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Introduction to Image Processing:

Global Operations: Wavelet Transform

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Wavelet Transform

Module 7281, CPVR1: Introduction to Computer Vision

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Wavelet Transform

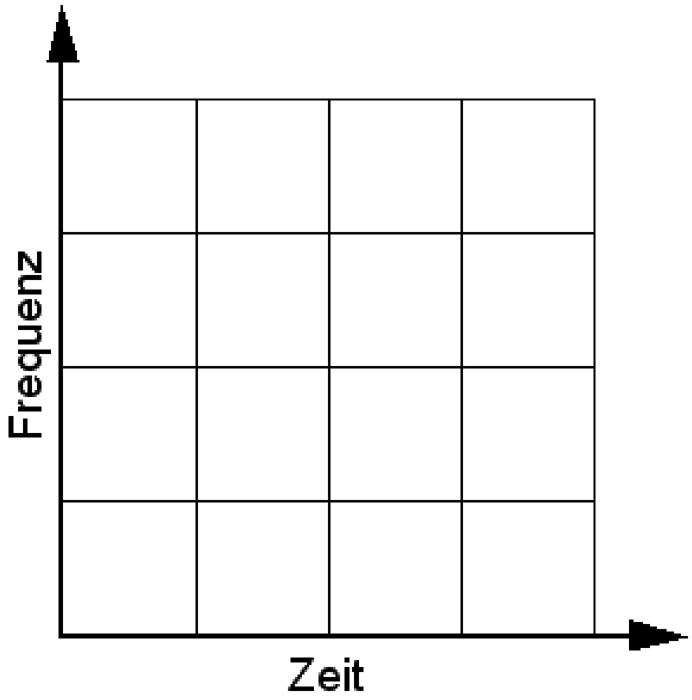
- The foundation of the **Wavelet theory** goes back to the 1930's.
- The today's wavelet transform was invented by the **French physicists Jean Morlet** and **Alex Grossmann** in the 1980's.
- A good introduction to wavelets is by **Barbara Burke**: „Wavelets: Die Mathematik der kleinen Wellen“ [Burke97].

  All images in this presentation are from this book.

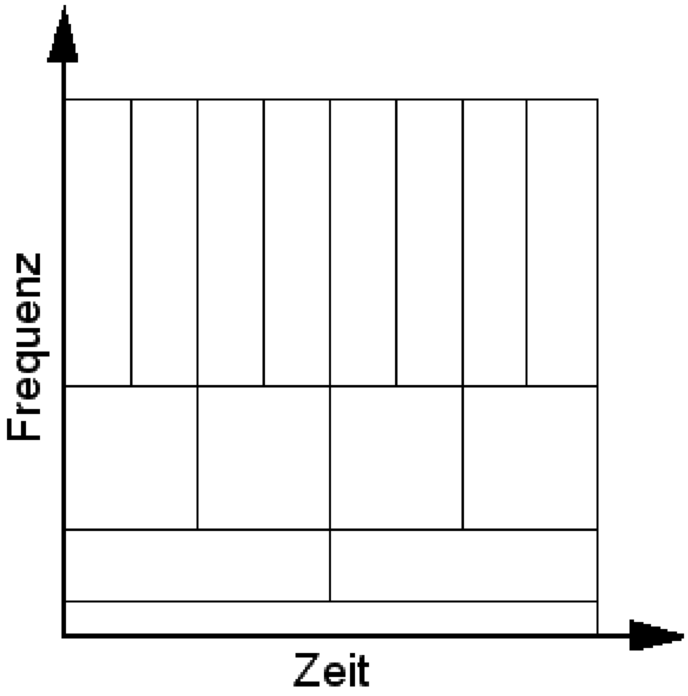
- The motivation behind wavelets where some problems with the FT.

Wavelet Transform vs. Windowowed Fourier Transform

- The discrete Windowed Fourier transform (WFT) remains static.
- The wavelet transform analyses the signal in multiple resolutions:

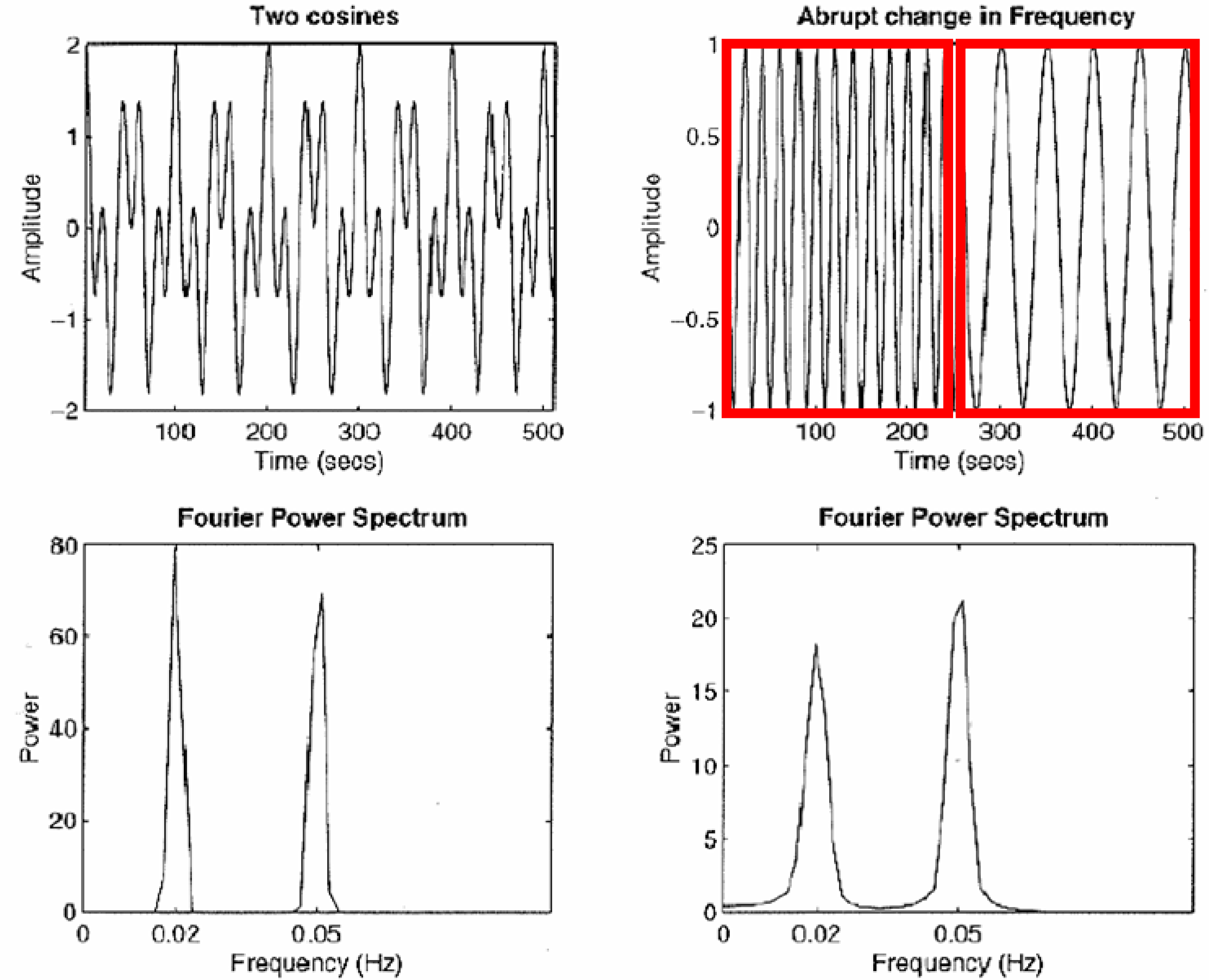


Static time-frequency resolution in FT



Multiple Resolution in WT

Wavelet Transform vs. Windowed Fourier Transform

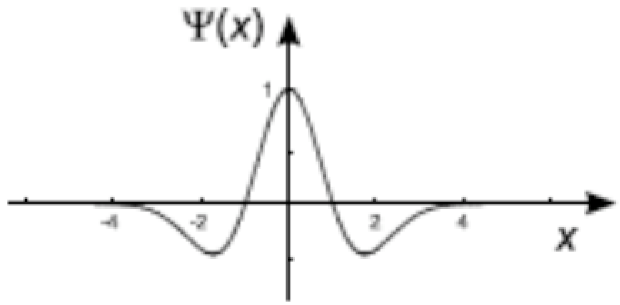


Wavelet Transform: Base or Mother Wavelets

- Wavelets are functions that are used for sampling.
- By sampling we calculate the cover ration with the underlying signal section.
- Wavelets can be scaled and shifted.
- The unscaled and unshifted wavelet is called **base** or **mother wavelet**.

