

# STATS310B – Lecture 9

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## 1 Uniform integrability

We ended last lecture with the statement of the following proposition,

**Proposition 1.** *Let  $\{X_n\}_{n \geq 1}$  be a sequence of random variables. The sequence  $\{X_n\}_{n \geq 0}$  is uniformly integrable if and only if the following both hold,*

1.  $\sup_n \mathbb{E}[|X_n|] < \infty$ .
2. For all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $A$  and  $n$ , if  $\mathbb{P}(A) < \delta$ , then  $\mathbb{E}[|X_n| \mathbf{1}_A] < \varepsilon$ .

Which we will now prove.

*Proof.* First suppose that  $\{X_n\}_{n \geq 1}$  is uniformly integrable. Fix any  $\varepsilon > 0$ , then there exists a  $k$  such that for all  $n$ ,

$$\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] < \varepsilon.$$

It follows that,

$$\mathbb{E}[|X_n|] = \mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| \leq k\}}] + \mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] < k + \varepsilon.$$

Thus,  $\sup_n \mathbb{E}[|X_n|] < \infty$ . Now let  $\varepsilon > 0$  be arbitrary and fix  $k$  so that for all  $n$ ,

$$\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] < \varepsilon/2.$$

Set  $\delta = \frac{\varepsilon}{2k}$  and suppose  $\mathbb{P}(A) < \delta$ . Then

$$\begin{aligned} \mathbb{E}[|X_n| \mathbf{1}_A] &= \mathbb{E}[|X_n| \mathbf{1}_{A \cap \{|X_n| > k\}}] + \mathbb{E}[|X_n| \mathbf{1}_{A \cap \{|X_n| \leq k\}}] \\ &\leq \mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] + \mathbb{E}[k \mathbf{1}_A] \\ &< \varepsilon/2 + k\mathbb{P}(A) \\ &< \varepsilon/2 + k\delta \\ &= \varepsilon. \end{aligned}$$

Now conversely suppose that the two conditions hold and note that,

$$\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] \geq \mathbb{E}[k \mathbf{1}_{\{|X_n| > k\}}] = k\mathbb{P}(|X_n| > k).$$

Thus, for every  $n$  and  $k$ ,  $\mathbb{P}(|X_n| > k) \leq \frac{\sup_m \mathbb{E}[|X_m|]}{k}$ . Thus, given  $\epsilon > 0$  choose  $\delta$  so that  $\mathbb{P}(A) < \delta$  implies that  $\mathbb{E}[|X_n| \mathbf{1}_A] < \epsilon$  for every  $n$ . If  $k > \frac{\sup_m \mathbb{E}[|X_m|]}{\delta}$ , then  $\mathbb{P}(|X_n| > k) < \epsilon$  for all  $n$  and hence

$$\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] < \epsilon. \quad \square$$

The above proposition has the following important corollary.

**Corollary 1.** *Let  $X$  be an integrable random variable. Then for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $\mathbb{P}(A) < \delta$ , then  $\mathbb{E}[X \mathbf{1}_A] < \epsilon$ .*

*Proof.* By proposition (1) it suffices to show that the sequence  $X_n = X$  is uniformly integrable. By the dominated convergence theorem

$$\lim_{k \rightarrow \infty} \mathbb{E}[|X| \mathbf{1}_{\{|X| > k\}}] = 0,$$

showing that for all  $\epsilon$  there exists a  $k$  such that  $\mathbb{E}[|X| \mathbf{1}_{\{|X| > k\}}] < \epsilon$ .  $\square$

## 2 Lévy downwards convergence theorem

With these results about uniform integrability we are ready to finish proving Lévy downwards convergence theorem. Recall the theorem's statement,

**Theorem 1** (Lévy's downwards convergence theorem). *Let  $X$  be an integrable random variable defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let  $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \dots$  be a decreasing sequence of sub- $\sigma$ -algebras of  $\mathcal{F}$ . Define  $\mathcal{F}^* = \bigcap_{n=0}^{\infty} \mathcal{F}_n$ , then*

$$\mathbb{E}(X|\mathcal{F}_n) \rightarrow \mathbb{E}(X|\mathcal{F}^*),$$

*almost surely and in  $L^1$ .*

*Proof.* In the previous lecture we use the upcrossing lemma to show that there exists an integrable random variable  $X^*$  such that if  $X_n = \mathbb{E}(X|\mathcal{F}_n)$ , then  $X_n \rightarrow X^*$  almost surely. It remains to show that  $X_n \rightarrow X^*$  in  $L^1$  and that  $X^* = \mathbb{E}(X|\mathcal{F}^*)$ . We will first show that  $\{X_n\}_{n \geq 0}$  is uniformly integrable. Thus, fix  $\epsilon > 0$  and let  $\delta$  be such that  $\mathbb{P}(A) < \delta$  implies that  $\mathbb{E}[|X| \mathbf{1}_A] < \epsilon$ . Now choose  $k$  so that  $\frac{\mathbb{E}[|X|]}{k} < \epsilon$ . Let  $A_n$  be the event  $\{\mathbb{E}(|X||\mathcal{F}_n) > k\}$ . Note that, by Markov's inequality,

$$\begin{aligned} \mathbb{P}(A_n) &\leq \frac{\mathbb{E}[\mathbb{E}(|X||\mathcal{F}_n)]}{k} \\ &= \frac{\mathbb{E}[|X|]}{k} \\ &< \delta. \end{aligned}$$

