds averaging decomposes each field q into the sum of a coarse-grained (in space or time) or statistical-ensemble-averaged field $\bar{q}$ and a perturbation q'. Note that $\overline{q'} = 0$ by definition. The coarse-graining operation is assumed to be linear, commute with differentiation and satisfy $\overline{\bar{p}q} = \bar{p}\bar{q}$ for any two fields p and q (Monin and Yaglom 2007). Following the example given by Christensen and Zanna (2022), consider the incompressible Navier-Stokes equations, which have the general form

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \sum f_i,$$
$$\frac{\partial u_i}{\partial x_i} = 0$$

where u_i are the components of the velocity, x_i are the coordinates and f_i are various forces per unit mass. Summation over repeated indices is implied. Applying the decomposition and coarse-graining both sides of the equations yields

$$\begin{aligned}
& \frac{\partial}{\partial x_i}\left(\overline{u_i} + u_i'\right) = 0 \\
\Rightarrow \qquad & \frac{\partial \overline{u_i}}{\partial x_i} + \underbrace{\frac{\partial \overline{u_i'}}{\partial x_i}}_{=0} = 0 \\
\Rightarrow \qquad & \frac{\partial \overline{u_i}}{\partial x_i} = \frac{\partial u_i'}{\partial x_i} = 0
\end{aligned}$$

and

$$\begin{aligned}
\sum \overline{f_i} &= \overline{\frac{\partial}{\partial t}\left(\overline{u_i} + u_i'\right)} + \overline{\left(\overline{u_j} + u_j'\right)\frac{\partial}{\partial x_j}\left(\overline{u_i} + u_i'\right)} \\
&= \frac{\partial \overline{u_i}}{\partial t} + \underbrace{\frac{\partial \overline{u_i'}}{\partial t}}_{=0} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j}\underbrace{\frac{\partial \overline{u_i'}}{\partial x_j}}_{=0} + \underbrace{\overline{u_j'}}_{=0}\frac{\partial \overline{u_i}}{\partial x_j} + \overline{u_j'\frac{\partial u_i'}{\partial x_j}} \\
&= \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_i'u_j'}}{\partial x_j} - \overline{u_i'\underbrace{\frac{\partial u_j'}{\partial x_j}}_{=0}} \\
\Leftrightarrow \qquad & \frac{\partial \overline{u_i}}{\partial t} + \overline{u_j}\frac{\partial \overline{u_i}}{\partial x_j} = \sum \overline{f_i} - \frac{\partial \overline{u_i'u_j'}}{\partial x_j}.
\end{aligned}$$

The physical meaning of this last equation is that the evolution of the coarse-grained velocity field depends not only on itself and the coarse-grained forces, but also on the perturbations via $\overline{u_i'u_j'}$, which does not necessarily vanish. This dependence is a consequence of nonlinearity.

The theoretical discussion in this section establishes that physical processes occurring at one place in the spectrum of temporal and spatial scales are coupled to all the other processes in the spectrum (Franzke et al. 2015). In particular, to ignore the effect of processes not explicitly resolved in numerical models would introduce unacceptable systematic biases, or errors, in the model forecasts. GCMs therefore require parametrisation schemes to estimate the effects of unresolved processes as functions of the available large-scale information (and the parameters of these functions must be chosen appropriately—hence the name "parametrisation").

1.1.2 Mathematical formulation of the parametrisation problem

In order to make any progress on the parametrization problem, it is necessary to formalise the notion of estimating the "effects" of unresolved processes. I will describe the general approach, which is well-

established (see, e.g., Hasselmann 1976; Palmer 2001; Demaeyer and Vannitsem 2018; Brajard et al. 2021), while acknowledging possible alternative conventions where they exist.

The Earth system may be considered as a dynamical system whose exact evolution is governed by equations of the form

$$\frac{\mathrm{d}\boldsymbol{z}}{\mathrm{d}t} = \boldsymbol{F}(\boldsymbol{z}, t), \tag{1.1.1}$$

where the vector $\boldsymbol{z}$ contains all the variables needed to fully specify the state of the system and $\boldsymbol{F}$ is a nonlinear differential operator. One then employs a change of variables $\boldsymbol{z} \to (\boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{x}$ contains the information or variables that are explicitly resolved by the model being studied and $\boldsymbol{y}$ contains the unresolved, 'sub-grid' information or variables. Practically, $\boldsymbol{x}$ and $\boldsymbol{y}$ can be thought of as projections of the full state $\boldsymbol{z}$ onto lower-dimensional subspaces (Brajard et al. 2021). Alternatively (and more concretely), $\boldsymbol{x}$ can be obtained by averaging the original variables over several adjacent grid points, with $\boldsymbol{y}$ being the corresponding residuals (Zacharuk et al. 2018; Alcala and Timofeyev 2021).

