STATS300A - Lecture 16

Dominik Rothenhaeusler
Scribed by Michael Howes

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1 Recap

Our current goal is to find uniformally most powerful unbiased (UMPU) tests for testing $H_0 : \theta \in \Omega_0$ against $H_1 : \theta \in \Omega_1$. Recall that a test function ϕ is unbiased at level α if

$$\mathbb{E}_{\theta_0}\phi \leq \alpha \text{ for all } \theta_0 \in \Omega_0,$$

and

$$\mathbb{E}_{\theta_1}\phi \geq \alpha \text{ for all } \theta_1 \in \Omega_1.$$

We also say a test ϕ was α-similar if for all $\theta \in W$ where $W = \overline{\Omega}_0 \cap \overline{\Omega}_1$. We previously proved the following theorem which relates unbiased and α-similar tests.

Theorem 1 (TSH 4.11)**.** *If $\theta \mapsto \mathbb{E}_\theta\phi$ is continuous for all tests ϕ and ϕ is uniformly most powerful among level α α-similar tests, then ϕ is UMPU at level α.*

Today we will find optimal unbiased tests in multiparameter exponential families. Specifically we will derive optimal one sided tests in the presence of nuisance parameters.

2 Multiparameter exponential families

Suppose we have a model $\{P_\gamma\}$ where $\gamma = (\theta, \lambda) \in \mathbb{R}^{k+1}$ is unknown and P_γ has density

$$p_\gamma(x) = p_{(\theta,\lambda)}(x) = h(x)\exp\left\{\theta U(x) + \sum_{i=1}^k \lambda_i T_i(x) - A(\theta,\lambda)\right\}.$$

We wish to test $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$. For a fixed θ, the family $\{p_{\theta,\lambda}\}$ is an exponential family with sufficient statistics $T = (T_1, \ldots, T_k)$ and so

$$P_{(\theta,\lambda)}(X|T) = P_\theta(X|T).$$

In particular we have $P_{(\theta,\lambda)}(U(X)|T(X)) = P_\theta(U(X)|T(X))$ and so $U(X)|T(X)$ has no λ dependence.

Remark 1. This observation is important. We have shown that conditioning eliminates the nuisance parameters. Thus we can fix $\gamma_0 = (\theta_0, \lambda_0) \in \Omega_1$ and $\gamma_1 = (\theta_1, \lambda_1)$ and construct a test based on $P_{\theta_0}(X|T)$ against $P_{\theta_1}(X|T)$ which has no λ dependence. Thus we can use tools from one-dimensional hypothesis testing. Even better, conditioning on T gives us a one-dimensional exponential family.

Lemma 1. *For each t, $U(X)|T = t$ forms a one-dimensional exponetial family in θ.*

Proof. We will only consider the discrete case. For all u and t let

$$A_{u,t} = \{x \in \mathcal{X} : U(x) = u, T(x) = t\} \quad \text{and} \quad A_t = \{x \in \mathcal{X} : T(x) = t\}.$$

$$\begin{aligned}
P_{\theta,\lambda}(U(X) = u|T(X) = t) &= \frac{P_{\theta,\lambda}(U(X) = u, T(X) = t)}{P_{\theta,\lambda}(T(X) = t)} \\
&= \frac{\sum\limits_{x \in A_{u,t}} p_{\theta,\lambda}(x)}{\sum\limits_{x \in A_t} p_{\theta,\lambda}(x)} \\
&= \frac{\sum\limits_{x \in A_{u,t}} \exp\left\{\theta u + \sum_{i=1}^k \lambda_i t_i\right\} h(x)}{\sum\limits_{x \in A_t} \exp\left\{\theta U(x) + \sum_{i=1}^k \lambda_i t_i\right\} h(x)} \\
&= \underbrace{\exp\{\theta u\}}_{\text{exponential tilt}} \times \underbrace{\sum_{x \in A_{u,t}} h(x)}_{g(t,u) = \text{base measure}} \times \underbrace{\frac{1}{\sum\limits_{x \in A_t} \exp\{\theta U(x)\} h(x)}}_{c(t,\theta) = \text{normalizing constant}}.
\end{aligned}$$

So $U(X)|T(X) = t$ is a one-dimensional exponential family with sufficient statistic U. $\square$

Our general recipe for one sided testing $\theta \leq \theta_0$ against $\theta > \theta_0$ is

(1) Fix an alternative $\theta = \theta_1 > \theta_0$ and $\lambda_1 \in \mathbb{R}^k$.