Thus,  $\mathbb{E}[|X| \mathbf{1}_{A_n}] < \epsilon$ . Furthermore, since  $A_n \in \mathcal{F}_n$  we have

$$\mathbb{E}[\mathbb{E}(|X||\mathcal{F}_n) \mathbf{1}_{A_n}] = \mathbb{E}[|X| \mathbf{1}_{A_n}] < \epsilon.$$

By Jensen's inequality  $|X_n| = |\mathbb{E}(X|\mathcal{F}_n)| \leq \mathbb{E}(|X||\mathcal{F}_n)$ . Thus,  $\mathbb{E}[|X_n| \mathbf{1}_{A_n}] < \epsilon$ . Furthermore, since  $|X_n| < \mathbb{E}(|X||\mathcal{F}_n)$ , we have  $\{|X_n| > k\} \subseteq A_n$  and so

$$\mathbb{E}[|X_n| \mathbf{1}_{\{|X_n| > k\}}] \leq \mathbb{E}[|X_n| \mathbf{1}_{A_n}] < \epsilon.$$

Thus,  $\{X_n\}_{n \geq 0}$  are uniformly integrable. Since  $X_n \rightarrow X^*$  almost surely, this implies that  $X_n \rightarrow X^*$  in  $L^1$ . It remains to show that  $X^* = \mathbb{E}(X|\mathcal{F}^*)$ . First note that  $X^*$  is  $\mathcal{F}^*$  measurable. This is because for every  $m$ ,  $X_n = \mathbb{E}(X|\mathcal{F}_n)$  is  $\mathcal{F}_m$  measurable for every  $n \geq m$ . Thus,  $X^* = \lim_{n \geq m} X_n$  is  $\mathcal{F}_m$  measurable. Since this holds for every  $m$  we must have that  $X^*$  is  $\mathcal{F}^* = \bigcap_{m=1}^{\infty} \mathcal{F}_m$  measurable. Now suppose that  $A \in \mathcal{F}^*$ . Then  $A \in \mathcal{F}_n$  for every  $n$  and hence

$$\mathbb{E}[X_n \mathbf{1}_A] = \mathbb{E}[X \mathbf{1}_A],$$

for every  $m$ . Furthermore,

$$|\mathbb{E}[X_n \mathbf{1}_A] - \mathbb{E}[X^* \mathbf{1}_A]| \leq \mathbb{E}[|X_n - X^*| \mathbf{1}_A] \leq \mathbb{E}[|X_n - X^*|].$$

Since  $X_n \rightarrow X^*$  in  $L^1$ , this implies that

$$\mathbb{E}[X^* \mathbf{1}_A] = \lim_{n \rightarrow \infty} \mathbb{E}[X_n \mathbf{1}_A] = \lim_{n \rightarrow \infty} \mathbb{E}[X \mathbf{1}_A] = \mathbb{E}[X \mathbf{1}_A].$$

Thus,  $X^* = \mathbb{E}(X|\mathcal{F}^*)$ . □

We will now discuss and later prove an application of this convergence theorem.

### 3 De Finetti's theorem

**Definition 1.** Let  $X_1, X_2, \dots$  be an infinite sequence of random variables. The sequence  $\{X_n\}_{n \geq 1}$  is *exchangeable* if for any permutation  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  that fixes all but finitely many values  $(X_{\pi(1)}, X_{\pi(2)}, \dots)$  has the same distribution as  $(X_1, X_2, \dots)$ .

