

# STATS310A - Lecture 16

Persi Diaconis  
Scribed by Michael Howes

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## 1 Announcements

Final homework uploaded to Canvas and due November 30.

- Read chapters 25, 26.
- Do 25 / 1, 3 and 26 / 1, 3, 12-14.
- The hints contain surprises.

## 2 The central limit theorem

We are in the process of proving Lindeberg's version of the central limit theorem. Recall that we have a triangular array of random variables  $\{X_{n,i}\}$  where  $i = 1, \dots, k_n$  and  $n = 1, 2, \dots$ . We assume that the array has independent rows. That is, for each  $n$ ,  $\{X_{n,i}\}_{i=1}^{k_n}$  are independent. Assume also that

$$\mathbb{E}[X_{n,i}] = 0 \quad \text{and} \quad \sigma_{i,n}^2 = \text{Var}(X_{n,i}) < \infty.$$

Define  $S_n = \sum_{i=1}^{k_n} X_{n,i}$  and  $s_n^2 = \sum_{i=1}^{k_n} \sigma_{i,n}^2 = \text{Var}(S_n)$ .

**Definition 1.** A triangular array with independent rows  $\{X_{n,i}\}$  is said to satisfy *Lindeberg's condition* if for all  $\varepsilon > 0$

$$\frac{1}{s_n^2} \sum_{i=1}^{k_n} \int_{\{|X_{n,i}| > \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P} \xrightarrow{n} 0.$$

Lindeberg's version of the central limit theorem is:

**Theorem 1** (Lindeberg). *Let  $\{X_{n,i}\}$  be a triangular array with independent rows. If  $\{X_{n,i}\}$  satisfy Lindeberg's condition, then for all  $x \in \mathbb{R}$ ,*

$$\mathbb{P}\left(\frac{S_n}{s_n} \leq x\right) \rightarrow \Phi(x),$$

where  $\Phi(x) = \mathbb{P}(Z \leq x)$  for  $Z \sim \mathcal{N}(0, 1)$ .

*Proof.* To prove this we will use the portmanteau theorem. Let  $C_c^\infty(\mathbb{R})$  be the class of infinitely differentiable functions on  $\mathbb{R}$  with compact support. By the portmanteau theorem it suffices to show that for all  $f \in C_c^\infty(\mathbb{R})$ ,  $\mathbb{E}[f(S_n/s_n)] \xrightarrow{n} \mathbb{E}[f(Z)]$  where  $Z \sim \mathcal{N}(0, 1)$ . Thus fix such an  $f$ . Define  $Z_{n,i}$  to be independent random variables such that  $Z_{n,i} \sim \mathcal{N}(0, \sigma_{n,i}^2)$ . Let  $Z_n = \sum_{i=1}^{k_n} Z_{n,i}$ . Then

$$Z = \frac{1}{s_n} Z_n \sim \mathcal{N}(0, 1).$$

The idea behind the proof is to swap out  $X_{n,i}$  for  $Z_{n,i}$  one at a time. With this in mind, define

$$T_{n,i} = X_{n,1} + \dots + X_{n,i-1} + Z_{n,i} + \dots + Z_{n,k_n}.$$

Note that  $X_i, Z_i$  are independent of  $T_{n,i}$  for each  $i$ . Furthermore we have

$$S_n = T_{n,k_n} + X_{n,k_n} \quad \text{and} \quad Z_n = T_{n,1} + Z_{n,1}.$$

And also

$$T_{n,i} + Z_{n,i} = T_{n,i-1} + X_{n,i-1},$$

for  $i = 2, \dots, k_n$ . Thus by telescoping we have

$$f\left(\frac{S_n}{s_n}\right) - f\left(\frac{Z_n}{s_n}\right) = \sum_{i=1}^{k_n} f\left(\frac{T_{n,i} + X_{n,i}}{s_n}\right) - f\left(\frac{T_{n,i} + Z_{n,i}}{s_n}\right).$$

And so

$$\left| \mathbb{E}\left[f\left(\frac{S_n}{s_n}\right)\right] - \mathbb{E}[f(Z)] \right| \leq \sum_{i=1}^{k_n} \left| \mathbb{E}\left[f\left(\frac{T_{n,i} + X_{n,i}}{s_n}\right)\right] - \mathbb{E}\left[f\left(\frac{T_{n,i} + Z_{n,i}}{s_n}\right)\right] \right| \quad (1)$$

