# STATS310B – Lecture 8

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## 1 Polya's Urn

Consider an urn of infinite capacity. The urn initially has one white ball and one black ball inside it. At each time step, a ball is picked uniformly at random from the urn and replaced back into the urn with another ball of the same color. Equivalently we choose a color with probability proportional to the number of balls of the same color and then but in an additional ball of the chosen color.

Let $W_n$ be the proportion of white ball at time $n$ with $W_0 = \frac{1}{2}$. We would like to understand the limiting behavior of $W_n$ as $n \to \infty$.

**Proposition 1.** *Let $\mathcal{F}_n = \sigma(W_1, \dots, W_n)$. Then the sequence $\{W_n\}_{n\geq 0}$ is a martingale with respect to $\{\mathcal{F}_n\}_{n\geq 0}$.*

*Proof.* Note that the total number of balls at time $n$ is $n+2$. Let $N_n$ be the number of white ball in the urn at time $n$. Thus, $W_n = \frac{1}{n+2}N_n$. It follows that,

$$\begin{aligned}
\mathbb{E}(W_{n+1}|\mathcal{F}_n) &= \frac{1}{n+3}\mathbb{E}(N_{n+1}|\mathcal{F}_n)\\
&= \frac{1}{n+3}\left((N_n+1)\frac{N_n}{n+2} + N_n\frac{n+2-N_n}{n+2}\right)\\
&= \frac{1}{n+3}\left(\frac{1}{n+2}N_n^2 + \frac{N_n}{n+2} + N_n - \frac{1}{n+2}N_n^2\right)\\
&= \frac{1}{n+3}\left(\frac{n+3}{n+2}N_n\right)\\
&= \frac{1}{n+2}N_n\\
&= W_n.
\end{aligned}$$

□

Note that $W_n \in [0,1]$ for every $n$ and thus $\sup_n \mathbb{E}[W_n^+] \leq 1 < \infty$. It follows that there exists an integrable random variable $W$ such that $W_n \to W$ almost surely. We will in fact prove that $W$ is uniformly distributed on $[0,1]$.

*Proof.* We will show by induction that for all $n$, $N_n$ is uniformly distributed on $\{1, 2, \ldots, n+1\}$, where $N_n$ is the number of white balls at time $n$. This is true when $n = 0$ since $N_0 = 1$. Now suppose that the result is true for some $n$. Then, for $k = 1, \ldots, n+2$,

$$\begin{aligned}\mathbb{P}(M_{n+1} = k) &= \sum_{j=1}^{n+1} \mathbb{P}(M_{n+1} = k | N_n = j)\mathbb{P}(N_n = j) \\ &= \frac{1}{n+1} \sum_{j=1}^{n+1} \mathbb{P}(M_{n+1} = k | N_n = j).\end{aligned}$$

Note that $\mathbb{P}(M_{n+1} = k | N_n = j) \neq 0$ only if $j = k-1$ or $j = k$. This is even true if $k = 1$ or $k = n+2$ although in these cases one $\mathbb{P}(M_{n+1} = k | N_n = k-1) = 0$ or $\mathbb{P}(M_{n+1} = k | N_n = k) = 0$ respectively which agrees with the calculations below. Thus,

$$\begin{aligned}\mathbb{P}(M_{n+1} = k) &= \frac{1}{n+1} \left(\mathbb{P}(M_{n+1} = k | N_n = k-1) + \mathbb{P}(M_{n+1} = k | N_n = k)\right) \\ &= \frac{1}{n+1} \left(\frac{k-1}{n+2} + \frac{n+2-k}{n+2}\right) \\ &= \frac{1}{n+1} \left(\frac{n+1}{n+2}\right) \\ &= \frac{1}{n+2}.\end{aligned}$$

Thus, $N_{n+1}$ is uniformly distributed on $\{1, \ldots, n+2\}$. Hence, $W_n$ is uniformly distributed on $\left\{\frac{1}{n+2}, \ldots, \frac{n+1}{n+2}\right\}$ which implies $W_n$ converges in distribution to $U[0, 1]$ but $W_n$ also converges almost surely (and this $W_n \to W$ in distribution). Thus, $W$ must be uniformly distributed on $[0, 1]$. □

## 2 Lévy's downwards convergence theorem

Our next convergence theorem is Lévy's downwards convergence theorem which is also called the backwards martingale theorem.

