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All models are wrong but some are useful.

Robustness in the Strategy of Scientific Model Building
George Box, 1979

Chapter 5

Assessment of the parametrised model

In Chapter 4, I analysed the subgrid tendencies and constructed a statistical model (4.6) to estimate them from the coarse model tendencies. In this chapter, I will assess the performance of the parametrised model obtained by coupling the statistical model into the coarse Rayleigh-Bénard model.

First, a "truth" solution was obtained by running the fine (i.e., 2048×256; see Table 4.1) model for 300 time units, saving the output at intervals of 0.2 time units. The last snapshot from the fine model simulation described in § 4.1 (Step 1) was used as the initial condition, thus keeping the data used to test the parametrisation separate from the data used to fit it. Every snapshot in the truth dataset was then coarse-grained using the method described in § 4.2; the aim is for the parametrised model to approximate this coarse-grained truth solution as closely as possible.

A control solution was obtained by running the unmodified coarse (i.e., 256×64) model, using the first coarse-grained snapshot of the truth solution as an initial condition. Output was saved at approximately the same interval of 0.2 time units.

The equations governing the parametrised model, given schematically by (4.5), can be written out explicitly as

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} &= \begin{pmatrix} 1 + f_u(z) & 0 \\ 0 & 1 + f_w(z) \end{pmatrix} \left[-\boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} - \boldsymbol{\nabla} \pi + \left(\frac{\mathrm{Pr}}{\mathrm{Ra}} \right)^{1/2} \left(\nabla^2 \boldsymbol{u} + \tilde{\nu} f(z) |\nabla^2 \boldsymbol{u}| \nabla^2 \boldsymbol{u} \right) + \theta \hat{\boldsymbol{z}} \right], \\
\frac{\partial \theta}{\partial t} &= (1 + f_\theta(z)) \left[-\boldsymbol{u} \cdot \boldsymbol{\nabla} \theta + (\mathrm{Ra}\,\mathrm{Pr})^{-1/2} \left(\nabla^2 \theta + \tilde{\kappa} f(z) |\nabla^2 \theta| \nabla^2 \theta \right) \right], \quad \text{and} \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0,
\end{aligned}$$

where the terms in red distinguish these modified equations from the originals (3.13)–(3.15). The parametrised model was run with the same resolution, time step, initial condition and output interval as the control.

Unfortunately, when the parametrised model was run with the f_θ, f_u and f_w that were fitted in § 4.4, it became unstable and crashed within 10 time units. Given the low coefficients of determination for the u and w fits (Table 4.2), I chose to set $f_u(z) = f_w(z) \equiv 0$—that is, to only parametrise the θ tendency and leave the momentum equation unmodified. This stabilised the model.

5.1 Short-term forecast performance

I first assess short-term accuracy using the root-mean-square error (RMSE) of u, w and θ, defined by

$$\mathrm{RMSE}_\chi(t) = \sqrt{\langle(\chi_{\mathrm{fc}}(t) - \chi_{\mathrm{truth}}(t))^2\rangle_{x,z}},$$

where $\chi = u, w, \theta$, χ_{fc} is the forecast generated by either the control or parametrised model and χ_{truth} is the *coarse-grained* truth solution. Figure 5.1 (a-c) compares the RMSE of the control and parametrised solutions for the first 70 time units of simulation.

Panel (a) shows that the θ RMSE of the parametrised model initially grows at less than half the rate seen in the control, remaining smaller than the control value for almost 10 time units before surpassing it. While both models reach a peak RMSE before settling to a steady equilibrium, the parametrised model takes approximately four times longer to do so. It seems likely that by reducing the magnitude of $\partial\theta/\partial t$ near the top and bottom of the domain, the parametrisation preserves the properties of the thermal boundary layers. These are relatively thick and slowly-evolving in the coarse-grained truth solution, but are thinned too quickly by the control model.

For u and w, the RMSE of the parametrised model initially increases at a rate much closer to that of the control model. However, by plotting the difference between the two and zooming the time axis (panels d-f), it can be seen that the parametrised model does indeed have a slightly smaller error.

