STATS 305A - Lecture 2

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1 Outline

Today we will do two things. A matrix and linear algebra overview. Covering

(a) Independence,

(b) Rank/ invertibility/ solving $Ax = b$,

(c) Decompositions.

We will also discuss some basic of optimisation. Detailed linear algebra notes will be put on the course webpage.

2 Linear algebra review

2.1 Vectors

Vector are $x \in \mathbb{R}^n$, we will write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x_i \in \mathbb{R}.$$

We will not use row vectors. All vectors are column vectors. The *norm* of a vector is its size. We will must commonly use the Euclidean or 2-norm. That is we will define

$$\|x\| = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x^T x}.$$

We will occassionally use p-norms for $p \in [1, \infty]$ given by

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}.$$

When $p = 1$, we have $\|x\|_1 = \sum_{i=1}^n |x_i|$. When $p = 2$, $\|x\|_2$ is the Euclidean norm and when $p = \infty$, $\|x\|_\infty = \max\{|x_i| : i = 1, \ldots, n\}$. These are the values of p we will most commonly use.

We will say that k-vectors $x_1, \ldots, x_k$ are *(linearly) independent* if for all $\alpha \in \mathbb{R}^k$,

$$\sum_{i=1}^k \alpha_i x_i = 0 \iff \alpha_i = 0 \text{ for } i = 1, 2, \ldots, k.$$

This is equivalent to requiring that for all $\alpha, \eta \in \mathbb{R}^k$,

$$\sum_{i=1}^{k} \alpha_i x_i = \sum_{i=1}^{k} \beta_i x_i \iff \alpha_i = \beta_i \text{ for } i = 1, 2 \ldots, k.$$

The *linear span* of the collection of vectors $\{x_i\}_{i=1}^k$ is the set

$$\mathrm{span}\{x_i\} = \left\{ \sum_{i=1}^{k} \alpha_i x_i : \alpha \in \mathbb{R}^k \right\}.$$

Note that a collection of vectors $\{x_i\}_{i=1}^k$ are independent if and only if for $i = 1, \ldots, k$, x_i is not in $\mathrm{span}\{x_j\}_{j \neq i}$.

2.2 Matrices

We will now look at solving and understanding the matrix equation $Ax = b$. This does not always have a solution. Suppose $A \in \mathbb{R}^{m \times n}$, then we have

$$\begin{aligned} A &= [a_1, \ldots, a_n], \quad a_i \in \mathbb{R}^m, \\ &= \begin{bmatrix} \widetilde{a}_1^T \\ \ldots \\ \widetilde{a}_m^T \end{bmatrix}, \quad \widetilde{a}_j \in \mathbb{R}^n. \end{aligned}$$

The vectors a_i are the columns of A and the transposed vectors $\widetilde{a}_j^T$ are the rows of A. The *rank* of A is

$$\#\text{ of independent columns of A} = \#\text{ of independent rows of A}.$$

The rank of a matrix is a very unstable quality. We may have rank 0, then add a small amount of noise and suddenly have rank n. This makes it not very useful in statistics but we will use it occassionally.

Given $A \in \mathbb{R}^{m \times n}$, we say that A has a *left inverse* $B \in \mathbb{R}^{n \times m}$ such that

$$BA = I_n = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix}.$$

The matrix I_n is called the $n \times n$ identity. We say that A has a right inverse $C \in \mathbb{R}^{n \times m}$ if $AC = I_m$. If A has a left inverse B and a right inverse C, then $B = C$ and we define $A^{-1} := B = C$. In this case we say that A is invertible with inverse A^{-1}. To see why $B = C$, note that

$$B = BI_m = B(AC) = (BA)C = I_nC = C.$$

Definition 1. The *range* or *column space* of a matrix A is equal to

$$\mathrm{span}\{a_i\}_{i=1}^n = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

We have equality in the above line since $Ax = \sum_{i=1}^n a_i x_i$.

Definition 2. The *null space* of A is

$$\mathrm{null}(A) = \{x \in \mathbb{R}^n : Ax = 0\} \subseteq \mathbb{R}^n.$$

The null space will be important in our linear models. If $Y = X\beta + \varepsilon$, then $\mathrm{null}(X)$ contains all the directions in β that we get no information about. Even if $\mathrm{null}(X) = \{0\}$, there may be problems if our matrix is "ill-conditioned". This means that there are vectors that are "close to" $\mathrm{null}(X)$. These are directions of β in which it is hard (but not impossible) to get information.