In principle, the change of variables splits (1.1.1) into two parts

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, t),\\ \frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} &= \boldsymbol{H}(\boldsymbol{x}, \boldsymbol{y}, t)\end{aligned}$$

that (still exactly) describe the coupled evolution of the resolved and unresolved variables via nonlinear differential operators $\boldsymbol{G}$ and $\boldsymbol{H}$. The goal of parametrisation is now to derive a new system of equations

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}t} = \tilde{\boldsymbol{G}}(\tilde{\boldsymbol{x}}, t) \tag{1.1.2}$$

whose solution $\tilde{\boldsymbol{x}}(t)$ approximates the true $\boldsymbol{x}(t)$ as well as possible without explicitly modelling $\boldsymbol{y}(t)$. The measure used to compare $\tilde{\boldsymbol{x}}$ and $\boldsymbol{x}$ depends on the intended function of the parametrised model; weather forecasting applications that prioritise short-term predictive skill might call for minimisation of the root-mean-square error (RMSE), while climate prediction would be more concerned with the closeness of their probability distributions (including both mean and extreme values).

Before proceeding, it is important to note two assumptions that are implicitly made in writing down (1.1.2). The first is that $\tilde{\boldsymbol{G}}$ is a *deterministic* function of $\tilde{\boldsymbol{x}}$. The second is that $\tilde{\boldsymbol{G}}$ depends only on the value of $\tilde{\boldsymbol{x}}$ at time t and not earlier times (that is to say, the parametrisation is *memoryless*). Recent research has shown that relaxing these assumptions has the potential to greatly improve the accuracy and reliability of the parametrised model, and this will be discussed in detail in § 1.3 and Chapter 2. However, for the purpose of this basic introduction, I will temporarily follow the traditional approach and retain these assumptions.

There is more than one possible interpretation of $\tilde{\boldsymbol{G}}(\boldsymbol{x}, t)$ and its relationship to $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, t)$. Hasselmann (1976) describes $\tilde{\boldsymbol{G}}(\boldsymbol{x}, t)$ as an ensemble average of $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, t)$, taken over the distribution of all $\boldsymbol{y}$ that are possible for a given $\boldsymbol{x}$, i.e.

$$\tilde{\boldsymbol{G}}(\boldsymbol{x}, t) = \langle \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, t) \rangle_{\boldsymbol{y}}.$$

Demaeyer and Vannitsem (2018) instead assert that the ultimate goal of parametrisation is to literally approximate $\boldsymbol{y}$ as a function $\boldsymbol{\xi}(\boldsymbol{x})$, in which case

$$\tilde{\boldsymbol{G}}(\boldsymbol{x}, t) = \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\xi}(\boldsymbol{x}), t).$$

In practice, $\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{y}, t)$ usually separates into a known resolved part $\boldsymbol{D}(\boldsymbol{x}, t)$ independent of $\boldsymbol{y}$ and an unresolved coupling term $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y}, t)$. It then suffices to approximate only the unresolved part by a parametrisation $\boldsymbol{P}(\boldsymbol{x}, t)$ using one of the above methods, so that the parametrised model reads

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{D}(\tilde{\boldsymbol{x}}, t) + \boldsymbol{P}(\tilde{\boldsymbol{x}}, t). \tag{1.1.3}$$

TODO: x = <z>, motivate data-driven method (see update presentation)

1.2 Practical background

TODO: introductory paragraph

1.2.1 Physical processes requiring parametrisation

Following the broad theoretical argument in the previous sections, I now turn to concrete examples of processes that are often represented by parametrisations, restricting the discussion to atmospheric processes for brevity. This section will give a basic introduction to each process, its effect on the larger-scale behaviour of the climate system and the role of its corresponding parametrisation scheme. The purpose of these examples is to provide real-world context and motivation for current parametrisation research in more idealised settings, which will form one of the main topics of this review.

