STATS305A - Lecture 7

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Contents

1 Announcements

- Etude 1 due 5pm Thursday sharp
  - On Thursday solutions to the etude will be posted at around 5pm.
  - Students grade their own etudes.
  - Students upload a new etude with corrections made to the orginal submission. You must upload a revised etude. This second submission is due 5pm Friday.
- Homework 2 will be out soon.

2 Fisherian Testing

2.1 Setting

We propose a null H_0. We collect data and compute a statistic T_n which is just some function of our data. Under the null H_0, T_n follows *some* distribution. Call this distribution T (this is not a T-distribution, just the distribution of T_n when the null holds).

We nest pick a *level* $\alpha \in (0, 1)$ at which to reject H_0. We choose a *rejection region* R where

$$\mathbb{P}_{H_0}(T \in R) \leq \alpha.$$

That is, under the null H_0, T_n is unlikely to fall in R. We then reject if $T_n \in R$.

2.2 P-values

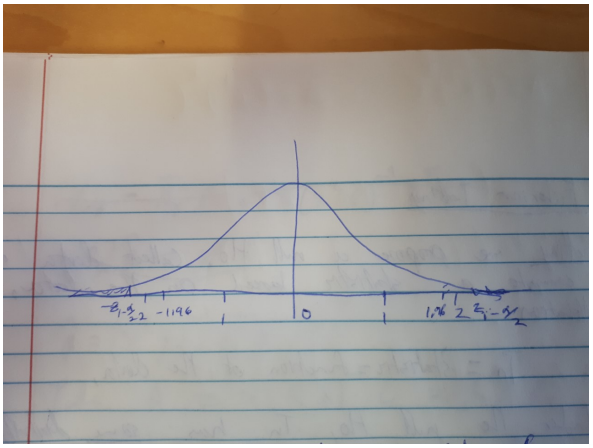
Often our rejection regions are nested. That is if R_α denotes the rejection region at level α, then

$$R_{\alpha_0} \subsetneq R_{\alpha_1}, \text{ if } \alpha_0 < \alpha_1.$$

For example if $T \sim \mathsf{N}(0,1)$ under H_0, then we often choose

$$R_\alpha = (-\infty, -z_{1-\alpha/2}) \cup (z_{1+\alpha/2}, \infty),$$

where $z_{1-\alpha}$ is the $1-\alpha$ quantile of a standard normal $Z \sim \mathsf{N}(0,1)$, that is $\mathbb{P}(Z \geq z_{1-\alpha}) = \alpha$. See below



The *p-value* of a statistic T_n is:

- The smallest level α at which we can reject the test ie $p = \inf\{\alpha : T_n \in R_\alpha\}$.
- Equivalently, the probability under H_0 of seeing a sample as strange/extreme as what we have observed.

Example 1. Consider the example above when H_0 implies that T_n is normally distributed. Set $t_n := T_n$ (the observed value), then our p-value is

$$p = \mathbb{P}_{H_0}(|T| \geq |T_n|).$$

Example 2. Assume that $Y = X\beta + \mathbf{1}\beta_0 + \varepsilon$ where $X \in \mathbb{R}^{n\times(d-1)}$ and $Z = [1, X]$. Let $H = Z(Z^TZ)^{-1}Z$ = projection onto full model. Let $H_0 = \frac{1}{n}\mathbf{1}\mathbf{1}^T$ = projection onto range of the submodel $Y = \mathbf{1}\beta_0 + \varepsilon$. Our null hypothesis is that the submodel is true. Define $S_n^2 = \frac{1}{n-d}\|(I-H)Y\|_2^2$. Under the null hypothesis we saw that

$$T_n := \frac{\frac{1}{d-1}\|(H-H_0)Y\|_2^2}{\frac{1}{n-d}\|(I-H)Y\|_2^2} \sim F_{d-1,n-d}.$$

We reject the null if T_n is large which means the larger model explains the data better than the smaller model.

2.3 Sampling/M-tests

Suppose that we decide to estimate β via

$$\widehat{\beta} = \arg\min_b \sum_{i=1}^n \left|Y_i - X_i^T b\right|.$$

Where we assume $Y = X\beta + \varepsilon$, $\varepsilon \sim \mathsf{N}(0, \sigma^2 I)$ where we assume σ^2 is known. We wish to test the null hypothesis $\beta = 0$.

Under the null hypothesis we can compute $\varepsilon^{(1)}, \ldots, \varepsilon^{(N)}$ independent samples from $\mathsf{N}(0, \sigma^2 I)$ where N is big. We then define $Y^{(i)} := \varepsilon^{(i)}$ and

$$\widehat{\beta}^{(i)} = \arg\min_b \sum_{j=1}^n \left|Y_j^{(i)} - X_j b\right|.$$

The samples $\widehat{\beta}^{(i)}$ gives us an approximation to the distribution of $\widehat{\beta}$ assuming that the null is true.

Abstractly all we need is some region in $\mathbb{R}^d$ that contians a $1-\alpha$ fraction of all $\widehat{\beta}^{(i)}$. Then we reject the null if $\widehat{\beta}$ is out of that region. Here is one way to do this.

