Time of flight refractometry of air, acrylic and water

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This report presents time-of-flight measurements of the speed of light and refractive index in air, poly(methyl methacrylate) (PMMA, commonly known as acrylic) and water using a simple teaching apparatus. The discrepancy between the results and values reported in literature is attributed to unavoidable misalignment of the apparatus, imperfect fabrication of the PMMA sample and water container, and drift in time measurements. An alternative method using refraction at the PMMA and water interfaces is proposed to eliminate these systematic errors.

INTRODUCTION

Measurements of the speed of light have a long history, with discussion of whether or not it was finite traced back to the ancient Greeks. Rømer (1676) is usually credited with making the first accurate measurements for his observation of variations in Io's eclipse durations behind Jupiter as the Earth approached and receded in its annual orbit [1].

More sophisticated experiments later increased the precision with which the speed of light was known, to the point that it could be given an exact value in SI units in a 1983 resolution of the seventeenth General Conference on Weights and Measures. Presently, the metre "is defined by taking the fixed numerical value of the speed of light in vacuum, c, to be 299 792 458 when expressed in the unit $\mathrm{m\,s^{-1}}$" [2].

The refractive index of a material is the factor by which it reduces the speed and wavelength of light relative to the speed and wavelength in vacuum. Refractometry, the measurement of refractive indices, has many applications in modern research, industry and medicine. These include concentration measurements of dissolved material in liquids (e.g., sugar in drinks), characterisation of glasses and plastics (e.g., for use in lenses) and analysis of biofluids (e.g., blood and urine) [3].

Modern Methods

Modern refractometry methods are numerous. Meeten [3] describes several, including:

Interferometry. An interferometer separates a laser into two beams, one of which travels through air and the other through the medium of interest, before recombining them to produce a pattern of interference fringes from which the phase difference and refractive index can be determined.

Deviation. A beam of light is passed at an angle through a flat sheet of the material with known thickness. The beam that emerges on the other side of the sheet is parallel to the incident beam but offset laterally due to refraction. The magnitude of the deviation can be related to the refractive index, thickness and angle of incidence using Snell's Law.

Critical angle (Abbe method). The incidence angle at which light, passing from the medium of interest into a medium of known lower refractive index, is refracted at 90° is found by measuring the reflectance or transmittance at the interface as a function of incidence angle. The refractive index is easily calculated using Snell's Law. This method is commonly used in commercial refractometers.

The Refractive Index of Air

Many experiments, including the one detailed here, are not conducted in vacuum and must therefore account for the refractive index of air, which is known to depend on wavelength, temperature, pressure and relative humidity. A commonly-used empirical formula was developed by Edlén in 1966 and later revised by Birch and Downs in 1994 to better match experimental results [4]. The revised formula calculates the refractive index in three steps: the first defines an intermediate quantity

$$n_{\mathrm{s}} = 1 + 10^{-8} \times \left(8342.54 + 2406147 \left[130 - \left(\sigma/\mathrm{\mu m^{-1}}\right)^2 \right]^{-1} + 15998 \left[38.9 - \left(\sigma/\mathrm{\mu m^{-1}}\right)^2 \right]^{-1} \right) \quad (1)$$

to account for dispersion (wavelength dependence); $\sigma = 1/\lambda$ is the wavenumber of the light in vacuum. The second step defines another intermediate quantity

$$n_{\mathrm{tp}} = 1 + \frac{(p/\mathrm{Pa})n_{\mathrm{s}}}{96095.43} \times \frac{\left[1 + 10^{-8}(0.601 - 0.00972T/{}^\circ\mathrm{C})p/\mathrm{Pa}\right]}{1 + 0.003661T/{}^\circ\mathrm{C}} \quad (2)$$

accounting for the air pressure p and temperature T. The final value is

$$n_{\mathrm{a}} = n_{\mathrm{tp}} - (f/\mathrm{Pa}) \times \left[3.7345 - 0.0401 \left(\sigma/\mathrm{\mu m^{-1}}\right)^2 \right] \times 10^{-10}, \quad (3)$$

where f is the partial pressure of water vapour.