Cloud microphysics parametrisations model the composition of clouds in terms of the amount of water in the solid (e.g., hail), liquid (cloud droplets and rain) and gaseous phases, and the rate of transitions between these phases. Accurate representation of cloud formation and evolution is crucial for several reasons. First, the interaction of clouds with solar and terrestrial radiation has a major influence on the overall energy balance of the atmosphere (McFarlane 2011). Second, cloud water phase transitions lead to the formation of precipitation, and are a source and sink of latent heat that drives convection (McFarlane 2011). Furthermore, the statistics of cloud formation are linked to global temperatures in a feedback loop; uncertainty in the sign and magnitude of this feedback effect is a major contributor to uncertainty in the sensitivity of global temperatures to increases in atmospheric CO_2 concentration (Andrews et al. 2012; Christensen and Zanna 2022; Stevens and Bony 2013). Microphysical processes occur on the spatial scales of single water droplets and ice particles (of order 10^{-6} m to 10^{-2} m; Lamb (2003)), among the smallest scales in the atmosphere. All atmospheric models must parametrise the amount of cloud water in each phase, the size distribution and concentration of liquid and ice particles, and the resulting rate and type (rain, hail, snow, etc.) of precipitation (Christensen and Zanna 2022).

Moist convection encompasses vertical motions in the atmosphere that are accompanied (and driven) by phase changes of water, and must be parametrised when it occurs on sub-grid scales. Moist convection is generally triggered by warming and moistening at low levels, which create convective instability, and is manifested by narrow updrafts and downdrafts that can rarely be resolved explicitly (McFarlane 2011). Convection transports heat and moisture vertically, removing the instability and generating storms where the condensed water falls out as precipitation; more broadly, it is a key component of the global atmospheric circulation in spite of its small spatial scale (Christensen and Zanna 2022).

It should be noted that parametrisation also encompasses estimation techniques for processes that are too complex to model exactly for reasons unrelated to spatial and temporal resolution (McFarlane 2011). A good example of such a process is *radiative transfer*. Air and its constituent gases, as well as clouds, absorb, emit, reflect and/or scatter solar and terrestrial radiation. This leads to differential heating and cooling that drives the atmospheric circulation. While the theory of radiative transfer is understood well enough to allow precise calculations in principle, the prohibitive computational cost of such calculations necessitates a parametrisation based on simplifying assumptions (the details of which are beyond the scope of this review) (Christensen and Zanna 2022).

1.2.2 Traditional solutions to the problem and their limitations

With the examples of cloud microphysics, moist convection and radiative transfer in mind, I now broadly review the traditional approaches used to construct parametrisation schemes in practice. In particular, this section will identify the key assumptions upon which many traditional schemes are founded, and the circumstances under which the assumptions may be violated. This will motivate research into novel approaches.

A parametrisation scheme is traditionally constructed by formulating a simplfied, easily solvable and

deterministic conceptual model of the physical process in question. The solution of this model is then used to estimate the effect of the process on the coarse-scale state of the parent model, known as the *unresolved tendency* (McFarlane 2011). For example, the earliest convective parametrisation was developed by Manabe, Smagorinsky, and Strickler (1965) and simply assumed that the net effect of convection is to relax the vertical structure of the atmosphere towards a neutrally stable state whenever it becomes convectively unstable. One obvious deficiency of simple conceptual models is that they cannot possibly capture the full range of variability in the processes they simulate. One major branch of modern parametrisation research therefore studies *data-driven* schemes that instead use observational or high-resolution simulated data to fit empirical models for the unresolved processes, naturally capturing a wider range of variability (Christensen and Zanna 2022). Data-driven parametrisation will be discussed in much further detail in the next section.

In general, a traditional parametrisation scheme aims to capture the net unresolved tendency due to all occurrences of the unresolved process (e.g., all convective updrafts and downdrafts) within each grid cell of the parent model. A deterministic prediction of this type is valid when each grid cell contains many independent realisations whose varying contributions may be expected to yield a reliable average tendency. This requires a *scale separation* between the unresolved process and the resolution of the parent model. Scale separation breaks down when model development and increases in available computing resources allow simulations at resolutions approaching the previously unresolved scales. In this case, knowledge of the coarse-scale state cannot be expected to uniquely determine the unresolved tendency because the process is only realised a few times in each grid cell. The resulting (seemingly) random nature of the true unresolved tendency motivates stochastic treatments (McFarlane 2011; Christensen and Zanna 2022; Berner et al. 2017). These will be discussed in the next section.

