STATS300B – Lecture 6

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Contents

1 The delta method

We ended last class with the statement of the higher order delta method. We stated the following.

Theorem 1 (Delta method 3 (higher order))**.** *Suppose that X_n are random k-vectors such that*

$$r_n(X_n - \theta) \xrightarrow{d} X,$$

where r_n is a deterministic function with $r_n \to +\infty$. Let ϕ be a real-valued function that is twice differentiable at θ with $\phi'(\theta) = 0$. Then,

$$r_n^2(\phi(X_n) - \phi(\theta)) \xrightarrow{d} \frac{1}{2}X^T\nabla^2\phi(\theta)T.$$

We will now prove the above.

Proof. Note that,

$$\begin{aligned}\phi(X_n) &= \phi(\theta) + \nabla\phi(\theta)^T(X_n - \theta) + (X_n - \theta)^T\nabla^2\phi(\theta)(T_n - \theta) + o(\|X_n - \theta\|_2^2)\\ &= \phi(\theta) + \frac{1}{2}(X_n - \theta)^T\nabla^2\phi(\theta)(X_n - \theta) + o(\|X_n - \theta\|_2^2).\end{aligned}$$

□

Note that $o(\|X_n - \theta\|_2^2) = o_p(r_n^{-2})$. Thus, by Slutsky's

$$\begin{aligned}r_n^2\left(\phi(X_n) - \phi(\theta)\right) &= \frac{1}{2}(r_n(X_n - \theta))^T\nabla^2\phi(\theta)(r_n(X_n - \theta)) + r_n^2 o_p(r_n^{-2})\\ &\xrightarrow{d} \frac{1}{2}X^T\nabla^2\phi(\theta)X.\end{aligned}$$

The central limit theorem gives a special one-dimensional case of the higher-order delta method. If $X_1, \dots$ are i.i.d. with mean μ and variance $\sigma^2 < \infty$, then $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} \mathsf{N}(0, \sigma^2)$. Thus, if g is a twice differentiable at μ, then

$$n(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g''(\mu)\sigma^2 Z^2,$$

where $Z \sim \mathsf{N}(0,1)$. So that, $\frac{n}{g''(\mu)\sigma^2}(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} \chi^2_1$.

2 Maximum likelihood estimation

Suppose we have data X in the form of n i.i.d. observations $X_1, \dots, X_n$ drawn from a distribution $\mathbb{P}_\theta$ with density p_θ. The likelihood $L(\theta) = p_\theta(X)$ is the density evaluated at X viewed as a function of θ. The value $\widehat{\theta}$ that maximizes $L(\theta)$ is called the maximum likelihood estimator (MLE) of θ. Much of this course will focus on properties of MLEs.

2.1 MLEs for one-dimensional exponential families

Suppose that X is generated from a one dimensional exponential family in canonical form. That is,

$$p_\eta(x) = \exp\{\eta T(x) - A(\eta)\} h(x),$$

where η is the natural parameter for the family, $T(X)$ are the sufficient statistics for the family and $A(\eta) = \log\left(\int \exp\{\eta T(x)\} h(x)dx\right)$ is the log-partition function for the family. The log-partition function is convex and differentiable with,

$$A'(\eta) = \mathbb{E}_\eta[T(X)],$$

and

$$A''(\eta) = \mathrm{Var}_\eta(T(X)).$$

Another name of the $A(\eta)$ is the *cummulant generating function of* $T(X)$. Suppose now that we have an i.i.d. sample of size n drawn from p_η. Let $l(\eta) = \log(p_\eta(X_1, \dots, X_n))$ be the log-likelihood function. Then,

$$\begin{aligned}
(l(\eta) &= \eta \sum_{i=1}^n T(X_i) - nA(\eta) + \log\left(\prod_{i=1}^n h(X_i)\right) \\
\therefore l'(\eta) &= \sum_{i=1}^n T(X_i) - nA'(\eta) = \sum_{i=1}^n T(X_i) - n\mathbb{E}_\eta[T(X)] \\
\therefore l''(\eta) &= -nA''(\eta) = -n\,\mathrm{Var}_\eta(T(X)) \le 0.
\end{aligned}$$

Thus, any solution to $l'(\eta) = 0$ is a local maximum of l. It follows that the MLE $\widehat{\eta}$ solves the equation,

$$\mathbb{E}_{\widehat{\eta}}[T(X)] = \frac{1}{n}\sum_{i=1}^n T(X_i) = \bar{T}_n$$