(2) Condition on T so that $X|T$ does not depend on λ and $U|T$ follows a one dimensional exponential family.

(3) Construct the MP test for the conditional distribution. That is

$$\phi_t(u) = \begin{cases} 1 & \text{if } u > k(t), \\ \rho(t) & \text{if } u = k(t), \\ 0 & \text{if } u < k(t). \end{cases}$$

where $k(t)$ and $\rho(t)$ are determined by the conditional level constraint

$$\mathbb{E}_{\theta_0}[\phi_t|T = t] = \alpha. \tag{1}$$

We will next argue that under some assumptions that test $\phi^*(u,t) = \phi_t(u)$ is the UMPU test for H_0 against H_1. Note that for every test ϕ

$$\mathbb{E}_\gamma \phi = \mathbb{E}_\gamma \left[\mathbb{E}_\gamma[\phi|T]\right] = \mathbb{E}_\gamma \left[\mathbb{E}_\theta[\phi|T]\right].$$

In particular if $\theta \leq \theta_0$, then

$$\mathbb{E}_\gamma \phi^* = \mathbb{E}_\gamma \left[\mathbb{E}_\theta[\phi|T]\right] \leq \mathbb{E}_\gamma[\alpha] = \alpha,$$

and we have equality if $\theta = \theta_0$. Thus ϕ^* is level α and α-similar.

By Neyman-Pearson, there is no test that satisfies the constraint (1) and has strictly large power than ϕ^* for any fixed t or $\theta_1 > \theta$. Thus ϕ^* is the most powerful test in the class of tests satisfying (1) for any fixed $\theta_1 > \theta$. Since ϕ^* does not depend on θ_1, the test ϕ^* is in fact the UMP test among tests satisfying the constrain (1).

Recall that we are trying to show that ϕ^* is the UMPU test. By theorem (1) it suffices to show that ϕ^* is uniformly most powerful among α-similar tests. We thus wish to relate the tests that satisfy condition (1) to the α-similar tests. With this in mind we make the following definition.

Definition 1. Let ϕ be a test for $H_0 : \gamma \in \Omega_0$ against $H_1 : \gamma \in \Omega_1$ and let $W = \overline{\Omega}_0 \cap \overline{\Omega}_1$. Suppose that T is a sufficient statistic for $\{P_\gamma : \gamma \in W\}$. For $\alpha \in [0, 1]$, a test ϕ is said to have *Neyman structure* if $\mathbb{E}_\gamma[\phi|T] = \alpha$ almost surely for all $\gamma \in W$.

Thus tests with Neyman structure are precisely those test that satisfy condtion (1). Note that all tests with Neyman structure are α-similar. This is because for $\gamma \in W$,

$$\begin{aligned}\mathbb{E}_\gamma \phi &= \mathbb{E}_\gamma \left[\mathbb{E}[\phi|T]\right] \\ &= \mathbb{E}_\gamma[\alpha] \\ &= \alpha.\end{aligned}$$

The converse is not true in general. Note that we can define the function $g(T) = \mathbb{E}_\gamma[\phi|T] - \alpha$ for some $\gamma \in W$. The function g is well defined because T is sufficient for $\{P_\gamma : \gamma \in W\}$.The ϕ has Neyman structure if $g(t)$ is almost surely 0. On the other hand suppose that ϕ is α-similar. Then, for all $\gamma \in W$, we have

$$\mathbb{E}_\gamma[g(t)] = \mathbb{E}_\gamma[\mathbb{E}_\gamma[\phi|T] - \alpha] = \mathbb{E}_\gamma[\phi] - \alpha = 0.$$

Thus ϕ being α-similar implies that $g(T)$ is first order ancillary. Thus for a converse we need completeness.