The conceptual models core to traditional parametrisations usually contain free parameters that are determined from the coarse-scale model state using additional assumptions called *closures*. Closures often postulate a state of quasi-equilibrium between the unresolved processes and their large-scale environment, such as a balance between the accumulation of convective available potential energy (CAPE) and its removal by convection, or between the horizontal convergence of moisture at low levels and its convective transport to higher levels (McFarlane 2011; Christensen and Zanna 2022; Palmer 2019). However, there is no guarantee that such an equilibrium exists; in fact, it has been demonstrated that the CAPE balance is violated by fluctuations on sub-diurnal time scales (Donner and Phillips 2003) and by midlatitude continental convection (Zhang 2002). Newer parametrisation schemes have allowed departures from equilibrium by representing the unresolved processes in a prognostic rather than diagnostic manner (i.e., allowing the processes to have their own self-governing time dependence rather than calculating them from the large-scale state at each time step independently of their values at the previous step) (Rio, Del Genio, and Hourdin 2019; Berner et al. 2017). This creates *memory* or *latency* in the parametrised tendencies, meaning that the tendencies have some nonzero response time when subjected to sudden changes in the large-scale state. Memory will be discussed further in the next section.

Parametrisation schemes commonly suffer from several other issues that I will briefly address here. Firstly, while the division of the general atmospheric dynamics into a set of separately parametrised processes (microphysics, convection, etc.) is physically motivated, it remains somewhat arbitrary because these processes, strictly speaking, form a continuum without well-defined boundaries (Christensen and Zanna 2022; McFarlane 2011). It is a goal of contemporary research to unify the parametrisations as much as possible. Secondly, given the importance of future climate projections, it is natural to ask whether the parametrisation schemes that have been developed and tuned on today's climate remain valid as the climate changes over decade- to century-long simulations. This is a matter of particular concern for data-driven parametrisations, since there is little reason to trust empirical models once they are extrapolated beyond the range of the data originally used to fit them (Christensen and Zanna 2022). Finally, unless very special care is taken, parametrisation schemes can cause the parent model to violate known physical conservation laws (e.g., mass, energy and momentum) (Christensen and Zanna 2022). Efforts to resolve this issue are ongoing.

1.3 Novel approaches to the parametrisation problem

Since the 1990s, the limitations identified in § 1.2.2 have prompted many to reconsider the principles upon which parametrisation schemes are founded. The main advance has been the development of stochastic parametrisations that incorporate random noise in the predicted tendencies. More recently, others have approached the problem from an entirely new direction, developing data-driven parametrisations that learn to predict the tendencies empirically. This section will introduce the stochastic and data-driven approaches in general terms while omitting the technical details of individual implementations, the aim being to contextualise and motivate the study of these approaches in more idealised frameworks in Chapter 2. Other methods (such as superparametrisation) exist but are beyond the scope of this review.

1.3.1 Stochastic parametrisation and memory

The potential value of stochasticity for climate modelling was first established by Hasselmann (1976), whose seminal paper sought to explain the characteristics of long-term climate variability. Knowing that climate depends on interactions between all components of the Earth system (atmosphere, ocean, cryosphere, biosphere, etc.), Hasselmann argued that the effect of the more rapidly-evolving atmosphere on the other, more slowly-varying components is that of a stochastic forcing. Owing to their long response time, the slowly-varying components effectively integrate this stochastic forcing, allowing long-term climate variability to be characterised as a type of random walk process akin to the Brownian motion of a massive particle in a fluid.

Further motivation for stochastic parametrisation in particular stems from the need to reliably estimate uncertainties in weather forecasts and climate projections. Weather forecasting centres typically propagate initial condition uncertatinties (due to imperfect observations) through to the final forecast in a Monte Carlo fashion, by initialising an ensemble of model runs with perturbed initial conditions and measuring the spread of the resulting forecasts. It has been observed that deterministic models produce systematically under-dispersed ensembles that fail to span the range of actual weather outcomes, indicating that these models are failing to capture additional sources of variability (Palmer et al. 2005; Berner et al. 2017; Palmer 2019). Knowing that poor scale separation and departures from quasi-equilibrium should preclude deterministic relationships between unresolved tendencies and the large-scale state (see § 1.2.2), it should seem highly likely that deterministic parametrisation contributes to this deficiency.