Thus, the MLE is a type of method of moments estimator in this example. In particular, we have $\widehat{\eta} = \phi(\bar{T})$ where ϕ can be thought of as $(A')^{-1}$. By the central limit theorem,

$$\sqrt{n}(\bar{T} - \mathbb{E}_\eta[T(X)]) \xrightarrow{d} \mathsf{N}(0, \mathrm{Var}_\eta(T(X))) = \mathsf{N}(0, A''(\eta)).$$

Thus, by the delta method,

$$\sqrt{n}(\widehat{\eta} - \eta) \xrightarrow{d} \mathsf{N}(0, \phi'(\eta)^2 A''(\eta)).$$

We know that $\phi'(\theta) = \frac{1}{A''(\eta)}$ and so, $\sqrt{n}(\widehat{\eta} - \eta) \xrightarrow{d} \mathsf{N}(0, A''(\eta)^{-1})$. We will see that similar results hold for distributions that are not exponential families.

2.2 Asymptotic efficiency

Recall that the Fisher information of a parameter θ is defined to be,

$$I(\eta) = \mathbb{E}_\eta[l'(\eta)^2] = -\mathbb{E}_\eta[l''(\eta)].$$

We have seen that for exponential families, $I(\eta) = -\mathbb{E}[-A''(\eta)] = A''(\eta)$. We also have the Cramer–Rao lower bound that states if $\widehat{\eta}$ is an unbiased estimator for η based on an i.i.d. sample of size n, then

$$\mathrm{Var}_\eta(\widehat{\eta}) \geq \frac{1}{nI(\eta)}.$$

For exponentially families, $\mathrm{Var}(\widehat{\eta}_{MLE}) \sim \frac{1}{nA''(\eta)} = \frac{1}{nI(\eta)}$. Thus, the MLE estimator asymptotically reaches the Cramer–Rao lower bound. Estimators with this property are called *asymptotically efficient*.

Definition 1. An estimator $\widehat{\eta}$ is *asymptotically efficient* if $\mathrm{Var}(\widehat{\eta}) \sim \frac{1}{nI(\eta)}$.

We can also compare the asymptotic efficiency of two estimators.

Definition 2. Let $\widehat{\eta}_1$ and $\widehat{\eta}_2$ be two estimators. The asymptotic relative efficiency (ARE) of $\widehat{\eta}_2$ compared to $\widehat{\eta}$ is,

$$\lim_{n\to\infty} \frac{\mathrm{Var}_\eta(\widehat{\eta}_2)}{\mathrm{Var}_\eta(\widehat{\eta}_0)}.$$

2.3 The ARE of the median

Let $X_1, X_2, \dots$ be i.i.d. with common CDF F. Let $\gamma \in (0,1)$ and let $\widetilde{\theta}_n$ be the $[\gamma n]^{th}$ order statistic for $X_1, \dots, X_n$, where $[y]$ denotes the ceiling of y. If $F(\theta) = \gamma$ and if $F'(\theta)$ exists and is strictly positive, then

$$\sqrt{n}(\widetilde{\theta}_n - \theta) \xrightarrow{d} \mathsf{N}\left(0, \frac{\gamma(1-\gamma)}{[F'(\theta)]^2}\right).$$

The idea is that the event $\sqrt{n}(\widetilde{\theta}_n - \theta) \leq a$ is equivalent to $\widetilde{\theta}_n \leq \theta + \frac{a}{\sqrt{n}}$. Which is in turn to at least $[\gamma n]$ of X_i's being less than $b = \theta + \frac{a}{\sqrt{n}}$. Thus,

$$\{\widetilde{\theta}_n \leq a\} = \left\{\sum_{i=1}^n \mathbf{1}_{\{X_i \leq b\}} \geq [\gamma n]\right\}.$$

Furthermore, the random variables $\mathbf{1}_{\{X_i \leq b\}}$ are i.i.d. and with mean $F(\theta + a/\sqrt{n})$ and variance $F(\theta + a/\sqrt{n})(1 - F(\theta + a/\sqrt{n}))$. Thus, we can apply the central limit theorem to and use the fact the CDF is differentiable at θ.