**Theorem 1** (Lévy's downards convergence theorem)**.** *Let $X$ be an integrable random variable defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \ldots$ be a decreasing sequence of sub-$\sigma$-algebras of $\mathcal{F}$. Define $\mathcal{F}^* = \bigcap_{n=0}^{\infty} \mathcal{F}_n$, then*

$$\mathbb{E}(X|\mathcal{F}_n) \to \mathbb{E}(X|\mathcal{F}^*),$$

*almost surely and in $L^1$.*

*Proof.* Let $X_n = \mathbb{E}(X|\mathcal{F}_n)$. We will first show that $\{X_n\}_{n \geq 0}$ has an almost sure limit $X^*$. We will then prove that $X_n$ converges to $X^*$ in $L^1$ and then finally we will show that $X^* = \mathbb{E}(X|\mathcal{F}^*)$. Fix $n \in \mathbb{N}$ and consider the time reversed finite sequence,

$$X_n, X_{n-1}, X_{n-2}, \ldots, X_0.$$

The above sequence is a martingale with respect to $\mathcal{F}_n, \mathcal{F}_{n-1}, \ldots, \mathcal{F}_0$. This is because $\mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ and thus

$$\mathbb{E}(X_{k-1}|\mathcal{F}_k) = \mathbb{E}(\mathbb{E}(X|\mathcal{F}_{k-1})|\mathcal{F}_k) = \mathbb{E}(X|\mathcal{F}_k).$$

Fix an interval $[a, b]$ and let $U_n$ be the number of complete upcrossings of $[a, b]$ by $X_n, X_{n-1}, \ldots, X_0$.

By the upcrossing lemma,

$$\begin{aligned}\mathbb{E}[U_n] &\le \frac{\mathbb{E}[(X_0-a)^+]-\mathbb{E}[(X_n-a)^+]}{b-a}\\ &\le \frac{\mathbb{E}[(X_0-a)^+]}{b-a}\\ &= \frac{\mathbb{E}[(\mathbb{E}(X|\mathcal{F}_0)-a)^+]}{b-a}\\ &\le \frac{\mathbb{E}[\mathbb{E}((X-a)^+|\mathcal{F}_0)]}{b-a}\\ &\le \frac{\mathbb{E}[(X-a)^+]}{b-a}.\end{aligned}$$

Note that $U_n \le U_{n+1}$ and thus $U_n \nearrow U$ for some random variable $U$. By the monotone convergence theorem, $\mathbb{E}[U] \le \frac{\mathbb{E}[(X-a)^+]}{b-a} < \infty$. As with Doob's sub-martingale convergence theorem, this implies that $X^* = \lim_{n\to\infty} X_n$ exists almost surely. We will now show that $X^*$ is integrable. Note that

$$\begin{aligned}\mathbb{E}[|X^*|] &= \mathbb{E}\left[\lim_{n\to\infty}|X_n|\right]\\ &\le \liminf_{n\to\infty}\mathbb{E}[|X_n|]\\ &= \liminf_{n\to\infty}\mathbb{E}[|\mathbb{E}(X|\mathcal{F}_n)|]\\ &\le \liminf_{n\to\infty}\mathbb{E}[\mathbb{E}(|X||\mathcal{F}_n)]\\ &= \liminf_{n\to\infty}\mathbb{E}[|X|]\\ &= \mathbb{E}[|X|] < \infty.\end{aligned}$$

To show that $X^* = \mathbb{E}(X|\mathcal{F}^*)$ and that $X_n \to X^*$ in $L^1$ we need to first review the concept of *uniform integrability.* □

# 3 Uniform integrability

**Definition 1.** A sequence of random variables $\{X_n\}_{n\ge1}$ is *uniformly integrable* if for $\epsilon > 0$, there exists $K > 0$ such that,

$$\sup_n \mathbb{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \varepsilon.$$

Uniform integrability allows one to calculate the expectation of a limit.

