Calculation and analysis of subgrid tendencies in a coarse model of Rayleigh-Bénard convection

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1. We can estimate small-scale effects using high-resolution data

Original system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{F}(z)$$

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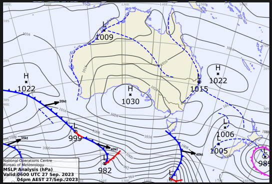
$z \to (\boldsymbol{x}, \boldsymbol{y})$



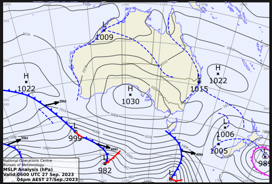
Large scales $\boldsymbol{x}$

Small scales $\boldsymbol{y}$

Original system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{F}(z)$$

$z \to (\boldsymbol{x}, \boldsymbol{y})$



Large scales $\boldsymbol{x}$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y})$$

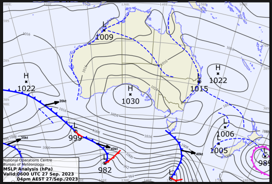
Small scales $\boldsymbol{y}$

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \dots$$

Original system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \boldsymbol{F}(z)$$

$z \to (\boldsymbol{x}, \boldsymbol{y})$



Large scales $\boldsymbol{x}$

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y})$$

Favourable or unfavourable conditions

Moisture, heat transports

Small scales $\boldsymbol{y}$

$$\frac{\mathrm{d}\boldsymbol{y}}{\mathrm{d}t} = \dots$$

But we can't solve

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y})!$$

We need a *parametrised* system

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \approx \boldsymbol{G}(\boldsymbol{x}) + \boldsymbol{P}(\boldsymbol{x}).$$

How can we estimate $\boldsymbol{C}(\boldsymbol{x}, \boldsymbol{y})$ without knowing $\boldsymbol{y}$?

The decomposition $z \to (x, y)$ is achieved via

$$x = \langle z \rangle, \qquad y = z - \langle z \rangle.$$

On the one hand,

$$\left\langle \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle = \langle F(z) \rangle.$$

On the other hand,

$$\left\langle \frac{\mathrm{d}z}{\mathrm{d}t} \right\rangle = \frac{\mathrm{d}\langle z \rangle}{\mathrm{d}t} = G(\langle z \rangle) + C(\langle z \rangle, y).$$

This means that

$$C(\langle z \rangle, y) = \langle F(z) \rangle - G(\langle z \rangle).$$

We can calculate the coupling using high-resolution data.

2. Putting it into practice: Rayleigh-Bénard convection

2D Rayleigh-Bénard convection:

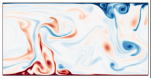
$$\frac{\mathrm{D}}{\mathrm{D}t}\begin{pmatrix} u \\ w \end{pmatrix} = -\boldsymbol{\nabla}\pi + \nu\nabla^2\begin{pmatrix} u \\ w \end{pmatrix} + \theta\hat{\boldsymbol{z}}$$

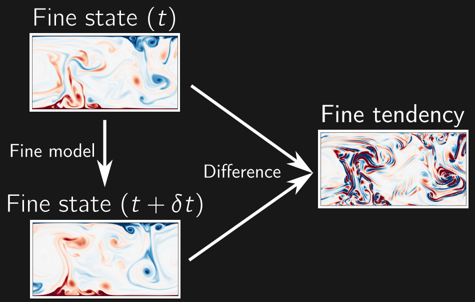
$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \kappa\nabla^2\theta$$

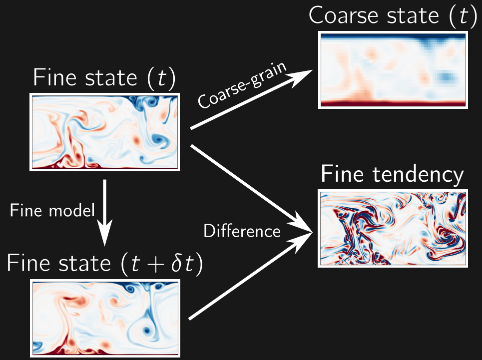
$$\boldsymbol{\nabla}\cdot\begin{pmatrix} u \\ w \end{pmatrix} = 0$$

