STATS 310A - Lecture 2

Persi Diaconis
Scribed by Michael Howes

09/23/21

Contents

1 Strong law of large numbers

Recall that we have $\Omega = (0, 1]$ and for $\omega \in \Omega$ we write $\omega = 0.d_1(\omega)d_2(\omega)\ldots$ where $d_i(\omega)$ is the i^{th} binary digit of Ω. If ω has two binary expansions, we pick the expansion that ends in all 1's. We also defined $r_i = 2d_i - 1 \in \{-1, 1\}$ and $S_n = \sum_{i=1}^n r_i$. Let $B \subseteq \Omega$ be the subset

$$B = \left\{\omega \in \Omega : \lim_{n\to\infty} |S_n(\omega)/n| = 0\right\}.$$

Recall also that a subset $A \subseteq \Omega$ is said to nelligable if for every $\varepsilon > 0$, there exists a countable collection of intervals $\{(a_i, b_i]\}_{i=1}^\infty$ such that $A \subseteq \cup_i(a_i, b_i]$ and $\sum_i b_i - a_i = \sum_i p((a_i, b_i]) < \varepsilon$. The strong law of large numbers (slln) states that B^c is nelligable.

Proof. Fix $\delta > 0$ and note that $\{|S_n/n| > \delta\} = \{S_n^4 > \delta^4 n^4\}$. By Markov's inequality we can thus conclude that

$$p(\{|S_n/n| > \delta\}) \le \frac{1}{\delta^4 n^4} \int_0^1 S_n(w)^4 d\omega.$$

Since $S_n = \sum_{i=1}^n r_i$ we know that $(S_n)^4 = \sum_{i,j,k,l=1}^{n} r_i r_j r_k r_l$. Note that there are fives possibilities for the term $r_i r_j r_k r_l$. These are

(a) The case r_i^4. This case occurs n times and when it occurs $\int_0^1 r_i(\omega)r_j(\omega)r_k(\omega)r_l(\omega)d\omega = \int_0^1 1 d\omega = 1$.

(b) The case $r_i^2 r_j r_k$ when i, j, k are distinct. In this case $\int_0^1 r_i r_j r_k r_l d\omega = \int_0^1 r_j r_k d\omega = 0$.

(c) The $r_i^2 r_j^2$ when $i \ne j$. This case occurs $3n(n-1)$ times and in this case $\int_0^1 r_i r_j r_k r_l d\omega = \int_0^1 1 d\omega = 1$.

(d) The case $r_i^3 r_j$ where $i \ne j$. In this case $r_i r_j r_k r_l = r_i r_j$ and thus $\int_0^1 r_i r_j r_k r_l d\omega = 0$.

(e) The case $r_i r_j r_k r_l$ and i, j, k, l are all distinct. In this case $\int_0^1 r_i r_j r_k r_l = 0$.

Combining all of the above we have

$$\int_0^1 S_n(\omega)^4 d\omega = \sum_{i,j,k,l=1}^{n} \int_0^1 r_i(\omega) r_j(\omega) r_k(\omega) r_l(\omega) d\omega = n + 3n(n-1) \leq 3n^2.$$

Thus we have $p(\{|S_n/n| > \delta\}) \leq \frac{3}{\delta^4 n^2}$. Now set $\delta_n = \frac{1}{n^{1/8}}$ so that $\delta_n \to 0$ and $\sum_{n=1}^{\infty} \frac{3}{\delta_n^4 n^2} < \infty$. Note that for all $m \in \mathbb{N}$, we have

$$\bigcap_{n=m}^{\infty} \{\omega : |S_n(\omega)/n| \leq \delta_n\} \subseteq B.$$

By taking complements we see

$$B^C \subseteq \bigcup_{n=m}^{\infty} \{\omega : |S_n(\omega)/n| > \delta_n\}.$$

The set $\{w : |S_n(\omega)/n| > \delta_n\}$ can be written as a finite union of disjoint intervals $\cup_{i=1}^{k_n} I_{n,i}$ such that $\sum_{i=1}^{k_n} p(I_{n,i}) < \frac{3}{\delta_n^4 n^2}$. Thus we have

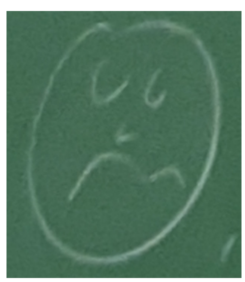
$$B^c \subseteq \bigcup_{n=m}^{\infty} \bigcup_{i=1}^{k_n} I_{n,i},$$

for all $m \in \mathbb{N}$. Thus given $\varepsilon > 0$ we can choose m such that $\sum_{n=m}^{\infty} \frac{3}{\delta_n^4 n^2} < \varepsilon$ and note that

$$\sum_{n=m}^{\infty} \sum_{i=1}^{k_n} p(I_{n,i}) \leq \sum_{n=m}^{\infty} \frac{3}{\delta_n^4 n^2} < \varepsilon.$$

Showing that B^c is nelligiable, $\square$

Thus we can say that $\lim_{n\to\infty} \frac{S_n(\omega)}{n} = 0$ for *almost all* $\omega \in \Omega$ or simply that $\lim_{n\to\infty} \frac{S_n}{n} = 0$ *almost surely*. But!