Basiswavelet :

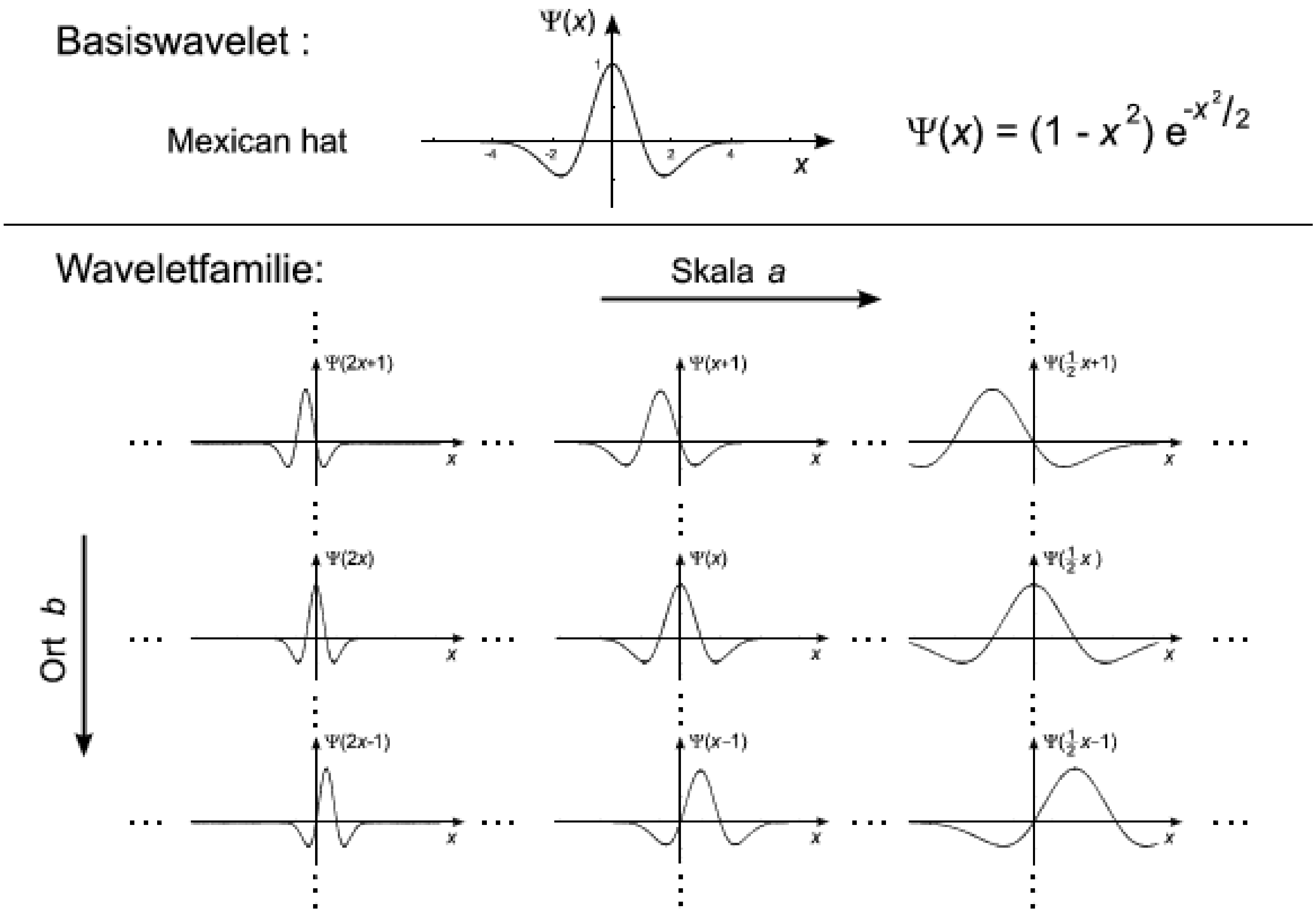
Mexican hat



$$\Psi(x) = (1 - x^2)\, e^{-x^2/2}$$

Wavelet Transform: Base Wavelet

- All versions of the base wavelet together are the **wavelet family**.



Wavelet Transform: Base Wavelet

Base wavelets Ψ must fullfill the following conditions:

- The area of the wavelet must be zero:

$$\int \psi(x)\,dx = 0$$

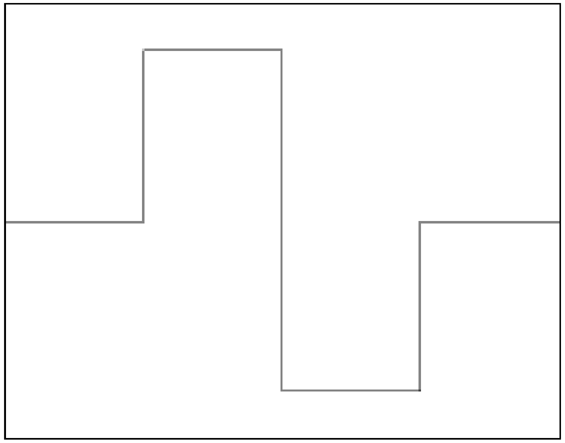
- A wavelet Ψ (a,b) of a wavelet family is defined as follows:

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$$

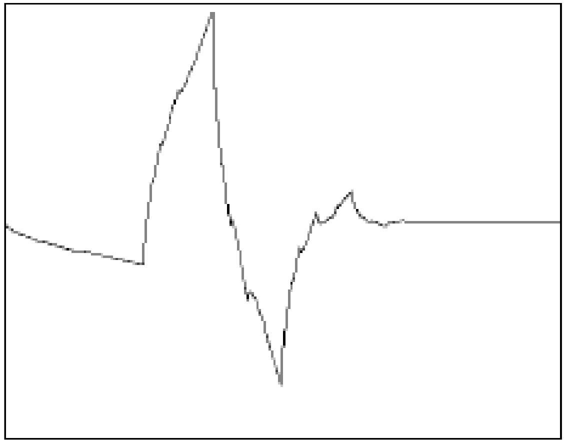
  where **a is the scale factor** and **b is the shift factor**.

