STATS310B – Lecture 4

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01/13/22

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1 Stopped σ-algebras

Recall the following definition from the previous lecture.

Definition 1. Given a filtration $\{\mathcal{F}_n\}_{n\geq 0}$ and a stopping time T. The *stopped σ-algebra* is defined to be

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n \text{ for all } n\}.$$

We saw that $\mathcal{F}_T$ is indeed a σ-algebra. Informally, a typical event in $\mathcal{F}_T$ is one that depends on $\mathcal{F}_n$ for $n \leq T$.

Example 1. Let S_n be a simple symmetric random walk on $\mathbb{Z}$ with $S_0 = 0$. Let $\mathcal{F}_n = \sigma(S_0, S_1, \ldots, S_n)$. We saw previously that for $a, b \in \mathbb{Z}$ with $a < 0 < b$, the random variable $T = \inf\{n : S_n = a \text{ or } S_n = b\}$ is a stopping time with respect to $\{F_n\}_{n\geq 0}$. We also claimed that the event $A = \{S_k \geq 0, \text{ for } k \leq T\}$ was in $\mathcal{F}_T$. To see why this is true, take any n. Then,

$$A \cap \{T = n\} = \left(\bigcap_{k=0}^{n} \{S_k \geq 0\}\right) \cap \{T = n\}.$$

Since S_k is $\mathcal{F}_n$ measurable for $n \leq k$ and since T is a stopping time, both of the above events are in $\mathcal{F}_n$. Thus, the above intersection is in $\mathcal{F}_n$ and so $A \in \mathcal{F}_T$.

Proposition 1. *If S and T are stopping times with respect to $\{F_n\}_{n\geq 0}$ and $S \leq T$ always, then $\mathcal{F}_S \subseteq \mathcal{F}_T$.*

Proof. Let $A \in \mathcal{F}_S$ and take any $n \in \mathbb{N}$. Then

$$\begin{aligned} A \cap \{T = n\} &= A \cap \{S \leq n\} \cap \{T = n\} \\ &= \left(\bigcup_{k=0}^{n} A \cap \{S = k\}\right) \cap \{T = n\}. \end{aligned}$$

For each k we have $A \cap \{S = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ and so $\bigcup_{k=0}^{n} A \cap \{S = k\} \in \mathcal{F}_n$. Also, $\{T = n\} \in \mathcal{F}_n$ and thus $A \cap \{T = n\} \in \mathcal{F}_n$ and so $A \in \mathcal{F}_T$. □

Note that for the above argument, we need $S \leq T$ always. The result may not be true if $S \leq T$ almost surely.

Example 2. Suppose that $N \in \{0, 1, \ldots\}$. Then the constant random variable $T = N$ is a stopping time with respect to any filtration $\{\mathcal{F}_n\}_{n\geq 0}$. The stopped σ-algebra is $\mathcal{F}_N$.

Definition 2. A stopping time T is *bounded* if there is an integer N such that $T \leq N$ always.

Note that if $T \leq N$ always, then by proposition 1 and example 2, we have $\mathcal{F}_T \subseteq \mathcal{F}_N$. We will also define stopped random variables.

Definition 3. Let $\{X_n\}_{n\geq 0}$ be a sequence of random variables adapted to $\{\mathcal{F}_n\}_{n\geq 0}$. Let T be a stopping time with respect to $\{\mathcal{F}_n\}_{n\geq 0}$ such that $\mathbb{P}(T < \infty) = 1$. The random variables X_T is defined on the set where $T < \infty$ by

$$X_T(\omega) = X_{T(\omega)}(\omega).$$

Note that we can define X_T on all of Ω by taking $X_T(\omega)$ equal to any value on the set $\{T = \infty\}$ which has probability zero. Note that the random variable X_T may be very different to each of the random variables X_n. For example if S_n is the SSRW on $\mathbb{Z}$ with $S_0 = 0$, then the support of S_n is the set of integers in $\{-n, -n+1, \ldots, n-1, n\}$ with the same parity as n. However, if $T = \inf\{n : S_n = a \text{ or } S_n = b\}$, then $S_T \in \{a, b\}$ almost surely (we will see shortly that $\mathbb{P}(T < \infty)) = 1$.

