

STATS310A - Lecture 8

Persi Diaconis
Scribed by Michael Howes

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1 Integration

1.1 Definition

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f : \Omega \rightarrow [-\infty, \infty]$ be measurable. We wish to define

$$\int f d\mu = \int_{\Omega} f d\mu = \int_{\Omega} f(\omega) \mu(d\omega).$$

Definition 1. The function f is a *simple function* if there exists a finite partition of Ω $\{A_i\}_{i=1}^n$ such that $A_i \in \mathcal{F}$ and values $x_i \in [-\infty, \infty]$ such that

$$f(\omega) = \sum_{i=1}^n x_i \delta_{A_i}(\omega),$$

where we use the conventions $0 \cdot \infty = \infty \cdot 0 = 0$ and $x \cdot \infty = \infty \cdot x = \infty$ if $x \in (0, \infty]$.

Lemma 1. If $f \geq 0$, then there exists a sequence of simple functions $f_n \geq 0$ such that $f_n(\omega) \nearrow f(\omega)$ for all $\omega \in \Omega$.

Proof. Fix n and define

$$f_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(\omega) < \frac{k}{2^n} \text{ for some } k = 1, \dots, n2^n, \\ n & \text{if } f(\omega) \geq n. \end{cases} \quad \square$$

Definition 2. Suppose $f(\omega) \geq 0$ for all ω . Define

$$\int_{\Omega} f(\omega) \mu(d\omega) = \sup \left(\sum_{i=1}^n \nu_i \mu(A_i) \right),$$

where the supremum is over all measurable partitions of Ω and $\nu_i = \inf_{\omega \in A_i} f(\omega)$. For a general f define

$$f_+(\omega) = \max\{f(\omega), 0\} \text{ and } f_-(\omega) = \max\{-f(\omega), 0\}.$$

Note that $f_+, f_- \geq 0$ and $|f| = f_+ + f_-$ and $f = f_+ - f_-$. We define $\int f d\mu$ based on cases.

- If $\int f_+ d\mu < \infty$ and $\int f_- d\mu < \infty$, then define $\int f = \int f_+ d\mu - \int f_- d\mu$.
- If $\int f_+ d\mu = \infty$ and $\int f_- d\mu < \infty$, then define $\int f d\mu = \infty$.
- If $\int f_+ d\mu < \infty$ and $\int f_- d\mu = \infty$, then define $\int f d\mu = -\infty$.
- If $\int f_+ d\mu = \infty$ and $\int f_- d\mu = \infty$, then $\int f d\mu$ is not defined.

1.2 Properties

Proposition 1. Suppose $f, g, f_n \geq 0$.

- (a) If $f(\omega) = \sum_{i=1}^n x_i \delta_{A_i}(\omega)$ is simple, then $\int f d\mu = \sum_{i=1}^n x_i \mu(A_i)$.
- (b) If $f(\omega) \leq g(\omega)$ for all ω , then $\int f d\mu \leq \int g d\mu$.
- (c) If $f_n(\omega) \nearrow f(\omega)$ for all n and ω , then $\int f_n d\mu \nearrow \int f d\mu$. (Monotone convergence theorem).
- (d) If $\alpha, \beta \geq 0$, then $\int \alpha f + \beta g d\mu = \alpha \int f d\mu + \beta \int g d\mu$.

Proof. (a) Fix a partition $\{B_i\}_{i=1}^m$ and let $\beta_j = \inf_{\omega \in B_j} f(\omega)$. If $\omega \in A_i \cap B_j$, then $\beta_j \leq f(\omega) = x_i$.

Thus

$$\begin{aligned}
 \sum_{j=1}^m \beta_j \mu(B_j) &= \sum_{j=1}^m \beta_j \sum_{i=1}^n \mu(A_i \cap B_j) \\
 &= \sum_{i=1}^n \sum_{j=1}^m \beta_j \mu(A_i \cap B_j) \\
 &\leq \sum_{i=1}^n \sum_{j=1}^m x_i \mu(A_i \cap B_j) \\
 &= \sum_{i=1}^n x_i \sum_{j=1}^m \mu(A_i \cap B_j) \\
 &= \sum_{i=1}^n x_i \mu(A_i).
 \end{aligned}$$

Thus $\int f d\mu \leq \sum_{i=1}^n x_i \mu(A_i)$. The other direction we get for free since $\{A_i\}_{i=1}^n$ is a partition and f equals x_i on A_i .