Wavelet Transform: Base Wavelet

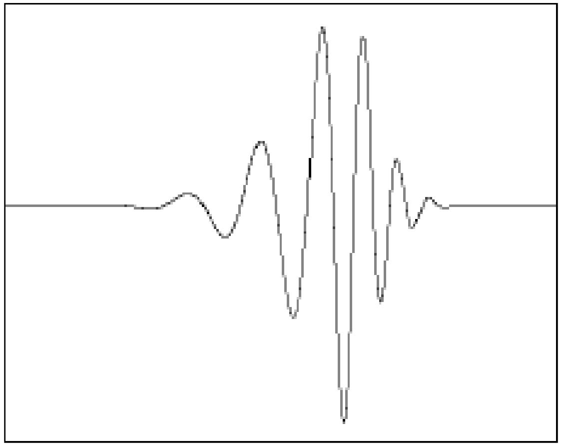
Some well known base wavelets:



Haar wavelet



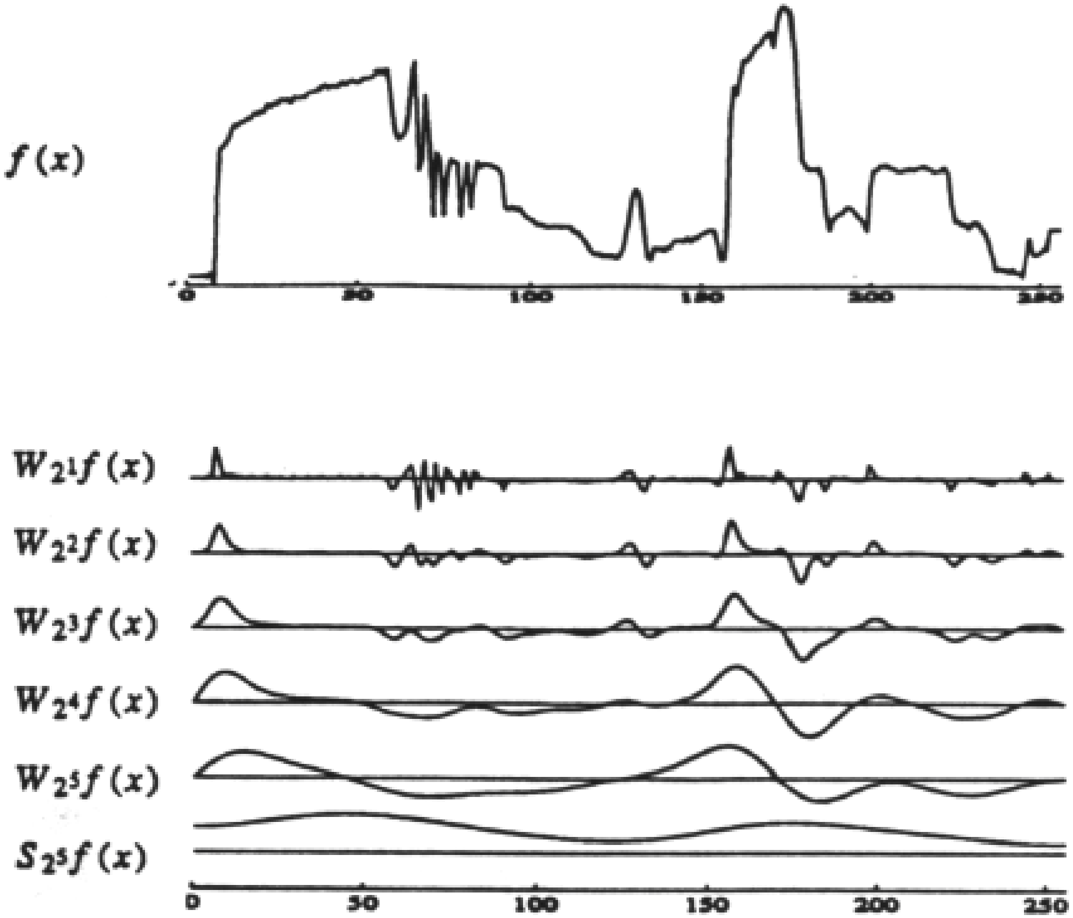
Daubechies 4 wavelet



Daubechies 20 wavelet

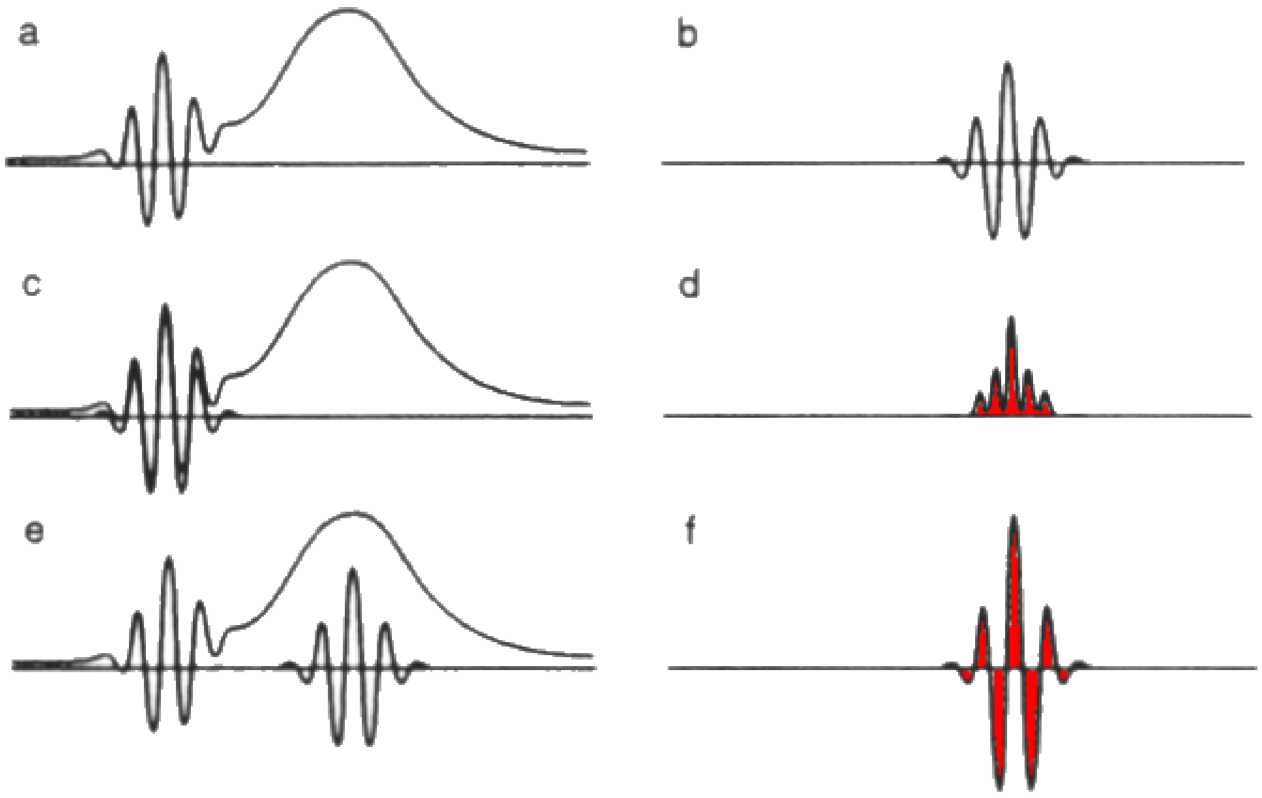
Wavelet Transform: Continuous WT

- By sampling the signal with scaled wavelets we get the wavelet transform.
- It corresponds to the cover ratio of the wavelets with the signal.
- The reconstruction is done as in the FT by summing up all wavelet coefficients.