The conditions measured in this experiment (using an inexpensive electronic weather station) were $T = (20.4 \pm 1.0)\,°\mathrm{C}$, $p = (1028 \pm 2)\,\mathrm{hPa}$ and relative humidity RH $= (47 \pm 10)\,\%$ with an estimated wavelength $\lambda = 630\,\mathrm{nm}$. When substituted into (1), (2) and (3), these yield

$$n_\mathrm{a} = 1.000275. \tag{4}$$

By comparison to experimental results, Birch and Downs estimated the 1σ uncertainty of the formula to be 10^{-8} [4], but given the uncertainty in T, p, RH and λ this is more likely to be on the order of 10^{-6} [5], which is nonetheless negligible in comparison to the uncertainties in the results presented in this report.

AIM

This experiment seeks to determine the speed of light and refractive index in air, poly(methyl methacrylate) (PMMA, or acrylic) and water.

METHODS

The experiment used a combined laser source and detector unit (PHYWE 11226-99), containing a red laser source whose intensity was modulated at 50 MHz. The laser passed along an optical bench to a reflector mounted at a variable position x, was reflected back to the unit and directed onto a photodiode by a retro-reflector [6], as shown in Figure 1. The modulation of the laser intensity created pulses in both the source signal and the intensity measured by the photodiode, allowing the time difference between them to be measured on a digital oscilloscope (Tektronix TBS-1072B) connected to the source/detector unit. An illustration of this time measurement is shown in the top panel of Figure 1.

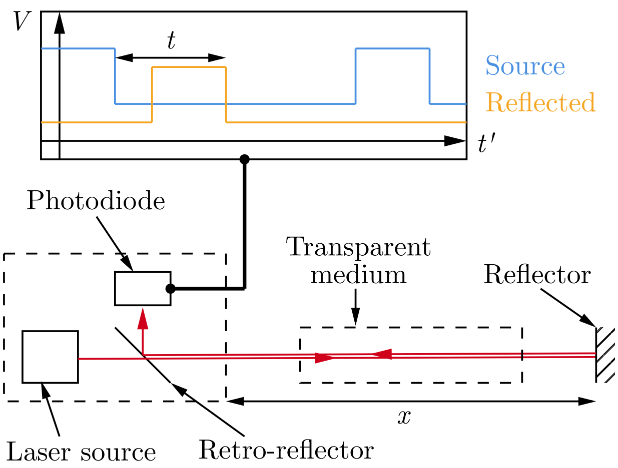

FIG. 1. Experimental setup, showing model oscilloscope trace (top).

The speed of light in air, c_a, was calculated by measuring the time difference t between the emitted and reflected beams as a function of the reflector's position x relative to a reference point and performing a linear regression using the simple relation

$$2x = c_\mathrm{a} t. \tag{5}$$

Two different approaches were compared for measuring the speed of light in PMMA and water.

Δx method. A cylinder of the transparent medium was placed in the path of the laser as shown in Figure 1 and the source/detector unit calibrated to show a zero time difference between the emitted and reflected beams. The cylinder was then removed, shortening the time taken by the light, and the reflector moved horizontally until the oscilloscope again showed a zero time difference, indicating that the initial and final optical path lengths were equal. Mathematically,

$$2n_\mathrm{a}x' = 2n_\mathrm{a}(x - l) + 2n_\mathrm{m}l,$$

where x and x' are the initial and final reflector positions, l is the length of the cylinder and n_m is the refractive index of the medium. Consequently, each measurement (x, x') gives a value

$$n_\mathrm{m} = \frac{n_\mathrm{a}(x' - x + l)}{l}. \tag{6}$$

An alternative is to calculate n_m from the y-intercept of the linear regression of x' on x, since it follows from (6) that

$$x' = x + l\left(\frac{n_\mathrm{m}}{n_\mathrm{a}} - 1\right). \tag{7}$$

Δt method. As before, the medium was placed in the optical path and the source/detector unit calibrated to show a zero time difference. The medium was removed and the resulting reduction in travel time Δt measured. Knowing that

$$\Delta t = 2l\left(\frac{1}{c_\mathrm{m}} - \frac{1}{c_\mathrm{a}}\right),$$

where c_m and c_a are the speeds of light in the medium and air respectively, it follows that

$$n_\mathrm{m} = \frac{c}{c_\mathrm{m}} = \frac{c\Delta t}{2l} + \frac{c}{c_\mathrm{a}}. \tag{8}$$