Theorem 1. [The rank-nullity theorem] *Let* $A \in \mathbb{R}^{n\times m}$*, then* $\mathrm{null}(A) = \mathrm{range}(A^T)^\perp$*, where for* $S \subseteq \mathbb{R}^n$*,* $S^\perp = \{y \in \mathbb{R}^n : y^T x = 0,\ \textit{for all } x \in S\}$*.*

Proof. *(sketch)* If $x \in \mathrm{null}(A)$, then $Ax = 0$. Thus for $y = A^T w \in \mathrm{range}(A^T)$ we have

$$y^T x = w^T (A^T)^T x = w^T (Ax) = w^T 0 = 0.$$

Thus $y \in \mathrm{range}(A^T)^\perp$. If $x \in \mathrm{range}(A^T)^\perp$, then for all $w \in \mathbb{R}^n$,

$$0 = (A^T w)^T x = w^T (Ax),$$

which implies $Ax = 0$. That is $x \in \mathrm{null}(A)$. □

2.3 Special matrices and matrix decompositions

Definition 3. A matrix $Q \in \mathbb{R}^{n\times n}$ is *orthogonal* if $Q^T Q = QQ^T = I_n$. If $Q = [q_1, q_2, \dots, q_n]$, then this is equivalent to

$$q_i^T q_j = \begin{cases} 1 & \text{if } i = j, 0^{\text{else.}} \end{cases}$$

That is the vectors $\{q_1, \dots, q_n\}$ are an orthonormal set. *[Aside: if* $Q \in \mathbb{R}^{n\times n}$ *and* $Q^* Q = I$*, then* Q *is said to be* orthonormal *or* unitary*]*.

If $U \in \mathbb{R}^{n\times m}$ where $m \geq n$ (more rows than columns, tall matrix), then U will also be called orthogonal if $UU^T = I_n$. Note that if $m > n$, then $U^T U$ cannot equal I_m. Again this is equivalent to the columns of U being orthogonal.

We now will examine a number of matrix factorisations. A lot of computational matrix calculations/statistics/ML involves reducing a matrix A to a product of simpler matrices. In particular we can use these to solve the equation $Ax = b$.

Definition 4. A matrix $R \in \mathbb{R}^{n\times n}$ is *upper (or right) triangular* if R is of the form

$$R = \begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,n} \\ 0 & r_{2,2} & \dots & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & r_{n,n} \end{bmatrix}.$$

That is R has all zeros below the main diagonal.

If we are asked to solve $Rx = b$ and R is upper triangular we can use the following "back-subsitution" algorithm. First write the equation as

$$\begin{bmatrix} r_{1,1} & r_{1,2} & \dots & r_{1,n-1} & r_{1,n} \\ 0 & r_{2,2} & \dots & r_{2,n-1} & r_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & r_{n-1,n-1} & r_{n-1,n} \\ 0 & 0 & \dots & 0 & r_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix}$$

Assume that $r_{i,i} \neq 0$ for all $i = 1, \dots, n$. Thus we can see that we must have $r_{n,n} x_n = b_n$. Thus $x_n = \frac{b_n}{r_{n,n}}$. We can then move up to the next row where we have

$$\begin{aligned} r_{n-1,n_1} x_{n-1} + r_{n-1,n} x_n &= b_{n-1} \\ \therefore x_{n-1} &= \frac{1}{r_{n-1,n-1}} \left(b_{n-1} - r_{n-1,n} x_n\right) \\ &= \frac{1}{r_{n-1,n-1}} \left(b_{n-1} - r_{n-1,n} \frac{b_n}{r_{n,n}}\right), \end{aligned}$$

since we previously showed $x_n = \frac{b_n}{r_{n,n}}$. Continuing in this way we can see that if we have solved for $x_n, x_{n-1}, \ldots, x_{k-1}$ we can solve for x_k. This takes approximately n^2 opperations to perform which is pretty much optimal.

What do we do if we have a matrix A that isn't upper triangular? We can use the QR factorisation.

Theorem 2. *Let $A \in \mathbb{R}^{m\times n}$ be a matrix with $m \geq n$ (A is a tall matrix), then there exist $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times n}$ such that R is upper triangular and invertible, $Q^TQ = I_n$ and $A = QR$. The matrices Q and R are called the* QR factorisation *of A.*

Note that if we have the QR factorisation of A, then we can easily solve $Ax = b$. Since this is because $Ax = b$ implies $Q^TAx = Q^Tb$. Since $A = QR$ we have $Q^TA = Q^TQR = R$ and thus the solution to $Rx = Q^Tb$ solves the original equation $Ax = b$. The QR factorisaation can be constructed iteratively via the Gram-Schmidt algorithm.

Given $A = [a_1, a_2, \ldots, a_n] \in \mathbb{R}^{m\times n}$ and $k \leq n$ we will produce

$$Q_k = [q_1, \ldots, q_k] \in \mathbb{R}^{m\times k} \text{ and } R_k \in \mathbb{R}^{k\times k},$$

such that

- R_k is upper triangular,
- $\text{span}\{q_i\}_{i=1}^k = \text{span}\{a_i\}_{i=1}^k$,
- $Q_k^TQ_k = I_k$, and
- $Q_kR_k = [a_1, a_2, \ldots, a_k]$.