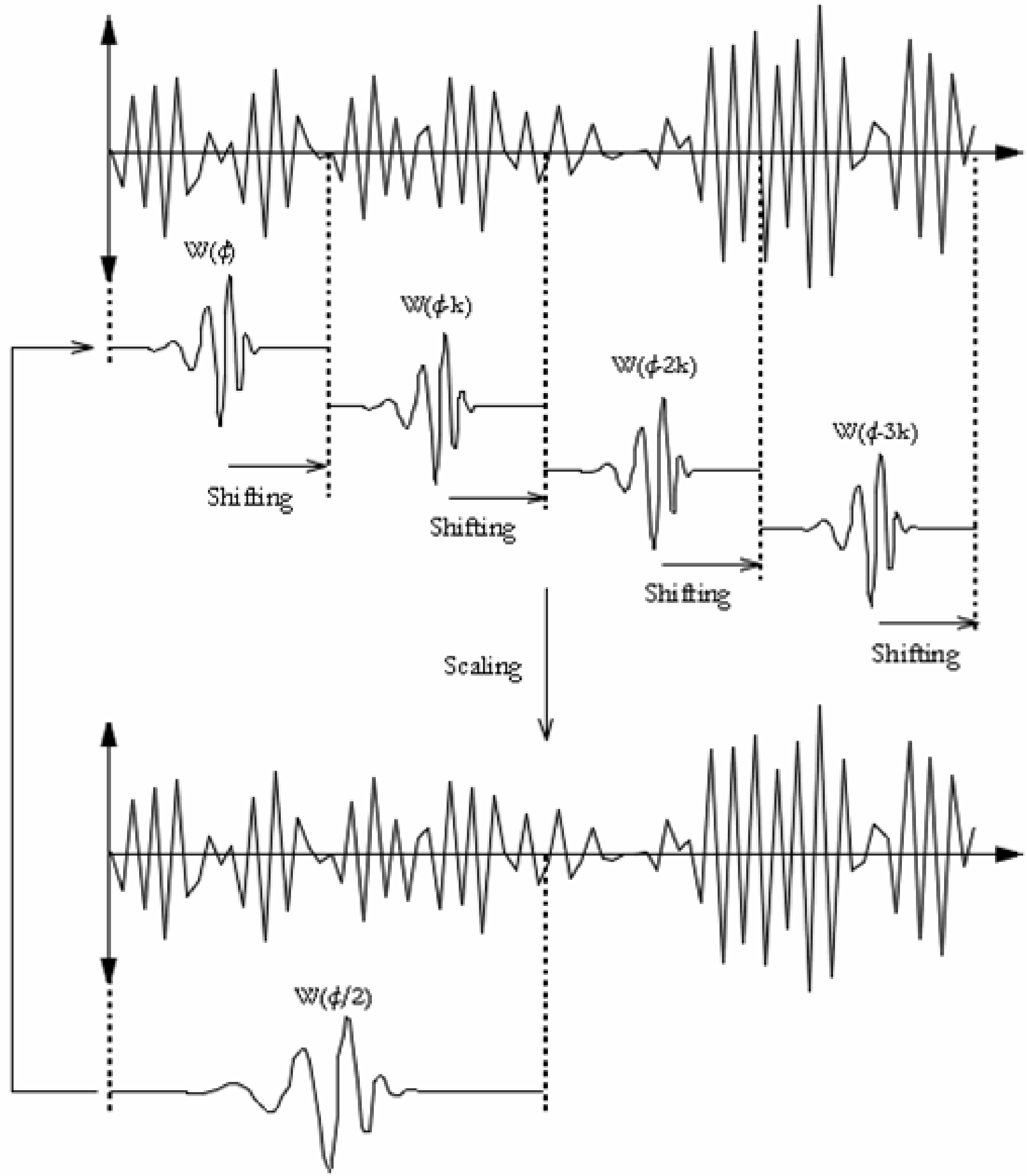
Wavelet Transform: Continuous WT

- How do we calculate the cover ratio of the wavelet with the signal?
- You multiply the wavelet function with the signal section.
- We calculate the area of the resulting function with integration.
- This value is called **Wavelet coefficient**.



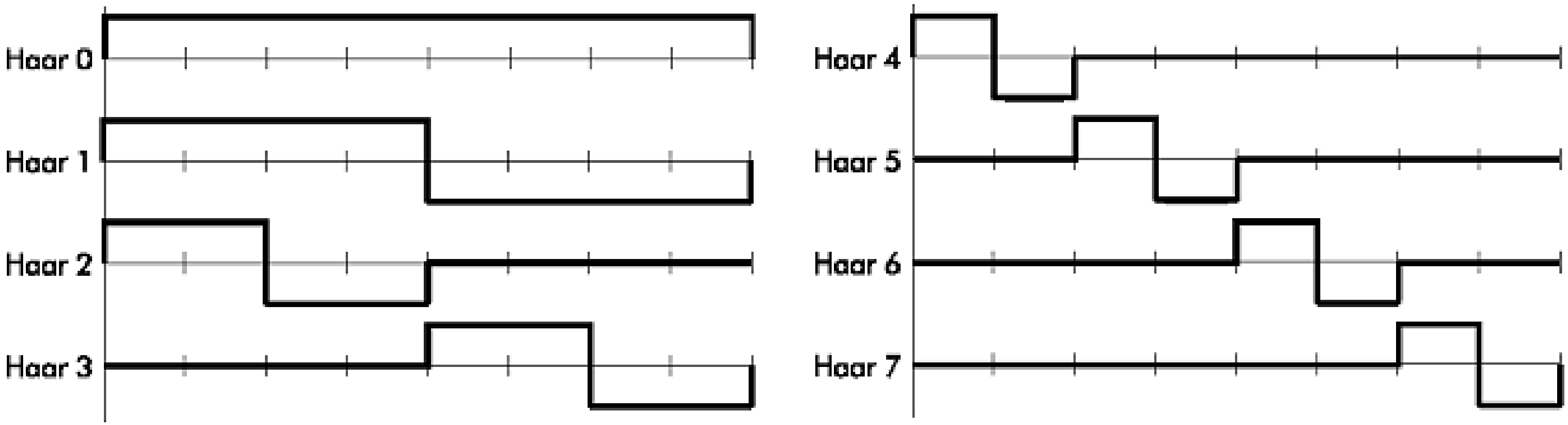
Wavelet Transform: Discrete WT

- The **discrete WT (DWT)** samples the signal in **discrete step distances**.
- The break through came in 1986 with the **Fast Wavelet Transform (FWT)** from *Stéphane Mallat* und *Yves Meyer*



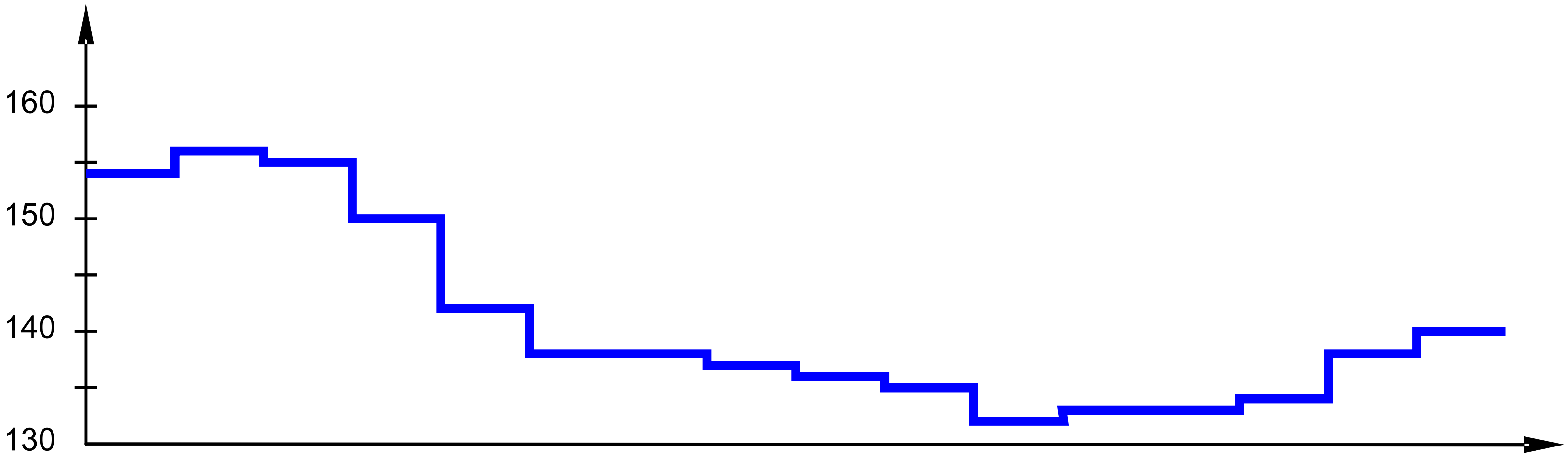
Wavelet Transform: 1D Haar Transform

- The simplest Wavelet transform is the ***Haar transform***.
- It is a simple and hard function.
- Through scaling and shifting we get the Haar Wavelet family:



Wavelet Transform: 1D Haar Transform

- We build the average of neigbouring cells (approximation).
- In the lines in between we write the difference (detail)



Signal	**154**	**156**	**155**	**150**	**142**	**138**	**138**	**137**	**136**	**135**	**133**	**134**	**134**	**135**	**138**	**140**

Wavelet Transform: 1D Haar Transform

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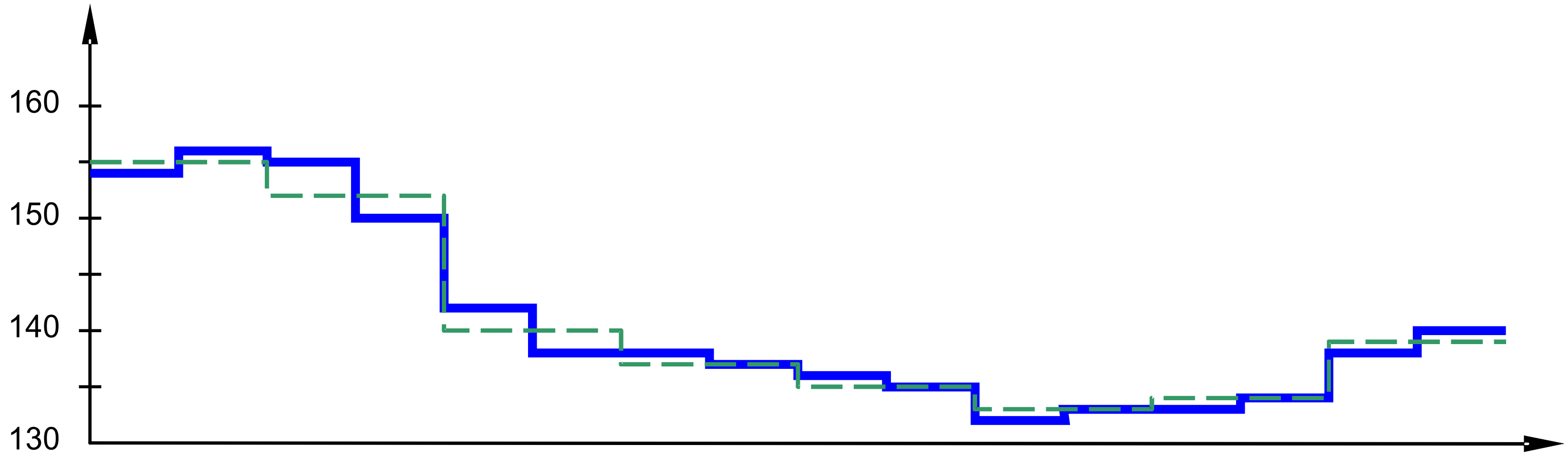

Signal	154	156	155	150	142	138	138	137	136	135	133	134	134	135	138	140
Detail 1	-1	1	2.5	-2.5	2	-2	0.5	-0.5	0.5	-0.5	-0.5	0.5	-0.5	0.5	-1	1
Approximation 1	155		152.5		140		137.5		135.5		133.5		134.5		139	

Wavelet Transform: 1D Haar Transform

- We build the average of neigbouring cells (approximation).
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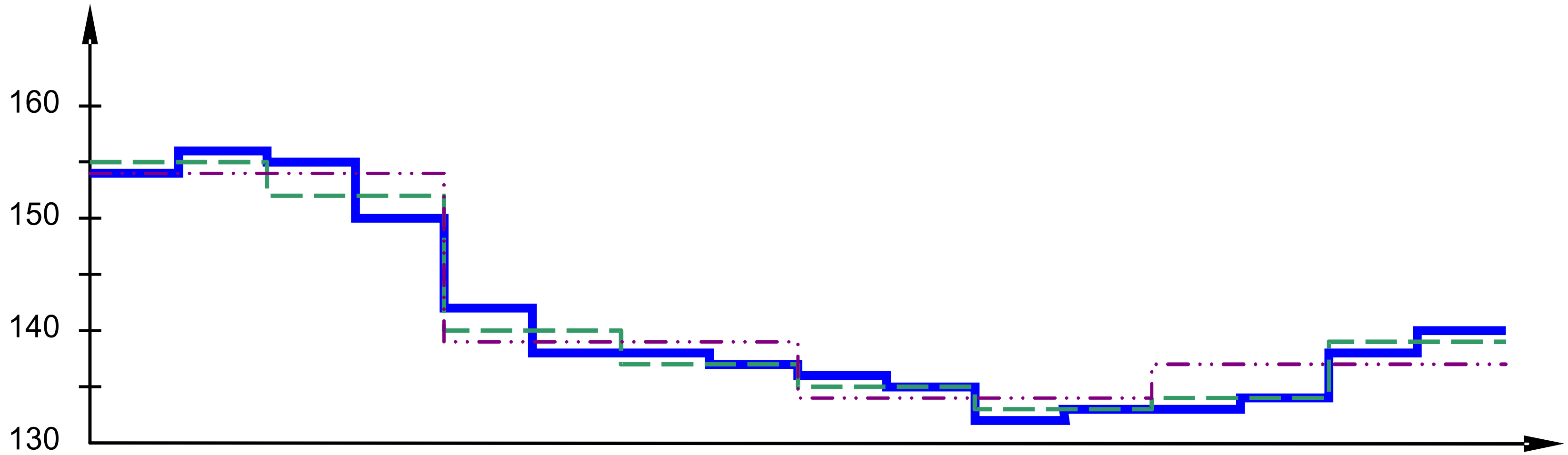
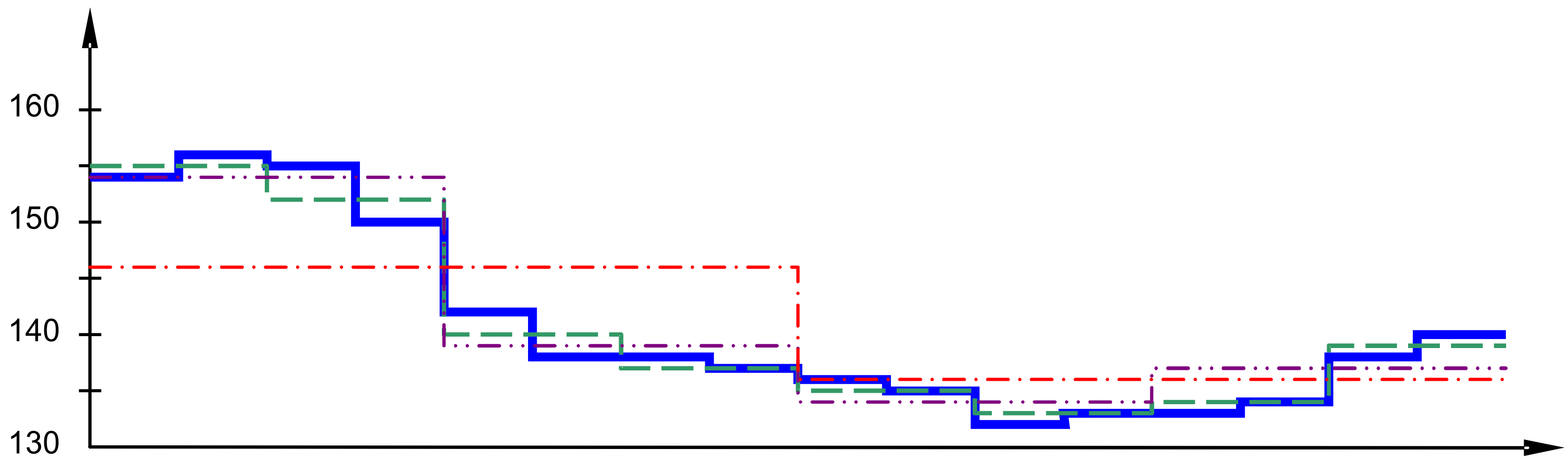