This statement does not have any qualitative bounds to it. Also how well does our model actually reflect coin flipping? A true model would have a lot of physics and observations of how people flip coins. Persi has written papers about such things. One is "Dynamic bias in the coin toss" written with collaborators.

2 Assigning Probabilities

See "Ten Great Ideas About Chane" by Persi.

Definition 1. Let Ω be a set. A collection $\mathcal{F}_0$ of subset of Ω is a *field* is

(a) The set Ω is in $\mathcal{F}_0$.

(b) If $A \in \mathcal{F}_0$, then $A^c \in \mathcal{F}_0$.

(c) If $A, B \in \mathcal{F}_0$, then $A \cup B \in \mathcal{F}_0$.

We say that $\mathcal{F}_0$ is closed under compliments and finite unions.

Definition 2. Let Ω be a set and $\mathcal{F}$ a collection of subsets of Ω. The collection $\mathcal{F}$ is a σ-field if it is a field and closed under countable unions. That is if $A_1, A_2, \ldots \in \mathcal{F}$, then $\cup_i A_i \in \mathcal{F}$.

Examples 1. (a) $\Omega = (0,1]$ and $\mathcal{F}_0 = \{$finite unions of disjoint intervals of the form (a,b]$\}$. This is a field since $(a,b]^c = (0,a] \cup (b,1]$. It is not a σ-field since $(0,1/2) = \cup_{i=1}^{\infty}(0,1/2-1/i] \notin \mathcal{F}_0$.

(b) The collection of all subsets of Ω is a σ-field.

(c) $\{\Omega, \emptyset\}$ is a σ-field.

(d) If $\mathcal{F}_i$ are σ-fields on Ω for all $i \in I$, then

$$\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i,$$

is a σ-field.

(e) If $\mathcal{C}$ is any collection of subsets of Ω, then define

$$\sigma(\mathcal{C}) := \bigcap_i \mathcal{F}_i,$$

where the intersection is of all σ-fields on Ω that contain $\mathcal{C}$. This is a σ-field and is called the σ-field generated by $\mathcal{C}$.

(f) The Borel set (in $(0,1]$) is the σ-field generated by all the intervals in $(0,1]$. For example our set

$$B = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{|S_n/n| < 1/k\},$$

is a Borel set.

Definition 3. Let Ω be a set and $\mathcal{F}$ a field. The pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability on $(\Omega, \mathcal{F})$ is a function

$$P : \mathcal{F} \to [0,1], \quad A \mapsto P(A),$$

such that

(a) $P(\Omega) = 1$, $P(\emptyset) = 0$.

(b) $P(A^C) = 1 - P(A)$.

(c) If $\{A_i\}_{i=1}^{\infty}$ are pairwise disjoint and $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$, then $P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

Remark 1. It is often "easy" to assign probabilities to a field $\mathcal{F}_0$ of sets and there is a standard way to uniquely extend these probabilites to the σ-field generated by $\mathcal{F}_0$.

Definition 4. Let P be a probability on a field $\mathcal{F}_0$. For all $A \subseteq \Omega$, define

$$P^*[A] := \inf \left\{ \sum_{i=1}^{\infty} P(B_i) : A \subseteq \bigcup_{i=1}^{\infty} B_i \text{ and } B_i \in \mathcal{F}_0 \right\}.$$

The function P^* is called the *outer measure associated to* $(\Omega, \mathcal{F}, P)$.

The function P^* has the following properties

(a) $P^*(\Omega) = 1$, $P^*(\emptyset) = 0$.

(b) P^* is countable subadditive. That if if $A = \cup_{i=1}^{\infty} A_i$, then $P^*[A] \leq \sum_{i=1}^{\infty} P^*[A_i]$.