We will now use Taylor's approximation to bound each of the terms in the above sum. For  $x, h \in \mathbb{R}$ , define

$$g(h) = \left| f(x+h) - f(x) - hf'(x) - \frac{h^2}{2}f''(x) \right|.$$

Since all derivatives of  $f$  are bounded, Taylor's approximation with remainder says that there exists  $k > 0$  such that for all  $h$  and  $x$

$$g(h) \leq k \min\{|h|^3, |h|^2\}.$$

Thus for all  $x, h_1, h_2 \in \mathbb{R}$  we have

$$\begin{aligned} \left| f(x+h_1) - f(x+h_2) - f'(x)(h_1-h_2) - \frac{1}{2}f''(x)(h_1^2-h_2^2) \right| &= |g(h_1) - g(h_2)| \\ &\leq |g(h_1)| + |g(h_2)|. \end{aligned}$$

We wish to apply this to equation (1) with  $x = \frac{T_{n,i}}{s_n}$ ,  $h_1 = \frac{X_{n,i}}{s_n}$  and  $h_2 = \frac{Z_{n,i}}{s_n}$ . Thus we need to add the high order terms  $f'(x)(h_1-h_2)$  and  $\frac{1}{2}f''(x)(h_1^2-h_2^2)$ . Since  $X_{n,i}$  and  $Z_{n,i}$  have the same mean

and variance and  $X_{n,i}, Z_{n,i}$  are independent of  $T_{n,i}$ , we have

$$\begin{aligned}
& \left| \mathbb{E} \left[ f \left( \frac{T_{n,i} + X_{n,i}}{s_n} \right) \right] - \mathbb{E} \left[ f \left( \frac{T_{n,i} + Z_{n,i}}{s_n} \right) \right] \right| \\
&= \left| \mathbb{E} \left[ f \left( \frac{T_{n,i} + X_{n,i}}{s_n} \right) \right] - \mathbb{E} \left[ f \left( \frac{T_{n,i} + Z_{n,i}}{s_n} \right) \right] - \mathbb{E} \left[ f' \left( \frac{T_{n,i}}{s_n} \right) \left( \frac{X_{n,i}}{s_n} - \frac{Z_{n,i}}{s_n} \right) \right] \right. \\
&\quad \left. - \frac{1}{2} \mathbb{E} \left[ f'' \left( \frac{T_{n,i}}{s_n} \right) \left( \frac{X_{n,i}^2}{s_n^2} - \frac{Z_{n,i}^2}{s_n^2} \right) \right] \right| \\
&\leq \mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) + g \left( \frac{Z_{n,i}}{s_n} \right) \right].
\end{aligned}$$

Thus combining this with equation (1), we have

$$\begin{aligned}
\left| \mathbb{E} \left[ f \left( \frac{S_n}{s_n} \right) \right] - \mathbb{E}[f(Z)] \right| &\leq \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) + g \left( \frac{Z_{n,i}}{s_n} \right) \right] \\
&= \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) \right] + \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{Z_{n,i}}{s_n} \right) \right] \\
&= (I) + (II).
\end{aligned}$$

We will deal with the sum (I) first. Recall that  $g(h) \leq k \min\{h^2, h^3\}$ . Thus we will split  $\mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) \right]$  into two regions where we will use two different bounds. For each  $\varepsilon > 0$ , we have

$$\begin{aligned}
(I) &= \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) \right] \\
&= \sum_{i=1}^{k_n} \int_{\{X_{n,i} \leq \varepsilon s_n\}} g \left( \frac{X_{n,i}}{s_n} \right) d\mathbb{P} + \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} g \left( \frac{X_{n,i}}{s_n} \right) d\mathbb{P} \\
&\leq k \sum_{i=1}^{k_n} \int_{\{X_{n,i} \leq \varepsilon s_n\}} \left| \frac{X_{n,i}}{s_n} \right|^3 d\mathbb{P} + k \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} \left| \frac{X_{n,i}}{s_n} \right|^2 d\mathbb{P} \\
&= \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} \leq \varepsilon s_n\}} \left| \frac{X_{n,i}}{s_n} \right| |X_{n,i}|^2 d\mathbb{P} + \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P} \\
&\leq \frac{k\varepsilon}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} \leq \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P} + \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P} \\
&\leq \frac{k\varepsilon}{s_n^2} \sum_{i=1}^{k_n} \text{Var}(X_{n,i}) + \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P} \\
&= k\varepsilon + \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{X_{n,i} > \varepsilon s_n\}} |X_{n,i}|^2 d\mathbb{P}.
\end{aligned}$$