Signal	**154**	**156**	**155**	**150**	**142**	**138**	**138**	**137**	**136**	**135**	**133**	**134**	**134**	**135**	**138**	**140**
Detail 1	-1	1	2.5	-2.5	2	-2	0.5	-0.5	0.5	-0.5	-0.5	0.5	-0.5	0.5	-1	1
Approximation 1	**155**		**152.5**		**140**		**137.5**		**135.5**		**133.5**		**134.5**		**139**	
Detail 2	1.25		-1.25		1.25		-1.25		1		-1		-2.25		2.25	
Approximation 2	**153.75**				**138.75**				**134.5**				**136.75**			

Wavelet Transform: 1D Haar Transform

- We build the average of neigbouring cells (approximation).
- In the lines in between we write the difference (detail)



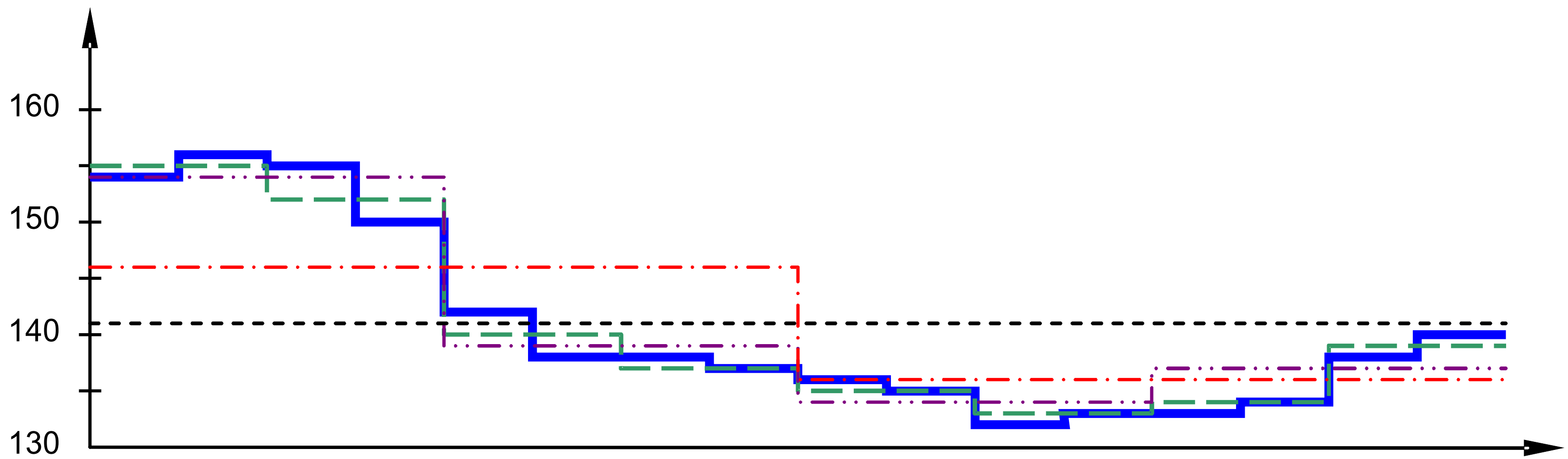
Signal	154	156	155	150	142	138	138	137	136	135	133	134	134	135	138	140
Detail 1	-1	1	2.5	-2.5	2	-2	0.5	-0.5	0.5	-0.5	-0.5	0.5	-0.5	0.5	-1	1
Approximation 1	**155**		**152.5**		**140**		**137.5**		**135.5**		**133.5**		**134.5**		**139**	
Detail 2	1.25		-1.25		1.25		-1.25		1		-1		-2.25		2.25	
Approximation 2	**153.75**				**138.75**				**134.5**				**136.75**			
Detail 3	7.5				-7.5				-1.125				1.125			
Approximation 3	**146.25**								**135.625**							

Wavelet Transform: 1D Haar Transform

- We 1 average value and 15 detail coefficiants:

  140.9375, 5.3125, 7.5, -1.125, 1.25, 1,25, 1, -2.25, -1, 2.5, 2, 0.5, 0.5, -0.5, -0.5, -1

- Because the details are symmetric, we store only one detail value:



Signal	154	156	155	150	142	138	138	137	136	135	133	134	134	135	138	140
Detail 1	-1	1	2.5	-2.5	2	-2	0.5	-0.5	0.5	-0.5	-0.5	0.5	-0.5	0.5	-1	1
Approximation 1	155		152.5		140		137.5		135.5		133.5		134.5		139	
Detail 2	1.25		-1.25		1.25		-1.25		1		-1		-2.25		2.25	
Approximation 2	153.75				138.75				134.5				136.75			
Detail 3	7.5				-7.5				-1.125				1.125			
Approximation 3	146.25								135.625							
Detail 4	5.3125								-5.3125							
Approximation 4	140.9375															

Wavelet Transform: 1D Haar Transform

- The forward transform is called **analysis**.
- The **approximation** and **details** can also be built with **filter masks**:
- This is a **1D convolution** (2 multiplies & 1 add):

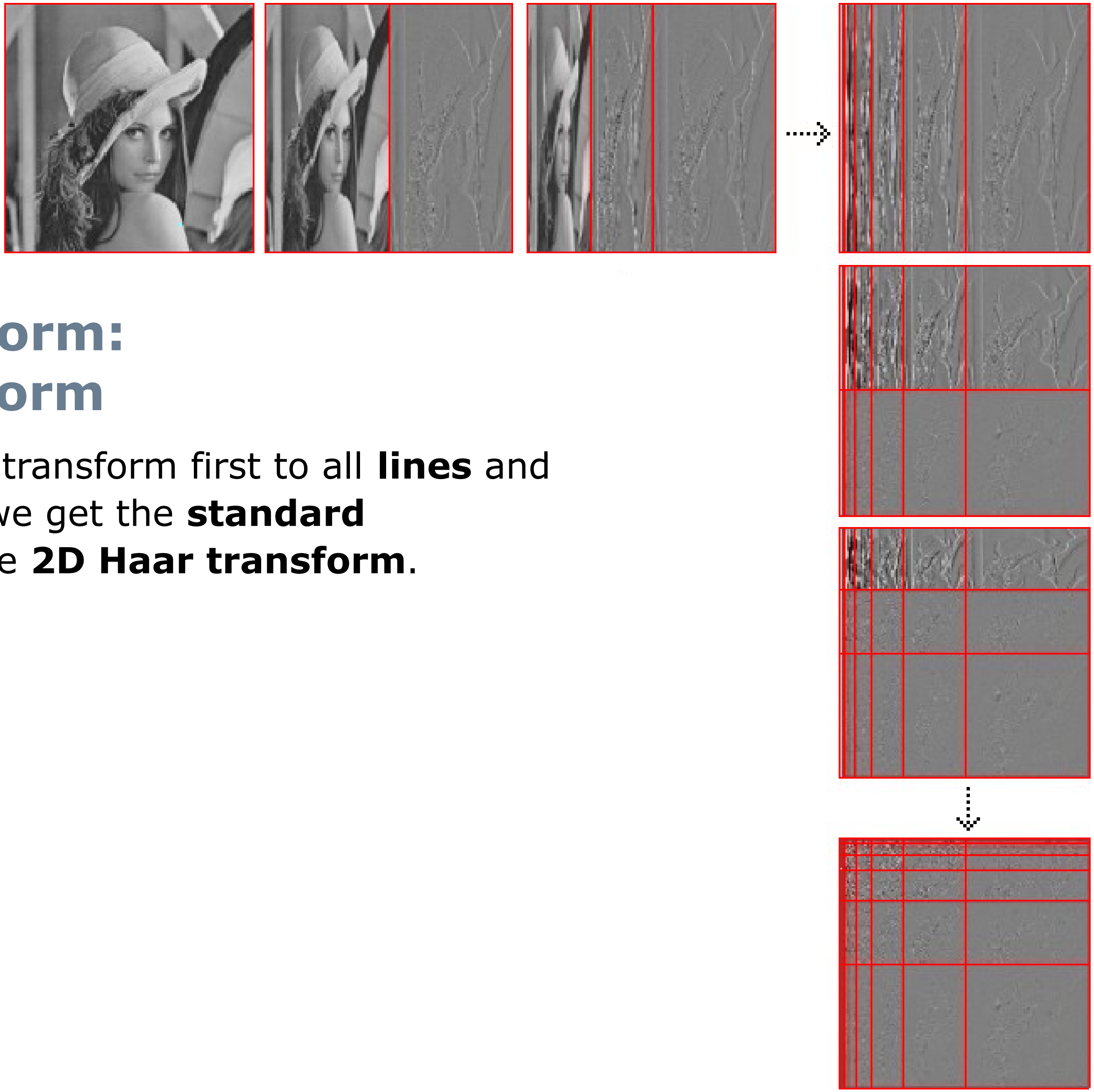
$$h_0 = \left[\frac{1}{2}, \frac{1}{2}\right]$$ low pass filter (approximation)