When we reach $k = n$ we will be done. We will show how to do this for $k = 1, 2$ and generalise.

<u>Start $k = 1$</u>: Define $q_1 = \frac{a_1}{\|a_1\|}$, $r_{11} = \|a_1 1\|$, then $r_{11}q_1 = a_1$ and $q_1^Tq_1 = \frac{a_1^Ta_1}{\|a_1\|^2} = 1$.

<u>Next step $k = 2$</u>: We already have q_1 and we want q_2. We want to preserve the span and so q_2 should be a linear combination of a_2 and q_1. We also want q_2 to be orthogonal to q_1. Thus we subtract off the q_1 part of a and define

$$v = a_2 - a_2^Tq_1q_1.$$

Then $v^Tq_1 = a_2^Tq_1 - a_2^Tq_1q_1^Tq_1 = 0$. We want q_2 to have norm 1 and thus define $q_2 = \frac{v}{\|v\|}$. Note that the equation $a_2 = \|v\|\, q_2 + a_2^Tq_1q_1$ shows that $r_{2,2} = \|v\|$ and $r_{1,2} = a_2^Tq_1$. Thus we have constructed R_2 as well.

<u>Inductive case</u>: Suppose that we have just finished the $k-1$ step. Generalising what we did before we can set

$$v = q_k - \sum_{i=1}^{k-1} a_k^Tq_iq_i.$$

Then $v^Tq_i = 0$ for $i = 1, \ldots, k-1$ and we can define $q_k = \frac{v}{\|v\|}$. Since again $a_k = \|v\|\, q_k + \sum_{i=1}^{k-1} a_k^Tq_iq_i$, we have that

$$r_{i,k} = \begin{cases} a_k^Tq_i & \text{if } i < k, \\ \|v\| & \text{if } i = k. \end{cases}$$

Thus we have construct R as well. Note that at each step we add a new column of R.

The QR algorithm is implemented on any scientific programming language you might use (R, python, julia, etc). You do not need to use Gram-Schmidt by hand. Indeed Gram-Schmidt is rarely used because it is numerical unstable and so other methods are used. Gram-Schmidt is good for the theoretical justification that the QR decomposition exists.

2.4 Eigen-decompositions

Let $A \in \mathbb{R}^{n\times n}$, a vector v is an eigenvector with eigenvalue λ if $v \neq 0$ and $Av = \lambda v$. Most of our analysis will involve symmetric matrices (those that satisfy $A^T = A$). This simplfies the analysis of eigenvalues a lot.

Theorem 3. [The Spectral Theorem] *If A is a symmetric matirx, then A has an orthonormal set of eigenvalues $v_1, \ldots, v_n$ with real eigenvalue $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. That is*

$$Av_i = \lambda_i v_i \text{ for } i = 1, \ldots, n,$$

and for $i, j = 1, \ldots, n$,

$$v_i^T v_j = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

If we set $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $V = [v_1, v_2, \ldots, v_n]$, then the spectral theorem states that $A = V\Lambda V^T$ and $V^T V = I_n$. To see why this holds, note that $A = V\Lambda V^T$ if and only if $AV = V\Lambda$. By matrix multiplication $AV = [Av_1, \ldots, Av_n]$ and $V\Lambda = [\lambda_1 v_1, \ldots \lambda_n v_n]$. Thus we have $AV = V\Lambda$ if and only if $Av_i = \lambda_i$ for each i. We also previously saw that the vector $\{v_i\}_{i=1}^n$ are orthonormal if and only if $V^T V = I_n$.

A proof of the spectral theorem will be given in the document that is to be put on the course webpage. The spectral theorem makes calculations easier since $A^k = V\Lambda^k V^T$ for $k = 1, 2, \ldots$ and if A is invertible $A^{-k} = V\Lambda^{-k}V^T$.

2.5 Singular value decomposition

The spectral theorem assumes that A is symmetric. The following decomposition works for all matrices. Each matrix $A \in \mathbb{R}^{m\times n}$ with $m \geq n$ (A is tall) has an SVD (singular value decomposition) into

$$A = U\Sigma V^T,$$

where $U \in \mathbb{R}^{m\times n}$, $\Sigma \in \mathbb{R}^{n\times n}$ is diagonal with diagonal entries $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, $V \in \mathbb{R}^{n\times n}$ and $U^T U = I_n = V^T V$. This decomposition tells us how A acts on any vector $x \in \mathbb{R}^n$.

Given $x \in \mathbb{R}^n$, we first change it to the basis of V, then we multiply by gains $\sigma_1, \ldots, \sigma_n$ and then give the ouput in terms of the U basis. This is what the equation $Ax = U\Sigma V^T x$ tells us.

Note that $A^{-1} = V\Sigma^{-1}U^T$ if $\sigma_i > 0$ for all i and $m = n$. Again see notes for the construction of the SVD.

3 Optimisation

We didn't have time to cover this today.