STATS310A - Lecture 14

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Contents

1 Recap

Today we will continue with Stien's method and Poisson approximation. As before $\mathcal{P}_\lambda$ will be used to denote the Poisson distribution with parameter $\lambda > 0$. That is, for $j \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have $\mathcal{P}_\lambda(\{j\}) = \frac{e^{-\lambda}\lambda^j}{j!}$. As before we will say that a random variable Z is Poisson(λ) to mean that Z has distribution $\mathcal{P}_\lambda$ and so

$$\mathbb{P}(Z \in A) = \mathcal{P}_\lambda(A).$$

Suppose we have a finite index set I and random variables $\{X_i\}_{i \in I}$ such that X_i takes values 0, 1. Suppose $\mathbb{P}(X_i = 1) = \mathbb{E}[X_i] = p_i$ and that $\mathbb{P}(X_i = 1, X_j = 1) = \mathbb{E}[X_i X_j] = p_{ij}$. Let

$$W = \sum_{i \in I} X_i,$$

and

$$\lambda = \sum_{i \in I} p_i = \mathbb{E}[W].$$

Also define $\mathbb{P}_W$ to be the probability measure $\mathbb{P}_W(A) = \mathbb{P}(W \in A)$ for $A \subseteq \mathbb{N}$. Suppose that we have a dependency graph Γ for $\{X_i\}_{i \in I}$. That is, for all subsets $A, B \subseteq I$, if A and B are disjoint and there are no edges between A and B in Γ, then

$$\{X_i\}_{i \in A} \text{ and } \{X_j\}_{j \in B},$$

are independent. For $i \in I$ we define N_i to be the neighbourhood of i in Γ. That is,

$$N_i = \{j \in I : \text{there is an edge from } i \text{ to } j \text{ in } \Gamma\} \cup \{i\}.$$

We wish to prove

Theorem 1. *With notation as above*

$$\|\mathcal{P}_\lambda - \mathbb{P}_W\|_{TV} \leq \min(3, \lambda^{-1}) \left(\sum_{i \in I} \sum_{i \in N_i \setminus \{i\}} p_{ij} + \sum_{i \in I} \sum_{j \in N_i} p_{ij} \right),$$

where $\|\cdot\|_{TV}$ *is the total variation distance.*

2 Stein's equation

The key idea is the following proposition:

Proposition 1. *A random variable is Poisson*(λ) *if and only if for all bounded* $f : \mathbb{N} \to \mathbb{R}$,

$$\mathbb{E}[Zf(Z)] - \lambda \mathbb{E}[f(Z+1)] = 0.$$

We'll need the following analytic lemma.

Lemma 1 $(**)$**.** *For all* $A \subseteq \mathbb{N}$ *and* $\lambda > 0$*, there exists a unique function* $f : \mathbb{N} \to \mathbb{R}$ *such that*

i. $f(0) = 0$,

ii. *For all* $j \in \mathbb{N}$, $\lambda f(j+1) - jf(j) = \delta_A(j) - \mathcal{P}_\lambda(A)$

iii. *For all* $j \in \mathbb{N}$, $|f(j)| \leq 1.25$.

iv. *For all* $j \in \mathbb{N}$, $|f(j+1) - f(j)| \leq \min(3, \lambda^{-1})$.

Proof. Starting at $j = 0$ we can set

$$f(j+1) = \frac{1}{\lambda} \left(jf(j) + \delta_A(j) - \mathcal{P}_\lambda(A) \right).$$