$$h_1 = \left[\frac{1}{2}, -\frac{1}{2}\right]$$ high pass filter (detail)

- The **reconstruction** called **synthesis** has the following filter masks:

$$f_0 = [1, 1]$$ left synthesis filter

$$f_1 = [1, -1]$$ right synthesis filter

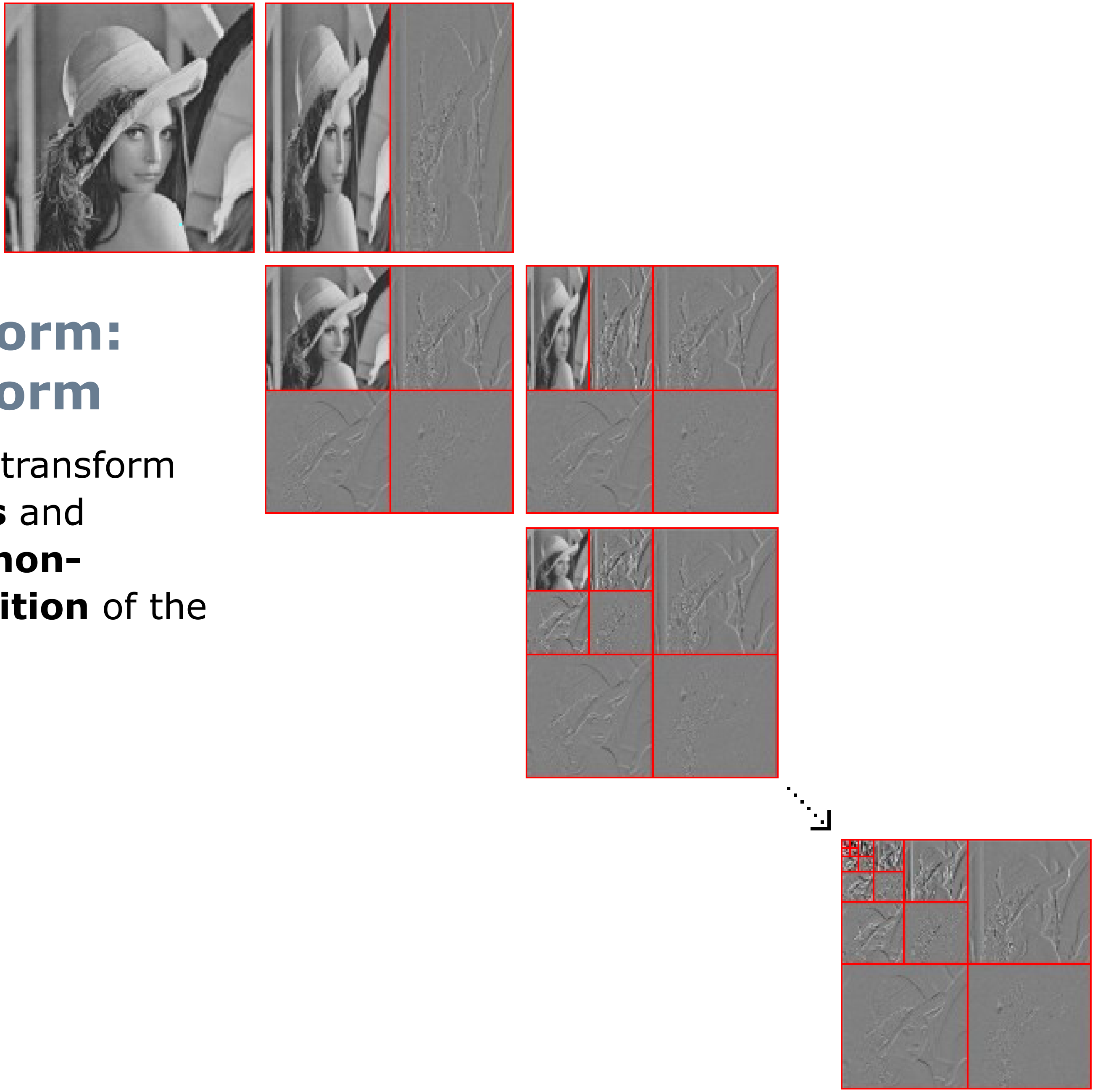


Wavelet Transform: 2D Haar Transform

- If you apply the Haar transform first to all **lines** and then to all **columns** we get the **standard decomposition** of the **2D Haar transform**.

Wavelet Transform: 2D Haar Transform

- If you apply the Haar transform alternately to all **lines** and **columns** we get the **non-standard decomposition** of the **2D Haar transform**.



Wavelet Transform: 2D Haar Transform

- Image transformed with this hard Haar transform tend to get block artefacts and to be noisy.
- The normalized Haar transform is better.

$$h_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$ low pass filter (approximation)

$$h_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$ high pass filter (detail)

The reconstruction has the following filter masks:

$$f_0 = \left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$ left synthesis filter

$$f_1 = \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$$ right synthesis filter

Wavelet Transform: Filter bank

- Other wavelet transforms have wider filter masks.
- We always need a low pass, a high pass and two synthesis filters.
- This set of filters is also called a **filter bank**.
- There is a specific relation between the filters.
- We therefore only need to define the low pass filter:

$$\begin{tikzcd}
\mathbf{h_0 = [\,a, b, c, d]} \arrow[r, "\text{order flip}"] \arrow[d, "\text{alternating flip}"'] \arrow[dr] & \mathbf{f_0 = [\,d, c, b, a]} \arrow[dl, "\text{alternating signs}"] \\
\mathbf{h_1 = [\,d, -c, b, -a]} & \mathbf{f_1 = [-a, b, -c, d]}
\end{tikzcd}$$