We will now state a useful theorem.

Theorem 1 (Optional stopping theorem)**.** *Let $\{X_n\}_{n\geq 0}$ be a martingale adapted to a filtration $\{\mathcal{F}_n\}_{n\geq 0}$. Let S and T be bounded stopping times with respect to $\{F_n\}$ such that $S \leq T$ always. Then X_S and X_T are integrable and $\mathbb{E}(X_T|\mathcal{F}_S) = X_S$ almost surely. In particular $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.*

Note that when T and S are both constant this reduces to the result that for all n, k such that $k \leq n$,

$$\mathbb{E}(X_n|\mathcal{F}_k) = X_k \quad \text{and} \quad \mathbb{E}[X_n] = \mathbb{E}[X_0].$$

The above results can be proved directly by using induction and the tower property. We will now prove the optional stopping theorem.

Proof. Let N be an integer such that $S \leq T \leq N$ always. Note that

$$|X_S|, |X_T| \leq |X_0| + |X_1| + \ldots + |X_N|.$$

Thus, X_S and X_T are both integrable. We will next show that X_S is $\mathcal{F}_S$ measurable. Fix any $c \in \mathbb{R}$, we wish to show that $\{X_S \leq c\} \in \mathcal{F}_S$ which requires $\{X_S \leq c\} \cap \{S = n\} \in \mathcal{F}_n$ for every n.

$$\{X_S \leq c\} \cap \{S = n\} = \{X_n \leq c\} \cap \{S = n\}.$$

Both $\{S = n\}$ and $\{X_n \leq c\}$ are in $\mathcal{F}_n$ and so $\{X_S \leq c\} \in \mathcal{F}_S$. Now let $A \in \mathcal{F}_S$. We wish to show

$$\mathbb{E}[X_T \mathbf{1}_A] = \mathbb{E}[X_S \mathbf{1}_A].$$

Note that

$$\begin{aligned}
\mathbb{E}[X_N \mathbf{1}_A] &= \mathbb{E}\left[X_N \sum_{n=0}^{N} \mathbf{1}_{\{T=n\}} \mathbf{1}_A\right] \\
&= \sum_{n=0}^{N} \mathbb{E}[X_N \mathbf{1}_{\{T=n\}} \mathbf{1}_A] \\
&= \sum_{n=0}^{N} \mathbb{E}[X_N \mathbf{1}_{\{T=n\}\cap A}].
\end{aligned}$$

We know that $\{T = n\} \cap A \in \mathcal{F}_n$ since $A \in \mathcal{F}_S \subseteq \mathcal{F}_T$. Thus

$$\begin{aligned}\sum_{n=0}^{N} \mathbb{E}[X_N \mathbf{1}_{\{T=n\}\cap A}] &= \sum_{n=0}^{N} \mathbb{E}[\mathbb{E}(X_N|\mathcal{F}_n)\mathbf{1}_{\{T=n\}\cap A}] \\ &= \sum_{n=0}^{N} \mathbb{E}[X_n \mathbf{1}_{\{T=n\}\cap A}] \\ &= \mathbb{E}\left[\sum_{n=0}^{N} X_n \mathbf{1}_{\{T=n\}} \mathbf{1}_A\right] \\ &= \mathbb{E}[X_T \mathbf{1}_A].\end{aligned}$$

By the same argument we have $\mathbb{E}[X_N \mathbf{1}_A] = \mathbb{E}[X_S \mathbf{1}_A]$. Thus, $\mathbb{E}[X_T \mathbf{1}_A] = \mathbb{E}[X_S \mathbf{1}_A]$ and thus

$$\mathbb{E}(X_T|\mathcal{F}_S) = X_S.$$

If we take $S = 0$, then we get

$$\mathbb{E}[X_T] = \mathbb{E}[\mathbb{E}(X_T|\mathcal{F}_0)] = \mathbb{E}[X_0].$$ □

The requirement that S and T are bounded stopping times may seem restrictive, but the result is not true if we do not have this assumption. Fortunately we can often approximate an unbounded stopping time by a sequence of stopping times.

