STATS310A - Lecture 4

Persi Diaconis
Scribed by Michael Howes

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1 The $\pi - \lambda$ Theorem

Let $(\Omega, \mathcal{F})$ be a measure space. Recall that a collection of sets $\mathcal{P}$ is a π-system if $\mathcal{P}$ is closed under finite intersection. A collection of sets L is a λ-system if $\Omega \in L$, L is closed under complements and L is closed under countable *disjoint* unions.

Theorem 1. *If $\mathcal{P}$ is a π-system, L is a λ-systam and $\mathcal{P} \subseteq L$, then $\sigma(\mathcal{P}) \subseteq L$.*

Example 1. Consider the following $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{A : A \subseteq \Omega\}$. Define two probabilities $P_1, P_2 : \mathcal{F} \to [0, 1]$ by $P_1(\omega) = \frac{1}{4}$ for all $\omega \in \Omega$ and $P_2(\omega) = \frac{1}{2}$ if $\omega = 2, 4$ and $P_2(\omega) = 0$ otherwise. The collection $\mathcal{G} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}\}$ is λ-system but not a σ-algebra (or even an algebra). The two measures P_1 and P_2 agree on $\mathcal{G}$ but not on $\sigma(\mathcal{G}) = \mathcal{F}$. Note that the collection

$$L = \{A \in \mathcal{F} : P_1(A) = P_2(A)\}$$

is always a λ-system for any probabilities P_1, P_2. The $\pi - \lambda$ theorem lets us conclude that L contains $\sigma(\mathcal{P})$ if P_1 and P_2 agree on $\mathcal{P}$ and $\mathcal{P}$ is a π-system.

Note the following facts.

Fact 0: If L is a λ system and $B_1, B_2 \in L$ with $B_1 \subseteq B_2$, then $B_2 \setminus B_1 \in L$. This is because $B_2 \setminus B_1 = B_2 \cap B_1^c = (B_2^c \cup B_1)^c$ and $B_2^c \cap B_1 = \emptyset$ since $B_1 \subseteq B_2$. Thus L is closed under relative complements as well as complements.

Fact 1: If L is both a π-system and a λ-system, then L is a σ-algebra. To see why this is, consider $(A_i)_{i=1}^\infty$ a countable collection of sets in L. Iteratively define $A_1' = A_1$ and $A_i' = A_i \cap (A_1' \cup \ldots \cup A_{i-1}')$. Since L is both a π-system and a λ-system, $A_i' \in L$. Also $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty A_i' \in L$ since the sets $(A_i')_{i=1}^\infty$ are disjoint and L is a λ-system.

We will now prove the $\pi - \lambda$ theorem.

Proof. Given $\mathcal{P}$ a π-system and L a λ-system with $\mathcal{P} \subseteq L$, define L_0 to be the λ-system generated by $\mathcal{P}$. We will show that L_0 is a π-system which by fact 1 will imply that L_0 is a σ-algebra and hence $\sigma(\mathcal{P}) = L_0 \subseteq L$.

For each $A \in L_0$, define $L_A = \{B : B \cap A \in L_0\}$. We will chose that L_A is a λ-system. Note first that $\Omega \in L_A$ since $A \in L_0$. Also if $B \in L_A$, then $A \cap B \in L_0$ and $A \cap B \subseteq A$. Thus since L_0

is a λ-system and closed under relative complements we have $A \setminus (A \cap B) = A \cap B^c$ is in L_0. Thus $B^c \in L_A$. Finally if $(B_i)_{i=1}^{\infty}$ is a countable collection of disjoint element of L_A, then $(A \cap B_i)_{i=1}^{\infty}$ is a countable collection of disjoint elements of L_0. Thus $\bigcup_{i=1}^{\infty} A \cap B_i = A \cap (\bigcup_{i=1}^{\infty} B_i) \in L_0$ and so $\bigcup_{i=1}^{\infty} B_i \in L_A$.

If $A \in \mathcal{P}$ and $B \in \mathcal{P}$, then $A \cap B \in \mathcal{P} \subseteq L_0$. Thus $B \in L_A$. Since B was arbitrary, this means that L_A is a λ-system that contains $\mathcal{P}$ and hence $L_0 \subseteq L_A$ for every $A \in \mathcal{P}$.

