

# STATS310A - Lecture 3

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9/28/21

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## 1 Homework

Read chapters 3 and 4. Do book questions 3.6, 3.7, 3.8, 4.5, 4.15 in the books. If  $\xi_n > 0$ ,  $\xi_n \rightarrow \infty$  and  $\frac{\xi_n}{\sqrt{n}} \rightarrow 0$ , then

$$P\{S_n > \sqrt{n}\xi_n\} \geq \exp(-(1 + o(1))\xi_n^2/2).$$

See canvas for more details. There are some hints for textbook questions in the back of the textbook.

## 2 Extending measures

Recall our set up:  $\Omega$  is a set,  $\mathcal{F}_0$  is an algebra of subsets of  $\Omega$ ,  $P$  is a probability measure on  $\mathcal{F}_0$ .

A set  $A \subseteq \Omega$  is measurable if

$$P^*(E) = P^*(E \cap A) + P^*(E \cap A^c),$$

for all  $E \subseteq \Omega$ . Where we defined

$$P^*(B) = \inf \left\{ \sum_{i=1}^{\infty} P(A_i) : A_i \in \mathcal{F}_0 \text{ and } B \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Let  $\mathcal{M}$  be the set of measurable subsets of  $\Omega$ . We have been proving the following:

**Theorem 1.** *Let  $P^*$  and  $\mathcal{M}$  be as above, then*

- $\mathcal{M}$  is  $\sigma$ -algebra.
- $P^*$  is a probability on  $\mathcal{M}$ .
- $P^*$  extends  $P$  on  $\mathcal{F}_0 \subseteq \mathcal{M}$ .
- $P^*$  is unique

Last time we showed that  $\mathcal{M}$  is an algebra. We used the key trick that  $A \in \mathcal{M}$  if and only if

$$P^*(E) \geq P^*(E \cap A) + P^*(E \cap A^c),$$

since the other inequality always holds by subadditivity.

Step 1 If  $A_i \in \mathcal{M}$  is a countable collection of disjoint sets, then for every  $E \subseteq \Omega$ ,

$$P^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} P^*(E \cap A_i).$$

*Proof.* We first prove the above for finite unions  $A_1, \dots, A_n$ . If  $n = 1$ , then we simply have

$$P^*(E \cap A_1) = P^*(E \cap A_1).$$

Suppose  $n = 2$ . Since  $A_1 \in \mathcal{M}$  and  $A_1 \cap A_2 = \emptyset$ , we have

$$\begin{aligned} P^*(E \cap (A_1 \cup A_2)) &= P^*(E \cap (A_1 \cup A_2) \cap A_1) + P^*(E \cap (A_1 \cup A_2) \cap A_1^c) \\ &= P^*(E \cap A_1) + P^*(E \cap A_2). \end{aligned}$$

Thus the result holds for  $n = 2$ . Since  $\mathcal{M}$  is a field we can use induction to conclude the result for general  $n \in \mathbb{N}$ . Now suppose  $A = \bigcup_{i=1}^{\infty} A_i$ , where the sets  $A_i$  are disjoint and measurable. Let  $F_n = \bigcup_{i=1}^n A_i$ , then

$$\begin{aligned} P^*(E \cap A) &\geq P^*(E \cap F_n) \\ &= \sum_{i=1}^n P^*(E \cap A_i). \end{aligned}$$

Thus  $P^*(E \cap A) \geq \sum_{i=1}^{\infty} P^*(E \cap A_i)$ . Also  $P^*(E \cap A) \leq \sum_{i=1}^{\infty} P^*(E \cap A_i)$  by countable subadditivity.  $\square$

Step 2 The collection  $\mathcal{M}$  is a  $\sigma$ -algebra and  $P^*$  is countably additive on  $\mathcal{M}$ .

*Proof.* If  $E = \Omega$  in step 1, then we can immediately see that  $P^*$  is countably additive on  $\mathcal{M}$ . To show  $\mathcal{M}$  is a  $\sigma$ -algebra, we need to show that  $\mathcal{M}$  is closed under countable unions.

Let  $A_i \in \mathcal{M}$  for each  $i \in \mathbb{N}$ . If we define  $A'_i = A_i \cap \left(\bigcup_{j=1}^{i-1} A_j^c\right)$ , then  $A'_i \in \mathcal{M}$ , the sets  $A'_i$  are disjoint and  $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A'_i$ . Thus we may assume that the sets  $A_i$  are disjoint.

As before, define

$$F_n = \bigcup_{i=1}^n A_i \text{ and } A = \bigcup_{i=1}^{\infty} A_i.$$

Note that  $A^c \subseteq F_n^c$ . We know that  $F_n \in \mathcal{M}$  and thus for any  $E \subseteq \Omega$ ,

$$\begin{aligned} P^*(E) &= P^*(E \cap F_n) + P^*(E \cap F_n^c) \\ &= \sum_{i=1}^n P^*(E \cap A_i) + P^*(E \cap F_n^c) \\ &\geq \sum_{i=1}^n P^*(E \cap A_i) + P^*(E \cap A^c). \end{aligned}$$

Letting  $n$  go to infinity we can conclude

$$P^*(E) \geq \sum_{i=1}^{\infty} P^*(E \cap A_i) + P^*(E \cap A^c) = P^*(E \cap A) + P^*(E \cap A^c),$$

thus  $A \in \mathcal{M}$ .  $\square$

Step 3  $\mathcal{F}_0 \subseteq \mathcal{M}$ .