Proposition 2. *Let T be a stopping time with respect to a filtration $\{F_k\}_{k\geq 0}$. Then for every $n \in \mathbb{N}$, the random variable*

$$T \wedge n = \min\{T, n\},$$

is a stopping time bounded by n. Furthermore, $T \wedge n \to T$ on the set where $T < \infty$.

Proposition 3. *To see that $T \wedge n$ is a stopping time, note that*

$$\{T \wedge n = k\} = \begin{cases} \{T = k\} & \text{if } k < n, \\ \{T \geq k\} & \text{if } k = n, \\ \emptyset & \text{if } k > n. \end{cases}$$

Thus, we immediately see that $\{T \wedge n = k\} \in \mathcal{F}_k$ when $k \neq n$. When $k = n$, note that

$$\{T \geq k\} = \{T < k\}^C = \left(\bigcup_{j=1}^{k-1} \{T = j\}\right)^C.$$

We know that $\{T = j\} \in \mathcal{F}_j \subseteq \mathcal{F}_k$ for all $j < k$ and so $\{T \geq k\} \in \mathcal{F}_k$.

2 Gambler's ruin

We will now study an example which shows the usefulness of the optional stopping theorem. Let $\{S_n\}_{n\geq 0}$ be a SSRW with $S_0 = 0$ and as before let $T = \inf\{n : S_n = a \text{ or } S_n = b\}$. We are interested in

$$\mathbb{P}(T < \infty, S_T = b).$$

We will soon see that $\mathbb{P}(T < \infty) = 1$. Thus, the probability we are interested in is

$$\mathbb{P}(S_T = b).$$

Remark 1. This question is related to the *gambler's ruin*. Consider a gambler with $x \in \{1, 2, \ldots\}$ dollars. The gambler repeatedly makes bets where they can either win or lose \$1. They continue playing until they make $y > x$ dollars, or they go broke. The probability that they don't go broke is equal to the probability that they end with y dollars. This is equal to $\mathbb{P}(S_T = b)$ when $a = -x$ and $b = y - x$.

Note that S_T only takes the values a and b. Thus,

$$\mathbb{E}[S_T] = a\mathbb{P}(S_T = a) + b\mathbb{P}(S_T = b) = a(1 - \mathbb{P}(S_T = b)) + b\mathbb{P}(S_T = b) = a + (b - a)\mathbb{P}(S_T = b).$$

Thus, if we can calculate $\mathbb{E}[S_T]$, then we will know $\mathbb{P}(S_T = b)$ since

$$\mathbb{P}(S_T = b) = \frac{\mathbb{E}[S_T] - a}{b - a}.$$

We wish to use the optional stopping theorem to conclude that $\mathbb{E}[S_T] = \mathbb{E}[S_0] = 0$. The stopping time $T \wedge n$ is bounded for each n, and so we know that $\mathbb{E}[S_{T\wedge n}] = 0$. We also have $|S_{T\wedge n}| \le \max\{-a, b\}$. Thus, it suffices to show that $T \wedge n \to T$ almost surely. To do this, we need to show that $\mathbb{P}(T < \infty) = 1$.