A special case is when $\gamma = 1/2$ and $\widetilde{\theta}_n$ is the median of the sample $X_1, \dots, X_n$. For a symmetric distribution we can look at the ARE of $\widetilde{\theta}_n$ compared to $\bar{X}_n$. If $X_i \sim \mathsf{N}(\theta, \sigma^2)$, then $\mathrm{Var}(\bar{X}_n) = \frac{\sigma^2}{n}$ and $\mathrm{Var}(\widetilde{\theta}_n) \sim \frac{1}{n} \cdot \frac{1}{4F'(\theta)}$. Note that,

$$F'(\theta) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2\sigma^2}(\theta - \theta)^2\right\} = \frac{1}{\sqrt{2\pi\sigma^2}}.$$

Thus, the ARE of the sample median compared to the sample mean is

$$\frac{2\pi\sigma^2}{4\sigma^2} = \frac{\pi}{2} \approx 1.577 > 1.$$

Thus, for Gaussian data, the sample mean has lower asymptotic variance than the sample median.

3 M-estimators and Z-estimators

The MLE is defined to be,

$$\widehat{\theta}_n = \operatorname*{argmax}_{\theta\in\Theta} L(\theta|X).$$

We would like to know the following about $\widehat{\theta}_n$,

1. Consistency,
2. Asymptotic distribution,
3. Optimality.

Since the MLE is defined as a maximizer it is an M-estimator. In many cases the MLE is also defined by solving the equation $l'(\widehat{\theta}) = 0$, this makes $\widehat{\theta}$ a Z-estimator as well. To prove consistency for M-estimators and Z-estimators, it helps to think of the log likelihood as a random function. We would like to say that the sample average of the log likelihood converges to the population log likelihood for all θ. Thus, we need to understand the convergence of random functions.

3.1 A weak law for random functions

Definition 3. Let K be a compact set and let $\mu : K \to \mathbb{R}$ be a continuous function. Define $\|\mu\|_\infty = \sup_{t\in K} |\mu(t)|$.

Lemma 1. *Let X be a random variable taking values in a compact set K. Let $h(t,x)$ be a function such that $h(\cdot,x)$ is a continuous function from K to $\mathbb{R}$. Define $W(t) = h(t,X)$ for $t \in K$. Thus, W is a random continuous function on K. Suppose that $\mathbb{E}\|W\|_\infty < \infty$, then*

1. *The function $\mu(t) = \mathbb{E}[W(t)]$ is continuous on K.*
2. *As $\varepsilon \searrow 0$,*

$$\sup_{t\in K} \mathbb{E}\left[\sup_{s:\|s-t\|\le\varepsilon} |W(s) - W(t)|\right] \to 0.$$

Proof. Suppose that $t_n \to t$, then $W(t_n) \to W(t)$. Furthermore, $|W(t_n)| \le \|W\|_\infty$ and $\|W\|_\infty$ is integrable. Thus, by the dominated convergence theorem,

$$\mu(t_n) = \mathbb{E}[W(t_n)] \to \mathbb{E}[W(t)] = \mu(t).$$

Thus, μ is continuous. For each x in our sample space, define

$$M_\varepsilon(t) = \sup_{\|s-t\|\le\varepsilon} |W(t) - W(s)|.$$

For fixed t, $M_\varepsilon(t) \to 0$ as $\varepsilon \to 0$ and $|M_\varepsilon(t)| \le 2\|W\|_\infty$. Thus, by the dominated convergence theorem for each $t \in K$ we have,

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{\|s-t\|\le\varepsilon} |W(t) - W(s)|\right] = 0.$$

The uniform result of the lemma is derived by the above point-wise result combined with the compactness of K □

We can use the above lemma to prove our weak law for random functions. We know from the regular weak law that $\left|\bar{W}_n(t) - \mu(t)\right| \xrightarrow{p} 0$, but we want a result that is uniform in t.

Theorem 2. *Let $X_1, X_2, \ldots$ be i.i.d. random variables and let $W_i(t) = h(t, X_i)$ where h is a function such that $h(\cdot, x) : K \to \mathbb{R}$ is a continuous function on a compact set K for all x. Suppose that $\mathbb{E}[\|W_1\|_\infty] < \infty$. Let $\mu(t) = \mathbb{E}[W_1(t)]$, then*

$$\|W_n - \mu\|_\infty \xrightarrow{p} 0,$$

as $n \to \infty$.