

STATS310B – Lecture 5

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1 Sub-martingales and super-martingales

We ended last lecture with the definition of two generalizations of martingales. They were,

Definition 1. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration and let $\{X_n\}_{n \geq 0}$ be an adapted sequence of integrable random variables.

1. The sequence $\{X_n\}_{n \geq 0}$ is a *sub-martingale* if for all n , $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ almost surely.
2. Likewise, the sequence $\{X_n\}_{n \geq 0}$ is a *super-martingale* if for all n , $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n$ almost surely.

Clearly $\{X_n\}_{n \geq 0}$ is a martingale if and only if, $\{X_n\}_{n \geq 0}$ is both a sub-martingale and a super-martingale.

1.1 New martingales from old

Jensen's inequality allows us to create many sub-martingales and super-martingales from a martingale.

Proposition 1. Let $\{X_n\}_{n \geq 0}$ be a martingale and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $Y_n = \phi(X_n)$ is integrable for all n . Then,

1. If ϕ is convex, then $\{Y_n\}_{n \geq 0}$ is sub-martingale.
2. If ϕ is concave, then $\{Y_n\}_{n \geq 0}$ is a super-martingale.

Proof. If ϕ is convex, then

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = \mathbb{E}(\phi(X_{n+1})|\mathcal{F}_n) \geq \phi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) = \phi(X_n) = Y_n.$$

If ϕ is concave, then $-\phi$ is convex and so $\{-Y_n\}_{n \geq 0}$ is a sub-martingale. This implies that $\{Y_n\}_{n \geq 0}$ is a super-martingale. \square

Example 1. If $\{X_n\}_{n \geq 0}$, then $|X_n|$, X_n^2 and $e^{\theta X_n}$ are all sub-martingales (provided the last two are integrable).

We can also get new sub-martingales from a sub-martingale.

Proposition 2. Let $\{X_n\}_{n \geq 0}$ be a sub-martingale and let $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing convex function. If $Y_n = \phi(X_n)$ is integrable from every n , then $\{Y_n\}_{n \geq 0}$ is a sub-martingale.

Proof. By convexity and Jensen's,

$$\mathbb{E}(Y_{n+1}|\mathcal{F}_n) \geq \phi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)).$$

Also, $\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n$ almost surely. Since ϕ is non-decreasing, this implies

$$\phi(\mathbb{E}(X_{n+1}|\mathcal{F}_n)) \geq \phi(X_n) = Y_n. \quad \square$$

Example 2. If $\{X_n\}_{n \geq 0}$ is a sub-martingale, then X_n^+ is a sub-martingale. If $\theta > 0$ and $e^{\theta X_n}$ is integrable for every n , then $e^{\theta X_n}$ is also a sub-martingale. The random variables X_n^2 and $|X_n|$ need not form sub-martingales even if they are integrable.

We can also get a martingale from a sub-martingale.

Proposition 3 (Doob's decomposition). Let $\{X_n\}_{n \geq 0}$ be a sub-martingale. Then we can write $X_n = X_0 + M_n + A_n$, where $\{M_n\}_{n \geq 0}$ is a martingale and $\{A_n\}_{n \geq 0}$ is a non-decreasing, predictable sequence.

The definition of a predictable sequence is given below.

Definition 2. Let $\{\mathcal{F}_n\}_{n \geq 0}$ be a filtration. An adapted sequence of random variables $\{A_n\}_{n \geq 0}$ is *predictable* if for every $n \geq 1$, A_n is \mathcal{F}_{n-1} -measurable.