Proposition 4. *With S_n and T as above, $\mathbb{P}(T < \infty) = 1$.*

Proof. We can divide the set $\{1, 2, 3, \ldots\}$ into infinitely many blocks B_j of consecutive integers such that B_j contains $|a| + b$ integers. Let A_j be the event that $S_k - S_{k-1} = 1$ for all $k \in B_j$. Each A_j has probability $\mathbb{P}(A_j) = 2^{-|a|-b}$ and the events A_j are independent since the sets B_j are disjoint. Thus, by the second Borel–Cantelli lemma,

$$\mathbb{P}(A_j, \text{ infinitely often}) = 1.$$

For each A_j we have $A_j \subseteq \{T < \infty\}$. To see this let k_1 be the minimum of B_j. Suppose that $\omega \in A_j$. If $T(\omega) > k_1$, then $S_{k_1}(\omega) \ge a$. Since $\omega \in A_j$,

$$S_{k_1+|a|+b}(\omega) = S_{k_1}(\omega) + |a| + b = b.$$

Thus, $T(\omega) \le k_1 + |a| + b < \infty$. We thus have $A_j \subseteq \{T < \infty\}$, and so

$$\mathbb{P}(T < \infty) \ge \mathbb{P}(A_j, \text{ infinitely often}) = 1.$$

□

Thus, $T \wedge n \to T$ almost surely and furthermore more $S_{T\wedge n} \to S_T$. Since $S_{T\wedge n}$ is uniformly bounded we have

$$\mathbb{E}[S_T] = \lim_{n\to\infty} \mathbb{E}[S_{T\wedge n}] = \lim_{n\to\infty} \mathbb{E}[S_0] = 0.$$

We thus have

$$\mathbb{P}(S_T = b) = \frac{-a}{-a + b}.$$

In terms of the gambler's ruin this equal $\frac{x}{y}$. So if the gambler starts with $x = 900$ dollars, they have a 90% of making $y = 1000$ dollars before going broke. But as we see, the gambler will have to be very patient. We can also use the optional stopping theorem to calculate the expected value of T as well. Recall that

$$M_n = S_n^2 - n,$$

is a martingale since $\mathrm{Var}(X_n) = 1$. And thus

$$\mathbb{E}[M_{T\wedge n}] = \mathbb{E}[S_{T\wedge n}^2] - \mathbb{E}[T \wedge n].$$

The random variables $S_{T\wedge n}^2$ are uniformly bounded by $\max\{a^2, b^2\}$ and converge almost surely to S_T^2. Furthermore, $T \wedge n$ are all non-negative, and they converge almost surely to T. Therefore, by the dominated and monotone convergence theorem, we have

$$\mathbb{E}[S_T^2] = \lim_{n\to\infty} \mathbb{E}[S_{T\wedge n}^2] = \lim_{n\to\infty} \mathbb{E}[T \wedge n] = \mathbb{E}[T].$$

We also know that

$$\begin{aligned}\mathbb{E}[S_T^2] &= a^2\mathbb{P}(S_T = a) + b^2\mathbb{P}(S_T = b)\\ &= \frac{a^2 b}{b-a} + \frac{-b^2 a}{b-a}\\ &= \frac{ab(a-b)}{b-a}\\ &= -ab.\end{aligned}$$

In the gambler's ruin, this equals $x(y-x)$. So to get from \$900 to \$1000 or go broke, the gambler will have to wait on average 90000 turns.

3 Sub-martingales and super-martingales

We will now state two definitions that generalize martingales.

Definition 4. Let $\{F_n\}_{n\geq 0}$ be a filtration and let $\{X_n\}_{n\geq 0}$ be a sequence of adapted, integrable random variables. Then,

1. $\{X_n\}_{n\geq 0}$ is a *sub-martingale* if for every n,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \geq X_n. \text{ a.s}$$

2. $\{X_n\}_{n\geq 0}$ is a *super-martingal*e if for every n,

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) \leq X_n. \text{ a.s}$$

Remark 2. One can use Jensen's inequality to make sub-martingales out of martingales. Let $\{M_n\}_{n\geq 0}$ be a martingale and let ϕ be a convex function such that $X_n = \phi(M_n)$ is integrable for every $n \geq 0$. Then $\{X_n\}_{n\geq 0}$ is a sub-martingale since, almost surely