Now if $A \in L_0$ and $B \in \mathcal{P}$, then $L_0 \subseteq L_B$ and hence $B \cap A \in L_0$ which implies $B \in L_A$. Thus L_A is a λ-system that contains $\mathcal{P}$ and hence $L_0 \subseteq L_A$. But this means that for all $A, B \in L_0$, $A \cap B \in L_0$, thus L_0 is a λ-system. $\square$

2 Independence

Definition 1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{C}_i \subseteq \mathcal{F}$ for each $i \in I$ (where I is any index set). The collection $\{\mathcal{C}_i\}_{i \in I}$ are said to be *independent* if for all finite subsets $J \subseteq I$ and every choice of $A_j \in \mathcal{C}_j$ for $j \in J$, we have

$$P\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P(A_j).$$

Theorem 2. *Suppose that $\{\mathcal{C}_i\}_{i \in I}$ are independent π-systems, then $\{\sigma(\mathcal{C}_i)\}_{i \in I}$ are independent.*

Proof. Since the definition of independence only use finite subsets of I we may assume that $I = \{1, \ldots, n\}$ for some $n \in \mathbb{N}$. Define $\mathcal{B}_i = \mathcal{C}_i \cup \{\Omega\}$, the collections $\mathcal{B}_i$ are still π-systems and $\sigma(\mathcal{C}_i) = \sigma(\mathcal{B}_i)$ so we will work with $\mathcal{B}_i$. Define

$$L = \left\{ B_1 \in \sigma(\mathcal{B}_i) : \prod_{i=1}^{n} P(B_i) = P\left(\cap_{i=1}^{n} B_i\right) \text{ for all } B_2 \in \mathcal{B}_2, \ldots, B_n \in \mathcal{B}_n \right\}.$$

We will show that L is a λ-system. By the $\pi - \lambda$ theorem we will be able to conclude that $\sigma(\mathcal{B}_1) \subseteq L$. First note that $\Omega \in L$ since $\{\mathcal{B}_i\}_{i=2}^{n}$ are independent. Suppose $B_1 \in L$, then

$$P(B_2 \cap \ldots \cap B_n) = P(B_1 \cap B_2 \cap \ldots \cap B_n) + P(B_1^c \cap B_2 \cap \ldots \cap B_n).$$

Rearranging we have

$$P(B_1^c \cap B_2 \cap \ldots \cap B_n) = P(B_2 \cap \ldots \cap B_n) - P(B_1 \cap B_2 \cap \ldots \cap B_n).$$

By our assumption that $\{\mathcal{B}_i\}_{i=2}^{n}$ are independent and that $B_1 \in L$, we have

$$\begin{aligned} P(B_1^c \cap B_2 \cap \ldots \cap B_n) &= P(B_2 \cap \ldots \cap B_n) - P(B_1 \cap B_2 \cap \ldots \cap B_n) \\ &= \prod_{i=2}^{n} P(B_i) - \prod_{i=1}^{n} P(B_i) \\ &= (1 - P(B_1)) \prod_{i=2}^{n} P(B_i) \\ &= P(B_1^c) \prod_{i=2}^{n} P(B_i). \end{aligned}$$

and thus $B_1^c \in L$. Finally if $(B_{1,j})_{j=1}^\infty$ are in L and disjoint, then set $B_1 = \bigcup_{j=1}^\infty B_{1,j}$ and note that

$$\begin{aligned}
P(B_1 \cap B_2 \cap \dots B_n) &= \sum_{j=1}^\infty P(B_{1,j} \cap B_2 \cap \dots \cap B_n) \\
&= \sum_{j=1}^\infty P(B_{1,j}) \prod_{i=2}^n P(B_i) \\
&= \left(\sum_{j=1}^\infty P(B_{1,j}) \right) \prod_{i=2}^n P(B_i) \\
&= \prod_{i=1}^n P(B_i).
\end{aligned}$$

Thus $B_1 \in L$, L is a λ-system and $\sigma(\mathcal{B}_1) = L$. We thus have that $\{\sigma(\mathcal{B}_1), \mathcal{B}_2, \dots, \mathcal{B}_n\}$ is an independent collection. We can repeat this argument with

$$L_2 = \left\{ B_2 \in \sigma(\mathcal{B}_2) : P\left(\bigcap_{i=1}^n B_2 \right) = \prod_{i=1}^n P(B_i) \text{ for all } B_1 \in \sigma(\mathcal{B}_1) \text{ and } B_i \in \mathcal{B}_i, i = 3, \dots, n \right\}.$$

We will then conclude that $\sigma(\mathcal{B}_2) = L$. By induction we will prove that $\{\sigma(\mathcal{B}_i)\}_{i=1}^n$ is an independent collection. □