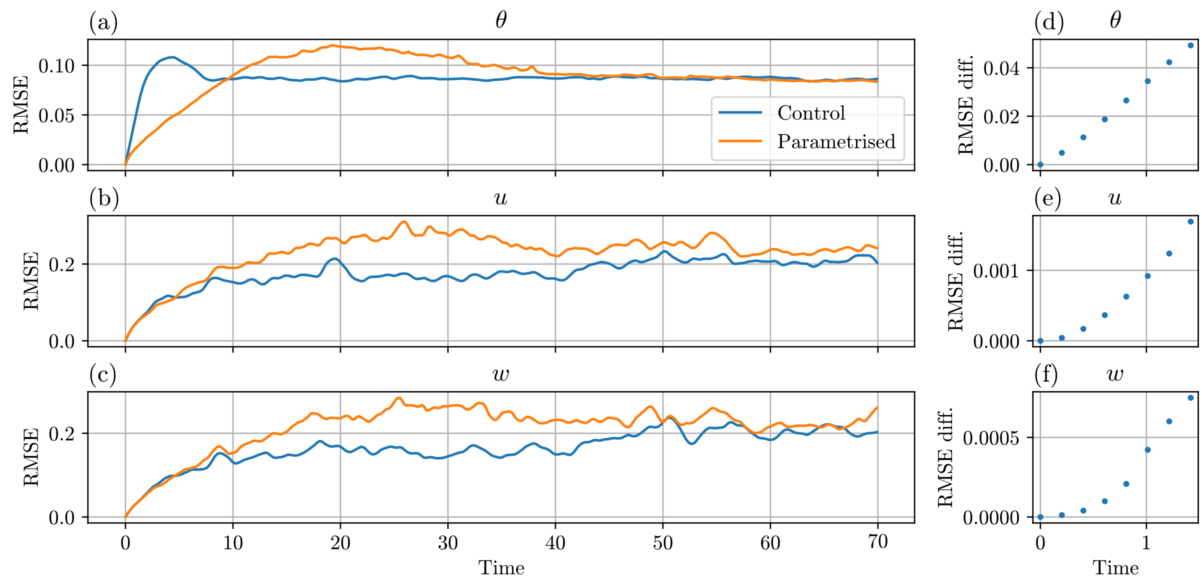

Figure 5.1: **(a-c):** Root-mean-square error of the control (blue) and parametrised (orange) solutions for θ, u and w over the first 70 time units of simulation. **(d-f):** The control RMSE minus the parametrised RMSE over the first 1.5 time units, confirming that the parametrised solution has lower RMSE in the short term.

Another way to assess short-term accuracy is to calculate domain-averaged quantities for the parametrised, control and truth solutions and compare them over time. As in § 3.6, I consider the Nusselt number Nu, thermal boundary layer thickness δ_θ, RMS speed u_{rms}, kinetic energy dissipation rate ϵ_k and thermal dissipation rate ϵ_θ, plotting their time series in Figure 5.2. Both the control and parametrised models make grossly inaccurate predictions for δ_θ and ϵ_θ, but the parametrised model's predictions are nonetheless closer to the truth for the first ~ 20 time units. Since these are purely thermal quantities, their improved prediction can be attributed to the nature of the parametrisation scheme in the same way as the θ RMSE. The parametrised model also makes more accurate initial predictions for the other three quantities, but only for the first ~ 5 time units.