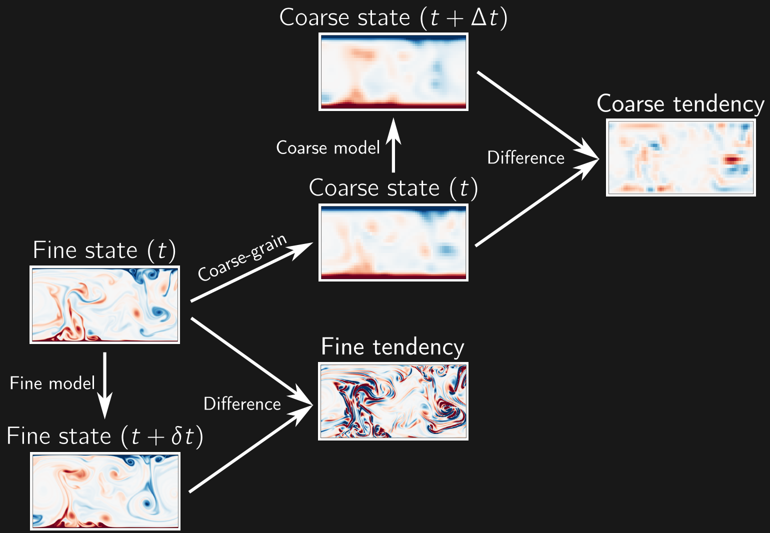
- No-slip, isothermal top/bottom boundaries
- Periodic lateral boundaries
- Solved using Dedalus (pseudospectral code in Python)

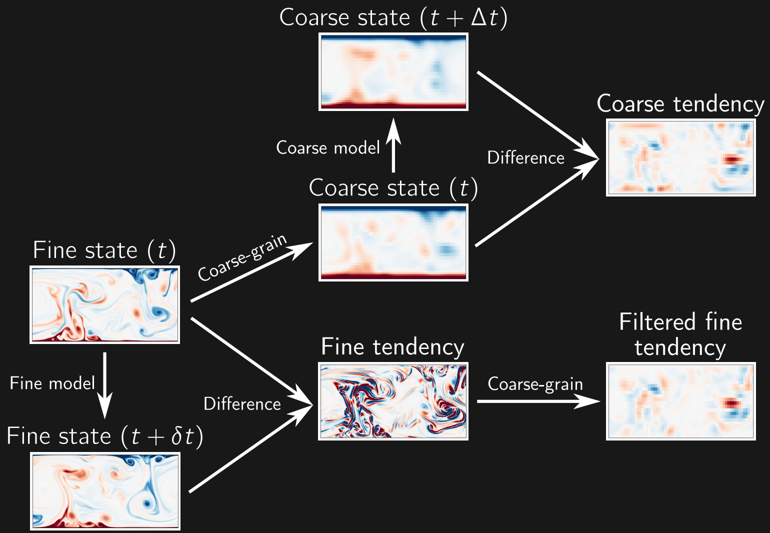
Fine state (t)