Wavelet Transform: Filter bank

- **Normalized Haar (filter length 2)**

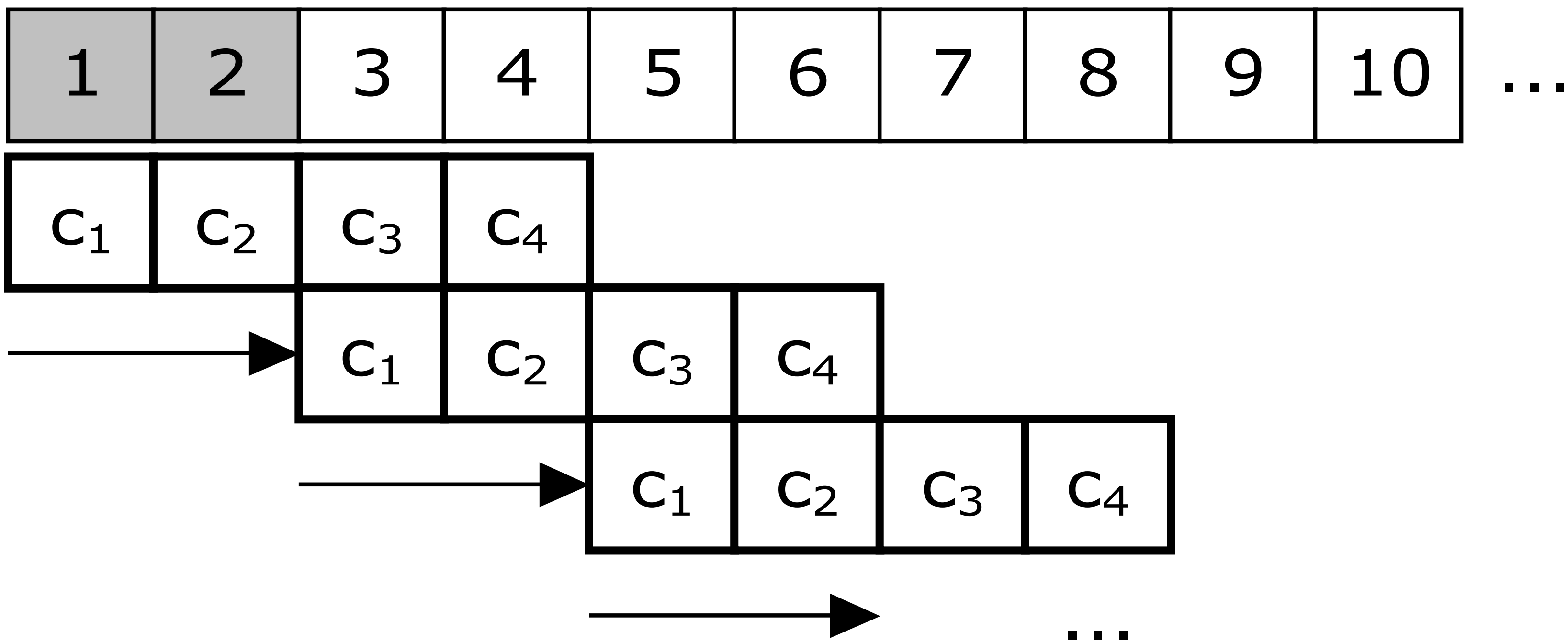
```
h0 = [ 0.70710678118655,  0.70710678118655]
h1 = [ 0.70710678118655, -0.70710678118655]
f0 = [ 0.70710678118655,  0.70710678118655]
f1 = [ 0.70710678118655,  0.70710678118655]
```

- **Daubechies 4: (filter length 8)**

```
h0 = [ 0.23037781330886,  0.71484657055254,  0.63088076792959, -0.02798376941698,
      -0.18703481171888,  0.03084138183599,  0.03288301166698, -0.010597401785]
h1 = [-0.010597401785,   -0.03288301166698,  0.03084138183599,  0.18703481171888,
       -0.02798376941698, -0.63088076792959,  0.71484657055254, -0.23037781330886]
f0 = [-0.010597401785,    0.03288301166698,  0.03084138183599, -0.18703481171888,
       -0.02798376941698,  0.63088076792959,  0.71484657055254,  0.23037781330886]
f1 = [-0.23037781330886,  0.71484657055254, -0.63088076792959, -0.02798376941698,
        0.18703481171888,  0.03084138183599, -0.03288301166698, -0.010597401785]
```

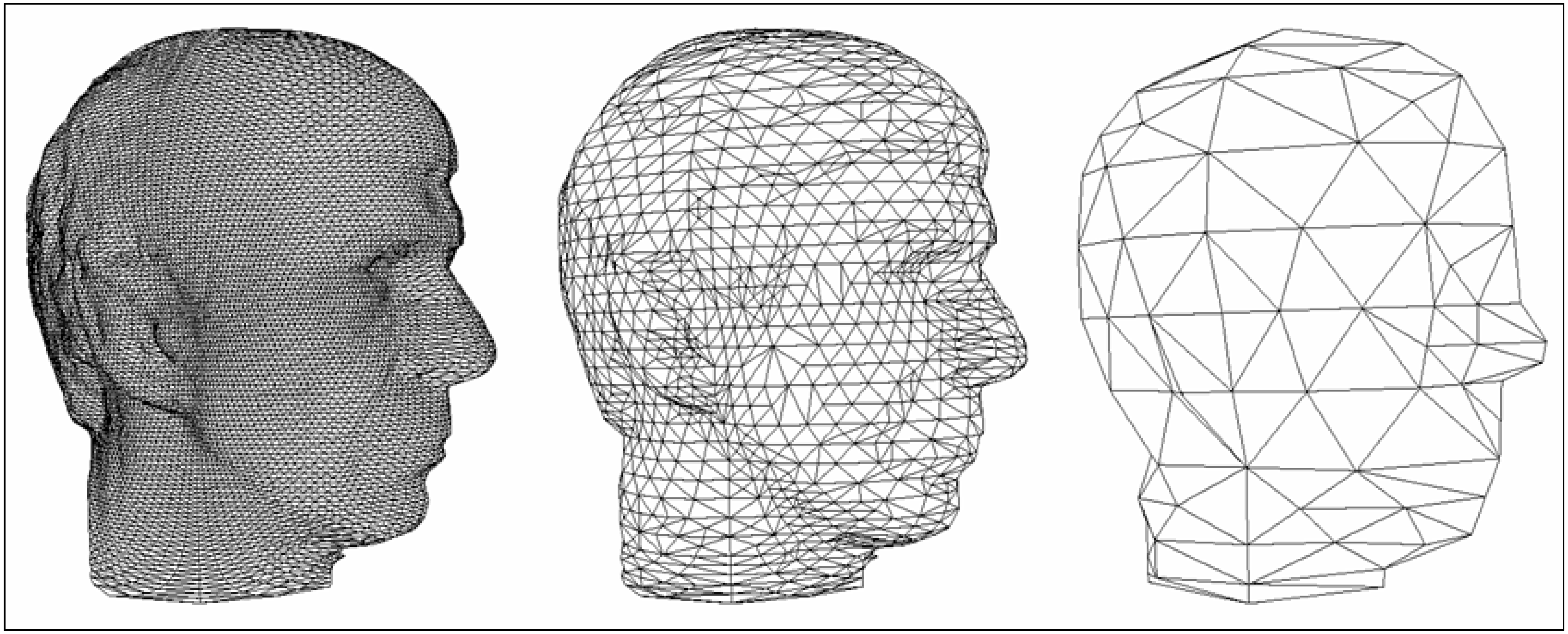
Wavelet Transform: Filter bank

- No matter how wide the filters are, they are move allways 2 cells.
- For filters wider than 2 we get a border problem:



Wavelet Transform: Applications

- Image compression
- Multi scalen resolution (level of detail)



- Audio analysis

Wavelet Transform: Applications

- Reconstruct the following 1D Haar transform:

  89.6875,

  1.6875, -5.875, 3.25, -4.5, -0.25, 2.25, 0.75, -1, -4 , -1, -0.5, 1.5, 1, 0.5, 0

- The first value is called DC. Why?