Lemma 2. *If T is sufficient and complete for $\{P_\gamma : \gamma \in W\}$, then every α-similar test has Neyman structure.*

Proof. As before let $g(T) = \mathbb{E}_\gamma[\phi|T]$ which is well defined by sufficiency. As noted above $g(T)$ is first order ancillary. Since T is complete, this implies that $g(T) = 0$ almost surely. Thus $\mathbb{E}_\gamma[\phi|T] = \alpha$ almost surely and so ϕ has Neyman structure. □

Combing what we have done this lecture with Theorem (1) we have

Theorem 2. *Suppose $\beta_\phi(\gamma) = \mathbb{E}_\gamma[\phi]$ is continuous for every test ϕ. If ϕ^* is UMP among level α tests with Neyman structure, then*

(1) The test ϕ is UMP among α-similar tests.

(2) The test ϕ is UMPU at level α.

3 A Poisson example

Consider data (X, Y) where $X \sim \text{Pois}(v)$, $Y \sim \text{Pois}(u)$ and X and Y are independent. For example Y could model the number of people who recovered from a disease after recieving a new medicine and X could model the number of people who recovered from the same disease in a control group. With this application in mind we would like to test the hypotheses

$$H_0 : u \leq v \text{ against } H_1 : u > v.$$

Thus rejecting the null would correspond to a belief that our drug increases the chance of recovery. So that we can work with an exponential family we will rewrite these hypotheses as

$$H_0 : \log\left(\frac{u}{v}\right) \leq 0 \text{ against } H_1 : \log\left(\frac{u}{v}\right) > 0.$$

The joint density of (X, Y) is

$$\begin{aligned} p_{u,v}(x,y) &= \frac{\exp\{-v\}\, v^x}{x!}\frac{\exp\{-u\}\, v^y}{y!} \\ &= \frac{1}{x!y!}\exp\{x\log(v) + y\log(u) - v - u\} \\ &\propto \frac{1}{x!y!}\exp\left\{y\log\left(\frac{u}{v}\right) + (x+y)\log(u)\right\}. \end{aligned}$$

If we use the notation of the previous example we have

$$\gamma = (\theta, \lambda) = \left(\log\left(\frac{u}{v}\right), \log(v)\right),$$

and

$$(U, T) = (Y, X + Y).$$

Our goal is to test $\theta < 0$ in the presence of the nuisance parameter λ. Our first step is to check that T is sufficient for fixed θ and that θ is complete on the boundary $W = \{(0, \lambda)\}$. One can check that when

$$Y|(X+Y=n) \sim \text{Binomial}\left(n, \frac{u}{u+v}\right).$$

Note that if $\log(\frac{u}{v}) = \theta$ and $\log(v) = \lambda$, then $v = \exp\{\lambda\}$ and $u = \exp\{\theta + \lambda\}$. Thus

$$\frac{u}{u+v} = \frac{\exp\{\theta+\lambda\}}{\exp\{\theta+\lambda\} + \exp\{\lambda\}} = \frac{\exp\{\theta\}}{\exp\{\theta\}+1}.$$

Thus for fixed θ, the distribution of $Y|X+Y$ does not depend on λ and so $T = X + Y$ is sufficient for Y. Furthermore on $W = \{(\theta, \lambda) : \theta = 0\}$ we have

$$p(x, y; \lambda) \propto \frac{1}{x!y!}\exp\{(x+y)\lambda\},$$

and so T is complete on the boundary by results on exponential famillies.

We next have the derive the UMP test with Neyman structure. For all n, the family model

$$\left\{\text{Binomial}\left(n, \frac{e^\theta}{1+e^\theta}\right) : \theta \in \mathbb{R}\right\},$$

has monotone likelihood ratio in Y. Furthermore when $\theta = 0$, the distribution of $X|X+Y=n$ is Binomial$(n, 0.5)$. Thus the optimal test for H_0 against H_1 is

$$\phi(k,n) = \begin{cases} 1 & \text{if } k > c_n, \\ \rho_n & \text{if } k = c_n, \\ 0 & \text{if } k < c_n. \end{cases}$$

where the constants ρ_n and c_n are chosen so that

$$\mathbb{P}(Z > c_n) + \rho_n \mathbb{P}(Z = c_n) = \alpha,$$

where $Z \sim \text{Binomial}(n, 0.5)$. The test ϕ is the UMP level α test with Neyman structure. Thus the results of the previous section imply that ϕ is UMPU at level α.