The function f is well-defined by recursion and unique by induction. Thus there exists a unique function f satisfying items *i.* and *ii.*. We wish to show that f satisfies items *iii.* and *iv.*. If we multiply the equation

$$\lambda f(j+1) - jf(j) = \delta_A(j) - \mathcal{P}_\lambda(A),$$

by $\frac{\lambda^j}{j!}$, we get the equation

$$\frac{\lambda^{j+1}}{j!} f(j+1) - \frac{\lambda^j}{(j-1)!} f(j) = \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A) \right),$$

for $j \geq 1$ and for $j = 0$ we have

$$\lambda f(1) = \lambda \left(\delta_A(0) - \mathcal{P}_\lambda(A) \right).$$

Thus

$$\begin{aligned}
\frac{\lambda^k}{(k-1)!} f(k) &= \lambda f(1) + \sum_{j=1}^{k-1} \left(\frac{\lambda^{j+1}}{j!} f(j+1) - \frac{\lambda^j}{(j-1)!} f(j) \right) \\
&= \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A) \right) \\
&= -\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A) \right).
\end{aligned}$$

The last equality hold because

$$\begin{aligned}\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A)\right) &= e^{\lambda} \sum_{j=0}^{\infty} \delta_A(j) \mathcal{P}_\lambda(\{j\}) - \mathcal{P}_\lambda(A) \sum_{j=0}^{\infty} \frac{\lambda^j}{j!} \\ &= e^{\lambda} \mathcal{P}_\lambda(A) - \mathcal{P}_\lambda(A) e^{\lambda} \\ &= 0.\end{aligned}$$

Taking absolute values we get

$$\begin{aligned}|f(k)| &= \frac{(k-1)!}{\lambda^k} \left| \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A)\right) \right| \\ &\leq \frac{(k-1)!}{\lambda^k} \sum_{j=0}^{k-1} \frac{\lambda^j}{j!} |\delta_A(j) - \mathcal{P}_\lambda(A)| \\ &= \frac{1}{\lambda} \sum_{j=0}^{k-1} \frac{(k-1)!}{\lambda^{k-j-1} j!} |\delta_A(j) - \mathcal{P}_\lambda(A)| \\ &\leq \frac{1}{\lambda} \sum_{j=0}^{k-1} \frac{(k-1)!}{\lambda^{k-j-1} j!}.\end{aligned}$$

The last equality holds since $\delta_A(j), \mathcal{P}_\lambda(A) \in [0,1]$. We will now perform a change of variables and sum over $j' = k - j - 1$ so that $j = k - j' - 1$. We thus have

$$\begin{aligned}|f(k)| &\leq \frac{1}{\lambda} \sum_{j=0}^{k-1} \frac{(k-1)!}{\lambda^{k-j-1} j!} \\ &= \frac{1}{\lambda} \sum_{j'=0}^{k-1} \frac{(k-1)!}{\lambda^{j'} (k-1-j')!} \\ &= \frac{1}{\lambda} \sum_{j'=0}^{k-1} \frac{(k-1)(k-2)\ldots(k-j')}{\lambda^{j'}} \\ &\leq \frac{1}{\lambda} \sum_{j'=0}^{k-1} \left(\frac{k-1}{\lambda}\right)^{j'} \\ &\leq \frac{1}{\lambda} \sum_{j'=0}^{\infty} \left(\frac{k-1}{\lambda}\right)^{j'}.\end{aligned}$$

If $k < \lambda + 1$, then above series is convergent and we have

$$\begin{aligned}|f(k)| &\leq \frac{1}{\lambda} \left(\frac{1}{1 - \frac{k-1}{\lambda}}\right) \\ &= \frac{1}{\lambda - k + 1}.\end{aligned}$$