The principle of stochastic parametrisation is, therefore, that unresolved tendencies should be randomly sampled from an appropriate distribution at each point in space and time, not simply set to the mean of the distribution (Franzke et al. 2015). This choice has now been theoretically justified using statistical mechanical arguments; most notably, Wouters and Lucarini (2012, 2013) showed that, assuming a weak coupling to the resolved variables, the effect of unresolved dynamics should be parametrised by a combination of deterministic and stochastic terms, as well as a *non-Markovian* memory term depending on the past states of the resolved variables.

The simplest and earliest approach to stochastic parametrisation is the method of *stochastically perturbed parametrisation tendencies* (SPPT), which takes an existing deterministic parametrisation and randomly scales its output with a multiplicative noise field (Palmer 2019; Christensen 2020). Re-using the notation of § 1.1.2, an SPPT model for the resolved variables $\boldsymbol{x}$ takes the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{D}(\boldsymbol{x}, t) + [I + \mathrm{diag}(\boldsymbol{e})]\boldsymbol{P}(\boldsymbol{x}, t),$$

where $\boldsymbol{P}$ is the existing deterministic parametrisation, $\boldsymbol{e}$ is a mean-zero random vector with the same length as $\boldsymbol{x}$ and I is the identity matrix. The choice of multiplicative (and thus inherently state-dependent) rather than additive noise is intuitively motivated by the expectation that variability in the effect of unresolved processes should be greatest when those processes are most active (Franzke et al. 2015). Christensen (2020) performed a comparison of high-resolution simulations to parametrised single-column model output which justified the use of multiplicative perturbations.

In GCMs, where the variables being modelled have spatial dependence, the multiplicative perturbation takes the form of a random *field* $e(x, y, t)$. It has been argued that this random field should be spatially and temporally correlated (in contrast to uncorrelated "white" noise) in order to emulate the organisation of unresolved processes on larger scales and their persistence in time (Christensen and Zanna 2022; Franzke et al. 2015). In particular, perturbations that explictly depend on their own past values constitute a type of memory, albeit distinct from the memory term advocated by Wouters and Lucarini (2012, 2013), which would instead couple the perturbations to the past values of the large-scale variables.

Stochastic parametrisations have several known advantages over their deterministic counterparts, and are now implemented in operational weather forecast models. They have been found to remedy the aforementioned issues of ensemble underdispersion and prediction unreliability (Palmer et al. 2005; Berner et al. 2017), and (as of 2009) even make the skill of five-day forecasts comparable to that of deterministically parametrised two-day forecasts (Palmer 2019). Despite the zero-mean nature of the noise, it has been shown that stochastic parametrisations can reduce systematic model biases ("noise-induced drift"; Palmer et al. (2005)) and stabilise the simulation of regime-based behaviour (such as the El Niño-Southern Oscillation) in the climate system (Berner et al. 2017).

Reviewing the field, Palmer (2019) identifies outstanding issues for further research. The main concern is the lack of rigour in most stochastic schemes; SPPT, for example, modifies existing deterministic schemes *ad hoc* rather than incorporating stochasticity *ab initio* in the development process. More objective approaches are yet to gain widespread acceptance. This motivates both data-driven methods and further testing using simpler dynamical systems where objectivity is more feasible.

1.3.2 Data-driven parametrisation and machine learning

The fitting of predictive statistical models to data is, of course, ubiquitous in the sciences, but attempts to use such models as parametrisation schemes and couple them into fully fledged GCMs are a relatively new phenomenon—certainly more so than stochastic parametrisations. The prerequisite for all data-driven parametrisations is obviously training data: mathematically, an approximate solution of (1.1.1) of sufficient accuracy and resolution to be considered "truth" for the application at hand. Training data are typically derived from high-resolution simulations, such as regional weather model runs or large eddy simulations (LES). The other ingredient is the imperfect low-resolution model that will later be augmented by parametrisation.

To generate an appreciation of how these data might be used to construct a parametrisation in practice, I roughly follow the argument and notation of Brajard et al. (2021). Define a map $\mathcal{M}$ that takes each state $\boldsymbol{z}(t)$ in the training dataset to the state $\boldsymbol{z}(t + \delta t)$ at the next time step. Similarly, denote by $\mathcal{M}^{\mathrm{r}}$ the low-resolution model (the superscript r meaning "reduced"), which maps a low-resolution state $\boldsymbol{x}(t)$ to a prediction for $\boldsymbol{x}(t+\Delta t)$, where Δt is not necessarily equal to δt (usually larger). The spaces of high- and low-resolution states, having different dimensions, may be linked by an operator $\langle\cdot\rangle$ that projects high-resolution states onto the low-resolution state space. Practically, the projection operation is simply a coarse-graining of $\boldsymbol{z}$ by averaging values at adjacent grid points to match the resolution of $\boldsymbol{x}$. Now, for each state $\boldsymbol{z}$ in the training dataset, one may compute the difference