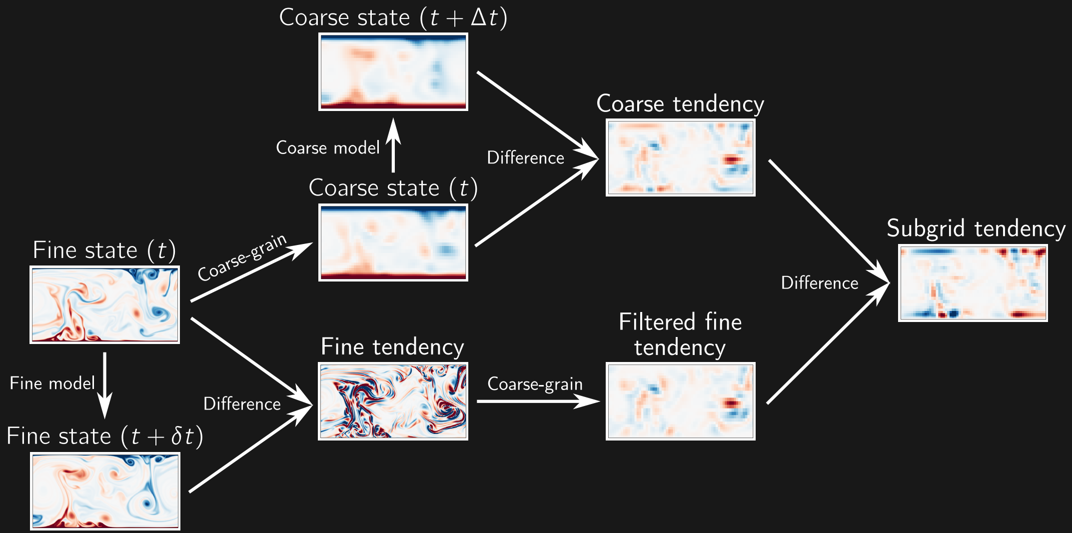



Fine state (t)
Fine model
Difference
Fine tendency
Fine state $(t + \delta t)$


Coarse state (t)
Fine state (t)
Coarse-grain
Fine tendency
Fine model
Difference
Fine state $(t + \delta t)$


Coarse state $(t + \Delta t)$
Coarse tendency
Coarse model
Difference
Coarse state (t)
Fine state (t)
Coarse-grain
Fine tendency
Fine model
Difference
Fine state $(t + \delta t)$


Coarse state $(t + \Delta t)$
Coarse model
Difference
Coarse tendency
Coarse state (t)
Fine state (t)
Coarse-grain
Fine model
Fine tendency
Filtered fine tendency
Difference
Coarse-grain
Fine state $(t + \delta t)$


Coarse state $(t+\Delta t)$
Coarse model
Difference
Coarse tendency
Coarse state (t)
Subgrid tendency
Fine state (t)
Coarse-grain
Difference
Filtered fine tendency
Fine tendency
Fine model
Coarse-grain
Difference
Fine state $(t+\delta t)$

3. Coarse-graining carefully

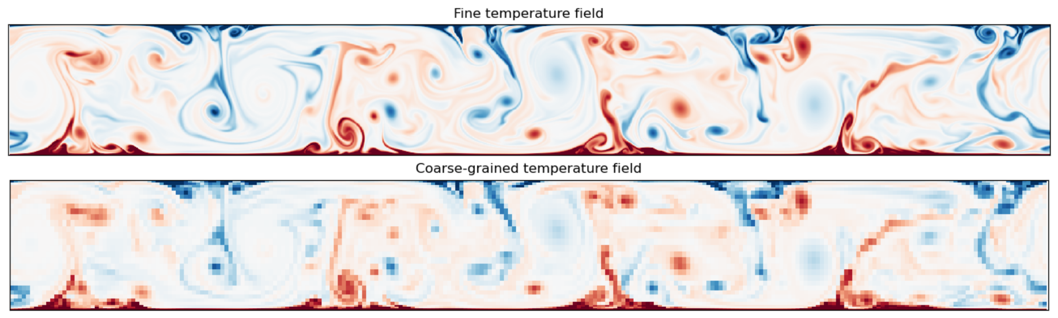
We need to map the state z of the full system to a coarse state $x = \langle z \rangle$ that the coarse model sees.

The definition of $\langle \cdot \rangle$ is up to us (!) and our choice can strongly affect the results.

The coarse-grained fields must:

- Be well-resolved on the coarse grid
- Respect physical constraints:
    - $u(z = 0, 1) = w(z = 0, 1) = 0$
    - $\theta(z = 0) = 1/2, \theta(z = 1) = -1/2$
    - $\nabla \cdot (u, w) = 0$

Obvious (but wrong) choice: average fine fields over each coarse grid-box.