In particular when $k \leq \lambda + \frac{1}{5}$,

$$|f(k)| \leq \frac{1}{4/5} = 1.25.$$

Using

$$\frac{\lambda^k}{(k-1)!} f(k) = -\sum_{j=k}^{\infty} \frac{\lambda^j}{j!} \left(\delta_A(j) - \mathcal{P}_\lambda(A)\right),$$

we also have

$$\begin{aligned}
|f(k)| &\leq \frac{(k-1)!}{\lambda^k} \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} |\delta_A(j) - \mathcal{P}_\lambda(A)| \\
&\leq \frac{(k-1)!}{\lambda^k} \sum_{j=k}^{\infty} \frac{\lambda^j}{j!} \\
&= \frac{1}{k} \sum_{j=k}^{\infty} \frac{\lambda^{j-k} k!}{j!} \\
&= \frac{1}{k} \sum_{m=0}^{\infty} \frac{\lambda^m k!}{(m+k)!} \\
&= \frac{1}{k} \sum_{m=0}^{\infty} \frac{\lambda^m}{(m+k)(m+k-1)\dots(k+1)} \\
&\leq \frac{1}{k} \sum_{m=0}^{\infty} \left(\frac{\lambda}{k+1}\right)^m
\end{aligned}$$

If $k > \lambda - 1$, then the above series is convergent and so

$$|f(k)| \leq \frac{1}{k} \left(\frac{1}{1 - \frac{\lambda}{k+1}} \right) = \frac{k+1}{k(k+1-\lambda)}.$$

In particular if $k > \lambda + 1/5$ and $k \geq 2$, then $\frac{k+1}{k} \leq \frac{3}{2}$ and so

$$|f(k)| \leq \frac{k+1}{k(1+1/5)} = \frac{5(k+1)}{6k} \leq 1.25.$$

For $k < 2$, we have $f(0) = 0$ and

$$|f(1)| = \frac{1}{\lambda} |\delta_A(1) - \mathcal{P}_\lambda(A)|,$$

which is maximized when $A = \{0\}$ or $A = \mathbb{N} \setminus \{0\}$. In both these cases,

$$|\delta_A(1) - \mathcal{P}_\lambda(A)| = 1 - e^{-\lambda},$$

and thus

$$|f(1)| \leq \frac{1}{\lambda} \left(1 - e^{-\lambda}\right) \leq 1.$$

Thus we have shown *iii.*. To show *iv.* we need to bound $|f(j+1) - f(j)|$. By the triangle inequality

$$|f(j+1) - f(j)| \leq |f(j+1)| + |f(j)| \leq 2 \times 1.25 \leq 3.$$

For homework show that

$$|f(j+1) - f(j)| \leq \lambda^{-1},$$

for $\lambda \geq \frac{1}{3}$. □

We are now ready to prove proposition 1.

Proof. First suppose that Z is Poisson(λ) and $f : \mathbb{N} \to \mathbb{R}$ is bounded, then

$$\begin{aligned}\mathbb{E}[Zf(Z)] &= \sum_{j=0}^{\infty} jf(j)\mathcal{P}_\lambda(\{j\}) \\ &= \sum_{j=0}^{\infty} jf(j)\frac{e^{-\lambda}\lambda^j}{j!} \\ &= \sum_{j=1}^{\infty} jf(j)\frac{e^{-\lambda}\lambda^j}{j!} \\ &= \sum_{j=1}^{\infty} f(j)\frac{e^{-\lambda}\lambda^j}{(j-1)!} \\ &= \sum_{k=0}^{\infty} f(k+1)\frac{e^{-\lambda}\lambda^{k+1}}{k!} \\ &= \lambda\sum_{k=0}^{\infty} f(k+1)\mathcal{P}_\lambda(\{k\}) \\ &= \lambda\mathbb{E}[f(Z+1)].\end{aligned}$$

Alternative one can note that the equation

$$\mathbb{E}[Zf(Z)] = \lambda\mathbb{E}[f(Z+1)],$$

is linear in f and thus reduce the result to the case when f is an indicator function (that is, use a (1),(2),(3) argument).