Proof of Doob's decomposition. Define,

$$M_n = \sum_{k=0}^{n-1} X_{k+1} - X_k - \mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k).$$

Also define,

$$A_n = \sum_{k=0}^{n-1} \mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k).$$

Then,

$$M_n + A_n = \sum_{k=0}^{n-1} X_{k+1} - X_k = X_n - X_0.$$

Thus, it remains to show that $\{M_n\}_{n \geq 0}$ is a martingale and that $\{A_n\}_{n \geq 0}$ is predictable and non-decreasing. Note that for every k , $\mathbb{E}(X_{k+1} - X_k|\mathcal{F}_k)$ is \mathcal{F}_k -measurable. Thus, A_n is \mathcal{F}_{n-1} -measurable. We also have,

$$A_{n+1} - A_n = \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - \mathbb{E}(X_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1}|\mathcal{F}_n) - X_n \geq 0.$$

Also note that,

$$\mathbb{E}(X_{n+1} - X_n - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)|\mathcal{F}_n) = \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = 0.$$

Thus,

$$\mathbb{E}(M_{n+1} - M_n|\mathcal{F}_n) = \mathbb{E}(X_{n+1} - X_n - \mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n)|\mathcal{F}_n) = 0.$$

Thus, $\mathbb{E}(M_{n+1}|\mathcal{F}_n) = \mathbb{E}(M_n|\mathcal{F}_n) = M_n$ and so $\{M_n\}_{n \geq 0}$ is a martingale. \square

Doob's decomposition result is important because a lot of properties about martingales are well known. Thus, if one can get a handle on the increasing predictable sequence $\{A_n\}_{n \geq 0}$, then the original sub-martingale $\{X_n\}_{n \geq 0}$ can be studied.

1.2 Optional stopping for sub and super-martingales

Proposition 4. Let $\{X_n\}_{n \geq 0}$ be an integrable sequence adapted to $\{\mathcal{F}_n\}_{n \geq 0}$. Let T and S be bounded stopping times for $\{\mathcal{F}_n\}_{n \geq 0}$. Then,

1. If $\{X_n\}_{n \geq 0}$ is a sub-martingale, then $\mathbb{E}(X_T | \mathcal{F}_S) \geq X_S$.
2. If $\{X_n\}_{n \geq 0}$ is a super-martingale, then $\mathbb{E}(X_T | \mathcal{F}_S) \leq X_S$.

The following proposition allows us to rigorously work with sequences $\{X_n\}_{n \geq 0}$ that are (sub/super)-martingales up to a stopping time T .

Proposition 5. Let $\{X_n\}_{n \geq 0}$ be a sequence of integrable random variables adapted to a filtration $\{\mathcal{F}_n\}_{n \geq 0}$. Let T be a stopping time with respect to $\{\mathcal{F}_n\}_{n \geq 0}$, such that on the event $\{T > n\}$, we have

$$X_n \leq \mathbb{E}(X_{n+1} | \mathcal{F}_n) \quad \text{a.s.} \quad (1)$$

By which we mean $\mathbb{P}(X_n > \mathbb{E}(X_{n+1} | \mathcal{F}_n), T > n) = 0$. Then the sequence $\{X_{n \wedge T}\}_{n \geq 0}$ is a sub-martingale with respect to $\{\mathcal{F}_n\}_{n \geq 0}$

Proof. As an exercise, one can show that $X_{n \wedge T}$ is \mathcal{F}_n -measurable and integrable. Now note that,

$$\begin{aligned} \mathbb{E}(X_{(n+1) \wedge T} | \mathcal{F}_n) &= \mathbb{E} \left(\sum_{i=0}^n X_{(n+1) \wedge T} \mathbf{1}_{\{T=i\}} + X_{(n+1) \wedge T} \mathbf{1}_{\{T>n\}} | \mathcal{F}_n \right) \\ &= \sum_{i=0}^n \mathbb{E}(X_{(n+1) \wedge T} \mathbf{1}_{\{T=i\}} | \mathcal{F}_n) + \mathbb{E}(X_{(n+1) \wedge T} \mathbf{1}_{\{T>n\}} | \mathcal{F}_n) \\ &= \sum_{i=0}^n \mathbb{E}(X_i \mathbf{1}_{\{T=i\}} | \mathcal{F}_n) + \mathbb{E}(X_{n+1} \mathbf{1}_{\{T>n\}} | \mathcal{F}_n) \\ &= \sum_{i=0}^n X_i \mathbf{1}_{\{T=i\}} + \mathbb{E}(X_{n+1} \mathbf{1}_{\{T>n\}} | \mathcal{F}_n), \end{aligned}$$