The PMMA sample was a cylinder of length $(0.491 \pm 0.001)\,\mathrm{m}$. The water was held in a plastic tube of length $(0.513 \pm 0.002)\,\mathrm{m}$ that was capped at both ends with clear plastic windows. Both lengths were measured using a tape measure.

RESULTS

Air

The speed of light and refractive index in air were determined according to (5) by performing an ordinary least-squares linear regression of t on $2x$ and taking the reciprocal of the result. This was necessary because ordinary least-squares regression only accounts for errors in the response variable; t was chosen as the response because its uncertainty (80 ps, typically 2%, from the manual cursor positioning on the oscilloscope) was more significant than that in $2x$ (5 mm, typically 0.3%, due to possible misalignment). The results were

$$c_a = (2.941 \pm 0.016) \times 10^8 \, \mathrm{m\,s^{-1}},$$
$$n_a = 1.019 \pm 0.005.$$

The uncertainty in c_a was obtained from the least-squares regression algorithm (`scipy.optimize.curve_fit` in Python), and this was propagated to n_a using the standard first-order Taylor expansion method. Despite realistic uncertainty estimates, these values are not consistent with the accepted value (4), deviating by 3.6σ or 1.9%.

Figure 2 shows the plot of $2x$ against t. The data points fall consistently below the dashed expected line with the magnitude of the deviation increasing with distance and time; given that the orange regression line fits the points to within their uncertainties, it is evident that a systematic error, rather than improper uncertainty estimates, are responsible for the discrepancy in speed and refractive index. Possible explanations are proposed in the Discussion.

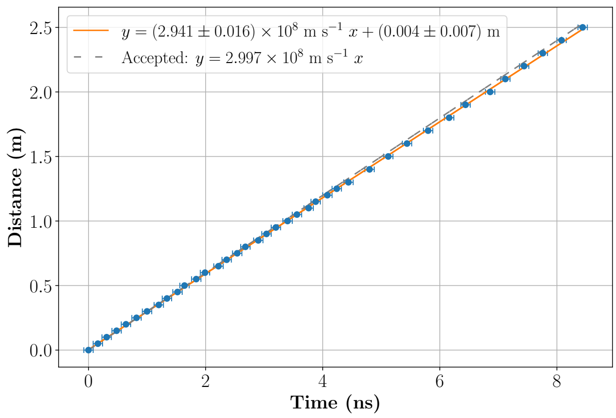

FIG. 2. Plot of $2x$ vs. t in air, with gradient c_a.

PMMA

The speed of light and refractive index in PMMA were first calculated two ways using the Δx measurements described previously: using (6) for each pair of measurements (x, x') and from the intercept of the linear regression of x' on x according to (7). The values were also calculated for each Δt measurement using (8). The results are shown in Figure 3; Δx measurements were performed on two different days but the result of intercept method on the second day are not shown because measurements were taken over a smaller range of x, creating an excessive uncertainty in the regression coefficients.

Four values for n_{PMMA} at $\lambda = 630\,\mathrm{nm}$ from experiments in existing literature were considered (data obtained from database [7]):

$$\begin{aligned}[8](20\,^\circ\mathrm{C}) &: 1.4888,\\ [9](\text{room temp., manufacturer 1}) &: 1.4831,\\ [9](\text{room temp., manufacturer 2}) &: 1.4909, \text{ and}\\ [10](20.1\,^\circ\mathrm{C}) &: 1.4909;\end{aligned}$$

these are shown as horizontal lines in Figure 3.

