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Chapter 3

Rayleigh-Bénard convection

TODO: introductory paragraph

3.1 Problem statement

Rayleigh-Bénard convection is the motion of a fluid confined between two horizontal isothermal plates, the temperature of the bottom plate being higher than that of the top plate. The governing equations for the flow follow from the Navier-Stokes equations of mass, energy and momentum conservation. The reader is referred to Chandrasekhar (1961) for a detailed derivation; I summarise the assumptions and approximations involved below.

The density, ρ, of the fluid is related to its temperature T by the linear equation of state

$$\rho = \rho_0[1 - \alpha(T - T_0)],$$

where α is the (constant) volume coefficient of thermal expansion and ρ_0 and T_0 are the base-state density and temperature such that $\rho = \rho_0$ when $T = T_0$. The key assumption is that density variations are small ($\alpha(T - T_0) \ll 1$), which allows the governing equations to be simplified under the *Boussinesq approximation*. The Boussinesq approximation involves first writing the pressure, p, of the fluid as

$$p = p_0 - \rho_0 g z + p',$$

where p_0 is an arbitrary constant, g is the acceleration due to gravity and z is the vertical coordinate. p' is the (time-varying) deviation from a hydrostatically balanced background profile $p_0 - \rho_0 g z$ in which the upward pressure gradient force per unit volume $\rho_0 g$ cancels the downward weight force per unit volume $-\rho_0 g$. Since $\alpha(T - T_0) \ll 1$, density variations are neglected everywhere except in their contribution to the weight force, leading to a net buoyant (background pressure gradient plus weight) force per unit mass

$$\frac{\rho_0 - \rho}{\rho_0} g = \alpha(T - T_0) g.$$

With these assumptions in mind, I adopt the governing equations as they are derived by Chandrasekhar (1961):

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\frac{1}{\rho_0} \boldsymbol{\nabla} p' + \alpha(T - T_0) g \hat{\boldsymbol{z}} + \nu \nabla^2 \boldsymbol{u} \quad \text{(momentum conservation)}, \tag{3.1.1}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} T = \kappa \nabla^2 T \quad \text{(energy conservation), and} \tag{3.1.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{(incompressibility)}. \tag{3.1.3}$$

$\boldsymbol{u}$ is the fluid velocity, t is time, $\hat{\boldsymbol{z}}$ is the upward unit vector, ν is the (constant) kinematic viscosity and κ is the thermal diffusivity (also constant). Notice that the aforementioned buoyancy term $\alpha(T - T_0)g$ appears on the right-hand side of (3.1.1).

The parametrisation test-bed developed in this work solves the governing equations in a two-dimensional domain $(x, z) \in [0, d] \times [0, L]$, subject to no-slip, isothermal boundary conditions on the top and bottom plates,

$$\boldsymbol{u} = \boldsymbol{0}, \quad T = T_0 + \frac{\delta T}{2} \qquad \text{at } z = 0 \text{ and} \tag{3.1.4}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad T = T_0 - \frac{\delta T}{2} \qquad \text{at } z = d, \tag{3.1.5}$$

and periodic boundary conditions in the horizontal,

$$\boldsymbol{u}(x = 0) = \boldsymbol{u}(x = L) \quad \text{and} \quad T(x = 0) = T(x = L). \tag{3.1.6}$$

δT is the constant temperature difference between the plates.

3.2 Nondimensionalisation and scale analysis

It is helpful to nondimensionalise the governing equations (3.1.1)–(3.1.6); this is not only useful for numerical work but also gives insight into the different flow regimes that are possible. A range of nondimensionalisations are used in fluid dynamics literature; I adopt a common one (see, e.g., Grötzbach 1983; Ouertatani et al. 2008; Stevens, Verzicco, and Lohse 2010) which is suitable for the turbulent convective regime.

For low-viscosity, turbulent flow, a suitable time scale is the *free-fall time* t_0, which is the time for a fluid element with constant temperature $T = T_0 - \delta T$ to fall from the top plate to the bottom plate under the influence of buoyancy $(-g\alpha\delta T)$ alone. It is simple to show that

$$t_0 \sim \left(\frac{d}{g\alpha\delta T}\right)^{1/2},$$

ignoring a factor of $\sqrt{2}$. The obvious length and temperature scales are the plate separation d and temperature difference δT, respectively.

Making the substitutions $p'/\rho_0 \to \pi$ and $T - T_0 \to \theta$ in (3.1.1)–(3.1.6) and expressing all variables in units of t_0, d and δT leads to the dimensionless equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla}\pi + \left(\frac{\mathrm{Pr}}{\mathrm{Ra}}\right)^{1/2} \nabla^2 \boldsymbol{u} + \theta \hat{\boldsymbol{z}}, \tag{3.2.1}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\theta = (\mathrm{Ra}\,\mathrm{Pr})^{-1/2}\, \nabla^2\theta, \quad \text{and} \tag{3.2.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0, \tag{3.2.3}$$

with boundary conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \theta = +\frac{1}{2} \qquad \text{at } z = 0, \tag{3.2.4}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \theta = -\frac{1}{2} \qquad \text{at } z = 1, \tag{3.2.5}$$

$$\boldsymbol{u}(x = 0) = \boldsymbol{u}(x = \Gamma) \quad \text{and} \quad \theta(x = 0) = \theta(x = \Gamma). \tag{3.2.6}$$

There are three dimensionless parameters: the aspect ratio of the domain

$$\Gamma \equiv \frac{L}{d},$$

the *Prandtl number*

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa}$$

which measures the relative importance of viscosity (momentum diffusivity) and thermal diffusivity, and the *Rayleigh number*

$$\mathrm{Ra} \equiv \frac{g\alpha d^3 \delta T}{\kappa\nu}.$$

The Rayleigh number can be interpreted as the ratio of the time scale for thermal transport by conduction to the time scale for thermal transport by convection. It determines the importance of diffusion for the evolution of $\boldsymbol{u}$ and θ; inspection of (3.2.1) and (3.2.2) indicates that low Ra implies strong diffusion and high Ra weak diffusion. Detailed theoretical analysis of the governing equations (see, e.g., Chandrasekhar (1961) and the seminal work by Lord Rayleigh (1916)) reveals that there exists a critical Rayleigh number (dependent on boundary conditions but of order 10^3), below which the equations have a stable equilibrium with the fluid at rest and a linear conductive temperature profile. Above the critical value, the equilibrium is unstable and small perturbations lead to the formation of a regular series of steady, rotating convection cells. If the Rayleigh number is increased much further (Le Quéré (1991) cites $\mathrm{Ra} \approx 2 \times 10^8$), the solution becomes unsteady and increasingly turbulent. This work is concerned with the turbulent regime, since Rayleigh numbers for atmospheric deep moist convection can be as large as 10^{22} (Chillà and Schumacher 2012).

3.3 Model configuration

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