since $X_i \mathbf{1}_{\{T=i\}}$ is \mathcal{F}_n -measurable. The event $\{T > n\} = \{T \leq n\}^c$ is in \mathcal{F}_n and thus, by our assumption (1),

$$\begin{aligned} \mathbb{E}(X_{(n+1) \wedge T} | \mathcal{F}_n) &= \sum_{i=0}^n X_i \mathbf{1}_{\{T=i\}} + \mathbf{1}_{\{T>n\}} \mathbb{E}(X_{n+1} | \mathcal{F}_n) \\ &\geq \sum_{i=0}^n X_i \mathbf{1}_{\{T=i\}} + \mathbf{1}_{\{T>n\}} X_n \\ &= \sum_{i=0}^{n-1} X_i \mathbf{1}_{\{T=i\}} + X_n \mathbf{1}_{\{T>n-1\}} \\ &= X_{n \wedge T}. \end{aligned} \quad \square$$

Remark 1. If the inequality in (1) is replaced with an equality, then $\{X_{n \wedge T}\}_{n \geq 0}$ is a martingale. Likewise, if the inequality in (1) is reversed, then $\{X_{n \wedge T}\}_{n \geq 0}$ is a super-martingale. The proofs are analogous.

The idea behind proposition (5) is that even we can ignore what happens after the stopping time T . This is useful in examples when the distribution of X_n changes after T occurs.

Example 3. Suppose a gambler starts with $x > a$ dollars. At each turn the gambler can win or lose a dollar with equal probability. However, if their total is greater than a , they have to pay b in tax each turn. We wish to know how long it will take for the gambler to have less than a . Let $\{X_n\}_{n \geq 0}$ be the total the gambler has at each turn and let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Let $T = \min\{n : X_n \leq a\}$. We wish to bound $\mathbb{E}[T]$. Note that the event $\{T > n\}$ equals $\{X_1 > a, \dots, X_n > a\}$. Thus, on the event $\{T > n\}$, X_{n+1} is either $X_n + 1 - b$ or $X_n - 1 - b$. Furthermore, $X_{n+1} - X_n$ is independent of \mathcal{F}_n . Thus, on $\{T > n\}$,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n + \mathbb{E}[X_{n+1} - X_n] = X_n - b \leq X_n.$$

Now define $Y_n = X_n + nb$. On $\{T > n\}$, we have $\mathbb{E}(Y_{n+1}|\mathcal{F}_n) = X_n - b + (n+1)b = Y_n$. Thus, $\{Y_{n \wedge T}\}_{n \geq 0}$ is a martingale. Since $0 \wedge T = 0$, we thus have

$$\mathbb{E}[Y_{T \wedge n}] = \mathbb{E}[Y_0] = \mathbb{E}[X_0] = \mathbb{E}[x] = x.$$

On the other hand,

$$\mathbb{E}[Y_{T \wedge n}] = \mathbb{E}[X_{T \wedge n}] + b\mathbb{E}[T \wedge n] \geq a + b\mathbb{E}[T \wedge n].$$

The inequality holds because if $T > n$, then $X_{T \wedge n} = X_n > a$ and if $T \leq n$, then $X_{T \wedge n} = X_T = a$. The random variables $n \wedge T$ are non-negative and increase up to T . Thus,

$$x \geq a + b\mathbb{E}[T],$$

which implies $\mathbb{E}[T] \leq \frac{x-a}{b}$. This agrees with our intuition, since the gambler loses b dollars each turn on average and T records the time it takes the gambler's total to go from x dollars to a dollars.

2 Optimal stopping problem

We will now consider a different kind of problem. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The finite horizon optimal stopping problem is as follows. Suppose we have integrable random variables $\{X_n\}_{n=1}^N$ and a filtration $\{\mathcal{F}_n\}_{n=1}^N$, can we find a stopping time T taking values in $\{1, \dots, N\}$ that maximizes $\mathbb{E}[X_T]$? Note that we have made no assumptions about $\{X_n\}_{n=1}^N$ beyond integrability. We have not assumed that $\{X_n\}_{n=1}^N$ is adapted to $\{\mathcal{F}_n\}_{n=1}^N$. Nonetheless, this problem has a solution in this very general set up!