Now conversely suppose that for all bound $f : \mathbb{N} \to \mathbb{R}$,

$$\mathbb{E}[Zf(Z)] = \lambda\mathbb{E}[f(Z+1)].$$

Let A be a subset of $\mathbb{N}$ and let f be as in Lemma ($**$). We then have

$$\begin{aligned}\mathbb{P}(Z \in A) - \mathcal{P}_\lambda(A) &= \sum_{j=0}^{\infty} \left(\delta_A(j) - \mathcal{P}_\lambda(A)\right)\mathbb{P}(Z=j) \\ &= \sum_{j=0}^{\infty} \left(jf(j) - \lambda f(j+1)\right)\mathbb{P}(Z=j) \\ &= \mathbb{E}[Zf(Z) - \lambda f(Z+1)] \\ &= 0.\end{aligned}$$

So $\mathbb{P}(Z \in A) = \mathcal{P}_\lambda(A)$ and Z is Poisson(λ). □

3 Proof of the Poisson approximation

We are now ready to prove

$$\|\mathbb{P}_W - \mathcal{P}_\lambda\|_{TV} \le \min(3, \lambda^{-1})\left(\sum_{i\in I}\sum_{i\in N_i\setminus\{i\}} p_{ij} + \sum_{i\in I}\sum_{j\in N_i} p_{ij}\right),$$

where the notation is as at the start of this lecture.

Proof. Since

$$\|\mathbb{P}_W - \mathcal{P}_\lambda\|_{TV} = \sup_{A\subseteq\mathbb{N}} |\mathbb{P}_W(A) - \mathcal{P}_\lambda(A)|,$$

it suffices to show that

$$|\mathbb{P}_W(A) - \mathcal{P}_\lambda(A)| \leq \min(3, \lambda^{-1}) \left(\sum_{i\in I} \sum_{i\in N_i\setminus\{i\}} p_{ij} + \sum_{i\in I} \sum_{j\in N_i} p_{ij} \right),$$

for all $A \subseteq \mathbb{N}$. Fix such an A and define $\Delta = \mathbb{P}_W(A) - \mathcal{P}_\lambda(A)$. Let f be as in Lemma ($**$). Then, as seen in the previous proof,

$$\begin{aligned}
\Delta &= \mathbb{P}(W \in A) - \mathcal{P}_\lambda(A) \\
&= \mathbb{E}[W f(W) - \lambda f(W+1)] \\
&= \mathbb{E}\left[\sum_{i\in I} X_i f(W) - p_i f(W+1)\right] \\
&= \sum_{i\in I} \mathbb{E}\left[X_i f(W) - p_i f(W+1)\right]
\end{aligned}$$

For every i, let $W_i = W - X_i$ and $V_i = \sum_{j\in N_i^c} X_j$. Note that by the definition of a dependency graph, V_i is independent of X_i. Note also that

$$\begin{aligned}
X_i f(W) &= \begin{cases} 0 & \text{if } X_i = 0, \\ f(W_i+1) & \text{if } X_i = 1. \end{cases} \\
&= X_i f(W_i + 1).
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta &= \sum_{i\in I} \mathbb{E}\left[(X_i - p_i) f(W_i+1) + p_i(f(W_i+1) - f(W+1))\right] \\
&= \sum_{i\in I} \mathbb{E}\left[(X_i - p_i)(f(W_i+1) - f(V_i+1))\right] + \mathbb{E}\left[p_i(f(W_i+1) - f(W+1))\right]. \qquad (1) \\
&= \sum_{i\in I} \mathbb{E}\left[(X_i - p_i)(f(W_i+1) - f(V_i+1))\right] + \sum_{i\in I} \mathbb{E}\left[p_i(f(W_i+1) - f(W+1))\right] \\
&= (I) + (II).
\end{aligned}$$

The equality in (1) holds because V_i and X_i are independent and thus for each i,

$$\begin{aligned}
E\left[(X_i - p_i) f(W_i+1)\right] &= \mathbb{E}\left[(X_i - p_i)(f(W_i+1) - f(V_i+1))\right] + \mathbb{E}[(X_i - p_i) f(V_i+1)] \\
&= \mathbb{E}\left[(X_i - p_i)(f(W_i+1) - f(V_i+1))\right] + \mathbb{E}[(X_i - p_i)]\mathbb{E}[f(V_i+1)] \\
&= \mathbb{E}\left[(X_i - p_i)(f(W_i+1) - f(V_i+1))\right].
\end{aligned}$$