- (b) This follows from the definition since

$$\inf\{f(\omega) : \omega \in A_i\} \leq \inf\{g(\omega) : \omega \in A_i\},$$

for any measurable set A_i .

- (c) By (b) we know that $\int f_n d\mu$ is an increasing sequence and that $\int f_n d\mu$ is bounded above by $\int f d\mu$. Thus $\lim_n \int f_n d\mu$ exists and $\lim_n \int f_n d\mu \leq \int f d\mu$. It remains to prove that $\int f d\mu \leq \lim_n \int f_n d\mu$. Thus we must show that for every partition $\{A_i\}_{i=1}^m$, we have

$$\sum_{i=1}^m \nu_i \mu(A_i) \leq \lim_n \int f_n d\mu,$$

where $\nu_i = \inf\{f(\omega) : \omega \in A_i\}$. Let $S = \sum_{i=1}^m \nu_i \mu(A_i)$. We will consider different cases.

- i. Suppose that S is finite and $0 < \nu_i < \infty$ and $0 < \mu(A_i) < \infty$ for all i . Choose ε such that $\varepsilon < \mu(A_i)$ for all i . Define

$$A_{n,i} = \{\omega \in A_i : f_n(\omega) \geq \nu_i - \varepsilon\}.$$

Since $f_n \nearrow f$, we know that $A_{n,i} \nearrow A_i$ as $n \rightarrow \infty$. Thus $\mu(A_{n,i}) \nearrow \mu(A_i)$. Note that

$$\int f_n d\mu \geq \sum_{i=1}^m (\nu_i - \varepsilon) \mu(A_{n,i}).$$

Thus

$$\lim_n \int f_n d\mu \geq \sum_{i=1}^n (\nu_i - \varepsilon) \mu(A_i) = \sum_{i=1}^m \nu_i \mu(A_i) - \varepsilon \sum_{i=1}^m \mu(A_i).$$

Letting $\varepsilon \rightarrow 0$, we see that $\lim_n \int f_n d\mu \geq \sum_{i=1}^n \nu_i \mu(A_i) = S$. Thus $\lim_n \int f_n d\mu \geq \int f d\mu$.

- ii. Now suppose that $S < \infty$ and ν_i or $\mu(A_i)$ is equal to 0 or ∞ for some i . By reordering we have $0 < \nu_i, \mu(A_i) < \infty$ for $i = 1, \dots, i_0$ and the rest of the terms are of the form $0 \cdot \infty$, $\infty \cdot 0$ or $0 \cdot 0$. We can then apply case 1 to the partition $\{A_i\}_{i=1}^{i_0} \cup \left\{ \left(\bigcup_{i=1}^{i_0} A_i \right)^c \right\}$.
- iii. Suppose that $S = \infty$. Then for some i_0 we have $\nu_{i_0} \mu(A_{i_0}) = \infty$. Thus either $\nu_{i_0} = \infty$ and $\mu(A_{i_0}) > 0$ or $\nu_{i_0} > 0$ and $\mu(A_{i_0}) = \infty$. Choose $x, y > 0$ so that $0 < x < \nu_{i_0}$ and $0 < y < \mu(A_{i_0})$. Let $B_n = \{\omega : f_n(\omega) > x\}$. Then $B_n \nearrow B = \{\omega : f(\omega) \geq x\}$. So $\mu(B_n) \nearrow \mu(B) \geq \mu(A_{i_0}) \geq y$. Thus by using the partition $\{B_n, B_n^c\}$, we see that

$$\int f_n d\mu \geq x \cdot \mu(B_n).$$

Thus $\lim_n \int f_n d\mu \geq xy$. The product xy can be arbitrary large and thus $\int f_n d\mu \geq \infty$.