By Lindeberg's condition we have that the second term goes to zero for all  $\varepsilon > 0$ . Thus

$$\lim_n \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{X_{n,i}}{s_n} \right) \right] \leq k\varepsilon.$$

And so the sum (I) goes to 0 as  $n$  goes to infinity. For sum (II), we can get a similar bound. If we split each expectation into two regions, then we again get

$$(II) = \sum_{i=1}^{k_n} \mathbb{E} \left[ g \left( \frac{Z_{n,i}}{s_n} \right) \right] \leq k\varepsilon + \frac{k}{s_n^2} \sum_{i=1}^{k_n} \int_{\{Z_{n,i} > \varepsilon s_n\}} |Z_{n,i}|^2 d\mathbb{P}.$$

Note that it thus suffices to prove that  $\{Z_{n,i}\}$  satisfy Lindeberg's condition. This is because if  $\{Z_{n,i}\}$  satisfies Lindeberg's condition, then sum (II) will go to zero by the same argument we used for (I).

We have only assumed that Lindeberg's condition holds of  $\{X_{n,i}\}$  but we can prove this implies Lindeberg's condition holds for  $\{Z_{n,i}\}$ . Note that for all  $\varepsilon' > 0$  we have

$$\begin{aligned} \frac{\sigma_{n,i}^2}{s_n^2} &= \frac{1}{s_n^2} \int |X_{n,i}|^2 d\mathbb{P} \\ &= \frac{1}{s_n^2} \int_{\{|X_{n,i}| \leq \varepsilon' s_n\}} |X_{n,i}|^2 d\mathbb{P} + \frac{1}{s_n^2} \int_{\{|X_{n,i}| > \varepsilon' s_n\}} |X_{n,i}|^2 d\mathbb{P} \\ &\leq (\varepsilon')^2 + \frac{1}{s_n^2} \int_{\{|X_{n,i}| > \varepsilon' s_n\}} |X_{n,i}|^2 d\mathbb{P}. \end{aligned}$$

Thus, by Lindeberg's condition on  $\{X_{n,i}\}$ , we have

$$\lim_n \max_{1 \leq i \leq k_n} \left\{ \frac{\sigma_{n,i}^2}{s_n^2} \right\} \leq \varepsilon'.$$

It follows that  $\max_{1 \leq i \leq k_n} \left\{ \frac{\sigma_{n,i}}{s_n} \right\}$  goes to zero as  $n$  goes to infinity. Let  $Z \sim \mathcal{N}(0, 1)$ . To show that  $\{Z_{n,i}\}$  satisfy Lindeberg's condition let  $\varepsilon > 0$  be given. We then have

$$\begin{aligned} \frac{1}{s_n^2} \sum_{i=1}^{k_n} \int_{\{Z_{n,i} > \varepsilon s_n\}} |Z_{n,i}|^2 d\mathbb{P} &= \frac{1}{s_n^2} \sum_{i=1}^{k_n} \int_{\{\sigma_{n,i}|Z| > \varepsilon s_n\}} |\sigma_{n,i}Z|^2 d\mathbb{P} \\ &\leq \frac{1}{s_n^2} \sum_{i=1}^{k_n} \int_{\{\sigma_{n,i}|Z| > \varepsilon s_n\}} |\sigma_{n,i}Z|^2 \frac{\sigma_{n,i}|Z|}{\varepsilon s_n} d\mathbb{P} \\ &\leq \frac{\mathbb{E}[|Z|^3]}{\varepsilon s_n^3} \sum_{i=1}^{k_n} \sigma_{n,i}^3 \\ &\leq \frac{\mathbb{E}[|Z|^3]}{\varepsilon s_n^3} \max_{1 \leq i \leq k_n} \{\sigma_{n,i}\} \sum_{i=1}^{k_n} \sigma_{n,i}^2 \\ &= \frac{\mathbb{E}[|Z|^3]}{\varepsilon} \max_{1 \leq i \leq k_n} \left\{ \frac{\sigma_{n,i}}{s_n} \right\} \\ &\rightarrow 0. \end{aligned}$$