*Proof.* Pick  $A \in \mathcal{F}_0$ ,  $E \subseteq \Omega$  and  $\varepsilon > 0$ . Find a collection  $(A_i)_{i=1}^\infty$  such that  $E \subseteq \bigcup_{i=1}^\infty A_i$ ,  $A_i \in \mathcal{F}_0$  and  $\sum_{i=1}^\infty P(A_i) \leq P^*(A) + \varepsilon$ . Let  $B_n = A_n \cap A$  and  $C_n = A_n \cap A^c$ . Then  $E \cap A \subseteq \bigcup_{n=1}^\infty B_n$  and  $E \cap A^c \subseteq \bigcup_{n=1}^\infty C_n$  and  $B_n, C_n \in \mathcal{F}_0$ . Thus

$$P^*(E \cap A) \leq \sum_{n=1}^\infty P(B_n) \text{ and } P^*(E \cap A^c) \leq \sum_{n=1}^\infty P(C_n).$$

Thus

$$\begin{aligned} P^*(E) &\geq \sum_{n=1}^\infty P(A_n) - \varepsilon \\ &= \left( \sum_{n=1}^\infty P(B_n) + P(C_n) \right) - \varepsilon \\ &= \left( \sum_{n=1}^\infty P(B_n) + \sum_{n=1}^\infty P(C_n) \right) - \varepsilon \\ &\geq P^*(E \cap A) + P^*(E \cap A^c) - \varepsilon. \end{aligned}$$

Letting  $\varepsilon$  go to 0, we see that  $P^*(E) \geq P^*(E \cap A) + P^*(E \cap A^c)$  so  $A \in \mathcal{M}$ . □

Step 4 If  $A \in \mathcal{F}_0$ , then  $P^*(A) = P(A)$ . (see proof in the textbook).

### 3 Uniqueness of the extension

We will use the  $\pi - \lambda$  theorem.

**Definition 1.** A collection of sets  $\mathcal{P}$  is a  $\pi$ -system if  $\mathcal{P}$  is closed under finite intersection.

**Definition 2.** A collection of sets  $L$  is a  $\lambda$ -system if

- (a)  $\Omega \in L$ ,
- (b)  $A \in L$  implies  $A^c \in L$ ,
- (c)  $L$  is closed under countable disjoint unions.

**Example 1.** If  $\Omega = \{1, 2, 3, 4\}$ , then

$$S = \{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

is a  $\lambda$ -system but not a  $\pi$ -system and not an algebra.

The following is due to Dynkin and is a clean substitute for monotone class arguments.

**Theorem 2.** If  $\mathcal{P}$  is a  $\pi$ -system and  $L$  is a  $\lambda$ -system and  $\mathcal{P} \subseteq L$ , then  $\sigma(\mathcal{P}) \subseteq L$ .

As an immediate application we have the following.

**Proposition 1.** Suppose  $\mathcal{F}_0$  is an algebra of subsets of  $\Omega$  and  $P$  is a probability on  $\mathcal{F}_0$ . If  $P'$  and  $P''$  are two probability measures on  $\sigma(\mathcal{F}_0)$  that extend  $P$ , then  $P' = P''$ .

*Proof.* Let  $L = \{A \in \sigma(\mathcal{F}_0) : P'(A) = P''(A)\}$ . Then  $L$  is a  $\lambda$ -system and since  $\mathcal{F}_0$  is a  $\pi$ -system contained in  $L$ , we have  $\sigma(\mathcal{F}_0) \subseteq L$  and thus  $P'$  and  $P''$  agree on  $\sigma(\mathcal{F}_0)$ . □

Note that in the case of section 2 we know that  $\sigma(\mathcal{F}_0) \subseteq \mathcal{M}$ .

**Corollary 1.** *Lebesgue measure ( $P^*$  on  $(0, 1]$ ) is the unique extension of length to the Borel sets.*

Important note: When extending from  $\mathcal{F}_0$  to  $\mathcal{M}$ , we started with a probability on  $\mathcal{F}_0$ . This means that if  $A \in \mathcal{F}_0$  and  $(A_i)_{i=1}^\infty$  is a countable disjoint collection of sets in  $\mathcal{F}_0$  such that  $A = \bigcup_{i=1}^\infty A_i$ , then  $P(A) = \sum_{i=1}^\infty P(A_i)$ . Proving this in the Lebesgue measure case requires the Heine-Borel theorem. We need to use compactness of closed, bounded intervals at some point.

## 4 Comments on homework

- 3.6-3.8 are similar to what we have done in class but for finitely additive measures.
- 4.5 is about limit superiors and limit inferiors.
- 4.15 is about showing that the square free numbers have density  $6\pi^2$ !