Example 2. Consider our coin tossing example $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}, \lambda)$ where $\mathcal{B}$ is the collection of Borel sets and λ is Lesbegue measure. The digits d_i are independent in that if we set

$$A_i = \{\Omega, \emptyset, \{\omega : d_i(\omega) = 1\}, \{\omega : d_i(\omega) = 0\}\},$$

then $\{A_i\}_{i=1}^\infty$ are independent. The collections $\bigcup_{i=1}^\infty A_{2i}$ and $\bigcup_{i=1}^\infty A_{2i-1}$ are also independent and thus from ω we can get two uniform $(0,1]$ samples

$$\omega_1 = \sum_{i=1}^\infty 2^{-i} d_{2i}(\omega) \text{ and } \omega_2 = \sum_{i=1}^\infty 2^{-i} d_{2i-1}(\omega).$$

We can repeat this infinitely often to get a countable sequence of indpendent uniform $(0,1]$ random variables. We could use this to get any other iid sequence of real valued random variables.

3 Borel-Cantelli Lemmas

Let $(\Omega, \mathcal{F}, P)$ be a probability space. For $(A_i)_{i=1}^\infty$ in $\mathcal{F}$ define

$$A_i \ i.o. = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m.$$

Thus $\omega \in A_i$ *i.o.* if and only if $\omega \in A_i$ for infinitely many i. That is ω is in A_i *infinitely often (i.o.)*.

Theorem 3. [Borel-Cantelli 1] *If* $\sum_{i=1}^\infty P(A_i) < \infty$*, then* $P(A_i \ i.o.) = 0$.

Proof. For every $n \in \mathbb{N}$, A_i *i.o.* $\subseteq \bigcup_{m=n}^\infty A_m$ and thus by countable subadditivity,

$$P(A_i \ i.o.) \le \sum_{m=n}^\infty P(A_m) \underset{n\to\infty}{\longrightarrow} 0.$$ □

Example 3. Let $(\Omega, \mathcal{F}, P) = ((0,1], \mathcal{B}, \lambda)$, let $l_n(\omega)$ be the length of the heads run starting at n. That is

$$\{\omega : l_n(\omega) = k\} = \{\omega : d_n(\omega) = d_{n+1}(\omega) = \ldots = d_{n+k-1}(\omega) = 1 \text{ and } d_{n+k}(\omega) = 0\}.$$

Thus $P(\{l_n = k\}) = \frac{1}{2^{k+1}}$. Let r_n be a sequence of positive integers and let $A_n = \{\omega : l_n(\omega) \geq r_n\}$. Then

$$P(A_n) = \sum_{k=0}^{\infty} \frac{1}{2^{r_n+k+1}} = \frac{1}{2^{r_n}}.$$

By Borel-Cantelli, if $\sum_{n=1}^{\infty} \frac{1}{2^{r_n}} < \infty$, then $P(A_n \; i.o) = 0$. For example this is the case if $r_n = (1+\varepsilon)\log_2(n)$.

Theorem 4. [Borel-Cantelli 2] *If* $\sum_{i=1}^{\infty} P(A_i) = \infty$ *and* $\{A_i\}_{i=1}^{\infty}$ *are independent, then*

$$P(A_i \; i.o.) = 1.$$

Proof. Note that $(A_i \; i.o.)^c = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c$. Thus it suffices to show that for each $n \in \mathbb{N}$, we have $P\left(\bigcap_{m=n}^{\infty} A_m^c\right) = 0$. Note that for all n and N, we have

$$\begin{aligned} P\left(\bigcap_{m=n}^{\infty} A_m^c\right) &\leq P\left(\bigcap_{m=n}^{N} A_m^c\right) \\ &= \prod_{m=n}^{N} P(A_m^c) \\ &= \prod_{m=n}^{N} (1 - P(A_m)) \\ &\leq \prod_{m=n}^{N} \exp(P(A_m)) \\ &= \exp\left(\sum_{m=n}^{N} P(A_m)\right) \\ &\underset{N\to\infty}{\longrightarrow} 0. \end{aligned}$$

Thus $P((A_i \; i.o.)^c) = 0$. □

Example 4. As before let l_n be the length of the heads run at n. For a sequnce of real number x_i, we have $\limsup_i x_i = l$ if and only if for all $\varepsilon > 0$, we have

(a) For infinitely many n, $x_n \geq l - \varepsilon$, and

(b) For sufficiently large n, $x_n \leq l + \varepsilon$.

That is $\limsup_i x_i$ is the largest limit point of the sequence x_i. Using B.C. one and two, it can be shown that

$$\limsup_n \frac{l_n(\omega)}{\log_2(n)} = 1,$$

with probability 1. We have already seen from B.C. 1, that with probability 1, for every ε, $l_n(\omega) \leq (1+\varepsilon)\log_2(n)$ for all but finitely many n.