Thus  $\{X_{n,i}\}$  satisfies Lindeberg's condition and we are done.  $\square$

### 3 Comments

#### 3.1 The main idea and generalizations

In some sense our proof was elementary but the idea is *very general*. Any random variable  $Z$  can be expressed as

$$Y = U(X_1, \dots, X_n),$$

where  $X_i$  are independent and  $U$  is a function. If  $U$  is smooth and we have bounds on the derivatives of  $U$ , then  $Y$  will be close to

$$U(Z_1, \dots, Z_n),$$

where  $Z_1, \dots, Z_n$  are independent normal. Sourav Chatterjee writes about this in “[A generalization of Lindeberg’s principle](#)” in the Annals of Probability. The only property of the normal distribution that we really used was that normals have finite third moment and that sums of independent normals are normal.

### 3.2 Comments on the theorem and proof

There is a converse to Lindeberg’s theorem (Lindeberg-Feller). Suppose  $s_n \rightarrow \infty$  and  $\frac{\sigma_{n,i}}{s_n} \rightarrow 0$ . Under this assumption, if  $\frac{S_n}{s_n}$  converges weakly to a normal random variable, then Lindeberg’s condition holds. This is proved in the textbook.

Our version of the central limit theorem is a limit theorem. It does not have an error bound or tell us anything about finite  $n$ . It is possible to get explicit error bounds and Lindeberg did do this. For notation, define

$$S(x) = \begin{cases} x^3 & \text{if } |x| \leq 1, \\ x^2 & \text{if } |x| \geq 1. \end{cases}$$

There exists a constant  $C > 0$  such that for all  $X_1, X_2, \dots$  independent with mean 0 and variance  $\sigma_i^2$ , then

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{S_n}{s_n} \leq x \right) - \Phi(x) \right| \leq C \left( \sum_{i=1}^n l_i \right)^{1/4},$$

where  $l_i = \mathbb{E} \left[ S \left( \frac{X_i}{s_n} \right) \right]$ . See S.D. Chatterji “[Lindeberg’s central limit theorem à la Hausdorff](#).”

The “right” convergence rate was proved in the Berry-Essen theorem.

**Theorem 2.** *There exists a constant  $C \in (0.4097, 0.4748)$  such that if  $X_i$  are i.i.d. with mean  $\mu$ , variance  $\sigma^2$  and finite third moment, then*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left( \frac{S_n}{s_n} \leq x \right) - \Phi(x) \right| \leq \frac{C \mathbb{E}[|X_1|^3]}{\sigma^3 \sqrt{n}},$$

where  $S_n = \sum_{i=1}^n (X_i - \mu)$  and  $s_n^2 = \text{Var}(S_n) = n\sigma^2$ .

Thus when  $X_i$  has a third moment, the convergence is at a rate of  $\frac{1}{\sqrt{n}}$ .

### 3.3 The normal heuristic

In our central limit theorem the rows were independent. In many cases if  $X_{n,i}$  are not too wild and not too dependent, then  $\frac{S_n}{s_n} \Rightarrow \mathcal{N}(0, 1)$ . This is the normal heuristic and multiple examples were discussed previously.

### 3.4 Different proof techniques

There are many ways to prove the central limit theorem. For example

- (a) Lindeberg coupling (our proof).
- (b) Stein’s method (showing  $\mathbb{E}[W f(W)] \approx \mathbb{E}[f'(W)]$  where  $W = \frac{S_n}{s_n}$ ).
- (c) The method of moments (Laplace’s proof). We need the result:

**Theorem 3.** *If  $Q_n$  are a sequence of probabilities on  $\mathbb{R}$  and*

$$\int x^j Q_n(dx) \rightarrow \mathbb{E}[Z^j],$$

*for  $j = 1, 2, 3, \dots$ , then  $Q_n((-\infty, x]) \rightarrow \Phi(x)$  for all  $x$ .*

These ideas are used in physics.

- (d) Fourier analysis and characteristic functions (we will discuss these ideas next week).
- (e) Entropy. On  $\mathbb{R}$  the distribution with a fixed variance  $\sigma^2$  and the largest entropy is  $\mathcal{N}(0, \sigma^2)$  where the entropy of a distribution  $Q$  with density  $q$  is defined to be

$$\mathbb{E}_Q[-\log(q(X))].$$

Convoluting two distributions increases entropy. Thus  $\frac{S_n}{s_n}$  has increasing entropy and fixed variance and so it should be approaching the normal distribution. Turning this idea into a proof is Linnik's argument.