We could compute $q_{1-\alpha}^{sim} = 1-\alpha$ quartile of $\left\|\widehat{\beta}^{(i)}\right\|_2$. Thus we have

$$\mathbb{P}_{H_0}\left(\left\|\widehat{\beta}\right\| \geq q_{1-\alpha}^{sim}\right) \leq \alpha + \frac{1}{N}.$$

This is because under H_0, $\left\|\widehat{\beta}^{(i)}\right\|_2$ has the same distribution as $\left\|\widehat{\beta}\right\|$. If we define $t_i = \left\|\widehat{\beta}^{(i)}\right\|_2$ and $T_n = \left\|\widehat{\beta}\right\|_2$, then $\left\|\widehat{\beta}\right\|_2 > q_{1-\alpha}^{sim}$ if and only if $T_n > t_i$ for a $1-\alpha$ fraction of $\{t_1, \ldots, t_n\}$. And $T_n > t_i$ for a $1-\alpha$ fraction of $\{t_1, \ldots, t_n\}$ occurs with probability $\alpha \pm \frac{1}{N}$.

The p-value in this case is

$$\begin{aligned}
&\inf\{\alpha : \left\|\widehat{\beta}\right\|_2 > t_i \text{ for a } 1-\alpha \text{ fraction of } \{t_i\}_{i=1}^N\} \\
&= \inf\{\alpha : \left\|\widehat{\beta}\right\|_2 > q_{1-\alpha}^{sim}\} \\
&\approx \text{the fraction of } \{t_i\}_{i=1}^N \text{ such that } \left\|\widehat{\beta}\right\|_2 \leq t_i.
\end{aligned}$$

2.4 Aside: power of a test

Choose an alternative H_1 such as $Y = X\beta + \varepsilon$, $\|\beta\| \geq \sqrt{\frac{d}{n}}$ or $Y = X\beta + \varepsilon$, $\beta_1 > \frac{1}{\sqrt{n}}$.

The *power* of a test is also written as β but now β is a probability not a parameter in a model. The power is defined as

$$\beta := \mathbb{P}_{H_1}(T_n \text{ rejects}).$$

Once could try to maximise the power of a test while keeping the level α constant but this is subtle and complicated. The rejection region with the most power will depend on H_1. More often in practice we focus on the level and choose the rejection region in a way that reflects the null/alternative hypotheses.

3 ANOVA (Analysis of variance)

Suppose we have the model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij},$$

where $i = 1, \ldots, k$ are different groups and $Y_{i,j}$ for $j = 1, \ldots, n_i$ are different samples from group i. We call μ the population mean and α_i the group effect. We are interested in testing/estimating the differences $\alpha_i - \alpha_j$. The structure of this model allows us to write cleaner/more direct computations and tests.

3.1 ANOVA Decomposition

Let $\overline{Y}_{\bullet\bullet} = \frac{1}{N}\sum_{i=1}^{k}\sum_{j=1}^{n_i} Y_{i,j}$ be the global mean ($N = n_1 + n_2 + \ldots + n_k$). Let $\overline{Y}_{i\bullet} = \frac{1}{n_i}\sum_{j=1}^{n_i} Y_{ij}$ be the mean for group i. Define

$$\begin{aligned}
SST &:= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{\bullet\bullet}\right)^2 \quad \text{(total sum of squares.)} \\
SSB &:= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \quad \text{(between groups sum of squares.)} \\
&= \sum_{i=1}^{k} n_i\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \\
SSW &:= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{i\bullet}\right)^2 \quad \text{(within groups sum of squares.)}
\end{aligned}$$

We then have

$$\begin{aligned}
SST &= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{\bullet\bullet}\right)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{i\bullet} + \overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{i\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \\
&= \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(Y_{i,j} - \overline{Y}_{i\bullet}\right)^2 + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left(\overline{Y}_{i\bullet} - \overline{Y}_{\bullet\bullet}\right)^2 \\
&= SSW + SSB
\end{aligned}$$

This is called the ANOVA decomposition.

Suppose our null is $H_0 : \alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$, $\varepsilon_{i,j} \sim \mathsf{N}(0, \sigma^2)$. Then (exercise) under H_0:

- $SSW \perp\!\!\!\perp SSB$.
- $SSW \sim \sigma^2\chi^2_{N-k}$.
- $SSB \sim \sigma^2\chi^2_{k-1}$.

Thus $\frac{1k-1SSB}{\frac{1}{N-k}SSW} \sim F_{k-1,N-k}$.

We reject H_0 when the above statistic is large which means the between group differences are large relative to the within-group differences.

3.2 Testing differences

Often we may care more about differences in between the mean treatments. For example we may be dosing a drug at different levels $i = 1, \ldots, k$. We care more about $\alpha_i - \alpha_j$ more than just "is there a difference in treament from $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$?"

This gives rise to testing *constrasts* which are vectors $c \in \mathbb{R}^k$ satisfying $c^T\mathbf{1} = 0$. For example $c = e_i - e_j$. If $\widehat{\alpha}$ is any least squares solution then $\frac{c^T\widehat{\alpha}}{S\sqrt{c^T(X^TX)^\dagger c}} \sim T_{n-(d-1)}$ (T-distribution), where $S^2 = \frac{1}{n-(d-1)}\sum_{i=1}^{n}(Y_i - X_i^T\widehat{\beta})^2$.

Exercise: If $c = e_i - e_j$ (ie $c^T\beta = \alpha_i - \alpha_j$), then $c^T(X^TX)^\dagger c = \frac{1}{n_i} + \frac{1}{n_j}$.

3.3 An issue

If we look at all pairs of differences $\alpha_i - \alpha_j$ for $i < j$. Then we are doing $\frac{k(k-1)}{2}$ tests. If we reject the nulls at a level α, then we "expect" to have roughly $\alpha\frac{k^2}{2}$ false rejections. Note that $\alpha\frac{k^2}{2} >> \alpha$. This is the issue of *multiple testing*.