- (d) Suppose f and g are both simple functions $f = \sum_{i=1}^n x_i \delta_{A_i}$ and $g = \sum_{j=1}^m y_j \delta_{B_j}$. Then

$$\alpha f + \beta g = \sum_{i=1}^n \sum_{j=1}^m (\alpha x_i + \beta y_j) \delta_{A_i \cap B_j}.$$

Thus

$$\begin{aligned} \int \alpha f + \beta g d\mu &= \sum_{i=1}^n \sum_{j=1}^m (\alpha x_i + \beta y_j) \mu(A_i \cap B_j) \\ &= \alpha \sum_{i=1}^n x_i \sum_{j=1}^m \mu(A_i \cap B_j) + \beta \sum_{j=1}^m y_j \sum_{i=1}^n \mu(A_i \cap B_j) \\ &= \alpha \sum_{i=1}^n x_i \mu(A_i) + \beta \sum_{j=1}^m y_j \mu(B_j) \\ &= \alpha \int f d\mu + \beta \int g d\mu. \end{aligned}$$

Now for general f and g , let $f_n \nearrow f$ and $g_n \nearrow g$ be approximating sequences of simple functions. Then $(\alpha f_n + \beta g_n) \nearrow \alpha f + \beta g$. Thus

$$\begin{aligned} \int \alpha f + \beta g d\mu &= \int \alpha f_n + \beta g_n d\mu \\ &= \alpha \int f_n d\mu + \beta \int g_n d\mu \\ &= \alpha \int f d\mu + \beta \int g d\mu. \end{aligned}$$

□

1.3 Remarks

- (a) If we have a continuous function $f : [a, b] \rightarrow \mathbb{R}$ and μ is Lebesgue measure on $[a, b]$, then $\int f d\mu$ agrees with the Riemann integral.
- (b) In the same setting as (a), the function $f = \delta_{\mathbb{Q} \cap [a, b]}$ is not Riemann integrable but $\int f d\mu$ does exist and $\int f d\mu = 0$.
- (c) We still need Riemann integrals for improper integrals. For example $\int_0^\infty \frac{\sin(x)}{x} dx$ is not Lebesgue integrable but it does have a convergent improper Riemann integral.
- (d) Riemann integration is also needed for calculations, for Brownian motion and many other things.
- (e) The Henstock integral generalises both the Riemann and Lebesgue integrals.

Example 1. In general $f_n \rightarrow f$, does not imply $\int f_n d\mu \rightarrow \int f d\mu$. Consider

$$f_n(\omega) = \begin{cases} n^2 & \text{if } \omega \in (0, 1/n) \\ 0 & \text{else.} \end{cases}$$

Then $f_n(\omega) \rightarrow 0$ for all ω but $\int f_n d\mu = n^2 \cdot \frac{1}{n} = n \nearrow \infty$.

The properties (a)-(d) hold if the hypotheses hold *almost surely*. For example for (b) we can prove

$$\mu(\{\omega : f(\omega) > g(\omega)\}) = 0 \implies \int f d\mu \leq \int g d\mu.$$

Proof. Let $G = \{\omega : f(\omega) \leq g(\omega)\}$. By hypothesis $\mu(G^c) = 0$. For every partition $\{A_i\}_{i=1}^m$, $\mu(A_i) = \mu(A_i \cap G)$. It follows that

$$\begin{aligned} \sum_{i=1}^n \inf_{A_i} f(\omega) \mu(A_i) &= \sum_{i=1}^m \inf_{A_i} f(\omega) \mu(A_i \cap G) \\ &\leq \sum_{i=1}^m \inf_{A_i \cap G} f(\omega) \mu(A_i \cap G) \\ &\leq \sum_{i=1}^m \inf_{A_i \cap G} g(\omega) \mu(A_i \cap G) \\ &= \sum_{i=1}^m \inf_{A_i \cap G} g(\omega) \mu(A_i \cap G) + \inf_{G^c} g(\omega) \mu(G^c) \\ &\leq \int g d\mu. \end{aligned}$$

□

Definition 3. If $(\Omega, \mathcal{F}, \mu)$ is a probability space and $X : (\Omega, \mathcal{F}) \rightarrow \mathbb{R}$ is a random variable, then we call $\int f d\mu$ the *expectation of f* and we write

$$\mathbb{E}[X] := \int X d\mu.$$