$$\boldsymbol{\epsilon}(\boldsymbol{z}) = \frac{\langle \mathcal{M}(\boldsymbol{z})\rangle - \langle \boldsymbol{z}\rangle}{\delta t} - \frac{\mathcal{M}^{\mathrm{r}}(\langle \boldsymbol{z}\rangle) - \langle \boldsymbol{z}\rangle}{\Delta t}$$

between the coarse-grained true tendency (on the left) and the tendency predicted by the coarse model when it sees the same state (on the right). This is the unresolved tendency. Knowledge of $\langle \boldsymbol{z}\rangle$ alone does not uniquely determine $\boldsymbol{\epsilon}$ because $\langle \mathcal{M}(\boldsymbol{z})\rangle$ depends on the original $\boldsymbol{z}$. However, if one can fit a statistical model $\boldsymbol{P}$ to the dataset of $(\langle \boldsymbol{z}\rangle, \boldsymbol{\epsilon})$, then $-\boldsymbol{P}(\boldsymbol{x}(t))\Delta t$ will be able to serve as an estimate of the error incurred by the coarse model in predicting $\boldsymbol{x}(t + \Delta t)$ given $\boldsymbol{x}(t)$. This motivates the construction of a parametrised model $\mathcal{M}^{\mathrm{p}}$, defined by

$$\mathcal{M}^{\mathrm{p}}(\boldsymbol{x}) = \mathcal{M}^{\mathrm{r}}(\boldsymbol{x}) + \boldsymbol{P}(\boldsymbol{x})\Delta t$$

that simply subtracts the estimated error at each time step of the coarse model. In theory, the solution $\boldsymbol{x}(t)$ obtained by iteration of $\mathcal{M}^{\mathrm{p}}$ will be more accurate than that obtained by iteration of the unparametrised $\mathcal{M}^{\mathrm{r}}$.

The current method of choice for constructing $\boldsymbol{P}$ in weather and climate modelling contexts is machine learning (ML). I will briefly introduce the key concepts, strengths and weaknesses of ML, following Beucler et al. (2022). ML encompasses a broad class of algorithms that programmatically and autonomously develop and apply rules for performing tasks, such as image classification, removing the need for a human to explicitly program the rules. They do this using supplied *training data* that exemplify the task to be performed. When it comes to data-driven parametrisation, the task is function approximation or regression, and the relevant algorithms usually require a training dataset of example inputs (i.e., coarse-scale variables) *labelled* with the desired outputs (i.e., unresolved tendencies). One commonly-used class of algorithms called *neural networks* consist of layers of inter-communicating calculation nodes called neurons and learn to approximate functions by optimising sets of weights associated with the neurons. Another type of algorithm is the *random forest*, which approximates functions by passing the inputs through a series of nested `if/else` decision trees.

The distinguishing advantage offered by ML data-driven parametrisations is their ability to systematically capture complex relationships without relying on artificially simplified physical models derived by humans (Irrgang et al. 2021; Beucler et al. 2022). In addition to constructing new parametrisations from scratch, ML has been used to emulate the action of existing parametrisation schemes in GCMs (e.g., Gentine et al. 2018). These ML emulators have the potential to reduce the computational burden that parametrisation usually imposes, leaving more resources for other tasks such as calling the parametrisation scheme more frequently or increasing the model resolution (Beucler et al. 2022). Furthermore, the ability to emulate multiple existing schemes at once is a step towards unifying the parametrisation of unresolved processes (a long-standing issue; see § 1.2.2).

ML parametrisations are not without their issues; they are known to be prone to overfitting and often cause their host models to become numerically unstable. They are also, by their very nature, far more difficult to interpret than traditional schemes. It is therefore very difficult to determine *a priori* whether or not they will continue to be valid in climate conditions outside the bounds of their training datasets (Irrgang et al. 2021; Beucler et al. 2022). "Interpretable" or "explainable" ML is an active area of research that aims to address these issues. Another active research topic is "physics-guided ML", the aim of which is to augment ML with known physical constraints and conservation laws that would otherwise be violated (e.g., Yuval, O'Gorman, and Hill 2021).

TODO: summary?

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