Proof. Fix $\varepsilon > 0$ and let $(B_{i,j})_{j=1}^{\infty}$ be a countable cover of A_i by intervals such that

$$\sum_{j=1}^{\infty} P[B_{i,j}] \leq P^*[A_i] + \varepsilon 2^{-i}$$

. Thus $\{B_{i,j}\}_{i,j=1}^{\infty}$ is a countable cover of A by elements of $\mathcal{F}_0$. Since $P[B_{i,j}] \geq 0$, we can rearrange the infinite series $\sum_{i,j=1}^{\infty} P[B_{i,j}]$ and thus conclude the following

$$\begin{aligned}
P^*[A] &\leq \sum_{i,j=1}^{\infty} P[B_{i,j}] \\
&= \sum_{i}^{\infty} \sum_{j=1}^{\infty} P[B_{i,j}] \\
&\leq \sum_{i=1}^{\infty} P^*[A_i] + \varepsilon 2^{-i} \\
&= \sum_{i=1}^{\infty} P^*[A_i] + \varepsilon \sum_{i=1}^{\infty} 2^{-i} \\
&= \sum_{i=1}^{\infty} P^*[A_i] + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we can conclude that $P^*[A] \leq \sum_{i=1}^{\infty} P^*[A_i]$. □

Definition 5. *[Caratheodory]* Let P be a probability measure on a field $\mathcal{F}_0$. A set $A \in \Omega$ is *measurable* if for all $E \subseteq \Omega$,

$$P^*[E] = P^*[E \cap A] + P^*[E \cap A^c].$$

We will use $\mathcal{M}$ to denote the set of all measurable sets.

We will prove:

- $\mathcal{M}$ is a σ-field that contains $\mathcal{F}_0$.
- P^* restricted to $\mathcal{M}$ is a probability (in particular P^* is countable additive on measurable sets).
- P^* restricted to $\mathcal{F}_0$ equals P.
- P^* is unique.

This goal will preoccupy us for the next lecture but we will start working on it today. With the notation as above we will show that $\mathcal{M}$ is a field.

Proof. The one main trick we will use is that to show $A \in \mathcal{M}$, it is enough to show

$$P^*(E \cap A) + P^*(E \cap A^c) \leq P^*(E),$$

for all $E \subseteq \Omega$. This is becuase we always have $P^*(E) \leq P^*(E \cap A) + P^*(E \cap A^c)$ by subadditivity.

We can see immediatie that $\Omega \in \mathcal{M}$ and that $\mathcal{M}$ is closed under complements. Thus suppose that $A, B \in \mathcal{M}$ and $E \subseteq \Omega$, then

$$\begin{aligned}
P^*(E) &= P^*(E \cap A) + P^*(E \cap A^c) \\
&= P^*(E \cap A \cap B) + P^*(E \cap A \cap B^c) + P^*(E \cap A^c \cap B) + P^*(E \cap A^c \cap B^c) \\
&\geq P^*(E \cap ([A \cap B] \cup [A \cap B^c] \cup [A^c \cap B])) + P^*(E \cap [A^c \cap B^c]) \\
&= P^*(E \cap [A \cup B]) + P^*(E \cap [A \cup B]^c).
\end{aligned}$$

Thus, by the main trick, $A \cup B \in \mathcal{M}$. □

3 Conclusion

One might ask why are we doing all this just to talk about probabilities? There are several reasons

- People want to work with infinite sequence spaces, random curves, Brownian motion and the set of all probabilities measures on $[0, 1]$. These are complicated spaces and it can be hard to assign probabilities on them by hand.
- We simply cannot assign a consistent notion of length to all subsets of $[0, 1]$.
- Keep an eye out for a halloween talk on non-measurable set.

A remark on finite vs countable additivity. Let $\mathbb{N} = \{1, 2, \ldots\}$. One would like to say that j choosen "at random" from $\mathbb{N}$ has a 50% chance of being even. We can make sense of this by defining

$$P_n(A) = \frac{|A \cap [n]|}{n},$$

where $|B|$ is the number of elements in B and $[n] = \{1, 2, \ldots, n\}$. If $\lim_{n\to\infty} P_n(A) = l$ exists we say that A has density $D(A) = l$. One can show

- $D(\text{multiples of } j) = \frac{1}{j}$,
- $D(\{\text{primes}\}) = 0$, and
- $D(\{\text{square free numbers}\}) = \frac{6}{\pi^2}$.

Not every set has a density. For example if

$$A = \{1, 10, 11, 100, 101, \ldots, 199, 1000, 10001, \ldots, 1999, 10000\}.$$

That is, A is the set of numbers that start with a 1 when written in decimal. Then $P_n(A)$ moves up and down between 1/9 and 5/9 infinitely often. The answer to this problem is to use the Hahn-Bananch theorem to extend D to all subsets of $\mathbb{N}$. This is gives us a measure that is finitely additive but not countably additive. This example is highly non-constructive and non unique since there are many possible Hahn Banach extensions and none of them are "natural."