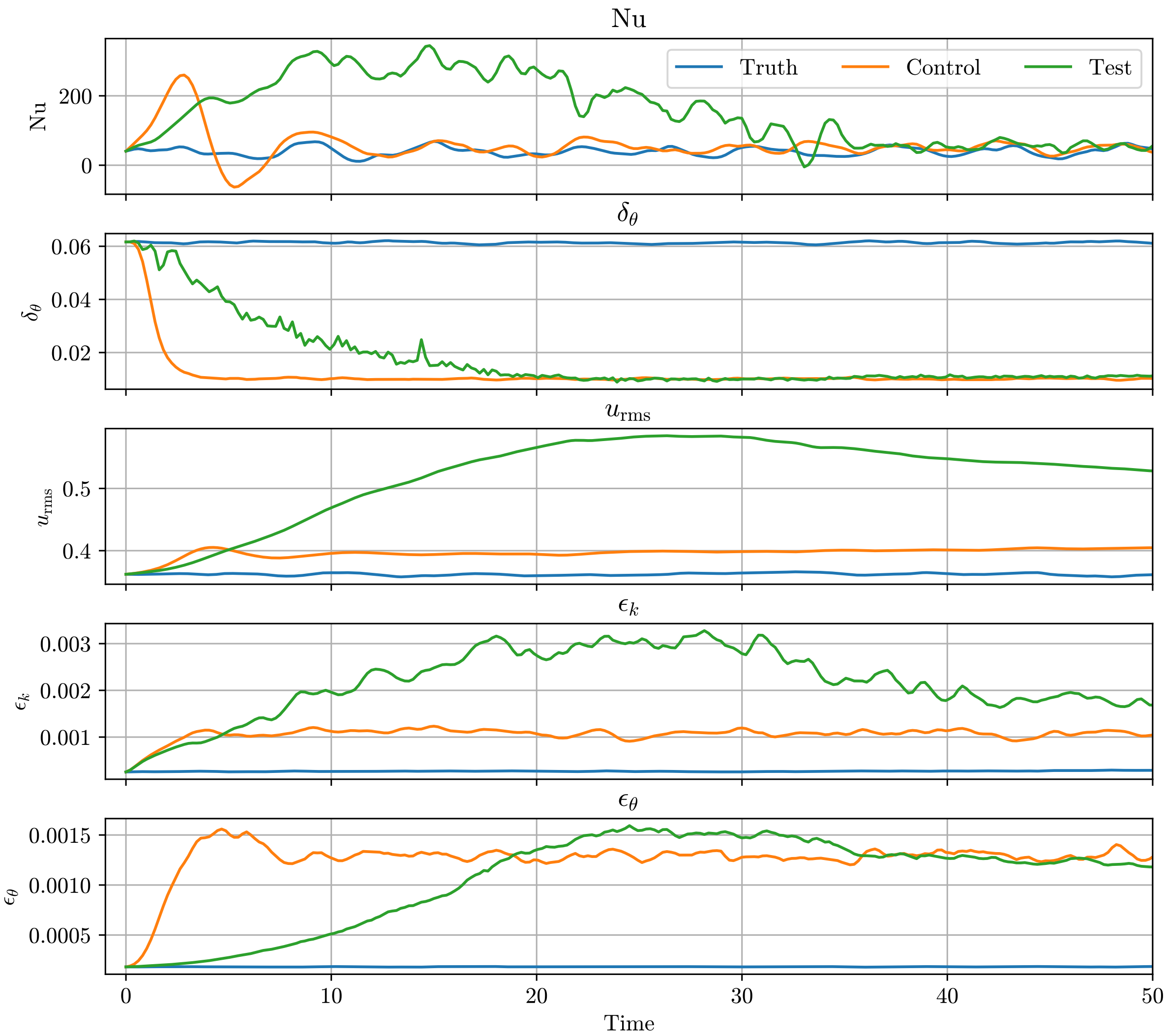

Figure 5.2: Time series of the Nusselt number Nu, thermal boundary layer thickness δ_θ, RMS speed $u_{\rm rms}$, kinetic energy dissipation rate ϵ_k and thermal dissipation rate ϵ_θ for the coarse-grained truth (blue), control (orange) and parametrised (green) solutions over the first 50 time units.

So far, I have shown that the parametrised model is capable of producing more accurate short-term forecasts than the control if the lead time is sufficiently short (on the order of a few time units)—a promising result.

5.2 Accuracy of long-term statistics

§ 5.1 showed that, while the parametrised solution is more accurate short-term forecasts, it is usually less accurate at longer lead times. I now ask whether it will produce more accurate long-term statistics if allowed to reach a statistically steady state.

The parametrised model was found to take approximately 800 time units to return to a statistically steady state (see Appendix B.2.1). It was therefore integrated for 1100 time units, giving 300 time units of data for calculating long-term statistics. For consistency, the control solution was also extended to 1100 time units.