- × Not well-resolved on coarse grid
- × Violates physical constraints

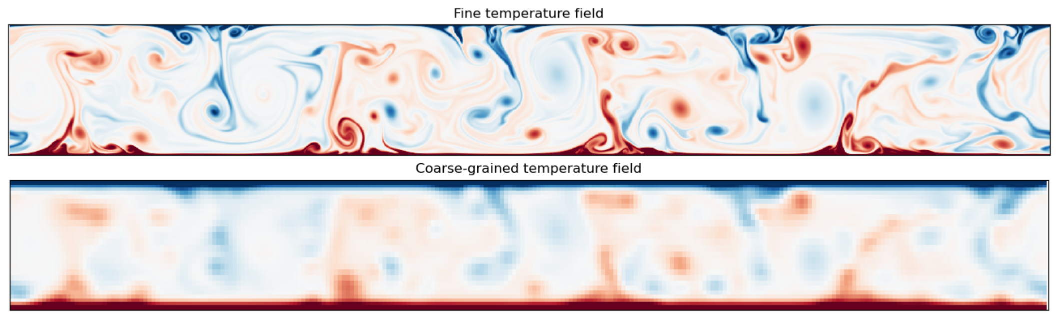
Solution: solve the diffusion equations

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ w \end{pmatrix} = -\boldsymbol{\nabla}\pi + \nu\nabla^2\begin{pmatrix} u \\ w \end{pmatrix}$$

$$\frac{\partial \theta}{\partial t} = \kappa\nabla^2\theta$$

$$\boldsymbol{\nabla}\cdot\begin{pmatrix} u \\ w \end{pmatrix} = 0$$

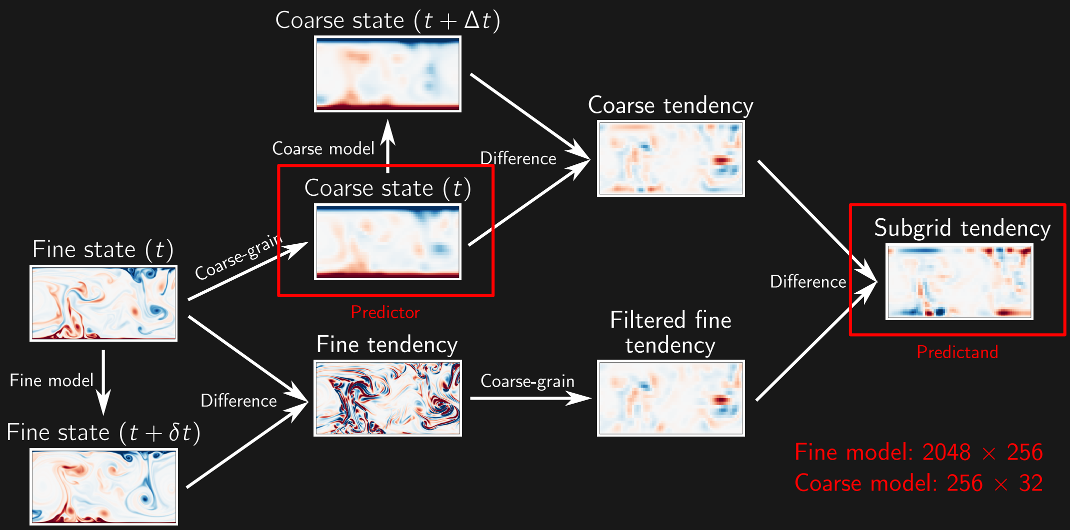
subject to the required boundary conditions.