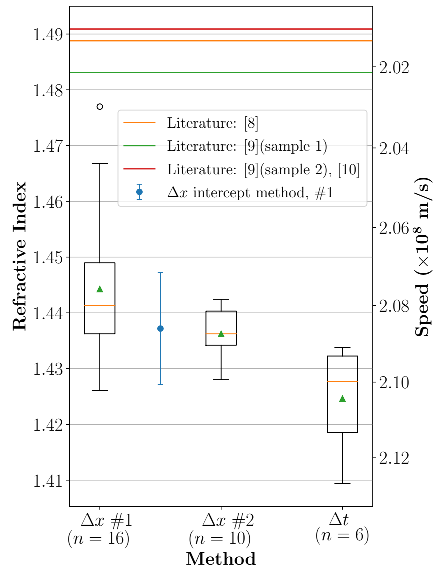

FIG. 3. Measured values of the speed of light in PMMA using the Δx and Δt methods. Green triangles are the mean values.

None of the measurements are consistent with the literature. Despite possible variations in manufacturing, the fact that the measurements are all consistent with each other (the uncertainties shown in Table I overlap) indicates again that the measurements were subject to a systematic error (see Discussion).

The consistency of the results with each other and the similar magnitudes of their uncertainties support the theoretical equivalence between the Δx and Δt methods, but does not suggest that one is more robust to systematic errors than the other.

Water

The same methods were applied for water as PMMA, with the results shown in Figure 4. The result of the intercept method on the second day is not shown for the same reason.

Four values for n_{H_2O} at $\lambda = 630\,\mathrm{nm}$ from existing literature were considered (data obtained from database [7]):

$$[11](20\,^{\circ}\mathrm{C}, \text{experimental}): 1.3320 \pm 0.0003,$$
$$[12](20\,^{\circ}\mathrm{C}, \text{experimental}): 1.3322,$$
$$[13](25\,^{\circ}\mathrm{C}, \text{review}): 1.3318, \text{ and}$$
$$[14](25\,^{\circ}\mathrm{C}, \text{review/theoretical}): 1.3316;$$

these are shown as horizontal lines in Figure 4.

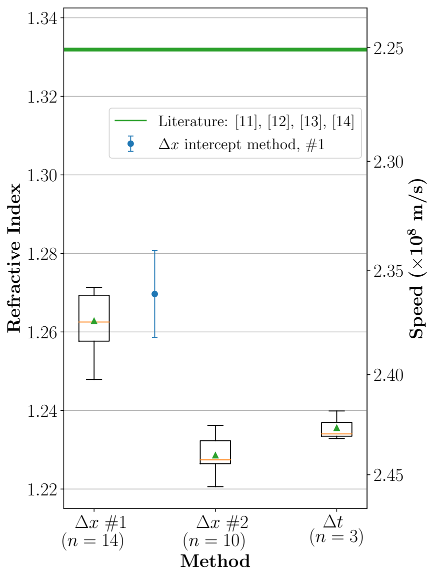

FIG. 4. Measured values of the speed of light in water using the Δx and Δt methods. Green triangles are the mean values.

As for PMMA, none of the results are consistent with the literature but the second Δx and Δt results are still consistent with each other. Similar conclusions can be drawn regarding the presence of systematic errors and the equivalence of the Δx and Δt methods.

Table I summarises all the numerical results of this report. The Δx and Δt methods in PMMA and water gave multiple values; those shown in the table are the averages, with the uncertainties taken to be the sample standard deviations.

Material (method)	n	c ($\times 10^8\,\mathrm{m\,s^{-1}}$)
Air	1.019 ± 0.005	2.941 ± 0.016
PMMA (Δx #1)	1.444 ± 0.013	2.076 ± 0.018
PMMA (Δx intercept #1)	1.437 ± 0.010	2.086 ± 0.015
PMMA (Δx #2)	1.436 ± 0.004	2.087 ± 0.006
PMMA (Δt)	1.425 ± 0.009	2.104 ± 0.014
Water (Δx #1)	1.263 ± 0.007	2.374 ± 0.013
Water (Δx intercept #1)	1.270 ± 0.011	2.361 ± 0.021
Water (Δx #2)	1.229 ± 0.004	2.440 ± 0.009
Water (Δt)	1.236 ± 0.003	2.426 ± 0.006

TABLE I. Summary of results.