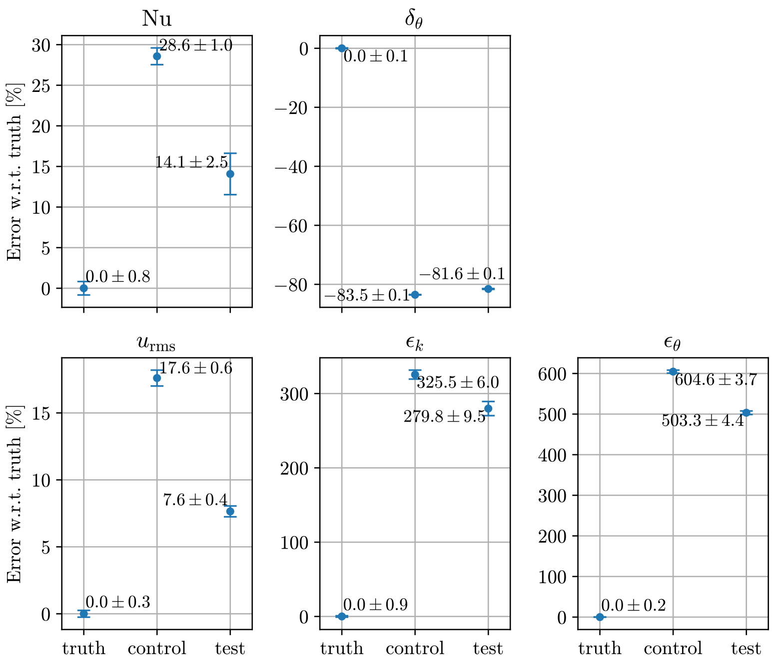

Figure 5.3: F

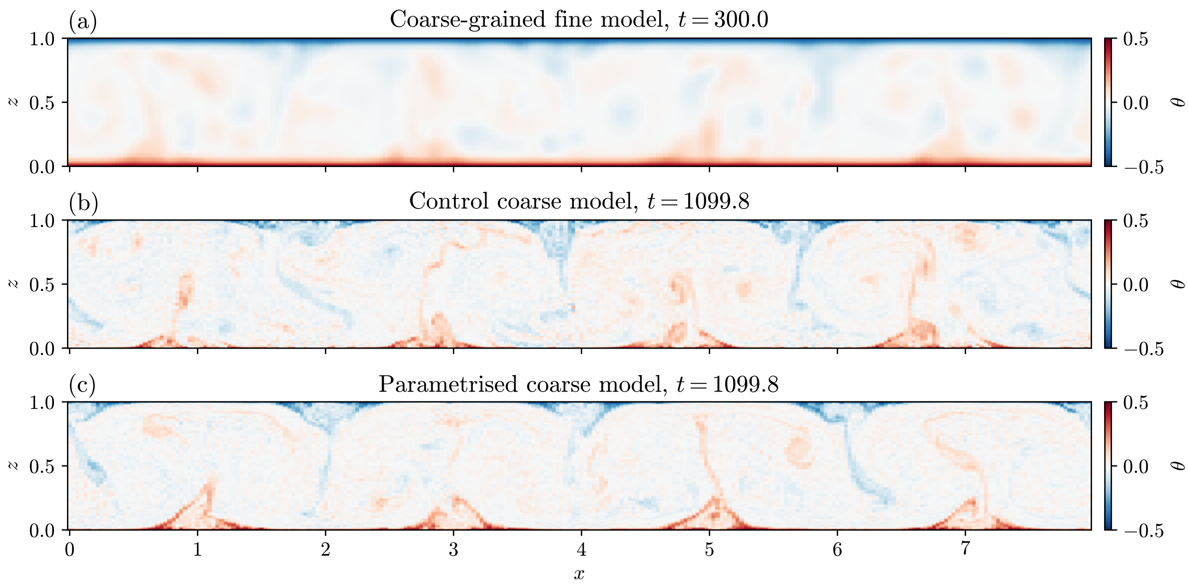

Figure 5.4: F

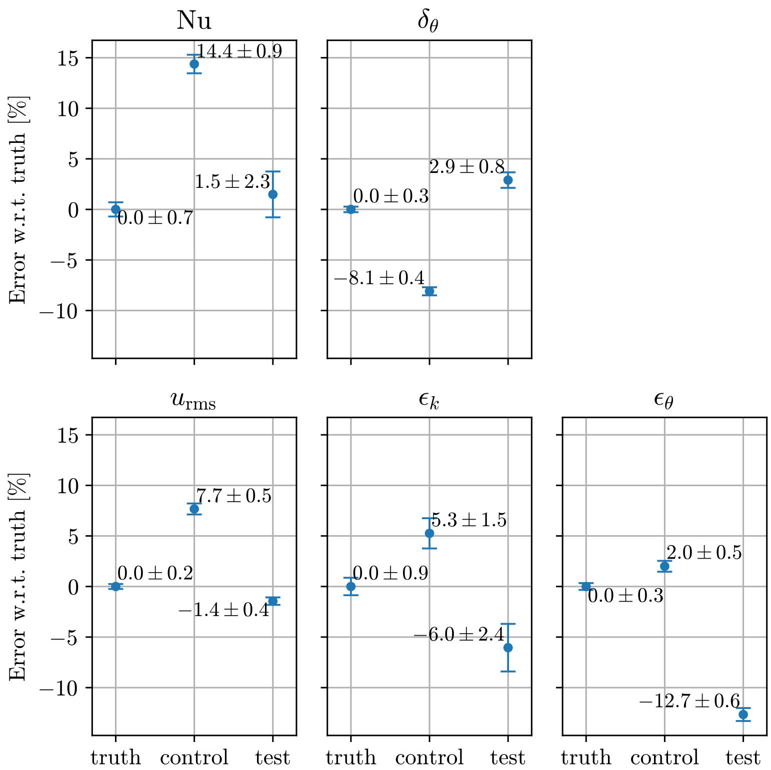
Nu
δ_θ
u_{rms}
ϵ_k
ϵ_θ
Error w.r.t. truth [%]
15
10
5
0
−5
−10
truth
control
test
14.4 ± 0.9
1.5 ± 2.3
0.0 ± 0.7
2.9 ± 0.8
0.0 ± 0.3
-8.1 ± 0.4
7.7 ± 0.5
0.0 ± 0.2
-1.4 ± 0.4
5.3 ± 1.5
0.0 ± 0.9
-6.0 ± 2.4
2.0 ± 0.5
0.0 ± 0.3
-12.7 ± 0.6

Figure 5.5: F