More precisely, we can think of the sequence  $\{X_n\}_{n \geq 1}$  as a single random variable  $X$  taking values in  $\mathbb{R}^{\mathbb{N}}$  given by

$$X(\omega) = (X_1(\omega), X_2(\omega), \dots).$$

The distribution of  $X$  is the probability measure it induces on  $(\mathbb{R}^{\mathbb{N}}, \mathcal{F}_{\mathbb{R}^{\mathbb{N}}})$  where  $\mathcal{F}_{\mathbb{R}^{\mathbb{N}}}$  is the product  $\sigma$ -algebra of countably many copies of the Borel  $\sigma$ -algebra. Thus, the law of  $X$  is the probability measure  $\mu_X$  given by

$$\mu_X(B) = \mathbb{P}(X^{-1}(B)).$$

For a permutation  $\pi : \mathbb{N} \rightarrow \mathbb{N}$  we can define another random variable  $X_\pi = (X_{\pi(1)}, X_{\pi(2)}, \dots)$ . The sequence  $\{X_n\}_{n \geq 0}$  is exchangeable if and only if for every  $\pi$  that fixes all but finitely many  $i \in \mathbb{N}$ ,  $\mu_{X_\pi} = \mu_X$ .

For example, if  $\{X_n\}_{n \geq 0}$  is an exchangeable sequence, then

$$\mathbb{P}(X_2 + X_3 + X_5^2 > 20 \text{ and } X_2 < 5) = \mathbb{P}(X_1 + X_3 + X_{10000}^2 > 20 \text{ and } X_1 < 5).$$

De Finetti's theorem is a theorem about exchangeable sequences. A special case of de Finetti's theorem concerns  $\{0, 1\}$ -valued exchangeable sequences.

**Theorem 2** (de Finetti's theorem for coin tosses.). *Let  $\{X_n\}_{n \geq 0}$  be an exchangeable sequence of  $\{0, 1\}$  valued random variables. Then there exists a probability measure  $\mu$  on  $[0, 1]$  such that for every  $n$  and every choice of  $x_1, \dots, x_n \in \{0, 1\}$ ,*

$$\mathbb{P}(X_1 = x_1, \dots, X_n = x_n) = \int_0^1 \theta^{\sum_{i=1}^n x_i} (1 - \theta)^{n - \sum_{i=1}^n x_i} \mu(d\theta).$$

Furthermore, the limit  $\Theta = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i$  exists almost surely and the random variable  $\Theta$  is distributed according to  $\mu$ .

Informally, conditional on the random variable  $\Theta$ ,  $X_1, \dots, X_n$  are i.i.d. random variables with  $\mathbb{P}(X_i = 1 | \Theta = \theta) = \theta$ .

### 3.1 The exchangeable $\sigma$ -algebra

Let  $X_1, X_2, \dots$  be any sequence of random variables. Recall that the  $\sigma$ -algebra  $\mathcal{G} = \sigma(X_1, X_2, \dots)$  consists of all set of the form  $X^{-1}(B)$  where  $X = (X_1, X_2, \dots)$  and  $B$  is a measurable subset of  $\mathbb{R}^{\mathbb{N}}$ .

**Definition 2.** A set  $B \subseteq \mathbb{R}^{\mathbb{N}}$  is *invariant under permutations of the first  $n$  coordinates* if for all permutations  $\pi$  of  $\{1, \dots, n\}$ , for all  $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$ , if  $x \in B$ , then  $(x_{\pi(1)}, \dots, x_{\pi(n)}, x_{n+1}, \dots) \in B$ .

For example  $\{x_1 + x_2 + x_3 > 5\}$  is invariant under permutations of the first 3 coordinates. The set  $\{x_1 + x_2 + x_3^2 > 5\}$  is invariant under permutations of the first 2 coordinates but not invariant under permutations of the first 2 coordinates.

**Definition 3.** Let  $\{X_n\}_{n \geq 0}$  be a sequence of random variables and let  $\mathcal{G} = \sigma(X_1, X_2, \dots)$ . Define  $\mathcal{E}_n$  be the  $\sigma$ -algebra of all  $A \in \mathcal{G}$  such that  $A = X^{-1}(B)$  for some  $B \subseteq \mathbb{R}^{\mathbb{N}}$  that is invariant under permutations of the first  $n$  coordinates. Also define  $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$ . The  $\sigma$ -algebra  $\mathcal{E}$  is called the *exchangeable  $\sigma$ -algebra* of  $\{X_n\}_{n \geq 0}$ .

We are now ready to state the general version of de Finetti's theorem.

**Theorem 3** (de Finetti's theorem). *Let  $\{X_n\}_{n \geq 0}$  be an exchangeable sequence of random variables and let  $\mathcal{E}$  be the exchangeable  $\sigma$ -algebra of  $\{X_n\}_{n \geq 0}$ . Then  $\{X_n\}_{n \geq 0}$  are i.i.d. given  $\mathcal{E}$  meaning that for every  $n$  and every choice of Borel sets  $A_1, \dots, A_n$ , we have*

$$\begin{aligned} \mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n | \mathcal{E}) &= \prod_{i=1}^n \mathbb{P}(X_i \in A_i | \mathcal{E}) \\ &= \prod_{i=1}^n \mathbb{P}(X_1 \in A_i | \mathcal{E}). \end{aligned}$$