We will now bound the absolute value of the sums (I) and (II). We'll start with (II) which is simplier. For each i, $f(W_i+1) = f(W+1)$ if $X_i = 0$ and otherwise $W_i + 1$ and $W + 1$ differ by 1. Thus,

$$|f(W_i+1) - f(W+1)| \leq X_i \min(3, \lambda^{-1}).$$

And so we have

$$\begin{aligned}\left|\sum_{i\in I}\mathbb{E}\left[p_i(f(W_i+1)-f(W+1))\right]\right| &\leq \sum_{i\in I} p_i\mathbb{E}\left[|f(W_i+1)-f(W+1)|\right]\\ &\leq \min(3,\lambda^{-1})\sum_{i\in I} p_i\mathbb{E}[X_i]\\ &= \min(3,\lambda^{-1})\sum_{i\in I} p_i p_i.\end{aligned}$$

The sum (I) is trickier but similar ideas can be used to bound it. For a fixed i, let $X_1',\dots,X_m'$ be an enumeration of the variables in $N_i\setminus\{i\}$. We then have

$$\begin{aligned}|f(W_i+1)-f(V_i+1)| &= \left|f\left(1+V_i+\sum_{k=1}^m X_k'\right)-f(1+V_i)\right|\\ &= \left|\sum_{j=1}^m X_j'\left(f\left(1+V_i+\sum_{k=1}^j X_k'\right)-f\left(1+V_i+\sum_{k=1}^{j-1}X_k'\right)\right)\right|\\ &\leq \sum_{j=1}^m X_j'\left|f\left(1+V_i+\sum_{k=1}^j X_k'\right)-f\left(1+V_i+\sum_{k=1}^{j-1}X_k'\right)\right|\\ &\leq \min(3,\lambda^{-1})\sum_{j=1}^m X_j'\\ &= \min(3,\lambda^{-1})\sum_{j\in N_i\setminus\{i\}} X_j.\end{aligned}$$

Thus we have that

$$\begin{aligned}\left|\sum_{i\in I}\mathbb{E}\left[(X_i-p_i)(f(W_i+1)-f(V_i+1))\right]\right| &\leq \min(3,\lambda^{-1})\sum_{i\in I}\sum_{j\in N_i\setminus\{i\}}\mathbb{E}[|X_i-p_i|X_j]\\ &= \min(3,\lambda^{-1})\sum_{i\in I}\sum_{j\in N_i\setminus\{i\}}\mathbb{E}[X_iX_j]+p_i\mathbb{E}[X_j]\\ &= \min(3,\lambda^{-1})\sum_{i\in I}\sum_{j\in N_i\setminus\{i\}} p_{ij}+p_ip_j.\end{aligned}$$

Thus combining our bounds on (I) and (II) we have

$$\begin{aligned}|\Delta| &\leq \min(3,\lambda^{-1})\left(\sum_{i\in I}\sum_{j\in N_i\setminus\{i\}} p_{ij}+p_ip_j+p_ip_i\right)\\ &= \min(3,\lambda^{-1})\left(\sum_{i\in I}\sum_{i\in N_i\setminus\{i\}} p_{ij}+\sum_{i\in I}\sum_{j\in N_i} p_{ij}\right) \qquad \square\end{aligned}$$

4 References

Three references are

- "Poisson Approximation and the Chen-Stein Method" by Arratia, Goldstein and Gord.
- "Exchangeable pairs and Poisson approximation" by Chatterjee, Diaconis and Meckes.
- "An Introduction to Stein's Method" by Barbour and Chen. This is a textbook which is available online through Stanford Libraries.