DISCUSSION

Systematic error sources

It is notable that the speeds measured in all three media were lower than the literature values, and by similar amounts (on the order of 5%). This supports the previous conclusions that a systematic error was present. Several possibilities are proposed:

Misalignment. There was a significant amount of play in the sliding mechanism of the reflector, making it necessary to support it with strips of paper to render the reflected beam horizontal. Furthermore, despite careful efforts to visually align the PMMA and water cylinders by observing the position of the laser spot at each end, it was very difficult to ensure that they were exactly parallel to the beam with the equipment available. Both factors made it difficult to obtain an accurate measurement of the path length. A worst-case uncertainty estimate can be derived using Pythagoras' theorem: if the beam deviates laterally from the axis of the cylinder by 2 cm at one end and the axis deviates from the ideal line by 1 cm, the increase in path length over two traversals of the 50 cm cylinder is

$$2\left(\sqrt{(3\,\mathrm{cm})^2 + (50\,\mathrm{cm})^2} - 50\,\mathrm{cm}\right) = 1.8\,\mathrm{mm}.$$

Imperfect cylinder shape. It is likely that the ends of the acrylic and water cylinders were not exactly parallel to each other, causing refraction and increasing the actual path length even with good alignment. The fact that the observed time difference between the emitted and reflected beams varied by up to 40 ps as the cylinders were rotated supports this hypothesis. An estimate of the resultant change in apparent refractive index can be derived from 8:

$$\delta n = \frac{c \times 40\,\mathrm{ps}}{2l} \approx 0.012.$$

This is a significant change. The amount of variation was limited by making a mark on the side of each cylinder so that their orientations did not change from one measurement to another.

Phase drift. During data collection, a systematic drift over time was observed in the phase difference of the signals from the source/detector unit, and although the unit was calibrated at the same reference position before every measurement, the effect was likely not completely eliminated.

Improvements

The three aforementioned sources of systematic error are all connected to the fact that the experiment uses a *time of flight* technique; in one way or another, all the techniques used relied on a measurement of distance and time. For this reason, a method that relies on the refraction of light at an interface, such as the deviation and critical angle methods described in the Introduction, may be more suitable for PMMA and water.

Consider a simple measurement of the incidence and refraction angles at an air-material interface; Snell's law $n_\mathrm{a} \sin i = n \sin r$ implies an approximate uncertainty, using the first-order Taylor series method and taking $i = 45°$, $r = 30°$, $\delta i = \delta r = 0.1°$, of

$$\begin{aligned}\delta\left(\frac{n}{n_\mathrm{a}}\right) &= \delta\left(\frac{\sin i}{\sin r}\right) \\ &= \sqrt{\left(\frac{\cos i}{\sin r}\delta i\right)^2 + \left(\frac{\sin i}{\sin^2 r}\cos r \delta r\right)^2} \\ &\approx 0.005.\end{aligned}$$

This is similar to the uncertainties presented in this report, but the method is not subject to unreliable path length or time measurements and could be inexpensively realised in a teaching laboratory with a laser source, prism and a target mounted to a goniometer. An uncertainty of 0.1° in the refraction angle would correspond to a motion of 1.7 mm at the end of a 1 m goniometer arm and is hence quite achievable.

CONCLUSION

The speed of light and refractive index in air, PMMA and water were successfully measured, but none were in agreement with literature. The discrepancy is attributed to a combination of misalignment of the apparatus, imperfect shape of the PMMA and water cylinders, and time-dependent drift in readings from the source/detector unit. The unsatisfactory nature of the result despite careful attempts to mitigate these errors prompts the proposal of an alternative method for PMMA and water that measures refraction angles at their interfaces with air.

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SUPPLEMENTS

The code, raw data and laboratory notebook associated with this report are available in full at `https://github.com/tschanzer/speed_of_light`.

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