STATS310A - Lecture 3

Persi Diaconis
Scribed by Michael Howes

9/28/21

Contents

1 Homework

Read chapters 3 and 4. Do book questions 3.6, 3.7, 3.8, 4.5, 4.15 in the books. If $\xi_n > 0$, $\xi_n \to \infty$ and $\frac{\xi_n}{\sqrt{n}} \to 0$, then

$$P\{S_n > \sqrt{n}\xi_n\} \geq \exp(-(1+o(1))\xi_n^2/2).$$

See canvas for more details. There are some hints for textbook questions in the back of the textbook.

2 Extending measures

Recall our set up: Ω is a set, $\mathcal{F}_0$ is an algebra of subsets of Ω, P is a probability measure on $\mathcal{F}_0$.

A set $A \subseteq \Omega$ is measurable if

$$P^*(E) = P^*(E \cap A) + P^*(E \cap A^c),$$

for all $E \subseteq \Omega$. Where we defined

$$P^*(B) = \inf\left\{\sum_{i=1}^\infty P(A_i) : A_i \in \mathcal{F}_0 \text{ and } B \subseteq \bigcup_{i=1}^\infty A_i\right\}.$$

Let $\mathcal{M}$ be the set of measurable subsets of Ω. We have been proving the following:

Theorem 1. *Let P^* and $\mathcal{M}$ be as above, then*

- *$\mathcal{M}$ is σ-algebra.*
- *P^* is a probability on $\mathcal{M}$.*
- *P^* extends P on $\mathcal{F}_0 \subseteq \mathcal{M}$.*
- *P^* is unique*

Last time we showed that $\mathcal{M}$ is an algebra. We used the key trick that $A \in \mathcal{M}$ if and only if

$$P^*(E) \geq P^*(E \cap A) + P^*(E \cap A^c),$$

since the other inequality always holds by subadditivity.

Step 1 If $A_i \in \mathcal{M}$ is a countable collection of disjoint sets, then for every $E \subseteq \Omega$,

$$P^*\left(E \cap \left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \sum_{i=1}^{\infty} P^*(E \cap A_i).$$

Proof. We first prove the above for finite unions $A_1, \ldots, A_n$. If $n = 1$, then we simply have

$$P^*(E \cap A_1) = P^*(E \cap A_1).$$

Suppose $n = 2$. Since $A_1 \in \mathcal{M}$ and $A_1 \cap A_2 = \emptyset$, we have

$$\begin{aligned} P^*(E \cap (A_1 \cup A_2)) &= P^*(E \cap (A_1 \cup A_2) \cap A_1) + P^*(E \cap (A_1 \cup A_2) \cap A_1^c) \\ &= P^*(E \cap A_1) + P^*(E \cap A_2). \end{aligned}$$

Thus the result holds for $n = 2$. Since $\mathcal{M}$ is a field we can use induction to conclude the result for general $n \in \mathbb{N}$. Now suppose $A = \bigcup_{i=1}^{\infty} A_i$, where the sets A_i are disjoint and measurable. Let $F_n = \bigcup_{i=1}^{n} A_i$, then

$$\begin{aligned} P^*(E \cap A) &\geq P^*(E \cap F_n) \\ &= \sum_{i=1}^{n} P^*(E \cap A_i). \end{aligned}$$

Thus $P^*(E \cap A) \geq \sum_{i=1}^{\infty} P^*(E \cap A)$. Also $P^*(E \cap A) \leq \sum_{i=1}^{\infty} P^*(E \cap A_i)$ by countable subadditivity. $\square$

Step 2 The collection $\mathcal{M}$ is a σ-algebra and P^* is countably additive on $\mathcal{M}$.

Proof. If $E = \Omega$ in step 1, then we can immediately see that P^* is countably additive on $\mathcal{M}$. To show $\mathcal{M}$ is a σ-algebra, we need to show that $\mathcal{M}$ is closed under countable unions.

Let $A_i \in \mathcal{M}$ for each $i \in \mathbb{N}$. If we define $A_i' = A_i \cap \left(\bigcup_{j=1}^{i-1} A_j^c\right)$, then $A_i' \in \mathcal{M}$, the sets A_i' are disjoint and $\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} A_i'$. Thus we may assume that the sets A_i are disjoint.