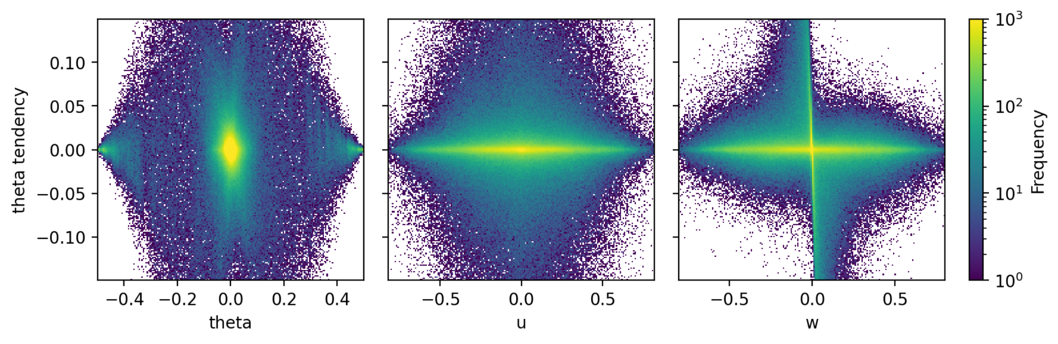
✓ Well-resolved on coarse grid

✓ Respects physical constraints

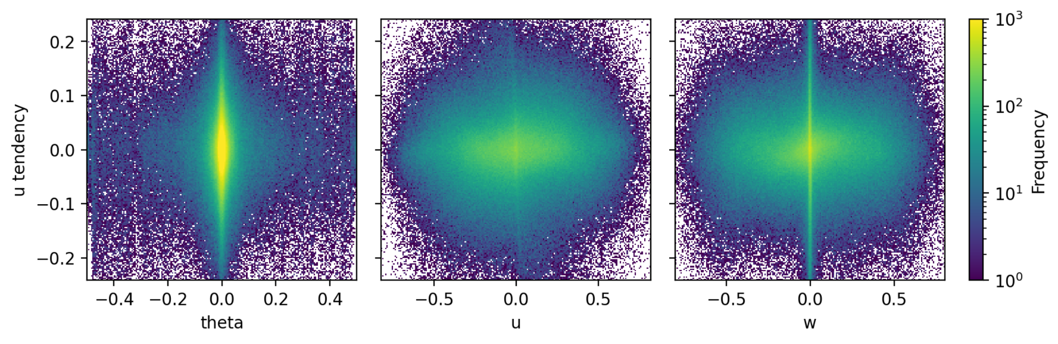
4. Can we predict the subgrid tendencies?


Coarse state $(t + \Delta t)$
Coarse model
Difference
Coarse tendency
Coarse state (t)
Coarse-grain
Fine state (t)
Predictor
Subgrid tendency
Difference
Fine tendency
Filtered fine tendency
Predictand
Fine model
Difference
Coarse-grain
Fine state $(t + \delta t)$
Fine model: 2048×256
Coarse model: 256×32

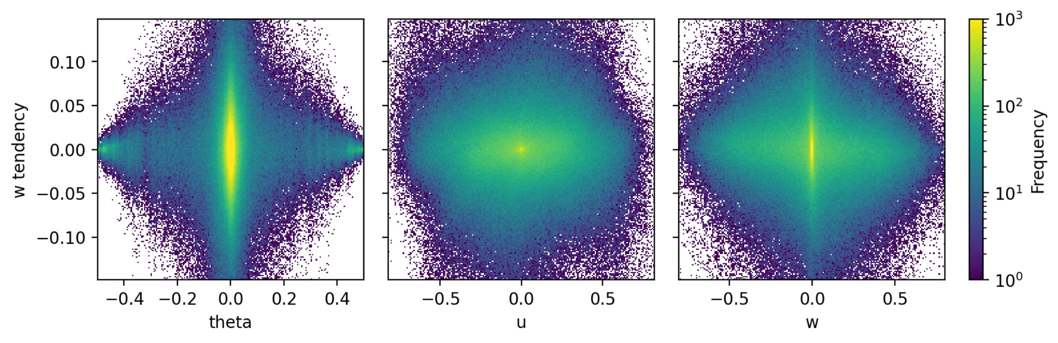
Subgrid temperature tendency



Subgrid horizontal velocity tendency



Subgrid vertical velocity tendency



Summary

1. In principle, one can quantify the effect of small-scale processes on the larger scales by coarse-graining high-resolution data.

2. We implemented this procedure in a 2D model of Rayleigh-Bénard convection.

3. The choice of coarse-graining method is important.

4. We found little correlation between subgrid tendencies and the coarse variables.

Next steps?