**Lemma 1.** *Suppose that $\{X_n\}_{n\ge0}$ is a uniformly integrable sequence and $X_n \to X$ almost surely. Then $X$ is integrable and $X_n \to X$ in $L^1$.*

*Proof.* Take any $\varepsilon > 0$ and take $k$ such that $\mathbb{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \varepsilon$. Then,

$$\mathbb{E}[|X_n|] \le \mathbb{E}[|X_n|\mathbf{1}_{\{X_n>k\}}] + \mathbb{E}[|X_n|\mathbf{1}_{\{X_n\le k\}}] \le \varepsilon + k.$$

Thus, $\mathbb{E}[|X|] \le \liminf_n \mathbb{E}[|X_n|] \le \varepsilon + k < \infty$. So $X$ is integrable. Note that for all $L > 0$, $|X|\mathbf{1}_{\{|X|>L\}} \le |X|$ and, almost surely

$$\lim_{L\to\infty}|X|\mathbf{1}_{\{|X|>L\}} = 0.$$

Thus, by the dominated convergence theorem,

$$\lim_{L\to\infty}\mathbb{E}[|X|\mathbf{1}_{\{|X|>L\}}] = 0.$$

This shows that given $\varepsilon > 0$, we can choose $k > 0$ so that $\mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] < \varepsilon$ and

$$\sup_n \mathbb{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] < \varepsilon.$$

Let $\varepsilon > 0$ be arbitrary and fix such a corresponding $k > 0$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by

$$\phi(x) = \begin{cases} -k & \text{if } x \leq -k, \\ x & \text{if } x \in (-k, k), \\ k & \text{if } x \geq k. \end{cases}$$

The function $\phi$ is bounded and continuous and $|\phi(x) - x| \leq |x|\mathbf{1}_{\{|x|>k\}}$ for all $x \in \mathbb{R}$. Thus,

$$\begin{aligned} \mathbb{E}[|X_n - X|] &\leq \mathbb{E}[|X_n - \phi(X_n)|] + \mathbb{E}[|\phi(X_n) - \phi(X)|] + \mathbb{E}[|\phi(X) - X|] \\ &\leq \mathbb{E}[|X_n|\mathbf{1}_{\{|X_n|>k\}}] + \mathbb{E}[|\phi(X_n) - \phi(X)|] + \mathbb{E}[|X|\mathbf{1}_{\{|X|>k\}}] \\ &< 2\varepsilon + \mathbb{E}[|\phi(X_n) - \phi(X)|]. \end{aligned}$$

The random variable $|\phi(X_n) - \phi(X)|$ is bounded above by $2k$ and goes to 0 almost surely. Thus, by the dominated convergence theorem,

$$\limsup_n \mathbb{E}[|X_n - X|] \leq 2\varepsilon + \limsup_n \mathbb{E}[|\phi(X_n) - \phi(X)|] = 2\varepsilon.$$

Thus, $X_n \to X$ in $L^1$. □

We will next state a characterization of uniform integrability that we will need in proving Lévy's downwards convergence theorem.

**Proposition 2.** *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables. The sequence $\{X_n\}_{n\geq 0}$ is uniformly integrable if and only if the following both hold,*

1. $\sup_n \mathbb{E}[|X_n|] < \infty$.
2. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $A$ and $n$, if $\mathbb{P}(A) < \delta$, then $\mathbb{E}[|X_n|\mathbf{1}_A] < \varepsilon$.*

We will prove this proposition in the next lecture.