As before, define

$$F_n = \bigcup_{i=1}^{n} A_i \text{ and } A = \bigcup_{i=1}^{\infty} A_i.$$

Note that $A^c \subseteq F_n^c$. We know that $F_n \in \mathcal{M}$ and thus for any $E \subseteq \Omega$,

$$\begin{aligned} P^*(E) &= P^*(E \cap F_n) + P^*(E \cap F_n^c) \\ &= \sum_{i=1}^{n} P^*(E \cap A_i) + P^*(E \cap F_n^c) \\ &\geq \sum_{i=1}^{n} P^*(E \cap A_i) + P^*(E \cap A^c). \end{aligned}$$

Letting n go to infinity we can conclude

$$P^*(E) \geq \sum_{i=1}^{\infty} P^*(E \cap A_i) + P^*(E \cap A^c) = P^*(E \cap A) + P^*(E \cap A^c),$$

thus $A \in \mathcal{M}$. $\square$

Step 3 $\mathcal{F}_0 \subseteq \mathcal{M}$.

Proof. Pick $A \in \mathcal{F}_0$, $E \subseteq \Omega$ and $\varepsilon > 0$. Find a collection $(A_i)_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty} A_i$, $A_i \in \mathcal{F}_0$ and $\sum_{i=1}^{\infty} P(A_i) \leq P^*(A) + \varepsilon$. Let $B_n = A_n \cap A$ and $C_n = A_n \cap A^c$. Then $E \cap A \subseteq \bigcup_{n=1}^{\infty} B_n$ and $E \cap A^c \subseteq \bigcup_{n=1}^{\infty} C_n$ and $B_n, C_n \in \mathcal{F}_0$. Thus

$$P^*(E \cap A) \leq \sum_{n=1}^{\infty} P(B_n) \text{ and } P^*(E \cap A^c) \leq \sum_{n=1}^{\infty} P(C_n).$$

Thus

$$\begin{aligned} P^*(E) &\geq \sum_{n=1}^{\infty} P(A_n)) - \varepsilon \\ &= \left(\sum_{n=1}^{\infty} P(B_n) + P(C_n) \right) - \varepsilon \\ &= \left(\sum_{n=1}^{\infty} P(B_n) + \sum_{n=1}^{\infty} P(C_n) \right) - \varepsilon \\ &\geq P^*(E \cap A) + P^*(E \cap A^c) - \varepsilon. \end{aligned}$$

Letting ε go to 0, we see that $P^*(E) \geq P^*(E \cap A) + P^*(E \cap A^c)$ so $A \in \mathcal{M}$. □

Step 4 If $A \in \mathcal{F}_0$, then $P^*(A) = P(A)$. (see proof in the textbook).

3 Uniqueness of the extension

We will use the $\pi - \lambda$ theorem.

Definition 1. A collection of sets $\mathcal{P}$ is a π-*system* if $\mathcal{P}$ is closed under finite intersection.

Definition 2. A collection of sets L is a λ-*system* if

(a) $\Omega \in L$,

(b) $A \in L$ implies $A^c \in L$,

(c) L is closed under countable disjoint unions.

Example 1. If $\Omega = \{1, 2, 3, 4\}$, then

$$S = \{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

is a λ-system but not a π-system and not an algebra.

The following is due to Dynkin and is a clean subsititute for monotone class arguments.

Theorem 2. *If $\mathcal{P}$ is a π-system and L is a λ-system and $\mathcal{P} \subseteq L$, then $\sigma(\mathcal{P}) \subseteq L$.*

As an immediate application we have the following.

Proposition 1. *Suppose $\mathcal{F}_0$ is an algebra of subsets of Ω and P is a probability on $\mathcal{F}_0$. If P' and P'' are two probability measures on $\sigma(\mathcal{F}_0)$ that extend P, then $P' = P''$.*

Proof. Let $L = \{A \in \sigma(\mathcal{F}_0) : P'(A) = P''(A)\}$. Then L is a λ-system and since $\mathcal{F}_0$ is a π-system contained in L, we have $\sigma(\mathcal{F}_0) \subseteq L$ and thus P' and P'' agree on $\sigma(\mathcal{F}_0)$. □

Note that in the case of section 2 we know that $\sigma(\mathcal{F}_0) \subseteq \mathcal{M}$.

Corollary 1. *Lebesgue measure (P^* on $(0,1]$) is the unique extension of length to the Borel sets.*

Important note: When extending from $\mathcal{F}_0$ to $\mathcal{M}$, we started with a probability on $\mathcal{F}_0$. This means that if $A \in \mathcal{F}_0$ and $(A_i)_{i=1}^{\infty}$ is a countable disjoint collection of sets in $\mathcal{F}_0$ such that $A = \bigcup_{i=1}^{\infty} A_i$, then $P(A) = \sum_{i=1}^{\infty} P(A_i)$. Proving this in the Lebesgue measure case requires the Heine-Borel theorem. We need to use compactness of closed, bounded intervals at some point.

4 Comments on homework

- 3.6-3.8 are similar to what we have done in class but for finitely additive measures.
- 4.5 is about limit superiors and limit inferiors.
- 4.15 is about showing that the square free numbers have density $6\pi^2$!